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On Weil restriction of reductive groups and a theorem of Prasad

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Abstract. Let G be a connected simple semisimple algebraic group over a local field F of arbitrary characteristic. In a previous article by the author the Zariski dense compact subgroups of $G(F)$ were classified. In the present paper this information is used to give another proof of a theorem of Prasad [8] (also proved by Margulis [3]) which asserts that, if G is isotropic, every non-discrete closed subgroup of finite covolume contains the image of $\tilde{G}(F)$, where $\tilde{G}$ denotes the universal covering of G . This result played a central role in Prasad's proof of strong approximation. The present proof relies on some basic properties of Weil restrictions over possibly inseparable field extensions, which are also proved here.

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1. Weil restriction of linear algebraic groups

Let F be a field and F' a subfield such that $[F/F'] < \infty$. In this section we discuss some properties of the Weil restriction $\mathcal{R}_{F/F'}G$ where G is a linear algebraic group over F . We are interested particularly in the case that F/F' is inseparable, where the Weil restriction involves some infinitesimal aspects. Thus the natural setting is that of group schemes. We assume that G is a connected affine group scheme of finite type that is smooth over F . The smoothness condition is equivalent to saying that G is reduced and “defined over F ” in the terminology of [11] Ch.11.

Throughout, we will speak of a scheme over a ring R when we really mean a scheme over $\mathbf{Spec} R$. Similarly, for any ring homomorphism $R' \rightarrow R$ and any scheme X' over R' we will abbreviate $X' \times_{R'} R := X' \times_{\mathbf{Spec} R'} \mathbf{Spec} R$. The basic facts on Weil restrictions that we need are summarized in [4] Appendix 2–3. Throughout the following we abbreviate

$$G' := \mathcal{R}_{F/F'}G.$$

By [4] A.3.2, A.3.7 this is a connected smooth affine group scheme over F' . The universal property of the Weil restriction identifies $G'(F')$ with $G(F)$.

Next, we fix an algebraic closure E' of F' and abbreviate $E := F \otimes_{F'} E'$. With $\Sigma := \text{Hom}_{F'}(F, E')$ there is then a unique decomposition $E = \bigoplus_{\sigma \in \Sigma} E_\sigma$, where each E_σ is a local ring with residue field E' and the composite map $F \rightarrow E_\sigma \twoheadrightarrow E'$ is equal to σ . The Weil restriction from any finite dimensional commutative E' -algebra down to E' is defined, and by [4] A.2.7–8 we have natural isomorphisms

$$\begin{aligned} G' \times_{F'} E' &\cong \mathcal{R}_{E/E'}(G \times_F E) \\ &= \mathcal{R}_{E/E'}\left(\bigsqcup_{\sigma \in \Sigma} G \times_F E_\sigma\right) \\ &\cong \prod_{\sigma \in \Sigma} G_\sigma \end{aligned} \quad (1.1)$$

with

$$G_\sigma := \mathcal{R}_{E_\sigma/E'}(G \times_F E_\sigma).$$

These isomorphisms are functorial in G and equivariant under $\text{Aut}(E'/F')$, which acts on the right hand side by permuting the factors according to its action on Σ . Next, for every $\sigma \in \Sigma$ we fix a filtration of E_σ by ideals

$$E_\sigma \supsetneq I_{\sigma,1} \supsetneq \cdots \supsetneq I_{\sigma,q-1} \supsetneq I_{\sigma,q} = 0$$

with subquotients of length 1. Here q is the degree of the inseparable part of F/F' . We also choose a basis of every successive subquotient. For every $1 \leq i \leq q$ there is a natural homomorphism

$$G_\sigma = \mathcal{R}_{E_\sigma/E'}(G \times_F E_\sigma) \longrightarrow \mathcal{R}_{(E_\sigma/I_{\sigma,i})/E'}(G \times_F (E_\sigma/I_{\sigma,i})).$$

