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On the order of the reduction of a point on an abelian variety

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Abstract. Consider a point of infinite order on an abelian variety over a number field. Then its reduction at any place v of good reduction is a torsion point. For most of this paper we fix a rational prime ℓ and study how the ℓ -part of this reduction varies with v . Under suitable conditions we prove various statements on this ℓ -part for all v in a set of positive Dirichlet density: for example that its order is a fixed power of ℓ , that its order is non-trivial for the reductions of finitely many points, or that its order is larger than a certain explicit value that varies with v . By similar methods we prove that for all v in a set of positive Dirichlet density the reduction of a given abelian variety possesses no non-trivial supersingular abelian subvariety.

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0. Introduction

Consider an abelian variety A over a number field K and a rational point of infinite order $a \in A(K)$. Then the reduction a_v of a at any place v of good reduction is defined over the finite residue field k_v and is therefore a torsion point. It is natural to ask how a_v varies with v . For most of this paper we fix a rational prime ℓ and study the ℓ -part of a_v . Since for $v \nmid \ell$ any ℓ -power torsion point over $\bar{k}_v$ possesses a unique ℓ -power torsion lift to $A(\bar{K})$, one can try to translate this question into one over $\bar{K}$. The main player in this game is the group

$$\ell^{-\infty}(\mathbb{Z}a) := \{x \in A(\bar{K}) \mid \exists n \geq 0 : \ell^n x \in \mathbb{Z}a\}.$$

This group is a natural extension of $\mathbb{Z}[1/\ell]$ with the group of ℓ -power torsion points

$$A[\ell^\infty] := \{x \in A(\bar{K}) \mid \exists n \geq 0 : \ell^n x = 0\}.$$

The latter group has been studied extensively by means of the Galois representation on the associated ℓ -adic Tate module $T_\ell(A)$. The former group also gives rise to a Tate module $T_\ell(A, a)$ which is an extension of $T_\ell(A)$ by $\mathbb{Z}_\ell$. It is a special case of the Tate modules of 1-motives introduced by Deligne [7, §10.1].

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Let $\Gamma_\ell \subset \text{Aut}_{\mathbb{Z}_\ell}(T_\ell(A))$ and $\tilde{\Gamma}_\ell \subset \text{Aut}_{\mathbb{Z}_\ell}(T_\ell(A, a))$ be the respective images of $\text{Gal}(\bar{K}/K)$.

In Section 1 we review some known general facts about Γ_ℓ and its Zariski closure. We also prove in Corollary 1.7 that for all v in a set of positive Dirichlet density the reduction of A possesses no non-trivial supersingular abelian subvariety. Although this statement has no direct relation with the results on a_v , the respective methods of proof have much in common.

General structural properties of $\tilde{\Gamma}_\ell$ are then discussed in Section 2. In particular we recall Theorem 2.8 from the Kummer theory of A which states that $\tilde{\Gamma}_\ell$ is an extension of Γ_ℓ by an open subgroup of $T_\ell(B)$, where B is the identity component of the Zariski closure of $\mathbb{Z}a$. This result is essentially due to Ribet [13], though in the case we need the proof was worked out only by Hindry [9, §2, Prop. 1].

In Section 3 we then show how the ℓ -part of a_v is determined by the action of the Frobenius element Frob_v on $\ell^{-\infty}(\mathbb{Z}a)$. Any question about this ℓ -part can thus be translated completely into a question on the group $\tilde{\Gamma}_\ell$.

In Section 4 we answer some of these questions. In all cases we prove that a certain behavior occurs for all places v of K in a set of Dirichlet density > 0 . For example, in Corollary 4.3 we show that under mild conditions every power of ℓ occurs as the order of the ℓ -part of a_v . In Theorem 4.4 we prove that for finitely many given points a_i of infinite order, the ℓ -parts of their reductions $a_{i,v}$ can be made simultaneously non-trivial on a set of positive Dirichlet density. Theorem 4.7 generalizes this result in another direction: Let $f(T) \in \mathbb{Z}[T]$ be any polynomial which is a product of cyclotomic polynomials and a power of T . Let p_v denote the residue characteristic at v . Then for suitable ℓ , the ℓ -parts of all $f(p_v)a_{i,v}$ can be made simultaneously non-trivial on a set of positive Dirichlet density.

In the final section 5 we use these theorems to derive two density results on the $a_{i,v}$ which no longer refer to any particular prime ℓ . These results as well as Corollary 1.7 are needed in joint work with Damian Roessler [12] and provided the motivation for the present paper.¹ Theorem 5.1 can also be deduced from work by Wong [16] who, instead of studying when the ℓ -part of a_v is zero, considers the dual question of when a_v lies in $\ell \cdot A_v(k_v)$. Related questions are addressed in work by Corrales-Rodríguez and Schoof [6], Khare and Prasad [10], and Larsen [11].

1. The ℓ -adic Galois group associated to an abelian variety

Let K be a number field and $\bar{K}$ an algebraic closure of K . Consider an abelian variety A of dimension g over K and a rational prime ℓ . Then

$$A[\ell^\infty] := \{x \in A(\bar{K}) \mid \exists n \geq 0 : \ell^n x = 0\}$$

¹ The author wishes to thank Damian Roessler for the very fruitful ongoing collaboration.

is a discrete group isomorphic to $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g}$ with a continuous action of $\text{Gal}(\bar{K}/K)$. One usually describes this action via the ℓ -adic Tate module

$$T_\ell(A) := \text{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, A[\ell^\infty]) \cong \mathbb{Z}_\ell^{2g},$$

which possesses a continuous Galois representation

$$\rho_\ell : \text{Gal}(\bar{K}/K) \longrightarrow \text{Aut}_{\mathbb{Z}_\ell}(T_\ell(A)) \cong \text{GL}_{2g}(\mathbb{Z}_\ell).$$

We are interested in its image $\Gamma_\ell := \rho_\ell(\text{Gal}(\bar{K}/K))$, which is a compact subgroup of $\text{GL}_{2g}(\mathbb{Z}_\ell)$. Much can be said about Γ_ℓ by means of its Zariski closure $G_\ell \subset \text{GL}_{2g, \mathbb{Q}_\ell}$. This is a linear algebraic group over $\mathbb{Q}_\ell$ with a natural faithful representation on the rational Tate module

$$V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong \mathbb{Q}_\ell^{2g}.$$

The following general facts are known about G_ℓ .

Theorem 1.1. (a) *The action of G_ℓ on $V_\ell(A)$ is semisimple and the natural homomorphism*

$$\text{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow \text{End}_{\mathbb{Q}_\ell, G_\ell}(V_\ell(A))$$

is an isomorphism.

(b) *G_ℓ is a reductive group.*

(c) *Γ_ℓ is an open subgroup of $G_\ell(\mathbb{Q}_\ell)$.*

Proof. By the definition of G_ℓ the statements in (a) are equivalent to the corresponding ones with Γ_ℓ in place of G_ℓ , which were proved by Faltings [8, Th. 3–4]. Part (b) follows from the first statement in (a). Part (c) is a theorem of Bogomolov [4], [3]. $\square$

By Galois theory every open subgroup of Γ_ℓ corresponds to a finite extension of K within $\bar{K}$, and replacing K by that extension amounts to replacing Γ_ℓ by the corresponding subgroup. In particular, let G_ℓ° denote the identity component of G_ℓ . Then replacing Γ_ℓ by any open subgroup of $\Gamma_\ell \cap G_\ell^\circ$ has the effect of replacing G_ℓ by G_ℓ° ; and thereafter G_ℓ will be connected.

