Isoperimetric Inequalities in a Boundary Value Problem in an Unbounded Domain

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$ and Ω^* the exterior of Ω . We consider the boundary value problem

$$\left. \begin{aligned} \Delta u &= \gamma^2 f(u) && \text{in } \Omega^* \\ u &= 1 && \text{on } \partial\Omega \\ u &\rightarrow 0 \text{ as } |x| \rightarrow \infty \end{aligned} \right\}. \quad (1.1)$$

Here x is a generic point of $\mathbb{R}^N$, γ is a given positive parameter and the nonlinearity is assumed to satisfy

$$f(0) = 0, \quad f'(0) > 0, \quad f''(u) \geq 0 \quad \text{for } u \geq 0. \quad (1.2)$$

The assumptions on $f(u)$ guarantee that the solution decays to zero when $|x| \rightarrow \infty$ at least as fast as the solution of the linearised problem. Two possible backgrounds for problem (1.1) are described in the following:

(a) *Poisson-Boltzmann problem:*

With this application in mind u represents the potential of a charge distribution. Important choices for $f(u)$ are then

$$f(u) = \sin h u \quad \text{or} \quad f(u) = u \quad (\text{Debye-Hückel approximation}).$$

The parameter γ then involves a number of physical constants (see e.g. Garrett & Poladian [2]).

(b) *Reaction-Diffusion:*

In this interpretation u is the concentration of a reactant. The reactant is fed into Ω and diffuses through $\partial\Omega$ into the reaction region Ω^* where a simultaneous diffusion-reaction process takes place. The concentration inside Ω is held constant by a continuous supply of reactant. An important case is again $f(u) = u$ (e.g. for a first order degradation process). Another frequent choice is (Michaelis-Menten kinetics)

$$f(u) = \frac{u}{a + u}, \quad \text{where } a \text{ is a positive constant.}$$

Especially with the second interpretation of problem (1.1) in mind one is mostly not interested in the values $u(x)$, but rather in some functional of the solution. If (1.1) describes a diffusion reaction process then one has the relation

$$\gamma^2 = \frac{k}{D},$$

where k is the reaction rate and D the diffusion coefficient. Very often then (see Aris [1]) one is interested in a pure number characterizing the influence of diffusion in the chemical reaction. One such possible number may be defined as

$$\eta := \frac{\oint_{\partial\Omega} |\nabla u| d\sigma}{\gamma^2 |\Omega|}. \quad (1.3)$$

Here $d\sigma$ is the surface element of $\partial\Omega$ and $|\Omega|$ the volume of Ω . We will concentrate mainly on optimal estimates for η in terms of geometrical data of Ω .

Note that for $\gamma = 0$ and the condition $u(x) = O(|x|^{2-N})$, for $|x| \rightarrow \infty$, $N \geq 3$, (1.1) is the classical electrostatic problem and $\oint_{\partial\Omega} |\nabla u| ds$ is up to normalization the electrostatic capacity of Ω .

2. Bounds derived from optimal sub- or supersolutions

Suppose we can find a function $\underline{u}(x)$ with the properties (subsolution)

$$\left. \begin{aligned} \Delta \underline{u} &\geq \gamma^2 f(\underline{u}) \text{ in } \Omega^* \\ \underline{u} &\leq 1 \text{ on } \partial\Omega, \underline{u}(x) \rightarrow 0 \text{ for } |x| \rightarrow \infty \end{aligned} \right\} \quad (2.1)$$

for the limiting case $\gamma = 0$ we have to replace the boundary condition at infinity by

$$\underline{u}(x) = O(|x|^{2-N}) \cdot |x|, \quad N \geq 3 \quad (N = \text{Number of dimensions}).$$

Then one has $\underline{u}(x) \leq u(x)$ for any $x \in \Omega^*$. For a supersolution $\bar{u}(x)$ one just has to reverse all inequality signs.

