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Angelsberg, Gilles

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A monotonicity formula for stationary biharmonic maps[★]

Gilles Angelsberg

Departement für Mathematik, ETH Zentrum, Rämistrasse 101, CH-8092 Zürich, Switzerland (e-mail: gilles@math.ethz.ch)

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Abstract. We give a rigorous proof of the monotonicity formula of S.-Y.A. Chang, L. Wang and P. Yang [3] for (extrinsically) stationary biharmonic maps of class $W^{2,2}$.

1. Introduction

For several decades regularity properties of weakly harmonic maps have been intensely studied. For a manifold M of dimension $m \leq 2$, C.B. Morrey [9] showed in 1948 that every minimizing map $u \in W^{1,2}(M, N)$ belongs to $C^\infty(M, N)$. The regularity problem for general critical points of the harmonic energy functional had remained open for a long time. In 1981, still for the case $m \leq 2$, M. Grüter [5] proved smoothness of conformal weakly harmonic maps. R. Schoen [12] introduced the notion of stationary harmonic maps and extended Grüter's result to this class. Finally, F. Hélein [6] showed that every weakly harmonic map in the case $m \leq 2$ is smooth. For $m \geq 3$, more complex phenomena show up. R. Schoen and K. Uhlenbeck [13] showed that if $u \in W^{1,2}(M, N)$ is energy minimizing, then u is smooth except on a closed subset of finite H^{m-3} -measure. This result is optimal since the radial projection from B^m into S^{m-1} is a minimizing map for $m \geq 3$, as shown by H. Brézis, J.-M. Coron and E. Lieb [2] for $m = 3$ and F.H. Lin [8] for $m \geq 3$. On the other hand, T. Rivière [11] proved existence of everywhere discontinuous weakly harmonic maps. For the intermediate class of stationary harmonic maps, L.C. Evans [4] showed partial regularity for maps into the sphere and F. Bethuel [1] generalized this result for arbitrary target manifolds. Their proofs rely on a monotonicity formula for stationary harmonic maps adapted from P. Price [10].

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Similar questions for (extrinsically) weakly biharmonic maps were studied by S.-Y.A. Chang, L. Wang and P. Yang in [3]. They showed smoothness for weakly biharmonic maps into the sphere and $m \leq 4$, and asserted partial regularity for stationary biharmonic maps into the sphere and $m \geq 5$. C.Y. Wang generalized these results for arbitrary target manifolds in [14] and [15]. Once again, a monotonicity formula derived from the stationarity assumption is crucial in the proof of partial regularity for $m \geq 5$. However, the derivation in [3, Proposition 3.2] is only given for sufficiently regular maps.

Here, we give a rigorous proof of this monotonicity formula in the case of stationary biharmonic maps of class $W^{2,2}(B_r, N)$.

Conceivably, this monotonicity formula may allow to study the singular behaviour of stationary biharmonic maps and especially minimizing biharmonic maps as suggested by M-C. Hong and C.Y. Wang in [7].

2. Stationarity Assumption

We recall some definitions and fix the notations used henceforth: Let B_r be the open ball of radius $r > 0$ in $\mathbb{R}^m$ centered at the origin and let N be a smooth Riemannian manifold isometrically embedded in $\mathbb{R}^n$. Recall that $W^{2,2}(B_r, N)$ is defined by

$$W^{2,2}(B_r, N) := \{u \in W^{2,2}(B_r, \mathbb{R}^n) : u(x) \in N \text{ for a.e. } x \in B_r\}.$$

We consider on $W^{2,2}(B_r, N)$ the Hessian energy functional

$$E(u) := \int_{B_r} |\Delta u|^2 dx.$$

Definition 1. A map u in $W^{2,2}(B_r, N)$ is called (extrinsically) weakly biharmonic if it is a critical point of the Hessian energy $E(\cdot)$ with respect to compactly supported variations on the target manifold, i.e. if

$$\frac{d}{dt} \Big|_{t=0} E(\pi \circ (u + t\xi)) = 0 \text{ for all } \xi \in C_0^\infty(B_r, \mathbb{R}^n),$$

where π denotes the nearest point projection onto N .

