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Author(s):

Kovacevic-Badstuebner, Ivana; Romano, Daniele; Antonini, Giulio; Lombardi, Luigi; Grossner, Ulrike

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Full-wave Computation of the Electric Field in the Partial Element Equivalent Circuit Method Using Taylor Series Expansion of the Retarded Green's Function

Ivana Kovacevic-Badstubner, *Member, IEEE*, Daniele Romano, Giulio Antonini, *Senior Member, IEEE*, Luigi Lombardi, and Ulrike Grossner, *Member, IEEE*

Abstract—This paper presents new analytical formulas for the efficient computation of the full-wave electric field generated by conductive, dielectric and magnetic media in the framework of the Partial Element Equivalent Circuit (PEEC) method. To this aim, the full-wave Green's function is handled by Taylor series expansion leading to three types of integrals for which new analytical formulas are provided in order to avoid slower numerical integration. An orthogonal (Manhattan type) tessellation of the geometries is assumed and the electrical quantities, i.e. currents, charges, and magnetization, are expanded in space through rectangular basis functions. The full-wave electric field radiated by charges, currents and magnetization is computed analytically in the post-processing step. The proposed closed-form computation of the electric field is tested using two examples, comparing the results obtained by the derived analytical formulas with the results from a finite element method solver.

Index Terms—Integral equations, electric field, magnetic field, partial element equivalent circuit (PEEC) method, Taylor series expansion.

I. INTRODUCTION

Integrated circuits and semiconductor devices are extremely vulnerable to Electromagnetic Interference (EMI) and certain measures are necessary to protect the control units from EMI disturbances. With the reduction of rise/fall times of current/voltage waveforms, the frequency spectrum of EMI noise sources extends above the hundreds of megahertz. Electronic system design for electromagnetic compatibility compliance is frequently based on a trial-and-error design approach and/or a solid engineering experience. Therefore, electromagnetic analysis is a highly useful tool for assisting the system development and predicting EMI noise generation in a wide frequency range early in the design. Besides conducted EMI disturbance, the prediction of radiated EMI based on the calculation of radiated electromagnetic fields is necessary in

order to prevent malfunction or even failure of the surrounding electronic systems. For various electronic applications, at the frequencies above several hundred MHz, the quasi-static hypothesis cannot be assumed for the evaluation of radiated fields and the full-wave computation taking into account the electromagnetic wave propagation is required. Among the numerical methods for electromagnetic analysis, the Partial Element Equivalent Circuit (PEEC) method [1], is particularly well suited. This comes from the nature of the PEEC-based modeling, where only the sources of radiated fields i.e. conducting/dielectric/magnetic regions, have to be discretized while the mesh of the background (air) region can be avoided in comparison to the finite element method (FEM) or the finite difference method (FDM).

Since its birth back in the 70's, the PEEC method has been significantly improved in direction of modeling non-orthogonal geometries [2], [3], a large variety of lossless and lossy, linear and non-linear materials [4], [5], [6], [7], [8], [9], [10], [11]. More recently, the electro-thermal PEEC formulation has also been proposed [12], [13]. Similarly to other integral equation-based techniques, the PEEC method requires the filling of large matrices and, as a consequence, the solution of large systems of equations. Significant improvements have been done also in the acceleration of the solvers in both the frequency and the time domain by using the fast multipole method [14], the multi-function approach [15], the multiscale compressed decomposition [16], [17]. The filling of the partial element matrices has also been improved by means of the geometrical signatures identification [18], [19] and the Taylor series expansion of the full-wave Green's function [20], [21].

The computation of radiated fields generated by electronic systems through the PEEC method has also been addressed. The first way to approach this problem is based on placing artificial electric and magnetic sensors, which are included as part of the PEEC model and solving process [22]. It is assumed that the sensors are small enough not to affect significantly the currents and charges and, thus, the electric and magnetic radiated fields. The second, and more rigorous method, is based on computing first currents and charges in the electrical system under analysis and then, electric and magnetic fields in the free space in the post-processing step. In this respect, analytical calculation of integrals involved in the computation of radiated fields is of interest as it allows to significantly

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Daniele Romano and Giulio Antonini are with the UAq EMC Laboratory, Department of Industrial and Information Engineering and Economics, University of L'Aquila, I-67100 L'Aquila, Italy, (e-mail: danielle.romano.vis@gmail.com; giulio.antonini@univaq.it).

Ivana Kovacevic-Badstubner, Ulrike Grossner are with the Advanced Power Semiconductor Laboratory, ETHZ, Zurich, Switzerland, (e-mail: kovacevic@aps.ee.ethz.ch, ulrike.grossner@ethz.ch).

Luigi Lombardi is with Micron Semiconductor Italia S.r.l, 67051 Avezzano (AQ), Italy (e-mail: luigilombardi89@gmail.com).

speed up the computation of radiated fields in comparison to numerical integration methods. The employment of analytical integrals directly implies the Manhattan type of volume mesh, and hence, a trade-off between computational complexity and modeling flexibility with respect to handling of more complex geometrical structures. In various applications, the accuracy is not compromised by using the Manhattan type of volume mesh, and the PEEC-based electromagnetic modeling can be used to predict the electromagnetic performance in a computationally efficient way. This paper addresses the analytical formulas for the calculation of radiated electric field, which is a step towards a computationally efficient and accurate full-wave PEEC EM-solver.

The analytical formulas for the computation of the full-wave partial element coefficients are presented in [20], [21], which allows to compute the currents and charges at high frequencies, which is necessary for an accurate calculation of electric and magnetic fields in the next step. The full-wave magnetic field can be analytically calculated based on the work presented in [21]. Regarding the calculation of electric field, some recent relevant examples can be found in [23], [24], [25], which refer to specific sources (dipoles or coils) radiating in particular condition (over half-space or stratified media). In the framework of the PEEC method, the analytical formulas for the calculation of the electric fields radiated by PEEC models are presented in [26] under the quasi-static assumption. Accordingly, the aim of this work is to extend the previous work [26] for the full-wave electric field, including all the sources, more specifically electrical currents, charges and magnetization. This requires as shown in the paper to analytically solve several specific integrals derived by using Taylor series expansion of the full-wave free-space Green's function. The solution of some of the presented integrals is not straightforward and cannot be gained by means of technical computing systems such as e.g. Mathematica or Maple. It is also to be pointed out that, for non-orthogonal geometries, if the elementary domains (volumes and surfaces) remain orthogonal, upon a pertinent 3D rotation, it is still possible to use the proposed formulas. Furthermore, meshing techniques can be adopted which preserve the orthogonal geometry for large parts of the structure under analysis. The proposed formulas can still be applied to this portion of the mesh.

The paper is organized as follows. Section II describes the computation of the electric field radiated by electrical currents, magnetization and charges, in the framework of the PEEC full-wave modeling, exploiting each contribution and assuming an orthogonal tessellation of volumes and surfaces. The analytical formulas of each contribution through Taylor series expansion are presented in Section III. In Section IV the accuracy and the range of applicability of the proposed formulas is shown. In addition, Section IV presents two examples: a current coil above a magnetic plate and a microstrip line with an array of square split-ring resonators, selected to validate the proposed analytical formulas in the presence of both magnetic and dielectric materials. A good agreement with the PEEC-based numerical calculation and a FEM-based simulation is demonstrated. The conclusions are summarized in Section V.

