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Elliptic and abelian period spaces

by

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1. Introduction. In his book on transcendental numbers Th. Schneider [7] presents a list of seven problems. Two of them, Problems 3 and 4, concern transcendence properties of values of elliptic integrals of the third kind and transcendence properties of abelian integrals respectively. In this note we consider mainly the first of those problems. Schneider himself worked on both questions in the case that the underlying differential forms are of the second kind ⁽¹⁾. In a series of papers based on his solution of Hilbert's seventh problem he went as far as the technical tools of the time allowed. It was clear, as he wrote, that the methods were exhausted and new methods were necessary to attack the problems. In 1966 Baker made a breakthrough by developing a new method to get transcendence results for linear forms in finitely many logarithms. In a series of papers he also extended Schneider's results about transcendence of values of elliptic functions in the case of the Weierstrass $\wp$ - and ζ -functions which contain also transcendence results on elliptic integrals of the second kind. His results were later extended in a very limited way to linear independence results for periods of integrals of the second kind in publications by Baker, Coates, Masser, Laurent and Bertrand.

This changed in 1983 when we introduced a new method far exceeding that of Baker and formulated and proved a fully general result now known as the Analytic Subgroup Theorem [12]. It deals with analytic subgroups of commutative algebraic groups defined over $\mathbb{Q}$ and shows that the smallest analytic subgroup defined over $\mathbb{Q}$ which contains a non-trivial algebraic point is an algebraic subgroup. An almost trivial application of the analytic subgroup theorem is the transcendence of π . Using this general result we

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⁽¹⁾ Note that we assume that the differential forms of the second kind include the differential forms of the first kind.

shall determine the dimension of the subspace of $\mathbb{C}$ generated over $\overline{\mathbb{Q}}$ by the periods of elliptic integrals attached to a fixed elliptic curve defined over $\overline{\mathbb{Q}}$. Depending on the nature of the differential forms this dimension can vary and this makes it necessary to determine possible relations and then to show that these relations are the only ones. The possible relations are shown to come from geometry, and to prove that these are all we apply the analytic subgroup theorem.

Until now the best result of this type has been published by Masser. He showed in [5, Chs. II and III] that the dimension of the vector space generated over $\overline{\mathbb{Q}}$ by 1, π and the periods of differential forms of the first and second kind on an elliptic curve defined over $\overline{\mathbb{Q}}$ is 4 or 6 depending on whether the elliptic curve has complex multiplication or not. The techniques at the time did not allow one to go further and include differential forms of the third kind as we do here.

Besides giving an extended answer to Schneider's third problem there is a very interesting relation to dynamics on an ellipsoid in $\mathbb{P}^3$. Classical problems dealing with twisted curvature lines and geodesics ask when the curvature lines are dense and when the geodesic flow is compact. In a paper in preparation we give a necessary and sufficient condition for a geodesic to be closed. The answer to the first question depends on irrationality of a quotient of two periods of the third kind or, stated in different terms, on the $\overline{\mathbb{Q}}$ -linear independence of two integrals involved. The second question leads one to consider periods on abelian surfaces. Here one has to prove linear independence results for periods of abelian integrals of the first kind. This is done in the present paper and was besides the Schneider problem one of the motivations for writing this note.

The restriction to ellipsoids in $\mathbb{P}^3$ is not really necessary to apply the transcendence machinery. If one considers an ellipsoid in an arbitrary finite dimension, elliptic integrals of the third kind have to be replaced by abelian integrals of the same kind. The endomorphism algebra of abelian varieties becomes more complex, but only technical difficulties appear which can be overcome.

The application of the subgroup theorem is quite subtle and involved. In Section 2 we introduce periods of the first, second and third kind for an elliptic curve and discuss the periods relation coming from endomorphisms of elliptic curves. We conclude with an example which is relevant to the billiards problem. In Section 3 the elliptic period spaces are introduced.

A prominent role in the proofs is played by generalized Jacobians which are extensions of the classical Jacobians of a curve by a commutative linear algebraic group. This transfers the problem to a problem on the period lattice of a commutative algebraic group. In our case it is the group $\text{Ext}^1(E, L)$, classifying extensions of the elliptic curve E by a linear algebraic group L up to isomorphism. We then recall the Analytic Subgroup Theorem and

determine the endomorphisms of such extensions which lead to geometric period relations. We end by stating our Theorem 2 and give in Theorem 3 an outlook without proof on possible generalizations. In Section 4 we prove Theorem 2. The proof splits into four cases depending on the properties of the point $[G]$ in $\text{Ext}^1(E, L)$ determined by the extension G of E .

In Section 5 we deal with an abelian surface coming up in the problem of closed geodesics. Again the period space is introduced and the geometric period relations are studied. They come from Albert's classification of algebras with involution (see for instance [6, IV.21, Theorem 2] or [4, Ch. 9], which is more explicit). The proof of Theorem 5 then splits into five cases; in each of them we have to modify the group G used in order to apply the analytic subgroup theorem.

In the Appendix we study Ext-groups more carefully and prove a number of auxiliary results needed for the proofs of Theorems 2, 3 and 5. We mention in particular Theorems 9 and 10 which are necessary in order to deal with degenerations ⁽²⁾.

2. Homology and cohomology for elliptic curves. The questions about compactness of curvature lines, geodesics or null-geodesics raised by Ronaldo Garcia lead to irrationality problems for periods of elliptic and hyperelliptic integrals. The question evolved from work of Garcia on foliations and the geodesic flow on ellipsoids already studied by Monge and Jacobi in the 19th century who established a conformal parametrization of the surface by two elliptic integrals on the basis of elliptic coordinates. Density of the leaves and closedness of geodesics is then determined by the existence or non-existence of linear period relations.

2.1. Rational differentials. Closed rational differential forms on a smooth hyperelliptic curve C given by an equation $y^2 = F(x)$ with F a polynomial of degree n with only simple roots can be written as a linear combination of a differential of the first, second, or third kind and an exact rational differential form. Here a rational differential form is called *of the first kind* if it is holomorphic, *of the second kind* if it is meromorphic without logarithmic singularities, and *of the third kind* if it has only logarithmic singularities. The residue sum for a differential form of the third kind is zero. In the case of an elliptic curve E with Weierstrass equation

$$y^2 = 4x^3 + g_2x + g_3$$

the differential form $\omega = dx/y$ is of the first kind, the differential form $\eta = x\omega$ is of the second kind, and differential forms ξ of the third kind are

⁽²⁾ Some of the results can be deduced with some effort from our recent joint paper with Annette Huber (arXiv:1805.10104). However, the proofs differ significantly.

parametrized by their polar divisor $D = (P) + (Q)$ which defines a point in $\text{Pic}^0(E)$ by mapping the divisor $(P) + (Q)$ to the point $P - Q$ in $\text{Pic}^0(E)$. We then write $\xi = \xi_D$. In general for an abelian variety A , the group of divisor classes on A which are algebraically equivalent to 0 is by definition the Picard group $\text{Pic}^0(A)$ which is canonically isomorphic to the dual abelian variety $A^\vee$ (see Section 6.6). A closed rational differential form on an elliptic curve E defined over $\overline{\mathbb{Q}}$ then takes the form

$$(1) \quad a\omega + b\eta + \sum_{i=1}^N c_i \xi_i + df$$

with $a, b, c_1, \dots, c_N \in \overline{\mathbb{Q}}$ and a rational function $f \in \overline{\mathbb{Q}}(C)$. Given a divisor $D = (P) + (Q)$ the existence of a differential form of the third kind is a consequence of the theorem of Riemann–Roch applied to the divisor D . One shows that $\dim H^0(E, \Omega_E^1(\log D)) = 2$, which implies that besides ω there is a form with polar divisor D . For a proof using Hodge theory see [2].

2.2. Homology. The endomorphism algebra $\text{End}(E)$ of an elliptic curve E is either $\mathbb{Z}$ or an order in an imaginary quadratic number field. In the latter case we say that E has complex multiplication, abbreviated by CM. It acts on the homology $H_1(E, \mathbb{Z})$ and on the de Rham cohomology $H_{\text{dR}}^1(E)$. The action on $H_{\text{dR}}^1(E)$ respects the Hodge filtration

$$0 \subset H^0(E, \Omega_E^1) \subset H_{\text{dR}}^1(E).$$

By Hodge theory there is a decomposition $H_{\text{dR}}^1(E) = H^{0,1} \oplus H^{1,0}$ with $\overline{H^{1,0}} = H^{0,1}$, $H^{0,1} = H^0(E, \Omega_E^1)$ and $H^{1,0} \simeq H^1(E, \mathcal{O}_E)$. This together shows that

$$(2) \quad \begin{aligned} \rho_{\text{dR}}(\alpha)[\omega] &= \alpha[\omega], \\ \rho_{\text{dR}}(\alpha)[\eta] &= \beta[\omega] + \overline{\alpha}[\eta] \end{aligned}$$

for the classes of ω, η with $\alpha, \beta \in \text{End}(E)$. The differential form of the third kind defined by the divisor $D = (P) + (Q)$ can be normalized by taking $Q = O$ where O denotes the neutral element of E ; we then write ξ_P instead of ξ_D . Clearly $\alpha^* \xi_P = \xi_{\alpha^\vee P}$ where $\alpha^\vee$ is the dual of α . Here we are referring to duality for abelian varieties. This means that if $\alpha \in \text{End}(A)$ for an abelian variety A then $\alpha^\vee \in \text{End}(A^\vee)$ denotes its dual.

The endomorphisms $\text{End}(E)$ also act on the homology $H_1(E, \mathbb{Z})$ through a rational representation $\rho_B : \text{End}(E) \rightarrow \text{End}(H_1(E, \mathbb{Z}))$ with $\mathcal{O}_E \ni \alpha \mapsto \rho_B(\alpha)$. We write $H_1(E, \mathbb{Z}) = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$; then $\alpha_* \gamma_i = \rho_B(\alpha) \gamma_i = a_i \gamma_1 + b_i \gamma_2$ for $i = 1, 2$ and $a_i, b_i \in \mathbb{Z}$.

2.3. Periods of differentials of the third kind. Since we are also dealing with differentials of the third kind on the elliptic curve $C = E$, we have to remove the points P and Q from C to obtain an open Riemann

surface $X = C \setminus \{P, Q\}$. Its homology $H_1(X, \mathbb{Z})$ has a basis $\gamma_1, \gamma_2, \gamma_3$ such that the cycles γ_1 and γ_2 become a basis in $H_1(C, \mathbb{Z})$, and γ_3 is a loop around P .

We calculate the periods of ω, η, ξ in terms of $\gamma_1, \gamma_2, \gamma_3$. As a result we get a period matrix

$$(3) \quad \begin{pmatrix} \langle \omega, \gamma_1 \rangle & \langle \omega, \gamma_2 \rangle & 0 \\ \langle \eta, \gamma_1 \rangle & \langle \eta, \gamma_2 \rangle & 0 \\ \langle \xi, \gamma_1 \rangle & \langle \xi, \gamma_2 \rangle & 2\pi i \operatorname{res}_P(\xi) \end{pmatrix}$$

with periods of the first kind in the first row, periods of the second kind in the second row, and periods of the third kind in the third row. Here $\langle \zeta, \gamma \rangle$ for a cycle γ and a differential form ζ is defined to be $\int_\gamma \zeta$, also written $\zeta(\gamma)$. The lemma below has been established by Masser [5, Lemma 3.1]. Our proof uses Hodge theory and has the advantage that it generalizes easily to similar period relations among differential forms of the second kind on abelian varieties with non-trivial endomorphism algebra. For simplicity we assume that the order is maximal and then $\mathcal{O} = \mathbb{Z} + \tau\mathbb{Z} \subset K = \mathbb{Q}(\sqrt{-d})$ for $d > 0$ squarefree where

$$\tau = \begin{cases} (1 + \sqrt{-d})/2, & d \equiv 1 \pmod{4}, \\ \sqrt{-d}, & d \equiv 2, 3 \pmod{4}. \end{cases}$$

This does not change the transcendence properties because we may replace E by an isogenous elliptic curve with maximal order. The ring $\mathcal{O}$ acts on the Betti homology group $H_1(E, \mathbb{Z})$ through a rational representation $\rho_B : \mathcal{O} \rightarrow \operatorname{End}(H_1(E, \mathbb{Z}))$ which makes it into a free module over $\mathcal{O}$ of rank one. We choose a generator γ_1 of $H_1(E, \mathbb{Z})$ and write $\gamma_2 = \rho_B(\tau)(\gamma_1)$.

LEMMA 1. *If the endomorphism algebra of an elliptic curve E is an order $\mathcal{O}$ of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ then there are relations*

$$\begin{aligned} \langle \omega, \gamma_2 \rangle &= \tau \langle \omega, \gamma_1 \rangle, \\ \langle \eta, \gamma_2 \rangle &= \beta \langle \omega, \gamma_1 \rangle + \bar{\tau} \langle \eta, \gamma_1 \rangle \end{aligned}$$

between the periods of the first and second kind with $\beta \in K$.

