Divide and conquer

a new parallel algorithm for the solution of a tridiagonal linear system of equations

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**Divide and Conquer:
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for the Solution of a
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Abstract

We describe a divide and conquer algorithm which solves linear tridiagonal systems with one right-hand side, especially suited for parallel computers. The algorithm is very flexible, permits multiprocessing or a combination of vector and multiprocessor implementations, and is adaptable to a wide range of parallelism granularities. This algorithm can also be combined with recursive doubling, cyclic reduction or Wang's partition method, in order to increase the degree of parallelism and vectorizability.

The divide and conquer method will be explained. Some results of time measurements on a CRAY X-MP/28, on an Alliant FX/8 and on a Sequent Symmetry S81b as well as comparisons with the cyclic reduction algorithm and Gaussian elimination will be presented. Finally some numerical results are given.

1 Introduction

The solution of tridiagonal systems of n linear equations $T\vec{x} = \vec{d}$ with one right-hand side $\vec{d}$ is a central part of many programs for scientific computation. To solve this problem on parallel computers an efficient algorithm should be available.

Divide and conquer is one of the most commonly used tools in the construction of algorithms. The basic idea underlying the technique is to solve a given problem of a certain size, by dividing the problem up into similar problems of smaller size, solving them, and then combining those solutions to form the solution of the original problem. Most commonly, the same technique is applied to the smaller problems, making the procedure recursive. Such divide and conquer algorithms are excellently suited for parallel computers.

For a scalar machine the best direct solution method is Gaussian elimination, specialized for tridiagonal systems. For vector computers, cyclic reduction which was originally proposed by Hockney and Golub [7], is more efficient, since all essential arithmetic operations can be vectorized [8, pp 286]. To solve very large systems on a vector computer, Wang's Partition Method [21] is also well suited.

On parallel computers with two CPU's the two-sided Gaussian elimination algorithm (twisted factorization [20]), discussed by Babuska [2], is very useful. The decomposition of the matrix T , as well as the subse-

quent solution process, can be performed in two parallel parts. Notable among parallel algorithms is also the recursive doubling method [17]. For the special case where T is symmetric and of Toeplitz form, Evans [6, pp 371] proposed a divide and conquer method (based on Cramer's Rule) to solve $T\vec{x} = \vec{d}$ on parallel computers.

In this paper, however, we describe a divide and conquer algorithm for solving general linear tridiagonal systems especially suitable for parallel computers (Multiple-Instruction-Multiple-Data systems [13, p 8]). The algorithm is very flexible, permitting multiprocessing or a combination of vector and multiprocessor implementations, and is adaptable to a wide range of the size of parallelism. (In the following sections, we refer to the new method as the "divide and conquer method" or $\mathcal{DAC}(p, k)$.)

For the following considerations let us assume that the matrix T is diagonally dominant (or at least positive definite), which is generally assumed for the algorithm as Gaussian elimination, cyclic reduction or Wang's partition method. This condition must be stated because of numerical stability. The algorithm which will be discussed needs, from a theoretical view, weaker assumptions.

The paper is arranged as follows: We will illustrate the algorithm with an example prior to formally presenting the algorithm. Then the divide and conquer method will be explained for any p , where p preferably is the number of CPUs of the parallel computer and $n=kp$ ($k, p \in \mathbb{N}$). In a third part, numerical results are reported for a FORTRAN program using the divide and conquer method. Then CPU-time measurements on a CRAY X-MP/28, on a Alliant FX/8, and on a Sequent Symmetry S81b are given and discussed. Finally we compare the CPU-times of the divide and conquer method with those of the cyclic reduction algorithm and Gaussian elimination.

2 Divide and Conquer Method

2.1 Divide and Conquer Method: three parallel tasks

Consider the tridiagonal linear system $T\vec{x} = \vec{d}$ with $n=3k$ equations ($1 < k \in \mathbb{N}$). We partition T into block tridiagonal form with each diagonal block a $k \times k$ tridiagonal matrix and each subdiagonal (superdiagonal) block a $k \times k$ null matrix, except for one single nonzero element on its

upper right (lower left) corner:

$$T = \left(\begin{array}{ccc|ccc|ccc} a_1 & c_1 & & & & & & & \\ & b_2 & a_2 & & & & & & \\ & & \ddots & & & & & & \\ & & & c_{k-1} & & & & & \\ & & b_k & a_k & c_k & & & & \\ & & & b_{k+1} & a_{k+1} & c_{k+1} & & & \\ & & & & b_{k+2} & a_{k+2} & & & \\ & & & & & \ddots & & & \\ & & & & & & c_{2k-1} & & \\ & & & & & b_{2k} & a_{2k} & c_{2k} & \\ & & & & & & b_{2k+1} & a_{2k+1} & c_{2k+1} \\ & & & & & & & b_{2k+2} & a_{2k+2} \\ & & & & & & & & \ddots \\ & & & & & & & & & c_{n-1} \\ & & & & & & & & & b_n & a_n \end{array} \right) \quad (1)$$

Let us suppose $b_{k+1}, b_{2k+1}, c_k, c_{2k} \neq 0$. We write the tridiagonal system in block form as

$$\left(\begin{array}{c|c|c} T_1 & C_1 & 0 \\ \hline B_2 & T_2 & C_2 \\ \hline 0 & B_3 & T_3 \end{array} \right) \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vec{x}_3 \end{pmatrix} = \begin{pmatrix} \vec{d}_1 \\ \vec{d}_2 \\ \vec{d}_3 \end{pmatrix} \quad (2)$$

where $\vec{x}$ and $\vec{d}$ are partitioned to conform with T .