In the following we construct an optimal subsolution in two steps. First we define $\varphi(\rho)$ as the solution of problem (1.1) for the exterior of an N -ball of radius R . It is easy to see that we can take $\varphi(\rho)$ as the solution of (prime denotes $\frac{d}{d\rho}$)

$$\left. \begin{aligned} \frac{1}{\rho^{N-1}} (\rho^{N-1} \varphi')' &= (\gamma R)^2 f(\varphi) \text{ for } \rho \in (1, \infty) \\ \varphi(1) &= 1, \varphi(\rho) \rightarrow 0 \text{ for } \rho \rightarrow \infty \end{aligned} \right\}. \quad (2.2)$$

The second important ingredient is the (exterior) distance function $d(x) = \min_{y \in \partial\Omega} |x - y|$, $x \in \Omega^*$. We then set

$$s(x) = \left(1 + H_0 d(x)\right)^{2-N}, \quad N \geq 3. \quad (2.3)$$

Here

$$H_0 = \max_{y \in \partial\Omega} \left\{ \frac{1}{N-1} \sum_{j=1}^{N-1} k_j(y) \right\},$$

where $k_j(y)$ denote the principal curvatures at a point y of $\partial\Omega$. Thus H_0 is the maximum of the mean curvature, which is assumed to be finite. Note that for the N -ball $H_0 = \frac{1}{R}$. The next result will enable us to derive optimal bounds for η as defined in Eq. (1.3).

Lemma 1 *Let Ω be a convex domain, i.e. $k_j \geq 0$ for $j = 1, \dots, N-1$. Then the function $\underline{u}(x) = \tilde{\varphi}(s(x))$, $s(x)$ defined by Eq. (2.3) and where $\tilde{\varphi}(s) = \varphi(\rho)$, $\varphi(\rho)$ being the solution of (2.2) and $s = \frac{1}{\rho^{N-2}}$, is a subsolution to problem (1.1).*

In order to prove Lemma 1 it is convenient to prove first an auxiliary result stated as:

Lemma 2 *Let Ω be a convex domain. Then the function $s(x)$ defined in Eq. (2.3) satisfies $\Delta s \geq 0$ in Ω^* .*

Proof: We have

$$\Delta s = \frac{(2-N)H_0}{(1+H_0 d)^{N-1}} \Delta d + \frac{(N-2)(N-1)}{(1+H_0 d)^N} H_0^2 \cdot |\nabla d|^2 .$$

But one has (see e.g. Gibarg-Trudinger [3], p. 355)

$$\Delta d = \sum_{i=1}^{N-1} \frac{k_i}{1+k_i d} , \quad |\nabla d| = 1 ,$$

so that

$$\Delta s = \frac{(N-2)H_0}{(1+H_0 d)^N} \{ (N-1)H_0 - (1+H_0 d) \Delta d \} .$$

We now rewrite the term $\{ (N-1)H_0 - (1+H_0 d) \Delta d \}$ putting the expression for Δd over a common denominator. Then we may write

$$\Delta d = \frac{A(d)}{B(d)} ,$$

with

$$A(d) = \sum_{j=0}^{N-1} j \cdot P_j d^{j-1} , \quad B(d) = \sum_{j=0}^{N-1} P_j \cdot d^j .$$

Here we have used the abbreviation P_j , where $P_0 \equiv 1$ and P_j = sum of all products with j different factors k_ℓ and $1 \leq j, \ell \leq N-1$.

With this notation we can write

$$\Delta s = \frac{(N-2)H_0}{(1+H_0 d)^N B(d)} \left\{ \underbrace{(N-1)H_0 B(d) - (1+H_0 d) A(d)}_{f(d)} \right\} .$$

The expression $f(d)$ can be put into the form

$$f(d) = \sum_{\ell=0}^{N-1} C_\ell \cdot d^\ell ,$$

where

$$C_\ell = H_0 P_\ell \left(1 - \frac{\ell}{N-1} \right) - (\ell+1) P_{\ell+1} .$$

We finally show that $C_\ell \geq 0$ for all ℓ . It is clear for $\ell = 0$ and $\ell = N-1$, noting that by definition

$$P_1 = \sum_{j=1}^{N-1} k_j \leq (N-1) H_0 .$$

We consider C_ℓ as a function of the $N - 1$ variables k_j . Since C_ℓ is symmetric in all variables the minimum is attained for values k_j such that

$$k_1 = k_2 = \dots = k_{N-1} = k^0 .$$

But

$$P_\ell(k^0, \dots, k^0) = \binom{N-1}{\ell} (k^0)^\ell ,$$

and hence

$$C_\ell(k, \dots, k_{N-1}) \geq C_\ell(k^0, \dots, k^0) = 0 ,$$

which implies that $\Delta s \geq 0$.

Remarks on Lemma 2:

(a) For the special case $\gamma = 0$ Lemma 2 shows that

$$s(x) = (1 + H_0 d(x))^{2-N}$$

is a subsolution in the “capacity problem”, i.e.

$$u(x) \geq (1 + H_0 d(x))^{2-N}, \quad N \geq 3 , \quad (2.4)$$

and the equality sign holds in (2.4) if Ω is the N -ball of radius $R = \frac{1}{H_0}$. Inequality (2.4) was derived by Payne & Philippin [4] by a different method.