Proof. Evaluating (2) at γ_1 and noticing that

$$\langle \rho_{\text{dR}}(\tau)(\eta), \gamma_1 \rangle = \langle \eta, \rho_B(\tau)(\gamma_1) \rangle = \langle \eta, \gamma_2 \rangle$$

gives

$$\langle \eta, \gamma_2 \rangle = \beta \langle \omega, \gamma_1 \rangle + \bar{\tau} \langle \eta, \gamma_1 \rangle,$$

as asserted. ■

2.4. An example from elliptic billiards. As already mentioned, periods of elliptic integrals play a fundamental role for dynamics on triaxial

ellipsoids. The quotient of two periods of an elliptic integral of the third kind becomes significant for the question whether orbits on ellipsoids for twisted curvature lines are non-compact. This question is studied in [13] and the answer is based on Theorem 2 of this note. The equation for the ellipsoid under consideration is given in diagonal form

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1$$

for real $a > b > c > 0$. The characteristic curve associated to the ellipsoid is

$$y^2 = (x - a)(x - b)(x - c),$$

which is an elliptic curve defined over the reals. The differential of the third kind here is

$$\xi = \sqrt{\frac{x}{(x - a)(x - b)(x - c)}} dx = \frac{x dx}{\sqrt{x(x - a)(x - b)(x - c)}},$$

which is a meromorphic differential on the curve

$$(4) \quad y^2 = x(x - a)(x - b)(x - c).$$

The right side of this equation has degree 4 and so ∞ is a singular point. When $x \rightarrow \infty$ the quotient y/x^2 tends to ± 1 and this means that the singularity at ∞ is a node. If we put $x = 1/u$ and $y = v/u^2$ then (4) becomes the equation

$$v^2 = (1 - au)(1 - bu)(1 - cu)$$

for the normalization of the curve and the two points at ∞ become $P = (0, -1)$ and $Q = (0, 1)$. The differential form ξ changes under the change of coordinate $(u, v) = (1/x, y/x^2)$ into

$$\xi_D = -\frac{1}{u} \frac{du}{v}$$

with du/v holomorphic. Since u is a uniformizing parameter, one sees that the polar divisor becomes $(\xi_D)_\infty = (P) + (Q)$ such that ξ_D has logarithmic singularities at P and Q with residues ± 1 . We conclude that either ξ_D is of the third kind or the point $P - Q$ has finite order.

3. Elliptic period space. One of our goals is to determine the vector space W over $\overline{\mathbb{Q}}$ generated by the entries of the period matrix (3) together with 1 for an elliptic curve E defined over $\overline{\mathbb{Q}}$. If the point $P - Q$ in E has infinite order the space W is generated by the periods

$$1, \langle \omega, \gamma_1 \rangle, \langle \omega, \gamma_2 \rangle, \langle \eta, \gamma_1 \rangle, \langle \eta, \gamma_2 \rangle, \langle \xi, \gamma_1 \rangle, \langle \xi, \gamma_2 \rangle, 2\pi i.$$

If in addition there is complex multiplication, we use again the period relations of Lemma 1 to reduce the above set of generators to

$$1, \langle \omega, \gamma_1 \rangle, \langle \eta, \gamma_1 \rangle, \langle \xi, \gamma_1 \rangle, \langle \xi, \gamma_2 \rangle, 2\pi i.$$

If P has finite order then we apply Theorem 10 of the Appendix and find that the generators are

$$1, \langle \omega, \gamma_1 \rangle, \langle \omega, \gamma_2 \rangle, \langle \eta, \gamma_1 \rangle, \langle \eta, \gamma_2 \rangle, 2\pi i$$

and

$$1, \langle \omega, \gamma_1 \rangle, \langle \eta, \gamma_1 \rangle, 2\pi i$$

respectively, depending on whether the elliptic curve has complex multiplication or not.

If a rational differential form as given in (1) is considered, the elliptic period space changes into the vector space over $\overline{\mathbb{Q}}$ generated by the set

$$1, \langle \omega, \gamma_1 \rangle, \langle \omega, \gamma_2 \rangle, \langle \eta, \gamma_1 \rangle, \langle \eta, \gamma_2 \rangle, \langle \xi_1, \gamma_1 \rangle, \langle \xi_1, \gamma_2 \rangle, \dots, \langle \xi_N, \gamma_1 \rangle, \langle \xi_N, \gamma_2 \rangle, 2\pi i.$$

The methods developed for calculating $\dim W$ generalize to this situation.

3.1. Generalized Jacobian. For our study of periods of rational integrals on a curve the concept of generalized Jacobians is indispensable. They provide the link between the cohomology of a curve and commutative algebraic groups. If X is a smooth projective algebraic curve defined over a field of characteristic 0 and D an effective divisor on X with support $|D|$, and if ξ is a rational differential form on X regular outside D , then there exists a morphism $\kappa : U = X \setminus |D| \rightarrow G$ into an extension of the Jacobian $J(X)$ of X by a commutative linear algebraic group and an invariant differential form $\omega \in \mathfrak{g}^\vee$ in the dual of the Lie algebra $\mathfrak{g}$ of G such that $\xi = \kappa^* \omega$ (see [8, Chs. I.1, V.8 and Proposition 5, Ch. V.10]). This makes it possible to transfer questions on transcendence properties of periods on an algebraic curve to the same questions on an algebraic group. For an elliptic curve E generalized Jacobians are extensions of the elliptic curve itself by a commutative linear algebraic group and the latter becomes a product of a vector group and a torus. In Section 6 we have put together the facts about group extensions which we shall need.

Using generalized Jacobians, periods of our differential forms ω , η of the second kind and ξ of the third kind on E can be written as periods of invariant differential forms on extensions of E by the additive group $\mathbb{G}_a$ and the multiplicative group $\mathbb{G}_m$ respectively. It is therefore natural to study the isomorphism classes $[G_a]$ of extensions

$$(5) \quad 0 \rightarrow \mathbb{G}_a \xrightarrow{\iota} G_a \xrightarrow{p} E \rightarrow 0$$

and the isomorphism classes $[G_m]$ of extensions

$$(6) \quad 0 \rightarrow \mathbb{G}_m \xrightarrow{\iota} G_m \xrightarrow{p} E \rightarrow 0,$$

which are groups denoted by $\text{Ext}^1(E, \mathbb{G}_a)$ and $\text{Ext}^1(E, \mathbb{G}_m)$ with the Baer sum as addition (see Section 6.2 for their construction and properties). In our application we have to consider extensions of E by $\mathbb{G}_a$, $\mathbb{G}_m$, $\mathbb{G}_m^2$, $\mathbb{G}_a \times \mathbb{G}_m$

or $\mathbb{G}_a \times \mathbb{G}_m^2$. The fiber product $G_a \times_E G_m$ is an extension

$$0 \rightarrow \mathbb{G}_a \times \mathbb{G}_m \xrightarrow{\iota} G \xrightarrow{p} E \rightarrow 0$$

of E by $L = \mathbb{G}_a \times \mathbb{G}_m$. A similar description applies to $\mathbb{G}_m^2$ with $G_m \times_E G_m$ replacing G_m and to $\mathbb{G}_a \times \mathbb{G}_m^2$ with $G_a \times_E G_m \times_E G_m$ replacing $G_a \times_E G_m$. These algebraic groups play later a central role.

Conversely, if such an extension is given and if p_a and p_m denote the projections of $\mathbb{G}_a \times \mathbb{G}_m$ onto $\mathbb{G}_a$ and $\mathbb{G}_m$ respectively then there exist extensions G_a and G_m up to isomorphism such that $G = G_a \times_E G_m$. For their construction see Section 6.2. The construction furnishes natural homomorphisms $(p_a)_* : G \rightarrow G_a$ and $(p_m)_* : G \rightarrow G_m$ induced from p_a and p_m . If $[G]$ denotes the extension class of G (for more details see Section 6.1) then the maps $[G] \mapsto [(p_a)_*G], [(p_m)_*G]$ and $[[G_a], [G_m]] \mapsto [G] = [G_a \times_E G_m]$ furnish an isomorphism $\text{Ext}^1(E, L) \simeq \text{Ext}^1(E, \mathbb{G}_a) \times \text{Ext}^1(E, \mathbb{G}_m)$. It is obvious that this property extends to products of arbitrary commutative linear algebraic groups L_1 and L_2 instead of $\mathbb{G}_a$ and $\mathbb{G}_m$ and we get the following

LEMMA 2. *Denote by p_1 and p_2 the canonical projections from $L_1 \times L_2$ onto the first and second factor respectively. There is a canonical isomorphism*

$$(7) \quad \begin{aligned} \text{Ext}^1(E, L_1 \times L_2) &\rightarrow \text{Ext}^1(E, L_1) \times \text{Ext}^1(E, L_2), \\ [G] &\mapsto [(p_1)_*G], [(p_2)_*G], \end{aligned}$$

with inverse $[[G_1], [G_2]] \mapsto [G_1 \times_E G_2]$.

Proof. The second statement follows directly from the definition of $(p_i)_*$. ■

3.2. Analytic Subgroup Theorem. As we have seen, the question whether curvature lines or geodesics are closed or dense depends on the arithmetic properties of period integrals for elliptic curves and abelian surfaces respectively. This leads us directly to the analytic subgroup theorem (see [12]). We briefly discuss the theorem and explain how it is applied in our cases. To begin, we let G be a connected commutative algebraic group defined over $\overline{\mathbb{Q}}$. Then $\mathfrak{g} = \text{Lie } G$ is in a natural way a vector space over $\overline{\mathbb{Q}}$. There is a natural exponential sequence

$$(8) \quad 0 \rightarrow \Lambda \rightarrow \mathfrak{g}_{\mathbb{C}} \xrightarrow{\exp_G} G(\mathbb{C}) \rightarrow 0$$

with $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ and $G(\mathbb{C})$ the complex Lie algebra and Lie group respectively and Λ the period lattice given by the kernel of the exponential map $\exp_G$. We consider a Lie subalgebra $\mathfrak{b} \subset \mathfrak{g}$ —which as a subspace of $\mathfrak{g}$ is a $\overline{\mathbb{Q}}$ -vector space as well—and denote by B the image of $\mathfrak{b}_{\mathbb{C}} = \mathfrak{b} \otimes \mathbb{C}$ under the exponential map $\exp_G$. This is an analytic subgroup of $G(\mathbb{C})$. We call a Lie subalgebra of $\mathfrak{g}$ *algebraic* if it is the Lie algebra of an algebraic subgroup of G . Let $\mathfrak{h} \subseteq \mathfrak{b}$ be the largest algebraic subalgebra and define

$B(\overline{\mathbb{Q}}) = B \cap G(\overline{\mathbb{Q}})$. Then trivially $H(\overline{\mathbb{Q}}) \subseteq B(\overline{\mathbb{Q}})$. The analytic subgroup theorem gives the converse.

THEOREM 1 (AST). $B(\overline{\mathbb{Q}}) = H(\overline{\mathbb{Q}})$.

The proof is given in [12] (see also [1]), and we only note that the case $B(\overline{\mathbb{Q}}) = 0$ is totally trivial.

3.3. Endomorphisms. To find relations between periods we have to determine the endomorphism algebra for an extension G of E by $\mathbb{G}_a$ and $\mathbb{G}_m$ respectively. Lemma 2 then determines extensions of E by arbitrary commutative linear algebraic groups. The groups $\text{End}(E)$, $\mathbb{G}_a$ and $\mathbb{Z}$ act on $\text{Ext}^1(E, \mathbb{G}_a)$ and $\text{Ext}^1(E, \mathbb{G}_m)$ respectively through pull-back $[G_a] \mapsto \alpha^*[G_a]$ and push-forward $[G_a] \mapsto [\nu_*G_a]$ and $[G_m] \mapsto [\alpha^*G_m]$ and $[G_m] \mapsto [\nu_*G_m]$ for $\alpha \in \text{End}(E)$ and $\nu \in \mathbb{G}_a$ and $\nu \in \mathbb{Z}$ respectively. Their construction is given for E replaced by an abelian variety in Section 6.2. To determine the endomorphism algebra of $[G] \in \text{Ext}^1(E, L)$ we have to find the stabilizer of $[G] \in \text{Ext}^1(E, L)$. This will be done in Section 6.4 in more generality and we quote here Theorem 7 only in the case of an elliptic curve. As a result we find that the endomorphisms of $[G] \in \text{Ext}^1(E, L)$ are given by

$$\text{End}(G_a) = \begin{cases} \mathbb{G}_a \times \text{End}(E) & \text{if } [G_a] \text{ is trivial,} \\ \text{End}(E) & \text{otherwise} \end{cases}$$

when $L = \mathbb{G}_a$. If $L = \mathbb{G}_m$ then

$$\text{End}(G_m) = \begin{cases} \text{finite index subgroup of } \text{End}(E) \times \mathbb{Z} & \text{if } [G_m] \text{ is torsion,} \\ \mathbb{Z} & \text{otherwise,} \end{cases}$$

and when $L = \mathbb{G}_a \times \mathbb{G}_m$ we have

$$\text{End}(L) = \text{End}((p_a)_*G) \times_{\text{End}(E)} \text{End}((p_m)_*G)$$

where p_a and p_m denote the projections from L onto $\mathbb{G}_a$ and $\mathbb{G}_m$ respectively.

