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Pérez-Escobar, José Antonio

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José Antonio Pérez-Escobar*

Showing Mathematical Flies the Way Out of Foundational Bottles: The Later Wittgenstein as a Forerunner of Lakatos and the Philosophy of Mathematical Practice

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Abstract: This work explores the later Wittgenstein's philosophy of mathematics in relation to Lakatos' philosophy of mathematics and the philosophy of mathematical practice. I argue that, while the philosophy of mathematical practice typically identifies Lakatos as its earliest of predecessors, the later Wittgenstein already developed key ideas for this community a few decades before. However, for a variety of reasons, most of this work on philosophy of mathematics has gone relatively unnoticed. Some of these ideas and their significance as precursors for the philosophy of mathematical practice will be presented here, including a brief reconstruction of Lakatos' considerations on Euler's conjecture for polyhedra from the lens of late Wittgensteinian philosophy. Overall, this article aims to challenge the received view of the history of the philosophy of mathematical practice and inspire further work in this community drawing from Wittgenstein's late philosophy.

Keywords: later Wittgenstein, Lakatos, philosophy of mathematical practice, apriorism, rule following, rule bending, rules of description

1 Introduction

The aim of this work is twofold. First, it argues that the later Wittgenstein is a forerunner of the philosophy of mathematical practice, thereby challenging the received view that Lakatos's work is the first in this direction. Second, it highlights features of the later Wittgenstein's philosophy of mathematics which are relevant to the philosophy of mathematical practice, in order to draw the attention of this community.

Sections 2 and 3 sketch the history and aims of the philosophy of mathematical practice. Section 2 focuses on the deficiencies of mainstream philosophy of mathematics that the philosophy of mathematical practice intended to address.

***Corresponding author: José Antonio Pérez-Escobar**, Chair of History and Philosophy of Mathematical Sciences, ETH Zurich, Zurich, Switzerland, E-mail: jose.perez@gess.ethz.ch

Section 3 describes how Lakatos improved on those deficiencies by addressing mathematical practices, hence becoming a “maverick” and, according to the received view, pioneering the philosophy of mathematical practice.

Section 4 argues that the later Wittgenstein’s philosophy of mathematics, which preceded Lakatos’ work, already constituted a shift from mainstream philosophy of mathematics and dealt with important aspects of how mathematics is used in practice. Subsection 4.1 describes one of these aspects, rule bending, and stresses the role of elements like culture and teaching in our use of mathematical rules. Subsection 4.2 is concerned with the use of mathematics as rules of description, and offers an interpretation of Lakatos’ considerations on Euler’s conjecture for polyhedra from the lens of Wittgenstein’s late philosophy of mathematics. This section intends to stimulate further work inspired by late Wittgensteinian philosophy in the context of the philosophy of mathematical practice.

2 The Inception of the Philosophy of Mathematical Practice

Both philosophy and mathematics are ancient academic disciplines and their presence in human affairs can be traced back to very early written remains. Eventually, mathematics itself became a subject of philosophical inquiry, and philosophy of mathematics came to be. It is precisely because of the early appearance of mathematics in the history of humankind that philosophy of mathematics is much older than the philosophy of other academic disciplines, like philosophy of biology. However, despite the long history of philosophy of mathematics, many of the ideas developed by Greek philosophers in philosophy of mathematics still prevail to this day. The legacy of Platonism, for example, is such that it remains one of the most popular metaphysical assumptions among mathematicians and in general (Cole 2010).

Yet, from its inception to this day, philosophy of mathematics has been characterized by an obvious and almost exclusive attention to the foundations of mathematics (Aspray and Kitcher 1988; Corfield 2003; Ferreirós and Gray 2006; Hersh 1979; Kitcher 1984; Mancosu 2008). Foundational studies, often dealing with questions on apriorism and logic or framed as ontological inquiries (for example, the nature of mathematical objects, and if they exist, how we can access them) have led to the creation of most major schools of thought in philosophy of mathematics: Platonism, empiricism, monism, logicism, formalism, among some others, all support a given metaphysical perspective on the most basic properties of mathematics. Platonism, for example, advocates that mathematics is real, ahistorical, perennial and transcends the mathematician. Hence, the Platonist

states that theorems are not invented or created, but discovered; it is implied that all mathematics is already somewhere in a realm of abstract ideas, waiting to be unveiled. Logicism dictates that all mathematics is reducible to logic, underscoring its a priori character. Empiricism, on the other hand, denies the a priori character of mathematics, that there is mathematical knowledge that is independent of experience, and claims instead that all mathematics is ultimately to be subjected to empirical testing, just like the other sciences.

However, a cohort of philosophers originated in the second half of the 20th century and pioneered by Lakatos (1976b), Kitcher (1988), Crowe (1988), Browder (1988) and Maddy (1992, 1997), among others, has noted that, in spite of the long track of philosophy of mathematics and the considerable number of schools of thought in the discipline, the focus of philosophers of mathematics has been quite limited. Up until the reaction of those philosophers, the practical aspects of mathematics, that is, how mathematics is practiced by different mathematical communities or cultures, had been neglected for the most part. On the other hand, the philosophy of empirical sciences like biology, even with a shorter history at its back, had enjoyed a broader range of topics among its repertoire. This is, in part, due to a number of particularities of mathematics as a field of knowledge, a lack of philosophical reflection on such a mathematical idiosyncrasy, and a passive history of mathematics that is limited in scope.

