A predicate calculus with control of derivations

Report**Author(s):**

Mey, Daniel

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Eidgenössische
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Institut für
Theoretische Informatik

Daniel Mey

**A Predicate Calculus with
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Author's address:

**Institut für Theoretische Informatik
ETH-Zentrum
CH-8092 Zürich, Switzerland**

e-mail: mey@inf.ethz.ch

Abstract A fragment of classical predicate calculus which does not contain rules for contraction is defined. It is shown to be decidable and yet propositionally complete. A semantics which reflects its constructive character is developed.

0. Notions *

For an introduction to mathematical logic, see e.g.

Y.I. MANIN. *A coarse in mathematical logic*. Springer (1977)

or

E. MENDELSON. *Introduction to mathematical logic*. Van Nostrand (1964)

or (German)

H-D. EBBINGHAUS. *Einführung in die mathematische Logik*. Wissenschaftliche Buchgemeinschaft Darmstadt (1986).

A **formal language** expresses properties of mathematical objects. A language of **first order predicate logic** consists of an alphabet with symbols for (individual) variables, constants, functions and relations, logical connectives ($\neg$ for negation, $\wedge$ for conjunction, $\vee$ for disjunction, $\rightarrow$ for implication), universal ($\forall$) and existential ($\exists$) quantification over individual variables (in contrast to quantification over higher order variables) and auxiliary symbols (commas and parantheses). The **formulas** of the language are finite strings built up inductively from these symbols. The language of **propositional logics** usually contains propositional variables and symbols for the connectives only (hence no quantifiers!).

The **semantics** of a predicate logic specifies the meaning of the formulas by assigning properties of given mathematical entities. For the semantics of classical first order logic, these entities are **mathematical structures** consisting of a universe of individual objects, mathematical functions, and relations, corresponding to the language considered. A formula is **valid in a structure** if the assigned properties hold (e.g. $\forall x \exists y (y > x)$ is valid in the structure N

* This section is supplementary and not a part of the original paper.

of natural numbers). A formula is **valid** if it is valid in every structure of the language considered.

A **calculus** for a logic consists of formal rules which can be combined in a defined way to form **derivations** (or **formal proofs**) for certain formulas which are called **provable**. A calculus is **sound** with respect to (wrt.) a semantics if every provable formula is valid wrt. the semantics; it is **complete** if every valid formula is provable. Sometimes, completeness can be established by showing that the **Lindenbaum-algebra** of the calculus (consisting of the classes of provably equivalent formulas) is a structure where the provable formulas are precisely the valid ones.

The rules of a **sequent calculus** are not directly concerned with formulas, but with sequents. Sequents are expressions of the form $\Gamma \supset \Pi$ where Γ and Π are finite sequences of formulas. Informally, $\Gamma \supset \Pi$ can be thought of the implication: conjunction of the formulas in Γ implies disjunction of the formulas in Π . A rule consists of zero hypotheses (**axioms**), or one or more hypotheses - and a conclusion. A derivation is a tree with rules as nodes and axioms as leaves. The familiar Gentzen sequent calculus **LK** for classical predicate logic e.g. contains as axioms the sequents $E \supset E$ for prime formulas E (formulas without connectives or quantifiers) and 19 other rules, one of them being the **cut rule** (see section 2) which can be eliminated from the calculus in the sense that every provable formula can be proved without the use of it (**cut-elimination**). The Gödel completeness theorem states soundness and completeness of **LK** wrt. the usual semantics.

Nevertheless, the set of these formulas is **undecidable**: there exists no decision procedure - an algorithm (or program), which determines whether or not an arbitrarily given formula is derivable. But there exist many mechanical methods which help finding proofs. Many of these methods rely on a **unification process**: two expressions containing unification variables are given and it is asked whether there exists a unifier. A unifier is a substitution containing expressions which, substituted for the variables of the given expressions, make them equal. A **most general unifier** is a unifier with the property that every other unifier can be obtained by another substitution.

1. Introduction

Sometimes it is very hard to guess what a derivation of a formula of first order predicate calculus could look like. Church has shown in 1936 that the problem of deciding whether there is a derivation at all is undecidable (see e.g. [BJ]). Krajicek and Pudlak demonstrated in 1988 that deciding whether the formula has a proof with a given skeleton (only terms are not determined) is also undecidable [KP].

The formal reason for these phenomena is that calculi for classical predicate calculus always include rules which are contracting in the sense that a hypothesis can be shorter than a conclusion (measuring e.g. the number of logical symbols of the formulas). In Gentzen's familiar calculus **LK**, these rules are the cut rule and the rules for contraction. If we drop them, we will demonstrate that we get a decidable system. The decision procedure basically evaluates all potential derivations and then computes whether a valid proof has been constructed. This is related to the procedure of determining membership of a word in a context sensitive language.