Now consider any finite place v of K and let p_v denote the characteristic of the finite residue field k_v . If $v \nmid \ell$ and A has good reduction at v , it is known that the restriction of ρ_ℓ to any inertia group above v is trivial. Let Frob_v be any element of a decomposition group at v which acts by taking $|k_v|^{\text{th}}$ powers modulo v . Then the conjugacy class of $\rho_\ell(\text{Frob}_v)$ depends only on v and is known to be semisimple, and its characteristic polynomial on $V_\ell(A)$ is known to have coefficients in $\mathbb{Z}$ and to be independent of ℓ .

Choose any semisimple element $t_v \in \text{GL}_{2g}(\mathbb{Q})$ whose characteristic polynomial is equal to that of $\rho_\ell(\text{Frob}_v)$. Let $T_v \subset \text{GL}_{2g, \mathbb{Q}}$ be the Zariski closure of the subgroup generated by t_v . The construction implies that the identity component

of T_v is a torus and its $\mathrm{GL}_{2g}(\mathbb{Q})$ -conjugacy class depends only on v . Following Serre [14] it is called the *Frobenius torus at v* . Moreover, for any $\ell \neq p_v$ there is a unique conjugate of $T_{v, \mathbb{Q}_\ell}$ by an element of $\mathrm{GL}_{2g}(\mathbb{Q}_\ell)$ which lies in G_ℓ , such that t_v is mapped to $\rho_\ell(\mathrm{Frob}_v)$. Serre [14, §5, pp.12–13] proves:

Theorem 1.2. *If G_ℓ is connected, then for all places v in a set of Dirichlet density 1 the group T_v itself is a torus and $T_{v, \mathbb{Q}_\ell}$ is conjugate under $\mathrm{GL}_{2g}(\mathbb{Q}_\ell)$ to a maximal torus of G_ℓ .*

Corollary 1.3. *There exists a set of rational primes ℓ of positive Dirichlet density for which G_ℓ splits over $\mathbb{Q}_\ell$.*

Proof. Let T_v be any Frobenius torus as in Theorem 1.2. Choose a finite extension F of $\mathbb{Q}$ such that $T_{v, F}$ splits. Then the set of rational primes ℓ which split completely in F has positive Dirichlet density, and for each of them $T_{v, \mathbb{Q}_\ell}$ splits. Since $T_{v, \mathbb{Q}_\ell}$ is conjugate to a maximal torus of G_ℓ , this shows that G_ℓ splits. $\square$

Next any polarization of A induces a Galois equivariant perfect alternating pairing $V_\ell(A) \times V_\ell(A) \rightarrow \mathbb{Q}_\ell(1)$, where $\mathrm{Gal}(\bar{K}/K)$ acts on $\mathbb{Q}_\ell(1)$ through the cyclotomic character. It follows that Γ_ℓ is contained in the group of symplectic similitudes $\mathrm{CSp}_{2g}(\mathbb{Q}_\ell)$. Let $\mu : \mathrm{CSp}_{2g} \rightarrow \mathbb{G}_m$ denote the multiplier map; then $\mu\rho_\ell : \mathrm{Gal}(\bar{K}/K) \rightarrow \mathbb{Z}_\ell^*$ is the cyclotomic character. The definition of G_ℓ implies that $G_\ell \subset \mathrm{CSp}_{2g, \mathbb{Q}_\ell}$; hence μ defines an algebraic character of G_ℓ .

Proposition 1.4. *Consider a maximal torus S_ℓ of G_ℓ and any weight χ of S_ℓ on $V_\ell(A)$. Then μ and χ are $\mathbb{Q}$ -linearly independent in the character group of S_ℓ .*

Proof. The perfect pairing implies that there exists a weight χ^* of S_ℓ on $V_\ell(A)$ such that $\chi\chi^* = \mu$. Both χ and χ^* are non-trivial, because the corresponding Frobenius eigenvalues have complex absolute value > 1 . Now by the Hodge-Tate decomposition there exists a cocharacter λ of S_ℓ whose weights on $V_\ell(A)$ are 0 and 1 and whose weight on $\mathbb{Q}_\ell(1)$ is 1; see for instance Serre [14, §5, pp.11–12]. For any such λ we have

$$\langle \chi, \lambda \rangle + \langle \chi^*, \lambda \rangle = \langle \chi\chi^*, \lambda \rangle = \langle \mu, \lambda \rangle = 1,$$

and one of the summands is 0 and the other 1. This implies that χ and χ^* cannot be non-zero rational multiples of each other. Since they are both non-trivial characters, they must be $\mathbb{Q}$ -linearly independent. Equivalently χ and $\mu = \chi\chi^*$ are $\mathbb{Q}$ -linearly independent, as desired. $\square$

Proposition 1.5. *Suppose that $A = A_1 \times \dots \times A_d$ for non-zero abelian varieties $A_1, \dots, A_d$. Consider a maximal torus S_ℓ of G_ℓ . Then there exist weights χ_i of S_ℓ on $V_\ell(A_i)$ so that μ is $\mathbb{Q}$ -linearly independent of $\chi_1, \dots, \chi_d$.*

Proof. By the Hodge-Tate decomposition, see [14, §5, pp.11–12], there exists a cocharacter λ of S_ℓ which on every $V_\ell(A_i)$ has the weights 0 and 1 with multiplicity $\dim A_i$ each, and whose weight on $\mathbb{Q}_\ell(1)$ is 1. So we can choose each χ_i such that $\langle \chi_i, \lambda \rangle$, the weight of the χ_i -eigenspace in the Hodge-Tate decomposition, is zero. Then for any weight χ which is a $\mathbb{Q}$ -linear combination of the χ_i , we still have $\langle \chi, \lambda \rangle = 0$. But $\langle \mu, \lambda \rangle = 1$; hence μ is not a $\mathbb{Q}$ -linear combination of the χ_i . $\square$

We finish this section with a first application of Proposition 1.4, which will not be used in the rest of the paper.