3.4. Period relations. Let E be an elliptic curve defined over $\overline{\mathbb{Q}}$. If the point $P = P_1 - P_2$ in E associated with the divisor $D = (P_1) + (P_2)$ of the differential of the third kind ξ on E is torsion, we apply Theorem 10 of Section 6.6. Accordingly the integrals $\langle \xi, \gamma_1 \rangle$ and $\langle \xi, \gamma_2 \rangle$ are periods of the first kind and $\langle \xi, \gamma_3 \rangle$ is in $2\pi i\mathbb{Z}$. This shows that if P has infinite order, the rank of the module generated over $\overline{\mathbb{Q}}$ by 1 together with the entries of the period matrix is in general at most 8 and at most 6 if E has CM. If the order of P is finite then 8 and 6 have to be replaced by 6 and 4. From Theorem 1 (see also [1]) we shall deduce that these numbers are the dimension of the vector space generated by the module over $\overline{\mathbb{Q}}$ which we call the *period space* in each case. At this place the calculation of the endomorphism algebra of G is needed. Our discussion leads us to the following

THEOREM 2. *If P is non-torsion, the dimension of the period space generated by 1 together with the entries of the period matrix is 8 if E has no complex multiplication and it is 6 in the CM-case. If P is a torsion point then the dimension is 6 if E has no complex multiplication and 4 if E has CM.*

Our Theorem 2 is an extension of results published in [10] and [11]. It deals with a situation which has not been considered before. It gives the solution of Schneider's 3rd problem raised in his book on transcendental numbers [7] in the case of periods in the classical sense.

COROLLARY 3.1. *Let*

$$\mathcal{B} \subseteq \{1, 2\pi i, \langle \omega, \gamma_1 \rangle, \langle \eta, \gamma_1 \rangle, \langle \xi, \gamma_1 \rangle, \langle \omega, \gamma_2 \rangle, \langle \eta, \gamma_2 \rangle, \langle \xi, \gamma_2 \rangle\}$$

be any set of elements linearly independent over $\mathbb{Q}$. Then the quotient of any pair of distinct elements of $\mathcal{B}$ is irrational.

Proof. If E has CM, there are no period relations defined over $\mathbb{Q}$. ■

In particular we deduce

COROLLARY 3.2. *The periods of the third kind*

$$\langle \xi, \gamma_1 \rangle = \int_c^b \sqrt{\frac{x}{(x-a)(x-b)(x-c)}} dx$$

and

$$\langle \xi, \gamma_2 \rangle = \int_b^a \sqrt{\frac{-x}{(x-a)(x-b)(x-c)}} dx$$

are transcendental and linearly independent over $\overline{\mathbb{Q}}$. The latter implies in particular that $\langle \xi, \gamma_1 \rangle / \langle \xi, \gamma_2 \rangle$ is transcendental.

The transcendence of $\langle \xi, \gamma_1 \rangle / \langle \xi, \gamma_2 \rangle$ is the key result needed in [13] to determine the closed geodesics on the ellipsoid. Let $\Lambda = \mathbb{Z}\omega(\gamma_1) + \mathbb{Z}\omega(\gamma_2)$ be the period lattice of the Weierstrass $\wp$ -function attached to E . Then the exponential map $\exp_E$ can be expressed in terms of the Weierstrass function as $\exp_E : z \mapsto (\wp(z), \wp'(z))$. The endomorphism algebra $\mathcal{O}$ of an elliptic curve is either $\mathbb{Z}$ or an order in an imaginary quadratic field. We define $\lambda(u, v) := v\zeta(u) - \eta(v)u$ where $\eta(v) := \zeta(z+v) - \zeta(z)$ ⁽³⁾. Instead of $\eta(\omega(\gamma))$ for $\gamma \in H_1(E, \mathbb{Z})$ we simply write $\eta(\gamma)$. The famous Legendre relation $\omega(\gamma_2)\eta(\gamma_1) - \omega(\gamma_1)\eta(\gamma_2) = 2\pi i$ can then be expressed as $\lambda(\omega(\gamma_1), \omega(\gamma_2)) = 2\pi i$ and shows that $2\pi i$ is a special value of the λ -function. If one takes $\gamma \in H_1(E, \mathbb{Z})$ then $\lambda(u, \omega(\gamma))$ is a period of an elliptic integral of the third kind for the differential ξ_P associated with $P = \exp_E(u)$ defined by $u \in \mathbb{C}$ as explained in Section 6.6.

⁽³⁾ See also [3, IV, Beispiel 3].

Let Γ be the module over $\text{End}(E)$ generated by $u_1, \dots, u_n \in \mathbb{C}$. The lattice Λ is also a module over $\text{End}(E)$. Let $\rho(\Lambda)$ and $\rho(\Gamma/\Gamma \cap \Lambda)$ be the ranks of Λ and $\Gamma/\Gamma \cap \Lambda$ as modules over $\text{End}(E)$, and let $r(\Gamma/\Gamma \cap \Lambda)$ be the rank of $\Gamma/\Gamma \cap \Lambda$ as a module over $\mathbb{Z}$. In [11] we proved that if $u_1, \dots, u_n$ are complex numbers such that $\wp(u_i) \in \overline{\mathbb{Q}}$ and if Γ is the group generated by $u_1, \dots, u_n$ then, for $\gamma \in H_1(E, \mathbb{Z})$, the dimension of the vector space generated over $\overline{\mathbb{Q}}$ by $1, \omega(\gamma), \eta(\gamma), \lambda(u_1, \omega(\gamma)), \dots, \lambda(u_n, \omega(\gamma))$ is $3 + r(\Gamma/\Gamma \cap \mathbb{Z}\omega(\gamma))$. This result also contains the periods $2\pi i$ since we may take $u_{n+1} = \omega(\gamma_1)$ for $\gamma_1 \in H_1(E, \mathbb{Z})$. Then $\lambda(\omega(\gamma_1), \omega(\gamma)) \in 2\pi i\mathbb{Z}$. It follows that the dimension of the vector space generated over $\overline{\mathbb{Q}}$ by $1, 2\pi i, \omega(\gamma), \eta(\gamma), \lambda(u_1, \omega(\gamma)), \dots, \lambda(u_n, \omega(\gamma))$ is $4 + r(\Gamma/\Gamma \cap \Lambda)$.

Theorem 2 can be generalized to any number of differentials of the third kind and to the module $\mathcal{M}$ generated over $\mathbb{Z}$ by

$$1, 2\pi i, \omega(\gamma_i), \eta(\gamma_i), u_j, \lambda(u_j, \omega(\gamma_i)), \quad 1 \leq i \leq 2, 1 \leq j \leq n.$$

A combination of the proof of Theorem 2 and of the theorem mentioned above leads to the following

THEOREM 3. *We have*

$$\dim(\mathcal{M} \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}) = 2 + 2\rho(\Lambda) + \rho(\Gamma/\Gamma \cap \Lambda) + 2r(\Gamma/\Gamma \cap \Lambda).$$

4. Proof of Theorem 2. Let G be the generalized Jacobian defined by the differential forms ω, η and ξ . We denote by D the divisor $D = (P_1) + (P_2) + 2(O)$ where $(P_1) + (P_2)$ and $2(O)$ are the polar divisors of ξ and η respectively as described in detail in Section 3.1; see also [8] and [3] (where generalized Jacobians are constructed explicitly). In this situation the generalized Jacobian is an extension

$$(9) \quad 0 \rightarrow \mathbb{G}_a \times \mathbb{G}_m \xrightarrow{\iota} G \xrightarrow{p} E \rightarrow 0$$

of E by $\mathbb{G}_a \times \mathbb{G}_m$. The extension G , a commutative algebraic group of dimension 3, has the property that there exists a section $\sigma : E \setminus |D| \rightarrow G$ of p such that the forms ω, η, ξ are the pull-backs by σ of invariant differential forms on G (see also Section 6.6 where the construction of differentials of the third kind is explained). We choose a basis f_1, f_2, f_3 for the Lie algebra $\mathfrak{g}$ of G in such a way that its dual basis $f_1^\vee, f_2^\vee, f_3^\vee$ satisfies $\sigma^* f_1^\vee = \omega, \sigma^* f_2^\vee = \eta, \sigma^* f_3^\vee = \xi$. Since

$$\langle \omega, \gamma_1 \rangle = \int_{\gamma_1} \omega = \int_{\gamma_1} \sigma^* f_1^\vee = \int_{\sigma_* \gamma_1} f_1^\vee,$$

and similarly for the other period integrals, we may work with $f_1^\vee, f_2^\vee$ and $f_3^\vee$ instead of ω, η and ξ . We identify $H_1(G, \mathbb{C})$ with the Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$

through the homomorphism

$$I : H_1(G, \mathbb{C}) \rightarrow \mathfrak{g}_{\mathbb{C}}, \quad \gamma \mapsto I(\gamma) := \int_{\gamma},$$

given by integration of invariant differential forms on G . The integrals $I(\gamma)$ are by definition linear forms on $\mathfrak{g}_{\mathbb{C}}^{\vee}$ and therefore I goes into $(\mathfrak{g}_{\mathbb{C}}^{\vee})^{\vee} = \mathfrak{g}_{\mathbb{C}}$. Then $I(\sigma_*\gamma_1)$, $I(\sigma_*\gamma_2)$ and $I(\sigma_*\gamma_3)$ form a basis for the image of $H_1(G, \mathbb{Z})$ in $\mathfrak{g}_{\mathbb{C}}$. In analogy to the construction (9) we get extensions

$$0 \rightarrow \mathbb{G}_a \xrightarrow{\iota} G_a \xrightarrow{p_a} E \rightarrow 0$$

of E by $\mathbb{G}_a$ defined by η and

$$0 \rightarrow \mathbb{G}_m \xrightarrow{\iota} G_m \xrightarrow{p_m} E \rightarrow 0$$

of E by $\mathbb{G}_m$ defined by ξ respectively. Then G can be written as the fiber product $G = G_a \times_E G_m$. From the construction of the fiber product we deduce that $\sigma = \sigma_a \times_E \sigma_m$ for $\sigma_a : E \setminus |D| \rightarrow G_a$ and $\sigma_m : E \setminus |D| \rightarrow G_m$. These maps induce the tangent maps $d\sigma_a$, $d\sigma_m$ and $d\sigma$ between the Lie algebras $\mathfrak{g}_a$, $\mathfrak{g}_m$ and $\mathfrak{g}$. As a convention and with abuse of notation we write all the exponential maps from Lie algebras to algebraic groups as $\exp$. In Section 6.1 we show how to associate with an extension G its class $[G]$ and in Section 6.2 how a group law on the set $\text{Ext}^1(A, L)$ of isomorphism classes of extensions of A by L for A an abelian variety and L a linear algebraic group is defined on the basis of Baer sums.

Coming back to our differential forms $\eta \neq 0$ and $\xi \neq 0$ we see that the extension class $[G_a]$ of G_a in $\text{Ext}^1(E, \mathbb{G}_a)$ is not zero since η is not of the first kind. In the case of ξ it might happen that the extension class $[G_m]$ of G_m becomes a torsion element in $\text{Pic}^0(E) = E^{\vee}$, the dual elliptic curve. If this happens we deduce from Theorem 10 (stated and proved in Section 6.6) that the periods of ξ are algebraic multiples of $2\pi i$ modulo periods of the first kind; in this case we replace G_m by $\mathbb{G}_m \times E$. This choice gives $G = G_a \times \mathbb{G}_m$.

For the proof we have to consider four cases depending on whether E has

- (a) no complex multiplication, $[G_m]$ not torsion,
- (b) no complex multiplication, $[G_m]$ torsion,
- (c) complex multiplication, $[G_m]$ not torsion,
- (d) complex multiplication, $[G_m]$ torsion.

We shall give the proof case by case. It would be nice to have a uniform proof working in all the cases.

4.1. Auxiliary lemmata. For the proof of Theorem 2 we need a lemma in group theory. Let G_1 and G_2 be commutative groups. Suppose that $H \neq 0$ is a proper subgroup of $G_1 \times G_2$, denote by $p_i : H \subset G_1 \times G_2 \rightarrow G_i$ the canonical projections and define $H_1 = p_1(H)$, $H_2 = p_2(H)$, $K_1 \times 0 = p_2^{-1}(0) \subseteq H$ and $0 \times K_2 = p_1^{-1}(0) \subseteq H$.

