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Interactive Equilibrium Modelling

A new approach
to the computer-aided exploration of structures
in architecture

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Dedicated to Insa

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Table of contents

Zusammenfassung

Die vorliegende Dissertation stellt neue Vorgehensweisen für den Entwurf von geometrisch komplexen und zugleich statisch effizienten Tragwerken vor. Computergestützte Entwurfssysteme gestatten es Architekten heute, Gebäude mit spektakulären, oft doppelt gekrümmten Formen zu entwerfen. Durch leistungsstarke numerische Methoden sind Bauingenieure fähig, entsprechend geformte Tragwerke statisch zu dimensionieren. Die häufig in der Planungspraxis disziplinär weitgehend unabhängig voneinander ablaufenden Entwurfs- und Bemessungsprozesse führen oft zu Bauwerken mit ungünstigem Lastabtrag und hohem Bedarf an natürlichen Ressourcen.

Das Ziel dieser Arbeit ist es, eine Methodik des Tragwerksentwurfs zu entwickeln, welche die Prinzipien des Kräftegleichgewichts mit Konzepten des computergestützten Entwerfens kombiniert. Dazu wird ein interaktiver Modellierungsprozess beschrieben, der dem Entwerfer eine schrittweise Annäherung zwischen Entwurfsidee und effizientem Tragwerk ermöglicht. Die Bedeutung der Arbeit liegt einerseits darin, den Graben zwischen freiem architektonischem Entwurf und wissenschaftlichem Tragwerksentwurf zu verringern. Andererseits besteht die Relevanz der Arbeit auch darin, den Entwurfsprozess für geometrisch komplexe Tragstrukturen mit effizientem Lastabtrag in frühen Entwurfsphasen zu systematisieren und zu erleichtern.

Die entwickelten Modellierungsverfahren basieren auf dem statischen Konzept der Fachwerkmodelle, kombiniert mit computergestützten Formfindungsmethoden. In einem ersten Schritt werden massgeschneiderte Methoden für die Modellierung ausgewählter Tragwerkstypologien entwickelt, zum Beispiel für gekrümmte Brücken oder baumartige Stützen. Jede dieser Methoden basiert auf einem klar definierten Konzept des Lastabtrags. Durch die fallweise Anpassung von Standard-Formfindungsmethoden werden die spezifischen Randbedingungen dieser Konzepte in die jeweiligen Modellierungsprozesse eingeschrieben. In einem zweiten Schritt wird eine allgemeine Methode entwickelt, welche es ermöglicht, Modelle mit generischen Randbedingungen zu erstellen, um damit unterschiedlichen Tragkonzepten gerecht zu werden.

Diese allgemeine Methode zum computergestützten Entwerfen von Tragwerken basiert auf einer neuen Technik zum Lösen von allgemeinen Formfindungsproblemen mit Randbedingungen. Zu diesem Zweck wird die Kraftdichtemethode auf neue Weise erweitert. Bisher bekannte Erweiterungen der Kraftdichtemethode haben grösstenteils das Ziel, Restriktionen aus Konstruktion und Fabrikation in den Formfindungsprozess zu integrieren. Hier werden allgemeine Randbedingungen dazu verwendet, kreative Prozesse für frühe Phasen des Tragwerksentwurfs mit einem hohen Mass an geometrischer Kontrolle auszustatten.

Durch neun prototypische Fallbeispiele wird die Flexibilität der Methode nachgewiesen. In einer ersten Gruppe von Beispielen wird gezeigt, wie es die Methode ermöglicht, die Formenvielfalt von allgemein bekannten

Tragwerkstypologien zu erschliessen. In einer zweiten Gruppe wird demonstriert, dass sich die Methode darüber hinaus für das Modellieren von neuartigen Tragwerkstypen eignet, die durch die Kombination und das Verschmelzen von bekannten Modellen entstehen.

Folgende Beiträge zum Stand der Forschung sind geleistet worden: massgeschneiderte Methoden zum Modellieren von ausgewählten Tragwerkstypen wurden entwickelt, eine allgemeine Modellierungsmethode basierend auf einer neuen Erweiterung der Kraftdichtemethode wurde formuliert und Fallbeispiele, die den Modellierungsprozess für ausgewählte Tragwerke darstellen, wurden präsentiert. Darüber hinaus stellt das neue Lösungsverfahren für Formfindungsprobleme mit Randbedingungen die Grundlage für die Entwicklung eines computergestützten „Entwurfstools“ für Tragwerke dar.

Schlüsselwörter:
Tragwerksentwurf, computergestützter Entwurf, geometrisches Modellieren, Architekturgeometrie, Gleichgewichtslösungen, Stabwerksmodelle, Formfindung unter Randbedingungen, Kraftdichtemethode

Abstract

In this dissertation, new approaches to the design of non-standard building structures with efficient static behaviour are presented. Computer-aided modelling tools enable architects to design spectacular buildings with complex geometry; through strong numerical methods, civil engineers are able to statically analyse and design corresponding structures. The often strong separation between architectural and structural design processes largely results in inefficient structural behaviour and high consumption of resources.

The aim is to develop a structural design methodology which combines principles of static equilibrium with computer-aided design concepts. For this purpose, an interactive modelling approach is described, enabling the designer to iteratively bring together design idea and efficient structural behaviour. On the one hand, the relevance of this dissertation lies in its contribution to reducing the gap between free architectural design and scientific structural design. On the other hand, the relevance lies in the systematisation and facilitation of the design process for geometrically complex building structures with efficient force flow.

The presented modelling approaches are built upon the conceptual framework of truss models, combined with computational form-finding methods. In a first step, tailored equilibrium modelling methods for selected typologies of structures, e.g. curved bridges or branching columns, are developed. Each of these modelling methods is based on well-defined structural concepts. Through customised, case-specific adaptation of standard form-finding techniques, the boundary conditions of the underlying structural concepts are inscribed in these interactive modelling processes. In a second step, a general equilibrium modelling approach is developed, which allows generic boundary conditions to be defined to address different structural concepts.

This general modelling approach is based on a new technique to solve form-finding problems with constraints both on the form and on the inner forces. For this purpose, the Force Density Method has been extended in a new way. Previously, several approaches were formulated in order to add boundary conditions to the Force Density Method, but these were largely driven by specific construction and fabrication properties. Here, general constraints are used to enable early-stage structural design explorations with a high degree of geometric control.

The flexibility of the new method is demonstrated in nine prototypical case studies. The first group of cases demonstrates that the method enables the exploration of the inherent geometric freedom of renowned structural typologies. The second group of cases demonstrates that the method is also suited for creating new typologies, by combining and merging given structural models.

The following contributions to the body of knowledge have been made: tailored modelling approaches for selected structural typologies were developed; a general equilibrium modelling approach based on a new extension of the Force Density Method was formulated; case studies illustrating the modelling process of selected structures were presented. Furthermore, the new solving technique for form-finding problems with boundary conditions establishes the conceptual basis for a new computer-aided structural design tool.

Keywords:
structural design, computer-aided design, geometric modelling, architectural geometry, equilibrium solutions, strut-and-tie models, constrained form finding, Force Density Method

1 Introduction

In the last two decades, computer-aided modelling software has enabled architects to design buildings with complex, often double-curved geometry. Through sophisticated structural analysis software, civil engineers were enabled to statically analyse almost any building shape. Mass customisation allows for the fabrication and construction of non-standard building envelopes and structures. As drawback, this seemingly limitless freedom in formal expression in architecture often leads to buildings with a high consumption of natural resources in construction. This dissertation provides the conceptual foundation for an alternative design methodology deeply rooted in structural principles, reaching out towards 'freeform' modelling. Furthermore, this new methodology enables an iterative, computer-aided design process of buildings with complex geometry *and* efficient structural performance.

The detachment of architectural and structural design has reached a new peak during recent decades, but the offspring of this development dates back to the Renaissance period. In Gothic architecture, boundaries between artist, architect and engineer did not yet exist. Master Builders of the Early Renaissance, such as Filippo Brunelleschi, were responsible for both design *and* construction, for the beauty *and* safety of buildings. In his book, *De re aedificatoria*, published in 1485 (Alberti, 1988), Leon Battista Alberti was the first to theoretically emphasise the separation between the practice of designing and the practice of building. With the emergence of modern science, namely the description of the strength of beams in the *Discorsi*, published in 1638 by Galileo Galilei (Galilei, 1974), and the formulation of the Law of Elasticity by Robert Hooke in 1678 (Gunther, 1931), the basis for a scientific theory of structures was established. By the foundation of polytechnic schools in the 19th century, the division in the building professions between architect and civil engineer became institutionalised (Rinke and Kotnik, 2013).

In the first half of the 20th century, the hierarchical division of tasks largely remained strict: in general, the architect was responsible for the formal, artistic design of the building, the civil engineer was responsible for "making it stand", for solving problems of statics. With the decline of the common belief in scientific progress during the second half of the 20th century, also the belief in the unlimited power of engineering methods faded. The new awareness of the limits of resources transformed engineering problems in many cases to questions of weighted interests, thus questions of design. According to the philosopher, Bruno Latour, *matters of fact* often became *matters of concern* (Latour, 2008). At the same time, materialistic perspectives gained influence in architectural design. Frei Otto, founder of the Institute of Lightweight Structures, states the following: "And even if a technical subject – lightweight surface structures – coined the name of an institute which is unique the world over, we did not follow a restriction such as this. Our aim was much greater. We wanted to attempt to contribute something to the field of architecture." (Burkhardt, Hennicke *et al.,* 1984: 6)

With the *digital turn* (Carpo, 2013) since the beginning of the 1990s, novel computerised tools for architecture and civil engineering have gained influence. Resulting from the change of industrialised planning and construction methods towards post-industrialised, information-driven design and fabrication processes, these novel tools have enabled design, structural analysis, fabrication, and construction of buildings with complex, double-curved geometry. The breakthrough in digital planning and construction processes in architecture was achieved by Frank Gehry, with the design of the Guggenheim Museum in Bilbao, Spain, completed in 1997.

In the past decade, a new awareness and interest has emerged in the potential of engineering concepts for design in architecture, with a focus on computational methods (Oxman and Oxman, 2010). Cutting-edge engineering practices emphasise their expertise as design consultants rather than as technical problem-solvers (Kara, 2008). The influential DETAIL magazine, specialised in architecture and construction, started to publish monographs of world-leading engineering companies, such as schlaich bergermann und partner (Bögle, Brensing *et al.,* 2011) and ARUP (Schittich and Brensing, 2012). However, also cutting-edge architectural practices, such as Zaha Hadid Architects, known for their extravagant freeform design, try to relate their work to innovative structural engineering, for instance by presenting the work of famous shell builders next to their own at the Architecture Biennale 2012 in Venice (Chipperfield, 2012: 158-159).

Despite such good intentions from both architects and engineers, a fundamental challenge in relating structure and form remains in the available models and descriptions of structural behaviour. The behaviour of complex, three-dimensional building shapes can barely be described with the typological models derived from classical analytical statics. These typologies that have dominated structural design education since the 1960s, such as for instance beam, arch or dome, are not sufficient for the design of buildings with complex spatial geometry. Due to this lack of a common language between architect and engineer, the collaboration is often reduced to a mere operational level, to data exchange and computerised dimensioning based on the finite element method (FEM) software. While FEM is a powerful tool for structural analysis of given structures, it provides only poor guidance for the process of finding reasonable structural form.

An alternative approach to the description of structural behaviour, besides classical analytical statics and FEM, is based on pin-jointed truss models and equilibrium solutions. This approach can be traced back to the development of graphic statics by Karl Culmann, the first Professor of Civil Engineering at the ETH Zurich (Maurer and Ramm, 1998). The academic, David Billington, emphasises the continuity between graphic statics and contemporary equilibrium design approaches, such as strut-and-tie models or stress fields (Bögle and Billington, 2009). Billington furthermore summarises this line of thinking of Swiss origin as "structural art", including, among others, the most famous structural designers of the 20th century, such as Robert Maillart, Pier Luigi Nervi, Heinz Isler and Jörg Schlaich.

Following the idea of structural design as synthetic practice based on pin-jointed equilibrium models, the goal of this dissertation is the development of a sound methodological foundation for geometrically flexible modelling approaches of efficient structures. These approaches enable designers to explore the inherent formal freedom of structures in an interactive, almost 'artistic' way, beyond the well-known typologies of classical analytical statics; this is especially relevant for the design of highly stressed structures, e.g. long-span bridges, roofs and towers. As main contribution, a novel algorithmic method for the creation of equilibrium forms with respect to external constraints is developed. Through this method, the designer is able to generate and refine equilibrium models in an iterative process.

The dissertation is structured as follows: Section 2 introduces a general definition of design as iterative search process driven by the designer. Section 3 describes the conception of structural design as iterative search, delimited from the narrow understanding of structural design as mere analysis and dimensioning. In Section 4, the problem of structural design modelling and related research goals are formulated. In Section 5, the conceptual framework of equilibrium design is defined. In Section 6, new approaches for the modelling of specific types of structures on the basis of truss models are presented. In Section 7, a novel general methodology for the iterative modelling of equilibrium structures is outlined; its core is constituted by a new form-finding method, allowing for the deliberate generation of mixed compression and tension structures. In Section 8, the power of the approach for the exploration of novel structures is demonstrated by a series of case studies. The dissertation is completed in Section 9 by a discussion of unique contributions, limitations, future work, and a formulation of consequential conclusions.

2 Design processes

In Section 2.1, a conceptual framework for the systematic approach to the activity of design in a general sense is established, based on the work of Herbert A. Simon. In Section 2.2, related approaches to the systematisation of the design process in architecture are presented.

2.1 Design as search process

In his book "Sciences of the Artificial"(first edition published in 1969), the economist and sociologist, Simon, founded the basis for a general theory of the activity of the *creation* of artificial things or organisations, as counterpart to the contrary activity of the *observation* of natural phenomena. The creation of artefacts in order to accomplish *goals* is the task of the engineer, or in more general terms, the designer. For Simon, the term 'designer' includes, besides 'hard' engineering disciplines, such as Mechanical and Electrical Engineering, and 'softer' disciplines, such as Architecture and Industrial Design, also Management and Politics, since those fields are concerned with the creation of companies and societies. Natural Sciences are concerned primarily with analytical and descriptive activities, while Design and Engineering are concerned instead with synthesis and normative activities. Simon describes the boundaries of the "sciences of the artificial" by specifying the essential properties of artefacts. Among these, three are emphasised here: artefacts are synthesised by human beings, they can be characterised in terms of goals and functions, and they are discussed both in terms of imperatives and descriptives (Simon, 1996: 5).

According to Simon, "everyone designs who devises courses of action aimed at changing existing situations into preferred ones" (*Ibid.*: 111). Based on this description, the intellectual activities of a medical doctor, an engineer, an architect and an entrepreneur are comparable, and essentially concerned with design processes. Simon bemoans the fact that "in this century the natural sciences almost drove the sciences of the artificial from professional school curricula […] engineering schools gradually became schools of physics and mathematics; medical schools became schools of biological science; business schools became schools of finite mathematics" (*Ibid.*). According to Simon, the reason for this development is the schools' desire for academic respectability, which, corresponding to general norms, is provided for by "subject matter that is intellectually tough, analytic, formalizable, and teachable" (*Ibid.:* 112). This development led to a loss of professional design competence, because the stronger universities taught almost no design competence, and the weaker technical institutions did not teach it on an appropriate intellectual level, beyond recipe-like application. In order to overcome this, professional schools have to research and teach the Science of Design, as "a body of intellectual tough, analytic, partly formalizable, partly empirical, teachable doctrine about the design process" (*Ibid.:* 113).

Simon describes Design as a problem-solving process consisting of a series of rational decisions. However, at the same time, he points out the limits of such a positivist notion in real world situations. Even if a problem is fully formalisable in theory, e.g. the maximisation of the profit of an enterprise within an economic system, in most cases it is not possible to obtain a decision via strictly logical reasoning. The reason for this is that either not all the information is available, or the amount of information is simply too large to be processed in a reasonable time, even for systems with comparatively low complexity. Simon therefore proposes the replacement of the idea of 'rational' decisions with the weaker term, 'bounded rational' decisions, which are derived in a heuristic manner from the incomplete information that is available to the designer. Based on this insight, Simon suggests generally replacing the goal of finding optimal solutions by the goal of finding satisfying solutions in design contexts.

According to Simon, a design as artefact with a certain complexity has a hierarchical internal structure in general, with multiple parts or levels that are more or less dependent on each other. A powerful approach to synthesise such an artefact is to divide it into several weakly dependent parts; each simpler part can then be designed almost independently. It is stated that, in general, there is more than one feasible decomposition. These may all result in satisfying solutions of different kinds (Simon, 1996: 128).

Generally, the design process is organised as a heuristic search for a satisfactory solution in the environment of all possible candidates. Based on the chosen decomposition of the problem, one way to organise the search process is the scheme of *generator-test cycles*. For example, if one decomposition of a design problem in two parts is given, one can set up a method that generates alternatives satisfying the requirements of one part, and then test them against the requirements of the other. Design processes for complex tasks may be organised in "nested series of such cycles" (*Ibid.:* 129). In one of his works on cognitive psychology, Simon describes the limitations of the general generator-test cycle scheme, and emphasises the power of a *directed search* scheme. A directed search is dependent on the nature of the problem, and allows the search to be steered specifically towards a solution: "The generality (and weakness) of the generate-and-test scheme lies precisely in the fact that the generation process and the test process are completely independent. Each has only to fulfil certain minimal conditions of its own. Conversely, the power (and specificity) of a method for the search formulation must lie in the dependence of the search process upon the nature of the object being sought and the process being made toward it" (Newell and Simon, 1972: 98).

One important perceptive aspect of Design is the choice of the representation of the problem. It has been emphasised by Simon, but also by other authors, that "the mind's eye"– the ability to visually comprehend dependencies – often efficiently replaces abstract, logical reasoning, if the right form of visual representation is chosen (Ferguson, 1977; Simon, 1996: 74). Especially in architecture or engineering, disciplines that deal with the arrangements of objects in three-dimensional space, the question of spatial representation is crucial. Those questions of representation attracted the attention of researchers concerned with Computer-Aided Design (CAD), which Simon describes as "the cooperation of human and computer in the design process" (Simon, 1996: 133). Ivan Sutherland is a pioneer in this field, having developed SKETCHPAD (Figure 2.1) already in the 1960s – the first version of a computer-aided drawing system. Among other functions, this system enabled the user to draw geometric shapes with a light pen, after which the computer performed automatic modifications of the shape according to user-defined constraints (Sutherland, 1963).

Simon points out that, since the goal of the design process is generally not optimal design, but satisfactory design within given constraints, multiple solutions with different appearances or 'styles' are possible. Those different styles are the results of different possible decompositions of the problem, and reflect the different internal structures of the design process (Simon, 1996: 130). By proposing the conception of a design process without final goals, Simon includes in his mainly technical notion of the design process the possibility of an artistic, creative design process. Therefore, the idea of a single predefined goal is replaced by a series of temporary goals that successively emerge during the process from intermediate states of the artefact. "In oil painting every new spot of pigment laid on the canvas creates some kind of pattern that provides a continuing source of ideas to the painter. The painting process is a process of cyclical interaction between painter and canvas in which current goals lead to new applications of paint, while the gradually changing pattern suggests new goals" (Simon, 1996: 163).

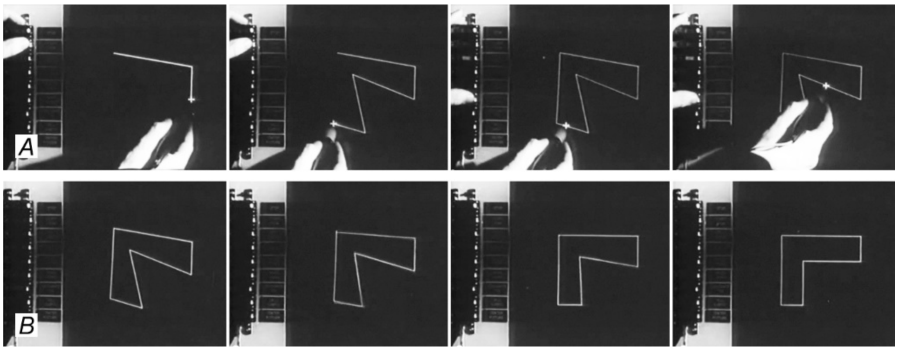

Figure 2.1: The first computer-aided drawing system SKETCHPAD in action: (*A*) the user draws a geometrical shape on the screen using a light pen, and (*B*) the automatic geometric regularisation of the shape (Kay 1987).

2.2 Formal approaches in architectural design

Different researchers and practitioners have dealt with the idea of a systematisation and formalisation of the design process in architecture. Early attempts in this direction date back to the beginning of the 19th century. The French architect, Jean-Nicolas-Louis Durand, described a method for the systematic development of architectural designs according to predefined typologies, based on strict geometric rules and grids (Durand, 1802). Since the 1960s, the idea of the formalisation of the architectural design process has been stimulated through developments in Information Technology and the availability of computers, firstly as mainframe computers at universities, and later as personal computers.

2.2.1 Diagrams and patterns

In his dissertation, "Notes on the Synthesis of Form", the architect, Christopher Alexander, presented a methodology for the design of buildings and urban settlements based on the idea of a mathematical decomposition of architectural design problems (Alexander, 1964). His notion of decomposition of design problems is almost congruent with the way Simon described it a few years later in "The Sciences of the Artificial", and was probably inspired by the earlier writing and lectures by Simon (Kühn, 2008). The following quote illustrates the idea of dependencies between different aspects of an architectural design problem: "Let us remind ourselves of the precise sense in which there is a system active in a form-making process. […] Its variables are the conditions which must be met by good fit between form and context. Its interactions are the causal linkages which connect the variables to one another. If there is not enough light in a house, for instance, and more windows are added to correct this failure, the change may improve the light but allow too little privacy; another change for more light makes the windows bigger, perhaps, but thereby makes the house more likely to collapse. These are examples of inter-variable linkage" (Alexander, 1964: 42). Alexander suggests detecting weakly dependent subsystems (Figure 2.2) with the aid of computers, to design them independently, and to synthesise the design from these almost independent parts.

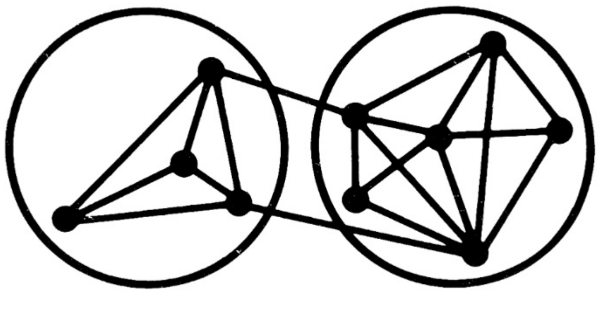

Figure 2.2: Decomposition of a system, represented by the network of lines and points, in two weakly dependent subsystems, represented by the two circles (Alexander, 1964: 43).

The question remains of how to approach the design of the basic subsystems. For this purpose, Alexander suggests choosing a diagrammatic representation of these subsystems, such that the solution to these elementary design problems becomes intuitively evident. Such diagrams are named "constructive diagrams". The diagram in Figure 2.3, for example, represents a new type of street crossing: "Here we have a street map with arrows of various widths on it, representing the number of vehicles per hour flowing in various directions at peak hours. In this form, the diagram indicates directly what form the new intersection must take. Clearly, a thick arrow requires a wide street, so that the overall pattern called for emerges directly from the diagram" (*Ibid.:* 88). In fact, Alexander's notion of "constructive diagrams" is directly inspired by structural engineering diagrams: "The engineer's primary sketch for a bridge structure is a diagram" (*Ibid.:* 86). In Footnote 8, referring to this passage, he mentions the engineers, Pier Luigi Nervi and Robert Maillart, and their use of diagrams in design (*Ibid.:* 209).

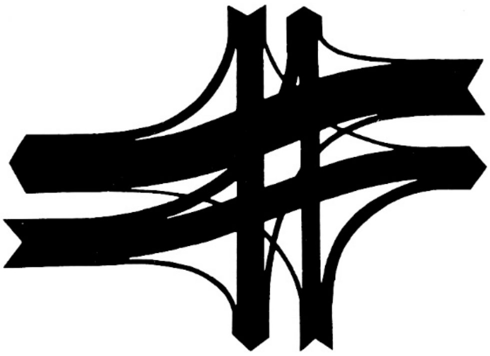

Figure 2.3: "Constructive diagram" of traffic flows, directly representing a possible design for a street intersection (Alexander, 1964: 88).

This idea of "constructive diagrams" is based on the concept of visual comprehension through "the mind's eye", as Simon describes it. A powerful aspect of the idea of the "constructive diagram" is its inherent geometric flexibility, while maintaining the main topological features. In his later writing, Alexander abandons the idea of the possibility of a formal, automatic decomposition of architectural design problems, and replaces the concept of "constructive diagrams" with the idea of "patterns". In his book, "A Pattern Language", published in 1977, he develops instead a catalogue of such elementary patterns, representing 'eternal' solutions for elementary design problems. Patterns, consisting of photographs and drawings with a short text, are numbered, and have self-explanatory titles, for instance, "Main Entrance", "Arcades", or "Vegetable Garden" (Alexander, Ishikawa *et al.,* 1977). The 'patterns' are related to each other with cross references, forming something in between a lexicon and a textbook. With his "Pattern Language", Alexander seeks to present one universal decomposition of all possible architectural design tasks.

2.2.2 Grammars and generate-test cycles

Another approach to formal design processes in architecture has been formulated by William J. Mitchell and George Stiny. In "The Logic of Architecture", Mitchell describes a triangular relation of reference that is in effect in the design process (Figure 2.4). Initially, the goals of a design are formulated in natural language, normally by the client. Mitchell claims that it is generally possible to formalise descriptions of design goals with the strict means of first-order logic ("critical language") (Mitchell, 1990: 59-71). Statements about design goals relate only indirectly to concrete changes in the built environment ("construction world"); the interventions are not yet precisely defined. Therefore, the design is developed in an abstract microcosm ("design world"), e.g. as a drawing, physical model, or CAD model. For instance, the critical language word *column* refers to a point with specific coordinates in the design world, which refers to a real vertical building component at a specific location in the "construction world".

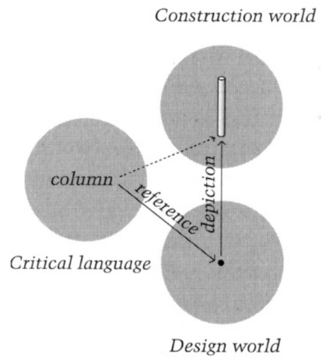

Figure 2.4: Relationship between critical language, design world, and construction world (Mitchell,1990: 108).