Definition 2. A weakly biharmonic map u in $W^{2,2}(B_r, N)$ is called (extrinsically) stationary biharmonic if, in addition, u is a critical point of the Hessian energy $E(\cdot)$ with respect to compactly supported variations on the domain manifold, i.e. if

$$\frac{d}{dt} \Big|_{t=0} E(u \circ (id + t\xi)) = 0 \text{ for all } \xi \in C_0^\infty(B_r, \mathbb{R}^m), \quad (1)$$

where id denotes the identity map.

In what follows we denote $u = (u^1, \dots, u^n)$, $u_j = \frac{\partial u}{\partial x_j}$ etc. We tacitly sum over repeated indices. We use the following notation

$$\int_{\partial B_r \setminus \partial B_\rho} f d\sigma := \int_{\partial B_r} f d\sigma - \int_{\partial B_\rho} f d\sigma.$$

We replace Lemma 3.1 in [3] by the following lemma.

Lemma 1. *If u is a stationary biharmonic map in $W^{2,2}(B_{2r}, N)$, then we have:*

$$\int_{B_{2r}} \left(4u_{kk}u_{ij}\xi_i^j + 2u_{kk}u_j\xi_{ii}^j - |\Delta u|^2 \nabla \cdot \xi \right) dx = 0$$

for every test function $\xi \in C_0^\infty(B_{2r}, \mathbb{R}^m)$.

Proof. We compute

$$\Delta u_t = (\Delta u)_t + t \left(2(u_{ij})_t \xi_i^j + (u_j)_t \xi_{ii}^j \right) + t^2 (u_{jk})_t \xi_i^k \xi_i^j,$$

where for any f we denote $f_t(y) := f(y + t\xi(y))$ for $\xi \in C_0^\infty(B_{2r}, \mathbb{R}^m)$. The change of variables $x = y + t\xi$ for $|t|$ sufficiently small induces a C^∞ -diffeomorphism of B_{2r} onto itself. The stationarity assumption (1) then implies:

$$\begin{aligned} 0 &= \frac{d}{dt} \Big|_{t=0} \int_{B_{2r}} |\Delta u_t|^2 (x - t\xi) \left(1 - t \nabla \cdot \xi + O(t^2) \right) dx \\ &= \int_{B_{2r}} \left(4u_{kk}u_{ij}\xi_i^j + 2u_{kk}u_j\xi_{ii}^j - |\Delta u|^2 \nabla \cdot \xi \right) dx. \end{aligned}$$

□

3. Monotonicity formula

We can now rigorously derive the monotonicity formula Proposition 3.2. in [3] for almost every radius. This is sufficient to prove partial regularity for (extrinsically) stationary biharmonic maps of class $W^{2,2}$.

Theorem 1 (Monotonicity formula). *For $K > 0$ and $u \in W^{2,2}(B_K, N)$ (extrinsically) stationary biharmonic, it holds for a.e. $0 < \rho < r \leq \frac{K}{2}$*

$$r^{4-m} \int_{B_r} |\Delta u|^2 dx - \rho^{4-m} \int_{B_\rho} |\Delta u|^2 dx = P + R,$$

where

$$\begin{aligned} P &= 4 \int_{B_r \setminus B_\rho} \left(\frac{(u_j + x^i u_{ij})^2}{|x|^{m-2}} + \frac{(m-2)(x^i u_i)^2}{|x|^m} \right) dx \\ R &= 2 \int_{\partial B_r \setminus \partial B_\rho} \left(-\frac{x^i u_j u_{ij}}{|x|^{m-3}} + 2 \frac{(x^i u_i)^2}{|x|^{m-1}} - 2 \frac{|\nabla u|^2}{|x|^{m-3}} \right) d\sigma. \end{aligned}$$

