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ON POLYNOMIALS WITH COEFFICIENTS OF MODULUS ONE

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Introduction

As in Littlewood [5], we let $\mathcal{G}_n$ be the class of all polynomials of the form

$$g_n(\theta) = \sum_{k=0}^n \exp(\alpha_k i) z^k,$$

where the α_k are arbitrary real constants and $z = \exp(2\pi i\theta)$. Clearly $\|g_n\|_{L^2} = (n+1)^{\frac{1}{2}}$ for all $g_n \in \mathcal{G}_n$, and the question "how close can such a g_n come to satisfying $|g_n| \equiv (n+1)^{\frac{1}{2}}?$ " has long been the object of intense study. In [5] Littlewood conjectured that there are positive absolute constants A_1 and A_2 such that, for arbitrarily large n , there exist $g_n \in \mathcal{G}_n$ with $A_1 n^{\frac{1}{2}} \leq |g_n(\theta)| \leq A_2 n^{\frac{1}{2}}$ for all θ . In [3] Erdős conjectured that there is a universal constant $c > 0$ such that for $n \geq 2$, $\|g_n\|_{\infty} \geq (1+c)n^{\frac{1}{2}}$ for all $g_n \in \mathcal{G}_n$.

It was shown by Littlewood [5] that the function

$$g(\theta) = \sum_{m=0}^n \exp(\tfrac{1}{2}m(m+1)\pi i(n+1)^{-1}) z^m$$

satisfies: (i) for any $\delta > 0$, $|g|n^{-\frac{1}{2}} \rightarrow 1$ uniformly in $n^{-\frac{1}{2}+\delta} \leq |\theta| \leq \frac{1}{2}$, and (ii) $|g| \leq 1.4n^{\frac{1}{2}}$ for all θ . In the first part of this paper we strengthen part (i) of the Littlewood result in two ways by producing polynomials g which yield an improved estimate for $|g|n^{-\frac{1}{2}}$ in a larger subset of the unit circle. In the second part we use the methods already developed to construct functions which are "almost" in $\mathcal{G}_n$ and which satisfy the Littlewood conjecture with (within the error) $A_1 = A_2 = 1$.

1. To begin our work, we require two elementary lemmas.

LEMMA 1. *Let*

$$F(x, T) = (1 - e^{2\pi i x}) \sum_{m=0}^{\infty} \frac{e^{2\pi i m x}}{m + T},$$

where $T > 1$ and F is defined by continuity when x is an integer. Then $|F(x, T)| < 3/T$ for all x .

Proof of Lemma 1. Let x be fixed and not an integer. Then

$$F(x, T) = \lim_{R \rightarrow \infty} (1 - e^{2\pi i x}) \sum_{m=0}^R \frac{e^{2\pi i m x}}{m + T} = \frac{1}{T} - \frac{1}{T} \sum_{m=1}^{\infty} \frac{e^{2\pi i m x}}{(m^2 - m)T^{-1} + 2m + T - 1},$$

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and

$$\sum_{m=1}^{\infty} \frac{1}{(m^2 - m)T^{-1} + 2m + T - 1} = \sum_1^{[T]} + \sum_{[T]+1}^{\infty} < 1 + T \int_T^{\infty} \frac{dx}{x^2} = 2,$$

completing the proof of Lemma 1.

LEMMA 2. *If r and x are not integers, then*

$$\sum_{m=-\infty}^{\infty} \frac{e^{2\pi i m x}}{m + r} = \frac{2\pi i e^{2\pi i k r}}{1 - e^{-2\pi i r}} e^{-2\pi i r x},$$

where $k = [x]$.

Proof of Lemma 2. Simply compute the Fourier Series of the function $F(x)$ of period 1 which, for $0 \leq x < 1$, is given by

$$F(x) = \frac{2\pi i}{1 - e^{-2\pi i r}} e^{-2\pi i r x}.$$

Employing these Lemmas, we are able to prove our basic result.

THEOREM 1. *Let N be a positive integer, and define the function*

$$P \in \mathcal{G}_{N^2-1} \text{ by } P(\theta) = \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \exp(2\pi i j k N^{-1}) z^{j+kN}, \quad z = \exp(2\pi i \theta).$$

Then

- (a) $\left| P\left(\frac{j}{N^2}\right) \right| = N$ for all integers j ,
- (b) For any ε , $N^{-1} < \varepsilon < \frac{1}{2}$, $|P(\theta)| = N + E$ for $-1 + \varepsilon \leq \theta \leq -\varepsilon$, where $|E| < 1 + 2\pi^{-1} + 5(\pi\varepsilon)^{-1}$,
- (c) For N odd, $P\left(\frac{1}{2N}\right) = O(1)$, while for N even, $P\left(\frac{N-1}{2N^2}\right) = O(1)$, and
- (d) $|P(\theta)| < \left(2 + \frac{3}{\pi}\right)N + O(1)$ for all θ .