Restricting philosophy of mathematics to the study of the basic foundations of mathematics had an unusual consequence: it became a place for testing large classical philosophical views in metaphysics and epistemology. For example, if someone were to discuss whether a priori knowledge is attainable, or whether the virtues of a realist or a constructivist approach are more appealing, the foundational work in philosophy of mathematics would be a good place to explore. This was noted by William Aspray and Philip Kitcher not very long ago in their insightful and influential “opinionated introduction”, in their book *History and philosophy of modern mathematics*:

Philosophy of mathematics appears to become a microcosm for the most general and central issues in philosophy—issues in epistemology, metaphysics, and philosophy of language—and the study of those parts of mathematics to which philosophers of mathematics most often attend (logic, set theory, arithmetic) seems designed to test the merits of large philosophical views about the existence of abstract entities or the tenability of a certain picture of human knowledge. (Aspray and Kitcher 1988, p. 17).

This happened at the cost of leaving out mathematics’ own topics and methods as a discipline. How mathematical knowledge grows, what makes some theories better than others to the eyes of mathematicians, how informal arguments are related to formal arguments, how diagrams are used, and questions surrounding the concepts of explanation and proof, are questions that were for the most part

disregarded. The philosophy of mathematics was conducted in a significantly different way from many other philosophies of the sciences.

As Ferreirós and Gray (2006) point out, an intuitive way to understand the aim of foundational studies in mathematics is via an architectural metaphor. Mathematics and its progress are regarded as a building and its construction. For the building to be built, it is first necessary to prepare a solid ground to support the ground floor, which in turn, will sustain further floors. This metaphor traces back to the 17th century analogies of sciences as buildings, and to the work of Hermann Weyl, who considered that it was a good match for foundational debates in mathematics. Weyl, after being trained under the supervision of the formalist bannerman David Hilbert, became heavily influenced by Brouwer's intuitionism and complained that foundational work in philosophy of mathematics was aimed to demonstrate that mathematics was built on stone (identifiable foundations, whatever their nature), and not on sand, although he believed otherwise and called formalism a "fake wooden structure" (Weyl 1994).¹ This is not to say that this purpose was not shared with the philosophy of the natural sciences, which also strived to study the foundations of physics and other fields. However, philosophy of science has also been concerned about how the building is being built, instead of focusing almost exclusively on the ground on which it is built. Even more, paying attention to this process we can gain insights about the role of foundations in science and other aspects of the character of foundations themselves, which would paradoxically be disregarded with a plain focus on foundations. Hilbert, being a faithful advocate of formalism and of the idea that mathematics is ultimately sustained by solid foundations, was by no means naïve, or oblivious to the fact that this applies to the philosophy of mathematics too, and that a commitment to the actual historical development of the discipline is imperative. According to him, the process of building the edifice is not linear and the foundations are often revisited and adapted to whatever needs arise:

The edifice of science is not raised like a dwelling, in which the foundations are first firmly laid and only then one proceeds to construct and to enlarge the rooms. Science prefers to secure as soon as possible comfortable spaces to wander around and only subsequently, when signs appear here and there that the loose foundations are not able to sustain the expansion of the rooms, it sets about supporting and fortifying them. This is not a weakness, but rather the right and healthy path of development. (Hilbert 1905, p. 102; Translation by Corry 1997, p. 130)²

¹ The Platonist may prefer a more archaeological metaphor of "excavating" and "discovering".

² Original text in German: "Das Gebäude der Wissenschaft wird nicht aufgerichtet wie ein Wohnhaus, wo zuerst die Grundmauern fest fundamentiirt werden und man dann erst zum Aufbau und Ausbau der Wohnräume schreitet; die Wissenschaft zieht es vor, sich möglichst schnell wohnliche Räume zu verschaffen, in denen sie schalten kann, und erst nachträglich, wenn es sich zeigt, dass hier und da die locker gefügten Fundamente den Ausbau der Wohnräume nicht zu

Therefore, keeping an eye on how the building is erected on the ground seems to be a very sensible thing to do. Most philosophers would agree that philosophical reflection should not be constrained to the foundations of knowledge. Philosophers of science quickly became aware of this, but philosophy of mathematics has historically lagged behind in this respect. It was not until the decades of the 70s and the 80s that a few philosophers, inspired by the pioneering line of philosophical inquiry of Imre Lakatos, became interested in this process of building the edifice, the so called mathematical practice. David Corfield, a philosopher concerned with the mathematical practice, controversially asserted that the philosophy of mathematical practice is, in fact, the only real philosophy of mathematics. In his book *Towards a Philosophy of Real Mathematics*, dedicated to support his stance but also heavily criticized for openly advocating anti-formalist views, he vindicated this still newborn line of enquiry in the philosophy of mathematics concerned with the mathematical practice:

Continuing Lakatos' approach, researchers here believe that a philosophy of mathematics should concern itself with what leading mathematicians of their day have achieved, how their styles of reasoning evolve, how they justify the course along which they steer their programmes, what constitutes obstacles to their programmes, how they come to view a domain as worthy of study and how their ideas shape and are shaped by the concerns of physicists and other scientists. (Corfield 2003, p. 10).