Proof. Choose in Lemma 1 the test functions $\xi(x) := \psi^\epsilon \left(\frac{|x|}{\tau} \right) x$, where $\psi = \psi^\epsilon : \mathbb{R}_+ \rightarrow [0, 1]$ is smooth with compact support on $[0, 1]$ and $\psi^\epsilon \equiv 1$ on $[0, 1 - \epsilon]$. Then, it follows with $\xi_i^j = \psi_i x^j + \psi \delta_{ij}$ and $\xi_{ii}^j = \psi_{ii} x^j + 2\psi_j$

$$\begin{aligned} 0 &= \int_{\mathbb{R}^m} \left((4-m) |\Delta u|^2 \psi - |\Delta u|^2 \psi_i x^i + 4u_{kk}u_{ij} \psi_i x^j \right. \\ &\quad \left. + 4u_{kk}u_j \psi_j + 2u_{kk}u_j \psi_{ii} x^j \right) dx. \end{aligned}$$

We have $\psi_i \left(\frac{|x|}{\tau} \right) = \frac{1}{\tau} \psi' \left(\frac{|x|}{\tau} \right) \frac{x^i}{|x|}$ and $\psi_{ii} \left(\frac{|x|}{\tau} \right) = \frac{1}{\tau^2} \psi'' \left(\frac{|x|}{\tau} \right) + \frac{(m-1)}{\tau} \psi' \left(\frac{|x|}{\tau} \right) \frac{1}{|x|}$.

Thus,

$$0 = (4 - m) \int_{\mathbb{R}^m} |\Delta u|^2 \psi dx - \frac{1}{\tau} \int_{\mathbb{R}^m} |\Delta u|^2 \psi' |x| dx + \frac{4}{\tau} \int_{\mathbb{R}^m} u_{kk} u_{ij} \psi' \frac{x^i x^j}{|x|} dx \\ + \frac{2(m+1)}{\tau} \int_{\mathbb{R}^m} u_{kk} u_j \psi' \frac{x^j}{|x|} dx + \frac{2}{\tau^2} \int_{\mathbb{R}^m} u_{kk} u_j \psi'' x^j dx.$$

This implies for $I^\epsilon(\tau) := \tau^{4-m} \int_{\mathbb{R}^m} |\Delta u|^2 \psi^\epsilon \left(\frac{|x|}{\tau} \right) dx$

$$\begin{aligned} \tau^{m-3} \frac{d}{d\tau} I^\epsilon(\tau) &= (4 - m) \int_{\mathbb{R}^m} |\Delta u|^2 \psi dx - \frac{1}{\tau} \int_{\mathbb{R}^m} |\Delta u|^2 \psi' |x| dx \\ &= -\frac{4}{\tau} \int_{\mathbb{R}^m} u_{kk} u_{ij} \psi' \frac{x^i x^j}{|x|} dx - \frac{2(m+1)}{\tau} \int_{\mathbb{R}^m} u_{kk} u_j \psi' \frac{x^j}{|x|} dx \\ &\quad - \frac{2}{\tau^2} \int_{\mathbb{R}^m} u_{kk} u_j \psi'' x^j dx. \end{aligned} \quad (2)$$

Furthermore, we have:

$$\begin{aligned} \int_\rho^r \psi'' \left(\frac{|x|}{\tau} \right) \tau^{1-m} d\tau &= - \int_\rho^r \frac{d}{d\tau} \left(\psi' \left(\frac{|x|}{\tau} \right) \right) \tau^{3-m} \frac{1}{|x|} d\tau \\ &= -\psi' \left(\frac{|x|}{r} \right) \frac{r^{3-m}}{|x|} + \psi' \left(\frac{|x|}{\rho} \right) \frac{\rho^{3-m}}{|x|} \\ &\quad + \frac{3-m}{|x|} \int_\rho^r \psi' \left(\frac{|x|}{\tau} \right) \tau^{2-m} d\tau. \end{aligned}$$

Thus, applying Fubini's theorem twice gives:

$$\begin{aligned} &\int_\rho^r \tau^{1-m} \int_{\mathbb{R}^m} u_{kk} u_j \psi'' \left(\frac{|x|}{\tau} \right) x^j dx d\tau \\ &= - \int_{\mathbb{R}^m} u_{kk} u_j x^j \psi' \left(\frac{|x|}{r} \right) \frac{r^{3-m}}{|x|} dx + \int_{\mathbb{R}^m} u_{kk} u_j x^j \psi' \left(\frac{|x|}{\rho} \right) \frac{\rho^{3-m}}{|x|} dx \\ &\quad + (3-m) \int_\rho^r \tau^{2-m} \int_{\mathbb{R}^m} u_{kk} u_j x^j \psi' \left(\frac{|x|}{\tau} \right) \frac{1}{|x|} dx d\tau. \end{aligned} \quad (3)$$