(b) The convexity assumption is needed in order to ensure that Δd is finite for all $x \in \Omega^*$. Inequality (2.4) however holds without the convexity assumption, as shown by Payne & Philippin.

Proof of Lemma 1: We have

$$\Delta \bar{u} = \frac{d\tilde{\varphi}}{ds} \cdot \Delta s + \frac{d^2\tilde{\varphi}}{ds^2} \cdot |\nabla s|^2 ,$$

and

$$|\nabla s|^2 = \frac{((N-2)H_0)^2}{(1+H_0 d)^{2N-2}} |\nabla s|^2 = \frac{((N-2)H_0)^2}{\rho^{2N-2}} . \quad (2.5)$$

Furthermore

$$\frac{d\tilde{\varphi}}{ds} = \frac{1}{N-2} \rho^{N-1} \frac{d\varphi}{d\rho} , \quad (2.6)$$

and hence

$$\begin{aligned} \Delta \underline{u} - \gamma^2 f(\underline{u}) &\geq \frac{((N-2)H_0)^2}{\rho^{2N-2}} \frac{d^2\tilde{\varphi}}{ds^2} - \gamma^2 f(\varphi) \\ &= \frac{H_0^2}{\rho^{N-1}} \frac{d}{d\rho} \left(\rho^{N-1} \frac{d\varphi}{d\rho} \right) - \gamma^2 f(\varphi) = 0 . \end{aligned}$$

On $\partial\Omega$ we have $\rho = 1$ and thus

$$\underline{u} = \varphi(1) = 1 \text{ on } \partial\Omega$$

as well as

$$u(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$

As a consequence of Lemma 1 we have

Theorem 3 *Let Ω be a finite convex domain in $\mathbb{R}^N$, $N \geq 3$. Denote by H_0 the maximum of the mean curvature of $\partial\Omega$ and by η_0 the effectiveness of a ball of radius H_0^{-1} (as defined by Eq. (1.3)). Then the effectiveness η of Ω satisfies*

$$\eta \leq \eta_0 \frac{|\partial\Omega|}{NH_0|\Omega|} \quad (2.7)$$

and the equality sign holds if Ω is a ball.

Proof: Since the subsolution $\underline{u}$ defined in Lemma 1 satisfies the boundary conditions required in problem (1.1) we have

$$\oint_{\partial\Omega} |\nabla u| d\sigma \leq \oint_{\partial\Omega} |\nabla \underline{u}| d\sigma. \quad (2.8)$$

But from Eq. (2.5) and (2.6) we find

$$\oint_{\partial\Omega} |\nabla \underline{u}| d\sigma = -H_0 \cdot \left. \frac{d\varphi}{d\rho} \right|_{\rho=1} \cdot |\partial\Omega|. \quad (2.9)$$

From (2.2) it follows that ($R = H_0^{-1}$)

$$\eta_0 = \frac{NH_0^2}{\gamma^2} |\varphi'(1)|. \quad (2.10)$$

Eqs. (2.9) and (2.10) then imply inequality (2.7).

Remarks on Theorem 1

- (a) An important case for applications is $N = 3$ and $f(u) = u$. Then the subsolution can be written down explicitly and one has at distance d from $\partial\Omega$

$$u(d) \geq \frac{e^{-\gamma d}}{1 + H_0 d}. \quad (2.11)$$

Inequality (2.7) then takes the form

$$\eta \leq \frac{(H_0 + \gamma)|\partial\Omega|}{\gamma^2|\Omega|}. \quad (2.12)$$

- (b) We can also derive an explicit lower bound for η from Lemma 1 in the case $N = 3, f(u) = u$. From the differential equation and the boundary conditions it follows that

$$\eta = \frac{1}{|\Omega|} \int_{\Omega^*} u \, dx \geq \frac{1}{|\Omega|} \int_{\Omega^*} \underline{u} \, dx .$$

We integrate over Ω^* using parallel surfaces to $\partial\Omega$. One has for the surface area $S(\delta)$ of a parallel surface at distance δ from $\partial\Omega$ (see e.g. Polya-Szegö [5], p.66)

$$S(\delta) = S(0) + 2M \cdot \delta + 4\pi \delta^2 . \quad (2.13)$$

Here M is the Minkowski constant of $\partial\Omega$ defined by

$$M = \int_{\partial\Omega} H \, d\sigma, \quad H = \text{mean curvature} , \quad (2.14)$$

and of course $S(0) = |\partial\Omega|$.