Theorem 1.6. *If G_ℓ is connected, the set of finite places v of K where the reduction of A does not possess a non-trivial supersingular abelian subvariety has Dirichlet density 1.*

Proof. By Theorem 1.2 it suffices to consider those places $v \nmid \ell$ of K for which $T_{v, \mathbb{Q}_\ell}$ is conjugate to a maximal torus S_ℓ of G_ℓ . Let v be such a place and suppose that the corresponding reduction A_v of A possesses a non-trivial supersingular abelian subvariety B_v . Then any eigenvalue of Frob_v on $V_\ell(B_v)$ has the form $\sqrt{|k_v|}$ times a root of unity, while the eigenvalue on $\mathbb{Q}_\ell(1)$ is $|k_v|$. Let χ be the weight of S_ℓ on $V_\ell(A)$ corresponding to that eigenvalue on $V_\ell(B_v) \subset V_\ell(A_v)$, and let n be the order of that root of unity. Then the values of χ^{2n} and μ^n on $\rho_\ell(\text{Frob}_v)$ coincide. But by the construction of the Frobenius torus the element $\rho_\ell(\text{Frob}_v)$ generates a Zariski dense subgroup of S_ℓ . Thus χ^{2n} and μ^n are equal as characters of S_ℓ , which contradicts their linear independence from Proposition 1.4. This shows that A_v does not possess a non-trivial supersingular abelian subvariety, as desired. $\square$

Corollary 1.7. *Let A be an abelian variety over a number field K . Then there exists a finite extension L of K such that for all finite places of L in a set of Dirichlet density 1 the reduction of A does not possess a non-trivial supersingular abelian subvariety.*

Proof. Choose an arbitrary rational prime ℓ and a finite Galois extension L of K over which G_ℓ becomes connected, and apply Theorem 1.6. $\square$

2. The ℓ -adic Galois group associated to an abelian variety with a point

Now fix a rational point of infinite order $a \in A(K)$ and set

$$\ell^{-\infty}(\mathbb{Z}a) := \{x \in A(\bar{K}) \mid \exists n \geq 0 : \ell^n x \in \mathbb{Z}a\}.$$

Then we have a natural short exact sequence of discrete groups

$$0 \longrightarrow A[\ell^\infty] \longrightarrow \ell^{-\infty}(\mathbb{Z}a) \xrightarrow{a \mapsto 1} \mathbb{Z}[1/\ell] \longrightarrow 0. \quad (2.1)$$

Any choice of a compatible system of ℓ -power roots of a determines a splitting $\lambda : \mathbb{Z}[1/\ell] \rightarrow \ell^{-\infty}(\mathbb{Z}a)$ satisfying $\lambda(1) = a$. We will call such a splitting *special*. Two special splittings differ by an element of

$$\mathrm{Hom}(\mathbb{Z}[1/\ell]/\mathbb{Z}, A[\ell^\infty]) \cong \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, A[\ell^\infty]) = T_\ell(A).$$

By contrast, two general splittings differ by an element of

$$\begin{aligned} \mathrm{Hom}(\mathbb{Z}[1/\ell], A[\ell^\infty]) &= \bigcup_{r \geq 0} \mathrm{Hom}(\mathbb{Z}[1/\ell]/\ell^r \mathbb{Z}, A[\ell^\infty]) \\ &\cong \bigcup_{r \geq 0} \ell^{-r} \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, A[\ell^\infty]) \\ &= \bigcup_{r \geq 0} \ell^{-r} T_\ell(A) = V_\ell(A). \end{aligned} \quad (2.2)$$

The sequence 2.1 is equivariant under the natural continuous action of $\mathrm{Gal}(\bar{K}/K)$, where the action on $\mathbb{Z}[1/\ell]$ is trivial. It is useful to describe this action via an associated Tate module. For this note that $\ell^{-\infty}(\mathbb{Z}a)/\mathbb{Z}a$ is isomorphic to $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g+1}$; hence

$$T_\ell(A, a) := \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \ell^{-\infty}(\mathbb{Z}a)/\mathbb{Z}a)$$

is isomorphic to $\mathbb{Z}_\ell^{2g+1}$ and sits in a short exact sequence

$$0 \longrightarrow T_\ell(A) \longrightarrow T_\ell(A, a) \longrightarrow \mathbb{Z}_\ell \longrightarrow 0. \quad (2.3)$$

Any special splitting of 2.1 determines a splitting of 2.3, i.e., an isomorphism $T_\ell(A, a) \cong T_\ell(A) \oplus \mathbb{Z}_\ell$. We will write any such decomposition in terms of column vectors. Then the natural Galois representation on $T_\ell(A, a)$ has the form

$$\tilde{\rho}_\ell = \begin{pmatrix} \rho_\ell & * \\ 0 & 1 \end{pmatrix} : \mathrm{Gal}(\bar{K}/K) \longrightarrow \begin{pmatrix} \mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell(A)) & T_\ell(A) \\ 0 & 1 \end{pmatrix} \cong \begin{pmatrix} \mathrm{GL}_{2g}(\mathbb{Z}_\ell) & \mathbb{Z}_\ell^{2g} \\ 0 & 1 \end{pmatrix}.$$

The construction implies that left multiplication by the same matrices also describes the Galois action on $\ell^{-\infty}(\mathbb{Z}a) \cong A[\ell^\infty] \oplus \mathbb{Z}[1/\ell]$. We are interested in the image

$$\tilde{\Gamma}_\ell := \tilde{\rho}_\ell(\mathrm{Gal}(\bar{K}/K)) \subset \begin{pmatrix} \Gamma_\ell & T_\ell(A) \\ 0 & 1 \end{pmatrix}.$$

Letting $N_\ell := \tilde{\Gamma}_\ell \cap T_\ell(A)$ denote its intersection with the upper right corner, we obtain a natural short exact sequence

$$0 \longrightarrow N_\ell \longrightarrow \tilde{\Gamma}_\ell \longrightarrow \Gamma_\ell \longrightarrow 1. \quad (2.4)$$

As with Γ_ℓ we will study $\tilde{\Gamma}_\ell$ with the help of its Zariski closure $\tilde{G}_\ell$, which is a linear algebraic group over $\mathbb{Q}_\ell$ with a natural faithful representation on

$$V_\ell(A, a) := T_\ell(A, a) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong \mathbb{Q}_\ell^{2g+1}.$$

By construction we have a natural short exact sequence

$$0 \longrightarrow U_\ell \longrightarrow \tilde{G}_\ell \longrightarrow G_\ell \longrightarrow 1$$

where U_ℓ is an algebraic subgroup of the vector group $V_\ell(A)$. Since G_ℓ is reductive by Theorem 1.1 (b), the subgroup U_ℓ is simply the unipotent radical of $\tilde{G}_\ell$.

Proposition 2.5. $\tilde{\Gamma}_\ell$ is open in $\tilde{G}_\ell(\mathbb{Q}_\ell)$ and N_ℓ open in $U_\ell(\mathbb{Q}_\ell)$.