Multiplying both sides of (2) with τ^{3-m} , integrating over τ from ρ to r and inserting (3) yield

$$\begin{aligned} I^\epsilon(r) - I^\epsilon(\rho) &= -4 \int_\rho^r \tau^{2-m} \int_{\mathbb{R}^m} u_{kk} u_{ij} \psi' \left(\frac{|x|}{\tau} \right) \frac{x^i x^j}{|x|} dx d\tau \\ &\quad - 8 \int_\rho^r \tau^{2-m} \int_{\mathbb{R}^m} u_{kk} u_j \psi' \left(\frac{|x|}{\tau} \right) \frac{x^j}{|x|} dx d\tau \\ &\quad + 2 \int_{\mathbb{R}^m} u_{kk} u_j x^j \psi' \left(\frac{|x|}{r} \right) r^{3-m} \frac{1}{|x|} dx \\ &\quad - 2 \int_{\mathbb{R}^m} u_{kk} u_j x^j \psi' \left(\frac{|x|}{\rho} \right) \rho^{3-m} \frac{1}{|x|} dx. \end{aligned}$$

For all Lebesgue points ρ and r of the function $g(s) := \int_{\partial B_s} \frac{u_{kk} u_j x^j}{|x|^{m-3}} d\sigma \in L^1_{\text{loc}}([0, K])$, as $\epsilon \rightarrow 0$, by applying Lemma 2 in the Appendix to the first two terms and the Lebesgue differentiation theorem to the last two terms, we obtain

$$\begin{aligned}
& r^{4-m} \int_{B_r} |\Delta u|^2 dx - \rho^{4-m} \int_{B_\rho} |\Delta u|^2 dx \\
&= \int_{B_r \setminus B_\rho} \left(4 \frac{u_{kk} u_{ij} x^i x^j}{|x|^{m-2}} + 8 \frac{u_{kk} u_j x^j}{|x|^{m-2}} \right) dx - 2 \int_{\partial B_r \setminus \partial B_\rho} \frac{u_{kk} u_j x^j}{|x|^{m-3}} d\sigma \\
&= 2 \int_{B_r \setminus B_\rho} \left(\frac{u_{kk} u_{ij} x^i x^j}{|x|^{m-2}} - \frac{u_{kki} u_j x^i x^j}{|x|^{m-2}} + \frac{u_{kk} u_j x^j}{|x|^{m-2}} \right) dx.
\end{aligned}$$

Here and henceforth, all third derivatives are interpreted in the sense of distributions. The monotonicity formula now follows by several integrations by parts. We proceed as in [3] and compute for a.e. ρ and r

