Information-Distilling Quantizers

Conference Paper**Author(s):**

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Publication date:

2018-02

Permanent link:

<https://doi.org/10.3929/ethz-b-000245059>

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Let X and Y be a pair of random variables with a given distribution P_{XY} . This work deals with the problem of quantizing Y into M values, under the objective of maximizing the mutual information between the quantizer's output and X . We denote the value of the mutual information attained by the optimal M -ary quantizer by

$$I(X; [Y]_M) \triangleq \sup_{\tilde{Y} \in [Y]_M} I(X; \tilde{Y}). \quad (1)$$

where $[Y]_M$ is the set of all (deterministic) M -ary quantizations of Y ,

$$[Y]_M \triangleq \{f(Y) : f : \mathcal{Y} \rightarrow [M]\}.$$

When X and Y are thought of as the input and output of a channel, respectively, the problem boils down to designing the M -level quantizer that maximizes the information rate, whereas (1) is the highest information rate attainable. It is therefore not surprising that this problem has received considerable attention [1]–[5]. For example, it is well known [6, Section 2.11] that when X is a BPSK input to an AWGN channel with output Y it holds that $I(X; [Y]_2) \geq 2I(X; Y)/\pi$ and this is achieved by taking $f(\cdot)$ to be the maximum a posteriori (MAP) estimator of X from Y .

Our focus is studying the fundamental properties of the function $I(X; [Y]_M)$, and in particular, identifying the joint distributions P_{XY} that are the most difficult to quantize, and characterizing the behavior of $I(X; [Y]_M)$ for these cases. Special attention is given to the symmetric binary case where $X \sim \text{Bernoulli}(1/2)$. In this setting, it may seem that the optimal binary quantizer should always retain a significant fraction of $I(X; Y)$, and that the MAP quantizer should be sufficient to this end. For large $I(X; Y)$, it is not difficult to see that this is indeed the case [7, Proposition 5]. However, if $Y \in \{0, 1, ?\}$ is the output of a binary erasure channel with input X , for large erasure probabilities the MAP quantizer may be arbitrarily inferior to the asymmetric quantizer $f(0) = 0$, $f(1) = f(?) = 1$ [7, Section III.c].

Furthermore, in certain cases, no binary quantizer can retain a significant fraction of $I(X; Y)$. Our main result is the following [7, Theorem 1]. Logarithms are taken w.r.t. base 2, with the exception of the $\ln$ function that is taken w.r.t. base e .

Theorem 1: If $X \sim \text{Bernoulli}(1/2)$ and $I(X; Y) = \beta > 0$, we have for binary quantization

$$I(X; [Y]_2) \geq \frac{1}{3e} \frac{\beta}{1 + \ln\left(\frac{1}{\beta}\right)}. \quad (2)$$

Furthermore, for any $\eta \in (0, 1)$ and any natural $M < \frac{12 \max\{\log(\frac{1}{\beta}), 1\}}{(1-\eta)^2}$

$$I(X; [Y]_M) \geq (M-1) \frac{\beta}{\max\{\log(\frac{1}{\beta}), 1\}} \frac{\eta(1-\eta)^2}{12}. \quad (3)$$

Finally, for any $0 < \beta \leq 1$, there exist distributions P_{XY} with $X \sim \text{Bernoulli}(1/2)$ and $I(X; Y) = \beta$, for which

$$I(X; [Y]_M) \leq 2M \frac{\beta}{\ln\left(\frac{e \log(e)}{2\beta}\right)}, \quad (4)$$

for every natural M .

Note that this is in stark contrast to the intuition from the binary AWGN channel. While for the former, two quantization levels suffice for retaining a $2/\pi$ fraction of $I(X; Y)$, Theorem 1 shows that there exist distributions for which at least $\Omega(\log(1/I(X; Y)))$ quantization levels are needed in order to retain a fixed fraction of $I(X; Y)$.

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