One modification of such a calculus, already being used as a core of a proof checking system, is studied by Ketonen and Weyhrauch [KW]. It proves the cut rule, but is even propositionally incomplete (e.g. $A \rightarrow A \wedge A$ is not derivable). In Girard's linear logic [Gi], the rules for weakening are also dropped, the only structural rules being those for exchange (see also e.g. [Av] and [GL]). In contrast, the calculus **LS** is designed for proving as many classically valid formulas as possible without including any contracting rules. In section 4, it is shown to be propositionally complete. Formulas B of the form $\exists^* C$ with $\exists^*$ a sequence of existential quantifiers and C quantifier-free are classically provable iff there is a natural number n for which the n -fold disjunction $B \vee \dots \vee B$ is **LS**-provable. This also holds for a calculus with further restrictions on the rules, having the additional property that in that case, there are terms $t_1, \dots, t_n$ for which the disjunction $Ct_1 \vee \dots \vee Ct_n$ is provable. This fragment will be shown to be complete wrt. a semantics developed in section 4. There, validity of a formula is not defined depending on an individual mathematical structure with functions, constants and relations; but depending on a class of structures having the same functions and constants. (A semantics for intuitionistic contraction-free logics is developed by Ono and Komori [OK].)

2. The calculus

The calculus **LS** is a Gentzen sequent calculus and obtained from the classical **LK** by dropping the cut rule and the rules for contraction and by changing the ($\rightarrow$ le)-rule.

Def. **LS** is a Gentzen sequent calculus given by the following axioms and rules. The following notations are used:

capital Greek letters	sequents of formulas
capital Roman letters	formulas (E prime)
$s, t, \dots$	terms
$u, v, \dots$	free variables
$x, y, \dots$	bounded variables
VC!	variable condition: eigenvariable u not in lower sequent
$\supset$	separation sign

AXIOMS

$$E \supset E$$

RULES

$$\frac{\Gamma \supset \Pi}{T, \Gamma \supset \Pi} \quad (\text{W le}) \qquad \frac{\Gamma \supset \Pi}{\Gamma \supset \Pi, T} \quad (\text{W ri})$$

$$\frac{\Gamma, T, Z, \Sigma \supset \Pi}{\Gamma, Z, T, \Sigma \supset \Pi} \quad (\text{E le}) \qquad \frac{\Gamma \supset \Pi, T, Z, \Omega}{\Gamma \supset \Pi, Z, T, \Omega} \quad (\text{E ri})$$

$$\frac{\Gamma \supset \Pi, T}{\neg T, \Gamma \supset \Pi} \quad (\neg \text{le}) \qquad \frac{T, \Gamma \supset \Pi}{\Gamma \supset \Pi, \neg T} \quad (\neg \text{ri})$$

$\frac{T, Z, \Gamma \supset \Pi}{T \wedge Z, \Gamma \supset \Pi} \quad (\wedge \text{le})$	$\frac{\Gamma \supset \Pi, T \quad \Gamma \supset \Pi, Z}{\Gamma \supset \Pi, T \wedge Z} \quad (\wedge \text{ri})$
$\frac{T, \Gamma \supset \Pi \quad Z, \Gamma \supset \Pi}{T \vee Z, \Gamma \supset \Pi} \quad (\vee \text{le})$	$\frac{\Gamma \supset \Pi, T, Z}{\Gamma \supset \Pi, T \vee Z} \quad (\vee \text{ri})$
$\frac{\Gamma \supset \Pi, T \quad Z, \Gamma \supset \Pi}{T \rightarrow Z, \Gamma \supset \Pi} \quad (\rightarrow \text{le})$	$\frac{T, \Gamma \supset \Pi, Z}{\Gamma \supset \Pi, T \rightarrow Z} \quad (\rightarrow \text{ri})$
$\frac{F(t), \Gamma \supset \Pi}{\forall x F(x), \Gamma \supset \Pi} \quad (\forall \text{le})$	$\frac{\Gamma \supset \Pi, F(u)}{\Gamma \supset \Pi, \forall x F(x)} \text{VC!} \quad (\forall \text{ri})$
$\frac{F(u), \Gamma \supset \Pi}{\exists x F(x), \Gamma \supset \Pi} \text{VC!} \quad (\exists \text{le})$	$\frac{\Gamma \supset \Pi, F(t)}{\Gamma \supset \Pi, \exists x F(x)} \quad (\exists \text{ri})$

According to Girard's terminology, the formulations of the three rules with two hypotheses are called **additive**; the corresponding **multiplicative** formulations are as follows:

$\frac{\Gamma \supset \Pi, T \quad \Sigma \supset \Omega, Z}{\Gamma, \Sigma \supset \Pi, \Omega, T \wedge Z} \quad (\wedge \text{ri.m})$
$\frac{T, \Gamma \supset \Pi \quad Z, \Sigma \supset \Omega}{T \vee Z, \Gamma, \Sigma \supset \Pi, \Omega} \quad (\vee \text{le.m})$
$\frac{\Gamma \supset \Pi, T \quad Z, \Sigma \supset \Omega}{T \rightarrow Z, \Gamma, \Sigma \supset \Pi, \Omega} \quad (\rightarrow \text{le.m})$

Modifications of **LS** are obtained by replacing one or several of the additive formulations by multiplicative ones. In the presence of contraction, however, all these modifications are equivalent: a proof in one calculus can be transformed into a proof in the other by weakenings and contractions.