Let $G_{\sigma,i}$ denote its kernel. By [4] A.3.5 we find that each $G_{\sigma,i}$ is smooth over F' and there are canonical isomorphisms

$$G_\sigma/G_{\sigma,1} \cong G \times_{F,\sigma} E' \quad (1.2)$$

and

$$G_{\sigma,i}/G_{\sigma,i+1} \cong \text{Lie } G \otimes_{F,\sigma} \mathbb{G}_{a,E'} \quad (1.3)$$

for all $1 \leq i \leq q-1$, where $\mathbb{G}_a$ denotes the additive group of dimension 1. Moreover, this description is functorial in G . Namely, let H be another smooth group scheme over F and define $H' := \mathcal{R}_{F/F'} H$, H_σ and $H_{\sigma,i}$ in the obvious way. Then any homomorphism $\varphi: H \rightarrow G$ induces homomorphisms $\mathcal{R}_{F/F'} \varphi: H' \rightarrow G'$, $H_\sigma \rightarrow G_\sigma$ and $H_{\sigma,i} \rightarrow G_{\sigma,i}$ and the resulting homomorphisms on subquotients are just

$$\varphi \times \text{id}: H \times_{F,\sigma} E' \longrightarrow G \times_{F,\sigma} E' \quad (1.4)$$

and

$$d\varphi \otimes \text{id}: \text{Lie } H \otimes_{F,\sigma} \mathbb{G}_{a,E'} \longrightarrow \text{Lie } G \otimes_{F,\sigma} \mathbb{G}_{a,E'}. \quad (1.5)$$

Recall that an *isogeny* of algebraic groups is a surjective homomorphism with finite kernel. An isogeny φ is *separable* if and only if its derivative $d\varphi$ is an isomorphism.

Proposition 1.6. *Let $\varphi: H \rightarrow G$ be a homomorphism of connected smooth linear algebraic groups over F .*

- (a) *If F/F' is separable, then $\mathcal{R}_{F/F'}\varphi: H' \rightarrow G'$ is an isogeny if and only if φ is an isogeny.*
- (b) *If F/F' is inseparable, then $\mathcal{R}_{F/F'}\varphi: H' \rightarrow G'$ is an isogeny if and only if φ is a separable isogeny.*

Proof. In the separable case we have $E' \xrightarrow{\sim} E_\sigma$, and assertion (a) follows directly from the decomposition 1.1 and the functoriality 1.4. So assume that F/F' is inseparable, i.e., that $q > 1$. First note that $\dim H' = [F/F'] \cdot \dim H$ and $\dim G' = [F/F'] \cdot \dim G$, by the successive extension above or by [4] A.3.3. Thus if either φ or $\mathcal{R}_{F/F'}\varphi$ is an isogeny, we must have $\dim H = \dim G$.

If $\mathcal{R}_{F/F'}\varphi$ is an isogeny, its kernel is finite; hence so is the kernel of its restriction $H_{\sigma,q-1} \rightarrow G_{\sigma,q-1}$. By 1.5 this means that $d\varphi$ is injective. For dimension reasons it follows that $d\varphi$ is an isomorphism; hence φ is a separable isogeny, as desired.

Conversely, suppose that φ is a separable isogeny. Then all the homomorphisms on subquotients 1.4 and 1.5 induced by $\mathcal{R}_{F/F'}\varphi$ are surjective. Using the snake lemma inductively one deduces that $\mathcal{R}_{F/F'}\varphi$ itself is surjective. For dimension reasons it is therefore an isogeny, as desired. $\square$

Theorem 1.7. *If G is reductive and F' infinite, then $G'(F')$ is Zariski dense in G' .*

Proof. If F/F' is separable, the isomorphism 1.1 shows that G' is reductive. In that case the assertion is well-known: see [11] Cor.13.3.12 (i).

We will adapt the argument to the general case.

Assume first that $G = T$ is a torus. Choose a finite separable extension F_1/F which splits T , and fix an isomorphism $\mathbb{G}_{m,F_1}^n \xrightarrow{\sim} T \times_F F_1$, where $\mathbb{G}_m$ denotes the multiplicative group of dimension 1. Combining this with the norm map yields a surjective homomorphism

$$\mathcal{R}_{F_1/F}\mathbb{G}_{m,F_1}^n \longrightarrow \mathcal{R}_{F_1/F}(T \times_F F_1) \xrightarrow{\text{Nm}} T.$$