Hence we have

$$\eta \geq \frac{1}{|\Omega|} \int_0^\infty \frac{e^{-\gamma\delta}}{1 + H_0 \delta} (|\partial\Omega| + 2M\delta + 4\pi \delta^2) \, d\sigma . \quad (2.15)$$

The evaluation of the integral gives after some routine calculation

$$\eta \geq \frac{1}{\nu} \left\{ \frac{4\pi}{\mu} + \lambda + \mu (H_0^2 |\partial\Omega| - \lambda) e^\mu \cdot E_1(\mu) \right\} , \quad (2.16)$$

where the following abbreviations have been used:

$$\nu = H_0^2 \gamma |\Omega|, \quad \mu = \frac{\gamma}{H_0}, \quad \lambda = 2M H_0 - 4\pi . \quad (2.17)$$

$E_1(\mu)$ denotes the exponential integral defined by

$$E_1(\mu) = \int_\mu^\infty \frac{e^{-t}}{t} \, dt .$$

Inequality (2.16) is again isoperimetric since the equality sign holds if Ω is a ball.

- (c) One could also derive a version of Theorem 1 valid for plane domains Ω . However then the limiting case $\gamma = 0$ does not make any sense. In addition problem (1.1) seems less interesting in the two dimensional case.

Next we derive an estimate for η which is based on the construction of an optimal supersolution $\bar{u}$. For this purpose we use as an auxiliary problem the “electrostatic problem” ($N \geq 3$)

$$\left. \begin{aligned} \Delta h &= 0 && \text{in } \Omega^* \\]h &= 1 && \text{on } \partial\Omega \\ h(x) &= O(|x|^{2-N}) && \text{as } |x| \rightarrow \infty \end{aligned} \right\} \quad (2.18)$$

In order to get an explicit inequality we restrict our attention to the special case $f(u) = u$.

Theorem 4 Let $f(u) = u$ in problem (1.1) and Ω be a finite domain with boundary of class $C^{2+\epsilon}$, but not necessarily convex. Let $\tau = \max_{\partial\Omega} |\nabla h|$, h being the solution of (2.18), and $C = w_N^{-1} \oint_{\partial\Omega} |\nabla h| d\sigma$ the “capacity” of $\partial\Omega$ ($w_N = (N-1)$ dimensional surface area of the unit N -sphere). Then we have

$$\eta \geq \left[1 + \frac{\kappa}{N-2} \frac{K_{\frac{N-4}{2}}(\kappa)}{K_{\frac{N-2}{2}}(\kappa)} \right] \frac{C \cdot w_N}{\gamma^2 |\Omega|} \quad (2.19)$$

where $\kappa = (N-2) \frac{\gamma}{\tau}$ and $K_p(\kappa)$ denote Bessel functions. The equality sign holds if Ω is the N -ball.

Proof: We use the solution of (1.1) with $f(u) = u$ and Ω the unit ball as an auxiliary function:

$$\left. \begin{aligned} \frac{1}{\rho^{N-1}} \left(\rho^{N-1} \varphi'(\rho) \right)' &= c^2 \varphi & \text{in } (1, \infty) \\ \varphi(1) &= 1, \varphi(\rho) \rightarrow 0 & \text{for } \rho \rightarrow \infty \end{aligned} \right\}. \quad (2.20)$$

The value of c will be chosen later. The solution of (2.20) is

$$\varphi(\rho) = \rho^{\frac{N-2}{2}} \cdot \frac{K_{\frac{N-2}{2}}(c\rho)}{K_{\frac{N-2}{2}}(c)}. \quad (2.21)$$

We then set $\rho = h^{-\frac{1}{N-2}}$, h being the solution of problem (2.18). It is convenient to write

$$\begin{aligned} \hat{\varphi}(h) &= \varphi(\rho) \text{ and to use the relation} \\ \frac{d}{dh} &= -\frac{1}{N-2} h^{-\frac{N-1}{N-2}} \frac{d}{d\rho} = -\frac{1}{N-2} \rho^{N-1} \frac{d}{d\rho}. \end{aligned} \quad (2.22)$$

We now choose $\bar{u}(x) = \varphi(\rho(x)) = \hat{\varphi}(h(x))$ and calculate

$$\Delta \bar{u} = \frac{d\hat{\varphi}}{dh} \cdot \Delta h + \frac{d^2 \hat{\varphi}}{dh^2} \cdot |\nabla h|^2 = \frac{d^2 \varphi}{dh^2} \cdot |\nabla h|^2. \quad (2.23)$$