Proof. By construction we have an inclusion of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & U_\ell(\mathbb{Q}_\ell) & \longrightarrow & \tilde{G}_\ell(\mathbb{Q}_\ell) & \longrightarrow & G_\ell(\mathbb{Q}_\ell) \longrightarrow 1 \\ & & \cup & & \cup & & \cup \\ 0 & \longrightarrow & N_\ell & \longrightarrow & \tilde{\Gamma}_\ell & \longrightarrow & \Gamma_\ell \longrightarrow 1. \end{array}$$

All these groups can be viewed as ℓ -adic Lie groups, and by a theorem of Chevalley [5, Ch. II, Cor. 7.9] the Zariski density of $\tilde{\Gamma}_\ell$ implies

$$[\text{Lie } \tilde{G}_\ell, \text{Lie } \tilde{G}_\ell] \subset \text{Lie } \tilde{\Gamma}_\ell.$$

On the other hand $V_\ell(A)$ does not contain the trivial representation of G_ℓ , because all Frobenius eigenvalues have complex absolute value > 1 . Thus $U_\ell \cong \text{Lie } U_\ell$ does not contain the trivial representation of G_ℓ , which implies that

$$\text{Lie } U_\ell = [\text{Lie } \tilde{G}_\ell, \text{Lie } U_\ell] \subset [\text{Lie } \tilde{G}_\ell, \text{Lie } \tilde{G}_\ell] \subset \text{Lie } \tilde{\Gamma}_\ell.$$

Since moreover $\text{Lie } \Gamma_\ell = \text{Lie } G_\ell$ by Theorem 1.1 (c), we deduce that $\text{Lie } \tilde{\Gamma}_\ell = \text{Lie } \tilde{G}_\ell$. Thus $\tilde{\Gamma}_\ell$ is open in $\tilde{G}_\ell(\mathbb{Q}_\ell)$, and therefore N_ℓ is open in $U_\ell(\mathbb{Q}_\ell)$, as desired. $\square$

Proposition 2.6. After replacing K by a suitable finite extension there exists a splitting of 2.1, not necessarily special, such that

$$\tilde{\Gamma}_\ell = \begin{pmatrix} \Gamma_\ell & N_\ell \\ 0 & 1 \end{pmatrix}.$$

Proof. Choose any Levi decomposition $\tilde{G}_\ell = G_\ell \ltimes U_\ell$ and consider the short exact sequence

$$0 \longrightarrow V_\ell(A) \longrightarrow V_\ell(A, a) \longrightarrow \mathbb{Q}_\ell \longrightarrow 0 \quad (2.7)$$

deduced from 2.3 by tensoring with $\mathbb{Q}_\ell$. As G_ℓ is reductive, acts trivially on $\mathbb{Q}_\ell$, and non-trivially on every non-zero subspace of $V_\ell(A)$, the sequence 2.7 possesses a unique splitting that is invariant under the Levi subgroup G_ℓ . On the other hand take any splitting λ of 2.1. Then the induced splitting of 2.7 differs from the Levi invariant splitting by some element of $V_\ell(A)$. Changing λ by the same element thus shows that the Levi invariant splitting of 2.7 comes from some splitting of

2.1, though not necessarily from a special one. With respect to this splitting the decomposition $\tilde{G}_\ell = G_\ell \ltimes U_\ell$ is the same as that in terms of formal matrices

$$\tilde{G}_\ell = \begin{pmatrix} G_\ell & U_\ell \\ 0 & 1 \end{pmatrix}.$$

Finally Proposition 2.5 implies that

$$(G_\ell(\mathbb{Q}_\ell) \cap \tilde{\Gamma}_\ell) \ltimes (U_\ell(\mathbb{Q}_\ell) \cap \tilde{\Gamma}_\ell)$$

is an open subgroup of $\tilde{G}_\ell(\mathbb{Q}_\ell)$ and hence of $\tilde{\Gamma}_\ell$. After replacing K by the corresponding finite extension $\tilde{\Gamma}_\ell$ itself is such a semidirect product, as desired. $\square$

Theorem 2.8. *Let B be the identity component of the Zariski closure of $\mathbb{Z}a$. Then N_ℓ is open in $T_\ell(B) \subset T_\ell(A)$ and we have $U_\ell = V_\ell(B) \subset V_\ell(A)$.*

Proof. This is a special case of a theorem essentially due to Ribet [13] on the Kummer theory of A , itself depending on results of Faltings [8] and Serre [15] as well as the Mordell-Weil theorem, and following a method first used by Bashmakov [1]. The case we need was formulated by Bertrand [2, Th. 2] and worked out by Hindry [9, §2, Prop. 1].

We begin with two technical reductions required by this reference. First, as the Mordell-Weil group $A(K)$ is finitely generated, the given element a is an integral multiple of an indivisible element $a' \in A(K)$. Replacing a by a' does not change B , and since $\ell^{-\infty}(\mathbb{Z}a) \subset \ell^{-\infty}(\mathbb{Z}a')$ is a subgroup of finite index prime to ℓ , it also changes neither $\tilde{\Gamma}_\ell$ nor N_ℓ nor U_ℓ . Thus without loss of generality we may, and do, assume that a itself is indivisible in $A(K)$. Next let d be the number of connected components of the Zariski closure of $\mathbb{Z}a$. To prove the theorem we may, and do, replace K by its finite extension $K(A[d])$.

Now for any two integers $r \geq s \geq 0$ consider the finite quotients

$$\begin{array}{ccccc} \tilde{\Gamma}_\ell & \longrightarrow & \tilde{\Gamma}_{\ell,r,s} & \subset & \mathrm{GL}_{2g}(\mathbb{Z}/\ell^r\mathbb{Z}) \ltimes T_\ell(A)/\ell^s T_\ell(A) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma_\ell & \longrightarrow & \Gamma_{\ell,r} & \subset & \mathrm{GL}_{2g}(\mathbb{Z}/\ell^r\mathbb{Z}). \end{array}$$

Then the short exact sequence 2.4 maps onto a short exact sequence

$$0 \longrightarrow N_{\ell,r,s} \longrightarrow \tilde{\Gamma}_{\ell,r,s} \longrightarrow \Gamma_{\ell,r} \longrightarrow 1$$

for some subgroup $N_{\ell,r,s} \subset T_\ell(A)/\ell^s T_\ell(A) \cong A[\ell^s]$. By [9, §2, Prop. 1] this group is a subgroup of $T_\ell(B)/\ell^s T_\ell(B) \cong B[\ell^s]$ whose index is bounded independently of r and s , provided that $r \geq \mathrm{ord}_\ell(d)$. Since N_ℓ is the projective limit of the $N_{\ell,r,s}$ as both r and s go to infinity, this implies that N_ℓ is an open subgroup of $T_\ell(B)$. The second statement follows from this and Proposition 2.5. $\square$

In particular, since a has infinite order by assumption, Theorem 2.8 implies that $N_\ell \neq 0$. Another direct consequence is:

Corollary 2.9. N_ℓ is open in $T_\ell(A)$ if and only if $U_\ell = V_\ell(A)$ if and only if $\mathbb{Z}a$ is Zariski dense in A .

3. The ℓ -part of the reduction at v

Now consider a place $v \nmid \ell$ of K where A has good reduction A_v . Then the restriction of $\tilde{\rho}_\ell$ to any inertia group above v is trivial, and so the conjugacy class of $\rho_\ell(\text{Frob}_v)$ depends only on v . We will show how this conjugacy class determines the ℓ -part of the reduction $a_v \in A_v$ of our fixed point a .

First the condition $v \nmid \ell$ implies that the reduction map induces an isomorphism

$$A[\ell^\infty] \xrightarrow{\sim} A_v(\bar{k}_v)[\ell^\infty].$$

Consider the composite homomorphism

$$\kappa_v : \ell^{-\infty}(\mathbb{Z}a) \subset A(\bar{K}) \longrightarrow A_v(\bar{k}_v) \longrightarrow A_v(\bar{k}_v)[\ell^\infty] \cong A[\ell^\infty],$$

where the first arrow is reduction modulo v , the second one is the projection to the ℓ -part, and the isomorphism on the right is the inverse of the reduction map. By construction its restriction to $A[\ell^\infty]$ is the identity, so κ_v induces a splitting of the sequence 2.1. It is important to note that κ_v does not in general correspond to a special splitting. Indeed, it does so if and only if $\kappa_v(a) = 0$, that is, if the ℓ -part of the reduction a_v vanishes.