II. ELECTRIC FIELD INTEGRALS

The PEEC method is based on the volume equivalence principle of Maxwell's equations and use the electric field integral equation (EFIE) and the continuity equation [1]. In the presence of conductors, dielectrics and magnetic materials, the magnetic vector potential is expressed by means of magnetization, electrical and polarization current densities, while the scalar electric potential is calculated from free and bound surface charge densities. Typically, rectangular basis functions are used both to expand these quantities and to test the EFIE and continuity equation, based on the Galerkin's testing procedure. It results in the following set of equations:

$$\begin{bmatrix} j\omega\mathbf{P}(\omega)^{-1} & -\mathbf{A}^T & \mathbf{0} \\ \mathbf{A} & \mathbf{Z}_s(\omega) + j\omega\mathbf{L}_p(\omega) & j\omega\mathbf{L}_m(\omega) \\ \mathbf{0} & \mathbf{D}(\omega) & \mathbf{T}(\omega) \end{bmatrix} \cdot \begin{bmatrix} \Phi(\omega) \\ \mathbf{I}(\omega) \\ \mathbf{M}(\omega) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_s(\omega) \\ -\mathbf{V}_s(\omega) \\ -\mathbf{G}(\omega)\mathbf{I}_s(\omega) \end{bmatrix} \quad (1)$$

where $\mathbf{P}(\omega)$ accounts for the coefficients of potential, $\mathbf{L}_p(\omega)$ is the partial inductance matrix, $\mathbf{L}_m(\omega)$ represents the induced effects due to time varying magnetization, $\mathbf{D}(\omega)$ and $\mathbf{T}(\omega)$ matrices describe the constitutive relation of the magnetic field and $\mathbf{A}$ is the connectivity matrix. Finally, matrix $\mathbf{Z}_s(\omega)$ is diagonal with the self impedances of elementary volumes, which reduce to resistances for conductors [27] and the impedances of the excess capacitance [4] for dielectrics. The derivation of (1) is detailed in [8]. The time domain counterpart is straightforward to be obtained.

Once system equation (1) has been solved at a given angular frequency ω or in a specified time window, electric potentials $\Phi(\omega)$, electrical currents $\mathbf{I}(\omega)$, and magnetization $\mathbf{M}(\omega)$ are computed and available for further computation. The surface charges can be recovered by

$$\mathbf{Q}(\omega) = \mathbf{P}(\omega)^{-1}\Phi(\omega). \quad (2)$$

At an observation point $\mathbf{r}$, the electric field (E-field) radiated by currents, charges and magnetization is given by

$$\begin{aligned} \mathbf{E}(\mathbf{r}, \omega) &= \mathbf{E}_s(\mathbf{r}, \omega) + \mathbf{E}_A(\mathbf{r}, \omega) + \mathbf{E}_C(\mathbf{r}, \omega) = \\ &= \mathbf{E}_s(\mathbf{r}, \omega) - j\omega\mathbf{A}(\mathbf{r}, \omega) - \nabla\Phi(\mathbf{r}, \omega). \end{aligned} \quad (3)$$

The first contribution on the right hand side represents an external electric field, the second one the contribution of current density and magnetization, the last one the contribution of the surface charges.

A. $\mathbf{E}_C(\mathbf{r}, \omega)$ contribution

Since the surface charges are assumed known as computed by the PEEC solver, the coulombian contribution to the electric field is given by

$$\mathbf{E}_C(\mathbf{r}, \omega) = - \sum_{n=1}^{N_s} \frac{1}{4\epsilon_0\pi S_n} \int_{S'} Q_n(\omega) \nabla' \left(\frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) ds'. \quad (4)$$

B. $E_A(\mathbf{r}, \omega)$ contribution

as:

The magnetic vector potential is given by

$$\mathbf{A}(\mathbf{r}, \omega) = \mathbf{A}_c(\mathbf{r}, \omega) + \mathbf{A}_m(\mathbf{r}, \omega) + \mathbf{A}_s(\mathbf{r}, \omega) \quad (5)$$

where

$$\mathbf{A}_c(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int_{V'_c} \frac{\mathbf{J}(\mathbf{r}', \omega) e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dv' \quad (6a)$$

$$\mathbf{A}_s(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int_{V'_s} \frac{\mathbf{J}_s(\mathbf{r}', \omega) e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dv' \quad (6b)$$

$$\mathbf{A}_m(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int_{V'_m} \mathbf{M}(\mathbf{r}', \omega) \times \nabla' \left(\frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) dv'. \quad (6c)$$

Assuming a volume tessellation of the Manhattan type, the three contributions are computed as

$$\mathbf{A}_c(\mathbf{r}, \omega) = \sum_{j=1}^{N_c} \frac{\mu_0}{4\pi a_{c,j}} \int_{V'_{c,j}} \frac{I_j(\omega) \mathbf{f}_j(\mathbf{r}') e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dv', \quad (7a)$$

$$\mathbf{A}_s(\mathbf{r}, \omega) = \sum_{i=1}^{N_s} \frac{\mu_0}{4\pi a_{s,i}} \int_{V'_{s,i}} \frac{I_{s,i}(\omega) \mathbf{f}_{cs,i}(\mathbf{r}') e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dv', \quad (7b)$$

$$\mathbf{A}_m(\mathbf{r}, \omega) = \sum_{\ell=1}^{3N_m} \frac{\mu_0}{4\pi} \int_{V'_{m,\ell}} M_\ell(\omega) \mathbf{b}_\ell(\mathbf{r}') \times \nabla' \left(\frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) dv' \quad (7c)$$

where $\mathbf{f}$ and $\mathbf{b}$ are generic basis functions. Since in the standard PEEC method electrical quantities, e.g. currents and charges, are assumed uniform over the elementary domains, $\mathbf{f}$ and $\mathbf{b}$ reduce to unit vectors. Finally, the electric field due to the time variations of currents and magnetization reads

$$\begin{aligned} \mathbf{E}_A(\mathbf{r}, \omega) = & - \sum_{j=1}^{N_c} \frac{\mu_0}{4\pi S_{c,j}} j\omega I_j(\omega) \int_{V'_{c,j}} \frac{\mathbf{f}_j(\mathbf{r}') e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dv' + \\ & - \sum_{i=1}^{N_s} \frac{\mu_0}{4\pi S_{s,i}} j\omega I_{s,i}(\omega) \int_{V'_{s,i}} \frac{\mathbf{f}_{cs,i}(\mathbf{r}') e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dv' + \quad (8) \\ & - \sum_{\ell=1}^{3N_m} \frac{\mu_0}{4\pi} \int_{V'_{m,\ell}} j\omega M_\ell(\omega) \nabla' \times \left(\frac{\mathbf{b}_\ell(\mathbf{r}') e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) dv'. \end{aligned}$$

C. Complex Integrals for E-Field Computation

1) *Integrals for E_C E-field Component:* From (4) it follows that it is required to compute a surface integral in the following form:

$$\mathcal{I}_{hc} = \int_{S'} -\nabla' \frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dS'. \quad (9)$$

From (9), three transfer functions hcx , hcy , hcz can be defined

$$\begin{aligned} \mathcal{I}_{hc} = & \hat{x} \underbrace{\int_{S'} -\frac{\partial}{\partial x'} \frac{e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dS'}_{hcx} + \\ & + \hat{y} \underbrace{\int_{S'} -\frac{\partial}{\partial y'} \frac{e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dS'}_{hcy} + \\ & + \hat{z} \underbrace{\int_{S'} -\frac{\partial}{\partial z'} \frac{e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dS'}_{hcz} \quad (10) \end{aligned}$$

where $R'(x', y', z') = \sqrt{\Delta_x'^2 + \Delta_y'^2 + \Delta_z'^2}$ with $\Delta_x' = x_o - x'$, $\Delta_y' = y_o - y'$, $\Delta_z' = z_o - z'$.