Below, two important properties of **LS** are proved.

PROPERTY 1. **LS** is properly weaker than **LK**.

PROOF. An **LS**-proof Q can be transformed into an **LK**-proof inductively on the depth of Q by inserting contractions after $(\rightarrow \text{le})$ -inferences. The converse is false: take e.g. $\exists x(R(a) \vee R(b) \rightarrow R(x))$ provable in **LK** but not in **LS**: The last rule applied in a proof would be $(\exists \text{ ri})$ with an upper sequent $\supset R(a) \vee R(b) \rightarrow R(t)$ and consequently both $R(a) \supset R(t)$ and $R(b) \supset R(t)$ would be provable for a term t , which is impossible.

•

PROPERTY 2. The cut rule
$$\frac{\Gamma \rightarrow \Pi, A \quad A, \Sigma \rightarrow \Omega}{\Gamma, \Sigma \rightarrow \Pi, \Omega} \text{ (CUT)}$$
 is not valid in **LS**.

PROOF. Let $B : \exists x(R(a) \vee R(b) \rightarrow R(x))$ as in the proof above. $B \vee B$ is provable: $\supset B \vee B$ is provable $\Leftarrow$
 $\supset R(a) \vee R(b) \rightarrow R(a), R(a) \vee R(b) \rightarrow R(b)$ is provable $\Leftarrow$
 $R(a) \vee R(b), R(a) \vee R(b) \supset R(a), R(b)$ is provable $\Leftarrow$
 $R(a) \vee R(b) \supset R(a), R(b)$ is provable, which is true.
 Furthermore, $B \vee B \supset B$ is clearly provable. If the cut rule was valid, a proof of B would follow, which was shown to be impossible in the proof above.

•

3. Decidability

The decision procedure for **LS** described below is divided into two steps:

1. Evaluate all potential derivations of a given formula from bottom to the top. This is possible because the upper sequent(s) of an inference are in a certain sense shorter than the lower one(s). However, the terms in $(\forall i e)$ and $(\exists i i)$ have to be replaced by special variables.
2. Compute whether the potential derivations of step 1 contain a correct one: in order to do this, determine the terms in $(\forall i e)$ and $(\exists i i)$ by a unification process.

The following notation is used.

Def. **LS'** is obtained from **LS** by:

- dropping exchange-rules
- allowing arbitrary permutations of formulas in the sequents of the rules.

REMARK. **LS'** and **LS** are equivalent: a proof in one of the calculi can be transformed into one in the other by inserting or deleting exchange-inferences.

Def. **LS(b)** is obtained from **LS'** by the following:

- language extended with **term-variables** $a_1, \dots, a_b$ and new free variables $v_1, \dots, v_b$
- axioms have the form $R(\dots) \supset R(\dots)$
- eigenvariables at $(\forall i i)$ and $(\exists i e)$ are among $v_1, \dots, v_b$. No variable conditions!
- terms t at $(\forall i e)$ and $(\exists i i)$ are among $a_1, \dots, a_b$.

LEMMA 1. Let S be a sequent, b natural number. Then all **LS(b)**-proofs of S can be computed.

PROOF. By induction on $n(S)$, the number of logical symbols plus number of predicate signs in S .

$n(S)=0$ The only possible proof of S is S itself (conclusions C of rules have $n(C)>0$). So just decide whether S is an axiom.

$n(S)\geq 0$ Determine all possible hypotheses of inferences with

conclusion S . Note that eigenvariables and terms of quantifier-inferences have to be chosen among $v_1, \dots, v_b$ and $a_1, \dots, a_b$ respectively. Now, for each of the determined hypothesis H , $n(H) < n(C)$. Therefore, by induction, all $LS(b)$ -proofs of H are obtained. These proofs, combined with the corresponding inferences, yield all $LS(b)$ -proofs of S .

.

Def. Let P^* be an $LS(b)$ -proof. The **unification problem for P^*** is the unification problem for the pairs of terms corresponding to each other in the axioms of P^* , where the only variables of the problem are the term-variables.

THEOREM 1. The following algorithm decides LS :

Input: sequent S with $n(S)=b$.

Output: an LS' -proof, if S is provable; else: message "not provable".

1. Compute all $LS(b)$ -proofs of S .
2. If there is a P^* computed in step 1 for which the unification problem is solvable by a most general unifier (mgu) σ and for which $P^*\sigma$ doesn't violate the variable conditions, return $P^*\sigma$; else: message.