Since F_1/F is separable, this morphism is smooth. By [4] A.2.4, A.2.12 it induces a smooth homomorphism

$$\mathcal{R}_{F_1/F'}\mathbb{G}_{m,F_1}^n \cong \mathcal{R}_{F/F'}\mathcal{R}_{F_1/F}\mathbb{G}_{m,F_1}^n \longrightarrow \mathcal{R}_{F/F'}T.$$

In particular, this morphism is dominant. On the other hand we have an open embedding $\mathbb{G}_{m,F_1}^n \hookrightarrow \mathbb{A}_{F_1}^n$ and hence, by [4] A.2.11, an open embedding $\mathcal{R}_{F_1/F'}\mathbb{G}_{m,F_1}^n \hookrightarrow \mathcal{R}_{F_1/F'}\mathbb{A}_{F_1}^n$. It is trivial to show that $\mathcal{R}_{F_1/F'}\mathbb{A}_{F_1}^n \cong \mathbb{A}_{F'}^{nd}$, where $d = [F_1/F']$. It follows that the F' -rational points in $\mathcal{R}_{F_1/F'}\mathbb{G}_{m,F_1}^n$ are Zariski dense, and so their images form a Zariski dense set of F' -rational points in $\mathcal{R}_{F/F'}T$, proving the theorem in this case.

If G is arbitrary let T be a maximal torus of G . As $\mathcal{R}_{F/F'}T$ is commutative, it possesses a unique maximal torus T' , which is smooth over F' by [11] Thm.13.3.6.

Lemma 1.8. $\mathcal{R}_{F/F'}T$ is the centralizer of T' in G' .

Proof. If F/F' is separable, this follows from the fact that $\mathcal{R}_{F/F'}T$ is a maximal torus of G' . So assume that F/F' is inseparable of characteristic p . Since $(\mathcal{R}_{F/F'}T)/T'$ is unipotent, we have $T' = (\mathcal{R}_{F/F'}T)^{p^n}$ for suitable $n \gg 0$. As T' is smooth and the rational points of $\mathcal{R}_{F/F'}T$ are Zariski dense, the centralizer of T' is equal to the centralizer of $(\mathcal{R}_{F/F'}T)(F')^{p^n}$. Note that the universal property of the Weil restriction identifies $(\mathcal{R}_{F/F'}T)(F')$ with $T(F)$.

Consider a scheme S' over F' and an S' -valued point $\varphi' : S' \rightarrow G'$. Via the universal property of the Weil restriction φ' corresponds to an $S' \times_{F'} F$ -valued point $\varphi : S' \times_{F'} F \rightarrow G$. We have seen that φ' factors through the centralizer of T' if and only if it commutes with $(\mathcal{R}_{F/F'}T)(F')^{p^n}$. This is equivalent to saying that φ commutes with $T(F)^{p^n}$. As T is a torus and F infinite, the subgroup $T(F)^{p^n}$ is Zariski dense in T . The condition therefore amounts to saying that φ factors through the centralizer of T . But this centralizer is equal to T . Therefore, translated back to G' , the condition says that φ' factors through $\mathcal{R}_{F/F'}T$. This proves the lemma. $\square$

By Lemma 1.8 the subgroup $\mathcal{R}_{F/F'}T$ is the centralizer of a maximal torus of G' , i.e., it is a Cartan subgroup of G' . Thus [11] Cor.13.3.12 implies that $G'(F')$ is Zariski dense in G' , proving Theorem 1.7. $\square$

Remark 1.9. If F' is a non-discrete complete normed field, Theorem 1.7 is true for arbitrary connected smooth algebraic groups G . This is an easy consequence of the implicit function theorem.

Next we turn to simple groups. To fix ideas, a smooth linear algebraic group over a field will be called *simple* if it is non-trivial and possesses no non-trivial proper connected smooth normal algebraic subgroup. It is called *absolutely simple* if it remains simple over the algebraic closure of the base field.

If G is simply connected semisimple and simple over F , it is isomorphic to $\mathcal{R}_{F_1/F}G_1$ for an absolutely simple simply connected semisimple group G_1 over some finite separable extension F_1/F (cf. [11] Ex.16.2.9). From [4] A.2.4 we then deduce that $G' \cong \mathcal{R}_{F_1/F'}G_1$. In this way questions about G' can be reduced to the case that G is absolutely simple.