Considering that an elementary source surface can be located in the $x-y$, $x-z$ or $y-z$ plane and adopting a surface Manhattan type tessellation, nine sub-functions hcx_{xy} , hcx_{xz} , hcx_{yz} , hcy_{xy} , hcy_{xz} , hcy_{yz} , hcz_{xy} , hcz_{xz} and hcz_{yz} from hcx , hcy , hcz can be defined:

$$hcx = \begin{cases} \underbrace{\int_{y'} \int_{x'} -\frac{\partial}{\partial x'} \frac{e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dx' dy'}_{hcx_{xy}} & \text{if the bar is in the } x-y \text{ plane} \\ \underbrace{\int_{z'} \int_{x'} -\frac{\partial}{\partial x'} \frac{e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dx' dz'}_{hcx_{xz}} & \text{if the bar is in the } x-z \text{ plane} \\ \underbrace{\int_{z'} \int_{y'} -\frac{\partial}{\partial x'} \frac{e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dy' dz'}_{hcx_{yz}} & \text{if the bar is in the } y-z \text{ plane} \end{cases} \quad (11)$$

$$hcy = \begin{cases} \underbrace{\int_{y'} \int_{x'} -\frac{\partial}{\partial y'} \frac{e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dx' dy'}_{hcy_{xy}} & \text{if the bar is in the } x-y \text{ plane} \\ \underbrace{\int_{z'} \int_{x'} -\frac{\partial}{\partial y'} \frac{e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dx' dz'}_{hcy_{xz}} & \text{if the bar is in the } x-z \text{ plane} \\ \underbrace{\int_{z'} \int_{y'} -\frac{\partial}{\partial y'} \frac{e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dy' dz'}_{hcy_{yz}} & \text{if the bar is in the } y-z \text{ plane} \end{cases} \quad (12)$$

$$hcz = \begin{cases} \underbrace{\int_{y'} \int_{x'} -\frac{\partial}{\partial z'} \frac{e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dx' dy'}_{hcz_{xy}} & \text{if the bar is in the } x-y \text{ plane} \\ \underbrace{\int_{z'} \int_{x'} -\frac{\partial}{\partial z'} \frac{e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dx' dz'}_{hcz_{xz}} & \text{if the bar is in the } x-z \text{ plane} \\ \underbrace{\int_{z'} \int_{y'} -\frac{\partial}{\partial z'} \frac{e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dy' dz'}_{hcz_{yz}} & \text{if the bar is in the } y-z \text{ plane} \end{cases} \quad (13)$$

2) *Integrals for E_A E-field Component:* Similarly, from (8), the solution of two volume integrals $\mathcal{I}_{ha}$ and $\mathcal{I}_{hm}$ is required.

$$\mathcal{I}_{ha} = \int_{V'} \frac{\mathbf{f}(\mathbf{r}') e^{-j\beta |\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} dV' \quad (14a)$$

$$\mathcal{I}_{hm} = \int_{V'} \nabla' \times \frac{\mathbf{f}(\mathbf{r}') e^{-j\beta |\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} dV' \quad (14b)$$

where $\mathbf{f}$ is a generic unit vector. Assuming a volume Manhattan type tessellation, from (14a) we can define a transfer function ha defined as

$$\mathcal{I}_{ha} = \begin{cases} \underbrace{\hat{x} \int_{V'} \frac{1}{R'(x', y', z')} e^{-j\beta R'(x', y', z')} dV'}_{ha} & \text{for } \mathbf{f} = [\hat{x}, 0, 0] \\ \underbrace{\hat{y} \int_{V'} \frac{1}{R'(x', y', z')} e^{-j\beta R'(x', y', z')} dV'}_{ha} & \text{for } \mathbf{f} = [0, \hat{y}, 0] \\ \underbrace{\hat{z} \int_{V'} \frac{1}{R'(x', y', z')} e^{-j\beta R'(x', y', z')} dV'}_{ha} & \text{for } \mathbf{f} = [0, 0, \hat{z}] \end{cases} \quad (15)$$

where R is defined in (10). In addition, from (14b), we can define 6 transfer functions $hm_{y,x}$, $hm_{z,x}$, $hm_{x,y}$, $hm_{z,y}$, $hm_{x,z}$ and $hm_{y,z}$ as follows

Fig. 1. A elementary surface in the $x-y$ plane, where $x_c = 0.5(x_1 + x_2)$ and $y_c = 0.5(y_1 + y_2)$.

$hm_{x,z}$ and $hm_{y,z}$ as follows

$$\mathcal{I}_{hm} = \begin{cases} \underbrace{\hat{y} \int_{V'} \frac{\partial}{\partial z'} \frac{e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dV'}_{hm_{y,x}} + \underbrace{\hat{z} \int_{V'} \frac{\partial}{\partial y'} \frac{-e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dV'}_{hm_{z,x}} & \text{for } \mathbf{f} = [\hat{x}, 0, 0] \\ \underbrace{\hat{x} \int_{V'} \frac{\partial}{\partial z'} \frac{-e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dV'}_{hm_{x,y}} + \underbrace{\hat{z} \int_{V'} \frac{\partial}{\partial x'} \frac{e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dV'}_{hm_{z,y}} & \text{for } \mathbf{f} = [0, \hat{y}, 0] \\ \underbrace{\hat{x} \int_{V'} \frac{\partial}{\partial y'} \frac{e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dV'}_{hm_{x,z}} + \underbrace{\hat{y} \int_{V'} \frac{\partial}{\partial x'} \frac{-e^{-j\beta R'(x', y', z')}}{R'(x', y', z')} dV'}_{hm_{y,z}} & \text{for } \mathbf{f} = [0, 0, \hat{z}] \end{cases} \quad (16)$$

where $R'(x', y', z')$ is defined as in (10). The full-wave computation through the Taylor series expansion of the transfer functions hm_{yx} , hm_{zx} , hm_{xy} , hm_{zy} , hm_{xz} and hm_{yz} is described in [21].

III. ANALYTICAL FORMULAS THROUGH TAYLOR SERIES EXPANSION FOR THE ELECTRIC FIELD

A. Computation of hcz_{xy} and hcx_{xy}

In this section, the analytical formulas to compute hcz_{xy} and hcx_{xy} , defined in (11) and (13), respectively, are provided. The other transfer functions defined in (11)–(13) can be then computed from hcz_{xy} and hcx_{xy} by exchanging the Cartesian axes as shown in Section III-A3. For the following analysis, an elementary surface (located in the $x-y$ plane) and an observation point are observed as shown in Fig. 1.

By using the notation given in Fig. 1, hc_{zy} and hc_{xy} defined in (11) and (13), respectively, can be rewritten as (17).

$$\begin{aligned} hc_{zy} &= \int_{y_1}^{y_2} \int_{x_1}^{x_2} -\frac{\partial}{\partial z'} \frac{e^{-j\beta R'(x', y')}}{R'(x', y')} dx' dy' = \\ &= (z_o - z_b) \int_{y_1}^{y_2} \int_{x_1}^{x_2} \left\{ \frac{1}{R'(x', y')^3} + \right. \\ &\quad \left. + \frac{j\beta}{R'(x', y')^2} \right\} e^{-j\beta R'(x', y')} dx' dy' \quad (17a) \end{aligned}$$

$$\begin{aligned} hc_{xy} &= \int_{y_1}^{y_2} \int_{x_1}^{x_2} -\frac{\partial}{\partial x'} \frac{e^{-j\beta R'(x', y')}}{R'(x', y')} dx' dy' = \\ &= \int_{y_1}^{y_2} \int_{x_1}^{x_2} (x_o - x') \left\{ \frac{1}{R'(x', y')^3} + \right. \\ &\quad \left. + \frac{j\beta}{R'(x', y')^2} \right\} e^{-j\beta R'(x', y')} dx' dy' \quad (17b) \end{aligned}$$

with

$$R'(x', y') = \sqrt{(x_o - x')^2 + (y_o - y')^2 + (z_o - z_b)^2}. \quad (17c)$$

1) hc_{zy} computation: Let us define R_{cc} as the center-to-center distance between the surface and the observation point (x_o, y_o, z_o) . The Taylor series expansion for the exponential term is adopted according to the analysis presented in [20]. The integral hc_{zy} (17a) can be rewritten into an equivalent form as

$$\begin{aligned} hc_{zy} &= (z_o - z_b) e^{-j\beta R_{cc}} \int_{y_1}^{y_2} \int_{x_1}^{x_2} \left\{ \frac{1}{R'(x', y')^3} + \right. \\ &\quad \left. + \frac{j\beta}{R'(x', y')^2} \right\} e^{-j\beta(R'(x', y') - R_{cc})} dx' dy'. \quad (18) \end{aligned}$$

In the next step, the term $e^{-j\beta(R'(x', y') - R_{cc})}$ in (18) is approximated with its third-order Taylor series expansion (19),

$$\begin{aligned} e^{-j\beta(R'(x', y') - R_{cc})} &\simeq 1 - j\beta(R'(x', y') - R_{cc}) + \\ &- \frac{1}{2}\beta^2(R'(x', y') - R_{cc})^2 + j\frac{1}{6}\beta^3(R'(x', y') - R_{cc})^3. \quad (19) \end{aligned}$$