With the first and the last unit vector $\vec{e}_1, \vec{e}_k \in \mathbb{R}^k$, the upper- and lower-diagonal blocks of (1) are given by

$$\begin{aligned} B_2 &= b_{k+1} \vec{e}_1 \vec{e}_k^T & B_3 &= b_{2k+1} \vec{e}_1 \vec{e}_k^T \\ C_1 &= c_k \vec{e}_k \vec{e}_1^T & C_2 &= c_{2k} \vec{e}_k \vec{e}_1^T \end{aligned} \quad (3)$$

With the notation introduced above expanding (2) and rearranging, we obtain

$$\vec{x}_1 = T_1^{-1} \vec{d}_1 - c_k T_1^{-1} \vec{e}_k \vec{e}_1^T \vec{x}_2 \quad (4)$$

$$\vec{x}_2 = T_2^{-1} \vec{d}_2 - b_{k+1} T_2^{-1} \vec{e}_1 \vec{e}_k^T \vec{x}_1 - c_{2k} T_2^{-1} \vec{e}_k \vec{e}_1^T \vec{x}_3 \quad (5)$$

$$\vec{x}_3 = T_3^{-1} \vec{d}_3 - b_{2k+1} T_3^{-1} \vec{e}_1 \vec{e}_k^T \vec{x}_2 \quad (6)$$

By defining

$$\begin{aligned}\vec{y}_1 &= T_1^{-1} \vec{d}_1 & \vec{y}_2 &= T_2^{-1} \vec{d}_2 & \vec{y}_3 &= T_3^{-1} \vec{d}_3 \\ \vec{z}_1 &= T_1^{-1} \vec{e}_k & \vec{z}_2 &= T_2^{-1} \vec{e}_1 & \vec{z}_3 &= T_2^{-1} \vec{e}_k & \vec{z}_4 &= T_3^{-1} \vec{e}_1\end{aligned}\quad (7)$$

we have the following implicit relations

$$\vec{x}_1 = \vec{y}_1 - c_k(\vec{e}_1^T \vec{x}_2) \vec{z}_1 \quad (8)$$

$$\vec{x}_2 = \vec{y}_2 - b_{k+1}(\vec{e}_k^T \vec{x}_1) \vec{z}_2 - c_{2k}(\vec{e}_1^T \vec{x}_3) \vec{z}_3 \quad (9)$$

$$\vec{x}_3 = \vec{y}_3 - b_{2k+1}(\vec{e}_k^T \vec{x}_2) \vec{z}_4. \quad (10)$$

For suitable $\alpha_i \in \mathbb{R}$, (8), (9) and (10) can be written in the form

$$\begin{aligned}\vec{x}_1 &= \vec{y}_1 + \alpha_1 \vec{z}_1 \\ \vec{x}_2 &= \vec{y}_2 + \alpha_2 \vec{z}_2 + \alpha_3 \vec{z}_3 \\ \vec{x}_3 &= \vec{y}_3 + \alpha_4 \vec{z}_4\end{aligned}\quad (11)$$

By substituting (11) in (4) and solving for $\vec{d}_1$

$$\begin{aligned}T_1 \vec{y}_1 + \alpha_1 T_1 \vec{z}_1 + c_k(\vec{e}_1^T \vec{y}_2) \vec{e}_k + \alpha_2 c_k(\vec{e}_1^T \vec{z}_2) \vec{e}_k + \alpha_3 c_k(\vec{e}_1^T \vec{z}_3) \vec{e}_k &= \vec{d}_1 \\ (\alpha_1 + c_k(\vec{e}_1^T \vec{y}_2) + \alpha_2 c_k(\vec{e}_1^T \vec{z}_2) + \alpha_3 c_k(\vec{e}_1^T \vec{z}_3)) \vec{e}_k &= \vec{0}.\end{aligned}$$

Since $\vec{e}_k \neq \vec{0}$, the last equation implies

$$\alpha_1 + c_k(\vec{e}_1^T \vec{y}_2) + \alpha_2 c_k(\vec{e}_1^T \vec{z}_2) + \alpha_3 c_k(\vec{e}_1^T \vec{z}_3) = 0. \quad (12)$$

Thus (5) is equivalent to

$$\begin{aligned}b_{k+1} \vec{e}_1 \vec{e}_k^T (\vec{y}_1 + \alpha_1 \vec{z}_1) + T_2 (\vec{y}_2 + \alpha_2 \vec{z}_2 + \alpha_3 \vec{z}_3) \\ + c_{2k} \vec{e}_k \vec{e}_1^T (\vec{y}_3 + \alpha_4 \vec{z}_4) &= \vec{d}_2 \\ (\alpha_2 + b_{k+1}(\vec{e}_k^T \vec{y}_1) + \alpha_1 b_{k+1}(\vec{e}_k^T \vec{z}_1)) \vec{e}_1 \\ + (\alpha_3 + c_{2k}(\vec{e}_1^T \vec{y}_3) + \alpha_4 c_{2k}(\vec{e}_1^T \vec{z}_4)) \vec{e}_k &= \vec{0}\end{aligned}$$

With the facts that $k > 1$ and $\vec{e}_1, \vec{e}_k \neq \vec{0}$, it follows

$$\alpha_2 + b_{k+1}(\vec{e}_k^T \vec{y}_1) + \alpha_1 b_{k+1}(\vec{e}_k^T \vec{z}_1) = 0 \quad (13)$$

$$\alpha_3 + c_{2k}(\vec{e}_1^T \vec{y}_3) + \alpha_4 c_{2k}(\vec{e}_1^T \vec{z}_4) = 0. \quad (14)$$