Theorem 1.10. Assume that G is simply connected semisimple and simple over F . Then G' is simple over F' .

Proof. By the above remarks we may assume that G is absolutely simple. Consider a non-trivial connected smooth normal algebraic subgroup $H' \subset G'$. Let

$$\bar{H}' \subset \prod_{\sigma \in \Sigma} G \times_{F, \sigma} E' \quad (1.11)$$

denote the image of $H' \times_{F'} E'$ under the composite of the natural maps

$$G' \times_{F'} E' \xrightarrow{1,1} \prod_{\sigma \in \Sigma} G_{\sigma} \twoheadrightarrow \prod_{\sigma \in \Sigma} G_{\sigma}/G_{\sigma,1} \xrightarrow{1,2} \prod_{\sigma \in \Sigma} G \times_{F, \sigma} E'.$$

Since H' is non-trivial and “defined over F' ”, by [11] Cor.12.4.3 we have $\bar{H}' \neq 1$. Since $H' \subset G'$ is a connected normal subgroup, so is $\bar{H}'$ in 1.11. It is therefore equal to the product of some of the factors on the right hand side. As $\bar{H}'$ is non-trivial, it contains at least one of these factors. But by construction it is also invariant under $\text{Aut}(E'/F')$, which permutes the factors transitively. We deduce that the inclusion 1.11 is in fact an equality. Now the following lemma implies that $H' \times_{F'} E' = G' \times_{F'} E'$; and hence $H' = G'$, as desired. $\square$

Lemma 1.12. *In the situation of Theorem 1.10, every normal algebraic subgroup $H \subset G' \times_{F'} E'$ which surjects to $\prod_{\sigma \in \Sigma} G \times_{F, \sigma} E'$ is equal to $G' \times_{F'} E'$.*

Proof. Using descending induction on i we will prove that $G_{\sigma, i} \subset H$ for all $\sigma \in \Sigma$ and $1 \leq i \leq q$. For $i = q$ the assertion is obvious, because $G_{\sigma, q} = 1$. Let us assume the inclusion for $G_{\sigma, i+1}$ and abbreviate

$$\text{gr}_i H_\sigma := \frac{H \cap G_{\sigma, i}}{G_{\sigma, i+1}} \subset \frac{G_{\sigma, i}}{G_{\sigma, i+1}} \stackrel{1.3}{\cong} \text{Lie } G \otimes_{F, \sigma} \mathbb{G}_{a, E'}. \quad (1.13)$$

By functoriality of the isomorphism 1.3, the conjugation action of $G'(E')$ on $G_{\sigma, i}$ corresponds to the adjoint representation of $G \times_{F, \sigma} E'$ on the right hand side. As H is a normal subgroup, all commutators between H and $G_{\sigma, i}$ must lie in H . It follows that

$$(\text{Ad}_h - \text{id})(\text{Lie } G) \otimes_{F, \sigma} \mathbb{G}_{a, E'} \subset \text{gr}_i H_\sigma \quad (1.14)$$

for every $h \in H(E')$. Since G is simply connected, it is known that the space of coinvariants of its adjoint representation is trivial (cf. [1], [2], or [5] Prop.1.11). On the other hand E' is algebraically closed, so by assumption $H(E')$ maps to a Zariski dense subgroup of $G \times_{F, \sigma} E'$. Thus, as h varies, the subgroups in 1.14 generate $\text{Lie } G \otimes_{F, \sigma} \mathbb{G}_{a, E'}$. The inclusion in 1.13 is therefore an equality, and so we have $G_{\sigma, i} \subset H$.

At the end of the induction we have $G_{\sigma, 1} \subset H$ for all $\sigma \in \Sigma$. Combining this with the fact that H surjects to $\prod_{\sigma \in \Sigma} G_\sigma / G_{\sigma, 1}$, we finally deduce $H = G' \times_{F'} E'$, as desired. This proves Lemma 1.12 and thereby finishes the proof of Theorem 1.10. $\square$

Remark 1.15. The analogue of Theorem 1.10 fails if G is not simply connected and both F/F' and the universal central extension $\pi : \tilde{G} \rightarrow G$ are inseparable. The reason is that by Proposition 1.6 (b) the homomorphism $\mathcal{R}_{F/F'} \varphi : \mathcal{R}_{F/F'} \tilde{G} \rightarrow G'$ is not surjective, so its image is a subgroup that makes G' not simple.