$$\begin{aligned}
\int_{B_r \setminus B_\rho} \frac{u_{kk} u_{ij} x^i x^j}{|x|^{m-2}} dx &= \int_{\partial B_r \setminus \partial B_\rho} \frac{u_k u_{ij} x^i x^j x^k}{|x|^{m-1}} d\sigma - \int_{B_r \setminus B_\rho} \frac{u_k u_{ijk} x^i x^j x^k}{|x|^{m-2}} dx \\
&\quad + \int_{B_r \setminus B_\rho} \left(-2 \frac{u_k u_{ik} x^i}{|x|^{m-2}} + \frac{(m-2) u_k u_{ij} x^i x^j x^k}{|x|^m} \right) dx \\
&= \int_{\partial B_r \setminus \partial B_\rho} \left(\frac{u_k u_{ij} x^i x^j x^k}{|x|^{m-1}} - \frac{u_k u_{ik} x^i}{|x|^{m-3}} \right) d\sigma \\
&\quad + \int_{B_r \setminus B_\rho} \left(\frac{u_k u_{ik} x^i}{|x|^{m-2}} + \frac{u_{ik} u_{jk} x^i x^j}{|x|^{m-2}} \right) dx \\
&\quad + (m-2) \int_{B_r \setminus B_\rho} \frac{u_k u_{ij} x^i x^j x^k}{|x|^m} dx, \\
\int_{B_r \setminus B_\rho} -\frac{u_{kki} u_j x^i x^j}{|x|^{m-2}} dx &= - \int_{\partial B_r \setminus \partial B_\rho} \frac{u_{ki} u_j x^i x^j x^k}{|x|^{m-1}} d\sigma \\
&\quad + \int_{B_r \setminus B_\rho} \left(\frac{u_{kj} u_j x^k}{|x|^{m-2}} + \frac{u_j u_{kk} x^j}{|x|^{m-2}} \right) dx \\
&\quad + \int_{B_r \setminus B_\rho} \left(\frac{(2-m) u_j u_{ik} x^i x^j x^k}{|x|^m} + \frac{u_{ik} u_{jk} x^i x^j}{|x|^{m-2}} \right) dx, \\
\int_{B_r \setminus B_\rho} \frac{u_{kk} u_j x^j}{|x|^{m-2}} dx &= \int_{\partial B_r \setminus \partial B_\rho} \frac{(u_k x^k)^2}{|x|^{m-1}} d\sigma - \int_{B_r \setminus B_\rho} \frac{|\nabla u|^2}{|x|^{m-2}} dx \\
&\quad + \int_{B_r \setminus B_\rho} \left(\frac{(m-2)(u_k x^k)^2}{|x|^m} - \frac{u_k u_{jk} x^j}{|x|^{m-2}} \right) dx.
\end{aligned}$$

Combining these equations leads to

$$\begin{aligned}
& r^{4-m} \int_{B_r} |\Delta u|^2 dx - \rho^{4-m} \int_{B_\rho} |\Delta u|^2 dx \\
&= 2 \int_{\partial B_r \setminus \partial B_\rho} \left(\frac{(u_i x^i)^2}{|x|^{m-1}} - \frac{u_j u_{ij} x^i}{|x|^{m-3}} \right) d\sigma \\
&\quad + 2 \int_{B_r \setminus B_\rho} \left(\frac{u_j u_{ij} x^i}{|x|^{m-2}} + 2 \frac{(u_j x^j)^2}{|x|^{m-2}} + \frac{u_i u_{jj} x^i}{|x|^{m-2}} - \frac{|\nabla u|^2}{|x|^{m-2}} + \frac{(m-2)(u_i x^i)^2}{|x|^m} \right) dx.
\end{aligned}$$

Moreover, it holds

$$0 = -4 \int_{\partial B_r \setminus \partial B_\rho} \frac{|\nabla u|^2}{|x|^{m-3}} d\sigma + 8 \int_{B_r \setminus B_\rho} \left(\frac{|\nabla u|^2}{|x|^{m-2}} + \frac{u_j u_{ij} x^i}{|x|^{m-2}} \right) dx$$

and

$$0 = 2 \int_{\partial B_r \setminus \partial B_\rho} \frac{(u_i x^i)^2}{|x|^{m-1}} d\sigma - 2 \int_{B_r \setminus B_\rho} \left(\frac{u_j u_{ij} x^i}{|x|^{m-2}} + \frac{|\nabla u|^2}{|x|^{m-2}} + \frac{u_i u_{jj} x^i}{|x|^{m-2}} + \frac{(2-m)(u_i x^i)^2}{|x|^m} \right) dx.$$

Adding the last three equations establishes the monotonicity formula. $\square$

Appendix

For $f, \phi \in L^1(\mathbb{R}, \mathbb{R})$, define

$$(\phi * f)(x) := \int_{\mathbb{R}} \phi\left(\frac{y}{x}\right) f(y) dy = \int_{\mathbb{R}} \phi(z) f(xz) x dz.$$

Then, there holds the following

Lemma 2. Suppose $f, (\phi_k)_{k \in \mathbb{N}} \in L^1(\mathbb{R}, \mathbb{R})$, with

$$\phi_k \geq 0, \int_{\mathbb{R}} \phi_k = 1 \text{ and } \text{supp } \phi_k \subset \left[1 - \frac{1}{k}, 1\right], \quad k \in \mathbb{N}.$$

Then, as $k \rightarrow \infty$ for $0 < \rho < r$

$$\phi_k * f \rightarrow id \cdot f \text{ in } L^1([\rho, r]),$$

where $(id \cdot f)(x) = xf(x)$.