In their “opinionated introduction” (Aspray and Kitcher 1988), Aspray and Kitcher identified these ideas with a departure from a tradition which can be traced back to Frege, focused on foundational work. The departing authors are named “the mavericks”, and the Hungarian philosopher Imre Lakatos was considered the first of them: “if the mainstream began with Frege, then the origin of the maverick tradition is a series of four papers by Lakatos, published in 1963–64 and later collected into a book (1976)” (p.17); “the general task of articulating an account of mathematical methodology, begun by Lakatos, has been tackled by writers from several different perspectives” (p. 19). This historical sketch became the received view, as constated, for instance, by Corfield (2003, p. 10), Ferreirós and Gray (2006), Carter (2019) and Krajewski (2020).

Overall, this direction leads to the acknowledgment of the importance of and a focus on the details of the mathematical practice(s). Today, a number of philosophers and historians (more or less influenced by the maverick tradition, depending

tragen vermögen, geht sie daran, dieselben zu stützen und zu befestigen. Das ist kein Mangel, sondern die richtige und gesunde Entwicklung.”

on the specific case) sharing an interest in the details of the mathematical practice(s) constitute the Association for the Philosophy of Mathematical Practice.³

3 Lakatos' Pioneering Philosophy of Mathematics

According to the received view, the first elaborated philosophical work concerned with the practical aspects of mathematics was crafted by Lakatos. Although he never defined himself as a social constructivist and it is not clear whether he qualifies as such (Ernest 1998, p. 135), his philosophy is believed to be influenced by this movement, and in turn influenced the movement itself (Ernest 1998, p. 160; Steffe 1992). More notably, his close friendship with Paul Feyerabend led to a mutual influence between the two philosophers (Lakatos and Feyerabend 1999). Feyerabend's epistemological anarchism is grounded in the idea that there are no universal, fixed rules that dictate the proceedings of science, and that much of the success of science is due to its methodological flexibility (Feyerabend 1975). In this line, Lakatos thought that mathematics allows for such a flexibility and that the historical development of mathematics shows it. Against previous formalist assumptions, he claimed that informal methods (visualization techniques, thought experiments, heuristics, etc.) were at the base of most of the developments in mathematics in the 19th century, that mathematics is in constant change and is revisable, and that therefore the practical aspects of mathematics should be granted more attention than they traditionally get (Lakatos 1976b).

Most of Lakatos' work in philosophy of mathematics was compiled in his book *Proof and Refutations*, posthumously published in 1976, two years after his early death at the age of 51. To support his controversial claims, he conducted a case study on Euler's conjecture for polyhedra (for all polyhedra, the number of vertices, minus the number of edges, plus the number of faces equals two [$V - E + F = 2$]). The case study takes the form of a fictional dialog in the context of a mathematics class, where the students provide counterexamples to Euler's conjecture for polyhedra. With this dialog, Lakatos aimed to illustrate how mathematics is done from a philosophical perspective while being faithful to the actual practice of mathematicians (hence its originality).

The title of his book, *Proof and Refutations*, is actually an allusion to Popper's *Conjectures and Refutations* (2014). Lakatos was deeply influenced by Popper's philosophy of science, borrowed much from it, and in a sense, his own ideas suppose a continuation of it. Popper's famous point that science is always preliminary or

³ <http://www.philmathpractice.org/>.

fallible – there are no true theories, principles or laws, only theories, principles or laws for which no counterexample has been yet found – was brought to mathematics by Lakatos, even when Popper never thought his philosophy would apply to this discipline (Gregory 2011). Lakatos claimed that Popper “made the mistake of reserving a privileged infallible status for mathematics” (Lakatos 1976b, p. 139). According to Lakatos, counterexamples for well-established mathematical theorems can appear at any point in the development of mathematics. But unlike Popper in regard to scientific knowledge, Lakatos claimed that a counterexample alone (or a given number of them) is not reason enough to discard a mathematical theorem – not even scientific knowledge, as he claimed in his own philosophy of science. In short, Lakatos agreed with Popper that proof does not warrant permanent and intrinsic truth value to knowledge, but he assured that refutation is not reason alone to discard knowledge either, and very importantly and to the disdain of a very dominant-at-the-time formalist tradition, that these two considerations apply not only to science but to mathematics as well.

Lakatos’ general philosophy of science will not be explored here. Regarding this topic it suffices to say that, according to him, when well established scientific theories are confronted with empirical counterexamples, scientists often opt to either reduce the scope of the theory in question, or modify auxiliary hypotheses of what he called “research programmes” (networks comprised by sequences of theories), instead of their hard core (Lakatos 1976a). But what is the role of proofs and refutations in his philosophy of mathematics? Let us come back to Euler’s Polyhedral Formula and the fictional dialog in *Proof and Refutations*. Lakatos claimed that informal mathematics is at the base of the discipline, and he intended to represent how mathematicians actually dealt with Euler’s conjecture. In the dialog, the teacher presents an informal proof – after Augustin-Louis Cauchy’s proof (Cauchy 1813) – based on the mathematical activity of visualization, a typical resource in the mathematical practice, for the conjecture at class. It is followed by a panoply of counterexamples posed by the students. For example, one of the students presents the case of a hollow cube, for which $V - E + F = 4$. Right away, another student remarks that, after Euler’s conjecture has been refuted, we can no longer rely on it, and it should be disregarded and discarded together with previous proof. He also claims that “a single counterexample refutes a conjecture as effectively as ten” (Lakatos 1976b, p. 13). The teacher, however, defends a particular idea of proof and its role in mathematics:

I agree with you that the conjecture has received a severe criticism by Alpha's counterexample. But it is untrue that the proof has 'completely misfired'. If, for the time being, you agree to my earlier proposal to use the word 'proof' for a 'thought-experiment which leads to decomposition of the original conjecture into subconjectures', instead of using it in the sense of a 'guarantee of certain truth', you need not draw this conclusion. My proof certainly proved Euler's conjecture in the first sense, but not necessarily in the second. You are interested only in proofs which 'prove' what they have set out to prove. I am interested in proofs even if they do not accomplish their intended task. (Lakatos 1976b, pp. 13–14)

In addition, Lakatos described a number of strategies that mathematicians employ in order to adjust and preserve the core integrity of a theorem when facing counterexamples. He took proofs and refutations away from the absolutist character ascribed to them by former traditions in philosophy of mathematics – that is, ahistorical, culturally invariable, universal judgments about the virtues of mathematical propositions. Instead, proofs and refutations constitute what Lakatos called a “dialectical unity”, ultimately depending not on formal criteria, but on the mathematical community – which has the last word on matters such as the validity of, and the reactions to, each proof and refutation. Lakatos put forward this dialectical unity of proofs and refutations as a key element of mathematical progress and refinement. We see how Lakatos' intent was to contrast how mathematics is assumed to work by a dominant formalist school of thought, against how he considered that it actually works. It is worth noting that Lakatos dealt with, and underscored the importance of, aspects of the mathematical practice that would later be studied in more detail, such as proving and explaining (different proofs for the same theorem may have unique epistemic properties), visualization (mental imagery techniques with special relevance in areas like geometry), the interrelations between informal and formal mathematics, and how mathematics “grows”. All these issues have shaped the development of mathematics, and therefore, a philosophical study of mathematics that overlooks them is either partial or lax.

Despite the potential of his ideas, Lakatos' philosophy of mathematics was not too warmly received by the academic community of his time. His quarrel with foundational studies and formalism (and particularly his disdain towards their conception of formal proofs) was met with strong opposition. In addition, it has been argued that he died too early for his ideas to have a greater impact on the philosophy of mathematics of his time (Ferreirós and Gray 2006; Mancosu 2008). Such an early death is also one of the main motivations of the special issue where this paper belongs, *Lakatos' Undone Work*. This abrupt interruption prevented Lakatos from establishing a new trend in philosophy of mathematics, at least in the short term. In fact, the next work following a similar line, Philip Kitcher's *The Nature of Mathematical Knowledge* (1984), did not appear until

almost two decades after Lakatos' pioneering papers, which culminated in *Proof and Refutations*.

4 Wittgenstein's Late Philosophy of Mathematics: An Early Shift of Focus from Foundational Work

In this section I intend to explore an overlapping topic to that of Lakatos' undone work, which may be termed "Lakatos' predone work", in the context of the later Wittgenstein's philosophy of mathematics. I do not mean "predone" in the sense that Wittgenstein comprehensively dealt with all that Lakatos worked on. This is far from my intention. Actually, the works of these two philosophers diverge in many substantial points in terms of scope and perspective. Rather, I intend to challenge the received view, described before, that Lakatos is the first of the "mavericks". Based on the descriptions in the sections above, I intend to clarify how the later Wittgenstein merits a place in the history of the philosophy of mathematical practice, concretely as an earlier figure than Lakatos, due to his challenges to mathematical apriorism and foundationalism.⁴ I also intend to show how the philosophy of mathematical practice may benefit from further exploration, development and application of his work.

First, it is worth noting that it is not completely clear to what extent Lakatos was influenced by Wittgenstein's philosophy of mathematics. The following memories of Donald Gillies, who was one of Lakatos' PhD students, are telling:

Lakatos was a most entertaining person with whom to discuss philosophy. I remember that at our first meeting, it was not long before I uttered the name: 'Wittgenstein'. Lakatos immediately replied: "Wittgenstein was the biggest philosophical fraud of the twentieth century." Naturally, as I was used to the Wittgenstein cult at Cambridge, this statement came as quite a shock to me. So I replied: "Dr Lakatos, what you say is truly surprising for me because I have just finished writing an essay in which I maintain that there are close links between your concept of mathematical proof and Wittgenstein's."

⁴ Wittgenstein and Lakatos also differ substantially in how they challenge apriorism and foundationalism. Some of these differences are too nuanced to be exposed here. For instance, while the later Wittgenstein's philosophy of mathematics has usually been characterized as anti-foundationalist (Floyd 2021, p. 1; Grève and Mühlhölzer 2014; Rodych 1999), some suggesting this trait carries over from his middle phase (Rodych 2018), it has also been proposed that it is pluri-foundationalist, while Lakatos deserves the anti-foundationalism label instead (Wagner 2019).

At my next meeting with Lakatos, he took up the Wittgenstein theme again. He said: “Regarding Wittgenstein, I looked through my copy of his *Remarks on the Foundations of Mathematics*, and I was surprised to find that I had written enthusiastic notes in the margins. But these notes were written in Hungarian which means that I must have written them ten years ago, just after I had arrived in England.”