PROOF. termination Lemma 1 implies that step 1 is computable. For each P^* computed in step 1, the decisions of step 2 are computable (use Robinson-algorithm for deciding the unification problem).

correctness $P^*\sigma$ returned Because σ is a unifier for the pairs of terms in the axioms of P^* , the initial sequents of $P^*\sigma$ are LS' -axioms. The inferences of P^* are transformed into inferences of $P^*\sigma$. Note that the variable conditions are satisfied. Finally, the last sequent of $P^*\sigma$ is S , because it does not contain any term-variable. Therefore, $P^*\sigma$ is an LS' -proof of S and S is LS -provable.

"not provable" returned Suppose S would be provable. It will be proved that no message would be returned, which is a contradiction.

Let Q be a regular LS' -proof of S (one with different eigenvariables, none of them contained in S). Such a proof can be easily obtained

from an arbitrary one by a renaming of variables). Replace the eigenvariables by variables among $v_1, \dots, v_b$, obtaining the LS' -proof P of S . Then, replace in P the terms $t_1, \dots, t_b$ of $(\forall \text{le})$ and $(\exists \text{ri})$ at these inferences, and above them, by term-variables among $a_1, \dots, a_b$, yielding an $LS(b)$ -proof P^* of S . The unification problem of P^* is solvable, $\alpha = \{a_1/t_1, \dots, a_n/t_n\}$ is a unifier. Let σ be an mgU. It remains to show that $P^*\sigma$ satisfies the variable conditions like $P^*\alpha = P$ does. Let H^*/C^* be a quantifier-inference of P^* with eigenvariable w . Suppose $C^*\sigma$ contains w . Let γ be a substitution with $\alpha = \sigma\gamma$. Then w is also contained in $C^*\alpha$. Therefore, $H^*\alpha / C^*\alpha$ violates the variable condition of $P^*\alpha$. Hence contradiction. So $P^*\sigma$ satisfies the variable conditions and thus no message will be returned. $\bullet$

REMARKS

1. The above algorithm is not sensitive to additive or multiplicative formulations of the rules. Hence, it is a decision procedure for the modifications of LS as well.
2. The algorithm demonstrates that an LS -proof can contain sequents with many symbols although it proves a sequent with few symbols. Take e.g.

$S: \forall x \forall z R(x, fxx, z) \supset \exists y R(fcc, y, fyy)$ which has the following $LS(b)$ -proof P^* :

$$\begin{array}{rcl}
 R(a_1 f a_1 a_1, a_3) \supset R(fcc, a_2 f a_2 a_2) & & \\
 \hline
 \forall z R(a_1 f a_1 a_1, z) \supset R(fcc, a_2 f a_2 a_2) & & (\forall\text{-li}) \\
 \hline
 \forall x \forall z R(x, fxx, z) \supset R(fcc, a_2 f a_2 a_2) & & (\forall\text{-li}) \\
 \hline
 \forall x \forall z R(x, fxx, z) \supset \exists y R(fcc, y, fyy) & & (\exists\text{-re})
 \end{array}$$

The unification problem is solved by the mgU

$\sigma = \{a_1/fcc, a_2/ffccfcc, a_3/f(ffccfccffccfcc)\}$. $P^*\sigma$ is an LS -proof of S . S contains 18 symbols (commas and paranthesises not counted), but the first sequent contains 56!

4. Propositional part

All rules for connectives of **LS** are reversible: the lower sequent is provable iff the upper one(s) is (are). This implies the provability of quantifier-free (qf.) contractions. As a consequence, the qf-parts of **LK** and **LS** are shown to be equivalent. This is used to establish a simple syntactic relation between the two calculi.

LEMMA 2. All **LS**-rules for connectives are reversible.

PROOF. The lemma will be shown for **LS'** (which is equivalent). Let P be a proof of the conclusion C of a rule for connectives. Proof by cases:

($\neg$ li) $C: \neg B, \Delta \rightarrow A$. Proof by induction on the depth of P :

$d(P)=0$ impossible.

$d(P)>0$ Subcases:

$$\frac{\Gamma \supset \Pi}{T, \Gamma \supset \Pi} \quad (\text{W le})$$

If T is not $\neg B$, then apply the ind. hyp. and a (W le)-inference. If T is $\neg B$, then replace the inference by a (W ri) with weakening formula T .

$$\frac{\Gamma \rightarrow \Pi, T}{\neg T, \Gamma \rightarrow \Pi} \quad (\neg \text{le})$$

If $\neg T$ is not $\neg B$, then apply the ind. hyp. and a ($\neg$ le)-inference. If $\neg T$ is $\neg B$, then drop the inference.

$$\frac{\neg B, \Gamma \supset \Pi, T \quad \neg B, \Gamma \supset \Pi, Z}{\neg B, \Gamma \supset \Pi, T \wedge Z} \quad (\wedge \text{ri})$$

By induction proofs of $\Gamma \supset \Pi, T, B$ and $\Gamma \supset \Pi, Z$ are obtained. Now apply ($\wedge$ ri).

other subcases... by induction.