By construction κ_v is equivariant under the action of Frob_v . Thus the following observation tells us that κ_v is completely determined by the element $\tilde{\rho}_\ell(\text{Frob}_v) \in \tilde{\Gamma}_\ell$.

Proposition 3.1. *For every place $v \nmid \ell$ of K where A has good reduction the homomorphism κ_v is the unique Frob_v -equivariant splitting of the sequence 2.1.*

Proof. Any other Frob_v -equivariant splitting $\ell^{-\infty}(\mathbb{Z}a) \rightarrow A[\ell^\infty]$ differs from κ_v by a Frob_v -invariant element of $\text{Hom}(\mathbb{Z}[1/\ell], A[\ell^\infty])$. By 2.2 the latter space is isomorphic to $V_\ell(A)$. Since all eigenvalues of Frob_v on $V_\ell(A)$ have complex absolute value > 1 , its subspace of Frob_v -invariants is zero. Thus κ_v is the unique Frob_v -invariant splitting. $\square$

To give a precise formula for $\kappa_v(a)$ we fix a special splitting λ of 2.1 and write

$$\tilde{\gamma}_v := \tilde{\rho}_\ell(\text{Frob}_v) = \begin{pmatrix} \gamma_v & n_v \\ 0 & 1 \end{pmatrix}$$

with $\gamma_v = \rho_\ell(\text{Frob}_v) \in \Gamma_\ell \subset \text{GL}_{2g}(\mathbb{Z}_\ell)$ and $n_v \in T_\ell(A) \cong \mathbb{Z}_\ell^{2g}$. Since γ_v does not have the eigenvalue 1, we can invert the matrix $\gamma_v - \text{id}$ over $\mathbb{Q}_\ell$ and thus define

$$m_v := (\gamma_v - \text{id})^{-1} n_v \in V_\ell(A) \cong \mathbb{Q}_\ell^{2g}.$$

Let π_ℓ denote the natural composite homomorphism

$$V_\ell(A) \twoheadrightarrow V_\ell(A) / T_\ell(A) \cong A[\ell^\infty].$$

Proposition 3.2. *We have $\kappa_v(a) = \pi_\ell(m_v)$. In particular the order of the ℓ -part of the reduction a_v is equal to the ℓ -part of the denominator of m_v .*

Proof. The splitting λ induces a decomposition

$$V_\ell(A, a) = V_\ell(A) \oplus \mathbb{Q}_\ell$$

which, as usual, we write in terms of column vectors. A direct calculation then shows that the eigenspace of $\tilde{\gamma}_v$ on $V_\ell(A, a)$ for the eigenvalue 1 is generated by the vector

$$\begin{pmatrix} -m_v \\ 1 \end{pmatrix}.$$

Thus again with respect to the decomposition induced by λ the map

$$\begin{aligned} \mathbb{Z}[1/\ell] &\longrightarrow \ell^{-\infty}(\mathbb{Z}a) = A[\ell^\infty] \oplus \mathbb{Z}[1/\ell], \\ x &\mapsto \begin{pmatrix} -\pi_\ell(xm_v) \\ x \end{pmatrix} \end{aligned}$$

defines a $\tilde{\gamma}_v$ -equivariant splitting of 2.1. The corresponding $\tilde{\gamma}_v$ -equivariant splitting in the other direction

$$A[\ell^\infty] \oplus \mathbb{Z}[1/\ell] = \ell^{-\infty}(\mathbb{Z}a) \longrightarrow A[\ell^\infty]$$

is given by

$$\begin{pmatrix} b \\ x \end{pmatrix} = \begin{pmatrix} b + \pi_\ell(xm_v) \\ 0 \end{pmatrix} + \begin{pmatrix} -\pi_\ell(xm_v) \\ x \end{pmatrix} \mapsto b + \pi_\ell(xm_v).$$

By Proposition 3.1 this map represents κ_v . Now since λ is a special splitting, the element $a = \lambda(1)$ corresponds to the vector

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It follows that $\kappa_v(a) = \pi_\ell(m_v)$, as desired. $\square$

4. Density results for the ℓ -part of the reduction

In this section we derive several statements on the Dirichlet density of the set of places v at which the ℓ -part of the reduction of a has certain properties. For all these statements we can disregard the finite set S of places dividing ℓ or where A has bad reduction.

Theorem 4.1. *Let A be an abelian variety over a number field K and $a \in A(K)$ a rational point of infinite order such that $\mathbb{Z}a$ is Zariski dense in A . Consider a rational prime and a point $b \in A[\ell^\infty]$. Then for all finite places v of K in a set of Dirichlet density > 0 the ℓ -part of the reduction of a is equal to the reduction of b .*

Proof. Choose a special splitting of 2.1 and let U denote the set of elements

$$\tilde{\gamma} = \begin{pmatrix} \gamma & n \\ 0 & 1 \end{pmatrix} \in \tilde{\Gamma}_\ell \subset \begin{pmatrix} \text{Aut}_{\mathbb{Z}_\ell}(T_\ell(A)) & T_\ell(A) \\ 0 & 1 \end{pmatrix}$$

satisfying $\det(\gamma - \text{id}) \neq 0$. Clearly this is an open subset of $\tilde{\Gamma}_\ell$. Next $\tilde{\gamma} \mapsto \pi_\ell((\gamma - \text{id})^{-1}n)$ defines a continuous function from U to the discrete set $A[\ell^\infty]$. It is therefore locally constant; hence

$$U_b := \left\{ \tilde{\gamma} \in \tilde{\Gamma}_\ell \mid \begin{array}{l} \det(\gamma - \text{id}) \neq 0, \text{ and} \\ \pi_\ell((\gamma - \text{id})^{-1}n) = b \end{array} \right\}$$

is an open subset of $\tilde{\Gamma}_\ell$.

Lemma 4.2. U_b is non-empty.

Proof. It suffices to show that the map

$$U \rightarrow V_\ell(A), \quad \begin{pmatrix} \gamma & n \\ 0 & 1 \end{pmatrix} \mapsto (\gamma - \text{id})^{-1}n$$

is surjective. This statement is invariant under conjugation by $V_\ell(A)$, and it suffices to prove it after replacing K by a finite extension. Thus using Proposition 2.6 we may without loss of generality assume that

$$\tilde{\Gamma}_\ell = \begin{pmatrix} \Gamma_\ell & N_\ell \\ 0 & 1 \end{pmatrix}.$$

The desired statement is then equivalent to

$$V_\ell(A) = \bigcup_{\substack{\gamma \in \Gamma_\ell \\ \det(\gamma - \text{id}) \neq 0}} (\gamma - \text{id})^{-1}N_\ell.$$