Proof. Observe that

$$\|f(h \cdot) - f\|_{L^1(\mathbb{R}, \mathbb{R})} \rightarrow 0 \text{ for } h \rightarrow 1 \quad (4)$$

and

$$(\phi_k * f)(x) - xf(x) = x \int_{\mathbb{R}} \phi_k(z) (f(xz) - f(x)) dz.$$

Thus, we have using Fubini's theorem and Hölder's inequality:

$$\begin{aligned} \|\phi_k * f - xf\|_{L^1([\rho, r])} &\leq \int_{\rho}^r |x| \int_{\mathbb{R}} |\phi_k(z)| |f(xz) - f(x)| dx dz \\ &\leq r \int_{\mathbb{R}} |\phi_k(z)| \left(\int_{\rho}^r |f(xz) - f(x)| dx \right) dz \\ &\leq r \|\phi_k\|_{L^1} \sup_{h \in \text{supp } \phi_k} \|f(h \cdot) - f\|_{L^1}. \end{aligned}$$

As $\text{supp } \phi_k \subset [1 - \frac{1}{k}, 1]$ and $\|\phi_k\|_{L^1} \leq 1$, assumption (4) implies that the right hand side converges to zero for $k \rightarrow \infty$. $\square$

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References

1. Bethuel, F.: On the singular set of stationary harmonic maps. *Manuscripta Math.* **78**, 417–443 (1993)
2. Brézis, H., Coron, J.-M., Lieb, E.: Harmonic maps with defects. *Comm. Math. Phys.* **107**, 649–705 (1986)
3. Chang, S.-Y.A., Wang, L., Yang, P.: A regularity theory for biharmonic maps. *Comm. Pure Appl. Math.* **52**, 1113–1137 (1999)
4. Evans, L.C.: Partial regularity for stationary harmonic maps into the sphere. *Arch. Rational Mech. Anal.* **116**, 101–113 (1991)
5. Grüter, M.: Regularity of weak H-surfaces. *J. reine angew. Math.* **329**, 1–15 (1981)
6. Hélein, F.: Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne. *C.R. Acad. Sci. Paris Sér. I Math.* **312**, 591–596 (1991)
7. Hong, M.-C., Wang, C.Y.: Regularity and relaxed problems of minimizing biharmonic maps into spheres. Preprint
8. Lin, F.H.: Une remarque sur l'application $x/|x|$. *C.R. Acad. Sci. Paris Sér. I Math.* **305**, 529–531 (1987)
9. Morrey C.B. Jr.: The problem of Plateau on a Riemannian manifold. *Ann. Math.* **49**, 807–951 (1948)
10. Price, P.: A monotonicity formula for Yang-Mills fields. *Manuscripta Math.* **43**, 131–166 (1983)
11. Rivière, T.: Everywhere discontinuous harmonic maps into spheres. *Acta Math.* **175**, 197–226 (1995)
12. Schoen, R.: Analytic aspects of the harmonic map problem. Seminar on Nonlinear Partial Differential Equations (S.S. Chern editor), MSRI Publications, vol. 2, Springer, New York, 1984
13. Schoen, R., Uhlenbeck, K.: A regularity theory for harmonic maps. *J. Differential Geometry* **17**, 307–335 (1982)
14. Wang, C.Y.: Biharmonic maps from $\mathbb{R}^4$ into a Riemannian manifold. *Math. Z.* (to appear)
15. Wang, C.Y.: Stationary biharmonic maps from $\mathbb{R}^m$ into a Riemannian manifold. *Comm. Pure Appl. Math.* **57**, 419–444 (2004)