($\neg$ ri), ($\wedge$ le), ($\wedge$ ri), ($\vee$ le), ($\vee$ ri), ($\rightarrow$ le), ($\rightarrow$ ri) similar.

REMARK. Observe that the proofs of the hypotheses are obtained from the proofs of its conclusions without incrementing their depths.

LEMMA 3. For qf. formulas, (C le) and (C ri) are provable.

PROOF The lemma is proved for (C le); it is similar for (C ri). By induction on the depth of a proof Q of $B, B, \Delta \supset \Lambda$ it is shown that $B, \Delta \supset \Lambda$ is provable as well, provided B is qf.

$d(Q)=0$ impossible!

$d(Q)>0$ cases:

$$\frac{\Gamma \supset \Pi}{T, \Gamma \supset \Pi} \quad (\text{W le})$$

If T is not B , then apply the ind. hyp. and a (W le)-inference. If T is B , then drop the inference.

$$\frac{\Gamma \supset \Pi, T}{\neg T, \Gamma \supset \Pi} \quad (\neg \text{le}), \neg T \text{ is not } B$$

Apply the ind. hyp. and a $(\neg \text{le})$ -inference.

$$\frac{\neg T, \Gamma \supset \Pi, T}{\neg T, \neg T, \Gamma \supset \Pi} \quad (\neg \text{le}), \neg T \text{ is } B$$

Lemma 2 yields a proof of $\Gamma \supset \Pi, T, T$ with the same depth as the proof of

$\neg T, \Gamma \supset \Pi, T$. Now, applying the ind. hyp., a proof for $\Gamma \supset \Pi, T$ and with $(\neg \text{le})$ one for $\neg T, \Gamma \supset \Pi$ is obtained.

(other cases) similar!

THEOREM 2. The quantifier-free parts of LS and LK are equivalent.

PROOF. Property 1 shows that if a qf. formula B is provable in LS, then also in LK. Lemma 3 now implies the converse: let Q be a classical proof of B . Applying cut-elimination, Q can be assumed cut-free. An LS-proof of B can be obtained from Q by inserting

weakenings at $(\rightarrow \text{le m})$ -inferences and by applying lemma 3 at contractions.

REMARK. Theorem 2 does not hold for modifications of **LS** where one or more of the additive rules are replaced by their multiplicative formulations. If the modification contains $(\wedge \text{ri})$, then it is easily verified that the classically valid formula $E \rightarrow E \wedge E$ is not provable. If it contains $(\vee \text{le})$, then $E \vee E \rightarrow E$ is not provable. Finally, if it contains $(\rightarrow \text{le})$, then $(\neg E \rightarrow E) \rightarrow E$ is not provable.

For formulas of the form $\exists^* C$ with $\exists^*$ a sequence of existential quantifiers and C quantifier-free, theorem 2 can be used to formulate a simple syntactic relation between the two calculi. Because every formula is classically equivalent to a formula in the above form, its classical provability can be reduced to a provability-problem in **LS**.

COROLLARY 1. Let B be of the form $\exists^* C$ with $\exists^*$ a sequence of existential quantifiers and C quantifier-free. Then B is **LK**-provable iff there is a natural number n for which the n -fold disjunction $B \vee \dots \vee B$ is **LS**-provable.

PROOF If $B \vee \dots \vee B$ is **LS**-provable, then, by reversibility of $(\vee \text{re})$, $\supset B, \dots, B$ is provable in **LS** and therefore in **LK**. Now, apply contractions and obtain a proof of B .

For the converse, take an **LK**-proof of B in Gentzen normal form (containing a qf. "midsequent" with no propositional inferences below it). The midsequent of this proof must have the form $\supset C(t_1), \dots, C(t_n)$ for tuples of terms $t_1, \dots, t_n$. Because of the propositional completeness of **LS**, this sequent is also **LS**-provable. $(\exists \text{ri})$ - and possibly exchange-inferences now provide an **LS**-proof of the n -fold disjunction $B \vee \dots \vee B$.

5. Semantics for a fragment

The cut rule is not valid in **LS** (see section 2). Related to that, $\equiv$: "LS-provable equivalent" is not transitive and therefore not an equivalence relation. E.g. if B is

$\exists x(Ra \vee Rb \rightarrow Rx)$ then $Rc \vee \neg Rc \equiv B \vee B$. $B \vee B \equiv B$ as well, but $Rc \vee \neg Rc \equiv B$ is false as can easily be seen. Therefore, we can't construct a Lindenbaum-algebra for **LS**, which could serve as a basis for an algebraic semantics.

The main motivation behind developing a semantics for a fragment of the calculus is not to have a tool for demonstrating a formula to be unprovable (this can be done without semantics because the calculus was shown to be decidable), but to give an example for a semantics which is propositionally complete and yet violates the cut rule.