Finally (6) and the definitions (7) imply

$$\begin{aligned} b_{2k+1}\vec{e}_1\vec{e}_k^T(\vec{y}_2 + \alpha_2\vec{z}_2 + \alpha_3\vec{z}_3) + T_3(\vec{y}_3 + \alpha_4\vec{z}_4) &= \vec{d}_3 \\ (\alpha_3 + b_{2k+1}(\vec{e}_k^T\vec{y}_2) + \alpha_2b_{2k+1}(\vec{e}_k^T\vec{z}_2) + \alpha_3b_{2k+1}(\vec{e}_k^T\vec{z}_3))\vec{e}_1 &= \vec{0} \end{aligned}$$

and since $\vec{e}_1 \neq \vec{0}$

$$\alpha_3 + b_{2k+1}(\vec{e}_k^T\vec{y}_2) + \alpha_2b_{2k+1}(\vec{e}_k^T\vec{z}_2) + \alpha_3b_{2k+1}(\vec{e}_k^T\vec{z}_3) = 0. \quad (15)$$

Equations (12), (13), (14) and (15) determine the following 4×4 tridiagonal system for the unknowns α_i :

$$\begin{pmatrix} (\vec{e}_k^T\vec{z}_1) & 1/b_{k+1} & 0 & 0 \\ 1/c_k & (\vec{e}_1^T\vec{z}_2) & (\vec{e}_1^T\vec{z}_3) & 0 \\ 0 & (\vec{e}_k^T\vec{z}_2) & (\vec{e}_k^T\vec{z}_3) & 1/b_{2k+1} \\ 0 & 0 & 1/c_{2k} & (\vec{e}_1^T\vec{z}_4) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = - \begin{pmatrix} (\vec{e}_k^T\vec{y}_1) \\ (\vec{e}_1^T\vec{y}_2) \\ (\vec{e}_k^T\vec{y}_2) \\ (\vec{e}_1^T\vec{y}_3) \end{pmatrix}$$

$$\mathcal{H}\vec{\alpha} = \vec{\beta} \quad (16)$$

From the solution of (16) we then obtain the desired solution $\vec{x}$ from (10). Summarizing the divide and conquer method, which divides the given tridiagonal system $T\vec{x} = \vec{d}$ into three parallel tasks of size $k = n/3$, can be depicted as

and will be denoted by $\mathcal{DAC}(3, k)$.

On a parallel computer the seven systems of order $n/3$ can be computed in parallel and independently. Actually, we have three groups of linear systems, where each group has the same coefficient matrix: two two-system groups and one three-system group. They can be treated also parallel to each other in three parallel tasks.

2.2 Divide and Conquer Method $DAC(p, k)$: p parallel tasks

Notation and Definitions

In general, for a system of n equations and for an arbitrary positive integer $p > 1$, we define $k = n/p$ (assuming that n is divisible by p and $k > 1$). The original tridiagonal system is then divided into square $k \times k$ blocks, analogous to (1). In block form we write

$$T\vec{x} = \vec{d} \quad (17)$$

$$\begin{pmatrix} T_1 & C_1 & & & \\ B_2 & T_2 & \ddots & & \\ & \ddots & \ddots & C_{p-1} & \\ & & B_p & T_p & \end{pmatrix} \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vdots \\ \vec{x}_{p-1} \\ \vec{x}_p \end{pmatrix} = \begin{pmatrix} \vec{d}_1 \\ \vec{d}_2 \\ \vdots \\ \vec{d}_{p-1} \\ \vec{d}_p \end{pmatrix}$$

where T_i are tridiagonal matrices: for $j = 0, 1, \dots, p-1$

$$T_{j+1} = \begin{pmatrix} a_{jk+1} & c_{jk+1} & & & \\ b_{jk+2} & a_{jk+2} & c_{jk+2} & & \\ & b_{jk+3} & \ddots & \ddots & \\ & & \ddots & a_{(j+1)k-1} & c_{(j+1)k-1} \\ & & & b_{(j+1)k} & a_{(j+1)k} \end{pmatrix}$$

and

$$\begin{aligned} B_{j+1} &= b_{jk+1} \vec{e}_1 \vec{e}_k^T \\ C_j &= c_{jk} \vec{e}_k \vec{e}_1^T \end{aligned} \quad j = 1, 2, \dots, p-1 \quad (18)$$

assuming $b_{jk+1}, c_{jk} \neq 0$ ($j = 1, 2, \dots, p-1$). $\vec{x}$ and $\vec{d}$ are partitioned to conform with the blocks of T : $\vec{x}_j, \vec{d}_j \in \mathbb{R}^k$ ($j = 0, 1, \dots, p-1$)

$$\vec{x}_j = \begin{pmatrix} x_{jk+1} \\ x_{jk+2} \\ \vdots \\ x_{(j+1)k-1} \\ x_{(j+1)k} \end{pmatrix} \quad \vec{d}_j = \begin{pmatrix} d_{jk+1} \\ d_{jk+2} \\ \vdots \\ d_{(j+1)k-1} \\ d_{(j+1)k} \end{pmatrix}$$

Derivation of the divide and conquer method

For $i = 1, 2, \dots, p$ we define

$$\vec{y}_i = T_i^{-1} \vec{d}_i \quad (19)$$

and for $i = 2, 3, \dots, p-1$

$$\begin{aligned} \vec{z}_m &= T_i^{-1} \vec{e}_1 \\ \vec{z}_{m+1} &= T_i^{-1} \vec{e}_k \end{aligned} \quad m = 2i - 2 \quad (20)$$

and

$$\vec{z}_1 = T_1^{-1} \vec{e}_k \quad \vec{z}_{2p-2} = T_p^{-1} \vec{e}_1 \quad (21)$$

$\vec{z}_m$ is the first and $\vec{z}_{m+1}$ is the last column of the inverse matrix of T_i .