Corollary 1.16. *If G is semisimple and simply connected, then G' is perfect.*

Proof. We may assume that G is simple. Then G is connected and non-commutative; hence so is G' . The commutator group of G' is therefore non-trivial connected and normal, and by [11] Cor.2.2.8 it is “defined over F ” and thus smooth. By Theorem 1.10 it is therefore equal to G' , as desired. $\square$

Theorem 1.17. *If G is simple isotropic and simply connected and F is infinite, then G' is generated by split tori.*

Proof. By assumption there exists a closed embedding $\mathbb{G}_{m,F'} \times_{F'} F \cong \mathbb{G}_{m,F} \hookrightarrow G$. The homomorphism $\mathbb{G}_{m,F'} \rightarrow G'$ corresponding to it by the universal property of the Weil restriction is again non-trivial; hence G' contains a non-trivial split torus. The algebraic subgroup of G' that is generated by all split tori in G' is therefore non-trivial. By construction it is normalized by $G'(F')$, so by Theorem 1.7 it is normal in G' . Being generated by smooth connected subgroups, it is itself smooth and connected by [11] Prop.2.2.6 (iii). By Theorem 1.10 it is therefore equal to G' , as desired. $\square$

2. Main results

In the following we consider a connected semisimple group G over a local field F . Let $\pi : \tilde{G} \rightarrow G$ denote its universal central extension. The commutator pairing $\tilde{G} \times \tilde{G} \rightarrow \tilde{G}$ factors through a unique morphism

$$[\cdot, \cdot]^\sim : G \times G \rightarrow \tilde{G}.$$

For any closed subgroup $\Gamma \subset G(F)$ we let $\tilde{\Gamma}'$ denote the closure of the subgroup of $\tilde{G}(F)$ that is generated by the set of generalized commutators $[\Gamma, \Gamma]^\sim$.

Theorem 2.1. *Let F be a local field, and let G be an isotropic connected simple semisimple group over F . Let $\Gamma \subset G(F)$ be a non-discrete closed subgroup whose covolume for any invariant measure is finite. Then $\tilde{\Gamma}'$ is open in $\tilde{G}(F)$.*

Before proving this, we note the following consequence (cf. [8], [3]).

Corollary 2.2. *Under the assumptions of Theorem 2.1 we have $\tilde{\Gamma}' = \tilde{G}(F)$. In particular, Γ contains $\pi(\tilde{G}(F))$.*

Proof. Since $G(F)$ is not compact and Γ is a subgroup of finite covolume, this subgroup is not compact. Thus $\tilde{\Gamma}'$ is normalized by an unbounded subgroup of $G(F)$, and it is open in $\tilde{G}(F)$ by Theorem 2.1. As in [6] Thm.2.2 one deduces from this that $\tilde{\Gamma}'$ is unbounded. Let $\tilde{G}(F)^+$ denote the subgroup of $\tilde{G}(F)$ that is generated by the rational points of the unipotent radicals of all rational parabolic subgroups. The Kneser-Tits conjecture, which is proved in this case (see [7] Thm. 7.6 or [10]), asserts that $\tilde{G}(F)^+ = \tilde{G}(F)$. On the other hand, a theorem of Tits [9] states that every unbounded open subgroup of $\tilde{G}(F)^+$ is equal to $\tilde{G}(F)^+$. Altogether this implies $\tilde{\Gamma}' = \tilde{G}(F)$, as desired. $\square$

Proof of Theorem 2.1. In the case $\text{char}(F) = 0$ the proof in [8] §2 cannot be improved. It covers in particular the archimedean case. We will give a unified proof in the non-archimedean case, beginning with a few reductions.