The semantics described below reflects the constructive character of the calculus: if $\exists xBx$ is **LS**-provable, then there is a term t such that Bt is provable. However, if $\Gamma \supset \Pi$, $\exists xBx$ is provable, one can not always find a t such that $\Gamma \supset \Pi, Bt$ is provable. As an example, $Ra \vee Rb \supset \exists xBx$ is provable, but $Ra \vee Rb \supset Bt$ is not provable for any term t . We therefore modify **LS** appropriately:

Def. **LS_r** : **LS** but

- no $(\forall \text{ ri})$ or $(\exists \text{ le})$
- rules with two hypotheses: all formulas except principal formula are quantifier-free
- in sequents, each quantifier bounds a different variable.

LS_r' is obtained from **LS_r** as **LS'** is defined from **LS**.

Validity of formulas are not defined wrt. a single mathematical structure, but wrt. a class of structures which is determined by a universe, functions and constants corresponding to the language L considered.

Def. L : $\{f_i\}$, $\{c_j\}$, $\{R_k\}$ first order language.

L-class $\underline{A}$ $\{A \mid A \text{ } L\text{-structure with } |A|=A, f_i^A = f_i^A, c_j^A = c_j^A\}$ with $A \neq \emptyset, \{f_i^A\}, \{c_j^A\}$ functions and constants corresponding to L .

L(e) : L plus **e-variables**: new free variables $e_1, e_2, \dots$

w: assignment of free variables of L

o: assignment of e-variables ("e-assignment").

B $L(e)$ -formula, $A \in \underline{A}$, w, o assignments of free and e-variables.

$[B]_{w,o,A}$ defined inductively:

- i) $[E]_{w,o}, [\neg C]_{w,o}, [C * D]_{w,o}$ (* is one of $\wedge, \vee$ or $\rightarrow$) defined classically
- ii) $[Qx_k C(x_k)]_{w,o} = [C(ek)]_{w,o}$ where Q is $\forall$ or $\exists$
- iii) $[A_1, \dots, A_n \supset B_1, \dots, B_m]_{w,o} = [\neg A_1 \vee \dots \vee B_1 \vee \dots]_{w,o}$
(defined classically).

B **valid in $\underline{A}$** (at w): exists o such that for all $A \in \underline{A}$: $[B]_{w,o,A} = \text{true}$

B **e-valid**: B $\underline{A}$ -valid in every L -class $\underline{A}$ at every assignment w .

MOTIVATION. An L -class determines all objects and operations of a mathematical structure, but not the properties of these. A formula is true in the class at an assignment iff there is an e-assignment such that the formula with deleted quantifiers is (classically) true in all structures of the class, provided the bounded variables are assigned to the corresponding components of o . (o can be regarded as "oracle" for the bounded variables, making the formula true however the predicates are interpreted.) If B is an existential formula where each quantifier bounds a different variable, then the definition becomes classical, if in the definition of validity in a class, "exists o " and "for all A " are changed.

EXAMPLES $B_k : \exists x_k (R(a) \vee R(b) \rightarrow R(x_k))$

1. B_k is not e-valid:

Let $\underline{A}$ be such that $a^A \neq b^A$, o arbitrary. Then either a^A or b^A is different from o_k . Assume without loss of generality that $o_k \neq a^A$. Let $A \in \underline{A}$ be such that $a^A \in R^A$ but not $o_k \in R^A$. This implies $[B_k] = [R(a) \vee R(b) \rightarrow R(ek)] = \text{false}$. Hence, the formula is not valid in $\underline{A}$ and thus not e-valid.

2. If A has one element, then B_k is valid in $\underline{A}$:

Because A has just one element c , o must be a constant assignment. Furthermore, $a^A = b^A = c$. Then $[B_k]_{w,o} = [R(a) \vee R(b) \rightarrow R(ek)]_{w,o} = \text{true}$ and the formula is valid in $\underline{A}$.

3. $B_1 \vee B_2$ is e-valid:

Let $\underline{A}$ be a class. Choose $o_1 = a^A$, $o_2 = b^A$. By definition, $[B_1 \vee B_2]_{w,o} = [B_1]_{w,o} \vee [B_2]_{w,o} = [R(a) \vee R(b) \rightarrow R(e_1)]_{w,o} \vee [R(a) \vee R(b) \rightarrow R(e_2)]_{w,o}$. If neither a^A nor b^A are in R^A , then both parts of the disjunction are true. If a^A is in R^A , then the left part is true. Finally, if b^A is in R^A then the right part is true.

4. $R(a) \vee R(b) \rightarrow \exists x_k R(x_k)$ is not e-valid:

Let $\underline{A}$ be the class of example 1. Then $[R(a) \vee R(b) \rightarrow \exists x_k R(x_k)] = [R(a) \vee R(b)] \rightarrow [R(e_k)] = [R(a) \vee R(b) \rightarrow R(e_k)] = \text{false}$ according to example 1.