The representation of a building in the abstract design world depicts the final design in a precise, but not unique way (Figure 2.5). For example, a point or circle in a plan drawing representing a column may refer to vertical building components of different materials and shapes.

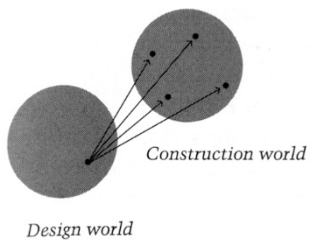

Figure 2.5: Multiple realisations of a design in the construction world (Mitchell, 1990: 68).

It is assumed that it is possible to judge the truth of a "critical language" sentence about the "construction world" based on its representation in the "design world," hence that the "design world" represents the "construction world" in sufficient detail for the designer to be able to reach reliable conclusions based on this representation. The design process itself is described as a path through the "design world". This path is defined through an *initial state* of the design, a representation of the topography of the site, for instance, and a set of "design operations" that subsequently transform the design, until the final state is reached in which all design goals are satisfied (Figure 2.6).

The nature of the "design operations" is highly dependent on the nature of the "design world": "In a design world populated by cardboard polygons, the basic operators might be a matte knife for shaping polygons, hands for translating and rotating polygons, and a glue gun for joining polygons together. In a design world of straight lines and arcs on a paper, the basic operations might be a pencil, a straightedge, compasses and an eraser. And in a computer-aided design system, the operators are programs that manipulate the data structure: these are often invoked by clicking on icons that depict more traditional tools" (Mitchell, 1990: 109). Here, Mitchell refers to a figure showing icons of a graphical user interface which represents, among others, a pen, an eraser and a brush.

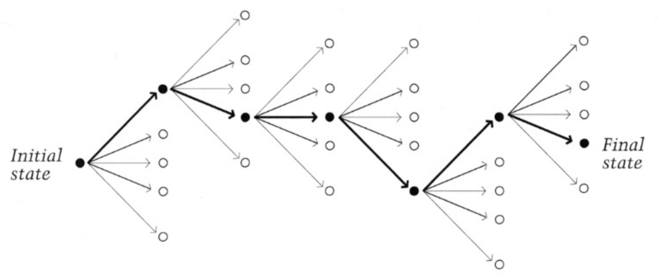

Figure 2.6: The design process as search path in the "design world", starting at an initial state, and reaching a final state. The nodes represent design states; the arrows represent the application of "design operations" (Mitchell, 1990: 56).

Mitchell presents a design world consisting of two-dimensional line drawings in detail, and illustrates the step-by-step generation of the plan of Villa Malcontenta by Palladio through the application of geometric design operators (*Ibid.:* 152-179). The example is based on the work of Georg Stiny, who developed the concept of "shape grammars", consisting of geometric rules for transformations, combinations and replacements, which allow for the derivation of complex geometric shapes from geometric primitives in an axiomatic way (Stiny, 1980). By using these two formal systems, logic-based critical language for the description and evaluation of design goals, and geometry-based rules as "design operators", Mitchell claims that it is possible to implement computational design processes as *generate-test cycles*, where either the generation, or the testing, or both, are automatically executed by a computer (Figure 2.7).

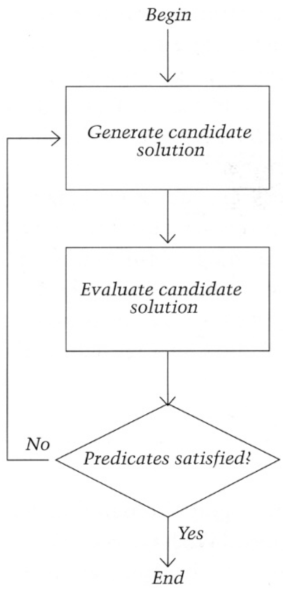

Figure 2.7: The design process as generate-test cycle (Mitchell, 1990: 180).

2.2.3 Interactive modelling and computational exploration

Patrik Schumacher is a theoretician and architect. Since 1988, he has been a partner in the office of Zaha Hadid Architects, and since 1996, Co-Director of the Design Research Laboratory at the Architectural Association, London. During the past 15 years, the office has established a unique design methodology, relying not only on the design skills of their employees and the intuition of the directors, but also on custom computational processes[1], which are partly developed at the Design Research Laboratory. In his latest book, "The Autopoiesis of Architecture", Schumacher picks up on Simon's conception of design as a problem-solving process and combines Simon's general terms with methods used in contemporary architectural design in general and specifically in the office of Zaha Hadid Architects. Schumacher intends "to probe and enhance both the creative productivity and the *rationality* of contemporary design processes. Accordingly, the design process is being theorized as *problem-solving process*. (The aesthetic and formal issues are also understood as problems to be solved.)" (Schumacher, 2012: 599). He emphasises that, especially for the description of the "micro structure of the design process, for the analysis of what goes on when design drawings or models are evolving" (*Ibid.),* the approaches developed by Newell and Simon in the field of cognitive information processing are suitable (Newell and Simon, 1972). Schumacher furthermore follows Simon's general description of the design process as a search through the space of possible solutions.

However, in architectural design processes, one design project is often explored both sequentially and parallel using multiple representations, for

[1] This was demonstrated at the lecture held by Shajay Bhooshan, Head of Computational Design Group, Zaha Hadid Architects, on 26th May, 2013 at ETH Hönggerberg (invited by Prof. Block)

instance, as sketches, drawings, reports, physical models and realistically rendered images. Each medium is used to represent different, but overlapping aspects of the design. The choice of representation is one aspect of design problem decomposition. Depending on the chosen decomposition, generate-test cycles might be established, or, by integration of constraints in the generation process, a directed search instead. Similarly to Simon, Schumacher rejects the idea of a fully formalisable design process, and states that formalisation, and hence computation, is only possible for "narrowly defined sub-problem[s]" (Schumacher, 2012: 603). Furthermore, he explains that "the conceptual apparatus presented here is […] construed to be more general in scope, encompassing both computational processes and processes based on human step-by-step modelling guided by language-based reasoning" (*Ibid.:* 292). In this sense, Sutherland´s SKETCHPAD not only represents the first prototype of a CAD system, but a design exploration system *avant la lettre*, allowing both human iterative modelling and automatic solving of narrowly constrained problems (Figure 2.1). Recently, the role of constraint modelling in the context of design innovation has been emphasised (Kilian, 2006); furthermore, a taxonomy of formal exploratory methods in architectural design based on their level of computational complexity has been established (Kotnik, 2010). Kotnik states that "Digital design, thus, is not about the formalisation of design processes or the automatization of decision making [...] but about the interaction of formal processes with architectural thinking" (*Ibid.:* 7).

2.2.4 Summary

A general notion of design as the *creation* of artefacts has been presented, together with three concepts for the systematisation and formalisation of the architectural design process.

Following Mitchell, design processes are defined as partly formalisable search processes; the phrase "partly formalisable" expresses that generally design processes cannot be fully formalised, hence cannot be entirely automated. The reason for this is the high complexity of non-trivial design problems. For the same reason, generally, the goal is to create *satisfying* design solutions, *optimal* design solutions are largely unfeasible. Usually a multitude of different search processes exist, based on different decompositions of the design problem. Different problems in decomposition lead to results with different 'styles'. In 'artistic' design processes, and also in architectural design processes, design goals may change during the search process. This might happen, when the designer is 'inspired' by an intermediate design state and therefore chooses to alter the initial design goals. This notion of the design process as a partly formalisable search process will serve as the foundation for the precise definition of the structural design process in Chapter 3.1.

Alexander proposes a catalogue of solutions for the elementary problems of architectural design in his book, "A Pattern Language". He has the ambition of thereby providing a universal collection of such elementary 'patterns' which enable the solution of the majority of all architectural design problems. Similar to his earlier concept of "constructive diagrams", each 'pattern' has the limited flexibility of adapting according to the properties of the individual project. Design approaches based on a universal catalogue of solutions to elementary design sub-problems are defined here as *combinatorial approaches*. In combinatorial design processes, the designer synthesises a design proposal by selecting and assembling such elementary solutions to standardised sub-problems.

Mitchell suggests the use of tailored sets of rules, also referred to as "shape grammars", for the formalised generation of architectural design proposals. These rules, or "design operations," enable the transformation and modification of abstract representations of buildings, such as drawings or CAD models. Within this notion of design, the designer creates a project by applying "design operators" in a specific sequence, starting with an initial design state, which could, for instance, be a representation of the site. The choice of a specific "shape grammar" determines the 'style' or "architectural language" of the possible design proposals. Such a 'grammar' might be combined with a test method for the evaluation of the design proposals due to criteria defined by the design goals. Mitchell suggests that the combination of formalisable "design operators" and formalisable test methods allows the automatisation of the search processes as a *generate-test cycle*. Design approaches based on the application of formal rules for the generation of design proposals are defined here as *generative approaches*. These approaches might either be executed by a human designer, or by a computer.

Schumacher follows Simon in the notion of design as a partly formalisable search process, and interprets Simon's ideas within the context of computational approaches in contemporary architectural design, such as *parametric modelling* and *scripting*.[2] Schumacher emphasises that automatisation in architectural design is feasible only for well-defined sub-problems. He proposes the notion of design in architecture as a multi-faceted process, encompassing both manual step-by-step modelling and the application of a computational process for the solution of specific, well-defined sub-problems. Design approaches consisting of alternating phases of manual modelling and computational solution are defined here as *interactive approaches*. The designer interacts with autonomous or formal processes, such as physical or digital computation processes, or methods of graphic calculus, in order to iteratively steer the search for a design solution.

[2] Both *parametric modelling* and *scripting* are common methods for end-user programming and CAD system automatisation in architectural design

3 Structural design processes

Simon's general notion of design as partly-formalisable search process has been described above, together with specific manifestations of the concept in architectural design. In Section 3.1, the term *structural design* is dissociated from *structural engineering*, and the notion of structural design as iterative search is introduced. In Section 3.2, different partly-formalised approaches to structural design are presented. And, finally, in Section 3.3, models and methods of one specific structural design approach, based on *form-finding* techniques, are illustrated in greater detail.

3.1 Structural design as search process

The development of *Structural Engineering* as a profession dates back to the early 19th century, and is tied to the establishment of polytechnic schools in Europe. Previously, the Architect or Master Builder had been responsible for the stability and safety of buildings. Structural design techniques were an integral part of architectural design knowledge, and passed on by word of mouth or textbooks, in the form of geometric rules and/or schematic drawings (Hauri, 1979). The first scientific approach to structural behaviour and the strength of materials dates back to experiments conducted by Galileo Galilei at the beginning of the 17th century; Henri Navier was the first to provide a general mathematical formulation for specific structural problems in the first half of the 19th century.

The development of structural theory can be divided into three main periods: establishment phase (1850-1875), classical phase (1875-1900) and consolidation phase (1900-1950) (Kurrer, 2008: 570-580). During these phases, practical design knowledge as inductive, synthetic approach was gradually replaced by a formal, analytical apparatus on a purely deductive basis. Structural theory is strong in analytical tasks, but rather weak and indifferent regarding design tasks (Polónyi, 1985). Building codes for structural engineering, such as for instance the *Tragwerksnormen* in Switzerland, defined by the codes 'SIA 260-269', represent the official 'design' methods derived from this apparatus, mainly dealing with questions of material-specific analysis and dimensioning.

One reason for this development, away from an integrative, holistic approach in the pre-scientific period of structural design, towards a narrowly defined deductive approach, is probably the "desire for academic respectability" as described by Simon, which led to a general loss of professional design competence during the 20th century in several disciplines (see Section 2.1). While *architectural design* represents the general problem of the holistic design of built artefacts, including all functional, technological, economical, ecological, cultural and aesthetic goals, *structural design* is often associated with the sub-problem dealing with the *dimensioning* of structural members and the verification of *usability* and *safety* from the perspective of statics. This condition is explicitly criticised by, among others, Mike Schlaich[3] (Schlaich, 2006) and Joseph Schwartz[4] (Schwartz, 2012). Here, the term *structural engineering* is used for approaches dealing with the narrow problem of dimensioning, while the term *structural design* is used for more general approaches that integrate additional architectural goals, e.g. aesthetic and economic questions with the goals of structural engineering.

Schlaich describes the structural design process as "iterative, cyclic, hopefully concentric and sometimes even slightly chaotic" (Schlaich, 2006). He furthermore outlines different phases of the structural design process, related to different representations and design spaces: conceiving, modelling, dimensioning, and detailing (Figure 3.1).

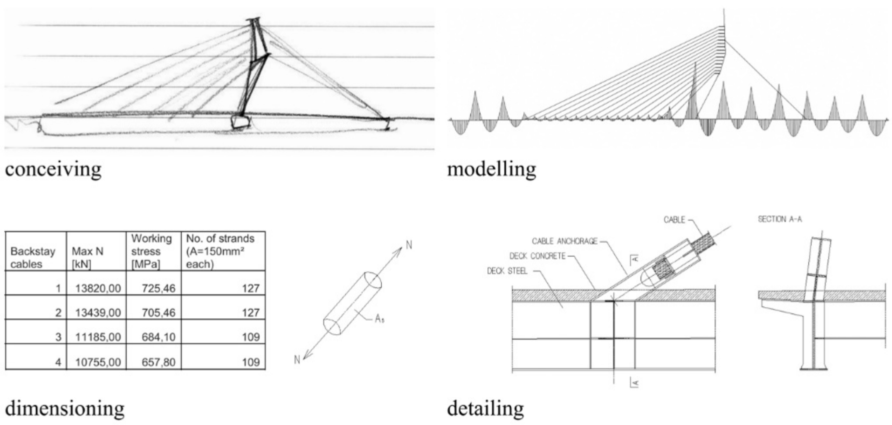

Figure 3.1: Different phases in the structural design process (Schlaich, 2006).

Heinz Hossdorf gives a diagrammatic description of the inner organisation of the design process, consisting of subsequent phases of intuitive changes and rational verification steps (Figure 3.2): "Das gegenständliche Entwerfen ist ein intellektueller Vorgang, in dem sich […] intuitive Einfälle mit Phasen von deren kritischer Überprüfung aneinander reihen, bis eine zufrieden stellende Lösung des Problems erreicht ist"[5] (Hossdorf, 2003: 131).

According to Hossdorf, the verification process itself is primarily concerned with the analytical proving of safety, but it might also direct the search process towards a solution through in-depth involvement with the problem: "Beim Entwurf von Tragwerken dreht sich die *Verifikation* vor allem um den möglichst objektiven Nachweis der Gebrauchssicherheit […] die vertiefte Auseinandersetzung mit den Problemen eines spezifischen Lösungsansatzes

3 "Structural design is reduced to dimensioning of sections"

4 "Structural calculations are still favoured over conceptual thinking – an approach that gets in the way of creative work"

5 "Design is an intellectual process consisting of alternating phases of intuitive ideas strung together with phases of critical assessment, eventually leading to a satisfying solution to the problem." (translation by the author)

[kann] u. U. auch richtungsweisende Anregungen für die weiter Suche bieten"[6] (*Ibid.*). The engineer Stefan Polónyi emphasises the need for such 'directing' methods in structural design; as an example, he mentions an analytical method that allows for the calculation of the optimal shape for thin concrete shells based on the boundary conditions and an intended stress distribution: "Die Wissenschaft ermöglicht es uns, Nachweise zu führen, aber zum Entwerfen brauchen wir *Vorweise*. […] Einen solchen Vorweis stellen […] die hautartigen Schalen dar, bei denen man den idealen Schnittkraftzustand formuliert und nach Annahme der Randbedingungen die Geometrie der Fläche errechnet"[7] (Polónyi, 1987: 146-147).

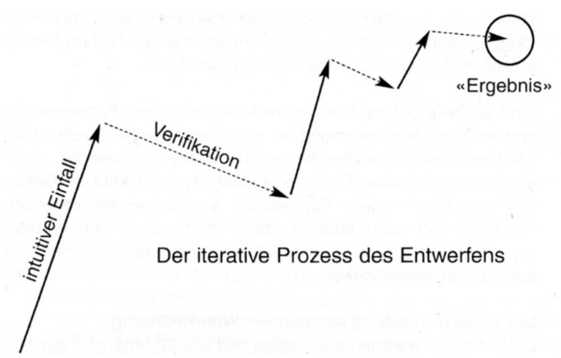

Figure 3.2: The iterative process of design ("der iterative Prozess des Entwerfens") consisting of alternating steps of design changes based on intuitive ideas ("intuitiver Einfall") and analytical verification of its feasibility ("Verifikation"), eventually leading to a final design state ("Ergebnis") (Hossdorf, 2003: 131).

3.2 Formal approaches in structural design

In this Section, different approaches to structural design as iterative search in the sense of Hossdorf are presented; all search processes incorporate a substantial body of formalised techniques. The approaches conceptually differ in the question of how the problem is decomposed, and which parts of the search process are formalised. These different approaches to structural design are categorised in three Sections, relating to *combinatorial*, *generative*, and *interactive* approaches as defined in Section 2.2.4 for architectural design.
The concepts presented in Section 3.2.1 are classified as *combinatorial design* approaches. Structural design based on modular systems forms the basis for a design methodology consisting of the selection and assembly of given structural elements, comparable to the use of Alexander's *pattern language*. Furthermore, Alexander's concept of *constructive diagrams* is inspired and directly related to the form-generating application of *structural diagrams*.

[6] "In the design of structures, verification methods are mainly used to objectively assess structural safety [...] in-depth involvement with the problems of a specific solving approach might provide indicative suggestions for further search though." (translation by the author)

[7] "Science enables us to analytically assess safety, but for design we need suggestive methods instead. Such a suggestive method exists, for example, for thin-shell structures: the geometry of the shell surface is calculated based on the assumption of the inner force distribution and the boundary conditions." (translation by the author)

The methods presented in Section 3.2.2 are classified as *generative design* approaches. Formal geometric rules are used for the creation of structural design proposals. Fully automated design processes, as a combination of generative rules and automatic structural analysis techniques, are described; these processes form *generate-test cycles* as defined by Mitchell.

The methods presented in Section 3.2.3 are classified as *interactive design* approaches. Methods based on physical, graphical and digital computational processes are incorporated within the manual step-by-step modelling workflow. These methods are used to solve sub-problems of structural design; here for instance the geometric adaptation of a structural system towards a state of static equilibrium.

3.2.1 Modular systems and structural diagrams

In his influential book, "Wendepunkt im Bauen", the architect Konrad Wachsmann describes the idea of an industrialised building process, based entirely on prefabricated elements that are assembled on site (Wachsmann, 1959). The proposed design process is determined by a rigid logic of spatial grids and proportions (Fig 3.3).

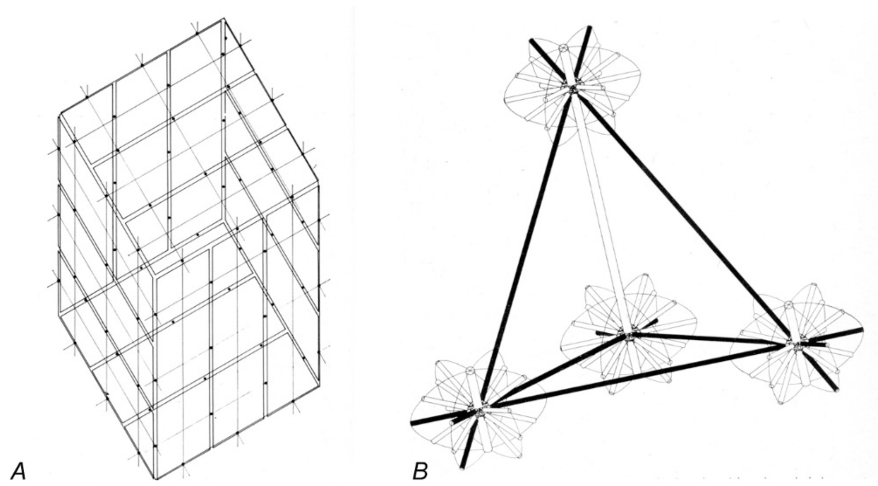

Figure 3.3: Building components aligned in (*A*) an orthogonal grid, and (*B)* an tetrahedral grid (Wachsmann, 1959: 66, 58).

The creation of architectural space is achieved through the combination of mass-produced, standardised elements in three dimensions; Wachsmann emphasises the "almost endless possibilities" in the combination of horizontal and vertical surfaces, within the grid, resulting from the logic of industrial assembly (Wachsmann, 1959: 10). The geometric systems of proportion and measurement for building parts are characterised by the logic of mass-production: "Die durch Massenproduktion gleicher Teile bedingten Ordnungssysteme bestimmen Flächen-, Körper- und Raummaße"[8]. (Wachsmann, 1959: 54). The goal of this formalised approach to architectural

[8] "The measures of areas, solids, and volumes are determined through the metric systems of mass production" (translation by the author)

design, based on standardised building components and geometric rules, is to create economic buildings with high standards in comfort and function, and thereby eventually enable a better life for all humans: "[Das Ziel ist] die zweckmäßigsten Materialien in der bestmöglichen Form und dem höchsten Leistungsstandard in der ökonomischsten Weise den berechtigten Ansprüchen aller Menschen gleichermaßen nutzbar zu machen"[9] (Wachsmann, 1959: 10).

3.2.1.1 *Modular systems*

In the 1950s, Wachsmann developed an innovative modular construction system for large span roofs for US Air Force hangars. The system forms a space frame structure based on a tetrahedral grid, constructed from bars of the same length, and one standardised connection node; the system allows large spaces of varying plan and section to be covered efficiently (Figure 3.3).

Wachsmann himself was working primarily as a researcher, and left the design and realisation of buildings based on his ideas and concepts to others. One prominent building in this context is the Kantonsschule Baden, Switzerland, built in 1964 by Fritz Haller, who also wrote the foreword for the 1989 edition of "Wendepunkt im Bauen".

In 1967, the book "Tragsysteme/Structure Systems" was published both in German and English, and soon reached wide international distribution among architects and students (Engel, 1967). It represents a catalogue of exemplary drawings that visualised structures assembled from basic elements in various combinations, mainly based on orthogonal, hexagonal and rotational grids (Figure 3.4). The author formulates the goals as follows: "While this book concerns itself with the systems of architectural structures, it is clearly focused on what is the prime reason for such systems: the creation of architectural form and space" (Engel, 1967: 9). His conception of architectural space is based on the repetitive combination of basic structural types such as trusses and more complex two-dimensional elements such as hyperbolic-paraboloid shells. In comparison to Wachsmann, a greater number of different grids and formal variations is presented, the focus is on structural aspects; questions related to fabrication and construction are not discussed.

[9] "[The goal is] to provide the most appropriate materials in the best possible form with the highest performance standards in the most economical way; and thereby satisfying the rightful demands of all people equally" (translation by the author)

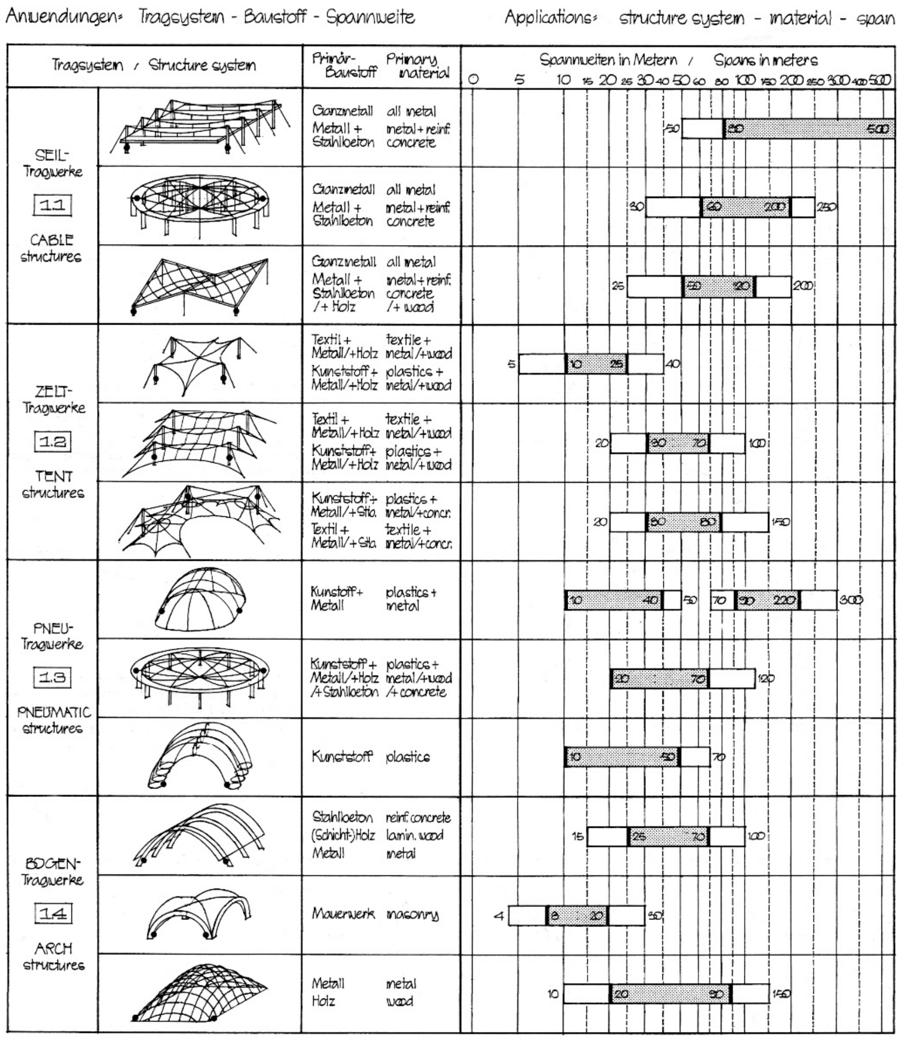

Figure 3.4: Different, clearly distinct types of structural systems for wide-span roofs (Engel,1997: 62)

3.2.1.2 Structural diagrams

Only one year after Wachsmann's book, another significant book was published: "Strukturformen der modernen Architektur", considered to be the first widely recognised approach to structural design as synthetic practice within architecture (Siegel, 1960). The book represents an interpretation of the formal vocabulary of post-war modernism from the perspective of structural theory, by means of diagrammatic drawings. The author emphasises, with a strong moral imperative, the desired close connection between art and technology in modern architecture: "Mies van der Rohe sagte: Funktion ist eine Kunst. Mit Absicht falsch gedeutet, wurden solche Worte oft genug ins Lächerliche verzerrt. Richtig verstanden, drücken sie die Einheit aus, die Kunst und Technik in der modernen Architektur bilden sollten. Aus dieser Einheit geborene Formen, deren Züge von der Technik des Bauens mitgeprägt sind, nennen wir

»Strukturformen«"[10] (*Ibid.:* 7). Siegel derives geometric rules for "good proportions" of buildings from the rules of classical analytic statics, in the sense of the efficient use of building materials (Figure 3.5).