Finally, by inserting the approximation (19) into (18) and after some algebraic manipulations, (20) is derived.

$$\begin{aligned} hc_{zy} &\simeq e^{-j\beta R_{cc}} \left[hc_{zy}^{QS} + j\beta hc_{zy}^{QS} R_{cc} + \right. \\ &\quad + \beta^2 \left(-\frac{1}{2} hc_{zy}^{QS} R_{cc}^2 + \frac{1}{2} hc_{zy}^{o2} \right) + \\ &\quad \left. + j\beta^3 \left(-\frac{1}{6} hc_{zy}^{QS} R_{cc}^3 + \frac{1}{2} hc_{zy}^{o2} R_{cc} - \frac{1}{6} hc_{zy}^{o3} \right) \right] \quad (20) \end{aligned}$$

where

$$\begin{aligned} hc_{zy}^{QS} &= (z_o - z_b) \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{1}{R'(x', y')^3} dx' dy' \\ hc_{zy}^{o2} &= (z_o - z_b) \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{1}{R'(x', y')} dx' dy' \\ hc_{zy}^{o3} &= 2(z_o - z_b) \int_{y_1}^{y_2} \int_{x_1}^{x_2} dx' dy' \quad (21) \end{aligned}$$

and $R'(x', y')$ defined in (17c). The third-order of Taylor series expansion is selected as an optimal trade-off between the accuracy and computational complexity. The analytical solution of hc_{zy}^{QS} is given in [26].

The integral inside hc_{zy}^{o3} is simply solved as follows

$$hc_{zy}^{o3} = 2(x_2 - x_1)(y_2 - y_1)(z_o - z_b). \quad (22)$$

The analytical solution for hc_{zy}^{o2} is given by (23), and the required analytical steps to obtain this solution are summarized in Appendix B in order to improve the readability of the paper.

$$hc_{zy}^{o2} = (z_o - z_b) \sum_{j=1}^2 \sum_{m=1}^2 (-1)^{(j+m)} \sum_{h=1}^5 [\gamma_h(x'_j, y'_m)] \quad (23)$$

where

$$\gamma_1 = -y'_m \quad (24a)$$

$$\gamma_2 = (z_o - z_b) \arctan \left(\frac{y'_m}{z_o - z_b} \right) \quad (24b)$$

$$\gamma_3 = -(z_o - z_b) \arctan \left(\frac{x'_j y'_m}{(z_o - z_b) \sqrt{x_j'^2 + y_m'^2 + (z_o - z_b)^2}} \right) \quad (24c)$$

$$\gamma_4 = y'_m \log \left(x'_j + \sqrt{x_j'^2 + y_m'^2 + (z_o - z_b)^2} \right) \quad (24d)$$

$$\gamma_5 = x'_j \log \left(y'_m + \sqrt{x_j'^2 + y_m'^2 + (z_o - z_b)^2} \right) \quad (24e)$$

where

$$\begin{aligned} x'_1 &= x_o - x_1, & x'_2 &= x_o - x_2, \\ y'_1 &= y_o - y_1, & y'_2 &= y_o - y_2. \end{aligned} \quad (25)$$

2) hc_{xy} computation: Based on the same strategy as applied for (20), hc_{xy} (17b) can be written as

$$\begin{aligned} hc_{xy} &\simeq e^{-j\beta R_{cc}} \left[hc_{xy}^{QS} + j\beta hc_{xy}^{QS} R_{cc} + \right. \\ &\quad + \beta^2 \left(-\frac{1}{2} hc_{xy}^{QS} R_{cc}^2 + \frac{1}{2} hc_{xy}^{o2} \right) + \\ &\quad \left. + j\beta^3 \left(-\frac{1}{6} hc_{xy}^{QS} R_{cc}^3 + \frac{1}{2} hc_{xy}^{o2} R_{cc} - \frac{1}{6} hc_{xy}^{o3} \right) \right] \quad (26) \end{aligned}$$

where

$$\begin{aligned} hc_{xy}^{QS} &= \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{x_o - x'}{R'(x', y')^3} dx' dy' \\ hc_{xy}^{o2} &= \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{x_o - x'}{R'(x', y')} dx' dy' \\ hc_{xy}^{o3} &= 2 \int_{y_1}^{y_2} \int_{x_1}^{x_2} (x_o - x') dx' dy'. \quad (27) \end{aligned}$$

and $R'(x', y')$ is defined in (17c). The analytical solution of hc_{xy}^{QS} is provided in [26]. The solution (28) of the integral within hc_{xy}^{o3} is straightforward.

$$hc_{xy}^{o3} = (x_o - x_1)^2(y_2 - y_1) + (x_o - x_2)^2(y_1 - y_2). \quad (28)$$

The analytical solution for hcx_{xy}^{o2} is given by (29), and the required analytical steps to obtain this solution are given in Appendix C.

$$hcx_{xy}^{o2} = \sum_{j=1}^2 \sum_{m=1}^2 (-1)^{(j+m)} \sum_{h=1}^2 [\gamma_h(x'_j, y'_m)] \quad (29)$$

where

$$\gamma_1 = \frac{1}{2} y'_m \sqrt{x_j'^2 + y_m'^2 + (z_o - z_b)^2} \quad (30a)$$

$$\gamma_2 = \frac{1}{2} [x_j'^2 + (z_o - z_b)^2] \log \left(y'_m + \sqrt{x_j'^2 + y_m'^2 + (z_o - z_b)^2} \right) \quad (30b)$$

where x'_1, x'_2, y'_1 and y'_2 are the same of those defined in (25).

3) *Computation of transfer functions hcx_{xz} , hcx_{yz} , hcy_{xy} , hcy_{xz} , hcy_{yz} , hcz_{xz} and hcz_{yz}* : The transfer functions hcx_{xz} , hcx_{yz} , hcy_{xy} , hcy_{xz} , hcy_{yz} , hcz_{xz} and hcz_{yz} , defined in (11)–(13), can be computed from the analytical formulas for hcz_{xy} (20) and hcx_{xy} (26) as

$$hcx_{xz} = hcx_{xy}^+ \quad (31a)$$

$$hcx_{yz} = hcz_{xy}^* \quad (31b)$$

$$hcy_{xy} = hcx_{xy}^\circ \quad (31c)$$

$$hcy_{xz} = hcz_{xy}^+ \quad (31d)$$

$$hcy_{yz} = hcx_{xy}^\diamond \quad (31e)$$

$$hcz_{xz} = hcx_{xy}^\S \quad (31f)$$

$$hcz_{yz} = hcx_{xy}^* \quad (31g)$$

where

- hcx_{xy}^+ and hcz_{xy}^+ have the same expression of (26) and (20), respectively, in which y_1 and y_2 are replaced with z_1 and z_2 , respectively, and z_b is replaced with y_b ;
- hcx_{xy}^* and hcz_{xy}^* have the same expression of (26) and (20), respectively, in which x_1 and x_2 are replaced with z_1 and z_2 , respectively, and z_b is replaced with x_b ;
- hcx_{xy}° have the same expression of (26) in which x_1 and x_2 are replaced with y_1 and y_2 , respectively, and y_1 and y_2 are replaced with x_1 and x_2 , respectively;
- $hcx_{xy}^\diamond$ have the same expression of (26) in which x_1 and x_2 are replaced with y_1 and y_2 , respectively, y_1 and y_2 are replaced with z_1 and z_2 , respectively and z_b is replaced with x_b ;
- $hcx_{xy}^\S$ have the same expression of (26) in which x_1 and x_2 are replaced with z_1 and z_2 , respectively, y_1 and y_2 are replaced with x_1 and x_2 , respectively and z_b is replaced with y_b .

Note that the notation used above is similar to the notation used in (20) and (26) (x_b and y_b , for the surface located in the $y-z$ and $x-z$ plane, respectively, are similar to z_b used for the surface located in the $x-y$ plane).