LEMMA 3. *The subgroup H induces an isomorphism $\varphi : H_1/K_1 \simeq H_2/K_2$ between subquotients of G_i which is zero only if $H = H_1 \times H_2$.*

Proof. Observe that the projections p_i induce isomorphisms $H/(0 \times K_2) \simeq H_1$ and $H/(K_1 \times 0) \simeq H_2$ which take $(K_1 \times K_2)/(0 \times K_2)$ into K_1 and $(K_1 \times K_2)/(K_1 \times 0)$ into K_2 . From standard isomorphism theorems in group theory we conclude that

$$(10) \quad H_1/K_1 \simeq H/(K_1 \times K_2) \simeq H_2/K_2.$$

If $\varphi = 0$ then $H_1 = K_1$ and $H_2 = K_2$. Since $K_1 \times K_2 \subseteq H$ we deduce from (10) that $H_1 \times H_2 = H$. ■

The following lemma characterizes connected algebraic subgroups of $G = G_a \times_E G_m$ in $\text{Ext}^1(E, \mathbb{G}_a \times \mathbb{G}_m)$.

LEMMA 4. *Let $0 \neq H \subset G$ be a connected algebraic subgroup of G . Then either $H \subseteq \mathbb{G}_a \times \mathbb{G}_m$ or $[G_a] = 0$ or $[G_m] = 0$.*

Proof. Suppose that H is not contained in $\mathbb{G}_a \times \mathbb{G}_m$. Then H projects onto E and it is either E or an extension of E by $\mathbb{G}_a$ or by $\mathbb{G}_m$, and G is the product of E by $\mathbb{G}_a \times \mathbb{G}_m$ or of G_a by $\mathbb{G}_m$ or of G_m by $\mathbb{G}_a$. An application of Theorem 8 of Section 6.2 gives the statement. ■

Before beginning with the proof of Theorem 2 we deal with the particular case $G_1 = G_a \times_E G_m$ and $G = G_1 \times L$ where $L = \mathbb{G}_a \times \mathbb{G}_m$. The Lie algebras of $\mathbb{G}_a$ and $\mathbb{G}_m$ have the invariant derivatives $\frac{d}{ds}$ and $t\frac{d}{dt}$ as a basis. Let e_1, e_2, e_3, e_a, e_m be the basis for $\mathfrak{g} = \text{Lie } G$ induced by pull-back from the basis f_1, f_2, f_3 of $\mathfrak{g}_1 = \text{Lie } G_1$ together with $f_a = \frac{d}{ds}$ and $f_m = t\frac{d}{dt}$.

Any $\gamma \in H_1(G(\mathbb{C}), \mathbb{C})$ gives some $I(\gamma) \in \mathfrak{g}_{\mathbb{C}}$ and can be expressed as

$$I(\gamma) = \langle \omega, \gamma \rangle e_1 + \langle \eta, \gamma \rangle e_2 + \langle \xi, \gamma \rangle e_3 + e_a + 2\pi i \cdot e_m,$$

which by abuse of notation we write as a coordinate vector

$$u = (\langle \omega, \gamma \rangle, \langle \eta, \gamma \rangle, \langle \xi, \gamma \rangle, 1, 2\pi),$$

and instead of $\exp_G(I(\gamma))$ we use $\exp_G(u)$. We shall use similar notation for other groups. Also if $\lambda \in \mathfrak{g}^{\vee}$ we use $\lambda(u)$ instead of $\lambda(I(\gamma))$. The following lemma, which follows easily from [11, Satz 2], serves as an induction step in the proof of our main proposition.

LEMMA 5. *For $0 \neq \gamma \in H_1(E, \mathbb{Z})$ the periods $\langle \omega, \gamma \rangle, \langle \eta, \gamma \rangle, \langle \xi, \gamma \rangle$ together with 1 and 2π are linearly independent over $\overline{\mathbb{Q}}$.*

For the general case we take $G = G_1 \times G_2$ with $G_1 = G_a \times_E G_m$, $G_2 = G_1 \times L$, $L = \mathbb{G}_a \times \mathbb{G}_m$ and $0 \neq u \in \mathfrak{g} = \text{Lie } G$ written as $u = (u_1, u_2, u_l)$ with $u_i = (\langle \omega, \gamma_i \rangle, \langle \eta, \gamma_i \rangle, \langle \xi, \gamma_i \rangle) \in \text{Lie } G_i$, $i = 1, 2$, and $0 \neq u_l = (1, 2\pi i) \in \text{Lie } L$. We assume that the Zariski closure H of $\exp(\mathbb{Q}u)$ is an algebraic subgroup of G which projects by Lemma 5 surjectively onto each factor of the form

G_1 and onto each factor of L . We further require that the images of u_1 and u_2 in the Lie algebra of E are $\mathbb{Q}$ -linearly independent.

PROPOSITION 1. *If H is a proper algebraic subgroup of G then either the extension class $[G_a]$ is zero or the extension class $[G_m]$ is zero or E has complex multiplication.*

Proof. Lemma 3 together with the assumption gives an isomorphism

$$\alpha : G_1/K_1 \simeq G_2/K_2$$

which is not zero since otherwise we could deduce that $G_1 \times G_2 = K_1 \times K_2 \subseteq H \subset G_1 \times G_2$, a contradiction to the hypothesis of the lemma. Therefore we may assume that $G_i \neq K_i$. Let ι_1 be the natural injection of $\mathbb{G}_a \times \mathbb{G}_m$ into G_1 , and p_i be the projection from G_i onto E . For K_1 there are two possibilities: either $p_1(K_1) = (0)$ or $p_1(K_1) = E$. In the first case $K_1 \subseteq \mathbb{G}_a \times \mathbb{G}_m$. This implies that G_1/K_1 takes the values E , G_m or G_a and we are left with the possibilities $\alpha \in \text{End}(E), \text{End}(G_m)$ or $\text{End}(G_a)$. Each of the alternatives implies that E has complex multiplication because α induces $\alpha_E \in \text{End}(E)$ with $d\alpha_E(dp_1(u_1)) = dp_2(u_2)$ at the Lie algebra level. By hypothesis the tangent vectors $dp_1(u_1)$ and $dp_2(u_2)$ are linearly independent over $\mathbb{Q}$ so that we may conclude that $\alpha_E \notin \mathbb{Z}$ and therefore E has complex multiplication. This settles the case $p_1(K_1) = (0)$.

Since K_1 is a proper subgroup of G_1 and projects onto E in the remaining case we conclude that G_1/K_1 is $\mathbb{G}_a$, $\mathbb{G}_m$ or $\mathbb{G}_a \times \mathbb{G}_m$. Then there are projections from G_1 onto one of the three and this readily implies the statement of the proposition. ■

4.2. Proofs. For the proof of Theorem 2 we have to go through the different cases. In each case it turns out to be necessary to modify the algebraic group to which we apply the analytic subgroup theorem.

4.2.1. Case (a). We take $G = G_1 \times G_1 \times L$ as above. Let p_1 and p_2 be the projection of G onto the first, respectively second factor G_1 , recall from (9) that p is the projection $p : G_a \times_E G_m \rightarrow E$ and write p_a and p_m for the projections from G onto G_a and G_m . We denote by $\mathfrak{g}$ the Lie algebra of G and take $e_1^\vee, e_2^\vee, e_3^\vee, g_1^\vee, g_2^\vee, g_3^\vee, e_a^\vee, e_m^\vee$ as a basis for the dual $\mathfrak{g}^\vee$ where $e_1^\vee = (p \circ p_1)^*\omega$, $e_2^\vee = (p \circ p_1)^*\eta$, $e_3^\vee = (p \circ p_1)^*\xi$, $g_1^\vee = (p \circ p_2)^*\omega$, $g_2^\vee = (p \circ p_2)^*\eta$, $g_3^\vee = (p \circ p_2)^*\xi$, $e_a^\vee = p_a^*dt$, $e_m^\vee = p_m^*d \log t$. If there is a non-trivial dependence relation over $\mathbb{Q}$

$$\begin{aligned} a_1 \langle \omega, \gamma_1 \rangle + b_1 \langle \eta, \gamma_1 \rangle + c_1 \langle \xi, \gamma_1 \rangle + a_2 \langle \omega, \gamma_2 \rangle + b_2 \langle \eta, \gamma_2 \rangle + c_2 \langle \xi, \gamma_2 \rangle \\ + d_a + d_m \pi i = 0, \end{aligned}$$

we consider the analytic subgroup $X \subset G(\mathbb{C})$ with Lie algebra defined in $\mathfrak{g}$ by $\lambda = \lambda_1 + \lambda_2 + \lambda_L \in \mathfrak{g}^\vee$ with $\lambda_1 = a_1 e_1^\vee + b_1 e_2^\vee + c_1 e_3^\vee$, $\lambda_2 =$

$a_2 g_1^\vee + b_2 g_2^\vee + c_2 g_3^\vee$ and $\lambda_L = d_a e_a^\vee + d_m e_m^\vee$ in $\mathfrak{g}^\vee$. It contains the point $Q = \exp_G(u)$ for $u = (u_1, u_2, 1, 2\pi i)$ with $u_1 = (\langle \omega, \gamma_1 \rangle, \langle \eta, \gamma_1 \rangle, \langle \xi, \gamma_1 \rangle)$, $u_2 = (\langle \omega, \gamma_2 \rangle, \langle \eta, \gamma_2 \rangle, \langle \xi, \gamma_2 \rangle)$, which is in $G(\overline{\mathbb{Q}})$ (see our convention in Section 4.1 before Lemma 5). By the analytic subgroup theorem, X contains the Zariski closure H of $\exp_G(\mathbb{Q}u)$. From Lemma 5 we deduce that H projects surjectively onto each factor G_1 and onto L since otherwise we would get linear dependence relations on these groups. If H projects onto $G_1 \times G_1$ then by Lemma 4 applied to $G_1 \times G_1$ and L we get an isomorphism $G_1 \times G_1/K_1 \simeq L/K_2$. Since L/K_2 is linear, there is a section $s : L/K_2 \rightarrow G_1 \times G_1$ with only trivial intersection with K_1 . We conclude that $G_1 \times G_2 \simeq K_1 \times L/K_2$ and deduce that G_a or G_m is trivial, which by assumption is not the case. Therefore H does not project onto $G_1 \times G_1$. Lemma 3 applied to G_1 and the image H_1 of H in $G_1 \times G_1$ furnishes an isomorphism $\alpha : G_1/K_1 \simeq G_1/K_2$. Lemma 4 shows that $K_1 \subseteq \mathbb{G}_a \times \mathbb{G}_m$ and so α induces an isomorphism $\bar{\alpha}$ from $E = G_1/\mathbb{G}_a \times \mathbb{G}_m$ to $E = G_1/\mathbb{G}_a \times \mathbb{G}_m$ which takes $\langle \omega, \gamma_1 \rangle$ to $\langle \omega, \gamma_2 \rangle$ and would not be in $\mathbb{Z}$. This was excluded in case (a).

4.2.2. Case (b). From Theorem 10 in Section 6.6 we know that the periods $\langle \xi, \gamma_1 \rangle$ and $\langle \xi, \gamma_2 \rangle$ are linear combinations of the periods $2\pi i$, $\langle \omega, \gamma_1 \rangle$ and $\langle \omega, \gamma_2 \rangle$ and then we take $G = G_a \times G_a \times L$. For the analytic subgroup X we take

$$\lambda = a_1 e_1^\vee + b_1 e_2^\vee + c_1 g_1^\vee + c_2 g_2^\vee + d_a e_a^\vee + d_m e_m^\vee = 0$$

with $e_1^\vee, e_2^\vee, g_1^\vee, g_2^\vee, e_a^\vee, e_m^\vee$ defined in a similar way to case (a), Q will be the image under $\exp$ of

$$u = (u_1, u_2, u_a, u_m) = (\langle \omega, \gamma_1 \rangle, \langle \eta, \gamma_1 \rangle, \langle \omega, \gamma_2 \rangle, \langle \eta, \gamma_2 \rangle, 1, 2\pi i),$$

and we proceed as in case (a).