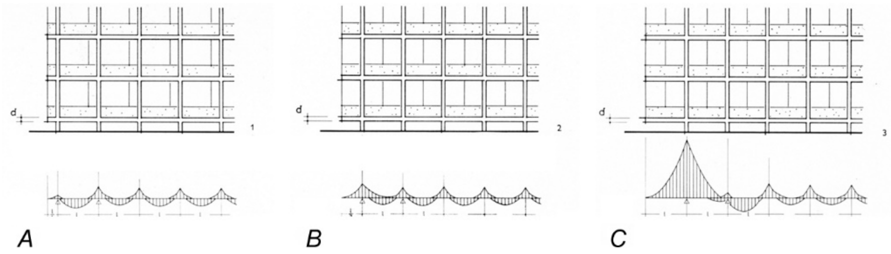

Figure 3.5: Proportion between main span and cantilever of a slab in a skeleton structure: (*A*) cantilever is too short and could carry more load, (*B*) cantilever has efficient span, (*C*) cantilever is too long and would require an adaptation of the section (Siegel, 1960: 48).

Furthermore, Siegel presents examples for efficient, hence 'good' shapes of structural elements, based on moment diagrams (Figure 3.6).

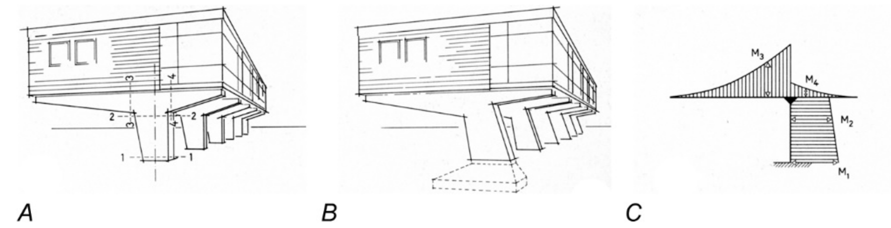

Figure 3.6: Form of a structural element: (A) initial proposal, (B) optimised shape, and (C) moment diagram (Siegel, 1960: 114).

This application of analytic diagrams for design purposes is almost congruent to Alexander's conception of the use of constructive diagrams in design. Like Alexander himself, Siegel also refers to the work of Nervi, for example, the form of the cantilever of the Florence Stadium, completed in 1932 (Siegel, 1960: 128-133). Another diagrammatic approach was applied by Nervi in the design of the Gatti Wool Mill, Rome, in 1953: "the ribs of the slab follow the isostatic lines of its principal bending moments. These lines depend exclusively on the loading conditions of the structure, and it was amazing to find out that by thus limiting our task to the interpretation of a purely physical phenomenon, we were able to discover unexpected and expressive new forms" (Nervi, 1956: 101). It is controversially discussed among engineers whether this strategy indeed improves structural behaviour (Lochner and Sobek, 2008). Nevertheless, a similar concept was proposed in 2007 by Meili Peter Architects, with Jürg

[10] "Mies van der Rohe used to say: function is an art. Deliberately misinterpreted, such words have often been ridiculed. If understood correctly, these words express the unity that Art and Science should form in modern Architecture. Forms emerging form this unity, shaped among others by building technologies, are referred to as »structural forms«" (translation by the author)

Conzett as engineer, in the competition project for the Art Gallery in Perm, Russia: façade openings are designed based on stress trajectories in a cantilevering structural wall (Conzett, 2008).

3.2.2 Rules, generate-test cycles and directed search

In 1985, the German engineer, Stefan Polónyi, criticised the predominant rigid distinction of structural systems: "There is a continual transition from the cable-strengthened beam to the frame girder. Yet, naturally it is not the done thing to mingle the two systems; a frame girder having a flexurally rigid top chord, for example, is generally avoided because it does not represent a distinct system within our systematics" (Polónyi, 1985: 40). He furthermore suggests that distinct elements be structurally combined, e.g. floor plates and walls, in order to create more efficient and elegant buildings (Polónyi, 1987: 143). Fifteen years later, Cecil Balmond, Chairman of Ove Arup & Partners, formulated the same idea more radically: "There is a lot more to structure than strict post and beam, slabs may fold and act as lines of vertical strength, beams may bifurcate and change shape, columns can serve as beams, the ingredients are all there to evolve form in fascinating ways." (Balmond, 2002: 14).

Since the 1990s, the use of new design and planning software has enabled architects to efficiently realise buildings with complex, often double-curved forms. Frank Gehry used design software from the aerospace and automotive industries for the construction of the Guggenheim Museum in Bilbao, Spain, which opened in 1997. Greg Lynn demonstrated the use of design tools from the animation film industry in architecture (Lynn, 1999), based on a unified mathematical description of freeform curves and surfaces, referred to as non-uniform rational Bézier-splines (NURBS). In 1999, Peter Cook and Colin Fournier won the competition for Kunsthaus Graz, Austria; the competition project was primarily developed using clay models complemented with computer models. Detailed CAD models based on NURBS surfaces were used for the construction documentation of the freeform building shape in later design stages (Figure 3.7) (Jonkhans, Argekunsthaus *et al.,* 2002).

The shift from industrial mass production to mass customisation with computer-controlled manufacturing processes enabled the affordable fabrication of façades and structures with complex geometries consisting of elements that all have individual geometry. The introduction of structural analysis software in the 1980s based on the Finite Element Method (FEM) enabled calculation and dimensioning of almost any form, without the necessity of distinguishing elementary structural systems. For the structural design process of the Kunsthaus Graz, Bollinger + Grohmann structural engineers used emerging digital design methods such as parametric modelling and end-user programming (also known as *scripting*) for a partly automatised generation of different layouts of the structure for the given freeform volume. (Rappaport, 2007: 64).

The goal of such complex building shapes is usually associated with the "Bilbao Effect": the media attention created by Gehry's Guggenheim Museum triggered a strong economic stimulus for the whole region, based both on increased tourism and direct investments. From this perspective, the higher building costs caused by the complex building geometry had already paid off within a few years (Jencks, 2011: 204-211).

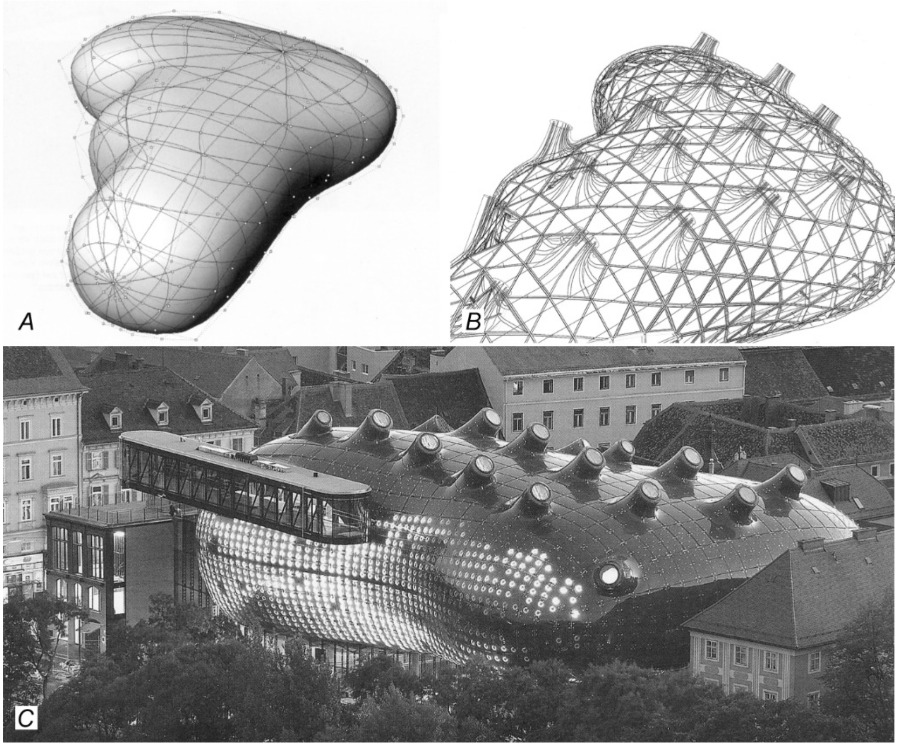

Figure 3.7: Kunsthaus Graz. (*A*) NURBS-based freeform model (Jonkhans, Argekunsthaus *et al.*, 2002: 126), (*B*) a detailed FE model of the steel structure, and (*C*) a photo of the building (Schittich and Cachola-Schmal, 2012: 11, 54)

The role of the engineer within the new logic of "attention economy" (Franck, 1998) is twofold: on the one hand, as structural engineer, he is still responsible for "making the building stand", possibly by small changes in the form through structural optimisation or rationalisation processes; on the other hand, since anything that attracts attention is desirable, as structural designer or design engineer, he might also act as creator of new forms, unexpected spatial concepts, or innovative constructions (Oxman and Oxman, 2010). The following three passages present structural design approaches that are related to Mitchell's concepts for formalised design processes.

3.2.2.1 *Generative rules*

In 2002, the architect Toyo Ito was selected to design the prestigious annual pavilion for the Serpentine Gallery in London, UK. The initial idea was to create a box with an irregular façade and roof. As structural designer, Balmond developed an algorithm for the generation of a geometrical grid that appears irregular, and which is applicable as structural grillage (Balmond, 2006: 62-67); the algorithm incorporates a structural logic that is distantly related to a reciprocal frame structure (Figure 3.8). The position of the grid's axes is obtained by a series of squares that are concentrically scaled and rotated. Through the decentred placement of the grid on the roof area, the impression of a highly irregular grid is achieved. Based on this grid, the structural grillage of the roof and the façades is developed; analysis and dimensioning was conducted with commercial structural analysis software in a subsequent step. Already in 1995 Balmond had formulated a similar pre-computational algorithm for the design of the Chemnitz Stadium roof in Germany, designed with the architects Peter Kulka and Ulrich Königs: "At competition stage, the multiple structural rings that provided our solution were placed by eye. Did intuition have any rationale to it; […] in other words, could the roof be self-generated by a chaotic rule?" (*Ibid.:* 66). Based on an initial idea of cantilevering circular segments supporting each other reciprocally, Balmond proposed a mathematical definition of a cycloid curve based on the given irregular outline of the roof and site. Such a functional or algorithmic definition of a shape based on given geometric and/or numeric input parameters is usually referred to as *parametric definition*. Contemporary CAD software packages often offer different ways, such as *scripting* or *visual programming*, to implement *parametric models*.

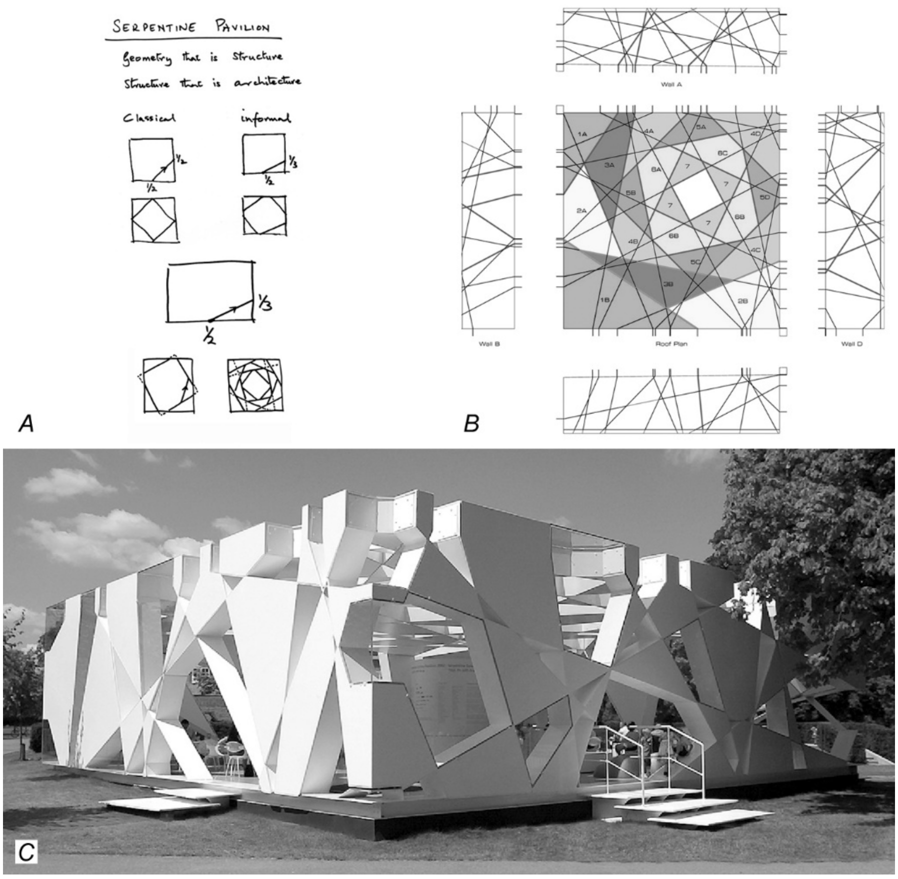

Figure 3.8: Serpentine Gallery Pavilion 2002. (*A*) Generative method, (*B)* resulting structural roof and façade layout, and (*C*) photo of the building (Balmond, 2006: 49, 52, 53).

3.2.2.2 Generate-test cycles

Bollinger et al. present an automated generate-test method for the creation of both structurally efficient and irregular truss structures for a competition of a pedestrian bridge in Reden, Germany (Bollinger, Grohmann et al., 2010a). The algorithm starts with a random placement of diagonals between given upper and lower chord; all diagonals are evaluated regarding structural performance using a standard analysis software. Subsequently, the positions of diagonals with bad performance are changed. These steps are repeated until a satisfying performance is obtained (Figure 3.9).

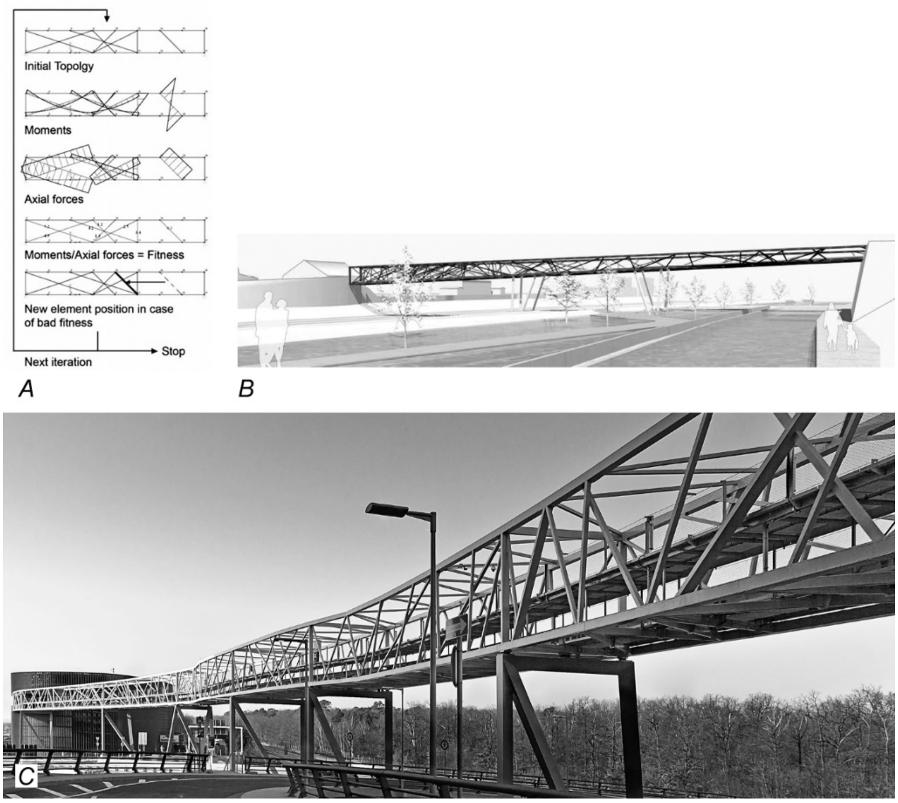

Figure 3.9: Competition project by FloSundK Architekten with Bollinger & Grohmann. (*A*) generate-test cycle for the design of a pedestrian bridge with irregular truss structure, and (*B*) rendering of the project (Bollinger, Grohmann *et al.,* 2010b: 194). (*C*) Photo of the Skylink bridge (Fahlbusch, Hofmann et al., 2012).

A few years later, Bollinger + Grohmann applied a similar generate-test method in the design of the Skylink bridge at Frankfurt Airport, Germany, completed in 2012 (Fahlbusch, Hofmann *et al.*, 2012). The authors describe their goal as the creation of an irregular structure, with no similarity to traditional structural systems: "Am Anfang stand das Bemühen, eine unregelmäßige Struktur zu erzeugen, die nicht auf traditionellen strukturellen Typologien beruht. […] Das Ergebnis weist meistens einen hohen Grad an Irregularität auf; die Tragwirkung wird komplex und damit nicht mehr ohne weiteres erklärbar"[11] (*Ibid.:* 640). Similar generate-test cycles in the context of structural design were previously presented by the third author, Tessmann, in his PhD thesis: a variety of automated processes were implemented, using different, project-specific

[11] "The initial idea was to create an irregular structure which is not based on a traditional structural typology. […] The result exhibits in most cases a high degree of irregularity, the structural behaviour of the structure becomes complex and hence is not self-explanatory" (translation by the author)

generative principles, combined with commercial FEM solutions for structural testing (Tessmann, 2008). Recently, Tessmann published a comprehensive summary of similar techniques developed in the office of Bollinger + Grohmann (Tessmann, 2013).

3.2.2.3 Directed search

The engineer, Mutsuro Sasaki, developed optimisation and generation methods for three-dimensional freeform structures as automated, directed search processes. He distinguishes two different approaches: "using the computer as correction tool", and "using the computer to create a shape" (Sakamoto and Ferre, 2008: 68). Sasaki presents two different methods for the two tasks: *Sensitivity Analysis* (Sasaki, 2007: 102-105) for the optimisation of given freeform shells in the sense of a minimisation of strain energy, and *3D Extended Evolutionary Structural Optimisation (Extended ESO)* (*Ibid.:* 106-109) for the generation of structures with efficient mechanic performance due to uniform stress distribution and minimised bending. Both methods are implemented as directed generate-test cycles, based on a custom FEM analysis software. The analysis results from the FEM component 'directs' the generative method in both approaches towards a better solution[12].

Sensitivity Analysis has been applied for several realised projects that incorporate freeform concrete shells: the Community Centre in Kitagata, Japan, by Arata Isozaki (2002), and the Island City Central Park in Fukuoka, Japan (2003-2005) and the Crematorium in Kakamigahara, Japan (2004-2006), both by Toyo Ito. For the application of the method, the architect provided an intended first shape of the freeform shell, and then the application of the Sensitivity Analysis method created an optimised or 'corrected' shape (Figure 3.10).

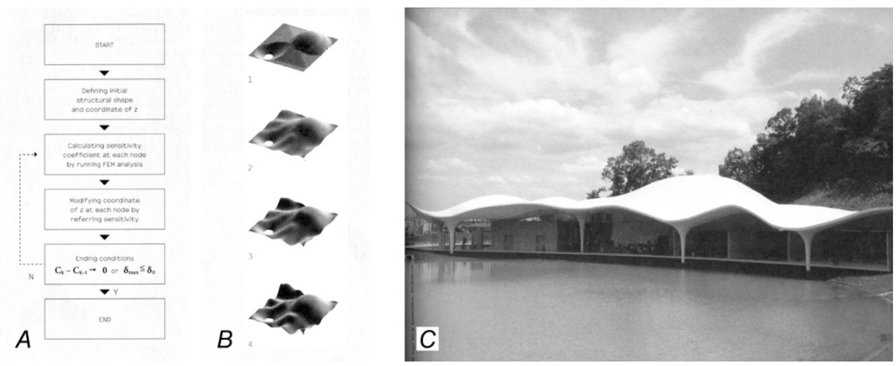

Figure 3.10: Sensitivity Analysis method. (*A*) Flow diagram (see Figure 9.10 for larger diagram), and (*B*) geometric case study (Sasaki, 2007: 105). (*C*) Photo of the Crematorium in Kakamigahara (Sakamoto and Ferre, 2008: 99).

The Extended ESO method combines a finite element analysis routine with a method of material addition and elimination, based on local stresses in the structure (Figure 3.11). The resulting organic shapes resemble bone or tree structures; Sasaki describes the qualities of the method for design as follows: "it

[12] The method uses the FEM software to calculate a gradient for the directed search

is possible to generate utterly unprecedented structural shapes, heretofore unseen and unimaginable, much like artificial life" (Rappaport, 2007: 171). Together with Isozaki, this method was applied in the design of a huge roof as part of a competition project for the new Florence Train Station in 2002. The project remained unbuilt, but a formally very similar project, Education City Convention Centre in Qatar, Doha, by Isozaki and Sasaki, was completed in 2011. However, the realisation of the project undermined the initial idea of the form: the structure consists of conventional welded polygonal steel tubes on huge concrete supports, and a cladding creating the 'optimised' shape was added afterwards (Sakamoto and Ferre, 2008: 108-115).

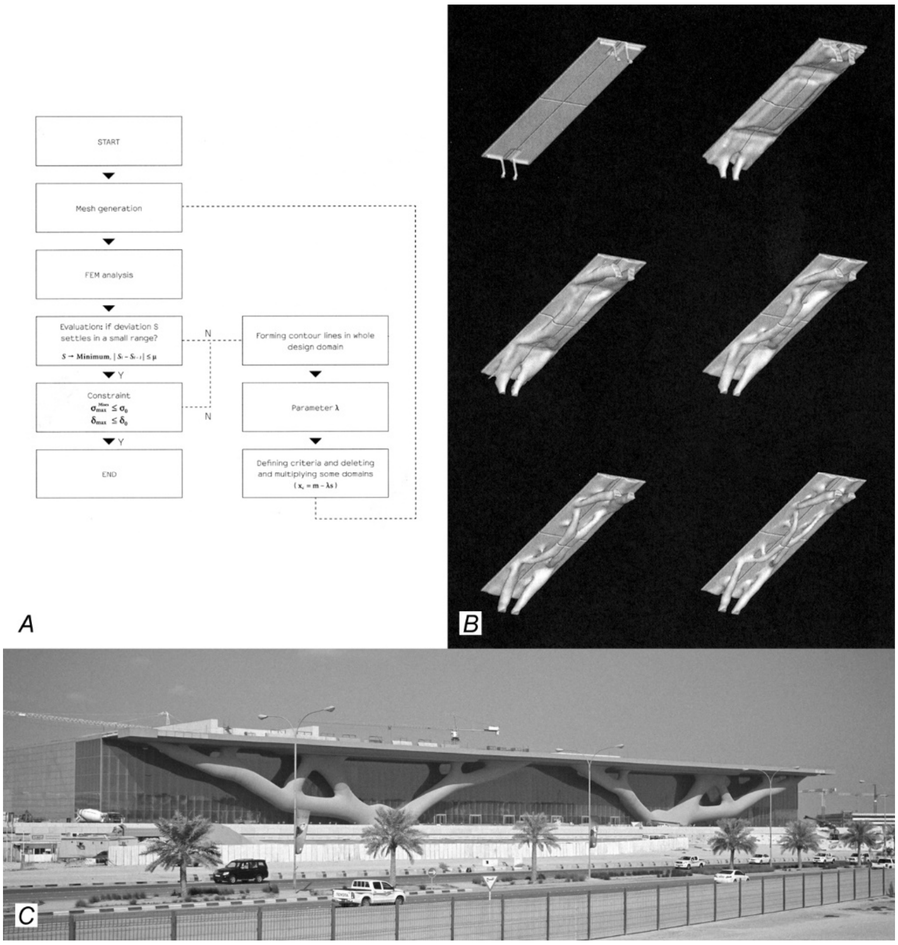

Figure 3.11: 3D Extended ESO method. (*A*) Flow diagram (see Figure 9.11 for larger diagram), and (*B*) geometric case study (Sasaki, 2007: 108, 109). (*C*) Photo of the Qatar Education City International Convention Centre (Agnew, 2013).

3.2.3 Interactive modelling and form finding

Simon, and also Schumacher, doubt the existence of formal, fully automatable approaches to non-trivial real-world design problems, due to their general complexity and multi-faceted character. The examples in the previous Section present formal methods for solving narrow, well-defined sub-problems of structural design, e.g. the optimisation of the form of a concrete shell, which is initially already close to a satisfying result, or finding a good solution within a set of parametrically defined trusses. Similarly, the example of the Education City Convention Centre in Qatar shows the generation of an efficient structure according to mechanical principles of uniform stress, incorporating only a minimum of architectural context.

Hossdorf advocates a complex understanding of the structural design process that incorporates both formalisable, analytic methods and non-formalisable, intuitive approaches (Figure 3.2): he denies the existence of statically 'correct' building shapes in a strictly scientific sense, and emphasises the importance of the engineer's intuition within the boundaries of physical feasible solutions and construction methods: "Es existiert keinerlei natur- bzw. ingenieurwissenschaftliche Gesetzmäßigkeit, aus der sich die statisch »richtige« Form einer Tragkonstruktion ableiten lässt … die »richtige« Formgebung ist […] innerhalb der Grenzen des physikalisch Möglichen eine Frage des *subjektiven Ermessens*. Jeder neue Baustoff, jedes neue Herstellungs- und Konstruktionsverfahren hat seine eigene Wesensart, die es zu entdecken, zu begreifen und für die jeweilige Bauaufgabe auszudeuten gilt"[13] (Hossdorf, 2003: 130). Hossdorf points out that the goals of structural design can never be described in a purely rational, formalisable way, and that the momentum of the design process might possibly alter the initial goals: "Das Resultat dieses iterativen Vorgangs ist – im Gegensatz zur Rechenaufgabe – in Anbetracht des subjektiven Ermessensspielraums der Lösungsschritte weder durch die angegangene Problemstellung vorherbestimmt, noch kann es […] je (objektiv) perfekt sein"[14] (*Ibid.:* 131). This conception of structural design as search process with possibly changing goals resembles Simon's description of the artistic, highly creative design process of, for example, oil painting (see Section 2.1).