Now N_ℓ is open in $T_\ell(A)$ by Corollary 2.9; hence $\ell^r T_\ell(A) \subset N_\ell$ for some integer r . On the other hand, for every integer $s > 0$ there exists $\gamma \in \Gamma_\ell$ with $\det(\gamma - \text{id}) \neq 0$ such that $\gamma \equiv \text{id} \pmod{\ell^s}$. Indeed, any power $\rho_\ell(\text{Frob}_v)^m$ for a place $v \notin S$ and m sufficiently divisible has these properties. For this element γ we then have

$$(\gamma - \text{id})T_\ell(A) \subset \ell^s T_\ell(A) \subset \ell^{s-r} N_\ell$$

and hence

$$\ell^{r-s} T_\ell(A) \subset (\gamma - \text{id})^{-1} N_\ell.$$

With $s \rightarrow \infty$ the desired equality follows. $\square$

Now take any element $\tilde{\gamma} \in U_b$. By openness there exists an open normal subgroup $\tilde{\Delta} \triangleleft \tilde{\Gamma}_\ell$ such that $\tilde{\gamma}\tilde{\Delta} \subset U_b$. As $\tilde{\Gamma}_\ell/\tilde{\Delta}$ is the Galois group of a finite extension of K , by the Chebotarev density theorem there exists a set of places $v \notin S$ of K of Dirichlet density > 0 for which

$$\tilde{\rho}_\ell(\text{Frob}_v) \equiv \tilde{\gamma} \pmod{\tilde{\Delta}}.$$

But for all these v we have $\tilde{\rho}_\ell(\text{Frob}_v) \in U_b$, which by Proposition 3.2 implies $\kappa_v(a) = b$. By the definition of κ_v this means that the ℓ -part of the reduction of a is equal to the reduction of b , as desired. $\square$

Corollary 4.3. *Let A be an abelian variety over a number field K and $a \in A(K)$ a rational point of infinite order such that $\mathbb{Z}a$ is Zariski dense in A . Consider a rational prime and an integer $r \geq 0$. Then for all finite places v of K in a set of Dirichlet density > 0 the ℓ -part of the reduction of a has order ℓ^r .*

Proof. Apply Theorem 4.1 to any point $b \in A[\ell^\infty]$ of order ℓ^r . (This was also partly proved by Khare and Prasad [10, §5, Lemma 4–5].) $\square$

Theorem 4.4. *For $1 \leq i \leq d$ let A_i be an abelian variety over a number field K and $a_i \in A_i(K)$ a rational point of infinite order. Let ℓ be a rational prime. Then for all finite places v of K in a set of Dirichlet density > 0 the ℓ -part of the reduction of a_i is non-trivial for every i .*

Proof. We apply the results of the preceding sections to $A := A_1 \times \dots \times A_d$ and $a := (a_1, \dots, a_d)$. Let $\text{pr}_i : A \rightarrow A_i$ denote the projection to the i^{th} factor. Then as in the proof of Theorem 4.1

$$U' := \left\{ \tilde{\gamma} \in \tilde{\Gamma}_\ell \mid \begin{array}{l} \det(\gamma - \text{id}) \neq 0, \text{ and} \\ \forall i : \text{pr}_i \pi_\ell((\gamma - \text{id})^{-1}n) \neq 0 \end{array} \right\}$$

is an open subset of $\tilde{\Gamma}_\ell$, and it suffices to prove: $\square$

Lemma 4.5. *U' is non-empty.*

Proof. We may replace K by a finite extension. Thus using Proposition 2.6 we may without loss of generality assume that there exists $m \in V_\ell(A)$ such that

$$\tilde{\Gamma}_\ell = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \Gamma_\ell & N_\ell \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} \gamma & n - (\gamma - \text{id})m \\ 0 & 1 \end{pmatrix} \mid \begin{array}{l} \gamma \in \Gamma_\ell \\ n \in N_\ell \end{array} \right\}.$$

We must therefore find $\gamma \in \Gamma_\ell$ and $n \in N_\ell$ such that $\det(\gamma - \text{id}) \neq 0$ and

$$\text{pr}_i \pi_\ell((\gamma - \text{id})^{-1}n - m) = \text{pr}_i \pi_\ell((\gamma - \text{id})^{-1}(n - (\gamma - \text{id})m)) \neq 0$$

for all i . This second condition is equivalent to

$$\text{pr}_i((\gamma - \text{id})^{-1}n) \not\equiv \text{pr}_i(m) \pmod{T_\ell(A_i)}.$$

Take any integer r so that $\ell^r m \in T_\ell(A)$. Then it suffices to have

$$\ell^r \operatorname{pr}_i((\gamma - \operatorname{id})^{-1}n) \notin T_\ell(A_i).$$

With $n = (n_1, \dots, n_d) \in N_\ell$ this is equivalent to

$$\ell^r n_i \notin (\gamma - \operatorname{id})T_\ell(A_i). \quad (4.6)$$

Now by functoriality the image $\operatorname{pr}_i(N_\ell) \subset T_\ell(A_i)$ is the unipotent part of the ℓ -adic Galois group attached to (A_i, a_i) . As a_i has infinite order, this image is non-trivial by Theorem 2.8. Since any finite number of non-trivial linear inequalities in a free $\mathbb{Z}_\ell$ -module can be simultaneously satisfied, we may therefore select $n = (n_1, \dots, n_r) \in N_\ell$ such that all $n_i \neq 0$. Then clearly 4.6 holds for any suitable $\gamma \in \Gamma_\ell$ that is sufficiently close to the identity. This proves that U' is non-empty, as desired. $\square$

Theorem 4.7. *For $1 \leq i \leq d$ let A_i be an abelian variety over a number field K and $a_i \in A_i(K)$ a rational point of infinite order. Then there exists a set of rational primes ℓ of Dirichlet density > 0 with the following property. Let $f(T) \in \mathbb{Z}[T]$ be any polynomial which is a product of cyclotomic polynomials and a power of T . For any finite place v of K let p_v denote the characteristic of the residue field and $a_{i,v}$ the reduction of a_i . Then for all finite places v of K in a set of Dirichlet density > 0 the ℓ -part of $f(p_v)a_{i,v}$ is non-trivial for every i .*

Proof. We apply the results of the preceding sections to $A := A_1 \times \dots \times A_d$ and $a := (a_1, \dots, a_d)$. By Corollary 1.3 there exists a set of rational primes ℓ of positive Dirichlet density for which the associated algebraic monodromy group G_ℓ splits over $\mathbb{Q}_\ell$. We will prove the theorem for any such ℓ .

Let $\mu : G_\ell \rightarrow \mathbb{G}_{m, \mathbb{Q}_\ell}$ be the multiplier character and let $\operatorname{pr}_i : A \rightarrow A_i$ denote the projection to the i^{th} factor. As in the proof of Theorem 4.1

$$U_f := \left\{ \tilde{\gamma} \in \tilde{\Gamma}_\ell \mid \begin{array}{l} \det(\gamma - \operatorname{id}) \neq 0, \text{ and} \\ \forall i : f(\mu(\gamma)) \operatorname{pr}_i \pi_\ell((\gamma - \operatorname{id})^{-1}n) \neq 0 \end{array} \right\}$$

is an open subset of $\tilde{\Gamma}_\ell$.

Lemma 4.8. U_f is non-empty.