4.2.3. Case (c). If E has complex multiplication by τ we let τ^*G_m be such that $[\tau^*G_m] = \tau^*[G_m]$ under the action of τ on $\text{Ext}^1(E, \mathbb{G}_m)$ (see Section 6.2). In case (c) our algebraic group will be $G = G_1 \times L$ with L as above and $G_1 = G_a \times_E G_m \times_E \tau^*G_m$ which projects onto the last factor $\tau^*\mathbb{G}_m$. Here and in the remaining case (d) the basis for $\mathfrak{g}^\vee$ has to be modified in an obvious way. The analytic subgroup X is defined in the Lie algebra $\mathfrak{g}$ by

$$\lambda = a_1 e_1^\vee + b_1 e_2^\vee + a_2 e_3^\vee + b_2 g_3^\vee + d_a e_a^\vee + d_m e_m^\vee = 0$$

where $g_3^\vee$ is the pull-back of $\tau^*\xi$ under the projection from G to τ^*G_m . We write $u_1 = (\langle \omega, \gamma_1 \rangle, \langle \eta, \gamma_1 \rangle, \langle \xi, \gamma_1 \rangle, \langle \tau^*\xi, \gamma_1 \rangle)$, $u_2 = (1, 2\pi i)$ and assume that $u = (u_1, u_2)$ satisfies $\lambda(u) = 0$. Now we proceed again as in case (a). This leads to the situation of Lemma 3 which we apply to G_1 , H and $G_2 = L$. If $H_1 \subseteq G_1$ denotes the image of H under the projection of G onto its first factor then if $H_1 \neq G_1$, Proposition 8 in Section 6.5 would imply that

$[G_a] = 0$ or that $[G_m]$ and $[\tau^*G_m]$ are linearly dependent over $\mathbb{Q}$. The latter means that there exist integers p, q not both 0 such that

$$0 = p[G_m] + q[\tau^*G_m] = (p + q\tau)^*[G_m]$$

and hence either $[G_m]$ is torsion or $p + q\tau = 0$. Both cases were excluded and therefore $H_1 = G_1$. According to Lemma 3 applied to G_1 and $G_2 = L$ there is an isomorphism $\alpha : G_1/K_1 \rightarrow L/K_2$. Since L is linear, we deduce that G_1/K_1 is linear and therefore K_1 is not contained in the linear part of G_1 . It is also different from G_1 since otherwise $K_2 = L$. This would lead to $G_1 \times L = K_1 \times K_2 \subseteq H \subset G_1 \times L$, which is false. Therefore $K_1 \subset G_1$ and we may apply Proposition 8 once again to $G = G_1$ and $H = K_1$ to conclude that contrary to the construction, $[G_a] = 0$ or $[G_m] = 0$ or $[G_m]$ and $[\tau^*G_m]$ are linearly dependent over $\mathbb{Q}$.

4.2.4. Case (d). This is a direct consequence of Proposition 5 where we have not made any assumption on $\text{End}(E)$. The proof of Theorem 2 is now complete.

5. Periods of hyperelliptic abelian integrals. In connection with closed geodesics we are led to determine the space of periods of the first kind on a hyperelliptic curve C defined over $\overline{\mathbb{Q}}$. Such a curve is given by an equation of the form

$$y^2 = P(x)$$

where $P(x)$ is a polynomial of degree 5 or 6 with coefficients in $\overline{\mathbb{Q}}$ and with non-zero discriminant. Let the curve C be embedded into the Jacobian $A = J(C)$ also defined over $\overline{\mathbb{Q}}$ by choosing a Weierstrass point. Similarly to the elliptic case we consider a basis f_1, f_2, f_3, f_4 for the Lie algebra $\mathfrak{a}$ of the generalized Jacobian constructed such that the dual basis $f_1^\vee, f_2^\vee, f_3^\vee, f_4^\vee$ pulls back to a basis of differentials $\omega_1 = dx/y, \omega_2 = x dx/y, \omega_3 = x^2 dx/y, \omega_4 = x^3 dx/y$ of the second kind on the curve. We keep using the convention of Section 4.1 before Lemma 5, modified suitably for the abelian case.

5.1. Hyperelliptic periods. The periods of the second kind are obtained by comparing the Betti homology

$$H_1(C, \mathbb{C}) = H_1(C, \mathbb{Z}) \otimes \mathbb{C}$$

with the de Rham cohomology $H_{\text{dR}}^1(C, \mathbb{C})$. Note that they are the same as $H_1(A, \mathbb{C}) = H_1(A, \mathbb{Z}) \otimes \mathbb{C}$ and $H_{\text{dR}}^1(A, \mathbb{C})$. Therefore also the periods are the same. We denote by e_1, e_2, e_3, e_4 a symplectic basis for $H_1(C, \mathbb{Z})$ with respect to the intersection pairing for cycles. Then the comparison isomorphism is given by

$$\omega_i = \langle \omega_i, e_1 \rangle e_1^\vee + \langle \omega_i, e_2 \rangle e_2^\vee + \langle \omega_i, e_3 \rangle e_3^\vee + \langle \omega_i, e_4 \rangle e_4^\vee$$

for $1 \leq i \leq 4$ and $e_1^\vee, e_2^\vee, e_3^\vee, e_4^\vee$ the dual basis where $\langle \omega_i, e_j \rangle = \int_{e_j} \omega_i$. This gives a period matrix

$$\begin{pmatrix} \langle \omega_1, e_1 \rangle & \langle \omega_1, e_2 \rangle & \langle \omega_1, e_3 \rangle & \langle \omega_1, e_4 \rangle \\ \langle \omega_2, e_1 \rangle & \langle \omega_2, e_2 \rangle & \langle \omega_2, e_3 \rangle & \langle \omega_2, e_4 \rangle \\ \langle \omega_3, e_1 \rangle & \langle \omega_3, e_2 \rangle & \langle \omega_3, e_3 \rangle & \langle \omega_3, e_4 \rangle \\ \langle \omega_4, e_1 \rangle & \langle \omega_4, e_2 \rangle & \langle \omega_4, e_3 \rangle & \langle \omega_4, e_4 \rangle \end{pmatrix}$$

with entries in $\mathbb{C}$. We consider the vector space W over $\overline{\mathbb{Q}}$ generated by the entries of the matrix together with 1 and $2\pi i$. Its dimension is at most 18 but can be smaller. This happens when there are endomorphisms of A not in $\mathbb{Z}$. Albert's classification of endomorphism algebras with involution shows that the dimension $[\text{End}_{\mathbb{Q}}(A) : \mathbb{Q}]$ of the $\mathbb{Q}$ -algebra $\text{End}_{\mathbb{Q}}(A) = \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ takes the values 1, 2, 2 and 4 depending on the four types of the algebra which are characterized in the following

THEOREM 4 (Albert [4, Proposition V.5.7], [6, IV.21, Theorem 2]). *The endomorphism algebra $\text{End}_{\mathbb{Q}}(A)$ of a polarized simple abelian surface A is either $\mathbb{Q}$, a real quadratic field, a CM-field of degree 4 or an indefinite quaternion algebra over $\mathbb{Q}$. If A is isogenous to $E \times E'$ then $\text{End}_{\mathbb{Q}}(A) = \text{End}_{\mathbb{Q}}(E) \times \text{End}_{\mathbb{Q}}(E')$ if E and E' are non-isogenous, and an order in the matrix algebra $M_2(\text{End}_{\mathbb{Q}}(E))$ otherwise.*

REMARK 5.1. We have treated separately the cases when $\text{End}_{\mathbb{Q}}(A)$ is $\mathbb{Z}$ and a real quadratic field. This is not necessary but we preferred to work out the simplest case separately. In Albert's classification this is Type I. In Theorem 4, Type III when $\text{End}_{\mathbb{Q}}(A)$ is a definite quaternion algebra is missing. The reason is that there are no simple abelian surfaces with this endomorphism algebra (see [4, Example IX.5.4]). One can show that Type III implies that the abelian surface is the product of an elliptic curve with complex multiplication with itself.

The endomorphism algebra $\text{End}_{\mathbb{Q}}(A)$ acts on $H_1(A, \mathbb{Q})$ by a rational representation of degree 4 and on $V = H^0(A, \Omega^1)$ by the algebraic de Rham representation of degree 2. The rational representation maps $u \in \text{End}_{\mathbb{Q}}(A)$ to $u_* : \gamma \mapsto u_* \gamma$, the push-forward of the homology class γ , and the algebraic de Rham representation acts on $H^0(A, \Omega_A^1)$ by pull-back $\omega \mapsto u^* \omega$ of the differential form. The period pairing $H^0(A, \Omega_A^1) \otimes H_1(A, \mathbb{Z}) \rightarrow \mathbb{C}$ given by $(\omega, \gamma) \mapsto \langle \omega, \gamma \rangle = \int_{\gamma} \omega$ satisfies

$$(11) \quad \langle \omega, u_* \gamma \rangle = \langle u^* \omega, \gamma \rangle$$

for $u \in \text{End}_{\mathbb{Q}}(A)$, and is non-degenerate. To determine the period relations on W we write

$$F := \text{End}_{\mathbb{Q}}(A) = \text{End}(A) \otimes \mathbb{Q}.$$

The following theorem shows that the relations coming from endomorphisms are the only non-trivial relations.

THEOREM 5. *The dimension of the F -module W satisfies the relation*

$$[F : \mathbb{Q}] \cdot (\dim(W) - 2) = 16.$$

We now restrict to the case when W is generated only by the entries of the first two rows of the period matrix without 1 and $2\pi i$. The general case needs only a small modification by taking the universal vector extension ⁽⁴⁾

$$(12) \quad 0 \rightarrow \operatorname{Ext}^1(A, \mathbb{G}_a)^\vee \rightarrow G \rightarrow A \rightarrow 0$$

and add $\mathbb{G}_a \times \mathbb{G}_m$ to the groups G^4 , G^2 or G appearing in the proof. In this case the factor $\dim(W) - 2$ should be replaced by $\dim(W)$.

5.2. Proof of Theorem 5. For the proof of the theorem we have to go through Albert's classification and to study more closely the endomorphism algebra and the representations ρ_{dR} and ρ_B . Our proof is decomposed according to the four cases in Albert's classification when A is simple and the extra case when A is split. This requires considering a different algebraic group G in each case.

5.2.1. The case $F = \mathbb{Q}$. We denote by $\mathfrak{a}$ the Lie algebra of A and write $w_i = (\langle \omega_1, e_i \rangle, \langle \omega_2, e_i \rangle) \in \mathfrak{a}_{\mathbb{C}} = \mathfrak{a} \otimes_{\mathbb{Z}} \mathbb{C}$. The vectors w_1, w_2, w_3, w_4 form a basis for the period lattice of A and $w = (w_1, w_2, w_3, w_4)$ is a period vector for A^4 which has the property that $\exp(\mathbb{Q}w) \subset A_{\text{tor}}^4(\mathbb{Q})$. This is needed in order to apply the analytic subgroup theorem. For the proof of Theorem 5 the abelian surface A will be replaced in this case by A^4 . A period relation on W is given by some $0 \neq \lambda = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \in \mathfrak{a}^\vee \oplus \mathfrak{a}^\vee \oplus \mathfrak{a}^\vee \oplus \mathfrak{a}^\vee$ such that $\lambda_i \in \operatorname{pr}_i^* \mathfrak{a}^\vee$ for pr_i the projection onto the i th summand of $\mathfrak{a}^\vee \oplus \mathfrak{a}^\vee \oplus \mathfrak{a}^\vee \oplus \mathfrak{a}^\vee$ and can be written as

$$(13) \quad \langle \lambda, w \rangle = \langle \lambda_1, w_1 \rangle + \langle \lambda_2, w_2 \rangle + \langle \lambda_3, w_3 \rangle + \langle \lambda_4, w_4 \rangle = 0.$$

We may assume that at least two of the λ_i are non-zero: otherwise only one is non-zero and this λ defines a proper analytic subgroup of A containing $\exp(\mathbb{Q}w_i)$; by the analytic subgroup theorem this analytic subgroup must be algebraic, which contradicts the assumption that A is simple.

We consider the analytic subgroup $X \subset A(\mathbb{C})^4$ defined in the Lie algebra by $\lambda = 0$. From (13) we deduce that $\exp(\mathbb{Q}w) \subseteq X$ and by the analytic subgroup theorem we find an algebraic subgroup H of A^4 such that $H(\mathbb{C}) \subseteq X$ and containing the 1-parameter subgroup $\exp(\mathbb{Q}w)$. The same arguments as above show that H projects onto each factor A . We assume that $\lambda_4 \neq 0$,

⁽⁴⁾ Universal in the sense that in Section 6.3 we show that any extension G of an abelian variety A by a vector group V is given by a homomorphism $d_G : H^1(A, \mathcal{O}_A)^\vee \rightarrow V$ and obtained as the push-out of (12) by d_G .

apply Lemma 3 to $H \subset G_1 \times G_2$ with $G_1 = A^3$ and $G_2 = A$ and obtain a homomorphism $\alpha : H_1 \rightarrow A/K_2$ where $K_2 \subset A$ and where H_1 is the image of H under the projection onto the first factor of G . Since $0 \times K_2 \subseteq H$ and $\lambda_4 \neq 0$ we see that $0 = \lambda(0, K_2) = \lambda_4(K_2)$ and $K_2 \subset A$. The abelian surface A is simple and therefore $K_2 = 0$. This means that $\alpha : H_1 \rightarrow A$ is surjective and its graph contains $\exp(\mathbb{Q}w)$ and is defined as $\lambda = 0$ for λ of the shape $\lambda(z_1, z_2, z_3, z_4)$ in $z_i = (z_{i,1}, z_{i,2}) \in \mathfrak{a}_{\mathbb{C}}$ with coefficients in $F = \mathbb{Q}$. We conclude that w_1, w_2, w_3 and w_4 are linearly dependent over $\mathbb{Q} = \text{End}_{\mathbb{Q}}(A)$. This contradiction leads to the following

PROPOSITION 2. *If $F = \mathbb{Q}$ then $\dim(W) = 8$.*

5.2.2. The real-quadratic case. In the case that F is real-quadratic and the involution is trivial we show first that $\dim(W) \leq 4$. To see this we write $F = \mathbb{Q} + \mathbb{Q}\sqrt{d}$ with $d > 0$ and squarefree. The field F acts on $H^0(A, \Omega^1)$ and decomposes $V = H^0(A, \Omega^1) \otimes \overline{\mathbb{Q}}$ as a direct sum $V = V^+ \oplus V^-$ of eigenspaces $V^{\pm} = \overline{\mathbb{Q}}\omega^{\pm}$ such that F acts on $V^{\pm}$ by scalar multiplication as $\alpha^*\omega^{\pm} = (a \pm b\sqrt{d})\omega^{\pm}$ for $\alpha = a \pm b\sqrt{d}$. The vector space $H_1(A, \mathbb{Q})$ of dimension 4 over $\mathbb{Q}$ is at the same time a vector space over F of dimension 2 with Betti representation $\rho_B : (a \pm b\sqrt{d}, e) \mapsto (a \pm b\sqrt{d})_*e$ for $e \in H^1(A_{\mathbb{C}}, \mathbb{Z})$. We take an F -basis e_1, e_2 and get a decomposition $H_1(A_{\mathbb{C}}, \mathbb{Q}) = Fe_1 \oplus Fe_2$. From (11) we deduce that

$$\langle \alpha_*e_i, \omega^{\pm} \rangle = \langle e_i, \alpha^*\omega^{\pm} \rangle = \alpha \langle e_i, \omega^{\pm} \rangle$$

and obtain four linearly independent period relations so that $\dim(W) \leq 4$.