Such a complex understanding of structural design as an 'artistic', creative process has been illustrated by the engineer and historian, David Billington: "The disciplines of structural art are efficiency and economy, and its freedom lies in the potential that it offers the individual designer for the expression of a personal style motivated by the conscious aesthetic search for engineering elegance" (Billington, 1983: 5). Billington furthermore identifies the most relevant building typologies of structural art: "the art of the structural engineer […] appears most clearly in bridges, tall buildings, and long-span roofs" (*Ibid.:* 4). One of the protagonists of Billington´s "structural art", the Swiss engineer Christian Menn, emphasises the partly-formalisable character of the structural design process in bridge building, referring to the uncertainty of cost, and the impossibility of formalising aesthetic judgment (Menn, 1990: 49). Billington highlights the strong Swiss roots of structural art (Billington, 2003), based on the legacy of graphic statics, extending to contemporary truss modelling

[13] "There is no Law of Nature or Law of Engineering that allows for the derivation of the statically »correct« form of a structure … the »correct« form is, […] within the boundaries of the physically feasible, a question of *subjective judgment*. Any new material, any new method of fabrication and construction has its own specific character; this character has to be discovered and interpreted individually for any building task." (translation by the author)

[14] "Contrary to an arithmetic problem, the result of this iterative process is, regarding the subjective freedom in choosing the steps for solution, neither determined by the initial goals, nor can it ever be regarded as (objectively) compulsory." (translation by the author)

approaches (Bögle and Billington, 2009) such as the strut-and-tie method (Schlaich, Schäfer *et al.*, 1987) and stress fields (Muttoni, Schwartz *et al.*, 1997).

In this Section, different formal methods are described that have been used to 'direct' the human step-by-step search process towards efficient forms, in the sense of Polónyi's "Vorweise"[15] (see Section 3.1). These methods are generally referred to as "form-finding" methods (Kotnik, 2011), and aim at the creation of almost bending-free structures. Three examples of iterative design processes based on alternating human modelling steps and 'directing' form- finding steps are presented below.

3.2.3.1 Hanging models: Antoni Gaudí

In the years from 1898 to 1908, the architect Antoni Gaudí designed the Church Colonia Güell near Barcelona, Spain, using hanging models (Tomlow, 1989: 134-137), based on a physical principle first described by Robert Hook:[16] the inversion of a hanging chain is the optimal form of an arch of uniform thickness under self-weight (Figure 3.12). Hanging strings or chains function purely by tension, and corresponding inverted and scaled real-size building structures function only by compression, which is a prerequisite for masonry buildings. The design process based on hanging models enabled the exploration of new building forms which could be constructed in masonry. In later stages, Gaudí replaced hanging chain models by weighted hanging networks of strings, in order to control the shape more precisely through freedom in the distribution of weights. In a real building, these weights are represented by the self-weight of walls and arches, and through the weight of additional crenels and pinnacles. This approach based on hanging models allowed Gaudí to explicitly integrate the constraints of compression-only forces in the modelling processes, thus 'freely' exploring form within these constraints. Possible "design operations" in the sense of Mitchell were a modification of the network such as changing the length, adding, or removing strings, and adapting the weight of sachets. With each local design operation, the model instantly adapted its form to a new equilibrium state, in an automatic 'self-forming' (Gaß, 1990: 0.16-0.17) or form-finding process. For architectural evaluation of these abstract hanging models, Gaudí rendered the building by painting over model photos (Tomlow, 1989: 64-71). Due to an economic crisis and other non-technical circumstances, only the basement and the crypt were realised, and were inaugurated as a church in 1915 (*Ibid.:* 18-19).

[15] "suggested methods" (translation by the author)

[16] "As hangs the flexible line, so but inverted will stand the rigid arch." Robert Hook 1675, quoted after (Heyman, 1997: 7)

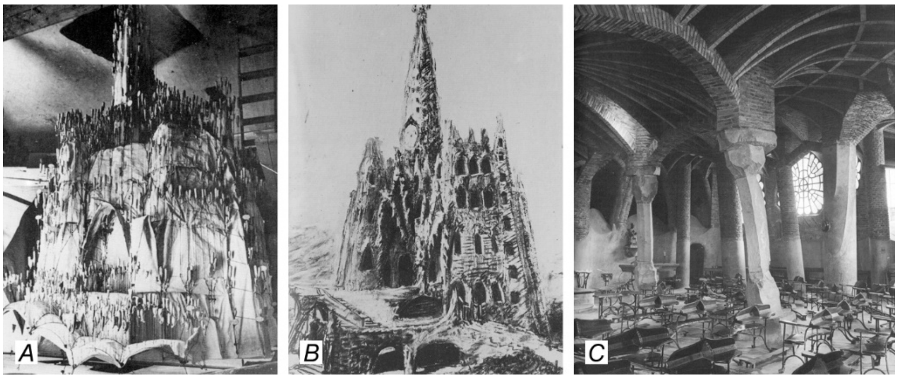

Figure 3.12: Colonia Güell Church. (*A*) Upside-down turned photo of a hanging model, (*B*) overpainted photo of the model, and (*C*) photo of the realised crypt (Tomlow, 1989: 118, 119, 83).

3.2.3.2 Graphic statics: Jürg Conzett

In 1999, the Swiss engineer, Jürg Conzett, used a form-finding method based on graphic statics for the design of a suspension bridge in the Grison Alps, Switzerland (Figure 3.13) (Mostafavi, Conzett *et al.,* 2006: 101-109). An earlier bridge at the same location had been destroyed by rockfall and, in order to find a safer solution for crossing the valley, it was decided to shift the position of the bridge. The topology at the new location led to an asymmetric, angled walkway. The available wire rope allowed for a maximal inner force of 393 kN. During the design process, an equilibrium shape of the suspension cable had to be determined; furthermore, under the maximal snow load, the inner cable force should be a constant 393 kN in all segments of the cable. This sub-goal was achieved by the application of methods from graphic statics. Graphic statics, a vector-based method for structural analysis and design using the means of drafting, is based on pairs of *reciprocal diagrams*, a *form diagram*, representing the structure's geometry, and a *force diagram*, representing the magnitudes of the inner forces in the members of the structure (for a more detailed description of graphic statics see Section 3.3.2). By first constructing the force diagram with a set of radii corresponding to the given maximum force of 393 kN in the segments of the suspension cable, the desired form of the cable could be derived from this force diagram and the given support points from the form diagram in an iterative manner. In subsequent design steps, diagonal cables between deck and hanger were introduced, to achieve higher stiffness, hence creating greater walking comfort.

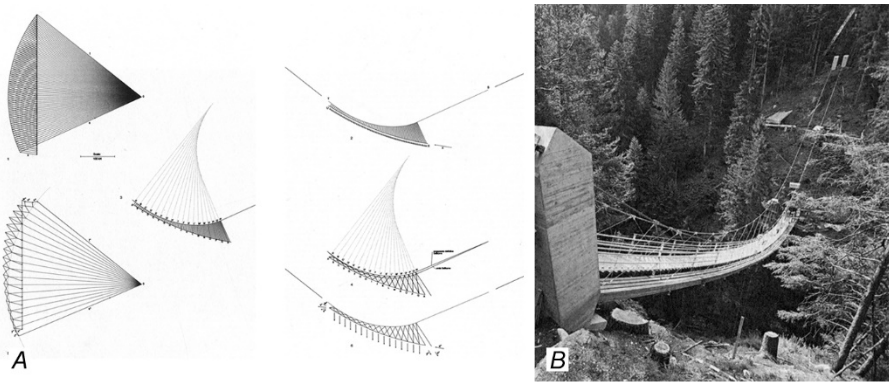

Figure 3.13: Second Traversiner Steg. (*A*) Graphic static drawings, and (*B*) the bridge during construction (Mostafavi, Conzett *et al.,* 2006: 102–103, 108).

3.2.3.3 Computational form finding: Schaich, Bergermann + Partner

The architects Gerkan, Marg and Partner, together with engineers, Schaich, Bergermann and Partner, designed the Moses Mabhida Stadium for the FIFA World Cup 2010 in Durban, South Africa (Figure 3.14), based on a winning competition entry. The competition project and the reworked final design were derived in a dialogic process between the architects and engineers; or, to put it in the words of Volkwing Marg: between "form finding and form fixing" (Jäger, 2011). Already during the competition phase, the "classical typology" of the wheel roof of pre-stressed spokes resting on columns (Bergermann and Göppert, 2000) had been adapted: in order to create a 'landmark' with a remarkable silhouette, a high steel arch was introduced, from which the spoked wheel roof could be suspended. Furthermore, the architects proposed replacing the initial simple arch by a splayed arch with one bifurcating end, referring to the 'Y' shape in South Africa's flag (Jäger, 2011). After the competition phase, the structural principle of the spoked wheel roof itself was altered, by breaking the continuous compression ring into two horizontal compression arches, and supporting them at the main arch (Figure 3.14 *A*, *B*). This decision "allowed the plan layout of the cable structure to be adapted perfectly to the depth of the grandstand" (*Ibid.:* 22). During the design phases, the goal of the intermediate form-finding steps was to minimise bending in the compression elements, and to find a set of pre-stressing forces that creates an equilibrium shape as close as possible to the modelled geometry: "The form-finding process was a semi-automatic step-by-step process in which analysis output data, deflections and element forces were evaluated and used, reformatted as an input for the next geometry input and pre-stress forces" (Figure 3.14 *C*, *D*) (Balz, Göppert *et al.,* 2009).

The third author, Roman Kemmler, describes in a different publication the general method for form finding of pre-stressed roofs and bridges used in the office of Schaich, Bergermann and Partner, based on pin-jointed truss models (Kemmler, 2012) (see Section 3.3.3). From the text it is not clear if this method was used in the form-finding process of the stadium in Durban. After a satisfying form had been found, "the structure was loaded and studied using typical environmental loads" (Balz, Göppert *et al.,* 2009), e.g. wind loads. Based on these studies, the final form, member dimensions and construction details were developed. The design goals of this project, the economy in material use due to a largely bending-free structural behaviour and the formal elegance due to the unity between form and construction, represent the general goals of "lightweight design" as formulated by Jörg Schlaich (Schlaich, 2003).

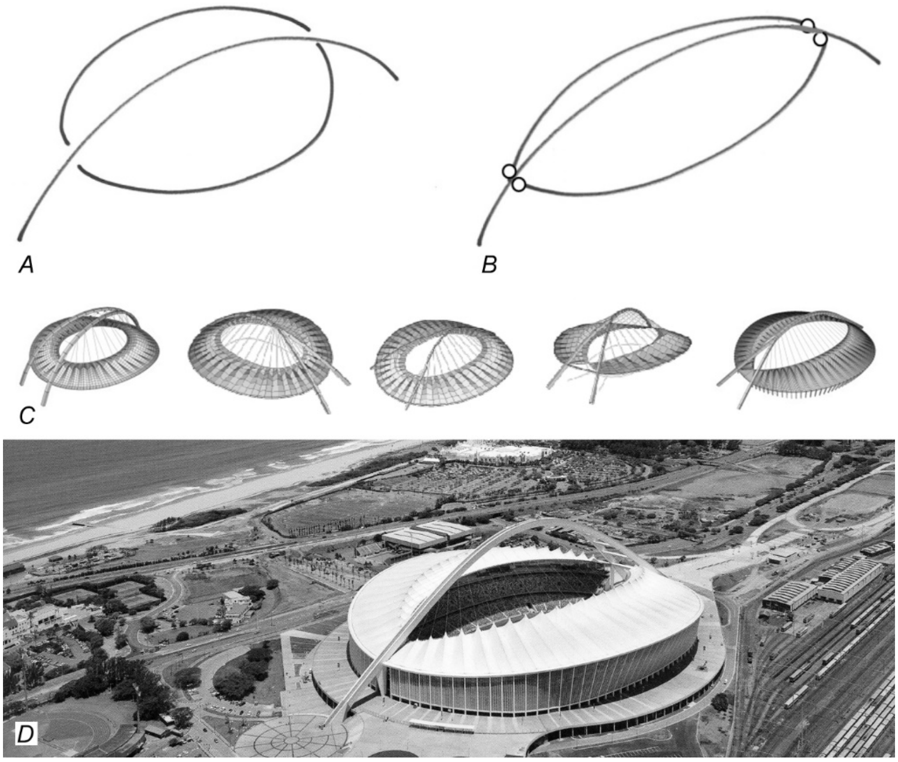

Figure 3.14: Moses Mabhida Stadium. (*A*) Initial structural concept, (*B*) final structural concept, (*C*) intermediate states of the analysis model during the design process (Balz, Göppert *et al.,* 2009: 1232, 1233), and (*D*) photo of the realised project (Bögle, Brensing *et al.,* 2011: 23).

3.2.4 Summary

Following Schlaich and Schwartz, the term *structural design* is delimited here from *structural engineering*. While *structural engineering* is concerned with the narrow technical goal of structural analysis and dimensioning, *structural design* describes a more holistic approach that is able also to include 'non-structural' aspects such as architectural space, formal appearance, as well as economic, ecological and social factors. These 'non-structural' aspects are usually already included at the very outset of the design process, during the development of the initial concept, referred to as the 'conceiving' phase (see Figure 3.1). Hossdorf gives an abstract description of the organisation of the structural design process (see Figure 3.2), which is interpreted here as a specific case of the design process as iterative search, as described by Mitchell (see Figure 2.6). All approaches presented in this Section deal with structural design in general, although they substantially differ in several aspects: the role of the designer in the process, the applied computational methods, and their formal/spatial flexibility.

The methodologies described by Wachsmann, Siegel and Engel have in common the fact that their conception of structural design is based on *combinatorial approaches*. The designer creates the structure in a highly controlled, direct modelling process, consisting of the manual selection and combination of pre-defined structural elements. The catalogue of elements can either be very restricted, sometimes consisting of only one type of node and one type of bar (Wachsmann´s space frame systems, Figure 3.3), or richer, consisting of a variety of different structural typologies (e.g. in Engel's book, "Structure Systems", Figure 3.4). The applied formal processes are based on classical analytical statics and enable rough estimations of dimensions and simple shape optimisation of standard elements based on structural diagrams. The designer is formally limited to spatial configurations based on planar or spatial regular grids; in this context, the most common grids are radial, rectangular, triangular or hexagonal.

The approaches described by Balmond, Bollinger + Grohmann and Sasaki have in common the fact that their conception of structural design is based on *generative methods*. The designer is responsible for the setup of a tailored algorithmic process, the process itself runs autonomously, and the result can be influenced only implicitly: through changing the initial configuration, the parameters or the algorithm. In certain cases, the generating process is simple and consists only of the recursive application of one single rule (Ito's pavilion together with Balmond, Figure 3.8), while, in other cases, rather complex generate-test cycles are implemented. Some approaches generate a large variety of design options, which are then automatically tested and selected (competition project together with Bollinger + Grohmann, Figure 3.9), while others methods are steered directly towards one solution (Sasaki's 3D Extended ESO method, Figure 3.10). The implementations of generate-test cycles generally use FEM analysis methods as formal test mechanism, while simpler methods, such as those described by Balmond, often confine themselves, for an intuitive structural reason, to the formulation of the generative rules. The design proposals created through generative methods generally exhibit a high degree of formal complexity, often without any relation to traditional structural typologies.

The approaches presented by Gaudi, Conzett and schlaich bergermann und partner have in common the fact that they are all based on *interactive modelling* methodologies. The search process is structured in iterative, alternating phases of modelling and computation; the designer directly 'interacts' with the model, while the model itself exhibits its own momentum driven by the underlying computational process. The designer judges the solution that emerges from the computational process regarding the design goals, and adapts the model accordingly; subsequently the computational process becomes operative again. Different computational form-finding methods have been applied in the presented examples. Conzett used graphic statics, in order to geometrically adapt the typology of a classical hanging bridge (Figure 3.13). Schlaich bergermann und partner used a computational form-finding method for the creation of a stadium design as crossover of a pre-tensioned wheel typology and an arch structure (Figure 3.14). Gaudí used a physical hanging model in order to create an entirely new interpretation of vaulted masonry structures (Figure 3.12). The design proposals created through these hybrid modelling approaches are kinematic structural systems in equilibrium, here called equilibrium structures, or funicular structures. The scope of equilibrium structures includes classical structural typologies such as arch, cable, or membrane structures, but, in addition to these, offers a great variety of adapted, extended and mixed typologies, beyond the classical cases.

3.3 Models and methods in form finding

In Section 3.2.4, three approaches for structural design as iterative, hybrid search processes based on alternating steps of manual modelling and form finding are presented. Form-finding methods are regarded as formal techniques that are able to 'direct' the design process towards almost bending-free, hence structurally efficient forms in the sense of Polónyi's "Vorweise"[17] (see Section 3.1). In this Section, an overview of the most common form-finding methods is given; in three subsections, physical methods, graphical methods, and computational methods are presented. As already emphasised by Billington (see Section 3.2.3), structural efficiency as design goal is essential for the building typologies associated with 'structural art', such as long-span roofs, bridges or high towers, hence the form-finding methods presented in this Section deal primarily with these typologies.

3.3.1 Physical methods

Since Galileo Galilei (1564-1642) to the present time, a variety of physical experiments and heuristic model-based methods have been used for form finding in structural design (Kotnik, 2011). These approaches are all based on the sufficiently direct analogy between the dominant physical behaviour of the structure and the forces acting within the experimental model. Depending on the form-finding method and the chosen construction of the real structure, the analogy is more or less direct. This analogy between structural behaviour and forces in the model is obvious in certain cases: for instance, for models consisting of tension networks used for the design of pre-stressed cable-net roofs, or soap bubble experiments applied in the design of membrane roofs. Already less evident is the application of hanging models for form finding of compression structures, since the phenomenon of buckling is not represented in the hanging model working in tension. The application of inflated cushions in shell design (as successfully employed by Heinz Isler, see below) is based on an analogy that is even more distant to the real structural behaviour, since the air pressure in the model acts perpendicularly to the membrane, while the shell's self-weight acts strictly vertically. In general, for the successful application of form-finding techniques based on a rather remote analogy, more experience by

[17] "suggesting methods" (translation by the author)

the designer is required, and more comprehensive structural analysis processes have to be employed.

In this Section, physical form-finding methods are presented in two groups. The first subsection, "Hanging models", includes techniques based on models which are externally loaded, mostly by hanging weights, and in Isler's case, by air pressure. The second subsection, "Pre-stressed models", summarises approaches based on internally pre-stressed materials, such as, for example, tension networks or soap films.

3.3.1.1 Hanging models

In 1675, the scientist and architect, Robert Hooke, published a new theorem of statics, written in Latin and as anagram, thus in a non-readable, encrypted form. The English translation of the original formulation is as follows: "As hangs the flexible line, so but inverted will stand the rigid arch." The theorem implies that inverted hanging forms working in tension correspond to geometrically analogous 'standing' compression structures, if their loads are proportionally related. A hanging chain represents the inversion of a compression structure; unreinforced masonry structures can only transfer compression forces, therefore hanging chains represent the inverse geometry of efficient masonry arches. Hooke's colleague and friend, Christopher Wren, knew about the theorem, and was the first to apply it in the design of a building (Addis, 2007: 198-209). Wren was commissioned to build the new St. Paul's Cathedral in London, UK (Figure 3.15), and he used the theorem for the design of a hidden masonry cone within the dome that would carry the heavy lantern efficiently. The cone is placed between the dome's outer roof, and the interior cupola. For finding the form of the cone, Wren used a hanging chain, with a heavy weight attached at the lowest point, representing the estimated weight of the lantern; the resulting shape was used as a section of the masonry cone.

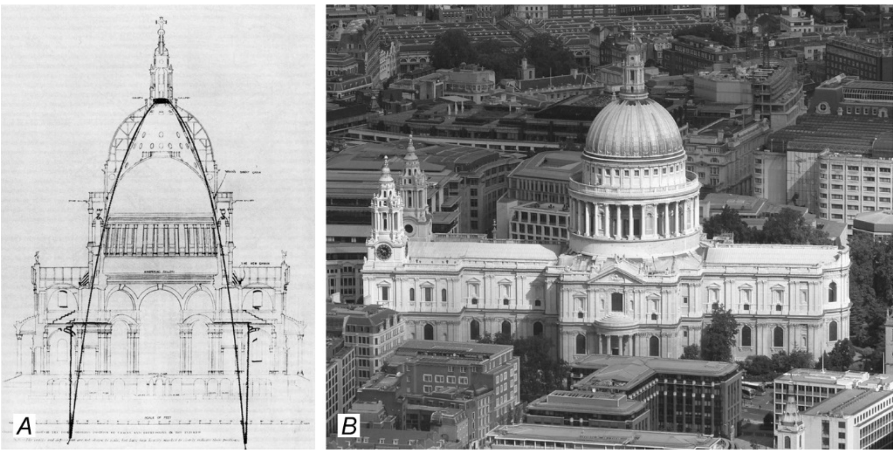

Figure 3.15: St. Paul´s Cathedral. (*A*) Inverted hanging chain in front of the section of the dome (Addis, 2007: 224), and (*B*) aerial view of the building (Fosh, 2013).

One prominent example of the application of Hooke's Theorem for structural analysis purposes was conducted by the mathematician Giovanni Poleni. In 1742, the dome of Saint Peter's in Rome, Italy, showed severe cracks, and Poleni was asked for an examination of its safety. Poleni analysed the dome structurally, based on a two-dimensional hanging string model. For this purpose, the mathematician discretised the mass of the dome's section, and used proportional weights for a hanging string model. The shape of the weighted hanging string did not exceed the section of the dome, thus Poleni concluded that the structure was safe. Nevertheless, he agreed with the recommendation of other experts to add horizontal iron ties (Heyman, 1997: 35-39).

Almost one century later, from 1833, the German architect Heinrich Hübsch systematically applied hanging string models in the design of vaulted masonry buildings (Graefe, 1983). His work goes beyond Wren´s and Poleni´s approaches, since he combines more than one string to create models of complex sections; for instance, he incorporates main and side aisles of a church section in one hanging model (Figure 3.16). This method allows Hübsch to deliberately counterbalance the horizontal thrust from the main vault with the thrust of the side vaults, thus to create longer spans or higher buildings, without using external flying buttresses. The architect describes his design method in detail: a hanging string model with small loops, allowing the attachment of weights, is fixed on a drawing board on top of the turned upside-down sections of the building. By alternatively changing the drawing of the section and adapting the weights of the model, a structurally feasible building section is derived in the iterative process. The method was successfully applied by Hübsch for the design of several realised churches in Germany, after conducting tests of his assumptions on 1:2 scale building mock-ups.

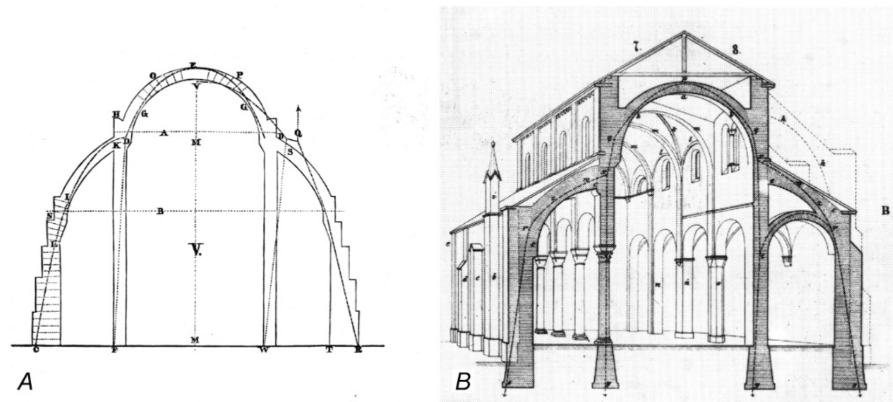

Figure 3.16: (*A*) Drawing for a mock-up building in scale 1:2, and (*B*) section drawing of a church with side aisles (Graefe, 1983: 188).

While Poleni applied a hanging model for the structural analysis of an existing building, Wren and Hübsch used hanging models in the design process of new buildings with innovative constructions. Nevertheless, both Wren and Hübsch designed buildings in the 'language' of classical architectural styles; Wren embossed the style of English Baroque, and Hübsch built in the manner of Historicism. Both architects used hanging models to achieve technical innovations, in order to be able to design buildings with unusual, ambitious proportions. However, the form-finding method remained 'hidden', and did not directly influence the formal language or the expression of their buildings. Gaudí was the first to use hanging models to create building designs with a new architectural language, inspired by the Art Nouveau movement (see Section 3.2.3). He was the first to explore the full spatial potential of hanging models, beyond the design of two-dimensional sections. Probably the most famous visual expression of Gaudí´s method are the inclined masonry columns; such columns were realised, for example, in the Colonia Güell Crypt (Moro, 2003: 207-215).

The *Institute for Lightweight Construction* (IL) in Stuttgart, Germany, founded by Frei Otto in 1964, has been dealing, among others, with problems of form finding using various physical experiments (Gaß, 1990). Hanging models played a crucial role in the work of the Institute: the documentation of Gaudí´s design process and the reconstruction of the hanging model of the Colonia Güell were conducted mainly at the IL (Tomlow, 1989). Furthermore, a large number of hanging models in different scales and dimensions were built for the design of the Multihalle Mannheim, Germany (Figure 3.17), as wooden grid shell (Burkhardt, 1976). The design process started with a 1:500 conceptual model built from wire mesh by the architect Mutschler & Partner with Otto. The mesh model served as prototype for a detailed 1:100 hanging model built at the IL (Bubner, 1976). The geometry of the hanging model was measured with stereophotogrammetry, and, based on these data, a computational validation and correction of the measured equilibrium form was executed by Klaus Linkwitz (see Section 3.3.3). Finally, an analytical safety proof and experimental load tests on the real structure were executed by Ove Arup & Partners (Happold and Liddell, 1975).