B. Computation of transfer function ha

Similarly as in the previous section, the transfer function ha (15) can be approximated through the third-order Taylor series

expansion as

$$\begin{aligned} ha \simeq & e^{-j\beta R_{cc}} \left[ha^{QS} + j\beta \left(ha^{QS} R_{cc} - ha^{o1} \right) + \right. \\ & + \beta^2 \left(-\frac{1}{2} ha^{QS} R_{cc}^2 + ha^{o1} R_{cc} - \frac{1}{2} ha^{o2} \right) + \\ & + j\beta^3 \left(-\frac{1}{6} ha^{QS} R_{cc}^3 + \frac{1}{2} ha^{o1} R_{cc}^2 - \frac{1}{2} L_{pij}^{o2} R_{cc} + \right. \\ & \left. \left. + \frac{1}{6} ha^{o3} \right) \right] \quad (32) \end{aligned}$$

$$\begin{aligned} ha^{QS} &= \int_{V'} \frac{1}{R'(x', y', z')} dV' \\ ha^{o1} &= \int_{V'} dV' \\ ha^{o2} &= \int_{V'} R'(x', y', z') dV' \\ ha^{o3} &= \int_{V'} R'(x', y', z')^2 dV' \quad (33) \end{aligned}$$

and $R'(x', y', z')$ defined by (15). The analytical solution of ha^{QS} is given in [26].

The solution for the integrals in (32) is derived by observing a elementary volume cell, i.e. a rectangular bar with 8 vertices P1 (x_1, y_1, z_1), P2(x_1, y_2, z_1), P3(x_2, y_1, z_1), P4(x_2, y_2, z_1), P5(x_1, y_1, z_2), P6(x_1, y_2, z_2), P7(x_2, y_1, z_2) and P8(x_2, y_2, z_2), where $x_1 < x_2$, $y_1 < y_2$ and $z_1 < z_2$. The E-field is calculated at the point $P_o(x_o, y_o, z_o)$ in the surrounding space. The integral ha^{o1} and ha^{o3} are simply solved by using (34) and (35), respectively, while the analytical solution for ha^{o2} is given by (37) and the required analytical steps to obtain (37) are summarized in Appendix D.

$$ha^{o1} = (x_2 - x_1)(y_2 - y_1)(z_2 - z_1). \quad (34)$$

$$\begin{aligned} ha^{o3} &= \sum_{j=1}^2 \sum_{m=1}^2 \sum_{n=1}^2 (-1)^{(j+m+n+1)} \frac{1}{3} (x_j'^3 y_m' z_n' + \\ & + x_j' y_m'^3 z_n' + x_j' y_m' z_n'^3) \quad (35) \end{aligned}$$

where

$$\begin{aligned} x'_1 &= x_o - x_1, & x'_2 &= x_o - x_2, \\ y'_1 &= y_o - y_1, & y'_2 &= y_o - y_2, \\ z'_1 &= z_o - z_1, & z'_2 &= z_o - z_2. \end{aligned} \quad (36)$$

$$ha^{o2} = \sum_{j=1}^2 \sum_{m=1}^2 \sum_{n=1}^2 (-1)^{(j+m+n)} \sum_{h=1}^8 [\gamma_h(x'_j, y'_m, z'_n)] \quad (37)$$

where

$$\gamma_1 = \frac{x'_j}{36} [5y'_m{}^3 + x'_j{}^2(5y'_m + 2z'_n)] - \frac{1}{4}x'_jy'_mz'_nR'(x'_j, y'_m, z'_n) \quad (38a)$$

$$\gamma_2 = \frac{1}{12}x'_j{}^4 \arctan\left(\frac{y'_mz'_n}{x'_jR'(x'_j, y'_m, z'_n)}\right) \quad (38b)$$

$$\gamma_3 = \frac{1}{12}y'_m{}^4 \arctan\left(\frac{x'_jz'_n}{y'_mR'(x'_j, y'_m, z'_n)}\right) \quad (38c)$$

$$\gamma_4 = \frac{1}{12}z'_n{}^4 \arctan\left(\frac{x'_jy'_m}{z'_nR'(x'_j, y'_m, z'_n)}\right) \quad (38d)$$

$$\gamma_5 = \frac{1}{12}(y'_m{}^4 - x'_j{}^4) \arctan\left(\frac{y'_m}{x'_j}\right) \quad (38e)$$

$$\gamma_6 = -\frac{1}{6}y'_mz'_n(y'_m{}^2 + z'_n{}^2) \log(x'_j + R'(x'_j, y'_m, z'_n)) \quad (38f)$$

$$\gamma_7 = -\frac{1}{6}x'_jz'_n(x'_j{}^2 + z'_n{}^2) \log(y'_m + R'(x'_j, y'_m, z'_n)) \quad (38g)$$

$$\gamma_8 = -\frac{1}{6}x'_jy'_m(x'_j{}^2 + y'_m{}^2) \log(z'_n + R'(x'_j, y'_m, z'_n)) \quad (38h)$$

$$R'(x'_j, y'_m, z'_n) = \sqrt{x'_j{}^2 + y'_m{}^2 + z'_n{}^2} \quad (38i)$$

and $x'_1, x'_2, y'_1, y'_2, z'_1$ and z'_2 are defined by (36).

IV. VERIFICATION

A set of numerical tests are performed to evaluate the range of applicability of the proposed formulas when using the double precision arithmetic and to compare the PEEC modeling based on the proposed analytical formulas to a FEM-based simulation of E -field.

A. Evaluation of the Range of Applicability

The range of applicability of the proposed formulas when using the double precision arithmetic is analyzed by calculating the integrals hc_{xy} and ha in the frequency range from 1 Hz to 1 THz (sampled by 26 logarithmically spaced frequency points). The distance between the elementary source of the electric field and the observation point is varied in the range from 1 μm to 1 m leading to 40 geometrical configurations. Such distance is calculated between the observation point and one edge of

- a square surface source for hc_{xy} ;
- a cubic volume source for ha .

Additionally, the tests are performed for the edge size of the elementary source in the range from 0.1 μm to 1 cm. In the presented analysis, the integrals are computed based on three approaches, DOUBLE-TAYLOR, QUAD-TAYLOR and NUMERICAL:

- DOUBLE-TAYLOR: the integrals are computed using the derived analytical formulas in double precision arithmetic;
- QUAD-TAYLOR: the integrals are computed using the derived analytical formulas in quadruple precision arithmetic;

- NUMERICAL: the integrals are calculated based on the adaptive numerical integration (*integral2* and *integral3*) in MATLAB with relative error tolerance threshold set equal to 10^{-6} .

The CPU times for the calculation of the hc_{xy} and ha integrals based on these methods are summarized in Table I, emphasizing the speed-up achieved by the DOUBLE-TAYLOR approach. In particular, the speed-up is computed as the ratio between the CPU time required for computing the integrals using the NUMERICAL method and the CPU time spent for computing the same integrals when using the derived analytical formulas i.e. the DOUBLE-TAYLOR or QUAD-TAYLOR method. From Table I it is evident the DOUBLE-TAYLOR method is extremely advantageous over the NUMERICAL method in terms of performance.

The accuracy of the proposed closed formulas are defined by a relative error calculated as

$$\epsilon = \frac{|h_{TAYLOR}| - |h_{NUMERICAL}|}{|h_{NUMERICAL}|}. \quad (39)$$

The relative errors of the hc_{xy} and $\mathcal{I}_{ha}$ computations through DOUBLE-TAYLOR approach are shown in Fig. 2 and Fig. 3, respectively, for the described test cases. The relative error for hc_{zy} is not reported since it shows the same behavior of hc_{xy} .

Fig. 2. Error evaluation for the transfer function hc_{xy} (26) for different surface sizes by using the DOUBLE-TAYLOR method.