Below we will prove that wrt. e-validity, LS_r is sound and complete. Showing completeness is much easier than showing soundness: If a sequent is valid, then it is valid in the so called canonical class under the canonical assignment described below. From this, a proof of the sequent may be obtained directly. For soundness, the quantifier inferences of the given proof must be encoded in a certain way.

LEMMA 4. B L -formula. If assignments w, w' agree on the components corresponding to free variables of B and if e-assignments o, o' agree on the components corresponding to the bounded variables of B , then $[B]_{w,o} = [B]_{w',o'}$.

PROOF By induction on B , the lemma is proven for B an $L(e)$ -formula with the additional assumption that o, o' also agree on the corresponding e-variables.

B prime w, w' and o, o' agree on the (free) variables of B which immediately implies the statement.

$B : \exists x_k C(x_k)$ $[\exists x_k C(x_k)]_{w,o} = [C(e_k)]_{w,o} = (\text{ind. hyp.}) [C(e_k)]_{w',o'} = [\exists x_k C(x_k)]_{w',o'}$.

$B : \forall x_k C(x_k)$ identically!

other cases ... with induction hypothesis.

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LEMMA 5. S L -sequent where each quantifier bounds a different variable. If S is an axiom or a conclusion of an LS_r' -inference with

hypotheses valid in $\underline{A}$, then S is valid in $\underline{A}$.

PROOF. Let $\underline{A}$ be a class and w an assignment. Cases:

S an axiom Then S is $E \supset E$. Choose o arbitrarily. Let A be in $\underline{A}$.

Then $[E \supset E]_{w,o} = [\neg E \vee E]_{w,o} = \text{true}$.

other cases The hypotheses H^1 (and H^2) of the inference are valid in $\underline{A}$. There exist o^1 (and o^2) such that for all $A \in \underline{A}$: $[H]_{w,o^1,A} = \text{true}$ (and $[H']_{w,o^2,A} = \text{true}$). Subcases:

$$\frac{\Gamma \supset \Pi}{T, \Gamma \supset \Pi} \quad (\text{W le})$$

$$[T, \Gamma \supset \Pi]_{w,o} = [\neg T]_{w,o} \vee [\Gamma \supset \Pi]_{w,o} = \text{true for } o = o^1.$$

(W ri) analogous.

$$\frac{\Gamma \supset \Pi, T}{\neg T, \Gamma \supset \Pi} \quad (\neg \text{le})$$

$$[\neg T, \Gamma \supset \Pi]_{w,o} = [\neg \neg T]_{w,o} \vee [\Gamma \supset \Pi]_{w,o} = [T]_{w,o} \vee [\Gamma \supset \Pi]_{w,o} = [H]_{w,o} = \text{true for } o = o^1.$$

($\neg$ ri) analogous.

$$\frac{T, Z, \Gamma \supset \Pi}{T \wedge Z, \Gamma \supset \Pi} \quad (\wedge \text{le}) \text{ clear.}$$

$$\frac{\Gamma \supset \Pi, T \quad \Gamma \supset \Pi, Z}{\Gamma \supset \Pi, T \wedge Z} \quad (\wedge \text{ri})$$

o is obtained from o^1 by replacing the components corresponding to bounded variables of Z by the same components of o^2 . Because $\Gamma \supset \Pi$ is quantifier-free, lemma 4 implies that $[\Gamma \supset \Pi]_{w,o} = [\Gamma \supset \Pi]_{w,o^1} = [\Gamma \supset \Pi]_{w,o^2}$. Because each quantifier bounds a different variable, the bounded variables of Z and T are different. Lemma 4 therefore implies $[T]_{w,o} = [T]_{w,o^1}$. Finally, because o and o^2 agree on the components corresponding to the bounded variables of Z , lemma 4 implies $[Z]_{w,o} = [Z]_{w,o^2}$. Together, $[\Gamma \supset \Pi, T \wedge Z]_{w,o} = [\Gamma \supset \Pi, T]_{w,o} \wedge [\Gamma \supset \Pi, Z]_{w,o} = [\Gamma \supset \Pi, T]_{w,o^1} \wedge [\Gamma \supset \Pi, Z]_{w,o^2} = \text{true}$.

other propositional rules similarly.

$$F(t), \Gamma \supset \Pi$$

($\forall$ le)

$$\frac{\forall x_k F(x_k), \Gamma \supset \Pi}{F(t), \Gamma \supset \Pi}$$

Let $o = o^1[e_k / [t]_w]$. Then, $[\neg \forall x_k F(x_k)]_{w,o} = [\neg F(e_k)]_{w,o} = [\neg F(t)]_{w,o}$. Because neither $\neg F(t)$ nor $\Gamma \supset \Pi$ contains x_k , lemma 4 implies $[\neg F(t)]_{w,o} = [\neg F(t)]_{w,o^1}$ and $[\Gamma \supset \Pi]_{w,o} = [\Gamma \supset \Pi]_{w,o^1}$. Therefore, $[\forall x_k F(x_k), \Gamma \supset \Pi]_{w,o} = [\neg \forall x_k F(x_k)]_{w,o} \vee [\Gamma \supset \Pi]_{w,o} = [\neg F(t)]_{w,o^1} \vee [\Gamma \supset \Pi]_{w,o^1} = [H]_{w,o^1} = \text{true}$.