According to (11), for suitable $\alpha_j \in \mathbb{R}$ the solution $\vec{x}$ can be expressed as

$$\vec{x}_i = \begin{cases} \vec{y}_1 + \alpha_1 \vec{z}_1 & \text{if } i = 1 \\ \vec{y}_i + \alpha_{2i-2} \vec{z}_{2i-2} + \alpha_{2i-1} \vec{z}_{2i-1} & \text{if } 2 \leq i \leq p-1 \\ \vec{y}_p + \alpha_{2p-2} \vec{z}_{2p-2} & \text{if } i = p \end{cases} \quad (22)$$

From the block system (17) and the definitions introduced above, we derive for all $j = 2, 3, \dots, p-1$

$$\begin{aligned} B_j \vec{x}_{j-1} P + T_j \vec{x}_j + C_j \vec{x}_{j+1} &= \vec{d}_j \\ B_j (\vec{y}_{j-1} + \alpha_{2j-4} \vec{z}_{2j-4} + \alpha_{2j-3} \vec{z}_{2j-3}) \\ + T_j (\vec{y}_j + \alpha_{2j-2} \vec{z}_{2j-2} + \alpha_{2j-1} \vec{z}_{2j-1}) \\ + C_j (\vec{y}_{j+1} + \alpha_{2j} \vec{z}_{2j} + \alpha_{2j+1} \vec{z}_{2j+1}) &= \vec{d}_j \end{aligned} \quad (23)$$

where $\alpha_0 = \alpha_{2p-1} = 0$.

Since $T_j \vec{y}_j = \vec{d}_j$ and (20), (21) respectively,

$$\begin{aligned} & (b_{(j-1)k+1}(\vec{e}_k^T \vec{y}_{j-1}) + \alpha_{2j-4} b_{(j-1)k+1}(\vec{e}_k^T \vec{z}_{2j-4}) \\ & \quad + \alpha_{2j-3} b_{(j-1)k+1}(\vec{e}_k^T \vec{z}_{2j-3})) \vec{e}_1 \\ & \quad + \alpha_{2j-2} \vec{e}_1 + \alpha_{2j-1} \vec{e}_k \\ & + (c_{jk}(\vec{e}_1^T \vec{y}_{j+1}) + \alpha_{2j} c_{jk}(\vec{e}_1^T \vec{z}_{2j}) + \alpha_{2j+1} c_{jk}(\vec{e}_1^T \vec{z}_{2j+1})) \vec{e}_k = \vec{0} \end{aligned}$$

With the facts that $k > 1$, $\vec{e}_1, \vec{e}_k \neq \vec{0}$, and $\vec{e}_1$ and $\vec{e}_k$ are linearly independent, we know the following equations hold

$$\begin{aligned} & (\vec{e}_k^T \vec{z}_{2j-4}) \alpha_{2j-4} + (\vec{e}_k^T \vec{z}_{2j-3}) \alpha_{2j-3} + \frac{1}{b_{(j-1)k+1}} \alpha_{2j-2} \\ & = -(\vec{e}_k^T \vec{y}_{j-1}) \end{aligned} \quad (24)$$

$$\begin{aligned} & \frac{1}{c_{jk}} \alpha_{2j-1} + (\vec{e}_1^T \vec{z}_{2j}) \alpha_{2j} + (\vec{e}_1^T \vec{z}_{2j+1}) \alpha_{2j+1} \\ & = -(\vec{e}_1^T \vec{y}_{j+1}) \end{aligned} \quad (25)$$

where $j = 2, 3, \dots, p-1$.

According to (12) the first block-equation $T_1 \vec{x}_1 + C_1 \vec{x}_2 = \vec{d}_1$ yields

$$\frac{1}{c_k} \alpha_1 + (\vec{e}_1^T \vec{z}_2) \alpha_2 + (\vec{e}_1^T \vec{z}_3) \alpha_3 = -(\vec{e}_1^T \vec{y}_2) \quad (26)$$

and the last block-equation $B_p \vec{x}_{p-1} + T_p \vec{x}_p = \vec{d}_p$ implies, like the equation (15)

$$\begin{aligned} & (\vec{e}_k^T \vec{z}_{2p-4}) \alpha_{2p-4} + (\vec{e}_k^T \vec{z}_{2p-3}) \alpha_{2p-3} + \frac{1}{b_{(p-1)k+1}} \alpha_{2p-2} \\ & = -(\vec{e}_k^T \vec{y}_{p-1}) \end{aligned} \quad (27)$$

The conditions (24), (25), (26) and (27) form a tridiagonal linear system of $2p-2$ equations for the unknowns $\alpha_1, \alpha_2, \dots, \alpha_{2p-2}$.