PROPOSITION 3. *If F is real-quadratic then $\dim(W) = 4$.*

Proof. Let w_i be the period vectors associated to e_i with coordinates $\langle \omega^{\pm}, e_i \rangle$. We consider the algebraic group $A \times A$. If the vector space W generated over $\overline{\mathbb{Q}}$ by the coordinates of w_i has dimension < 4 , there are linear forms λ_1 and λ_2 in the dual $\mathfrak{a}^{\vee}$ of the Lie algebra such that

$$\lambda_1(w_1) + \lambda_2(w_2) = 0.$$

As above we may assume that $\lambda_i \neq 0$. The analytic subgroup of $A \times A$ defined by $\lambda_1 + \lambda_2 = 0$ is by the analytic subgroup theorem an algebraic subgroup $H \subset A \times A$. Its Lie algebra contains (w_1, w_2) and surjects onto each factor $\mathfrak{a}$ since A is simple such that H is isogenous to A . Again an application of Lemma 3 shows that H defines an endomorphism $\alpha \in F$ whose differential $d\alpha$ maps w_1 to w_2 , which is not possible since w_1 and w_2 are linearly independent over F . ■

5.2.3. The quaternionic case. An indefinite quaternion algebra with center $\mathbb{Q}$ is a central simple algebra Q such that there exists an algebra K of dimension 2 over $\mathbb{Q}$ together with an invertible element $\theta \in \mathbb{Q}$ such that $Q = K + Kv$ where $v^2 = \theta$ and $mv = v\overline{m}$ for all $m \in K$ where $m \mapsto \overline{m}$

is the non-trivial automorphism of K over $\mathbb{Q}$. We write $K = \mathbb{Q} + \mathbb{Q}u$ and $Q = \mathbb{Q} + \mathbb{Q}u + \mathbb{Q}v + \mathbb{Q}uv$ (see [9, I.1 Definition]). The algebra Q acts on $H_1(A_{\mathbb{C}}, \mathbb{Q})$ by a rational representation of degree 4 and on $V = H^0(A, \Omega^1)$ by the algebraic de Rham representation of degree 2. The rational representation maps $u \in Q$ to $u_* : w \mapsto u_*w$. The decomposition of Q can be used to define a basis for $H_1(A_{\mathbb{C}}, \mathbb{Q})$ by choosing some $0 \neq e \in H_1(A(\mathbb{C}), \mathbb{Q})$ and defining $e_1 = e$, $e_2 = u_*e$, $e_3 = v_*e$ and $e_4 = u_*e_3 = u_*v_*e$. The restriction to K of the algebraic de Rham representation given as $\omega \mapsto u^*\omega$ can be diagonalized with two eigenvectors ω_1 and ω_2 which we take as a basis for V over K . Now we observe that (11) can be written as

$$\langle \omega_i, w_*e \rangle = \langle w^*\omega_i, e \rangle$$

for $i = 1, 2$ and $w \in Q$. Since $w^*\omega$ for $\omega \in V$ is a linear combination of ω_1 and ω_2 with algebraic coefficients and since $H_1(A, \mathbb{Q}) = Q \cdot e$ we conclude that

$$\langle \omega_i, w_*e \rangle \in \overline{\mathbb{Q}}\langle \omega_1, e \rangle + \overline{\mathbb{Q}}\langle \omega_1, e \rangle.$$

PROPOSITION 4. *If A has indefinite quaternionic multiplication then $\dim(W) = 2$.*

Proof. We assume that there is $0 \neq \lambda \in \mathfrak{a}^\vee$ such that $\lambda(w) = 0$ where $w \in \mathfrak{a}_{\mathbb{C}}$ is the vector with components $\langle \omega_i, e \rangle$. We apply the analytic subgroup theorem to A , w and λ and deduce that A is not simple, contrary to our assumption. ■

5.2.4. The CM-case. In the CM-case we write $K = \text{End}_{\mathbb{Q}}(E)$ as $K \supset K_0 \supset \mathbb{Q}$ with K_0 a real quadratic field over $\mathbb{Q}$ fixed under the canonical involution. We obtain a decomposition $K = \mathbb{Q} + \mathbb{Q}u + \mathbb{Q}v + \mathbb{Q}uv$ as in the quaternionic case with the difference that $uv = vu$. We use the same basis for $H^1(A, \mathbb{Q})$ as in the quaternionic case and obtain $\langle e_i, \omega_j \rangle \in \overline{\mathbb{Q}}\langle e_1, \omega_1 \rangle + \langle e_1, \overline{\mathbb{Q}}\omega_2 \rangle$ as above.

PROPOSITION 5. *If A has complex multiplication then $\dim(W) = 2$.*

Proof. The proof is along the same lines as in the quaternionic case. ■

5.2.5. The case of a split abelian surface. Suppose now that $\text{End}(A)$ is isogenous to a product $E_1 \times E_2$ of elliptic curves. Then we have to consider two cases: the factors are either non-isogenous or isogenous. In the second case Theorem 2 gives the answer and we may confine ourselves to the first case. In this case there are three subcases: either there is no complex multiplication, or one of the factors has complex multiplication, or both factors have. If the latter happens then the imaginary quadratic fields are disjoint.

This gives for $\text{End}(A)$ the following options:

$$\text{End}_{\mathbb{Q}}(A) = \begin{cases} \text{End}_{\mathbb{Q}}(E_1) \times \text{End}_{\mathbb{Q}}(E_2) & \text{if } E_1 \text{ and } E_2 \text{ are non-isogenous,} \\ \text{M}_2(\text{End}_{\mathbb{Q}}(E_1)) & \text{otherwise,} \end{cases}$$

and we deduce that

$$\dim(W) \leq \begin{cases} 4, & \text{no CM,} \\ 3, & \text{one factor CM,} \\ 2, & \text{factors disjoint CM-fields.} \end{cases}$$

The following theorem is an extension of what we need in applications and we state it for completeness.

THEOREM 6. *Let $E_1, \dots, E_n$ be elliptic curves and W be the vector space defined over $\mathbb{Q}$ by the periods of the first kind. Then*

$$\dim(W) = \sum \frac{2}{\dim(\text{End}_{\mathbb{Q}}(E))}$$

where the sum is taken over the isogeny classes E of the E_i .

We shall apply the theorem only in the case $n = 2$ and give the proof only in this case. In the general case the proof is an easy extension of the proof for $n = 2$.

Proof. If E_1 and E_2 are isogenous we have to prove that $\dim(W) = 1$ or 2, depending on whether there is complex multiplication or not. In this case the result follows directly from Theorem 2.

In the case that E_1 and E_2 are non-isogenous we let $e_{i,1}, e_{i,2}$ for $i = 1, 2$ be generators of $H_1(E_i, \mathbb{Q})$ and $0 \neq \omega_i \in H^0(E_i, \Omega_{E_i}^1)$. We first consider the case when there is no complex multiplication and write $w = (w_1, w_2)$ and $w_i = (w_{i,1}, w_{i,2})$ where $w_{i,k} = \langle e_{i,k}, \omega_k \rangle$. The Lie algebra of the group $G = G_1 \times G_2$ with $G_1 = E_1 \times E_2 = G_2$ contains w and if there is some $0 \neq \lambda \in (\text{Lie } G)^\vee$ such that $\lambda(w) = 0$ then again by the analytic subgroup theorem, G contains a proper algebraic subgroup H with $w \in \text{Lie } H$. The Lie algebra of the projection H_i of H to G_i contains w_i and if $H_i \subset G_i$ then H_i induces an isogeny from E_1 to E_2 which is excluded and as a consequence we have $H_i = G_i$. We introduce the subgroups $K_i \subseteq G_i$ given by $0 \times K_2 = H \cap (0 \times G_2)$ and $K_1 \times 0 = H \cap (G_1 \times 0)$. Lemma 3 gives an isomorphism $\kappa : G_1/K_1 \simeq G_2/K_2$. If G_1/K_1 is trivial then so is G_2/K_2 and hence $G_1 \times 0 = K_1 \times 0 \subseteq H$, $0 \times G_2 \subset H$ and then $G_1 \times G_2 \subseteq H \subset G_1 \times G_2$. This cannot be and so G_i/K_i is isogenous either to G_i , E_1 or E_2 for $i = 1, 2$. In the first case κ induces an isogeny α from G_1 to G_2 such that $d\alpha(w_1) = w_2$, which is of the form $\alpha_1 \times \alpha_2$ with $\alpha_i : E_i \rightarrow E_i$ and then $d\alpha_i(w_{i,1}) = w_{i,2}$. Then α_i is an isogeny not in $\mathbb{Z}$ for $i = 1, 2$ and $\dim(\text{End}_{\mathbb{Q}}(E_i)) = 2$. In the second case when G_1/K_1 is isogenous to E_1 then so is G_2/K_2 and κ induces an isogeny $\kappa : E_1 \rightarrow E_1$ not in $\mathbb{Z}$ because in this case $d\kappa(w_{1,1})$ and

$w_{1,2}$ are linearly dependent over $\overline{\mathbb{Q}}$ and then $\dim(\text{End}_{\mathbb{Q}}(E_1)) = 2$; similarly $\dim(\text{End}_{\mathbb{Q}}(E_2)) = 2$. Both cases are excluded and this proves the statement in the case when there is no CM.

If exactly one factor has CM, say E_1 , then our group G will be $G_1 \times E_2$. A period relation $\lambda(w) = 0$ for $w = (w_1, w_{2,2})$ with $w_1 = (w_{1,1}, w_{1,2})$ and with $w_{1,2}$ and $w_{2,2}$ linearly independent periods of E_2 leads as above to the existence of an algebraic subgroup $H \subset G$ with $w \in \text{Lie } H$ and as above we may assume that H projects onto G_1 as well as onto E_2 . By Lemma 3 this gives an isomorphism from G_1/K_1 onto E_2 and since G_1/K_1 is isogenous to E_2 we get an endomorphism of E_2 which is not in $\mathbb{Z}$ since its differential maps $w_{1,2}$ to a rational multiple of $w_{2,2}$. Therefore E_2 has CM as well, contrary to our assumption. We conclude that $\dim(W) = 3$ in this case.

It remains to establish the claim when E_1 and E_2 have complex multiplication with disjoint CM-fields. Here we put $G = E_1 \times E_2$ and follow the above arguments which produce a non-zero isogeny between E_1 and E_2 . This was excluded, therefore $\dim(W) = 2$ and Theorem 6 is proved. ■

An arbitrary abelian variety A over $\overline{\mathbb{Q}}$ decomposes up to isogeny into a product of isotypical components

$$A = A_1^{e_1} \times \cdots \times A_r^{e_r}$$

where the A_i are the simple pairwise non-isogenous factors of A . It can be shown that the vector space W generated by the periods over $\overline{\mathbb{Q}}$ is the direct sum of the vector spaces W_i generated over $\overline{\mathbb{Q}}$ by the periods of A_i and that

$$\dim(W) = \sum \frac{2 \dim(B)^2}{\dim(\text{End}_{\mathbb{Q}}(B))}$$

where the sum is taken over B in the set $A_1, \dots, A_r$. The proof does not need any extra ingredients up to technical details and is based on an extension of Lemma 3.