Proof of Theorem 1. A straightforward calculation shows that for integers m, r with $0 \leq m, r < N$ we have

$$P\left(-\frac{mN+r}{N^2}\right) = N \exp\left(-2\pi i \frac{mN+r}{N^2} r\right),$$

and (a) follows.

To obtain (b) we define the functions

$$G(\theta) = \frac{iN}{2\pi} (1 - z^N) \sum_{k=-\infty}^{\infty} \frac{z^{kN}}{k + N\theta}$$

and

$$H(\theta) = \frac{iN}{2\pi} (1 - z^N) \sum_{k=0}^{N-1} \frac{z^{kN}}{k + N\theta},$$

and the error functions

$$R(\theta) = P(\theta) - H(\theta) \text{ and } S(\theta) = H(\theta) - G(\theta).$$

To estimate $R(\theta)$ we let

$$\phi(x) = \frac{\sin x}{2(1 - \cos x)} - \frac{1}{x} \quad \text{and} \quad \psi(x) = \frac{1}{1 - \exp(ix)} - \frac{i}{x} = \frac{1}{2} + i\phi(x).$$

We can now write

$$R(\theta) = (1 - z^N)\psi(2\pi\theta) + \sum_{k=0}^{N-1} z^{(k+1)N} \left\{ \psi \left(2\pi \left(\frac{k+1}{N} + \theta \right) \right) - \psi \left(2\pi \left(\frac{k}{N} + \theta \right) \right) \right\},$$

which, together with the facts that $\phi'(x) < 0$, $-2\pi < x < 2\pi$, and $-1 + \varepsilon \leq \theta \leq -\varepsilon$, $\varepsilon < \frac{1}{2}$ yields

$$\begin{aligned} |R(\theta)| &\leq 2|\psi(2\pi\theta)| + \sum_{k=0}^{N-1} \left| \psi \left(2\pi \left(\frac{k+1}{N} + \theta \right) \right) - \psi \left(2\pi \left(\frac{k}{N} + \theta \right) \right) \right| \\ &= 2|\psi(2\pi\theta)| + \sum_{k=0}^{N-1} \left\{ \phi \left(2\pi \left(\frac{k}{N} + \theta \right) \right) - \phi \left(2\pi \left(\frac{k+1}{N} + \theta \right) \right) \right\} \\ &= 2|\psi(2\pi\theta)| + \frac{1}{2\pi} \left(\frac{1}{1+\theta} - \frac{1}{\theta} \right) \\ &< 2\left\{ \frac{1}{2} + \phi(2\pi(-1+\varepsilon)) \right\} + \frac{1}{2\pi\varepsilon(1-\varepsilon)} \\ &< 1 + \frac{2}{\pi} + \frac{2}{\pi\varepsilon}. \end{aligned} \quad (1)$$

To estimate $S(\theta)$, we observe that $1 - N\theta \geq \varepsilon N > 1$ and $N(1 + \theta) \geq \varepsilon N > 1$, so that we may apply Lemma 1 with $x = \pm N\theta$ and $T = \varepsilon N$ to get

$$\begin{aligned} |S(\theta)| &= \frac{N}{2\pi} |1 - z^N| \left| \sum_{k=-\infty}^{-1} \frac{z^{kN}}{k + N\theta} + \sum_{k=N}^{\infty} \frac{z^{kN}}{k + N\theta} \right| \\ &\leq \frac{N}{2\pi} |1 - e^{2\pi i N\theta}| \left\{ \left| \sum_{k=0}^{\infty} \frac{e^{-2\pi i k N\theta}}{k + 1 - N\theta} \right| + \left| \sum_{k=0}^{\infty} \frac{e^{2\pi i k N\theta}}{k + N(1 + \theta)} \right| \right\} < \frac{3}{\pi\varepsilon}. \end{aligned} \quad (2)$$

Next we apply Lemma 2 with $x = r = N\theta$ to conclude that

$$|G(\theta)| = N \text{ for all } \theta. \quad (3)$$

Finally, combining (1), (2) and (3) with the fact that $P(\theta) = G(\theta) + R(\theta) + S(\theta)$, we obtain (b).