The Final Algorithm

The divide and conquer method $\mathcal{DAC}(p, k)$ for solving a tridiagonal system of $n = pk$ equations on a parallel computer with p processors may be stated schematically as follows:

The algorithm consists of three steps:

First step: Solve in parallel the linear tridiagonal systems of size $k = n/p$:

$$\left. \begin{aligned}
 T_i \vec{y}_i &= \vec{d}_i & \text{for } i = 1, 2, \dots, p \\
 T_1 \vec{z}_1 &= \vec{e}_k \\
 T_i \vec{z}_m &= \vec{e}_1 \\
 T_i \vec{z}_{m+1} &= \vec{e}_k \\
 T_p \vec{z}_{2p-2} &= \vec{e}_1
 \end{aligned} \right\} \quad \text{for } i = 2, 3, \dots, p-1, \quad m = 2i-2 \quad (28)$$

Second step: Compute α_i by solving the tridiagonal system $H \vec{\alpha} = \vec{\beta}$

of $2p - 2$ equations given by

$$\begin{pmatrix} s_1 & t_1 & & & \\ r_2 & s_2 & \ddots & & \\ & \ddots & \ddots & t_{2p-3} & \\ & & r_{2p-2} & s_{2p-2} & \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{2p-2} \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{2p-2} \end{pmatrix} \quad (29)$$

where the elements of $\mathcal{H}$ are defined as

$$\begin{aligned} s_i &= \begin{cases} (\vec{e}_k^T \vec{z}_i) & \text{if } i \text{ odd} \\ (\vec{e}_1^T \vec{z}_i) & \text{if } i \text{ even} \end{cases} & i = 1, 2, \dots, 2p-2 \\ r_i &= \begin{cases} (\vec{e}_k^T \vec{z}_{i-1}) & \text{if } i \text{ odd} \\ 1/c_{\frac{1}{2}k} & \text{if } i \text{ even} \end{cases} & i = 2, 3, \dots, 2p-2 \\ t_i &= \begin{cases} 1/b_{\frac{i+1}{2}k+1} & \text{if } i \text{ odd} \\ (\vec{e}_1^T \vec{z}_{i+1}) & \text{if } i \text{ even} \end{cases} & i = 1, 2, \dots, 2p-3 \end{aligned}$$

and those of $\vec{\beta}$ are given by

$$u_i = \begin{cases} -(\vec{e}_k^T \vec{y}_{\frac{i+1}{2}}) & \text{if } i \text{ odd} \\ -(\vec{e}_1^T \vec{y}_{\frac{i+2}{2}}) & \text{if } i \text{ even} \end{cases} \quad i = 1, 2, \dots, 2p-2$$

Third step: Compute the solution $\vec{x} = \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vdots \\ \vec{x}_p \end{pmatrix}$ by the SAXPY-

operations

$$\vec{x}_i = \begin{cases} \vec{y}_1 + \alpha_1 \vec{z}_1 & \text{if } i = 1 \\ \vec{y}_i + \alpha_{2i-2} \vec{z}_{2i-2} + \alpha_{2i-1} \vec{z}_{2i-1} & \text{if } 2 \leq i \leq p-1 \\ \vec{y}_p + \alpha_{2p-2} \vec{z}_{2p-2} & \text{if } i = p \end{cases}$$

as p parallel vector operations of length k .

3 Remarks

Several observations pertinent to the divide and conquer method $\mathcal{DAC}(p, k)$ can be made:

1. The divide and conquer method $\mathcal{DAC}(p, k)$ works theoretically for tridiagonal matrices T with the property that every single T_i is regular. Notice that e.g. the matrices T_1 and T_2 of the regular matrix

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

with $\det T = -1$ are singular ($T_1 = T_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$). This means that the smaller systems (28) of size $n/p = 2$ can not be solved, and so the divide and conquer method breaks down.

This cannot happen however if we assume that T is diagonally dominant. Then every T_i is also diagonally dominant and thus the smaller systems can be solved.

2. To solve the system $\mathcal{H}\vec{\alpha} = \vec{\beta}$ we can again use the divide and conquer method, which makes $\mathcal{DAC}(p, k)$ recursive.
3. By using Gaussian Elimination¹ to solve the smaller systems (28) and $\mathcal{H}\vec{\alpha} = \vec{\beta}$ (29), the total number of scalar operations for $\mathcal{DAC}(p, k)$ is $18n + 7p - 10k - 21$ (see next table). This quantity is minimized for $p = 2$ and $k = n/2$.

¹Gaussian Elimination, to solve a tridiagonal linear system of n equations, requires $3n - 3$ additions, $3n - 3$ multiplications and $2n - 1$ divisions [11].

Scalar Operation Counts for $\mathcal{DAC}(p, k)$: $n = pk$			
	Additions	Multiplications	Divisions
$T_i = L_i R_i$ $i = 1, 2, \dots, p$	$n - p$	$n - p$	$n - p$
$L_i \vec{u}_i = \vec{d}_i$ $R_i \vec{y}_i = \vec{u}_i$ $i = 2, 3, \dots, p$	$2n - 2p$	$2n - 2p$	n
$L_i \vec{v}_i = \vec{e}_1$ $R_i \vec{z}_m = \vec{v}_i$ $i = 2, 3, \dots, p$	$n - p - k + 1$	$2n - 2p - 2k + 2$	$n - k$
$R_i \vec{z}_{m+1} = \vec{e}_k$ $i = 1, 2, \dots, p - 1$		$n - p - k + 1$	$n - k$
$1/c_{\frac{1}{2}k}$ $1/b_{\frac{i+1}{2}k+1}$			$2p - 2$
$\mathcal{H}\vec{\alpha} = \vec{\beta} \in \mathbb{R}^{2p-2}$	$6p - 9$	$6p - 9$	$4p - 5$
$\vec{x}_i$ $i = 1, 2, \dots, p$	$2n - 2k$	$2n - 2k$	