At this point we use a result of Payne & Philippin [4] stating that in N dimensions the solution h of (2.18) satisfies (the smoothness of $\partial\Omega$ is used here!)

$$|\nabla h|^2 \leq \tau^2 \cdot h^{\frac{2(N-1)}{N-2}}. \quad (2.24)$$

Hence

$$\Delta \bar{u} - \gamma^2 \bar{u} \leq \tau^2 \frac{d^2 \hat{\varphi}}{dh^2} \cdot h^{\frac{N-2}{2(N-1)}} - \gamma^2 \hat{\varphi},$$

but because of Eq. (2.22) this can be put into the form

$$\Delta \bar{u} - \gamma^2 \bar{u} \leq \frac{\tau^2}{(N-2)^2} \frac{1}{\rho^{2(N-1)}} \rho^{N-1} \frac{d}{d\rho} \left(\rho^{N-1} \frac{d\varphi}{d\rho} \right) - \gamma^2 \varphi(\rho) = 0,$$

if we choose $c = \frac{(N-2)\gamma}{\tau}$ in problem (2.20). It is easy to see that $\bar{u}$ satisfies the boundary conditions. We therefore have

$$\oint_{\partial\Omega} |\nabla u| d\sigma \geq \oint_{\partial\Omega} |\nabla \bar{u}| d\sigma . \quad (2.25)$$

The relations

$$\bar{u} = \sqrt{h} \frac{K_{\frac{N-2}{2}}(\kappa h^p)}{K_{\frac{N-2}{2}}(\kappa)}, \quad \kappa = \frac{(N-2)\gamma}{\tau}, \quad p = -\frac{1}{N-2}$$

together with well known identities for Bessel functions then show after some routine calculations that inequality (2.25) leads to the statement of Theorem 2.

Remarks on Theorem 2

(a) Inequality (2.19) is still not quite explicit. It is not hard to check that the function

$$s \frac{K_{\frac{N-4}{2}}(s)}{K_{\frac{N-2}{2}}(s)}$$

is increasing for $s > 0$. Hence we need an upper bound for τ and a lower bound for the capacity C . It was shown by Payne & Philippin [4] that

$$\tau \leq (N-2) H_0 . \quad (2.26)$$

For the capacity C one has the classical result of Poincaré-Szegö (see [6]) that

$$C \geq (N-2) \left(\frac{N|\Omega|}{w_N} \right)^{\frac{N-2}{N}} . \quad (2.27)$$

Inequality (2.19) therefore implies the weaker but still isoperimetric inequality

$$\eta \geq \left(1 + \frac{\gamma}{H_0} \cdot \frac{K_{\frac{N-4}{2}}\left(\frac{\gamma}{H_0}\right)}{K_{\frac{N-2}{2}}\left(\frac{\gamma}{H_0}\right)} \right) \frac{(N-2)}{\gamma^2 \left(\frac{N|\Omega|}{w_N}\right)^{2/N}} \quad (2.28)$$

which for $N = 3$ reduces to

$$\eta \geq 3 \left(\frac{1}{\gamma^2} + \frac{1}{\gamma H_0} \right) \left(\frac{4\pi}{3|\Omega|} \right)^{2/3} . \quad (2.29)$$

(b) For $N = 3$ the supersolution $\bar{u}$ can be written as

$$\bar{u}(x) = h(x) \exp \left\{ -\frac{\gamma}{\tau} \left(\frac{1}{h(x)} - 1 \right) \right\} . \quad (2.30)$$

An explicit upper solution $\bar{h}(x)$ for $h(x)$ would then lead to an explicit upper bound for $\bar{u}(x)$. However for general domains it seems difficult to give an optimal choice of $\bar{h}(x)$.

3. Concluding Remarks

- (a) The method of integration over level surfaces (see Sperb [7]) could also be used in order to derive an optimal inequality for the “effectiveness” η . One can show then that for given N -volume of Ω η is a minimum for the N -ball. In order to keep this note at a reasonable length we mention this result without proof.

- (b) A number of additional bounds could be proven by exploiting the fact that the function

$$P := |\nabla u|^2 - 2\gamma^2 F(u) + \beta u ,$$

where $F(u) = \int_0^u f(v)dv$ and β is chosen appropriately, is non positive in Ω^* . Here techniques described in Sperb [7] can be used.

- (c) The case $N = 2$ has been excluded here since it is of less interest in applications. It is however not hard to extend Theorem 1 (but not Theorem 2) to this case. Also for the remarks (a),(b) above the two-dimensional case is no exception.

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