6. Appendix on Ext-groups. In this section we give a short introduction to Ext^1 in the category of commutative algebraic groups defined over $\overline{\mathbb{Q}}$. Some of the material can be found in Serre's book [8] as already mentioned earlier.

6.1. Group extensions. Let A be an abelian variety and L a commutative linear algebraic group. A commutative algebraic group G which fits into an exact sequence of group homomorphisms of the form

$$(14) \quad 0 \rightarrow L \xrightarrow{\iota} G \xrightarrow{\pi} A \rightarrow 0$$

is called an *extension* of A by L . To give an extension G of A by L is the same as to give a pair $(\iota, \pi) \in \text{Hom}(L, G) \times \text{Hom}(G, A)$ such that (14) is exact. A homomorphism φ between extensions G and G' , the latter given by the

pair (ι', π') , induces a pair $(\nu, \alpha) \in \text{End}(L) \times \text{End}(A)$ such that $\varphi \circ \iota = \iota' \circ \nu$ and $\alpha \circ \pi = \pi' \circ \varphi$ making the diagram

$$(15) \quad \begin{array}{ccccccc} 0 & \longrightarrow & L & \xrightarrow{\iota} & G & \xrightarrow{\pi} & A \longrightarrow 0 \\ & & \downarrow \nu & & \downarrow \varphi & & \downarrow \alpha \\ 0 & \longrightarrow & L' & \xrightarrow{\iota'} & G' & \xrightarrow{\pi'} & A \longrightarrow 0 \end{array}$$

commutative. The set of isomorphism classes $[G]$ of extensions makes up a commutative group $\text{Ext}^1(A, L)$. Addition is given by the Baer sum which will be introduced in Section 6.2. By abuse of notation we often write G for $[G]$. It is obvious that Lemma 2 extends to abelian varieties.

6.2. Functorial properties and the Baer sum. The bi-functor Ext^1 which associates with a pair (A, L) the group $\text{Ext}^1(A, L)$ is contravariant in the first variable and covariant in the second variable. This means that if $A' \xrightarrow{\alpha} A$ is a homomorphism between abelian varieties then there is a uniquely determined homomorphism $\text{Ext}^1(A, L) \xrightarrow{\alpha^*} \text{Ext}^1(A', L)$, and a homomorphism $L \xrightarrow{\lambda} L'$ induces a homomorphism $\text{Ext}^1(A, L) \xrightarrow{\lambda_*} \text{Ext}^1(A, L')$. The two homomorphisms α^* and λ_* commute in the sense that

$$(16) \quad \begin{array}{ccc} \text{Ext}^1(A, L) & \xrightarrow{\alpha^*} & \text{Ext}^1(A', L) \\ \downarrow \lambda_* & & \downarrow \lambda_* \\ \text{Ext}^1(A, L') & \xrightarrow{\alpha^*} & \text{Ext}^1(A', L') \end{array}$$

commutes. This is a general property of Ext-groups in abelian categories. Conversely, since L and L' are linear, there is a section σ of λ . By functoriality, σ induces a section $\sigma_* : \text{Ext}^1(A, L') \rightarrow \text{Ext}^1(A, L)$ of λ_* since $\lambda \circ \sigma = \text{id}_L$ and then $\lambda_* \circ \sigma_* = (\text{id}_{L'})_*$. The extension α^*G is the fiber product $G \times_A A'$ and the extension λ_*G is the quotient $(L' \times G)/L$ where L acts on G by ι and on L' by λ . If $[G]$ and $[G']$ are in $\text{Ext}^1(A, L)$ then their Baer sum $[G] + [G']$ is $[\Delta^* s_*(G \times G')]$, which makes $\text{Ext}^1(A, L)$ into a commutative group with neutral element $0 = [A \times L]$; here Δ is the diagonal map from A into $A \times A$ and $s : L \times L \rightarrow L$ is addition on L .

The bi-functor Ext^1 is additive in both variables, which means that $\text{Ext}^1(A \times A', L) = \text{Ext}^1(A, L) \times \text{Ext}^1(A', L)$ and $\text{Ext}^1(A, L \times L') = \text{Ext}^1(A, L) \times \text{Ext}^1(A, L')$. For $L = \mathbb{G}_a$ or $L = \mathbb{G}_m$ and A an abelian variety the Ext-group is the vector space $H^1(A, \mathcal{O}_A)$ which has the same dimension as A , and $\text{Pic}^0(A) = A^\vee$ respectively. The latter is isomorphic to a subgroup $H^1(A, \mathcal{O}_A^*)$ (see [8, VII.7.16]). We deduce that multiplication by an integer n can be defined inductively using addition. Another way would be to take $n[G] = [\Delta^* s_*(G^n)]$ where $\Delta = \Delta_n$ is the diagonal from A to A^n and $s = s_n$ is n -fold addition $L^n \rightarrow L$ on L .

The abelian variety A has up to isogeny a decomposition

$$A = A_1^{e_1} \times \cdots \times A_n^{e_n}$$

with $A_1, \dots, A_n$ simple and pairwise non-isogenous. Similarly a commutative linear algebraic group L can always be decomposed as $L = V \times T$ with V a vector group and T a torus. Since the field of definition is algebraically closed, they can be written as $V = \mathbb{G}_a^k$ and $T = \mathbb{G}_m^l$ and we get

PROPOSITION 6. *Up to isogeny there are decompositions*

$$\mathrm{Ext}^1(A, L) \simeq \mathrm{Ext}^1(A_1, L)^{e_1} \times \cdots \times \mathrm{Ext}^1(A_n, L)^{e_n}$$

and if A is simple then

$$\mathrm{Ext}^1(A, L) = \mathrm{Ext}^1(A, \mathbb{G}_a)^k \times \mathrm{Ext}^1(A, \mathbb{G}_m)^l = H^1(A, \mathcal{O}_A)^k \times \mathrm{Pic}^0(A)^l.$$

In our applications we can assume that the abelian variety is a product of pairwise non-isogenous simple abelian varieties and this means that $e_1 = \cdots = e_n = 1$. Then

$$\mathrm{Ext}^1(A, L) = \prod_i H^1(A_i, \mathcal{O}_{A_i})^k \times \prod_i \mathrm{Pic}^0(A_i)^l.$$

If $k = l = 1$ the class $[G] \in \mathrm{Ext}^1(A, L)$ can be written as

$$[G] = ([G'_1], \dots, [G'_n], [G''_1], \dots, [G''_n])$$

with $[G'_i] \in H^1(A_i, \mathcal{O}_{A_i})$ and $G''_i \in \mathrm{Pic}^0(A_i)$.

6.3. Intrinsic description of $\mathrm{Ext}^1(A, T)$ and $\mathrm{Ext}^1(A, V)$. Extensions of an abelian variety A by a torus T of rank r with character group $X(T)$ are classified by $\mathrm{Hom}(X(T), \mathrm{Pic}^0(A)) = \mathrm{Hom}(X(\mathbb{G}_m), \mathrm{Pic}^0(A))^r$. To see this we first note that in view of Proposition 6 it suffices to give the proof in the case $T = \mathbb{G}_m$ and in the case $V = \mathbb{G}_a$. We apply the functor $\mathrm{Hom}(-, \mathbb{G}_m)$ to the exact sequence (14) with $L = \mathbb{G}_m$ to obtain a long exact sequence

$$\cdots \rightarrow \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \xrightarrow{d_G} \mathrm{Ext}^1(A, \mathbb{G}_m) \xrightarrow{\pi^*} \mathrm{Ext}^1(G, \mathbb{G}_m) \rightarrow \cdots$$

(see [8, VII.3, Proposition 7]). The construction shows that an extension G of A by $\mathbb{G}_m$ gives a homomorphism $d_G : X(\mathbb{G}_m) \rightarrow \mathrm{Pic}^0(A) = A^\vee$ with $d_G(\lambda) = \lambda_* G$ for $\lambda \in X(\mathbb{G}_m)$. The homomorphism d_G is injective since $d_G(\lambda) = 0$ implies that $\lambda_* G = \mathbb{G}_m \times A$. From (15) we deduce a homomorphism $\varphi : G \rightarrow \lambda_* G = \mathbb{G}_m \times A$. Followed by the projection onto $\mathbb{G}_m$, this gives a homomorphism $G \rightarrow \mathbb{G}_m$ which implies that $G = \mathbb{G}_m \times A$ is trivial.

In the case of a vector group $V = \mathbb{G}_a$ the same long exact sequence

$$\cdots \rightarrow \mathrm{Hom}(\mathbb{G}_a, \mathbb{G}_a) \xrightarrow{d_G} \mathrm{Ext}^1(A, \mathbb{G}_a) \xrightarrow{\pi^*} \mathrm{Ext}^1(G, \mathbb{G}_a) \rightarrow \cdots$$

with $\mathbb{G}_m$ replaced by $\mathbb{G}_a$ gives a connecting homomorphism $d_G : V^\vee \rightarrow \mathrm{Ext}^1(A, \mathbb{G}_a) = H^1(A, \mathcal{O}_A)$.

Together this gives homomorphisms

$$\mathrm{Ext}^1(A, T) \rightarrow \mathrm{Hom}(X(T), A^\vee), \quad [G] \mapsto d_G,$$

$$\mathrm{Ext}^1(A, V) \rightarrow \mathrm{Hom}(V^\vee, H^1(A, \mathcal{O}_A)), \quad [G] \mapsto d_G,$$

and we can state the following

PROPOSITION 7. *We have*

$$\mathrm{Ext}^1(A, T) \simeq \mathrm{Hom}(X(T), A^\vee) \quad \text{and} \quad \mathrm{Ext}^1(A, V) \simeq \mathrm{Hom}(V^\vee, H^1(A, \mathcal{O}_A)).$$

Proof. It remains to show that the two homomorphisms are surjective in the cases $T = \mathbb{G}_m$ and $V = \mathbb{G}_a$. We confine ourselves to the $\mathbb{G}_m$ case. Let θ be in $\mathrm{Hom}(X(\mathbb{G}_m), A^\vee)$ and $\lambda \in X(T)$. Then $\theta(\mathrm{id})$ is in $\mathrm{Pic}^0(A) = \mathrm{Ext}^1(A, \mathbb{G}_m)$ and if λ is written as $\lambda = n \mathrm{id}$ then

$$\theta(\lambda) = \theta(n \mathrm{id}) = n\theta(\mathrm{id}) = n \mathrm{id}_* \theta(\mathrm{id}) = \lambda_* \theta(\mathrm{id}) = d_{\theta(\mathrm{id})}(\lambda).$$

Therefore $\theta = d_{\theta(\mathrm{id})}$ and surjectivity is established. ■

If G is an extension of A by a torus T , there exists a universal vector extension $G^\natural$ of G . It is given by a natural exact sequence

$$0 \rightarrow H^1(A, \mathcal{O}_A)^\vee \rightarrow G^\natural \rightarrow G \rightarrow 0$$

and has the property that any vector extension of G by a vector group V is obtained by push-out by $d_G^\vee : H^1(A, \mathcal{O}_A)^\vee \rightarrow V$, the dual of d_G .

6.4. Endomorphisms. The endomorphism algebra of an extension G is the stabilizer of its isomorphism class $[G]$ in $\mathrm{Ext}^1(A, L)$. We have seen in Section 6.1 in the case $L = \mathbb{G}_m$ that for $G \in \mathrm{Ext}^1(A, \mathbb{G}_m)$ with A a simple abelian variety there exists a homomorphism $\mathrm{End}(G) \xrightarrow{\iota} \mathbb{Z} \times \mathrm{End}(A)$, $\varphi \mapsto (\nu, \alpha)$, with $\nu \in \mathbb{Z} \subseteq \mathrm{End}(A)$ and $\alpha \in \mathrm{End}(A)$. From [8, VII.1.1(b), equation (7)], we know that a pair (ν, α) in $\mathrm{End}(L) \times \mathrm{End}(A)$ as described in loc.cit. stabilizes $[G]$ if and only if

$$\alpha^*(G) \simeq \nu_*(G)$$

in $\mathrm{Ext}^1(A, L)$. Suppose that G is given as $\kappa : \mathbb{Z} \rightarrow A^\vee$ in $\mathrm{Hom}(X(\mathbb{G}_m), \mathrm{Pic}^0(A))$ as above. Then we obtain a short exact sequence

$$(17) \quad 0 \rightarrow \mathrm{End}(G) \xrightarrow{\iota} \mathbb{Z} \times \mathrm{End}(A) \xrightarrow{\delta} M(\kappa) \rightarrow 0$$

with $M(\kappa) = \mathrm{End}(A) \circ \kappa$ the left $\mathrm{End}(A)$ -submodule of $\mathrm{Hom}(X(\mathbb{G}_m), \mathrm{Pic}^0(A))$ generated by κ , and with $\delta(\nu, \alpha) = (\alpha^* - \nu_*) \circ \kappa$ ⁽⁵⁾. From (17) we deduce that (ν, α) is in the image of ι if and only if it is in $\ker(\delta)$. There are two cases: Either κ has infinite order and then we see that $\ker(\delta) = \Delta(\mathbb{Z})$ where $\Delta : \mathrm{End}(G) \supseteq \mathbb{Z} \ni m \mapsto (m, m) \in \mathbb{Z} \times \mathrm{End}(A)$. Since also $\mathrm{End}(G)$ is the kernel of δ we conclude that $\mathrm{End}(G) = \mathbb{Z}$. In the other case $M(\kappa)$ is a torsion module and this implies that $M(\kappa) \simeq \mathrm{End}(A)/\mathrm{Ann}(\kappa)$ with $\mathrm{Ann}(\kappa)$

⁽⁵⁾ In other terms: the pair $(\iota, \mathrm{End}(G))$ is the equalizer of the morphisms α^* and ν_* .

the annihilator of κ . We conclude that $\text{End}(G)$ is a subalgebra of $\mathbb{Z} \times \text{End}(A)$ of finite index $|M(\kappa)|$ and deduce that if $\kappa' = n\kappa$ and G, G' are the extensions defined by κ and κ' respectively then $\text{End}(G')$ is a subgroup of $\text{End}(G)$ of index n .