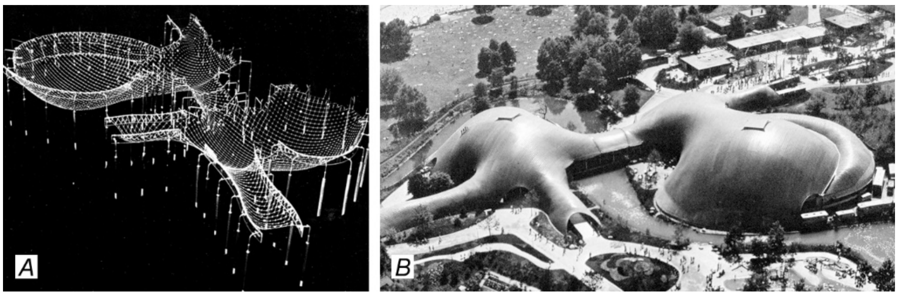

Figure 3.17: (*A*) Hanging model of Multihalle Mannheim (Burkhardt, 1976: 32), and (*B*) aerial view of the building (Addis, 2007: 557)

For the design and engineering of reinforced concrete shells, the Swiss engineer, Heinz Isler, used a highly refined process based on physical experiments and models in different scales (Chilton, 2000: 32-47). He applied different techniques of form finding, mostly using inflated and hanging membranes (Figure 3.18). Heinz Isler´s famous 'bubble shells' were generated by an inflated 'cushion-like' membrane model. The form-generating principle is that, for a shallow shell form, forces that are perpendicular to the membrane surface are sufficiently close to the shell´s vertically acting dead load. To create durable representations of these forms, the models were cast in plaster. The three-dimensional shape of these plaster models was measured by a mechanical device developed by Isler himself, in order to produce accurate construction documentation plans for the wooden scaffolding of the double-curved shells. For the assessment of structural safety and usability, loaded scale models made from resin were used. Their deformations were measured with electrical strain gauges. Until the early 1990s, Isler realised some public buildings, such as the Flieger Flab Museum[18], in Dübendorf, Switzerland, and the swimming pools[19] of the Health & Racquets Club, in Norfolk, UK. However, besides such unique projects, Isler built hundreds of similar thin-shell concrete roofs for industrial and commercial buildings, with spans of up to 55 m, such as, for instance, the Bürgi Garden Centre near Berne, Switzerland (Chilton, 2010).

Figure 3.18: (*A*) Creation of a plaster cast hanging model (Billington, 2003: 146), and (*B*) photo of the Bürgi Garden Centre (Chilton, 2000: 85)

In the last two decades, physical experiments have been gradually replaced by computer-based form-finding methods. For the design of certain structural typologies, physical models are still used, for instance in bridge design. Mike Schlaich explained[20] that for the design of the curved arch bridge in Ripshorst, Germany,[21] a physical form-finding model was built during the initial design phases. Also the engineer Laurent Ney used a weighted string model in the design process of the College Bridge in Kortrijk, Belgium[22] (Figure 3.19). The physical model of the curved suspension bridge served as starting point for the computational form-finding routine based on the Force Density Method (see Section 3.3.3) (Brunetta, Patteeuw *et al.*, 2005: 36-39).

[18] Completed in 1987, by Heinz Isler with Haus + Herd architects

[19] Completed in 1991, by Heinz Isler with Copeland Associates and Haus + Herd architects

[20] This conversation took place during a public panel discussion after a lecture by the author at the Technical University in Berlin, Germany, on 7 December 2011, by invitation of Prof. Dr. Lordick

[21] Completed in 1997, by Diekmann and Lohaus Architects with schlaich bergermann und partner

[22] Completed in 2009, by Ney + Partners

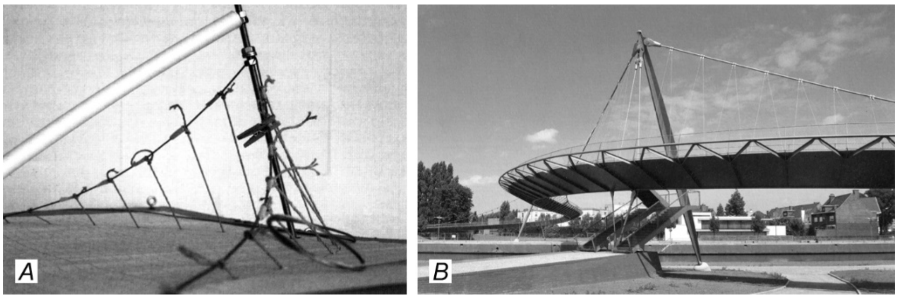

Figure 3.19: College Bridge in Kortrijk. (*A*) Form-finding model (Brunetta, Patteeuw *et al.*, 2005: 36), and (*B*) photo of the realised bridge (Adriaenssens, Devoldere *et al.*, 2010: 81)

3.3.1.2 Pre-stressed models

At the Institute for Lightweight Structures (IL), internally pre-stressed models were systematically used for the design and form finding of various membrane and cable-net structures. Immediately after the foundation of the IL in 1964, Rolf Gutbrodt and Frei Otto won the competition for the German pavilion at EXPO 1967 in Montreal, France[23] (Figure 3.20) with a proposal for a tensioned net roof (Burkhardt, 2005). For this project, various models were built: tulle models for the conceptual design and presentation, soap film models, detailed tensioned net models, and wooden models for wind tunnel tests. Specifically for the Montreal project, a three-dimensional measurement bench was installed at the IL, in order to be able to measure precisely the contour lines of double-curved surfaces. For the form finding and cutting pattern generation of the roof of the Montreal pavilion, a 1:75 model was built with fine steel wires. Custom devices were furthermore developed to measure the pre-stress forces in the model.

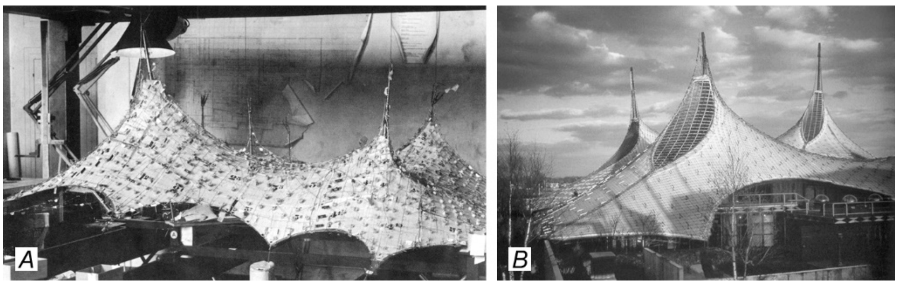

Figure 3.20: German pavilion at Montreal EXPO 1967. (*A*) Form-finding model (Glaeser, 1972: 42), and (*B*) photo of the pavilion (Nerdinger, 2005: 234).

Before the completion of the Montreal pavilion in 1967, an experimental cable-net roof was erected on the University of Stuttgart campus, to test the construction and assembly process of the pavilion. This structure later became the home of the IL (Figure 3.21). Several soap film experiments and steel wire models led to the development of the iconic 'eyelet' openings as solution for the construction of point supports holding the pre-stressed net. The use of stronger loops as edges of the eyelet openings avoids concentrated stresses within the net.

[23] Designed by Frei Otto and Rolf Gutbrod, with Leonhardt & Andrä

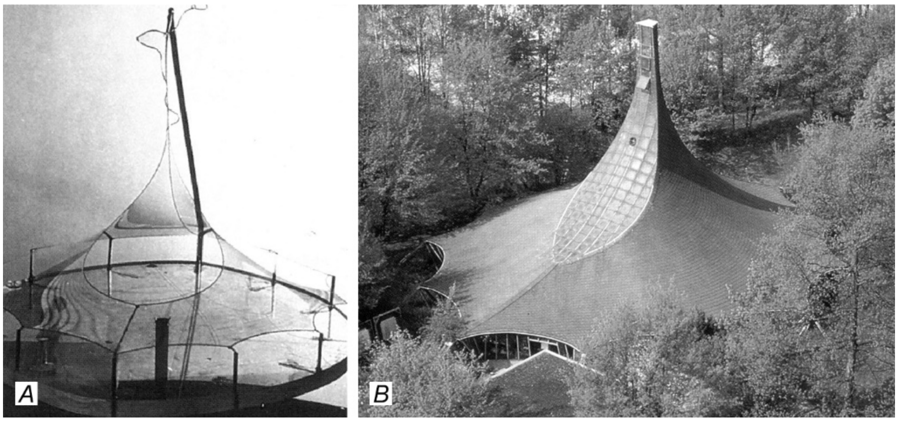

Figure 3.21: Prototypical roof at the Stuttgart University campus. (*A*) Soap film model (Otto and Rasch, 1996: 100), and (*B*) photo of the realised building (Otto and Rasch, 1996: 103).

In 1968, shortly before the completion of the IL building, which recused the experimental roof structure erected at the University of Stuttgart campus, Germany, the architects Behnisch & Partner, together with Jürgen Jordicke and Heinz Isler, won the competition for the buildings of the 1972 Olympic Games in Munich, Germany (Burkhardt, 2005). Inspired by Otto's work, they proposed a series of pre-stressed cable-net roof structures, with stocking material used for their competition model. In the further process of the validation of the feasibility of the design, Otto served first as consultant, and later became part of the official planning team of the roofs, that consisted officially of Behnisch & Partners, Frei Otto, and the engineers Leonhardt & Andrä. At the IL, the unique expertise in model building and measurement gained during the design process of the Montreal pavilion, could be directly employed for the Olympic roofs (Figure 3.22). Besides numerous experimental and detailed models on different scales, a precise 1:125 steel-wire network model was built. Jörg Schlaich, project leader at Leonhardt & Andrä, doubted the precision of the physical models, and pushed for the development of computational methods for the analysis and cutting pattern generation (Möller, 2005). In addition to the measuring bench, this time the geometry of the model was measured photogrammetrically by Klaus Linkwitz and his team. The measured geometry served as starting point for a computational optimisation and validation of the cutting pattern, using the *Force Density Method* (see Section 3.3.3).

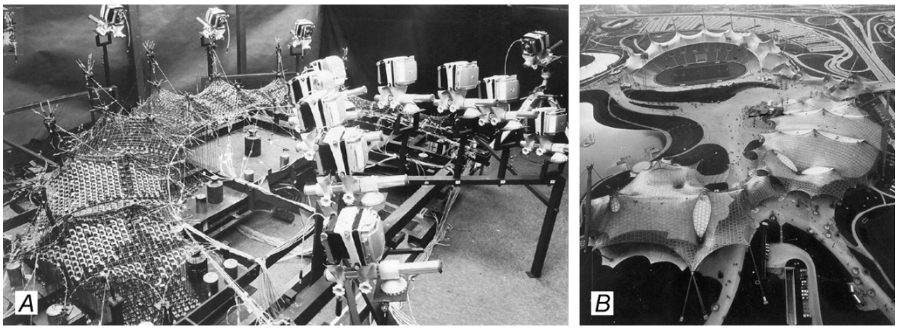

Figure 3.22: Olympic roofs in Munich. (*A*) Photogrammetric measurement of the 1:125 steel wire model (Otto and Rasch, 1996: 107), and (*B*) aerial view of the Olympic roofs (Nerdinger, 2005: 261).

Already in 1959, the Italian engineer Sergio Musmeci had applied physical experiments for the conceptual studies of a bridge consisting of a horizontal deck supported by thin shells (Neri, 2012). The structural concept was to exactly balance the forces induced by the dead load of the bridge deck, while neglecting the self-weight of the supporting shells working in compression. For this, Musmeci used soap film models on wire frames, and pre-stressed rubber cloths. Both types of models represent pre-stressed membranes in tension. Similarly to the duality between the hanging chain and compression arch, the geometry of a tensioned membrane also functions as efficient compression shell, if the loads correspond proportionally, and are flipped in sign. In this sense, a minimal soap bubble model is a valid geometry for a thin concrete shell working in compression, if the shell is loaded primarily at its supports, as is the case for the supporting shells of the bridge. For his masterpiece, the Basento Viaduct in Potenza, Italy (Figure 3.23), completed in 1976, Musmeci developed the form iteratively through a series of models: soap film and rubber cloth models for the form finding, and resin models on the scale of 1:100 for the initial structural analysis, and, finally, a 1:10 micro-concrete model for final usability and safety tests.

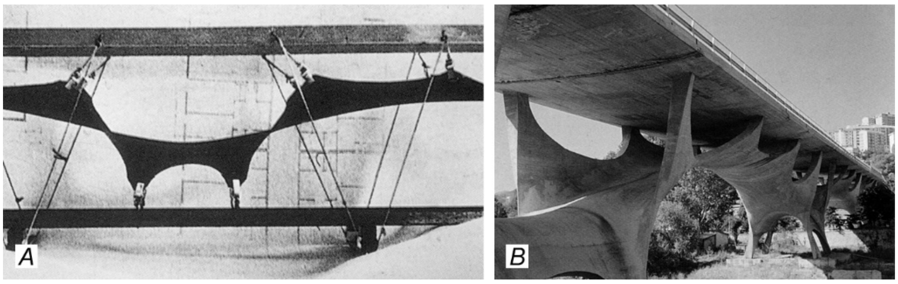

Figure 3.23: Basento Viaduct. (*A*) Rubber cloth model of the supporting shell, and (*B*) photo of the bridge (Nicoletti, 1999: 63, 64).

Today, pre-stressed models are occasionally used in early design stages. The recently completed Khan Shatyr Entertainment Centre in Astana, Kazakhstan,[24] was designed by using physical models for the design and form finding in initial stages (Figure 3.24). The Entertainment Centre consists of an asymmetric tent

[24] Completed in 2010, by Foster + Partners, with Buro Happold

roof, supported by an inclined 150 m high column, covering the facilities of the Centre. The physical models served as starting point for a refined computer-aided modelling and form-finding process of the tensioned roof (Mangelsdorf, 2010).

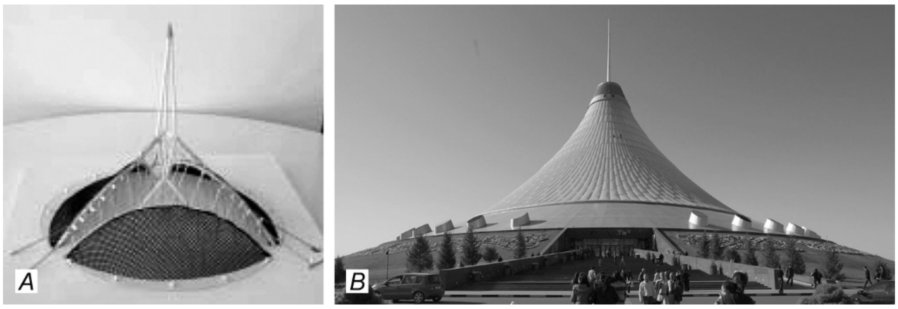

Figure 3.24: Khan Shatyr Entertainment Centre. (*A)* Early conceptual model (Mangelsdorf, 2010: 43), and (*B*) photo of the realised building (Ken and Nyetta, 2013).

3.3.2 Graphical methods

Since the Gothic period, a variety of heuristic, geometrical methods and "rules of thumb" have been used by Master Builders, especially for arch and vault designs (Addis, 1990: 137-152). These have been passed on by word of mouth or textbooks. Here, the author focuses on methods based on early scientific theories of statics, namely *graphic statics*.

Graphic Statics was mainly developed by Karl Culmann, Professor of Civil Engineering at the Swiss Federal Institute of Technology (ETH), Zurich, Switzerland (Maurer and Ramm, 1998). The method is built upon earlier research related to graphical calculus, by scientists such as Pierre de Varignon, James C. Maxwell and Luigi Cremona. Culmann's unique contribution was to systematically adapt and apply these methods to problems in statics. In his book, "Die grafische Statik", published in two parts, 1864 and 1866, and in his lectures, Culmann disseminated his approach to structural design and analysis based on vector calculus and drafting: "Central to Culmann´s philosophy was the importance of making visible in the method of calculation or analysis the workings of the inherently invisible stresses and forces inside structures" (Addis, 2007: 373).

Graphic Statics represents structures as planar pin-jointed models, and their static equilibrium state is depicted using vector diagrams. This allows the modification and construction of both the inner forces and the form of the structure by means of drafting. The method involves two diagrams, the *form diagram,* representing the geometry of the structure together with its external loads, and the *force diagram*, also known as the *Cremona Plan*, representing the equilibrium of forces (Figure 3.25).

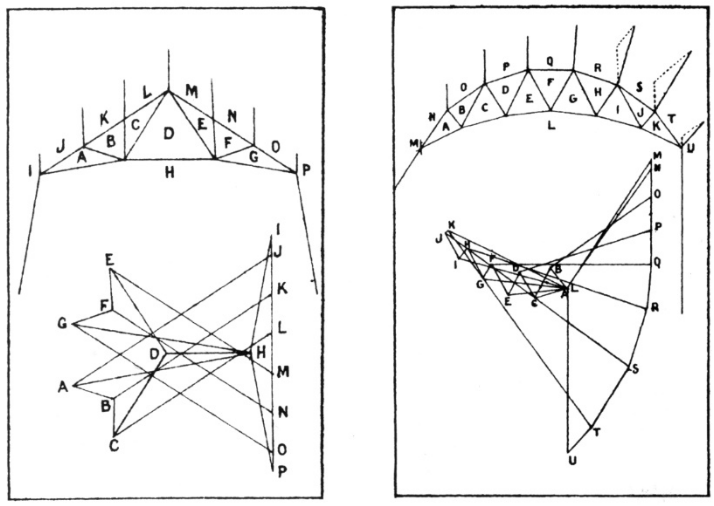

Figure 3.25: Form and force diagrams of loaded trusses by Robert H. Bow (Addis, 2007: 372).

These diagrams have the following properties: each edge in the form diagram is represented by an edge in the force diagram, its length is proportional to the magnitude of the inner force; corresponding edges in both diagrams have to be parallel; and topologically the two diagrams are *dual graphs*. Two diagrams with these properties are also called *reciprocal diagrams*. This specific representation allows the use of graphic statics in two 'directions': structural analysis of determinate structures is conducted by drawing the form diagram, and subsequently constructing the force diagram; structural design or form finding is conducted by constructing the geometry of the structure from chosen constraints of the force diagram, as used for instance in the design of the Second Traversiner Steg (see Section 3.2.3.2).

Culmann's work was continued at the ETH Zurich by his successor, Wilhelm Ritter, from 1882 (Billington, 1980). Ritter himself was then succeeded by Emil Mörsch in 1904, who switched to the University of Stuttgart, Germany, a few years later. Mörsch was the first to use truss models in the design and analysis of reinforced concrete structures, which finally led to the development of strut-and-tie models at the University of Stuttgart (Schlaich, Schäfer *et al.,* 1987) and stress fields at the ETH Zurich during the late 1980s (Muttoni, Schwartz *et al.,* 1997), based on the rigorous theory of plasticity. Bögle and Billington emphasise the continuity of thought, ranging from graphic statics to contemporary truss models for concrete structures (Bögle and Billington, 2009).

Already at the beginning of the 20th century, the application of graphic statics for structural analysis was largely replaced by numerical calculation techniques. Nevertheless, the strength of graphic statics for design purposes has been repeatedly emphasised. Nervi, for instance, underlined its value in design education: "I believe that graphical statics should play an important role in [the] last education phase, since its procedure gives a direct understanding – much better than that afforded by analytical methods – of force systems and their composition, decomposition and equilibrium." (Nervi, 1956: 21). In the last two decades, graphic statics has been used increasingly in structural design education for architects, especially at leading international schools of

Architecture: in Switzerland at the ETH Zurich (Schwartz, 2008; Block, Gengnagel *et al.,* 2013) and the EPFL Lausanne (Muttoni, 2011), in Germany at the RWTH Aachen (Gerhardt, 1989; Pichler, Eisenloffel *et al.,* 1997), and in the United States at the Massachusetts Institute of Technology and Yale University (Zalewski and Allen, 1998; Allen and Zalewski, 2009).

3.3.2.1 Planar funicular polygons

The key concept of graphic statics is the *funicular polygon*. It represents a method that allows the construction of the form of a weighted string for a set of given loads. In contrast to a hanging string model, the funicular polygon enables the designer to constrain the forces to given axes, called *lines of action*, which provides more control over the global form. Furthermore, the force diagram precisely represents the inner force in each segment of the funicular polygon. The concept of the funicular polygon was already described by Varignon as a kind of paper-based simulation of the weighted string. The funicular polygon has furthermore been used as general graphical tool, for instance, in the design of arches and cables, as well as for the construction of bending moment diagrams of beams and the calculation of the centre of mass for a given profile (Maurer and Ramm, 1998).

One of the most famous students of Culmann was Maurice Koechlin, who became Chief Analyst and Designer of Gustave Eiffel's construction company in 1879 (Addis, 2007: 373). Koechlin used graphic statics for the design of Eiffel's most famous buildings, such as the Garabit Viaduct in Ruynes-en-Margeride, France, completed in 1884, and the Eiffel Tower in Paris, France, completed in 1887 (Figure 3.26). The arch of the Garabit Viaduct was designed to contain all possible compressive lines resulting from moving loads; the silhouette of the Eiffel tower was derived from a funicular polygon based on wind forces (Allen, 2004).

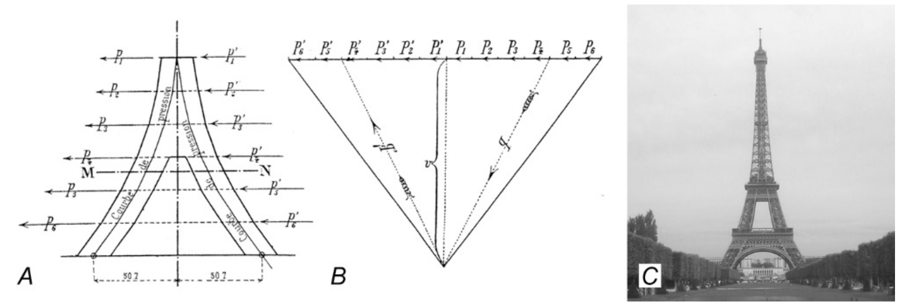

Figure 3.26: Graphical construction used in the design of the Eiffel Tower. (*A*) Form diagram, (*B*) force diagram (Allen, 2004: 72), and (*C*) photo of the building (Katie, 2013).

Rafael Guastavino Sr., architect and builder of Spanish origin, emigrated from Barcelona to New York in 1881, and founded a company specialised in the construction of masonry vaults using the traditional Catalan thin-tile vaulting technique. During his lifetime, until 1908, he realised more than 1000 vaulted structures in the United States, most of them as floors, ceilings and stairs within buildings designed by other, mostly well-known, architects (Allen, 2004). Guastavino's constructions were elegant and decorative, often realised with colourful glazed tiles, but also economical in construction. Graphic statics played an important role in the design of these tile vaults, on the one hand, for the determination of the form of his arch and dome sections, and on the other hand, for the determination of the required thickness of these structures based on inner stresses (Figure 3.27) (Ochsendorf, 2010: 162).

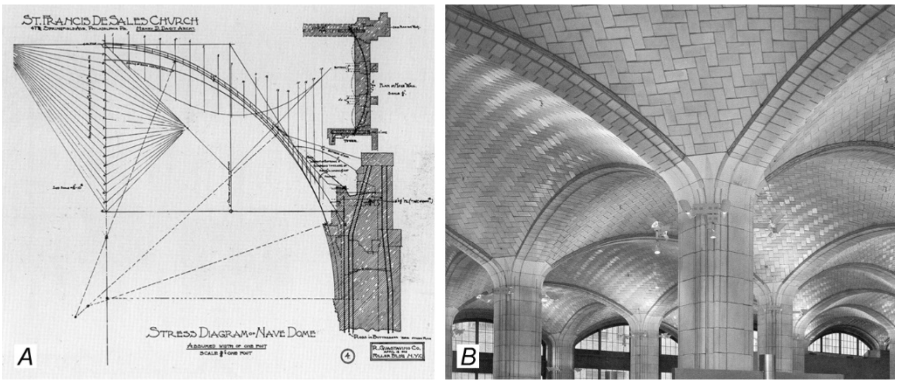

Figure 3.27: *(A)* Graphical analysis of the inner stresses in one of Guastavino´s domes based on two funicular polygons , and (*B*) vaults by Guastavino covering a public market at Queensboro Bridge, New York (Ochsendorf, 2010: 165, 93).

Gaudí, who was a contemporary of Guastavino, also used graphic statics. In contrast to Guastavino, who searched for economical formal solutions, Gaudí used graphic statics to explore novel, expressive forms. One prominent example of the application of graphic statics by Gaudí is the design of the retaining wall and pergola in the Park Güell, completed in 1914, in Barcelona, Spain (Figure 3.28). The section of the pergola is constructed as a funicular polygon in compression, based on the assumed soil pressure and soil weight (Moro, 2003: 207-215).

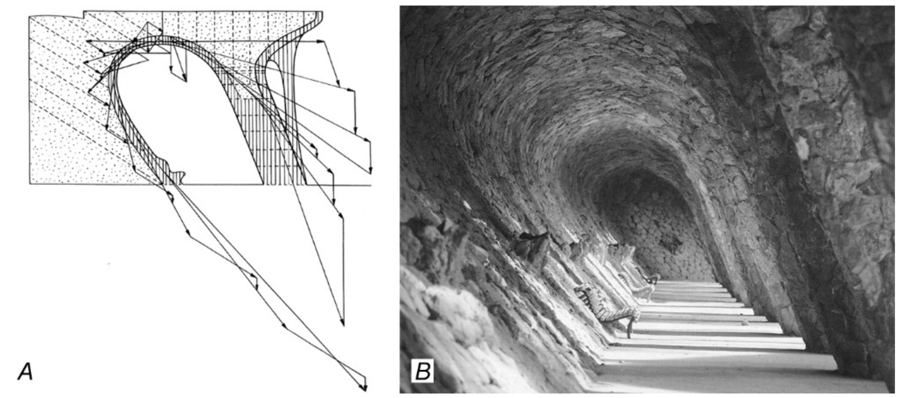

Figure 3.28: Retaining wall and pergola in Park Güell. (*A*) Construction of the form based on graphic statics, and (*B*) photo of the pergola (Moro, 2003: 211, 208).

The Swiss engineer, Robert Maillart, who was a student of Ritter, used graphic statics both for analytical and design purposes (Billington, 2003: 30-72). The shape of the arch of the famous Salginatobel Bridge in Schiers, Switzerland (Figure 3.29), completed in 1930, for instance, is designed such as to contain all possible compression lines (Allen, 2004). These compression lines were studied using funicular polygons constructed by use of the method of graphic statics (Figure 3.29).