Let Γ^{ad} denote the image of Γ in the adjoint group G^{ad} of G . Then $\tilde{\Gamma}'$ depends only on Γ^{ad} . On the other hand, all the assumptions in 2.1 are still satisfied for $\Gamma^{\text{ad}} \subset G^{\text{ad}}(F)$. Namely, since the homomorphism $G(F) \rightarrow G^{\text{ad}}(F)$ is proper with finite kernel, the subgroup Γ^{ad} is still non-discrete and closed. On the other hand, as the image of $G(F)$ in $G^{\text{ad}}(F)$ is cocompact, the covolume of Γ^{ad} in $G^{\text{ad}}(F)$

is again finite. To prove the theorem, we may therefore replace G by G^{ad} and Γ by Γ^{ad} . In other words, we may assume that G is adjoint.

Next, since G is connected simple and adjoint, it is isomorphic to $\mathcal{R}_{F_1/F} G_1$ for some absolutely simple connected adjoint group G_1 over a finite separable extension F_1/F . If $\tilde{G}_1$ denotes the universal covering of G_1 , we then have $\tilde{G} \cong \mathcal{R}_{F_1/F} \tilde{G}_1$. By the definition of Weil restriction we have $G(F) \cong G_1(F_1)$ and $\tilde{G}(F) \cong \tilde{G}_1(F_1)$; and since G is isotropic, so is G_1 . Thus after replacing F by F_1 and G by G_1 we may assume that G is absolutely simple.

For the next preparations note that F is non-archimedean, so $G(F)$ possesses an open compact subgroup. Its intersection with Γ is an open compact subgroup of Γ ; let us call it Δ . Let $\tilde{\Delta}'$ denote the closure of the subgroup of $\tilde{G}(F)$ that is generated by the set of generalized commutators $[\Delta, \Delta]^\sim$.

We will study the relation between these subgroups and various Weil restrictions of G . Consider any closed subfield $F' \subset F$ such that $[F/F']$ is finite. Note that in the case $\text{char}(F) = 0$ there is a unique smallest such F' , namely the closure of $\mathbb{Q}$. But in positive characteristic the extension F/F' may be arbitrarily large and, what is worse, it may be inseparable.

Set $G' := \mathcal{R}_{F/F'} G$ and $\tilde{G}' := \mathcal{R}_{F/F'} \tilde{G}$, and let $\pi' : \tilde{G}' \rightarrow G'$ be the homomorphism induced by π . From Proposition 1.6 we know that π' is not necessarily an isogeny. Identifying $G(F)$ with $G'(F')$ via the universal property of the Weil restriction, we can view Γ as a non-discrete closed subgroup of finite covolume of $G'(F')$. Similarly, we can view $\tilde{\Delta}'$ as a subgroup of $\tilde{G}'(F')$.

Lemma 2.3. $\tilde{\Delta}'$ is Zariski dense in $\tilde{G}'$.

Proof. Let $H' \subset G'$ and $\tilde{H}' \subset \tilde{G}'$ be the Zariski closures of Δ and $\tilde{\Delta}'$, respectively. By [11] Lemma 11.2.4 (ii) these groups are “defined over F' ”, i.e., smooth over F' . The intersection of Δ with the identity component of H' is open in Δ and thus again an open compact subgroup of Γ . After shrinking Δ we may therefore assume that H' is connected. For any $\gamma \in \Gamma$ the subgroup $\gamma \Delta \gamma^{-1}$ is again an open compact subgroup of Γ , so it is commensurable with Δ . Thus $\gamma H' \gamma^{-1}$ is commensurable with H' . Since H' is connected, they must be equal; hence H' is normalized by Γ . It is therefore also normalized by the Zariski closure of Γ .

Under the assumptions of 2.1, a theorem of Wang [12] implies that the Zariski closure of Γ in G' contains all split tori of G' . Thus, in particular, it contains the images under π' of all split tori in $\tilde{G}'$. Since G is simple isotropic, so is $\tilde{G}$; hence by Theorem 1.17 these tori generate $\tilde{G}'$. It follows that H' is normalized by the image of $\tilde{G}'$. By construction $\tilde{H}'$ is the algebraic subgroup of $\tilde{G}'$ that is generated by the image of the connected variety $H' \times_{F'} H'$ under $[\cdot, \cdot]^\sim$. It is therefore connected and normalized by $\tilde{G}'$.