THEOREM 7. *The endomorphisms of $G \in \text{Ext}^1(A, L)$ are given by*

$$\text{End}(G) = \begin{cases} \mathbb{G}_a \times \text{End}(A) & \text{if } [G] \text{ is trivial,} \\ Z(\text{End}(A)) & \text{otherwise,} \end{cases}$$

when $L = \mathbb{G}_a$ and where $Z(\text{End}(A))$ denotes the center of $\text{End}(A)$. If $L = \mathbb{G}_m$ then

$$\text{End}(G) = \begin{cases} \text{End}(A) \times \mathbb{Z} & \text{if } [G] = 0, \\ \mathbb{Z} & \text{otherwise,} \end{cases}$$

and in the case when $L = \mathbb{G}_a \times \mathbb{G}_m$ we have

$$\text{End}(G) = \text{End}((p_a)_*G) \times_{\text{End}(A)} \text{End}_{\mathbb{Q}}((p_m)_*G)$$

where p_a and p_m denote the projections from L onto $\mathbb{G}_a$ and $\mathbb{G}_m$ respectively.

Proof. It remains to discuss the case $L = \mathbb{G}_a$ and $\kappa : \mathbb{G}_a^\vee \rightarrow H^1(A, \mathcal{O}_A)$. Here the same arguments lead to a short exact sequence

$$0 \rightarrow \text{End}(G) \xrightarrow{L} \text{End}(\mathbb{G}_a) \times \text{End}(A) \xrightarrow{\delta} M(\kappa) \rightarrow 0$$

where $\delta : (\nu, \alpha) \mapsto (\alpha^* - \nu_*) \circ \kappa$. Both $\mathbb{G}_a = \text{End}(\mathbb{G}_a)$ and the algebra $\text{End}(A)$ act on $H^1(A, \mathcal{O}_A)$. An endomorphism $\varphi \in \text{End}(G)$ has to fix the extension class κ and this means that $\delta(\nu, \alpha) \circ \kappa = 0$. It follows that either $\kappa = 0$, which gives the first alternative in the theorem, or $\alpha = \nu \in \text{End}(\mathbb{G}_a) = \mathbb{G}_a$. The latter leads to the second alternative, namely that α is in the center of $\text{End}(A)$. To see the last statement one simply needs to note that by Lemma 2 we may write $G = G_a \times_A G_m$ where $G_a = (p_a)_*G$ and $G_m = (p_m)_*G$ and then to chase cartesian diagrams. ■

6.5. Subgroups and Ext^1 . Let G be in $\text{Ext}^1(A, L)$. The following theorem gives a characterization of algebraic subgroups of G which map onto A in terms of $\text{Ext}^1(A, L)$.

THEOREM 8. *If G contains a non-trivial proper algebraic subgroup H different from G and not contained in L which maps onto A then there exists a projector $\lambda \in \text{End}(L)$ such that $\lambda_*([G]) = 0$.*

Proof. We may assume that $L_H = L \cap H$ has a complement L' in L . Since H maps onto A , the decomposition of L as $L' \times L_H$ leads to $G = G/H \times G/L' = L' \times H$. We take λ to be the projector of L onto L' , and λ_* will have the desired property. The linear group $L = L' \times L_H$ acts on L' through λ and on G through multiplication, and we see that for $l = (l', l_H) \in L$ we

get $l \cdot L' = l' \cdot L'$ and $l \cdot G = l \cdot (L' \times H) = l' \cdot L' \times l_H \cdot H$. This leads to

$$l \cdot (L' \times G) = l \cdot (L' \times (L' \times H)) = l' L' \times (l' L' \times l_H H)$$

and shows that

$$\begin{aligned} \lambda_*(G) &= (L' \times G)/L = (L' \times (L' \times H))/(L' \times L_H) \\ &= (L' \times L')/L' \times (H/L_H) \simeq L' \times A. \end{aligned}$$

It follows that $\lambda_*([G]) = [L' \times A] = 0$ in $\text{Ext}^1(A, L')$. ■

Let G be as in (9) with $A = E$ and $L = T \times \mathbb{G}_a$ where $T = \mathbb{G}_m \times \mathbb{G}_m$. Then according to Lemma 7 we can write G as $G = G_1 \times_E G_2 \times_E G_a$ for $[G_a] \in \text{Ext}^1(E, \mathbb{G}_a)$, $[G_i] \in \text{Ext}^1(E, \mathbb{G}_m)$ and $i = 1, 2$. The following proposition characterizes connected algebraic subgroups of G .

PROPOSITION 8. *The group G contains a proper algebraic subgroup not in L if and only if $[G_a] = 0$ or $m_1[G_1] + m_2[G_2] = 0$ in $\text{Pic}^0(E)$ with $m_1, m_2 \in \mathbb{Z}$ not both zero.*

Proof. This follows easily from the theorem. Note that

$$\text{Ext}^1(E, T \times \mathbb{G}_a) = \text{Ext}^1(E, \mathbb{G}_m) \times \text{Ext}^1(E, \mathbb{G}_m) \times \text{Ext}^1(E, \mathbb{G}_a)$$

and the right hand side is isomorphic to $E \times E \times H^1(E, \mathcal{O}_E)$. We write $[G] = ([G_1], [G_2], [G_a])$. Looking at the various projectors in $\text{End}(\text{Ext}^1(E, T \times \mathbb{G}_a))$ gives our assertion. ■

COROLLARY 6.1. *Suppose that $[G_1]$ and $[G_2]$ generate a subgroup of $\text{Ext}^1(E, \mathbb{G}_m)$ of rank 2 and that $[G_a]$ is non-zero. Then the only connected algebraic subgroups of G are contained in $\mathbb{G}_a \times T$.*

6.6. Differentials of the third kind. We briefly indicate how to associate to a differential form of the third kind an extension of E by $\mathbb{G}_m$ and how to recover the differential form from the extension. Let ξ_D be a form of the third kind with associated divisor $D = (P) + (Q)$. It defines a homomorphism

$$\begin{aligned} \rho : \mathbb{Z} \times \mathbb{Z} &\simeq \mathbb{Z}[D] \xrightarrow{\rho} \text{Pic}(E), \\ (m, n) &\mapsto \mathcal{O}(m \cdot (P) + n \cdot (Q)), \end{aligned}$$

from the free abelian group $\mathbb{Z}[D]$ generated by (P) and (Q) into the Picard group, the group of divisors on E modulo principal divisors. The latter composed with the homomorphism

$$\begin{aligned} \text{Pic}(E) &\xrightarrow{c_1} \text{NS}(E) \simeq H^2(E, \mathbb{Z}) \simeq \mathbb{Z}, \\ \mathcal{O}(m \cdot (P) + n \cdot (Q)) &\mapsto mc_1(\mathcal{O}(P)) + nc_1(\mathcal{O}(Q)), \end{aligned}$$

into the Néron–Severi group $\text{NS}(E)$, which associates to an invertible sheaf its Chern class, leads to a homomorphism

$$\tilde{\rho} = c_1 \circ \rho : \mathbb{Z} \times \mathbb{Z} \rightarrow H^2(E, \mathbb{Z}) \simeq \mathbb{Z}.$$

The kernel of $\tilde{\rho}$ is a subgroup $L \simeq \mathbb{Z}$ of $\mathbb{Z} \times \mathbb{Z}$ with image in $\text{Pic}^0(E)$. Its restriction to L gives a homomorphism $\kappa : \mathbb{Z} \rightarrow \text{Pic}^0(E)$ which depends on the divisor D . Proposition 7 furnishes an extension G of E by $\mathbb{G}_m$ in $\text{Ext}^1(E, \mathbb{G}_m)$ associated with ξ_D , as can be seen as follows: Let $|D|$ be the support of D . Over $U = E \setminus |D|$ the map

$$\mathbb{Z} \xrightarrow{\kappa} \text{Pic}^0(E) \rightarrow \text{Pic}^0(U)$$

is zero since $\mathcal{O}(m \cdot (P) + m \cdot (Q))|_U \simeq \mathcal{O}_U$ and the extension becomes $G_U = G \times_E U \simeq \mathbb{G}_m \times U$. This leads to a section $s : U \rightarrow G_U \subseteq G$ and $\xi_D = s^*\vartheta$ for an invariant differential form ϑ on G which is uniquely determined (see [3, VI.3]). This is summarized in the following

THEOREM 9. *To give $[G] \in \text{Ext}^1(E, \mathbb{G}_m)$ is the same as giving a homomorphism $\kappa : \mathbb{Z} \rightarrow \text{Pic}^0(E)$ which also amounts to giving a differential ξ of the third kind with polar divisor $D = (P) + (Q)$ for some D .*

If $D = (P) + (O)$ with O the neutral element of A we also write $\xi_D = \xi_P$. Multiplication by n in $\text{Ext}^1(A, \mathbb{G}_m)$ is given by the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{G}_m & \xrightarrow{\iota} & G & \xrightarrow{\pi} & E \longrightarrow 0 \\ & & \downarrow n & & \downarrow p & & \downarrow \text{id} \\ 0 & \longrightarrow & \mathbb{G}_m & \xrightarrow{\iota(n)} & n_*G & \xrightarrow{\pi} & E \longrightarrow 0 \end{array}$$

which induces $[G] \mapsto [G(n)] = [n_*G]$ and corresponds to an unramified covering $p : G = G(1) \rightarrow G(n)$ of degree n over E . The morphism p induces a cotangent map $p^* : \mathfrak{g}(n)^\vee \rightarrow \mathfrak{g}(1)^\vee$ between the duals of the Lie algebras. Since $\mathbb{G}_m$ has a canonical invariant differential form $d \log t$, we can choose $\vartheta \in \mathfrak{g}(1)^\vee$ and $\vartheta(n) \in \mathfrak{g}(n)^\vee$ such that $\iota(n)^*\vartheta(n) = d \log t = \iota^*\vartheta$. This choice gives $p^*\vartheta(n) = n \cdot \vartheta + \alpha p^*\omega$ with $\alpha \in \mathbb{Q}$. Since the covering p is over E , we have $p^*\omega = \omega$ and then we may replace $\vartheta(n)$ by $\vartheta(n) - \alpha\omega$ and assume therefore that $p^*\vartheta(n) = n\vartheta$. For $e \in \mathfrak{g}(1)$ we get

$$\langle \vartheta(n), p_*e \rangle = \langle p^*\vartheta(n), e \rangle = \langle n \cdot \vartheta, e \rangle = n \langle \vartheta, e \rangle.$$

If $[G(n)] = 0$ we can take $G(n) = \mathbb{G}_m \times E$ and $\vartheta(n) = p_1^*d \log t$ where p_1 is the projection onto $\mathbb{G}_m$. This leads to

$$\begin{aligned} \langle \vartheta(n), p_*e \rangle &= \langle p_1^*d \log t, p_*e \rangle = \langle p^*(p_1^*d \log t), e \rangle \\ &= n \langle p_1^*d \log t, p_*e \rangle = n \langle d \log t, (p \circ p_1)_*e \rangle. \end{aligned}$$

If $[G]$ corresponds to the differential ξ_G with polar divisor $D = (P) + (Q)$ then it is of the form $\xi_G = s^*\vartheta$ with ϑ normalized as above and then for any $\gamma \in H_1(E(\mathbb{C}) \setminus |D|, \mathbb{Z})$ we have $\langle \xi_D, \gamma \rangle = \langle s^*\vartheta, \gamma \rangle = \langle \vartheta, e \rangle$ with $e = s_*\gamma$.

THEOREM 10. *We have $\langle \vartheta, e \rangle \in \overline{\mathbb{Q}} \langle \vartheta(n), p_*e \rangle + \overline{\mathbb{Q}} \langle \omega, \gamma \rangle$. In particular $\langle \vartheta(n), p_*e \rangle \in 2\pi i\mathbb{Z}$ if the extension class $[G]$ is a torsion class.*

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