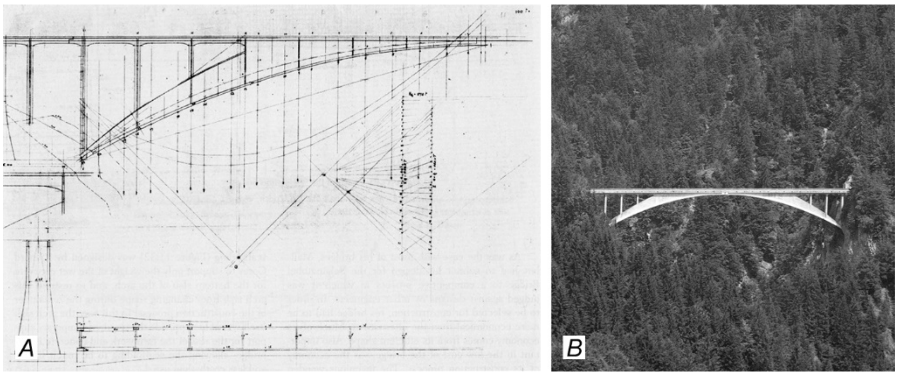

Figure 3.29: Salginatobel Bridge. (*A*) Construction of the form of the arch based on graphic statics (Allen, 2004: 73), and (*B*) photo of the bridge (Billington, 2003: 61).

3.3.2.2 Spatial funicular polygons

During the 1990s, through the availability of three-dimensional computer-aided drawing environments (CAD), it became possible to create spatial constructions directly in the virtual drawing space, without using projections. Some methods of graphic statics, such as the construction of the funicular polygon, can be directly extended to the three-dimensional drawing space, for instance, by replacing the concept of the *lines of action* by analogous *planes of action*. In his PhD thesis, Massimo Laffranchi presents a method for the form finding of curved bridges, based on such an extended understanding of graphic statics (Laffranchi, 1999: 23-30). He states that any loaded, spatially curved bridge axis can be balanced by two spatial funicular polygons (Figure 3.30). These funicular polygons can be interpreted literally as cables or arches. Furthermore, within the conceptual framework of contemporary truss models, the funiculars can be interpreted as compression zones, or as elements of reinforcement within concrete structures.

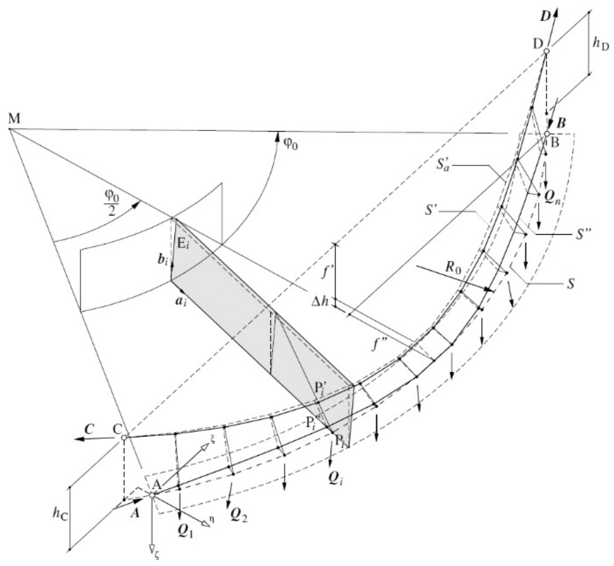

Figure 3.30: The axis of a curved bridge deck balanced by two funicular polygons (Laffranchi, 1999: 21).

Already in 1998, the historian and theorist, Karl-Eugen Kurrer, demanded the development of a method of "computer-aided graphostatics" as a contemporary "hinge between design and construction" (Kurrer, 1998). In recent years, several computer-aided approaches towards structural design and form finding based on three-dimensional extensions of graphic statics have been developed at MIT Cambridge (Block and Ochsendorf, 2007) and at ETH Zurich (Van Mele, Lachauer *et al.,* 2012; Akbarzadeh, Van Mele *et al.,* 2013; Schrems and Kotnik, 2013).

3.3.3 Computational methods

Computational form-finding methods often directly relate to experimental physical models. In fact, one of the first computational approaches to form finding was developed as a complementary technique for the validation and correction of the photogrammetrically measured geometry of physical form-finding models built at the IL (see Section 3.3.1.2). During the rapid development of the personal computer as design tool in architecture since the 1990s, the significance of computational form-finding methods has increased compared to the physical approaches for practical reasons: to use computer-based form-finding techniques is generally faster, cheaper, and more accurate. A comprehensive technical overview of various computational form-finding methods was published by Veenendaal and Block (Veenendaal and Block, 2012).

3.3.3.1 Force Density Method

For the form-finding process of the pre-stressed cable net roofs of the Olympic Stadium in Munich, large physical models were built. In addition, computational methods for the optimisation of the forms derived from the models were developed and applied. These were used for the critical task of the calculation of the exact construction geometry. The general formulation of the static equilibrium problem as mathematical relation between geometry and inner forces is non-linear and has to be solved iteratively (Linkwitz and Schek, 1971; Linkwitz, Schek *et al.,* 1974). Through a mathematical 'trick', specifically, the replacement of forces by force/length ratios, called *force densities*, the equilibrium problem was formulated as a system of linear equations (Schek, 1974). Through the definition of additional geometric constraints, the problem becomes non-linear again, and is generally solved using gradient-based methods, such as the Gauss-Newton Method. This approach, called the *Force Density Method* (FDM), was used for the rationalisation of the model geometries of the Olympic roofs in Munich. Furthermore, a detailed description of the application of FDM for the calculation of the construction geometry of the Multihalle Mannheim, Germany (Figure 3.31), is given by Gründig *et al.* (Gründig, Hangleiter *et al.*, 1976). As input, the method received measured node coordinates from the physical models. The approach enabled the calculation of equilibrium states of the network close to the measured model geometry, with the possibility of enforcing additional geometrical constraints, such as, for instance, equal cable lengths (Figure 3.31).

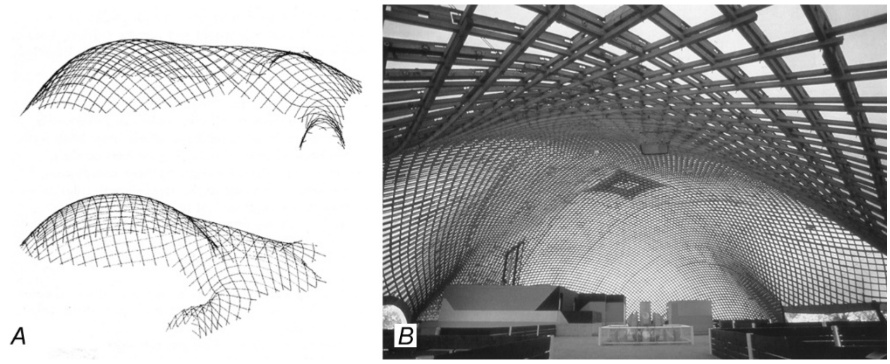

Figure 3.31: Multihalle Mannheim. (*A*) Computationally optimised equilibrium forms of the main dome (Gründig, Hangleiter *et al.*, 1976: 48), and (*B*) interior view (Otto and Rasch, 1996: 142) .

The linear version of the FDM can be used for the generation of tension networks in equilibrium from scratch, based on the support coordinates, the nodal load vectors, a given network connectivity, and a force-density value per strut. Each set of force-density values creates a unique equilibrium form, and the potential of this approach for form finding on the computer screen was already pointed out by Schek. During the 1970s, before the availability of personal computers, and the very limited possibilities of direct user interaction, the computer enabled the optimisation of equilibrium geometry, but did not yet offer an alternative for the model-based design and form-finding process.

Already during the early 1990s, the first commercial software packages for membrane design based on FDM were released. In the past decade, further extensions to FDM have been presented, dealing with the challenges of mixed compression tension structures. FDM in its original formulation by Schek enables the generation of equilibrium states for mixed compression/tension structures, negative force-density values result in compression struts; the challenge in using mixed force densities lies in the formal unpredictability of the resulting equilibrium networks. The recent extensions to FDM deal with form finding and interactive design of tensegrity structures[25] (Zhang and Ohsaki, 2006; Tachi, 2012) and with form finding of more general mixed tension and compression structures (Miki and Kawaguchi, 2010; Bahr and Kotnik, 2011). The latter two are lacking the possibility to impose geometric constraints on the free nodes, though. Newly, a form-finding method based on the FDM, with the possibility to define general geometric constraints on free nodes and force-density bounds, has been published (Tamai 2013).

3.3.3.2 Dynamic relaxation and particle-spring systems

An alternative approach to computational form finding, besides directly using the mathematical description of static equilibrium, is based on the dynamic simulation of physical form-finding experiments, such as soap bubble models or tensioned net models. Hence, a digital version of these physical models, based on discrete linear or planar elements with simulated elastic behaviour, is created. Starting with a given initial geometry, the process of form finding is employed as dynamic movement of the elastic digital model in time. The force vectors for each node are calculated based on the inner stresses of the adjoining elements. These forces are then used to iteratively update the position and velocity of the network nodes over time, eventually converging to an equilibrium state.

Michael Barnes was the first to apply this method, known as *dynamic relaxation* (DR), to the form finding of tension structures (Barnes, 1975). In the late 1990s, Barnes described an implemented computer-aided 'expert system' for the professional design of tensioned membranes on the basis of DR, with various options for the user to control the network layout, material properties, boundary conditions and internal pre-stress, and it was applied in the design of the aviary at the Munich Zoo, Germany (Figure 3.32) (Barnes, 1999).

Such tools allowed for the interactive design of membrane structures on screen, although physical models were still a useful tool for conceptual design, demonstration purposes, and for wind tunnel testing. However, at that time, models had already lost their crucial role as generator of the starting geometry for the computational processes, as was the case in the design process of the Olympic roofs in Munich.

[25] Tensegrity structures are pre-stressed, pin-jointed structures consisting of compression struts and tension cables with the condition that compression struts are never directly connected to each other

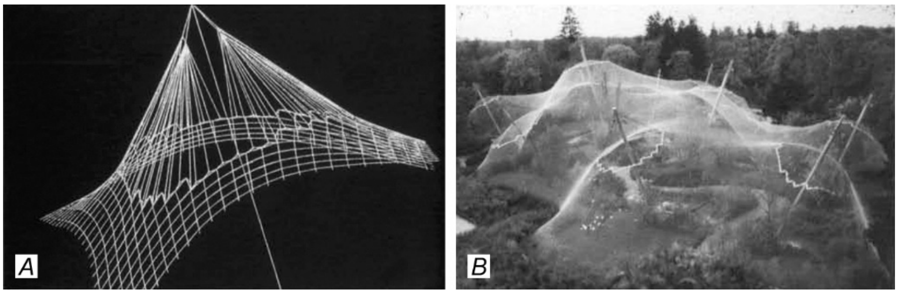

Figure 3.32: Aviary at the Munich Zoo. (*A*) CAD model of the roof designed with DR, and (*B*) photo of the aviary (Barnes, 2000: 34).

Another computer-aided modelling environment based on physical simulation was developed by Axel Kilian. The system, directly inspired by Gaudí´s method, enables the interactive design of virtual hanging models (Figure 3.33) (Kilian and Ochsendorf, 2005). In contrast to Barnes´s sophisticated 'expert system', Kilian´s tool was created with the intention of providing an intuitive and direct modelling system for architects and designers, allowing the rapid invention of new compression structures with unexpected forms. The simulation is based on a *particle-spring* (PS) system, which is a digital representation of a model consisting of weighted nodes connected by a network of elastic linear elements, such as, for instance, flexible strings or rubber bands. Similarly to DR, the process of form finding is based on physical simulation; PS uses a more advanced mathematical solver though, with faster convergence and higher solving stability, derived from methods developed for computer animation. Kilian explicitly frames the tool in the context of 'steering' design methods. He refers to such methods as 'design drivers' (Kilian, 2006). The tool is freely available and very popular among designers; it has been used in various contexts, ranging from furniture design to architectural design.

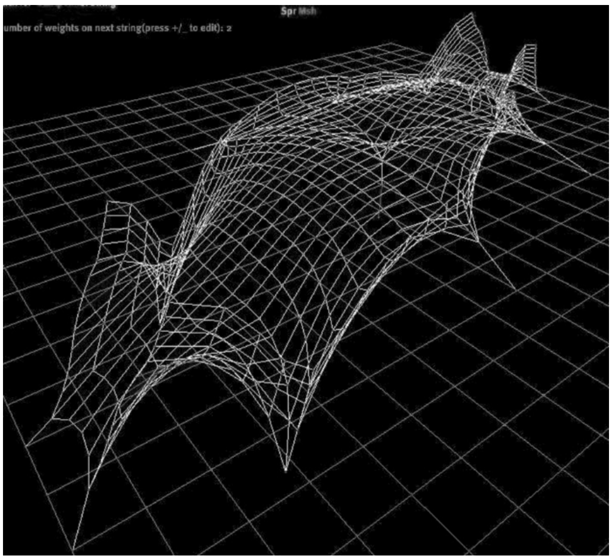

Figure 3.33: Particle-spring system for the interactive design of hanging models (Kilian and Ochsendorf, 2005: 82).

3.3.3.3 Form finding based on finite element models

Different computational methods for form finding using a general *finite-element* (FE) formulation of the equilibrium problem have been developed since the 1990s (Ramm, Bletzinger *et al.*, 1993; Bletzinger and Ramm, 2001; Sasaki, 2007). These approaches are based on a comprehensive continuum mechanics formulation of the inner forces in the discrete elements, including bending moments, normal forces and shear forces, which has resulted in a detailed definition of the material properties of the elements being required. These methods are largely independent of the chosen layout of the discretisation (besides numerical precision), while the equilibrium form of a pin-jointed network is highly dependent on the network connectivity. FE models consisting of planar elements can be seen as the computational analogy to continuous soap bubble experiments, while pin-jointed models rather correspond to discrete tensioned networks.

Recently, several approaches to the optimisation of structures based on generate-test cycles using commercial FE analysis packages have been presented. Other methods use a custom formulation of FE in order to implement directed search algorithms (see Section 3.2.2.3). The biggest conceptual drawback of using FE in early conceptual design is the necessity to define detailed material properties. Thus, FE-based form finding has been applied successfully in the design of structures consisting of well-defined and uniform material. Sasaki, for example, used such methods for the optimisation of freeform concrete shells. Furthermore, FE is also used for advanced membrane form finding, enabling the user to optimise the fibre layout and cutting pattern of the membrane during the design process (Dieringer, Bletzinger *et al.,* 2011). This optimisation method was applied in the design of the membrane roofs of the Norway pavilion at EXPO 2010 in Shanghai, China (Figure 3.34). Furthermore, FE-based methods are used for the optimisation of technical components in the automotive and aviation industries.

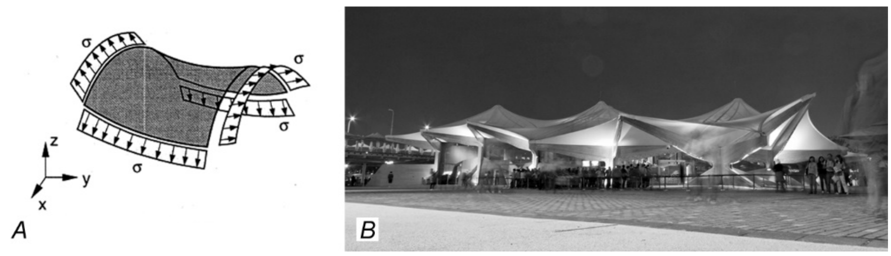

Figure 3.34: Norway pavilion at EXPO 2010. (*A*) Equilibrium equation of a tension membrane (Bletzinger and Ramm, 2001), and (*B*) photo of the membrane roof (Berlin, 2013).

3.3.3.4 Thrust network analysis

Philippe Block has developed another approach to the design of compression structures using discrete equilibrium networks (Block and Ochsendorf, 2007). One important weakness of the classical hanging model method, as developed by Gaudí, and its computational simulations, as developed by Kilian, is the lack of direct control of the geometry. This is especially relevant for the design of compression structures consisting of discrete blocks, such as masonry structures. Block's method, *Thrust Network Analysis* (TNA), enables the explicit control of the plan and the inner force distribution of the equilibrium network. For this, existing concepts of Graphic Statics and FDM are combined; a pair of reciprocal diagrams is used to define and modify the structure in plan and its inner force distribution, and the linearised formulation of static equilibrium is used to find the corresponding equilibrium network efficiently (Figure 3.37 *A*). For a given plan of the structure (referred to as *form diagram*), an equilibrated inner force distribution (referred to as *force diagram*), support heights, and a set of vertical nodal loads, the equilibrium network (referred to as *thrust network*) is uniquely defined.

Recently, the author contributed to the development and implementation of a method which allowed the creation of the interactive design tool *RhinoVAULT* based on TNA (see Section 4.2.4). The tool has been used in the design of the temporary project, "Brick-topia" in Barcelona, Spain (Figure 3.35).

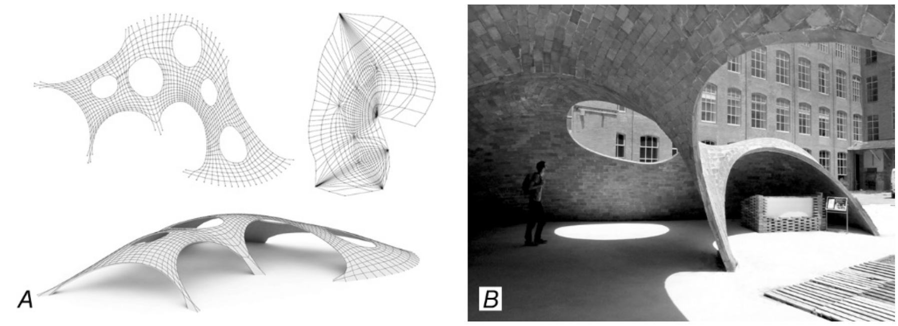

Figure 3.35: (*A*) Exemplary form diagram, force diagram, and thrust network generated with RhinoVAULT (Rippmann, Lachauer *et al.,* 2012). (*B*) The project Brick-topia designed by Map13 architects using RhinoVAULT (Lózar and Barba, 2013)

3.3.3.5 Statically-geometrically coupled method

Roman Kemmler, form-finding specialist at schlaich bergermann und partner, described a powerful method for the form finding of geometrically constrained, mixed compression and tension structures, on the basis of pin-jointed models (Kemmler, 2012). The approach is named *statically-geometrically coupled* (SGC) method, and examples for both stadium design (Figure 3.36) and bridge design are presented. As FDM, the SGC method is based on static equilibrium only, without the necessity of defining material properties. SGC allows the solution of equilibrium forms defined through constraints imposed on geometry and internal forces. However, the method requires an in-depth mathematical understanding of the mechanical conditions of equilibrium, since the constraints have to be consistent and must have a unique solution: "die formulierten Kraft- und Geometriebedingungen [dürfen] sich nicht widersprechen und nicht mehrdeutig sein"[26] (*Ibid.:* 478). In this sense, the method is a computational system for engineers with specialist knowledge, and the method hardly supports an iterative search process in the sense of structural design as intuitive exploration. The method is used internally in the offices of

[26] "the formulated conditions for forces and geometry have to be consistent and must have a unique solution" (translation by the author)

schlaich bergermann und partner, and technical details have to date not been disclosed.

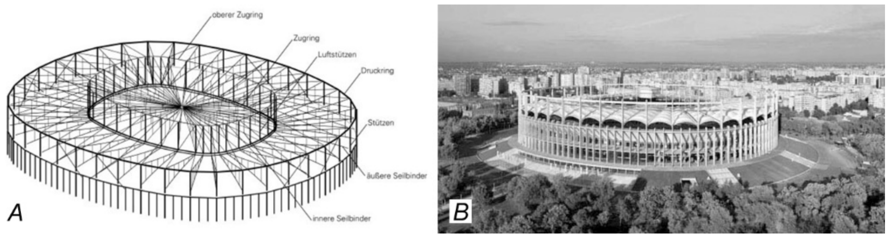

Figure 3.36: National Stadium in Bucharest, Romania. (*A*) Equilibrium model of the suspension cable roof , and (*B*) aerial view of the stadium (Kemmler, 2012, 479).

3.3.4 Summary

All form-finding methods presented here are aimed at the generation of structures that function with axial forces only for a dominant, design-driving loading case. The methods differ substantially in their media, or, to use Mitchell's terms, in their *design spaces*, hence also in their modelling methodologies (modelling in physical space, on paper, or on a computer screen). Furthermore, the methods also differ regarding their flexibility in dealing with specific types of equilibrium structures, as well as regarding their ability to enforce additional constraints on geometry and inner forces. Through constraints, the designer is able to control the form-finding process, and to interactively steer the design in a specific direction.

The presented physical methods are very tangible and direct, since the designer gets a direct feedback of the experimental model through 'material computation': physical laws e.g. gravity, in the cases of hanging models, or surface tension, in the case of soap films, drive the process. Form-finding approaches based on physical methods are very flexible regarding topology and geometry, and they work equally well for planar cases and for spatial cases. These methods are mainly applied in the design of compression-only structures, such as masonry vaults, or tension-only structures, such as tensioned cable-nets, since physical models of mixed compression-and-tension structures are very vulnerable to instabilities. Furthermore, physical methods in general lack the possibility of defining external constraints on form and forces. After a satisfying equilibrium structure is found, the geometry of the physical model has to be measured, which is often time-consuming, and the internal forces have to be determined in an additional process. For this reason, today physical methods are mainly used for early design studies; the determination of the exact form and force distribution is mostly achieved using computer-aided methods.

The presented graphical methods are very powerful for the form finding of planar structures, for instance for arches, cables and beams. Through the manual execution of the algorithms of graphic statics on a drawing board, the designer has a high degree of control over geometry and inner forces; both geometry and forces are explicitly represented in the form and force diagrams. Furthermore, the designer is able to enforce constraints on the geometry, by concepts such as the *line of action*, and to enforce constraints on the inner forces, by defining certain parts of the force diagram in advance. The concepts of graphic statics are not limited to two-dimensional cases, and the generalisation of the methodology to three-dimensional space has been demonstrated within the digital drawing space of CAD systems. However, all extensions to spatial cases increase the complexity of the method and the diagrams severely, reducing the intuitiveness of the approach. Recent approaches combine graphical algorithms with computational modelling and solving strategies, in order to automatise certain repetitive parts of the approach.

Computational form-finding methods were initially developed for the rationalisation of geometric data measured from physical models. Two main approaches for computational form finding of pin-jointed networks were developed in parallel: the *Force Density Method* and *dynamic relaxation*. While the Force Density Method is based on the mathematical description of static equilibrium, Dynamic Relaxation instead simulates the dynamic behaviour of physical form-finding processes. Like the physical processes, Dynamic Relaxation is also vulnerable to instabilities, due to mixed compression and tension elements. The Force Density Method works better for such structures, although the form-finding process is still difficult to control if both compression and tension forces are acting within one structure. Computational methods based on the finite element formulation allows form finding for sophisticated structural models beyond pin-jointed networks to be conducted, for example, by using continuous surface elements. Such methods are applied, among others, to high-end tensioned membrane design. The *thrust network analysis* method combines concepts from graphics statics with the force density approach, in order to provide a highly controlled form-finding process for compression-only structures.

And finally, the *statically-geometrically coupled method* provides a general framework for constraint form finding of pin-jointed structures with mixed compression and tension forces. The limitations of this powerful method lies in its preconditions: each form-finding problem has to be formulated such that the constraints on geometry and forces are consistent and have a mathematically unique solution. This requires a structural designer who has an intimate knowledge of the problems of spatial equilibrium, and limits the possibilities of playful, free explorations of structures.

Today computational form-finding methods are increasingly replacing physical approaches, even in early design stages, since digital approaches are faster, cheaper, and directly provide the geometry and the inner force distribution in a numerical form.

4 The problem of structural design modelling

Structural design as iterative, almost 'artistic' search process has been outlined by Schlaich and Hossdorf (see Section 3.1). The goal of this dissertation is to enable and facilitate the deliberate exploration of the inherent formal freedom of well-known structural typologies. Furthermore, the aim is to enable and foster design explorations beyond the boundaries of such well-known typologies. Inspired by experimental structural design processes based on physical, graphical, and computational form-finding techniques, the problem is stated as follows:

Develop interactive modelling methods which enable the creation and iterative modification of efficient structural systems in early stages of the design process.

Existing approaches do not provide a general solution to this problem. Earlier methods, based on the combinatorial assembly of given structural typologies within rigid spatial grids, lack the desired geometrical flexibility (see Section 3.2.1). Other approaches, based on generative methods and computational optimisation, provide a greater freedom of form, but mostly lack the possibility for the designer to influence the search process directly (see Section 3.2.2). Interactive approaches based on physical, graphical, or existing computational modelling and form-finding procedures (see Section 3.2.3) are conceptually closest to the desired goal, although these methods suffer from limitations based on their underlying form-finding techniques (see Section 3.3). These limitations are rooted in their specific modelling methods: physical models lack the possibility of defining certain boundary conditions, graphical methods are largely limited to the two-dimensional drawing space, and computational methods are either limited, or counter-intuitive, in handling constraints.

With the general availability of computational modelling tools, the need for geometrically flexible structural concepts has emerged, and the significance of interactive modelling methods for structures has greatly increased. Interactive structural design approaches are especially relevant for the design of buildings that require formally new and unexpected, yet highly efficient, structural systems. This is often the case for prominent buildings which belong to the typologies that Billington associates with "structural art", such as long-span bridges, large roofs and high towers (see Section 3.2.3). Implemented as computer-aided design tools, interactive modelling methods may furthermore serve as important technological advancements in bridging the gap between the design of expressive freeform architecture and the design of efficient and elegant structural systems.

5 Equilibrium solutions

In this Section, the fundamental concepts of equilibrium solutions and lower bound design are summarised, and exemplary structural stiffening schemes are presented.

5.1 Lower-bound theorem of plasticity theory

The approaches developed here are based on the conceptual framework of equilibrium solutions, rooted in plasticity theory and limit state design. Within the context of the theory of plasticity, the rigid-plastic material model is generally assumed for structures. This allows material stiffness and deflections to be neglected; the transfer of loads can be described by only considering the static equilibrium of forces.

In this dissertation, the structures are modelled as pin-jointed, trussed structures, resulting in solely axial forces in the members. According to the lower-bound theorem of the plasticity theory, such trussed structures are safe for a given loading case, if the member dimensions for the given state of equilibrium are sufficient for axial stresses and are not in the danger of buckling. Lower-bound solutions have been successfully applied in the design of a variety of different construction types: in the design of steel structures (Baker, Horne *et al.*, 1956), structures in reinforced concrete, using strut-and-tie models (Schlaich, Schäfer *et al.*, 1987) or stress fields (Muttoni, Schwartz *et al.*, 1996), and masonry structures (Ganz and Thürlimann, 1984), using thrust lines (Heyman, 1995) or thrust networks (Block and Ochsendorf, 2007).