4. The following table gives the number of scalar operations for several known methods, that solve a single tridiagonal system of n equations:

	Additions	Mul.	Divisions
Gaussian Elimination	$3n-3$	$3n-3$	$2n-1$
Cyclic Reduction [10] $n = 2^m - 1$	$6n-4m-2$	$8n-6m-2$	$3n-2m$
Partition Method [21] $n = pk$	$7n-4p-3k-1$	$9n-6p-4k-1$	$5n-2p-k-2$
$\mathcal{DAC}(p, k)$	$6n+2p-3k-8$	$8n-5k-6$	$4n+5p-2k-7$

Although $\mathcal{DAC}(p, k)$ requires nearly twice the number of operations as Gaussian Elimination. $\mathcal{DAC}(p, k)$ is highly parallelizable, whereas the Gauss algorithm must be executed in scalar mode. The operation count for $\mathcal{DAC}(p, k)$ is nearly the same as for the classical cyclic reduction algorithm, even less than for Wang's partition method.

5. On a vector machine, an efficient implementation of $\mathcal{DAC}(p, k)$ applies Gaussian Elimination to all the smaller systems of order i

simultaneously. Details, of solving simultaneously many tridiagonal systems by Gaussian elimination, and a program, written in FORTRAN, are discussed in [15].

6. The greater we choose p , the smaller will be the size of the smaller systems; but the larger the size of $\mathcal{H}\vec{\alpha} = \vec{\beta}$. We should choose p so that on the one hand all processors of the parallel computer work as long as possible, and that on the other hand the overhead to compute $\vec{\alpha}$ is small. Furthermore we must consider that each processor can be equipped with a vector facility.

They are several possibilities to choose both the parameter p and the algorithm for computing the smaller systems (28):

- (a) Since $3p - 2$ linear systems of size n/p must be solved, we can fix the parameter p so that $3p - 2$ equals the number of CPUs of the parallel computer available. Now, three systems with every matrix T_i ($i = 2, \dots, p - 1$) must be solved, and so we prefer to combine the computation of $T_i \vec{y}_i = \vec{d}_i$, $T_i \vec{z}_m = \vec{e}_1$ and $T_i \vec{z}_{m+1} = \vec{e}_k$.
 - (b) The smaller systems can be solved by Gaussian Elimination where first we would compute the LR -decomposition of every T_i in p parallel tasks. In a second step the forward-substitution for the vectors $\vec{d}_i$ and $\vec{e}_1$ will be carried out in $2p - 1$ parallel tasks (Note that the forward-substitution for $\vec{e}_k$ is not necessary because $L\vec{e}_k = \vec{e}_k$). Finally, we arrive at the solutions by $3p - 2$ parallel backward-substitutions.
 - (c) The overhead to solve $T_i \vec{z}_m = \vec{e}_1$ and $T_i \vec{z}_{m+1} = \vec{e}_k$ in addition to $T_i \vec{y}_i = \vec{d}_i$ with cyclic reduction is small: in each reduction step we need only two elements more (one of $\vec{e}_k$ and one of $\vec{e}_1$). The back-substitutions, which calculate $\vec{z}_m$ and $\vec{z}_{m+1}$, can be performed in parallel to the one that computes $\vec{y}_i$.
7. Cyclic reduction is the best we can do on a vector computer with a direct solution method for a single tridiagonal system [16, p 216]. Therefore, on a parallel computer where each CPU supports vector arithmetic (such as the Alliant FX/8), the smaller systems (28) and $\mathcal{H}\vec{\alpha} = \vec{\beta}$ (29) should be solved by using cyclic reduction. The following table lists the scalar operation counts for this case, while

the next table lists the number of vector operations required for each step of the divide and conquer method $\mathcal{DAC}(p, k)$.

Scalar Operation Counts for $\mathcal{DAC}(p, k)$ by using Cyclic Reduction $n = pk$ $p = 2^{q-1}$ $k = 2^h - 1$			
	Additions	Multiplications	Divisions
$T_i \vec{y}_i = \vec{d}_i$ $i = 1, 2, \dots, p$	$6n - 4ph - 2p$	$8n - 6ph - 2p$	$3n - 2ph$
$T_i \vec{z}_m = \vec{e}_1$ (supplementary to $T_i \vec{y}_i = \vec{d}_i$) $i = 2, 3, \dots, p$	$2n - 2p - 2k + 2$	$2n - 3p - 2k + ph - h + 3$	$n - k$
$T_i \vec{z}_{m+1} = \vec{e}_k$ (supplementary to $T_i \vec{y}_i = \vec{d}_i$) $i = 1, 2, \dots, p-1$	$2n - 2p - 2k + 2$	$2n - 3p - 2k + ph - h + 3$	$n - k$
$1/c_{\frac{1}{2}k}$ $1/b_{\frac{1}{2}k+1}$			$2p - 2$
$\mathcal{H}\vec{\alpha} = \vec{\beta} \in \mathbb{R}^{2p-2}$	$12p - 4q - 14$	$16p - 6q - 18$	$6p - 2q - 6$
$\vec{x}_i$ $i = 1, 2, \dots, p$	$2n - 2k$	$2n - 2k$	
Total numbers of scalar operations	$31n + (22 - 10h)p - 14k - 12q - 30$		