Since Γ is non-discrete, the group Δ is not finite, and so H' is non-trivial. Let H denote the image of $H' \times_{F'} F$ under the canonical adjunction morphism $G' \times_{F'} F \rightarrow G$. By construction H is just the Zariski closure of Δ in G , so by the above arguments in the case $F' = F$ it is normalized by the image of $\tilde{G}$. But $\pi : \tilde{G} \rightarrow G$ is surjective, so H is a non-trivial connected normal subgroup of G . As G is absolutely simple, this implies $H = G$. As G is perfect, it follows that $\tilde{H}' \times_{F'} F$ surjects to G .

All in all we now deduce that $\tilde{H}'$ is a non-trivial connected smooth normal algebraic subgroup of $\tilde{G}'$. By Theorem 1.10 this implies $\tilde{H}' = \tilde{G}'$, as desired. $\square$

Note that Lemma 2.3 in the case $F' = F$ says that $\tilde{\Delta}'$ is Zariski dense in $\tilde{G}$. In particular Δ is compact and Zariski dense in G , so we can apply [5] Main Theorem 0.2. It follows that there exists a closed subfield $E \subset F$ such that $[F/E]$ is finite, E is absolutely simple and simply connected semisimple algebraic group $\tilde{H}$ over E , and an isogeny $\tilde{\varphi}: \tilde{H} \times_E F \rightarrow \tilde{G}$ with non-vanishing derivative, such that $\tilde{\Delta}'$ is the image under $\tilde{\varphi}$ of an open subgroup of $\tilde{H}(E)$.

Lemma 2.4. $E = F$.

Proof. Via the universal property of the Weil restriction the isogeny $\tilde{\varphi}$ corresponds to a homomorphism $\tilde{\varphi}': \tilde{H} \rightarrow \mathcal{R}_{F/E}\tilde{G}$, which satisfies

$$\tilde{\Delta}' \subset \tilde{\varphi}'(\tilde{H}(E)) \subset (\mathcal{R}_{F/E}\tilde{G})(E) = \tilde{G}(F).$$

By Lemma 2.3 in the case $F' = E$ we know that $\tilde{\Delta}'$ is Zariski dense in $\mathcal{R}_{F/E}\tilde{G}$. It follows that $\tilde{\varphi}'$ is dominant. This implies

$$\dim \tilde{H} \geq \dim \mathcal{R}_{F/E}\tilde{G} = [F/E] \cdot \dim \tilde{G} = [F/E] \cdot \dim \tilde{H};$$

hence $[F/E] = 1$, as desired. $\square$

Lemma 2.5. $\tilde{\varphi}$ is an isomorphism.

Proof. As $\tilde{\varphi}$ is an isogeny between simply connected groups, it is an isomorphism if and only if it is separable. In characteristic zero this is automatically the case. (Since $d\tilde{\varphi} \neq 0$, this is actually true whenever $\text{char}(F) \neq 2, 3$ (cf. [5] Thm.1.7), but we do not need that fact.) So for the rest of the proof we may suppose that $p := \text{char}(F)$ is positive. Set $F' := \{x^p \mid x \in F\}$; then F/F' is an inseparable extension of degree p . Consider the induced homomorphism

$$\tilde{\psi} := \mathcal{R}_{F/F'}\tilde{\varphi}: \mathcal{R}_{F/F'}\tilde{H} \longrightarrow \mathcal{R}_{F/F'}\tilde{G}.$$

By construction it satisfies

$$\begin{array}{ccc} \tilde{\Delta}' & \subset & \tilde{\psi}((\mathcal{R}_{F/F'}\tilde{H})(F')) \subset (\mathcal{R}_{F/F'}\tilde{G})(F') \\ & \parallel & \parallel \\ & \tilde{\varphi}(\tilde{H}(F)) & \subset \tilde{G}(F). \end{array}$$

Since $\tilde{\Delta}'$ is Zariski dense in $\mathcal{R}_{F/F'}\tilde{G}$ by Lemma 2.3, we deduce that $\tilde{\psi}$ is dominant. So for dimension reasons it is an isogeny. Proposition 1.6 (b) now shows that $\tilde{\varphi}$ is separable, as desired. $\square$

Combining Lemmas 2.4 and 2.5, we now deduce that $\tilde{\Delta}'$ is open in $\tilde{G}(F)$. Thus $\tilde{\Gamma}'$ is open in $\tilde{G}(F)$, completing the proof of Theorem 2.1. $\square$

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