Proof. As in the proof of Lemma 4.5, after replacing K by a finite extension we may assume that

$$\tilde{\Gamma}_\ell = \left\{ \begin{pmatrix} \gamma n - (\gamma - \operatorname{id})m \\ 0 & 1 \end{pmatrix} \mid \begin{array}{l} \gamma \in \Gamma_\ell \\ n \in N_\ell \end{array} \right\}$$

for some $m \in V_\ell(A)$. We must therefore find elements $\gamma \in \Gamma_\ell$ and $n \in N_\ell$ such that $\det(\gamma - \operatorname{id}) \neq 0$ and

$$f(\mu(\gamma)) \operatorname{pr}_i \pi_\ell((\gamma - \operatorname{id})^{-1}n - m) \neq 0$$

for every i . This second condition is equivalent to

$$f(\mu(\gamma)) \operatorname{pr}_i((\gamma - \operatorname{id})^{-1}n) \not\equiv f(\mu(\gamma)) \operatorname{pr}_i(m) \pmod{T_\ell(A_i)}.$$

Taking any integer r so that $\ell^r m \in T_\ell(A)$, it suffices to have

$$\ell^r f(\mu(\gamma)) \operatorname{pr}_i((\gamma - \operatorname{id})^{-1}n) \notin T_\ell(A_i). \quad (4.9)$$

Now by the assumption on ℓ there exists a split maximal torus $S_\ell \subset G_\ell$. Every character χ of S_ℓ is then defined over $\mathbb{Q}_\ell$. For any representation W of S_ℓ let $\operatorname{pr}_\chi : W \rightarrow W_\chi$ denote the projection to the weight space associated to χ . Recall from Proposition 2.5 that N_ℓ is open in U_ℓ , which is an algebraic representation of G_ℓ and hence of S_ℓ . Thus $N_{\ell,\chi} := V_\ell(A)_\chi \cap N_\ell$ is open in the weight space $U_{\ell,\chi}$. For every χ we want to select an element $n_\chi \in N_{\ell,\chi}$ such that for all i we have $\operatorname{pr}_i(n_\chi) \neq 0$ whenever $\operatorname{pr}_i(U_{\ell,\chi}) \neq 0$. This is possible, because any finite number of non-trivial linear inequalities in a free $\mathbb{Z}_\ell$ -module can be simultaneously satisfied. We will show the desired assertions with $n := \sum_\chi n_\chi \in N_\ell$ and a suitable element $\gamma \in S_\ell(\mathbb{Q}_\ell) \cap \Gamma_\ell$. To satisfy 4.9 it suffices to have

$$\forall i \exists \chi : \ell^r f(\mu(\gamma)) \operatorname{pr}_i((\gamma - \operatorname{id})^{-1}n_\chi) \notin \operatorname{pr}_\chi(T_\ell(A_i)).$$

As n_χ is an eigenvector of γ for the eigenvalue $\chi(\gamma) \in \mathbb{Z}_\ell$, this element is equal to

$$\frac{\ell^r f(\mu(\gamma))}{\chi(\gamma) - 1} \cdot \operatorname{pr}_i(n_\chi).$$

Fix an integer s so that for all i and χ with $\operatorname{pr}_i(n_\chi) \neq 0$ we have

$$\operatorname{pr}_i(n_\chi) \notin \ell^s \operatorname{pr}_\chi(T_\ell(A_i)).$$

By construction this affects all pairs (i, χ) with $\operatorname{pr}_i(U_{\ell,\chi}) \neq 0$. Thus it suffices to prove the following assertion, from which the n_χ have vanished.

Sublemma 4.10. *There exists an element $\gamma \in S_\ell(\mathbb{Q}_\ell) \cap \Gamma_\ell$ satisfying $\det(\gamma - \operatorname{id}) \neq 0$ such that for every i there exists a character χ with $\operatorname{pr}_i(U_{\ell,\chi}) \neq 0$ and*

$$\operatorname{ord}_\ell(\chi(\gamma) - 1) \geq r + s + \operatorname{ord}_\ell(f(\mu(\gamma))).$$

Proof. For every i let $B_i \subset A_i$ be the identity component of the Zariski closure of $\mathbb{Z}a_i$. Applying Proposition 1.5 to $B := B_1 \times \dots \times B_d$ shows that there exist weights χ_i of S_ℓ on $V_\ell(B_i) \subset V_\ell(A_i)$ so that μ is $\mathbb{Q}$ -linearly independent of $\chi_1, \dots, \chi_d$. The functoriality and Theorem 2.8 together imply that $\operatorname{pr}_i(U_\ell) = V_\ell(B_i)$. Since the projection map pr_i is S_ℓ -equivariant, we deduce that

$$\operatorname{pr}_i(U_{\ell,\chi_i}) = V_\ell(B_i)_{\chi_i} \neq 0.$$

It remains to find an element $\gamma \in S_\ell(\mathbb{Q}_\ell) \cap \Gamma_\ell$ with $\det(\gamma - \operatorname{id}) \neq 0$ and for all i

$$\operatorname{ord}_\ell(\chi_i(\gamma) - 1) \geq r + s + \operatorname{ord}_\ell(f(\mu(\gamma))). \quad (4.11)$$

The inequality 4.11 means that $\chi_i(\gamma)$ is much closer to the identity than $\mu(\gamma)$. To be precise let us first shrink Γ_ℓ so that Γ_ℓ acts trivially on $T_\ell(A)/\ell^2 T_\ell(A)$. Then for every element $\gamma \in S_\ell(\mathbb{Q}_\ell) \cap \Gamma_\ell$ we have $\mu(\gamma) \equiv 1 \pmod{\ell^2}$. On the other hand choose an integer $k > 0$ such that all non-zero roots of $f(T)$ are roots of unity of order dividing k and have multiplicity $\leq k$. Then after multiplying $f(T)$ by some more cyclotomic polynomials we may assume that $f(T) = T^{k'}(T^k - 1)^k$ for some $k' \geq 0$. A standard calculation now shows that

$$\begin{aligned} \text{ord}_\ell(f(\mu(\gamma))) &= k' \cdot \text{ord}_\ell(\mu(\gamma)) + k \cdot \text{ord}_\ell(\mu(\gamma)^k - 1) \\ &= k \cdot \text{ord}_\ell(k) + k \cdot \text{ord}_\ell(\mu(\gamma) - 1). \end{aligned}$$

Setting $t := r + s + k \cdot \text{ord}_\ell(k)$ we thus need to find an element $\gamma \in S_\ell(\mathbb{Q}_\ell) \cap \Gamma_\ell$ with $\det(\gamma - \text{id}) \neq 0$ and for all i

$$\text{ord}_\ell(\chi_i(\gamma) - 1) \geq t + k \cdot \text{ord}_\ell(\mu(\gamma) - 1). \quad (4.12)$$

To achieve this let S_ℓ^1 denote the identity component of $\text{Ker}(\mu|_{S_\ell})$, which is a subtorus of codimension 1. Since μ is $\mathbb{Q}$ -linearly independent of $\chi_1, \dots, \chi_d$ and S_ℓ splits over $\mathbb{Q}_\ell$, there exists a subtorus S_ℓ^2 of dimension 1 inside $\bigcap_{i=1}^d \text{Ker}(\chi_i|_{S_\ell})$ on which μ is non-trivial. We will take $\gamma = \gamma_1 \gamma_2$ with $\gamma_1 \in S_\ell^1(\mathbb{Q}_\ell) \cap \Gamma_\ell$ and $\gamma_2 \in S_\ell^2(\mathbb{Q}_\ell) \cap \Gamma_\ell$. Then the left hand side of 4.12 depends only on γ_1 , while the right hand side depends only on γ_2 .