What were Lakatos’ opinions at the time of our meeting? (Gillies 2019, p. 96)

I will now recapitulate references to Wittgenstein in the context of the philosophy of mathematical practice, which are not many. Most quotations are of the early, tractarian Wittgenstein, in the context of logical positivism and the “non-maverick” philosophy of mathematics. I will skip those, as I am concerned with Wittgenstein’s late philosophical work, namely, *Philosophical Investigations* (PI; Wittgenstein 1953), *Lectures on the Foundations of Mathematics* (LFM; Wittgenstein 1976) and *Remarks on the Foundations of Mathematics* (RFM; Wittgenstein 1978). All of these works preceded Lakatos’ work on philosophy of mathematics, since they were obviously written (or lectured) before Wittgenstein’s death in 1951.

One of the few mentions of Wittgenstein’s late philosophy in the context of the philosophy of mathematical practice is by Ferreirós (2016). Ferreirós, in the context of mathematical knowledge being a prominent mystery of philosophy, writes: “a mystery that some twentieth-century philosophers (Wittgenstein prominent among them) tried to dispel like a simple fog, but quite unconvincingly. For reasons why mathematics is not a game of tautologies, nor a mere calculus of symbol transformations, see especially Chapters 4 and 6” (Ferreirós 2016, p. 1). It is not clear to which of Wittgenstein’s works he is referring. However, he later quotes a fragment from the *Lectures on the Foundations of Mathematics* in the following manner:

(One tendency in the development of mathematics has been) the drift toward reduction of mathematics to a purely symbolic system; notable instances can be found in twentieth-century strict formalism, but also in Lagrange around 1800, Peano around 1900, and many others. A colorful quotation, expressing views akin to logical positivism and strict formalism, was provided by Wittgenstein in 1939: “The only meaning they (certain parts of mathematics, involving simple calculations but considered deep) have in mathematics is what the calculation gives them ... if you think you’re seeing into unknown depths —that comes from a wrong imagery” (Wittgenstein 1976, p. 254). This seems to presuppose a reduction of mathematics to merely the symbolic frameworks discussed above. (Ferreirós 2016, pp. 89–90).

Instead of seeing Wittgenstein as an ally to the cause of the philosophy of mathematical practice, Ferreirós seems to interpret the later as if he still maintained his early and middle views (characterized by embracing different kinds of formalism; see Rodych 1997). However, in the quote above, Wittgenstein does not ground the meaning of mathematics in a “strict formalism” but in the technique of calculation,

in its use⁵ (more on meaning as use will come later), faithfully to the spirit of his later philosophy. Not only that, the later Wittgenstein challenges formalism extensively in the *Lectures on the Foundations of Mathematics* and the *Remarks on the Foundations of Mathematics* (more on this will be discussed in Section 4.1). However, Ferreirós swiftly admits that “whether this reflects Wittgenstein’s mature opinion (probably it does not) is a question I must leave to experts in the topic.” (Ferreirós 2016, p. 90).

Maddy, despite a number of differences with the mavericks (see Mancosu 2008, pp. 10–13), works on the philosophy of mathematical practice. However, in her work on Wittgenstein, she did not use the later Wittgenstein’s philosophy of mathematics as a framework for the analysis of mathematical practices, but rather, interpreted his philosophy (which she qualifies as antiphilosophical) and compared it to her naturalist approach (Maddy 1993, 1997, pp. 162–171). According to Maddy, Wittgenstein advocates “the excision of all traditional philosophy, and in its place, recommending careful attention to the details of the practice itself”, which is understood as a radical formulation of naturalism (Maddy 1997, p. 171). More broadly, Wittgenstein’s late philosophy of mathematics has a certain synergy with “second philosophy”, a naturalist position endorsed by Maddy (2014, chapters 5, 6 and 7). However, Maddy finds the specifics of Wittgenstein’s ideas damaging to her main research interest, set theory (Maddy 1997, p. 168), and disagrees with them at several fronts (for instance, the extent to which pure mathematics should be “pruned”). Although Maddy’s exegesis of the later Wittgenstein’s philosophy of mathematics is very insightful, it has been challenged (Dawson 2014) and an alternative one featuring an extended conception of meaning as use (more on this concept later) may be better suited to understand pure mathematics (Pérez-Escobar and Sarikaya 2022).⁶

Avigad (2008) draws from the philosophy of language of the *Philosophical Investigations* in the context of describing language as a community practice and understanding mathematical proofs and natural language.⁷ However, he only makes a minor remark of Wittgenstein’s philosophy of mathematics (concretely, of the *Remarks on the Foundations of Mathematics*). He claims that “one finds similar views on the role of specifically mathematical concepts in Wittgenstein’s other

5 If such a use is only intra-mathematical, however, the later Wittgenstein would consider mathematics to be meaningless (see Maddy 1993, and Pérez-Escobar and Sarikaya 2022).

6 This does not necessarily mean that Maddy’s exegesis is wrong, and another is right. Instead, it means that a certain exegesis may provide a more useful theoretical framework to understand pure mathematics when compared to another. Which one reflects Wittgenstein’s views more faithfully is a different matter.