For (c) we assume that $|\theta| < \frac{1}{4}$ and apply the techniques of the proof of (b) to get

$$\begin{aligned} P(\theta) &= \frac{iN}{2\pi} (1 - e^{2\pi i N \theta}) \left(\sum_{k=0}^{\lfloor \frac{1}{2}N \rfloor} \frac{e^{2\pi i k N \theta}}{k + N\theta} + \sum_{k=\lfloor \frac{1}{2}N \rfloor + 1}^{N-1} \frac{e^{2\pi i k N \theta}}{k - N + N\theta} \right) + O(1) \\ &= \frac{iN}{2\pi} (1 - e^{2\pi i N \theta}) \left(\sum_{k=0}^{\infty} \frac{e^{2\pi i k N \theta}}{k + N\theta} + e^{2\pi i N^2 \theta} \sum_{k=-\infty}^{-1} \frac{e^{2\pi i k N \theta}}{k + N\theta} \right) + O(1) \end{aligned} \quad (4)$$

If N is odd, (4) immediately implies that

$$P\left(\frac{1}{2N}\right) = O(1).$$

For N even, (4) yields

$$\begin{aligned} P\left(\frac{N-1}{2N^2}\right) &= \frac{2iN}{\pi} \left(\sum_{k=0}^{\infty} \frac{(-1)^k e^{-\pi i(k/N)}}{2k+1} - e^{(\pi i/N)} \sum_{k=0}^{\infty} \frac{(-1)^k e^{\pi i(k/N)}}{2k+1} \right) + O(1) \\ &= \frac{2iN}{\pi} (1 - e^{\pi i/N}) \sum_{k=0}^{\infty} \frac{(-1)^k \cos(k\pi/N)}{2k+1} + \frac{4N}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k \sin(k\pi/N)}{2k+1} \\ &\quad + O(1) = O(1), \end{aligned}$$

where the final estimate follows, for example, from Gradshteyn and Ryzhik [4], p. 38, formulas 1.442-3 and 4, and (c) is proven.

To establish (d) we may assume that $|\theta| < \frac{1}{4}$, and we let $N\theta = M + t$, where $M = [N\theta]$. For $M \leq -1$ we have, by Lemmas 1 and 2,

$$\left| (1 - e^{2\pi i N \theta}) \sum_{k=0}^{\infty} \frac{e^{2\pi i k N \theta}}{k + N\theta} \right| = \left| (1 - e^{2\pi i t}) \left(\sum_{k=-\infty}^{\infty} \frac{e^{2\pi i k t}}{k + t} - \sum_{k=-\infty}^{M-1} \frac{e^{2\pi i k t}}{k + t} \right) \right| < 2\pi + 3. \quad (5)$$

It is equally trivial to obtain an identical estimate for $M \geq 0$, and the same method also yields

$$\left| (1 - e^{2\pi i N \theta}) \sum_{k=-\infty}^{-1} \frac{e^{2\pi i k N \theta}}{k + N\theta} \right| < 2\pi + 3 \quad \text{for } |\theta| < \frac{1}{4}. \quad (6)$$

Finally, (d) follows from (4), (5), and (6), and the proof of Theorem 1 is complete.

We point out several immediate consequences of this theorem. First, if α is a fixed real number and if we define Q by $Q(\theta) = P(\theta + \alpha)$, it is obvious that $Q \in \mathcal{G}_{N^2-1}$. Therefore, the bad interval in (b) can be shifted to any interval of length 2ε .

Second, if we are interested in a fixed subinterval of the unit circle, the estimate in (b) becomes quite remarkable. For example, setting $\varepsilon = \frac{1}{4}$ we obtain

COROLLARY 1. *On the unit semicircle $-\frac{3}{4} \leq \theta \leq -\frac{1}{4}$ we have $|P(\theta)| = N + E$, where $|E| < 1 + 22\pi^{-1} < 9$.*

Third, by employing, for example, a result of Beller [1], we are able to extend Theorem 1 to the case of polynomials of arbitrary degree. We have

COROLLARY 2. Let n be a positive integer. Then there is a $g \in \mathcal{G}_n$ satisfying

(e) For any ε , $[n^{\frac{1}{2}}]^{-1} < \varepsilon < \frac{1}{2}$, $|g(\theta)| = n^{\frac{1}{2}} + E$ for $-1 + \varepsilon \leq \theta \leq -\varepsilon$, where $|E| < 2 + 2\pi^{-1} + 5(\pi\varepsilon)^{-1} + 2n^{\frac{1}{2}}$, and

(f) $|g(\theta)| < (2 + 3/\pi)n^{\frac{1}{2}} + 2n^{\frac{1}{2}} + O(1)$ for all θ .