The methods presented in Sections 5 and 6 have the objective of creating kinematic equilibrium solutions for one dominant loading case, which is often the structure's dead load. These equilibrium solutions generated by form-finding processes are the starting point for further refinement of the structural system. Schlaich and Schäfer, for instance, present an example for the detailing of a cast steel component as bridge deck anchorage based on a strut-and-tie model (Schlaich and Schäfer, 1991). The equilibrium modelling process is part of the early stages of the structural design process. In later stages, a rigid structural system has to be derived from the kinematic equilibrium solution. Such a rigid, determinate or indeterminate, structural system is a precondition for the preliminary dimensioning of the structure, considering live loads. The creation of rigid structures based on equilibrium solutions, preliminary structural dimensioning, and structural detailing are beyond the scope of this work. Menn gives a detailed description of the preliminary design and dimensioning of arch bridges using classical analytical statics (Menn, 1990: 387-393). For the preliminary dimensioning of geometrically complex, spatial structures, the application of commercial analysis software based on FEM is useful. In the next Section, exemplary concepts of stiffening schemes for kinematic models are outlined.

5.2 Kinematic models and stiffening schemes

For one given kinematic equilibrium model, generally a variety of different *stiffening schemes* exists. The choice of a stiffening scheme often has strong aesthetic and constructive implications. Figures 5.1 and 5.2 illustrate different conceptual approaches to the stiffening of a simple two-dimensional arch bridge. These approaches are related to three of Robert Maillart's iconic arch bridges in Switzerland.

The form-finding process of a symmetric arch bridge, based on a discretised, uniform dead load, is straightforward (Figure 5.1 *A*); the arch geometry is obtained for instance by graphic statics, through the construction of a *funicular polygon* (see Section 3.3.2). In this state, the arch is modelled as pin-jointed structure; also the deck is initially modelled as pin-jointed structure, and the overall structural system forms a mechanism (Figure 5.1 *B*).

Figure 5.1: Arch geometry resulting from a form-finding process (*A*), and the pin-jointed kinematic structure consisting of both the arch and the deck (*B*).

In order to create a stiff system from the kinematic model shown in Figure 5.1 *(B)*, three approaches are illustrated in Figure 5.2: introducing diagonals, introducing a bending-stiff deck, or introducing a bending-stiff arch. By introducing diagonals or braces, the kinematic system is transformed to a truss that is able to withstand varying load cases. Within the framework of strut-and-tie models, Maillart's bridge in Zuoz, completed in 1901, can be interpreted as such a truss system. Alternatively, it is sufficient only to construct the deck as a stiff beam, supported by a pin-jointed arch. This strategy had be applied by Maillart in the design of the Valtschiel Bridge near Donat, completed in 1925. Furthermore, instead of the deck, also the arch can be constructed as bending-resistant element, in this case as three-hinged arch. This structural system was also applied in Maillart's Salginatobel Bridge near Schiers, completed in 1930. For geometrically complex, three-dimensional structures, there obviously is a greater variety of different stiffening schemes, but the fundamental concepts remain similar.

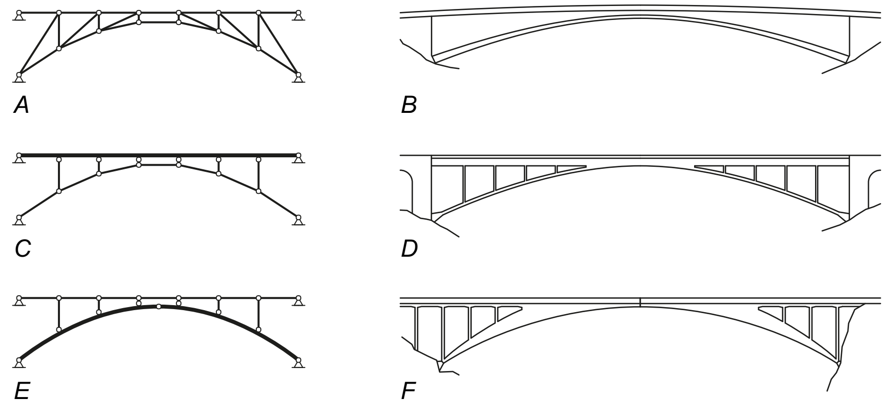

Figure 5.2: Different stiffening strategies for arch bridges, and corresponding realised examples by Robert Maillart: stiffening through diagonal members and the bridge in Zuoz (*A*, *B*); a deck-stiffened system and the Valtschiel Bridge (*C*, *D*); an arch-stiffened system and the Salginatobel Bridge (*E*, *F*).

All modelling methods presented in this dissertation deal exclusively with the question of static equilibrium, additional design steps in later phases have to follow for the creation of a realisable proposal. The designer has to choose an appropriate construction, and a lateral stiffening scheme has to be developed that is able to withstand loading cases other than the design load. Furthermore, subsequent steps of structural modelling, analysis, dimensioning, and detailing are needed, in order to obtain a structure which is safe and usable.These additional steps are beyond the scope of the thesis.

6 Type-specific equilibrium modelling

The problem of structural design modelling as stated in Section 4 is solved for specific, well-defined typologies of structures. Based on the conceptual framework of equilibrium solutions (Section 5), four different interactive modelling methods tailored to specific structural typologies are presented.

The examined typologies are: a roof based on trussed beams, a spatial arch bridge, a slab supported by branching columns, and a compression vault. The geometrically regular, standard cases of these types are well known, and are displayed in the standard references of structural design, as e.g. in Engel's "Structure Systems". Here, methods for modelling of non-standard manifestations of these typologies are presented.

The case studies are organised as follows: firstly, a brief description of the intended typology is given; secondly, the underlying structural concept and the resulting decomposition of the equilibrium problem is explained; and finally, the implementation of the form-finding technique and setup of the modelling process are presented.

6.1 Trussed roofs

Inspired by the curved glass roofs covering sections of the tracks in train stations, such as the roof of the Lehrter Bahnhof in Berlin, Germany,[27] or the roof of the Waterloo Station in London, UK,[28] the idea is to develop a method for the automatic generation of constrained trussed roof structures, based on a given NURBS freeform surface representing the roof geometry.

The surface is sliced into a number of vertical planes, resulting in profile curves, which are perpendicular to the tracks' direction in plan. Assuming a constant weight per square metre for the roof, an individual load distribution for each profile curve is calculated, based on an estimation of the tributary areas. The profile curve, together with the load distribution, form the basis for the generation of the efficient truss geometries. The idea is that the profile curve acts as flexurally rigid compression chord, supported by a tension cable with constant inner force. Using a method from graphic statics, similar to the one Conzett used in the design of the Second Traversina Bridge (see Section 3.2.3), the cable geometry and the strut layout are generated for a given number of divisions, and a given magnitude of the tension force along the truss' bottom chord (Figure 6.1).

[27] Completed in 2006, by Gerkan, Marg and Partner with schlaich bergermann und partner
[28] Completed in 1993, by Grimshaw Architects with Anthony Hunt Associates

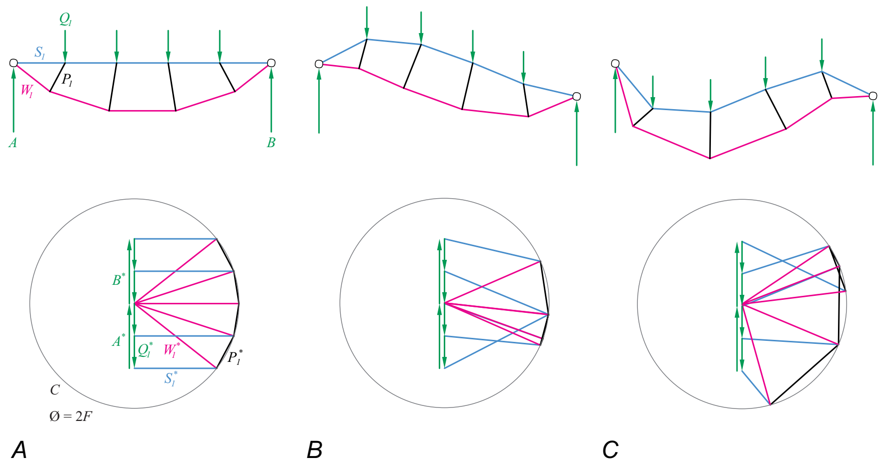

Figure 6.1: Geometry of the truss structure (above), and force diagram (below), for three different upper chord profiles: (*A*) the symmetric case, (*B*) a sligthly deformed profile, and (*C*) a heavily deformed profile (Lachauer and Kotnik, 2010: 197).

6.1.1 Form-finding method[29]

In order to demonstrate the method of graphic statics and its application to design, a geometric method for the procedural construction of a planar truss will be presented. The technique is based on a design method for the *constant chord force truss* (Zalewski and Allen, 1998: 275–300). This method generates a truss form with the top chord in pure compression and the bottom chord in pure tension for dead load. Additionally, the tension forces in the bottom chord are all equal. While Waclaw Zalewsky and Edward Allen describe the application of this method for specific top chord shapes, this paper explores the possibility of this method for arbitrary top chord forms.

The truss form is constructed from a given discrete curve S consisting of the segments $S_1, S_2, \ldots, S_n$, defining the geometry of the top chord, and chord force F. For each node of the top chord, S, a dead load component, Q_i, is assumed. The first step is to construct the reciprocal diagram from the chord segments S_i , the nodal weights $Q_1, Q_2, \ldots, Q_{n-1}$,, and F. The second step is to construct the bottom chord of the truss.

The construction of the force diagram is straight forward: The nodal loads Q_i^* in the force diagram are graphically added. The support forces A and B are derived either by the lever rule or graphically by a trial funicular (Schwartz, 2008: I 13–14). The circle C is then constructed around the tip of the force vector A^*, with radius F. The absolute value of F must be large enough such that the reciprocal load components Q_i^* are entirely located inside the circle. Next, construct the rays S_i^* in the direction of the top chord segments S_i. (Figure. 6.1). The connecting lines between the intersection points $I_i = C \cap S_i^*$ between the circle and the rays are the reciprocal representations P_i^* of the truss members connecting the top and the bottom chord P_i. The representation of

29 This Section has previously been published in Advances in Architectural Geometry 2010 (Lachauer and Kotnik, 2010: 196–197)

the force vectors W_i^* in the bottom chord are constructed by the connection of the intersection points I_i on the circle with the center of C. To construct the geometry of the bottom chord in the form diagram, start at support A and continue from left to right to the successive intersection of rays in the direction of P_i^* and W_i^*, which are the nodes of the chord.

6.1.2 Modelling approach

The method was implemented as a parametric model, allowing the interactive design of the structure based on modifications of the NURBS surface in real time, by changing the positions of the *control points*[30] of the surface (Figure 6.2). One additional, numerical parameter allows the control of the maximal inner force in the tension chords. By increasing the inner force in the tension cables, the distance between compression and tension chords is reduced, and vice versa. Other parameters define the number of divisions in both directions. As is the case with all generative approaches based on parametric models, the geometry of the structure can conveniently be controlled by the input parameters, such as the shape of the freeform surface and the numerical parameters. On the other hand, its general typology, defined by topological characteristics such as the overall linear shape, and the sequential order of vertical trusses, cannot be changed.

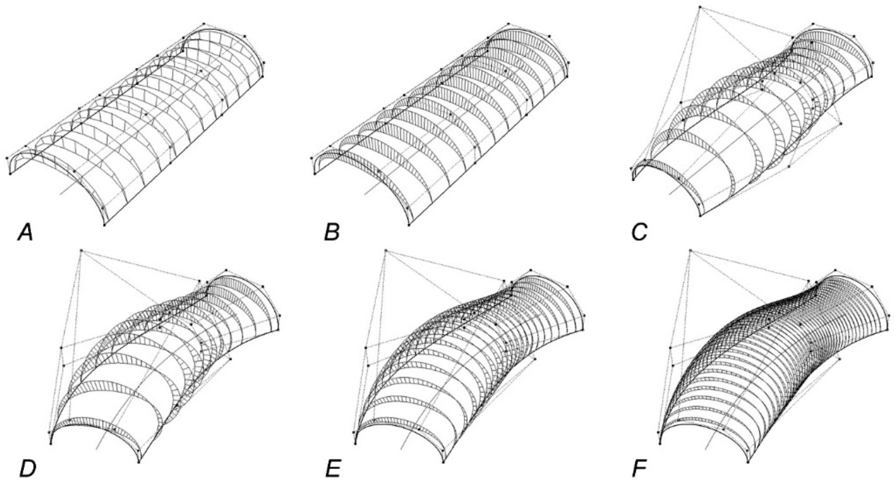

Figure 6.2 (*A*)-(*F*): Interactive, step-wise refinement of the model of the efficient roof structure based on a given freeform NURBS surface, segment numbers in both direction are controlled by numerical parameters (Lachauer and Kotnik, 2010: 201).

[30] 'Control points' are mathematical parameters controlling the shape of a NURBS surface; in a CAD modelling environment, they represent 'handles' which enable the user to modify the shape of the surface by changing the points' positions

6.2 Curved bridges

Fascinated by the surprising spatial quality of curved cable and arch bridges, such as, for example, the Liberty Bridge in Greenville, USA,[31] or the Campo Volantin Bridge in Bilbao, Spain,[32] the idea was to propose a systematic method for designing and form finding such structures. The design parameters are one planar freeform curve as given deck axis, and two support points for the main cable or arch. Inspired by Keil (2004), the design concept is based on the decomposition of the spatial structural problem into two sub-problems: finding a spatial funicular that balances the vertical forces induced by the dead load, and constructing a system in the plane of the bridge deck that balances the remaining horizontal force components (Figure 6.3).

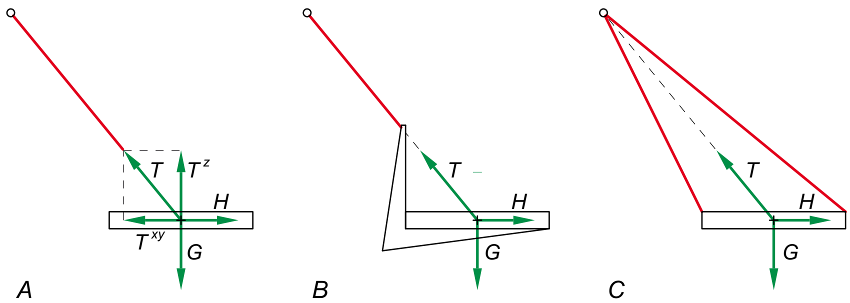

Figure 6.3: Decomposition of the equilibrium problem for a vertical section perpendicular to the bridge deck's axis: the vertical component $\boldsymbol{T^z}$ of the hanger force $\boldsymbol{T}$ balances the dead load $\boldsymbol{G}$, the horizontal component $\boldsymbol{T^{xy}}$ is balanced by the force $\boldsymbol{G}$. This force $\boldsymbol{H}$ is induced by a horizontal system lying within the plane of the bridge deck (*A*). In most cases, the hanger is not fixed to the centre of the mass of the bridge deck, but is attached with an additional tee element (*B*), or (*C*) the hanger is split into two hangers (Lachauer and Kotnik, 2011: 147).

6.2.1 Form-finding method[33]

In this section, a computational method for the generation the funicular polygon in space is described. It is based on a form-finding technique for tension structures using dynamic relaxation (DR) (Barnes, 1999). Here, DR is not explained to full extend, only the differences to Barnes's method are identified. The anchor points $\boldsymbol{S}_1 \ldots\ \boldsymbol{S}_n$, on the axis of the bridge deck are equally spaced, so one can assume the dead-load $\boldsymbol{G}_i = \boldsymbol{G}$ for all i. The nodes of the funicular at time t are named $\boldsymbol{X}_0^t \ldots \boldsymbol{X}_{n+1}^t$, the factor r controls the *rise* of the funicular (Figure 6.4). The supports $\boldsymbol{X}_0$ and $\boldsymbol{X}_{n+1}$ are input parameters.

[31] Completed in 2004, by Rosales + Partners with schlaich bergermann und partner

[32] Completed in 1997, by Santiago Calatrava

[33] This Section has previously been published in Computational Design Modeling (Lachauer and Kotnik, 2011: 147–148)

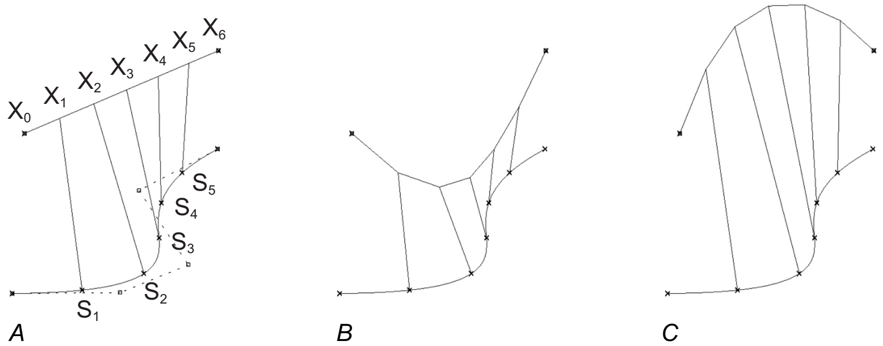

Figure 6.4: Initial state of the form-finding process (*A*), the funicular as tension cable for a rise $r > 0$ (*B*), the funicular as arch, acting in compression for a rise $r < 0$ (*C*) (Lachauer and Kotnik, 2011: 148).

As illustrated in Figure 6.3, the funicular has to fulfill $-\boldsymbol{T}_i^z = \boldsymbol{G}_i$ for all connecting elements between funicular and deck. In order to reach this condition the form-finding method DR is adapted. The residual force at node $\boldsymbol{X}_i^t$, for $0 < i < n + 1$, is considered as $\boldsymbol{R}_i^t = \boldsymbol{F}_i^t + \boldsymbol{F}_{i+1}^t + \boldsymbol{C}_i^t$.

Similar to DR, the forces in the funicular at node X_i^t are determined in relation to their initial length at time $t = 0$: $\boldsymbol{F}_i^t = (\boldsymbol{X}_{i-1}^t - \boldsymbol{X}_i^t)/\|\boldsymbol{X}_{i-1}^0 - \boldsymbol{X}_i^0\|$ and $\boldsymbol{F}_{i+1}^t = (\boldsymbol{X}_{i+1}^t - \boldsymbol{X}_i^t)/\|\boldsymbol{X}_{i+1}^0 - \boldsymbol{X}_i^0\|$. The difference to DR is the definition of the forces $\boldsymbol{C}$ in the connecting elements between deck and funicular:

$$\boldsymbol{C}_i^t = \frac{\boldsymbol{Q}}{\|\boldsymbol{Q}^z\|} r, \text{ with } \boldsymbol{Q} = \boldsymbol{S}_i - \boldsymbol{X}_i^t \tag{6.1}$$

Equation (6.1) ensures that the magnitudes of the vertical force components of all $\boldsymbol{C}_i^t$ are r. The solving procedure for the solution $\boldsymbol{X}_1^* \dots \boldsymbol{X}_n^*$ is straightforward using DR (*Ibid.*). The forces $\boldsymbol{T}^*$ are finally scaled by the factor $\|\boldsymbol{G}\|/r$ in order to balance the dead load vertically: $\boldsymbol{T}_i^* = \boldsymbol{C}_i^* \frac{\|\boldsymbol{G}\|}{r} = \frac{\boldsymbol{Q}}{\|\boldsymbol{Q}^z\|} \|\boldsymbol{G}\|$.

6.2.2 Modelling approach

As dead load, vertical forces are evenly distributed along the bridge axis. The first step is finding a spatial funicular attached to the two given support points, either as compression arch or as tension cable, that precisely balances the vertical forces due to dead load (Figure 6.5 *A*). The inclined hangers furthermore induce horizontal forces in the deck. In a second step, a horizontal system within in the plane of the bridge deck is found, which is able to balance these forces (Figure 6.5 *B*). Depending on the shape of the deck in plan, the horizontal forces can either be balanced by a funicular system, such as, for instance, by an arch or a cable in the bridge deck, or by a combined system formed by a compression chord and a tension chord (Figure 6.5 *C*), similar to the trusses in the roof of the previous example. Obviously, a framework lying within the bridge deck would also be able to balance the horizontal force components. However, for the construction of the deck as reinforced concrete element, a system based on the two funiculars could be more elegant and efficient.

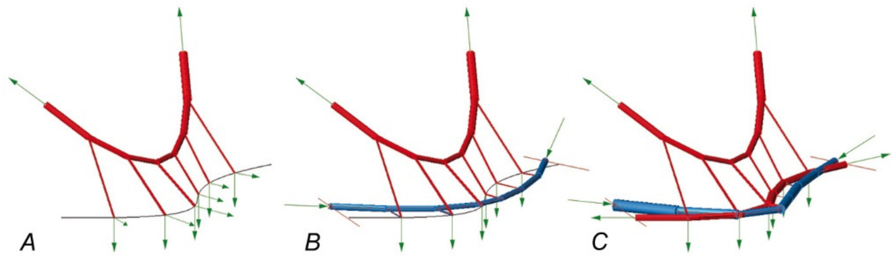

Figure 6.5: The spatial funicular induces horizontal and vertical force components in the deck axis via the inclined hangers; the vertical forces are balanced by the dead load (*A*). The horizontal force components can be balanced either by a horizontal funicular, e.g. an arch in compression (*B*), or by a combined truss system, consisting of the deck axis and an additional funicular (*C*) (Lachauer and Kotnik, 2011: 150).

For the generation of the spatial funicular, a method based on Dynamic Relaxation was developed, using custom, non-linear springs for the hangers. Their inner forces are defined such that the magnitude of the vertical components of the hanger forces are always equal to the dead load. For finding the shape of the horizontal system, custom parametric methods derived from graphic statics have been applied; these methods incorporate an automated generate-test cycle for finding a best-fitting position of the horizontal tension cable to the deck axis. Similar applications of parametric tools in structural design were earlier described by the author, especially the generation of planar funicular polygons close to a given axis (Lachauer, Jungjohann *et al.,* 2011). The methods described in this passage enable the designer to find an equilibrium shape for a curved arch or suspension bridge in two steps: firstly, finding the form of the spatial funicular and, secondly, constructing a horizontal system in the bridge deck. Both methods are implemented as interactive tools, solving the two sub-problems in real time. These methods provide high geometric flexibility for designing exactly this linear bridge typology (Figure 6.6), but again changes to the topology of the structure, such as, for instance, using a splayed arch instead of a linear arch, are not possible.

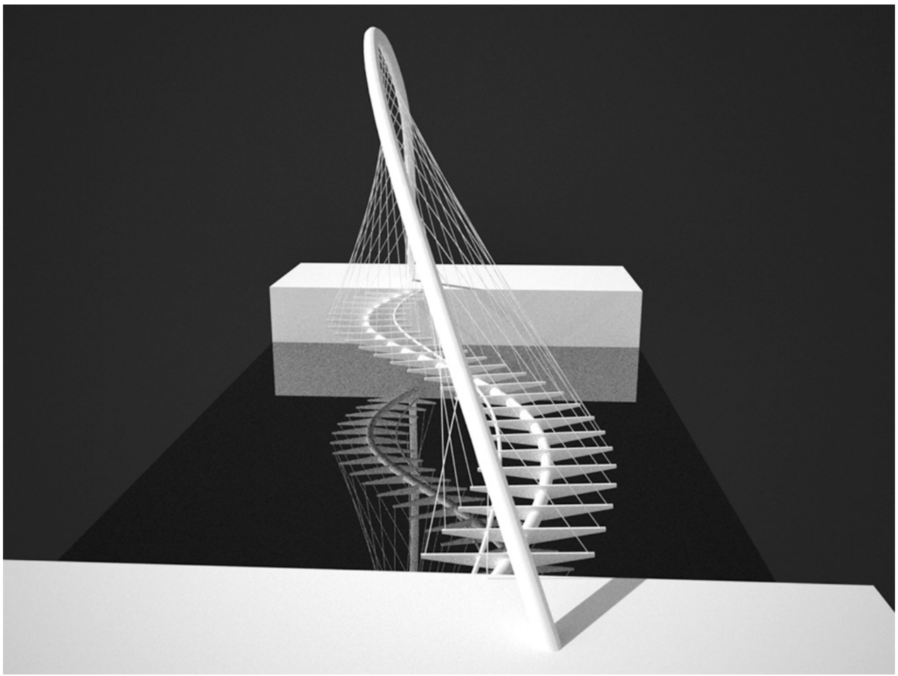

Figure 6.6: A design example for a curved arch bridge. The spatial arch bears the vertical force components, the inclined hangers the generating horizontal force components, which are balanced by the horizontal system. This system is formed by the combination of the deck axis acting in compression, and an additional tension cable lying within the bridge deck. (Lachauer and Kotnik, 2011: 151).

6.3 Branching structures

The goal of this research project has been to extend the concepts developed for the design of curved bridges towards more complex structures, beyond linear topology. By replacing the bridge deck with a two-dimensional plate, and the arch with a general compression network, one obtains a structural typology consisting of a flat slab, supported by a 'tree structure' or branching structure. Examples of such structures are, for example, the tree-columns supporting the roof of Terminal 3, Stuttgart Airport, Germany,[34] and the branching concrete structure of the new foyer of the Building Academy in Salzburg, Austria.[35] The design problem is decomposed in a comparable way to that of the curved bridges project (see Section 6.2). The first step is finding an equilibrium network balancing the dead load of the plate, and the second step is the development of an in-plate system, which is able to balance the horizontal force components resulting from the inclined columns.