7 Regarding this aspect of the later Wittgenstein’s philosophy of language, Avigad’s work has influenced others methodologically; see Carl et al. 2021.

works” which stand “in contrast to the traditional view of mathematics as a collection of definitions and theorems” (Avigad 2008, p. 327). This is not surprising; after all, Wittgenstein’s late philosophy of mathematics, in spite of a number of exegeses and discussions by other philosophers (see for instance: Bloor 1973; Bloor 1983; Bloor 1997; Dummett 1959; Floyd 2005; Frascolla 1994; Putnam and Conant 1996; Slater 1979; Wrigley 1977) is generally more unknown than the *Philosophical Investigations* (some reasons for this will be mentioned in the second next paragraph).

Finally, one of the few implementations of Wittgenstein’s late philosophy for the analysis of mathematical practices is by Secco and Pereira (2017). They insightfully relate the philosophical discussions about the proof of the four-color theorem with both early and late Wittgensteinian philosophy, especially with respect to the difference between proofs and experiments.

Hence, it seems that Wittgenstein’s late philosophy of mathematics has not received a lot of attention in the community of philosophers of mathematical practice. Considering the overlap and potential synergy between the two (discussed next), and Wittgenstein’s belief that his most important work was his philosophy of mathematics (Monk 1990, p. 466), this is strange. However, there are some reasons that may explain a relative disregard of Wittgenstein’s late philosophy of mathematics by this community. One reason might be the one that Ferreirós and Gray (2006) propose regarding the reception of the ideas of Lakatos’ himself: “being a good polemicist, and delighting in it, is not always the best way of promoting one’s ideas” (Ferreirós and Gray 2006, p. 17). The later Wittgenstein, just like Lakatos, was a polemical figure (and so were his ideas; see Engel 1967). Yet Lakatos’ ideas were much more influential and credited in the philosophy of mathematics (and especially in the philosophy of mathematical practice, as described before) than those of the later Wittgenstein. Other reasons include the reception of Wittgenstein’s work by the philosophical community: most of his work is comprised of isolated notes, notebooks, short essays and lecture materials (like notes from his students), written in a peculiar dialogical format extensively involving a voice of personal confession and temptation, and another voice urging to correct the first (for an overview on the early reception of Wittgenstein’s late philosophy, and the virtues and issues of Wittgenstein’s writing style, see Cavell 1962). As Cavell notes, it may be that Wittgenstein’s style fits the kind of philosophy he espouses. However, such a style is likely to have hindered the influence of his work in mainstream academia, especially given that they presented novel and controversial philosophical ideas. Furthermore, a sizeable part of his later work on mathematics was not publicly available until the early 2000s (Rodych 2018, Section 3; See Stern (1996) for a more comprehensive overview on the availability of Wittgenstein’s work), which may help explain why Wittgenstein’s philosophy of

mathematics is considered the least influential and least understood part of his later philosophy (Hacker 1996, p. 466). It has also been claimed that Wittgenstein's posthumously published work has been subjected to a series of questionable editorial decisions (Floyd and Mühlhölzer 2020, preface). Finally, it has been suggested that his philosophy is difficult to classify within the framework of academic schools (Bangu 2018).

Wittgenstein's late philosophy of mathematics explores a varied range of topics (for a recent overview in tractarian format, see Wagner 2020). Here, due to limitations of scope and space, I will focus on the two that I believe are most encompassing and relevant for the philosophy of mathematical practice, and which are not featured in Lakatos' work: rule bending and mathematics as rules of description. Both are connected to the idea of "meaning as use", a key notion in late Wittgensteinian philosophy of language and mathematics, ubiquitous in the *Lectures on the Foundations of Mathematics* and the *Remarks on the Foundations of Mathematics*. According to the later Wittgenstein, symbols, utterances and even images do not have intrinsic and univocal meaning; they are not stable representations of other kinds of phenomena. In order to elucidate their meaning, one should see how symbols, utterances and images are used (for instance, as implemented in more or less systematic techniques). Likewise, the meaning of an order is the outcome that the receiver of the order fulfills and that satisfies the speaker. Symbols, utterances and images, in a void deprived of uses, people, conventions or tacit knowledge, can be neither understood nor misunderstood. For the later Wittgenstein, these elements cannot be regarded as self-sufficient elements either. In the context of mathematics, this implies that the meaning of mathematics does not reside in inner properties of symbols, but in how they are (typically) put to use. The notion of meaning as use is itself very broad, and here I will only be concerned with rule bending and mathematics as rules of description.

4.1 Rule Bending

In their "opinionated introduction", Aspray and Kitcher claim that "closest to Frege's original approach was the program undertaken by Russell and Whitehead in *Principia Mathematica*" (Aspray and Kitcher 1988, p. 6). That is, Russell and Whitehead carry on with Frege's logicist approach, the provision of the logical foundations of mathematics. As Kitcher (1988) rightfully notes, such an enterprise relies on a commitment to mathematical apriorism. Aspray and Kitcher further claim that "both Lakatos and Kitcher proceed to the enterprise of exploring the methodology of mathematics after arguing that there is no a priori foundation for

mathematics” (Aspray and Kitcher 1988, p. 19), which makes them qualify as “mavericks”.

However, the later Wittgenstein deeply breaks with the Fregean tradition and extensively criticizes Russell’s and Whitehead’s *Principia Mathematica* and mathematical apriorism much earlier than Lakatos and Kitcher (very explicitly in the *Lectures on the Foundations of Mathematics*, so at least as of 1939).⁸ As an example among many, a simple and blunt quote is the following:

This definition of two: “This is two” I I is as good a definition as Russell’s, as good a definition as there is in the world. It can be misinterpreted, but so what? So can all definitions. (LFM, p. 24)

He attacks Russell’s and Whitehead’s endeavor from several angles, but here I will refer only to the notion of rule bending.