Proof of Corollary 2. Let $N = [n^{\frac{1}{2}}]$, $m = n - N^2$, and choose $P(\theta)$ as in Theorem 1. By Beller's result [1], we can choose $f \in \mathcal{G}_m$ such that $|f(\theta)| < 1.172m^{\frac{1}{2}} < 2n^{\frac{1}{2}}$ for all θ . If we now let $g(\theta) = P(\theta) + z^{N^2}f(\theta)$, the required estimates follow immediately from the Theorem, and Corollary 2 is proven.

Finally we observe that if we choose ε in Corollary 2 to be, for example, $n^{-\frac{1}{2}} \log n$, we obtain the improvement of Littlewood's result mentioned in the introduction.

2. We now proceed with our construction of functions $G(\theta)$ which are almost in $\mathcal{G}_n$, and which satisfy $|G(\theta)| = n^{\frac{1}{2}} + O(n^{\frac{1}{2}})$ for all θ . Toward this end, we have

THEOREM 2. Let n be a positive integer, and let N be the even positive integer satisfying $N^2 \leq n < (N+2)^2$. Then there exist functions f and g such that

(A) $z^{N^2}f + g \in \mathcal{G}_n$, and

(B) $|f(\theta) + g(\theta)| = n^{\frac{1}{2}} + O(n^{\frac{1}{2}})$, where the error is uniform in θ and n .

Remark. It will be seen from the following construction that g consists of two parts; a polynomial in $\mathcal{G}_{n-2n^{3/4}+O(n^{1/2})}$, plus z to an integral power multiplied by a polynomial with coefficients of modulus $\frac{1}{2}$ and degree $4n^{\frac{1}{2}} + O(n^{\frac{1}{2}})$ in $z^{\frac{1}{2}}$. Also, f is a function of precisely the same type as the second part of g , just described. Thus we see that, except for a relatively small number (i.e., $O(n^{\frac{1}{2}})$) of terms, $f + g \in \mathcal{G}_n$, and so we can use $f + g$ as the function $G(\theta)$ mentioned above.

Proof of Theorem 2. Define δ by $\delta N = [N^{\frac{1}{2}}]$, let $z = \exp(2\pi i\theta)$, $-\frac{1}{2} \leq \theta \leq \frac{1}{2}$, let $m = n - N^2$, and choose $G_1 \in \mathcal{G}_m$ such that $|G_1(\theta)| = O(n^{\frac{1}{2}})$ (see the proof of Corollary 2). Define the functions

$$f(\theta) = \frac{1}{2} z^{(\frac{1}{2}N - \delta N)N} \sum_{j = -\frac{1}{2}N - \delta N}^{-\frac{1}{2}N + \delta N - 1} z^{jN} \sum_{k=0}^{2N-1} \exp k\pi i(jN^{-1} + \theta),$$

$$F(\theta) = z^{(\frac{1}{2}N - \delta N)N} \left\{ \frac{1}{2} \sum_{j = \frac{1}{2}N - \delta N}^{\frac{1}{2}N + \delta N - 1} z^{jN} \sum_{k=0}^{2N-1} \exp k\pi i(jN^{-1} + \theta) \right. \\ \left. + \sum_{j = -\frac{1}{2}N + \delta N}^{\frac{1}{2}N - \delta N - 1} z^{jN} \sum_{k=0}^{N-1} \exp 2k\pi i(jN^{-1} + \theta) \right\},$$

and

$$g(\theta) = F(\theta) + z^{N^2} G_1(\theta).$$

A straightforward calculation yields (A). To establish (B) we proceed as in the proof of Theorem 1(b), and we obtain

$$\begin{aligned} f(\theta) + g(\theta) &= z^{(\frac{1}{2}N - \delta N)N} (1 - z^N) \frac{i}{2\pi} \sum_{j=-\frac{1}{2}N - \delta N}^{\frac{1}{2}N + \delta N - 1} \frac{z^{jN}}{(j/N) + \theta} + O(\delta^{-1}) + O(n^{\frac{1}{2}}) \\ &= z^{(\frac{1}{2}N - \delta N)N} (1 - z^N) \frac{iN}{2\pi} \sum_{j=-\infty}^{\infty} \frac{z^{jN}}{j + N\theta} + O(n^{\frac{1}{2}}). \end{aligned}$$

Therefore, by Lemma 2, $|f+g| = N + O(n^{\frac{1}{2}}) = n^{\frac{1}{2}} + O(n^{\frac{1}{2}})$, and the proof of Theorem 2 is complete.

In conclusion we mention a result with a similar flavour to Theorem 2 by Beller and Newman [2], who prove the Littlewood conjecture, with A_1 and A_2 both much smaller than 1, for polynomials whose coefficients are bounded by 1 in modulus.

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