($\exists$ ri) similarly.

THEOREM 3. (soundness) S L -sequent where each quantifier bounds a different variable. If S is LS_r -provable, then it is e -valid.

PROOF. By induction on Q where Q is a proof of S .

$d(Q)=0$ S is an axiom. Lemma 5 implies it is e -valid.

$d(Q)>0$ S is conclusion of an inference. Its upper sequent(s) are provable. The ind. hyp. implies e -validity of them, lemma 5 implies e -validity of the conclusion.

Def. i) A_{kan} canonical L -class: A_k : L -terms, $f_i A$, $c_j A$ canonical.
 ii) **w_{kan} canonical l -assignment:** $w_{kan}(u)=u$.

LEMMA 6. If the conclusion of an LS_r -inference (not a weakening) is valid in A_{kan} at w_{kan} , then its hypotheses are also valid for appropriate terms if the inference is ($\forall$ le) or ($\exists$ ri).

PROOF. Cases:

($\neg$ le), ($\neg$ ri), ($\wedge$ le), ($\vee$ ri), ($\rightarrow$ ri) compare corresponding cases of lemma 5: $[C]_{w,o} = [H]_{w,o}$ for arbitrary w, o .

$$\frac{\Gamma \supset \Pi, T \quad \Gamma \supset \Pi, Z}{\Gamma \supset \Pi, T \wedge Z} \quad (\wedge \text{ ri})$$

$[\Gamma \supset \Pi, T \wedge Z]_{w,o} = [\Gamma \supset \Pi, T]_{w,o} \wedge [\Gamma \supset \Pi, Z]_{w,o}$ for arbitrary w and o .

$$\frac{\Gamma \supset \Pi, F(t)}{\Gamma \supset \Pi, \exists x F(x)} \quad (\exists \text{ ri})$$

$w = w_{kan}$. $[\Gamma \supset \Pi, \exists x F(x)]_{w,o} = [\Gamma \supset \Pi]_{w,o} \vee [F(e_k)]_{w,o}$. Because w is canonical, $[F(e_k)]_{w,o} = [F(t)]_{w,o}$ where $t = o(e_k)$ is a term. Therefore the hypothesis of the inference is valid for an appropriate term in A_{kan} at w_{kan} , provided the conclusion is.

$(\forall \text{ le})$ analogously.

THEOREM 4. (completeness) L -sequent S is valid in A_{kan} at w_{kan} iff it is $LS_{\mathbf{r}}$ -provable.

PROOF. $\Leftarrow$ Theorem 3.

$\Rightarrow$ By induction on $|S|$ where $|S|$ is number of logical symbols in S .

$n=0$ $S: A_1, \dots, A_n \supset B_1, \dots, B_m$ only consists of prime formulas. If every A_i would be different from every B_j , then an $A \in \underline{A_{kan}}$ could be constructed which would make all A_i 's true and all B_j 's false. Therefore, $[S]_A = \text{false}$ and S would not be valid in A_{kan} at w_{kan} . It follows that an A_i and a B_j are identical. Hence, S can be proven with an axiom followed by weakenings.

$n \geq 0$ S is conclusion of an inference except weakening with hypotheses H_1 (and H_2).

Lemma 6 implies that they are valid in A_{kan} at w_{kan} . Because they have at least one logical symbol less than S , the induction hypothesis provides proofs for them which can be extended to a proof of S by combining them with the inference.

COROLLARY 2 Sequents are e-valid iff they are $LS_{\mathbf{r}}$ -provable.

PROOF Because $LS_{\mathbf{r}}$ and $LS_{\mathbf{r}}'$ are equivalent, it suffices to prove that a sequent S is e-valid iff it is $LS_{\mathbf{r}}'$ -provable. Let S be e-valid. Then S is valid in A_{kan} at w_{kan} . Theorem 4 implies the provability of S . If S is provable, then there exists a proof where every quantifier of a sequent bounds a different variable (rename the bounded variables of the proof). Theorem 3 now implies the validity of S .

REMARK Example 4 above demonstrated that $R(a) \vee R(b) \rightarrow \exists x_k R(x_k)$ is not e-valid. Soundness of $\mathbf{LS}_r$ implies that the formula is not $\mathbf{LS}_r$ -provable. Nevertheless, as mentioned before, the formula is provable in $\mathbf{LS}$. But the proof uses an $(\vee \text{le})$ -inference with upper sequents $R(a) \rightarrow \exists x_k R(x_k)$ and $R(b) \rightarrow \exists x_k R(x_k)$, thereby violating the $\mathbf{LS}_r$ -condition for the rules with two hypotheses.

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