Rule bending is a notion present in the *Philosophical Investigations* that is extensively applied to mathematics in the *Lectures on the Foundations of Mathematics* and the *Remarks on the Foundations of Mathematics*. Wittgenstein holds that rules, however they may be formulated, do not fully determine possible uses or practices derived from them. Uses and practices are constrained by the formulations of rules, but also by sociological and psychological factors. Hence, rule following, uses and practices are underdefined by rules, including mathematical rules (for instance, concerning symbol manipulation). Just like a legal code is not enough to determine how to elaborate a verdict in court, alleged foundations of mathematics do not determine mathematical practices. Consider the following quotes:

“Does the formula “ $y = X^2$ ” determine what is to happen at the 100th step?” This may mean, “Is there any rule about it?”—Suppose I gave you the training below 100. Do I mind what you do at 100? Perhaps not. We might say, “Below 100, you must do so-and-so. But from 100 on, you can do anything.” This would be a different mathematics. (Wittgenstein 1976, p. 29)

Might it not be said that the *rules* lead this way, even if no one went it? (Wittgenstein 1978, IV-49)

As for the latter quote, Wittgenstein resists an affirmative answer. It is problematic to decide what the right application of a rule is and what violates it (even for seemingly intractable rules, like in the form of contradictions; see LFM, p. 175). However, shared cultural, social and psychological factors, which together with the formulation of rules co-determine how rules are followed, contribute to keeping this problem in. Also, dissimilarities in cultural, social and psychological

⁸ There is an instance of Wittgenstein being qualified as a maverick (Marion 2011), but without connecting to the philosophy of mathematical practice.

factors may lead to unexpected uses of rules, to “bending” of the rules, relative to a reference use. Hence, Wittgenstein’s late philosophy of mathematics moves the emphasis away from the foundations of mathematics (and when he talks about foundations, it is mostly to criticize foundational endeavors) and closer to mathematical practices and ways of living. For this reason, Wittgenstein’s late philosophy is a powerful tool in ethnomathematics (François and Vandendriessche 2016). Perhaps also for this reason the philosophy of mathematical practice requires a strong empirical focus (Kant, Pérez-Escobar, and Sarikaya 2021). Wittgenstein did not think that mathematical foundations should or could determine mathematical practices, and a focus on mathematical foundations may limit philosophical clarifications of actual practices, and perhaps the potential of the practices themselves. In the spirit of the rest of his late philosophy, just like the aim of his philosophy is to “to show the fly the way out of the fly-bottle” (PI, 309), the aim of his philosophy of mathematics is to show mathematical flies the way out of foundational bottles.

4.2 Rules of Description

As said before, the notion of meaning as use is very broad and a comprehensive discussion of all its implications is out of place here. However, in the context of mathematics, a very interesting use is framing descriptions of observations. The later Wittgenstein considers mathematics to be rules of descriptions of the world, rather than descriptions of the world. For instance, we have rudimentary mathematical models based on addition of discrete objects for techniques, like counting, which are usually reliable and associated with “success” in many practices. In this context, $2 + 2 = 4$ is a mathematical model that we use to count apples.⁹ If we have two apples in a basket, and we bring to that basket two more apples which we took from somewhere else, we would expect to have four apples. However, if $2 + 2 = 4$ does not properly describe the situation (for instance, if we only count three apples in the basket), we would not be keen to question the mathematics, but the situation. We would suggest that something “went wrong”, for example “I must have dropped one apple when I took the apples to the basket”, or “someone stole an apple from the basket while I was busy bringing more apples”, and so forth. More rarely, we could instead question the fit of the model to the situation, claiming something like “apples have a “disappearing” or “merging” property hitherto

⁹ A tractarian formulation of this would be “ $2 + 2 = 4$ corresponds to the reality of apples”. This version, however, indicates that $2 + 2 = 4$ is a useful component in our practices, for instance, the technique of counting.

unknown”. Moreover, we could also claim we did not follow the technique correctly (in this case, we did not count correctly).

$2 + 2 = 4$ is a framework rule, and therefore directs our description of reality, organizing our experiences of the world. Wittgenstein describes this role of mathematics in several places in the *Lectures on the Foundations of Mathematics* and the *Remarks on the Foundations of Mathematics*, and some commentators have noticed it (see, for instance: Dawson 2014; Fogelin 1987, p. 214; Steiner 2009; Wagner 2020). For instance, regarding arithmetics, Wittgenstein claims:

Arithmetic is set up such that it is applicable in this way, and if we want to talk of a reality corresponding to arithmetic then we should say that this reality is to be seen in our using arithmetic in the way that we do. (Wittgenstein 1976, pp. 248–249)

This idea, like many others from Wittgenstein’s late philosophy of mathematics, is also present in the philosophy of language of the *Philosophical Investigations*, and applies to ordinary language very broadly. For instance, Wittgenstein makes the following remark on the sentence “there is no reddishgreen” (to which he refers as a rule of description, not as a description) as a simple illustration of the role of mathematics that I have just described:

The correspondence is between this rule and such facts as that we do not normally make a black by mixing a red and a green; that if you mix red and green you get a colour which is “dirty” and dirty colours are difficult to remember. All sorts of facts, psychological and otherwise. (LFM, p. 245)

Probably, one area where the later Wittgenstein could have elaborated more is the relative resilience of different pieces of mathematics. Certain pieces of mathematics, like those of basic arithmetics in the context of counting simple, discrete objects like apples, are very resilient: it would take a lot of “going wrong” in the above sense for us to revise the mathematics. However, other pieces are not so resilient. It would take less “going wrong” for us to revise less established mathematical models which are not so deeply engrained in our practices.