Vector Operation Counts for $\mathcal{DAC}(p, k)$ by using Cyclic Reduction $n = pk$ $p = 2^{q-1}$ $k = 2^h - 1$			
	Additions	Multiplications	Divisions
$T_i \vec{y}_i = \vec{d}_i$ $i = 1, 2, \dots, p$	$6p(h - 1)$	$8p(h - 1)$	$3p(h - 1)$
$T_i \vec{z}_m = \vec{e}_1$ (supplementary to $T_i \vec{y}_i = \vec{d}_i$) $i = 2, 3, \dots, p$	$(2p - 2)(h - 1)$	$(2p - 2)(h - 1)$	$(p - 1)(h - 1)$
$T_i \vec{z}_{m+1} = \vec{e}_k$ (supplementary to $T_i \vec{y}_i = \vec{d}_i$) $i = 1, 2, \dots, p-1$	$(2p - 2)(h - 1)$	$(2p - 2)(h - 1)$	$(p - 1)(h - 1)$
$1/c_{\frac{1}{2}k}$ $1/b_{\frac{1}{2}k+1}$			2
$\mathcal{H}\vec{\alpha} = \vec{\beta} \in \mathbb{R}^{2p-2}$	$6(q - 1)$	$8(q - 1)$	$3(q - 1)$
$\vec{x}_i$ $i = 1, 2, \dots, p$	$4p - 4$	$4p - 4$	
Total	$10ph - 6p + 6q - 4h - 6$	$12ph - 8p + 8q - 4h - 8$	$5ph - 5p + 3q - 2h + 1$
Total numbers of vector operations	$27ph - 19p + 17q - 10h - 13$		

8. In the following table, we list operation counts for some typical sizes of n :

Sample operation counts of $\mathcal{DAC}(p, k)$ by using					
			Gaussian El.	Cyclic Reduction	
n	p	k	Scalar count	Scalar count	Vector count
62	2	31	799	1378	203
126	2	63	1631	2894	247
124	4	31	1929	3232	452
254	2	127	3295	5946	291
252	4	63	3913	6712	550
248	8	31	4189	6952	933
510	2	255	6623	12070	335
508	4	127	7881	13712	648
504	8	63	8477	14360	1139
496	16	31	8709	14404	1878
1020	4	255	15817	27752	746
1016	8	127	17053	29256	1345
1008	16	63	17605	29668	2300
992	32	31	17749	29320	3751
2044	4	511	31689	55872	844
2040	8	255	34205	59128	1551
2032	16	127	35397	60356	2722
2016	32	63	35861	60296	4605
4092	4	1023	63433	112152	942
4088	8	511	68509	118952	1757
4080	16	255	70981	121892	3144
4064	32	127	72085	122568	5459
4032	64	63	72373	121564	9198

Note that $\mathcal{DAC}(p, k)$ using Cyclic Reduction requires approximately 1.7 times more scalar operations than when combined with Gaussian Elimination (however in the second case the vector facility on each CPU can be used). We conclude also that by fixing n and increasing p the vector count increases rapidly.

4 Time Measurements

It is assumed that the reader is familiar with the architecture of the CRAY X-MP/28, the Alliant FX/8 and the Sequent Symmetry S81b.

For a description see [9, pp 123], [9, pp 52] or [1], [18]. The time measurements have been obtained under the following conditions:

- The $\mathcal{DAC}(p, k)$ method has been coded in FORTRAN 77 and compared to both the classical cyclic reduction algorithm ($n = 2^k - 1$) [12] and Gaussian elimination, which were also coded in FORTRAN 77. (The compiler on the CRAY X-MP/28 is CFT77 VERSION 3.1, on the Alliant FX/8 "FORTRAN, version 4.1.35" is installed and on the Sequent Symmetry S81b: FORTRAN 77 V3.0.1 (C) Copyright 1981, 1987 Silicon Valley Software Inc.)
- To force explicit parallelism and/or explicit vectorization on the CRAY X-MP/28 and on the Alliant FX/8, compiler directives were used. More detailed descriptions can be found for CRAY computers and for the Alliant in [4] and [19] respectively.
- Every smaller system-triple of the $\mathcal{DAC}(p, k)$ method was combined in one single task and solved using cyclic reduction: Each processor computes

$$T_i \vec{y}_i = \vec{d}_i, \quad T_i \vec{z}_m = \vec{e}_1, \quad T_i \vec{z}_{m+1} = \vec{e}_k$$

(Only on the CRAY and the Alliant computers.)

- To parallelize the code on the Sequent Symmetry S81b we used functions in the Parallel Programming Library DYNIX. For more details see [14, pp 5-12].
- Since the CPUs of the Sequent Symmetry S81b do not support vector arithmetic, the smaller system-triples of $\mathcal{DAC}(p, k)$ were computed using Gaussian Elimination (combined into one single task analogous to the case above).
- The system $\mathcal{H}\vec{\alpha} = \vec{\beta}$ was solved using Gaussian Elimination on all three computers.
- $\mathcal{DAC}(p, k)$ always runs on p processors. The number of CPU's available to the program is variable:

CRAY X-MP/28:	$p = 1, 2$
Alliant FX/8:	$1 \leq p \leq 6$
Sequent Symmetry S81b:	$1 \leq p \leq 25$

If $t(n, p)$ denotes the elapsed CPU time for the solution of a tridiagonal system of n linear equations using p processors always using $\mathcal{DAC}(p, k)$, then we define *efficiency* $E(n, p)$ as a percentage

$$E(n, p) = \frac{100 * t(n, 1)}{p * t(n, p)}$$

If one processor takes an amount of time to do a task, then p processors can do the task in the p th part of the time. Only for very special situations this perfect efficiency of 100% can be achieved.

The following tables and figures show the high efficiency of $\mathcal{DAC}(p, k)$ for an increasing number p of processors.