From the results at Fig. 2 and Fig. 3, it can be seen that the relative error is higher than 1% for the frequencies above

TABLE I
CPU TIMES REQUIRED FOR CALCULATING THE INTEGRALS BASED ON THE DOUBLE-TAYLOR, QUAD-TAYLOR AND NUMERICAL APPROACHES FOR THE DESCRIBED TEST CASES OF 40 GEOMETRICAL CONFIGURATIONS EVALUATED IN THE FREQUENCY RANGE BETWEEN 1 Hz AND 1 THz AT 26 FREQUENCY SAMPLES.

	edge length					
	0.1 μm	1 μm	10 μm	100 μm	1 mm	1 cm
hc_{xy}						
NUMERICAL time	2 s	2 s	2 s	3 s	7 s	57 s
DOUBLE-TAYLOR time	< 10 ms	< 10 ms	< 10 ms	< 10 ms	< 10 ms	< 10 ms
DOUBLE-TAYLOR SPEED-UP	217x	224x	230x	337x	786 x	6404x
QUAD-TAYLOR time	5 s	5 s	5 s	5 s	5 s	5 s
QUAD-TAYLOR SPEED-UP	0.4x	0.4x	0.4x	0.6x	1.4 x	11.4x
ha						
NUMERICAL time	34 s	34 s	40 s	50 s	65 s	2797 s
DOUBLE-TAYLOR time	< 10 ms	< 10 ms	< 10 ms	< 10 ms	< 10 ms	< 10 ms
DOUBLE-TAYLOR SPEED-UP	3738x	3819x	4345x	5546x	7323x	310770x
QUAD-TAYLOR time	32 s	32 s	32 s	32 s	32 s	32 s
QUAD-TAYLOR SPEED-UP	1x	1x	1.2x	1.5x	2 x	87x

Fig. 3. Error evaluation for the transfer function $\mathcal{I}_{ha}$ (32) for different volume sizes by using the DOUBLE-TAYLOR method.

10 GHz and/or larger distances between the source and the observation point. It is important to underline that the higher relative errors for the size of elementary source smaller than 1 mm originates from the limited numerical precision used for the computation of the analytical formulas (21), (27) or (33) and not from the Taylor series expansion of the exponential term. This is confirmed by Fig. 4 showing the relative error of computing the ha -integrals (33) analytically using the quadruple precision arithmetic (as hc_{xy} integrals have the

same behavior they are not reported here).

Fig. 4. Error evaluation for the transfer function $\mathcal{I}_{ha}$ (32) for different volume sizes by resorting to the quadruple precision arithmetic computation for ha^{QS} , ha^{o2} and ha^{o3} ; the comparison to Fig. 3 shows an improved accuracy due to higher precision arithmetic.

However, from Table I it is evident that higher precision computation will be extremely slow for large problem sizes, and therefore a special care has to be taken to overcome the inaccuracy of analytical formulas related to the machine precision, which will be addressed in course of future work. For a larger edge size of the elementary source, i.e. 1 mm - 1 cm, the accuracy of analytical formulas is decreasing at

higher frequencies, which can be ascribed to the Taylor series expansion of the exponential term. However, the elementary mesh cells in practical applications are typically much smaller at higher frequencies i.e. above 10 GHz, which is required to accurately capture the high frequency effects.

B. Example 1: Coil above a linear magnetic plate

The effectiveness of the proposed approach and the analytical formulas derived for the computation of electric field are evaluated further using an example a coil above a linear magnetic plate. The coil is modeled as a copper horseshoe with squared cross-section. The geometry is shown in Fig. 5 and the geometric parameters are given in Fig. 6 where the blue line represents the lumped current port. The conductivity of the plate is $\sigma = 0.75 \cdot 10^7$ S/m and its relative magnetic permeability is $\mu_r = 200$. The port is excited by a $I = 1$ A current at the frequency $f = 1$ GHz. At this frequency, the quasi-static hypothesis is not valid for this structure. The analysis is performed by using a uniform mesh resulting in 6248 nodes, 15224 inductive branches and 2916 magnetic cells, with the edge size of elementary sources in the range (0.5 mm, 1.96 mm). The maximum mesh size to wavelength ratio is 0.00653. Accordingly, the selected test case is in the range of applicability of the proposed formulas as previously described.

Fig. 5. Test example: a coil above a linear magnetized plate.

Figure 7 shows the electric field observed on a line above the plate ($y = 0$ and $z = 30$ mm) computed by using:

- the PEEC method implementing the proposed approach (denoted as PEEC TAYLOR-FW) in which the computations are performed in double precision arithmetic.
- the numerical Gaussian integration (denoted as PEEC GAUSSIAN-FW) with 7 points of integration in each direction for $\mathcal{I}_{ha}$, $\mathcal{I}_{hm}$ and $\mathcal{I}_{hc}$, defined in (15), (16) and (10), respectively.
- the 3D FEM-solver of ANSYS HFSS (denoted as HFSS-FW), where the results were obtained by using a mesh with 769940 tetrahedral 3D elements for modeling the air boundary region and a mesh of 15934 triangular 2D elements for the conductors and magnetics objects.

The post-processing step for computing the electric field required 52 s and 13 min for PEEC TAYLOR-FW and PEEC GAUSSIAN-FW, respectively.

A very good agreement between all the methods is achieved as shown in Fig. 7. The accuracy of the FEM-based simulations is determined by the size of boundary region and the mesh size in the region of electric field computations, which in turn determines the memory requirements. In addition, to

Fig. 6. Geometry specifications for the test example of a coil above magnetized plate: $w_1 = 100$, $w_2 = 50$, $d = 19$, $\tau = 1$ and $t = 1$ (all dimensions are in mm).

prove that the quasi-static hypothesis is not valid at 1 GHz, the total electric field is calculated by the PEEC method implementing the quasi-static method presented in [26] (denoted as PEEC-QS) and compared to the full-wave PEEC method based on the proposed analytical formulas.

C. Example 2: SRRs Array

A microstrip line with a square split-ring resonators (SRRs) array is modeled similarly to [28] to further demonstrate the accuracy of the proposed approach. The geometry of one single SRR unit cell is shown in Fig. 9. The entire geometry of the SRRs array, together with its geometric parameters, is described in Fig. 10. A copper microstrip line is connected to the ground via two 50Ω ports over a dielectric substrate characterized by relative permittivity of $\epsilon_r = 9.7$, with four SRR elements (see Fig. 9) placed as shown in Fig. 10.

For the analysis, one port is excited by a $I = 1$ A current at the frequency $f = 1$ GHz, i.e. so that the quasi-static hypothesis is not valid. The PEEC simulation is performed by using a uniform mesh resulting in 8874 nodes, 22819 inductive branches, with the edge size of elementary sources in the range ($17.5 \mu\text{m}$, 0.6 mm). The maximum mesh size to wavelength ratio is 0.002. Hence, the selected test case is in the range of applicability of the proposed formulas as previously described.

Figure 11 shows the electric field observed along a line above the array at the position, $x = 10$ mm, $y = (-1 \text{ mm}, 10 \text{ mm})$, $z = 2$ mm, computed by using PEEC TAYLOR-FW, PEEC GAUSSIAN-FW and HFSS-FW methods (defined in the same way of Section IV-B). HFSS-FW has been run with a mesh of 853875 tetrahedral 3D elements,

Fig. 7. E -field components at 1 GHz observed on a line above the plate ($y = 0$ mm and $z = 30$ mm).

Fig. 8. Total electric field at 1 GHz observed on a line above the plate ($y = 0$ mm and $z = 30$ mm).

where 285542 tetrahedrons were used for modeling the air boundary region.

A good agreement between the E -field components cal-

Fig. 9. Top view of a single SRR unit made of copper (it is symmetric in the xy plane and all dimensions are in mm). Its thickness along the z -axis is $35 \mu\text{m}$.

Fig. 10. Top and side view of the SRRs array symmetric in the xy plane; all dimensions are given in mm).

culated by using all three specified methods is achieved. The post-processing step for computing the electric field required 78 s and 20 min for PEEC TAYLOR-FW and PEEC GAUSSIAN-FW, respectively, which confirms a great CPU time saving by using the proposed analytical calculation approach.