Theorem 1.1 (c) implies that Γ_ℓ contains an open subgroup of $S_\ell(\mathbb{Q}_\ell)$. Thus if we first select any non-trivial γ_2 , the inequality 4.12 will hold for every γ_1 that is sufficiently close to the identity. Furthermore, none of the weights of S_ℓ on $V_\ell(A)$ is zero, e.g., by Proposition 1.4. Thus in any neighborhood of the identity γ_1 can be chosen such that $\gamma = \gamma_1 \gamma_2$ does not have the eigenvalue 1 on $V_\ell(A)$, which means that $\det(\gamma - \text{id}) \neq 0$. Thus all requirements can be simultaneously satisfied, finishing the proof of Sublemma 4.10 and hence of Lemma 4.8. $\square$

Now we return to the proof of Theorem 4.7. Since $U_f \subset \tilde{\Gamma}_\ell$ is a non-empty open subset, as in the proof of Theorem 4.1 we conclude that there exists a set of places $v \notin S$ of K of Dirichlet density > 0 for which $\tilde{\rho}_\ell(\text{Frob}_v) \in U_f$. We may also assume that the associated residue fields k_v have prime order, because the remaining places form a set of Dirichlet density 0. For these places we have $\mu \rho_\ell(\text{Frob}_v) = |k_v| = p_v$. The definition of U_f and Proposition 3.2 thus imply that $f(p_v) \text{pr}_i \kappa_v(a) \neq 0$ for every i . By the definition of κ_v this means that the ℓ -part of $f(p_v) a_{i,v}$ is non-trivial for every i , as desired. $\square$

Remark 4.13. Theorem 4.7 is not true in general for every rational prime ℓ , even for a single abelian variety A and a single rational point $a \in A(K)$. For a counterexample suppose that A is an elliptic curve with complex multiplication over K . Then $\text{End}_K(A)$ is an order in an imaginary quadratic number field F , and for any rational prime ℓ the image of Galois is an open compact subgroup of $(F \otimes \mathbb{Q}_\ell)^*$.

Thus G_ℓ splits over $\mathbb{Q}_\ell$ if and only if ℓ splits in F , and in this case the proof of Theorem 4.7 goes through.

If ℓ does not split in F , we will show that the theorem is false. It is known that for every finite place $v \notin S$ with $|k_v| = p_v$ the element $\alpha_v := \rho_\ell(\text{Frob}_v)$ is an algebraic integer in F with $\alpha_v \bar{\alpha}_v = p_v$ and that the cardinality of $A_v(k_v)$ is equal to $(\alpha_v - 1)(\bar{\alpha}_v - 1)$. In particular the integer $(\alpha_v - 1)(\bar{\alpha}_v - 1)$ annihilates the reduction of a . Now the fact that F has only one prime above ℓ implies that

$$\text{ord}_\ell(\alpha_v - 1) = \text{ord}_\ell(\bar{\alpha}_v - 1) \leq \text{ord}_\ell(\alpha_v \bar{\alpha}_v - 1) = \text{ord}_\ell(p_v - 1).$$

Thus with $f(T) := (T - 1)^2$ we deduce that

$$\text{ord}_\ell((\alpha_v - 1)(\bar{\alpha}_v - 1)) \leq 2 \cdot \text{ord}_\ell(p_v - 1) = \text{ord}_\ell(f(p_v)).$$

This implies that $f(p_v)$ annihilates the ℓ -part of the reduction of a . Since this is so for every $v \notin S$, we conclude that in this example Theorem 4.7 is true precisely for ℓ in a set of Dirichlet density $1/2$.

5. Density results for the full reduction

In this section we derive some consequences of the density results of the preceding section which no longer refer to any particular prime ℓ .

Theorem 5.1. *For $1 \leq i \leq d$ let A_i be an abelian variety over a number field K and $a_i \in A_i(K)$ a rational point. Assume that for all finite places v of K in a set of Dirichlet density 1 the reduction of at least one a_i is annihilated by a power of the residue characteristic p_v . Then at least one $a_i = 0$.*

Proof. Suppose that some a_i is a torsion point of order n . If $n = 1$, we are done. Otherwise the order of the reduction of a_i at any finite place $v \nmid n$ is still n , and therefore not a power of p_v . Thus after removing A_i and a_i from the list the assumptions still hold. After iterating this we may assume that all a_i have infinite order; we must then derive a contradiction. Select any rational prime ℓ . Then by Theorem 4.4 for all finite places $v \nmid \ell$ of K in a set of Dirichlet density > 0 the ℓ -part of the reduction of every a_i is non-trivial. In particular, these reductions are not annihilated by a power of p_v , contradicting the given assumption. $\square$

Remark 5.2. Damian Roessler pointed out to the author that Theorem 5.1 can also be deduced from a theorem of Wong [16]. To sketch this set $A := A_1 \times \dots \times A_d$. For any prime ℓ let $\Gamma_{\ell,1}$ denote the image of $\text{Gal}(\bar{K}/K)$ in its action on the ℓ -torsion subgroup $A[\ell]$. By a theorem of Serre, which for example follows from [15, Th. 2], the group cohomology $H^1(\Gamma_{\ell,1}, A[\ell])$ vanishes for all $\ell \gg 0$. We temporarily fix any such $\ell > d$.

The assumptions in Theorem 5.1 imply that for all v in a set of Dirichlet density 1 the reduction of at least one a_i has trivial ℓ -part. Since multiplication by

ℓ induces an automorphism on the prime-to- ℓ part of $A_v(k_v)$, the reduction of a_i then lies in $\ell A_v(k_v)$. Wong [16, Th. 2] deduces from this that at least one a_i is contained in $\ell A(K)$. Since this is true for every $\ell \gg 0$, and the Mordell-Weil group $A(K)$ is finitely generated, this implies that at least one a_i is torsion. As in the proof of 5.1 we now deduce that at least one $a_i = 0$, as desired.

Theorem 5.3. *For $1 \leq i \leq d$ let A_i be an abelian variety over a number field K and $a_i \in A_i(K)$ a rational point. Let $f(T) \in \mathbb{Z}[T]$ be any polynomial which is a product of cyclotomic polynomials and a power of T . For any finite place v of K let p_v denote the characteristic of the residue field and $a_{i,v}$ the reduction of a_i . Assume that for all finite places v of K in a set of Dirichlet density 1 at least one $a_{i,v}$ is annihilated by $f(p_v)$. Then at least one a_i is a torsion point.*

Proof. Suppose that every a_i has infinite order. Then by Theorem 4.7 there exists a rational prime ℓ such that for all finite places v of K in a set of Dirichlet density > 0 the ℓ -part of every $f(p_v)a_{i,v}$ is non-trivial. In particular, these $a_{i,v}$ are not annihilated by $f(p_v)$, contradicting the given assumption. Thus the order of at least one a_i is finite. $\square$

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