CRAY X-MP/28					
n $E(n, 2)$		n $E(n, 2)$		n $E(n, 2)$	
62	84.74%	126	86.63%	254	89.22%
510	94.55%	1022	94.77%	2046	94.23%
4094	94.99%	8190	96.84%	16382	98.02%

Alliant FX/8					
n $E(n, 2)$	n $E(n, 3)$	n $E(n, 4)$	n $E(n, 5)$	n $E(n, 6)$	
126 96.21%	93 86.17%	124 84.31%	75 78.11%	90	78.93%
254 97.01%	189 86.23%	252 85.44%	155 81.08%	186	80.63%
510 97.95%	381 86.34%	508 87.15%	315 83.02%	378	82.14%
1022 98.22%	765 85.76%	1020 87.27%	635 83.95%	762	82.56%
2046 98.33%	1533 85.59%	2044 86.95%	1275 84.99%	1530	82.82%
4094 97.66%	3069 84.68%	4092 79.86%	2555 80.36%	3066	75.81%
8190 89.17%	6141 76.81%	8188 67.24%	5115 74.43%	6138	64.05%

Figure 1: Efficiency $E(n, p)$ of $\mathcal{DAC}(p, k)$ on the CRAY X-MP/28 and Alliant FX/8.

Figure 2: Efficiency $E(n, p)$ of $\mathcal{DAC}(p, k)$ on the Sequent Symmetry S81b for $n = 16800$ and $n = 25200$.

The efficiency on the CRAY X-MP/28 grows with an increasing size n until reaching a maximum value of 96.84%. On the Alliant FX/8 the maximum value of $E(n, p)$ decreases slowly with an increasing number of CPUs p . This is attributed to the fact that the size of the system $\mathcal{H}\vec{\alpha} = \vec{\beta}$ increases (It was computed on one single CPU). For this same reason the efficiency on the Sequent decreases rapidly for $p > 10$. The maximum value on this machine is around 95%.

The speedup reported in the following figures is given by

$$\text{Speedup} = \frac{t_{\text{algorithm}}(n, 1)}{t_{\mathcal{DAC}(p, k)}(n, p)}$$

where $t_{\text{algorithm}}(n, 1)$ denotes the measured time for the cyclic reduction or Gaussian elimination on one processor.

Figure 3: Speedup of $\mathcal{DAC}(p, k)$ on the Alliant FX/8 (above) and on the Sequent Symmetry S81b (below).

The maximum speedup of $\mathcal{DAC}(p, k)$ and cyclic reduction on the Alliant FX/8 is approximately 1.4 using 2 CPUs and 2.1 using 4 CPUs – this occurs at $n = 2^{11} - 1 = 2047$. So we gain about a factor of 2 on the method that was rated best until now.

The speedup of $\mathcal{DAC}(p, k)$ and Gaussian Elimination on the Sequent grows with an increasing number of CPUs until reaching its maximum value at about 10.8 (on 25 CPUs).

5 Numerical examples

Finally, we present a few examples that show the accuracy of the divide and conquer method $\mathcal{DAC}(p, k)$ by using Gaussian Elimination to solve the smaller system-triples.

The algorithm has been coded in FORTRAN and the observed error norms are given for execution on a CRAY X-MP/28 on only one processor (which is equivalent to the accuracy on more than one processor). All computations have been carried out in floating-point arithmetic with a relative accuracy of 48 bits (approximately 14 decimal digits of precision [5, pp 3-6]).

We set $n = 4000$, $a_i = a$, and $b_i = c_i = -1$. The right-hand side $\vec{d}$ is chosen such that the exact solution is $x_i = i/20$. The computed solution is denoted by $\vec{\hat{x}}$. The entries in the following table are the relative maximum errors ε_p given by

$$\varepsilon_p = \max_{1 \leq i \leq 4000} \frac{|\hat{x}_i - x_i|}{|x_i|}$$

where p equals the number of system-triples of $\mathcal{DAC}(p, k)$.

p	$a = 4.0$	$a = 2.0$	$a = 1.9$
25	0.201587E-13	0.515926E-08	0.145070E-08
16	0.184357E-13	0.528623E-08	0.145288E-09
10	0.194060E-13	0.516068E-08	0.243210E-08
8	0.182081E-13	0.513520E-08	0.454429E-08
4	0.182081E-13	0.501056E-08	0.439423E-08
2	0.141446E-13	0.475870E-08	0.286127E-08
Gauss:	0.142053E-13	0.535147E-08	0.117544E-07
Cycl. Red.: for $n=4095$	0.464028E-13	0.751175E-08	0.413403E-09

The last two rows show the errors, occurring in the solution of the tridiagonal linear system by using both Gaussian Elimination and Cyclic Reduction.

We observe that for the model problem $a = 4.0$, the relative error norms are much smaller than for the ill conditioned cases $a = 2.0$ and $a = 1.9$. Other values for $|a| > 2$ lead to results similar to those in the first column of the table above.

Strikingly, for T diagonally dominant, the relative maximum error by the Cyclic Reduction algorithm is more than twice that of $\mathcal{DAC}(p, k)$.

In the first two examples $\mathcal{DAC}(p, k)$ and Gaussian Elimination have nearly the same error. For the case $a=1.9$, however, $\mathcal{DAC}(p, k)$ is a little better.

The accuracy of the solution $\vec{x}$ is mainly determined by the computed solutions $\vec{y}_i$ and $\vec{z}_i$ of the smaller systems, and for values $|a| < 2$, pivoting may be necessary!

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