V. CONCLUSIONS

This paper presents a fast computation of the electric field radiated by electrical currents, magnetization and charges, in the framework of the PEEC full-wave modeling, assuming an orthogonal tessellation of volumes and surfaces. The integrals involved are analytically computed exploiting each contribution through Taylor series expansion of the full-wave free space Green's function. The derivation of the analytical formulas is provided, which can be useful as well as for other

Fig. 11. E field components at 1 GHz along a line above the array at the position: $x = 10$ mm, $y = (-1$ mm, 10 mm).

numerical methods requiring the computation of similar integrals. The numerical results validate the proposed analytical formulas showing a good agreement between the PEEC-based and FEM-based simulations.

APPENDIX

In this section, a list of indefinite integrals used to derive the analytic expressions for the transfer functions hcz_{xy}^{o2} , hcx_{xy}^{o2} and ha^{o2} is provided. In addition, the steps performed to achieve the analytical solution of hcz_{xy}^{o2} , hcx_{xy}^{o2} and ha^{o2} are summarized as well.

A. List of Indefinite Integrals

The integrals are summarized by (40). The solutions of $\mathcal{I}_a$ (40a), $\mathcal{I}_b$ (40b), $\mathcal{I}_c$ (40c) and $\mathcal{I}_d$ (40d), are derived using (41a) and (41b) from [26] and (42a) from [29].

$$\begin{aligned} \mathcal{I}_a &= \int (x^2 + c^2) \log(b + \sqrt{x^2 + b^2 + c^2}) dx = \\ &= -\frac{2}{3}c^2x + \frac{1}{9}x^3 + \frac{1}{6}bx\sqrt{x^2 + b^2 + c^2} + \\ &+ \frac{2}{3}c^3 \arctan\left(\frac{x}{c}\right) - \frac{2}{3}c^3 \arctan\left(\frac{bx}{c\sqrt{x^2 + b^2 + c^2}}\right) + \\ &+ c^2x \log(b + \sqrt{x^2 + b^2 + c^2}) + \\ &+ \frac{1}{3}x^3 \log(b + \sqrt{x^2 + b^2 + c^2}) + \\ &+ bc^2 \log(x + \sqrt{x^2 + b^2 + c^2}) + \\ &- \frac{1}{6}b(b^2 + 3c^2) \log(x + \sqrt{x^2 + b^2 + c^2}) \quad (40a) \end{aligned}$$

$$\begin{aligned} \mathcal{I}_b &= \int x^2 \log(u + \sqrt{x^2 + u^2 + w^2}) dx = \frac{1}{3}w^2x + \\ &- \frac{1}{9}x^3 + \frac{1}{6}ux\sqrt{x^2 + u^2 + w^2} - \frac{1}{3}w^3 \arctan\left(\frac{x}{w}\right) + \\ &+ \frac{1}{3}w^3 \arctan\left(\frac{ux}{w\sqrt{x^2 + u^2 + w^2}}\right) + \\ &+ \frac{1}{3}x^3 \log(u + \sqrt{x^2 + u^2 + w^2}) + \\ &- \left(\frac{1}{6}u^3 + \frac{1}{12}uw^2\right) \log(x + \sqrt{x^2 + u^2 + w^2}) \quad (40b) \end{aligned}$$

$$\begin{aligned} \mathcal{I}_c &= \int x^3 \arctan\left(\frac{u}{x}\right) dx = \\ &= -\frac{1}{4}u^3x + \frac{1}{12}ux^3 + \frac{1}{4}(x^4 - u^4) \arctan\left(\frac{u}{x}\right) \quad (40c) \end{aligned}$$

$$\begin{aligned} \mathcal{I}_d &= \int x^3 \arctan\left(\frac{u}{w\sqrt{x^2 + w^2}}\right) dx = \\ &= \left(-\frac{5}{12}u - \frac{1}{4}\frac{u^3}{w^3} + \frac{1}{12}\frac{ux^2}{w}\right) \sqrt{x^2 + w^2} + \\ &+ \frac{1}{4}(x^4 - w^2) \arctan\left(\frac{u}{w\sqrt{x^2 + w^2}}\right) + \\ &+ \frac{1}{2}u^2 \left(\frac{u^2}{2w^4} + \frac{1}{w}\right) \arctan\left(\frac{w\sqrt{x^2 + w^2}}{u}\right). \quad (40d) \end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{p_1} &= \int \log \left(u + \sqrt{x^2 + u^2 + w^2} \right) dx = \\
&= w \arctan \left(\frac{x}{w} \right) + w \arctan \left(\frac{w}{u} \right) + \\
&- w \arctan \left(\frac{ux}{w\sqrt{x^2 + u^2 + w^2}} \right) + \\
&+ x \log \left(u + \sqrt{x^2 + u^2 + w^2} \right) + \\
&+ u \log \left(x + \sqrt{x^2 + u^2 + w^2} \right) - x \quad (41a)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{p_2} &= \int x \arctan \left(\frac{uw}{x\sqrt{x^2 + u^2 + w^2}} \right) dx = \\
&= \frac{x^2}{2} \arctan \left(\frac{uw}{x\sqrt{x^2 + u^2 + w^2}} \right) + \\
&- \frac{w^2}{2} \arctan \left(\frac{ux}{w\sqrt{x^2 + u^2 + w^2}} \right) + \arctan \left(\frac{w}{u} \right) + \\
&- \frac{u^2}{2} \arctan \left(\frac{wx}{u\sqrt{x^2 + u^2 + w^2}} \right) + \arctan \left(\frac{u}{w} \right) + \\
&+ uw \log \left(x + \sqrt{x^2 + u^2 + w^2} \right) \quad (41b)
\end{aligned}$$

$$\mathcal{I}_{p_3} = \int \frac{x^2}{x^2 + w^2} dx = x - w \arctan \left(\frac{x}{w} \right). \quad (41c)$$

$$\mathcal{I}_{\ell_1} = \int \frac{x}{\sqrt{x^2 + a}} dx = \sqrt{x^2 + a} \quad (42a)$$

$$\mathcal{I}_{\ell_2} = \int \frac{1}{\sqrt{G(x)}} dx = \begin{cases} \frac{1}{\sqrt{c}} \log \left(\frac{2\sqrt{cG(x)+2cx+b}}{\sqrt{\Delta}} \right) & \text{if } c > 0 \\ \frac{1}{\sqrt{c}} \operatorname{arcsinh} \left(\frac{2cx+b}{\sqrt{\Delta}} \right) & \text{if } c > 0, \Delta > 0 \\ \frac{1}{\sqrt{c}} \log (2cx + b) & \text{if } c > 0, \Delta = 0 \\ \frac{-1}{\sqrt{-c}} \operatorname{arcsinh} \left(\frac{2cx+b}{\sqrt{-\Delta}} \right) & \text{if } c > 0, \Delta < 0 \end{cases} \quad (42b)$$

$$\begin{aligned}
\mathcal{I}_{\ell_3} &= \int \sqrt{G(x)} dx = \frac{2cx + b\sqrt{G(x)}}{4c} + \\
&+ \frac{\Delta}{8c} \int \frac{1}{\sqrt{G(x)}} dx \quad (42c)
\end{aligned}$$

where $G(x) = cx^2 + bx + a$ and $\Delta = 4ac - b^2$.