As it has been sketched so far, the idea that mathematics are used as rules of description has substantial potential for the analysis and understanding of practices in pure mathematics too. One does not need to go very far to find examples: Euler’s conjecture for polyhedra, the very example discussed by Lakatos, can be considered as follows. $V - E + F = 2$ is a rule of description of, among other things, what counts as a polyhedron, or as a face in polyhedra, or what is the proper way of counting faces. It is ...

as if we had hardened an empirical proposition into a rule and now experience is compared with it and it is used to judge experience (RFM VI-22).

Typical polyhedra (the ones that mathematicians or people like highschool students typically imagine, encounter in textbooks and so on) are likely to conform to the regularity. This regularity, in turn, is entangled with techniques and other elements of the mathematical practice, like how to count faces or how to construct a polyhedron. However, in the advent of new, niche examples of polyhedra, there is disagreement between mathematicians as for the appropriateness of Euler's conjecture regarding one or more of those aspects (or others). The relative resilience of Euler's conjecture as a rule of description depends not just on the regularities of typical polyhedra, but on how central it is in the practices of mathematicians (scope of the techniques with which it is intertwined, the usefulness of those techniques, etc.), in their ways of life, broadly speaking. This resilience may vary across communities.

The late Wittgensteinian view of mathematics as rules of description is therefore an insightful alternative to account for the counterexamples in mathematics with which Lakatos was concerned, in a way that also involves careful attention to mathematical practices. It also lays a ground to investigate how mathematics has normative power but also can be tailored to the needs of specific social practices, like scientific practices. Mathematics is intertwined with other non-mathematical components of scientific practices, and is usually adapted to the specific needs of science, including the special sciences (Pérez-Escobar 2020). A more comprehensive understanding of feedback loops between mathematical and non-mathematical components in scientific practices is necessary to understand the “unreasonable effectiveness of mathematics” (Wigner 1967), in a way that renders it not so “unreasonable”. It is not unreasonable that if mathematics is the result of regularities that “harden” within the frames socio-cultural practices, and then have normative power in those contexts, mathematics is effective: it is tightly integrated within practices in a way that renders it effective; there is nothing supernatural here. It is not “unreasonable” that Euler's conjecture describes polyhedra well, if it arises as a hardened regularity based on our observations of typical polyhedra, and is subsequently entangled with practices, so that it then influences our technique of counting faces, for instance. Hence, Wigner's claim that mathematics “is a wonderful gift which we neither understand nor deserve” (Wigner 1967, p. 237) features an aura of mystery that may be cleared with Wittgensteinian philosophy, just like mathematical flies could be shown the way out of the foundational bottles in Section 4.1. Perhaps the aura of mystery surrounding the unreasonable effectiveness of mathematics is a bewitchment of language, including mathematical language. Perhaps such a bewitchment stems mainly from the prose that accompanies mathematics, from what mathematicians and users of mathematics in general say that they do (Dawson 2014; Kant and Sarikaya 2021; Maddy 1993; Pérez-Escobar and Sarikaya 2022). The angle I propose here

would complement those of other, related lines of research (see, for instance, Molinini (2020) on the “weak objectivity” of mathematics and its “reasonable effectiveness”).

5 Conclusion

This work has been concerned with two main objectives: (1) to justify a place for the later Wittgenstein as a hitherto unrecognized “maverick” (perhaps the first of them) in the context of the philosophy of mathematical practice, and (2) to show how the philosophy of mathematical practice may benefit from the framework that the later Wittgenstein’s philosophy of mathematics provides.

Mathematics, for a very long time already, has enjoyed a privileged position among the sciences. The famous saying that “Mathematics is the queen of all sciences”, ascribed to Gauss (von Waltershausen 1856), is but a token of this lasting conception of mathematics. Indeed, mathematics has been considered a source of immaculate knowledge, more resilient to “contaminating” factors that can be found in the sciences from time to time. Mathematical knowledge is presented to us as true by its own virtue and, if there is something at all worth double-checking in mathematics, it is the nature of its foundations. While the received view in the community of philosophers of the mathematical practice is that Lakatos was the first to break from this traditional philosophy of mathematics, I have argued that the later Wittgenstein preceded him in rejecting apriorism, moving the focus away from the elucidation of foundations, and paying attention to mathematical practices. The problematic availability of his late work on philosophy of mathematics, his literary style and the novelty and polemic character of his ideas are likely causes of a relative lack of attention to the later Wittgenstein’s philosophy of mathematics. I have further argued that two of the main features of the later Wittgenstein’s philosophy of mathematics, namely rule bending and mathematics as rules of description, remain relatively underexplored in the philosophy of mathematical practice in spite of their potential in this context.

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