B. $hc z_{xy}^{o2}$ computation

It is important to note that $hc z_{xy}^{o2}$ (21) is equal to 0 if $z_o = z_b$. By applying formula (42c) from [29], a new expression for $hc z_{xy}^{o2}$ (43) is obtained. The integral (43) can be then analytically calculated by using the solution of $\mathcal{I}_{p_1}$ (41a), which leads to the final solution for $hc z_{xy}^{o2}$ (23).

$$\begin{aligned}
hc z_{xy}^{o2} &= -(z_o - z_b) \cdot \\
&\cdot \int_{y_1}^{y_2} \left\{ \log \left(\sqrt{x^2 + (y_o - y')^2 + (z_o - z_b)^2} \right) \right\}_{x=x_o-x_1}^{x=x_o-x_2} dy'. \quad (43)
\end{aligned}$$

C. $hc x_{xy}^{o2}$ computation

Similarly as for $hc z_{xy}^{o2}$, the integral $hc x_{xy}^{o2}$ (27) can be rewritten as (44) using (42a) from [29], which is in turn evaluated by using the solution of $\mathcal{I}_{\ell_3}$ (42c) from [29] to obtain $hc x_{xy}^{o2}$ (29).

$$\begin{aligned}
hc x_{xy}^{o2} &= \\
&= - \int_{y_1}^{y_2} \left\{ \sqrt{x^2 + (y_o - y')^2 + (z_o - z_b)^2} \right\}_{x=x_o-x_1}^{x=x_o-x_2} dy'. \quad (44)
\end{aligned}$$

D. ha^{o2} computation

The integral ha^{o2} (33) can be expressed as (45) using $\mathcal{I}_{\ell_3}$ (42c).

$$\begin{aligned}
ha^{o2} &= - \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left\{ \frac{1}{2} z \sqrt{z'^2 + (x_o - x')^2 + (y_o - y')^2} + \right. \\
&+ \frac{1}{2} \left[(x_o - x')^2 + (y_o - y')^2 \right] \cdot \\
&\cdot \log \left(z + \sqrt{z^2 + (x_o - x')^2 + (y_o - y')^2} \right) \left. \right\}_{z=z_o-z_1}^{z=z_o-z_2} dy' dx'. \quad (45)
\end{aligned}$$

Then, by applying (42c) for the second time and the solution of integral $\mathcal{I}_a$ (40a) into (45), the expression (46) for ha^{o2} is obtained. The final solution of ha^{o2} (37) is then derived using (42c) and (40).

$$\begin{aligned}
ha^{o2} &= \int_{x_1}^{x_2} \left\{ \left\{ \frac{1}{3} (x_o - x')^2 y + \frac{1}{18} y^3 + \right. \right. \\
&- \frac{1}{3} z y \sqrt{z^2 + (x_o - x')^2 + y^2} + \\
&+ \frac{1}{3} (x_o - x')^3 \arctan \left(\frac{zy}{(x_o - x') \sqrt{z^2 + (x_o - x')^2 + y^2}} \right) + \\
&- \frac{1}{6} y (3(x_o - x')^2 + y^2) \log(z + \sqrt{z^2 + (x_o - x')^2 + y^2}) + \\
&- \frac{1}{4} z ((x_o - x')^2 + z^2) \log(y + \sqrt{z^2 + (x_o - x')^2 + y^2}) + \\
&- \frac{1}{12} z (3(x_o - x')^2 - z^2) \log(y + \sqrt{z^2 + (x_o - x')^2 + y^2}) + \\
&- \left. \left. \frac{1}{3} (x_o - x')^3 \arctan \left(\frac{y}{x_o - x'} \right) \right\}_{z=z_o-z_1}^{z=z_o-z_2} \right\}_{y=y_o-y_1}^{y=y_o-y_2} dx'. \quad (46)
\end{aligned}$$

E. Tips for integrals evaluation

In this section, the formulas helpful for solving the singularities of the integrals for $hc z_{xy}$ (20), $hc x_{xy}$ (26), and ha (32) are provided similarly to [26]. The singularity of type $x \log \left(\sqrt{x^2 + y^2 + z^2} \right)$ is solved using (47),

$$\lim_{x \rightarrow 0} \left[x \log \left(\sqrt{x^2 + y^2 + z^2} \right) \right] = 0. \quad (47)$$

The function $x^2 \tan^{-1}\left(\frac{yz}{x\sqrt{x^2+y^2+z^2}}\right)$ is evaluated as $x|x|\arctan2\left(\frac{yz}{|x|\sqrt{x^2+y^2+z^2}}\right)$, where the function $\arctan2$ is defined by (48), with $\tan^{-1}(q)$ representing the inverse tangent of q computed in the range $[-\pi/2, \pi/2]$.

$$\arctan2\left(\frac{k}{m}\right) = \begin{cases} \tan^{-1}\left(\frac{k}{m}\right) & \text{if } m > 0 \\ \tan^{-1}\left(\frac{k}{m}\right) + \pi & \text{if } k \geq 0 \ \& \ m < 0 \\ \tan^{-1}\left(\frac{k}{m}\right) - \pi & \text{if } k < 0 \ \& \ m < 0 \\ \frac{\pi}{2} & \text{if } k > 0 \ \& \ m = 0 \\ -\frac{\pi}{2} & \text{if } k < 0 \ \& \ m = 0 \end{cases} \quad (48)$$

F. E_C evaluation: summary

Let us consider the computation of the E_C component of the electric field for a surface Manhattan type tessellation. Hence, considering that an elementary source surface can be located in the $x-y$, $x-z$ or $y-z$ plane, nine sub-functions hcx_{xy} , hcx_{xz} , hcx_{yz} , hcy_{xy} , hcy_{xz} , hcy_{yz} , hcz_{xy} , hcz_{xz} and hcz_{yz} can be defined as shown in equations (11)–(13). For these, it is sufficient evaluate only hcz_{xy} and hcx_{xy} because all the others can be computed from hcz_{xy} and hcx_{xy} by resorting to a Cartesian axes exchange as reported in Section III-A3. hcz_{xy} is analytically evaluated as reported into equations (20)–(25), where the exponential term is approximated though the Taylor series expansion for the exponential term. Similarly, hcx_{xy} is analytically evaluated as reported into equations (26)–(30). In addition, the required analytical formulas for the quasi-static integrals hcz_{xy}^{QS} and hcx_{xy}^{QS} , defined in (21) and (27), respectively, are provided in [26].

G. E_A evaluation: summary

Let us consider the computation of the E_A component of the electric field for a surface Manhattan type tessellation. In this case the transfer function h_a can be defined as shown in (15). Its analytical evaluation can be performed as reported into equations (32)–(38) in which, also in this case, the exponential term is approximated through the third-order Taylor series expansion. Finally, the analytical formula for the quasi-static term ha^{QS} , defined in (33), is provided in [26].

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Ulrike Grossner earned the Dipl.-Phys. and the Dr. rer. nat. degree from the Friedrich-Schiller-University Jena, Germany in 1997 and 2000, respectively. In 2014, she was appointed Full Professor at ETH Zurich, Switzerland, where she established the Advanced Power Semiconductor Laboratory (APS), working on devices and packaging for advanced power semiconductors.

Ivana Kovacevic-Badstubner received the Ph.D. degree from the ETH Zurich, Switzerland, in 2012. From 2008 to 2015, she worked in the Power Electronics Systems (PES) Laboratory at ETH Zurich focusing on the prediction of electromagnetic behavior of power electronics systems based on the developed numerical techniques and the lifetime modeling of power semiconductor modules. In March 2016, she joined the Advanced Power Semiconductor (APS) Laboratory at ETH Zurich. Her research interests

include novel packaging technologies for SiC power devices, the optimization of power module layout with respect to electromagnetic interference, and multi-domain modeling of power semiconductor devices and their modules.

Daniele Romano was born in Campobasso, Italy, in 1984. He received the Laurea degree in computer science and automation engineering in 2012 from University of L'Aquila, L'Aquila, Italy. He received the Ph.D. degree in 2018. Since 2012 he has been with the UAq EMC Laboratory, University of L'Aquila, focusing on EMC modeling and analysis, algorithm engineering and speed-up techniques applied to EMC problems.

Giulio Antonini (M94 - SM05) received the Laurea degree (*cum laude*) in electrical engineering from the University of L'Aquila, L'Aquila, Italy, in 1994 and the Ph.D. degree in electrical engineering from University of Rome "La Sapienza" in 1998. Since 1998, he has been with the UAq EMC Laboratory, University of L'Aquila, where he is currently a Professor. He has coauthored the book "*Circuit Oriented Electromagnetic Modeling Using the PEEC Techniques*", (Wiley-IEEE Press, 2017). His scientific interests are in the field of computational electromagnetics.

Luigi Lombardi received the Laurea (MD) in electronic engineering and the PhD, both *cum laude*, from University of L'Aquila, L'Aquila, Italy. In early 2018 he was a visiting researcher with the department of electronics in Carleton University, Ottawa, Ontario, Canada. Since November 2018 he works as a design engineer with the Non Volatile Memory (NVM) design team at Micron Semiconductor, Avezzano, AQ, Italy.