6.3.1 Form-finding method[36]

The main challenges in finding an equilibrium solution for a support structure with complex geometry for a flat and heavy slab are illustrated schematically for

[34] Completed in 2004, by Gerkan, Marg and Partner with schlaich bergermann und partner

[35] Completed in 2012, by soma architecture with Brandstatter ZT

[36] This Section has previously been published in Advances in Architectural Geometry 2012 (Lachauer and Block, 2012: 138–141)

a simple two-dimensional case with a pin-jointed branching column (Figure 6.7). Assuming that the position of the supports is given, one can determine a set support forces $\boldsymbol{F}_1^v$, $\boldsymbol{F}_1^v$ for the dead load Q of the slab (Figure 6.7 *A*). The self-weight of the support structure is not considered, as it is assumed to be small in comparison to the weight of the slab. For the branching support structure with arbitrary geometry, shown in Figure 6.7 (*B*), the forces in the strut elements connected to the slab are statically defined: using trigonometry, the axial member forces, $\boldsymbol{F}_i$ and horizontal components in them, or in other words the horizontal support forces, $\boldsymbol{F}_i^h$, can be directly found from the vertical support forces, $\boldsymbol{F}_i^v$, as indeed $\boldsymbol{F}_i = \boldsymbol{F}_i^v + \boldsymbol{F}_i^h$. Considering now the free[37] node $\boldsymbol{N}$, the resultant, or sum of forces in the struts attached to the slab, $\boldsymbol{F}^r = \boldsymbol{F}_1 + \boldsymbol{F}_2$, is not necessarily acting in the direction of the strut connected to the ground, which means that node $\boldsymbol{N}$ is not in equilibrium. Furthermore, the reaction force $-\boldsymbol{F}^h = \boldsymbol{F}_1^h + \boldsymbol{F}_2^h$, acting horizontally in the plate is also not balanced yet. With these observations as premise, the form finding problem can thus be divided in two categories:

Finding the geometry of the supporting structure of horizontally restrained slabs (e.g. projecting or cantilevering roofs attached to a building), such that all of its free nodes are in equilibrium for the given loads of the slab. This is possible for any given support position $\boldsymbol{G}$ to the ground, by just moving the free nodes $\boldsymbol{N}$, because the remaining horizontal reaction force $\boldsymbol{F}^h$ can be taken by the horizontal restraint of the slab (Figure 6.7 *C*).

Finding the geometry of the supporting structure of horizontally unrestrained slabs (e.g. roofs of free standing pavilions), such that all of its free nodes are in equilibrium for the given loads of the slab *and* the remaining horizontal force components are balanced within the slab. Therefore both the position of free nodes *and* ground supports have to be modified (Figure 6.7 *D*). Both categories of problems will be addressed in the next section.

[37] A *free* node is neither a ground support, nor a support of the slab

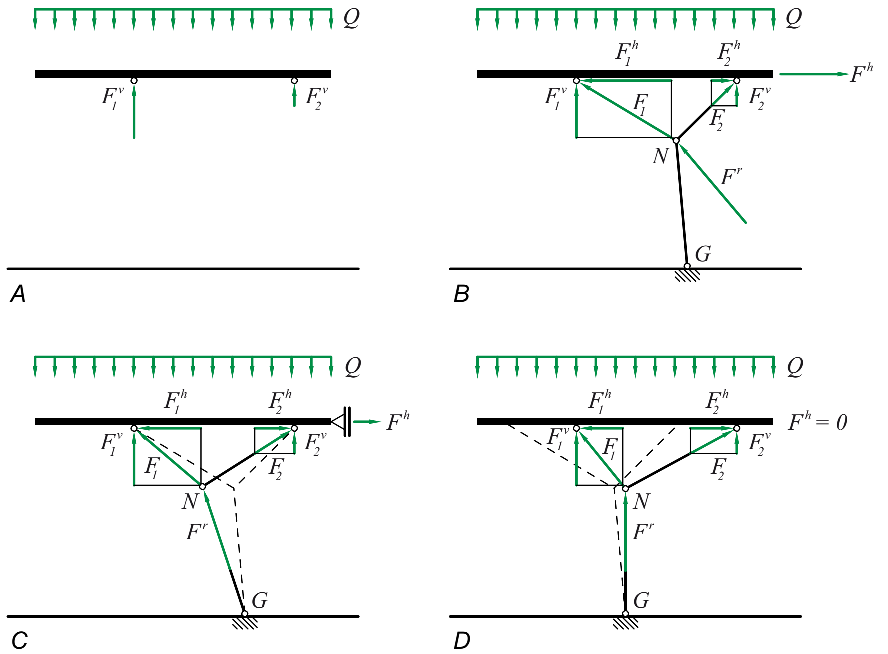

Figure 6.7: Given (A) dead load Q of the slab and support forces, (B) for an arbitrary geometry, the support structure in not in equilibrium in node $\boldsymbol{N}$, as both forces $\boldsymbol{F}^r$ and $\boldsymbol{F}^h$ are not balanced; (C) shows a support structure in equilibrium with a horizontally restrained slab; and (D) a support structure in equilibrium with the slab without horizontal restraint (Lachauer and Block, 2012: 138).

The method for form-finding will be described for two categories, for horizontally restraint slabs and for unrestraint slabs. In the first section, the generation of support structures that balance the vertical force components in the slab, is described. This method can be used for designing horizontally restraint slabs. In the second section, additional equilibrium conditions are formulated as extension to the method, in order to solve for both vertical and horizontal force components simultaneously.

The form-finding method is applied to a slab with given self-weight and support positions. Furthermore, a set of vertical support forces $\boldsymbol{F}_1^v \ldots \boldsymbol{F}_n^v$ balancing the dead load of the slab is assumed as given[38] (Figure 6.8).

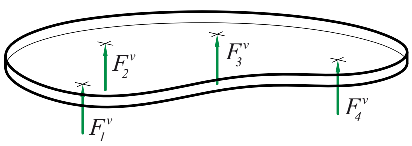

Figure 6.8: A generic plate with given supports and support forces (Lachauer and Block, 2012: 139).

[38] Note that within the lower-bound theorem of theory of plasticity, these can be chosen, if it is indeed assumed that the slab has enough bending stiffness to distribute the forces in those proportions to the supports; one possible set of support forces could be obtained using an finite element analysis tool

Additionally, a network of struts in space is given, that determine the connectivity of the structure and the starting point of the iterative form-finding procedure. The only restriction to the connectivity of the network is that each support of the slab has to be connected to exactly one strut, as otherwise the member forces of these struts cannot be uniquely defined; the nodes at the supports of the slab would again become statically indeterminate.

6.3.1.1 Finding Vertical Equilibrium

As described in above for a horizontally supported slab (Figure 6.7 *C*), the challenge is to find the position of the free nodes $\boldsymbol{N}_i$ such that those nodes are in equilibrium, and that the slab's support forces $\boldsymbol{F}_i^v$ are vertically balanced. If the slab is restrained in at least two points, the resulting horizontal force components $\boldsymbol{F}_i^h$ in the slab can either be equilibrated within the plane of the slab, with a tension or compression funicular or a by a truss.

Two conditions have to be satisfied for the equilibrium of the structure:

For each support i of the slab, a horizontal force $\boldsymbol{F}_i^h$ has to exist such that the support force $\boldsymbol{F}_i^v$ can be balanced by the force $\boldsymbol{F}_i$ in the supporting strut:

$$\boldsymbol{F}_i = \boldsymbol{F}_i^v + \boldsymbol{F}_i^h \tag{6.2}$$

For each free node $\boldsymbol{N}_i$, the forces $\boldsymbol{S}_i$, in the m neighboring struts have to be in equilibrium:

$$\boldsymbol{R}_i = \sum_{i=1}^{m} \boldsymbol{S}_i = 0 \tag{6.3}$$

In the two-dimensional example (Figure 6.7 *C* and *D*), for node $\boldsymbol{N}$, the forces in the neighboring struts would be $\boldsymbol{S}_1 = \boldsymbol{F}_1$, $\boldsymbol{S}_2 = \boldsymbol{F}_2$, and $\boldsymbol{S}_1 + \boldsymbol{S}_2 = \boldsymbol{F}^r$. In order to achieve the two equilibrium conditions (6.2) and (6.3), the structure is solved as a tension network consisting of zero length springs (Harding and Shepherd, 2011), and subsequently the sign of the forces is switched, resulting in a compression-only solution.

Starting with the provided initial geometry of the network of struts and the vertical force components $\boldsymbol{F}_1^v \ldots \boldsymbol{F}_n^v$, the initial forces $\boldsymbol{F}_1 \ldots \boldsymbol{F}_n$ in the struts are computed using trigonometry (as described below in step I). Subsequently the scalar c is calculated, defined as the inverse average magnitude of initial forces $\boldsymbol{F}_1 \ldots \boldsymbol{F}_n$ in the struts connected to the slab. This scalar relates the level of pre-stress of the springs to the magnitude of given vertical force components. In each time step t of the form-finding process, the following steps are performed:

I. The forces in the struts connected to the slab are calculated as $\|\boldsymbol{F}_i\| = c \cdot \|\boldsymbol{F}_i^v\| \cdot \sin^{-1} \alpha$, hereby Eq. (6.2) is directly satisfied; α is the angle between the strut and the slab.
II. All struts that are not connected to the slab supports are modeled as zero length springs; the magnitudes of forces $\boldsymbol{S}_i^t$ are proportional to the strut length, with an initial level of pre-stress of 1: $\|\boldsymbol{S}_i^t\| = l_i^t / l_i^0$; l_i^t is the length of the strut i at time t, and l_i^0 the initial length of this strut.

III. The position of each free node is updated: $\boldsymbol{N}^{t+1} = \boldsymbol{N}^{t} + d \cdot \boldsymbol{R}_i^t$; d is a small scalar, defining the step size of the procedure (e.g. $d = 0.1$).

This procedure is iteratively repeated, until the sum of residual forces $\sum \boldsymbol{R}_i^t$ is smaller than a given threshold ε. After convergence, all forces have to be divided by c in order to calculate the real member stresses and support forces that are in equilibrium with the given vertical loads $\boldsymbol{F}_1^v \ldots \boldsymbol{F}_n^v$. In order to accelerate convergence, one might introduce velocities to solve the problem with a *dynamic relaxation* formulation (Barnes, 1999) or apply more advanced *Runge-Kutta* solving strategies (Kilian and Ochsendorf, 2005). Instabilities in the solving procedure occur due to vanishing spring lengths in the case of very diverge magnitudes of the given vertical forces.

6.3.1.2 Finding Horizontal Equilibrium

In order to find an equilibrium state of the support structure, with the additional constraint that all horizontal force components in the plate should be in equilibrium, hence enabling a free standing structure, two more equilibrium conditions are added to Eq. (6.2) and (6.3). A set of in-plane forces $\boldsymbol{F}_i^h$ is in equilibrium if and only if their sum is zero:

$$\boldsymbol{R}_i = \sum_{i=1}^{n} \boldsymbol{F}_i^h = 0 \tag{6.4}$$

and if the sum of the moments they induce, around any point in the plate, is zero:

$$\boldsymbol{M} = \sum_{i=1}^{n} \boldsymbol{M}_i = \sum_{i=1}^{n} \boldsymbol{F}_i^h h_i = 0 \tag{6.5}$$

with h_i being the perpendicular distances of the line of action of the force $\boldsymbol{F}_i^h$ to a chosen point $\boldsymbol{O}$ (Figure 6.9).

To integrate these two additional constraints in the form finding process, the idea is to translate and rotate the roof, inclusive roof supports, at each iteration, such that the slab position moves towards a balanced position. Therefore one has initially to define an arbitrary point $\boldsymbol{O}$ to calculate the resulting moment $\boldsymbol{M}$. In each step of the solving procedure described in the previous section, two more steps are performed subsequently after step III:

VI. The system of the slab, together with supports and point $\boldsymbol{O}$, is rotated in plane by the angle $d_1 \cdot \boldsymbol{M}^t$ degrees.

V. The system of the slab, with slab supports and point $\boldsymbol{O}$, is translated horizontally by the vector $d_2 \cdot \boldsymbol{R}^t$.

Both d_1, d_2 are small constants defining the step size of the solving procedure. Experience has shown that d_1, d_2 being two magnitude smaller than *d* brings good results.

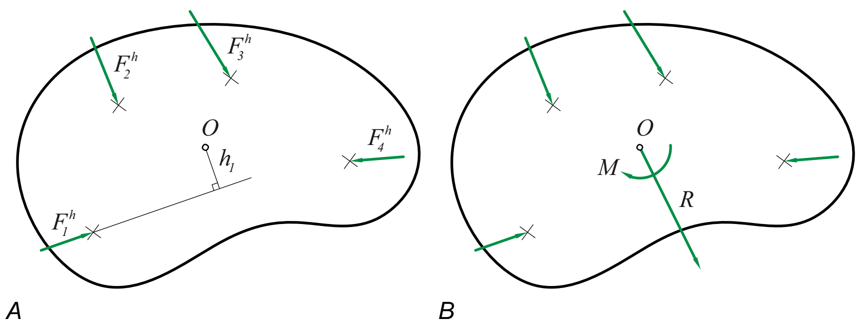

Figure 6.9: Slab with horizontal force components $\boldsymbol{F}_1^h \ldots \boldsymbol{F}_n^h$ acting at the supports, and the arbitrary chosen point $\boldsymbol{O}$ (*A*); slab with the equivalent actions $\boldsymbol{M}$ and $\boldsymbol{R}$ (*B*) (Lachauer and Block, 2012: 141).

6.3.2 Modelling approach

For the general case of a horizontally unsupported plate resting on inclined columns, the horizontal components generally induce a rotational moment and a translational force (Figure 6.10 *A*). For the case where the plate is horizontally supported in at least two points, such as, for instance, attached to another building, a generic truss or funicular structure can be constructed within the plate, which is able to transfer the horizontal force components to these supports (Figure 6.10 *B*). In order to create a balanced freestanding solution, the plate shown in (*A*) is rotated and shifted in plane during the form-finding process of the support structure (Figure 6.10 *C*, *D*).

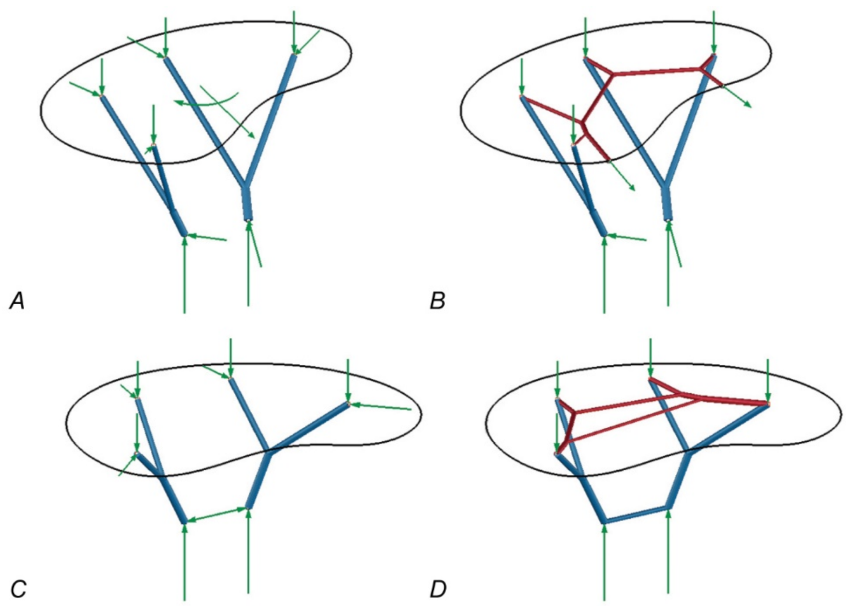

Figure 6.10: A horizontally unbalanced state of a flat slab on branching columns with an acting translational force and rotational moment (*A*). The slab is horizontally supported in two anchor points; a funicular system within the plate balances the force and the moment (*B*). A horizontally balanced state of the flat slab, achieved through translation and rotation of the slab (*C*). A tension system balancing the horizontal force components (*D*) (Lachauer and Block, 2012: 142).

The example shown in Figure 6.11 shows the form finding of a bending-free support structure for a concrete slab, based on a given irregular layout of possible structural support zones. An initial state of the structure is modelled by the designer using standard CAD tools (Figure 6.11 *A*). In this example, the slab is supported by a regular quadratic grid of supports. Below this, a regular horizontal grid of struts forms a zone of structural 'transition' and, finally, the lowest zone of the structure consists of vertical elements, either single columns, or clustered bundles of columns, for transferring the loads of the slab in the ground. The form-finding process transforms the structure towards an equilibrium form, the position of the slab is rotated and shifted slightly in-plane, to achieve global equilibrium (Figure 6.11 *B*). This method could obviously also be used for the design of an irregular ground-floor solution for a classical slab and column construction. The presented approach is flexible regarding the form and connectivity of the supporting structure. The designer is furthermore able to adjust the structure by adding struts, deleting struts, and changing the initial position of the nodes. Through a repetitive application of the form-finding method, an iterative search process is possible. Nevertheless, the general typology is restricted to one horizontal, planar slab; it is not possible, for instance, to stack multiple slabs, or to use an alternative, non-planar surface instead of the slab.

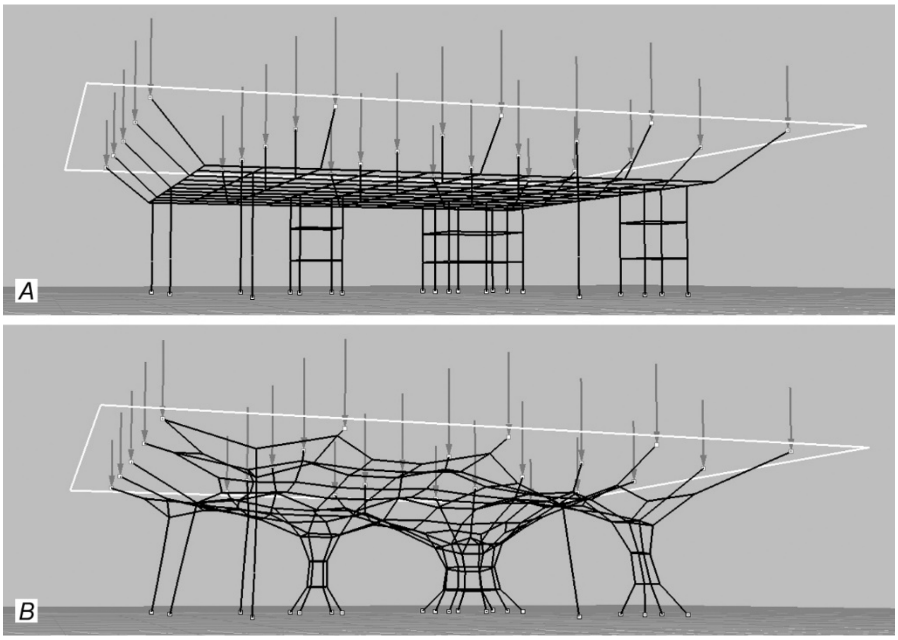

Figure 6.11: Compression-only suppport structure for a flat slab. Input of the model (*A*) and final equilibrium state (*B*) (Lachauer and Block, 2012: 143).

6.4 Compression vaults

As member of the BLOCK Research Group at ETH Zurich, the author contributed to the development of an interactive structural design tool for compression-only equilibrium networks (Rippmann, Lachauer *et al.,* 2012). The tool has been implemented as freely available, award winning,[39] plugin[40] for a commercial CAD software.[41] It is based on Thrust Network Analysis (Block and Ochsendorf, 2007) (see Section 3.3.3). The method addresses the difficulty in controlling the geometry of hanging models or their digital simulations. Often the plan of the network heavily deforms while converging towards an equilibrium state. Thrust Network Analysis (TNA) as design tool allows the control of both the structure's plan and its inner force distribution explicitly during the form-finding process, using planar diagrams. These diagrams are derived from graphic statics: in TNA the *form diagram* represents the plan of the structure, and the *force diagram* represents the equilibrium of the horizontal force components in the struts of the spatial network. The two diagrams, together with the given heights of the support points, uniquely define the geometry of the equilibrium network for one set of vertical loads (Figure 6.12).

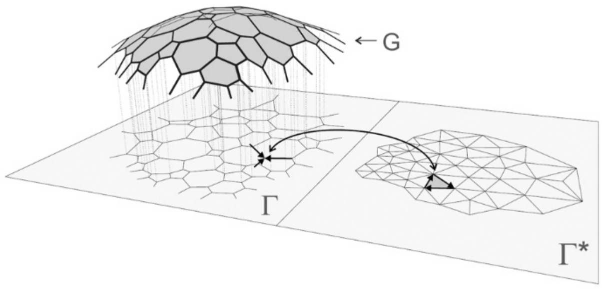

Figure 6.12: The *thrust network* G, its horizontal projection or *form diagram* Γ, and the corresponding *force diagram* Γ* (Block and Ochsendorf, 2007).

In the beginning of the design process, the designer is able to draw a planar network freely, and a first force diagram is generated automatically. In the following steps, the iterative search process for the spatial equilibrium network is directed by the designer by alternately modifying the geometry of form and force diagrams. Therefore, the spatial form-finding problem is decomposed into two sub-problems: automatic adjustment of user-defined form and force diagrams, such that the force diagram represents a valid equilibrium state of the force diagram,[42] and automatic generation of the equilibrium network based on the geometry of both diagrams. Both sub-problems are solved using computational relaxation approaches, Rippmann, Lachauer *et al.* (2012) give a detailed description of the underlying algorithms. The height of the network's support points are directly defined by the designer, but the overall shape of the network can only be influenced implicitly, by modifying the form and force diagrams. A further limitation of the tool is the fixed network topology during the search process. For topological changes, the designer has to start from scratch. Nevertheless, the interactive tool is a powerful and popular design tool, working also for large networks (Figure 6.13); it has been applied successfully in the design of several masonry shell prototypes built in full scale.

[39] ALGODeQ award 2014: http://algodeq.org/
[40] RhinoVAULT: http://www.block.arch.ethz.ch/brg/tools/rhinovault
[41] McNeel – Rhinoceros: http://www.rhino3d.com/
[42] As in graphic statics, corresponding branches of form and force diagrams have to be parallel

Furthermore, non-linear extensions to TNA have been developed, which enable the generation of closest-fit compression networks for given input geometries. These approaches have applications in the fields of structural design and modelling, as well as in the field of structural analysis of masonry structures (Block and Lachauer, 2011; Block and Lachauer 2014a; Block and Lachauer, 2014b).

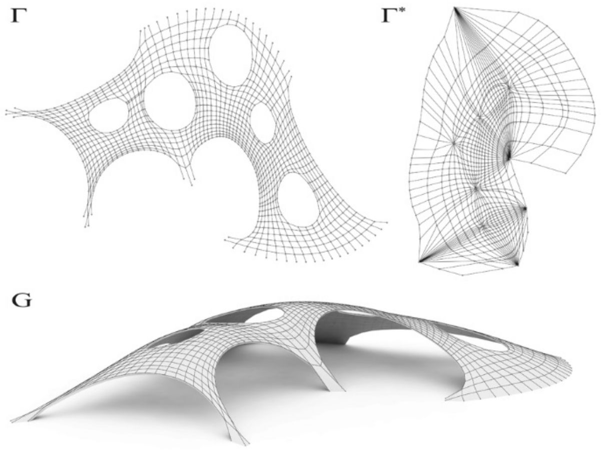

Figure 6.13: Form diagram Γ, force diagram Γ*, and thrust network G of an example with more than 2000 branches, generated with RhinoVAULT (Rippmann, Lachauer *et al.,* 2012: 226).

6.5 Conclusions

The research projects above deal with the development of interactive modelling methods for specific structural typologies. Each structural typology is defined through a well-defined structural concept, defining certain preconditions and constraints regarding their geometry, topology or inner force distribution. The trussed beams in the free-form roof project (section 6.1), for instance, are defined mainly through the assumption of a constant chord force, geometric preconditions (the structure lies within a vertical plane), and a specific topology (upper and lower chord connected with single members). Such typology-specific constraints on form and forces are regarded as equilibrium problems to be solved during the design process. Equilibrium problems are decomposed into sub-problems, and, based on the chosen decomposition, a tailored modelling strategy is developed. A modelling strategy consists of one or more form-finding techniques, which are able to solve the corresponding sub-problems. In the curved bridge project, for instance, in the first step, form finding of the arch or cable geometry is conducted, and, in a second step, the funicular system within the bridge deck is modelled (section 6.2).

As common contribution of all research projects above, new tailored methodologies for the design and modelling of specific equilibrium structures are presented. These methodologies are implemented based on contemporary computational approaches such as parametric modelling and scripting. All methodologies enable a new *geometrical flexibility* in modelling and, in the cases of the branching structures and the compression vaults, also *typological freedom* (Sections 6.3 and 6.4).

The main limitation of the four research projects is rooted in the lack of a common methodology. The presented approaches are using a mix of different form-finding techniques, including digital implementations of graphical methods, as well as customized versions of Dynamic Relaxation and the Force Density Method. Each modelling approach is based on a type-specific structural concept, and the constraints resulting from each concept are embedded in the modelling approach. Therefore, there is generally no possibility to alter these 'hard' constraints of the underlying structural concepts during the design process, the designer is still 'trapped' in the typology, even if this typology has a certain geometric or topologic flexibility.

This limitation of the modelling methodology restrict the freedom in the structural design process (as outlined by Hossdorf, Section 3.2.3). A truely 'artistic' design approach would require the possibility of freely changing initial design goals, preconditions, and constraints during the process (see also Simon's definition of the artistic process, Section 2.1).

7 General equilibrium modelling

In this Section, the problem of structural design modelling as stated in Section 4 is solved in a general manner, overcoming the limitations of type-specific modelling approaches. The approaches presented in Section 6 are interactive and geometrically flexible equilibrium modelling methods tailored to specific structural typologies. However, these modelling methods lack a common, unified technique, and provide only limited possibilities to topologically adapt the predefined structural types beyond mere geometric modification.

In this Section, a novel approach to equilibrium modelling based on a new computational form-finding method is presented. In Section 7.1, the modelling approach is conceptualised as an interactive process between human designer and computational form-finding. In Section 7.2, the technical core of the method, the new form-finding method, is explained in detail. And, finally in Section 7.3, an implemented prototype is briefly illustrated.

7.1 Modelling methodology

In this Section, the organisation of the general modelling process is presented in detail. Section 7.1.1 illustrates the decomposition of the process in phases of human reasoning, conducted by the designer, and phases of formal reasoning, conducted by the computer. Section 7.1.2 conceptualises the modelling process as iterative search, consisting of alternating steps of designer-driven model alterations and computational (re-)establishment of equilibrium.

7.1.1 Interactive modelling process

Following Schumacher and Kotnik's notion of digital architectural design as interaction between human reasoning and formal, computational processes (see Section 2.2.3), an analogous understanding of computer-aided structural design is developed (Figure 7.1). Inspired by Hossdorf's description of structural design as iterative search consisting of alternating phases of intuitive ideas and analytical thinking (see Section 3.1), the modelling process is decomposed such that the difficult and formalisable goal of finding geometrically constrained equilibrium solutions is conducted using computational processes, while all other goals of structural design, including, for instance, economic, ecological, aesthetic and social aspects are addressed by the designer through human, non-formal reasoning. Human reasoning is understood here as the entirety of different kinds of cognitive techniques, ranging from intuitive, maybe even poetic reasoning to rigorous analytical thinking. This might possibly include additional means beyond the computer-aided modelling environment, such as hand sketches or approximate calculations. This decomposition enables the designer to make use of his or her experience and intuition.

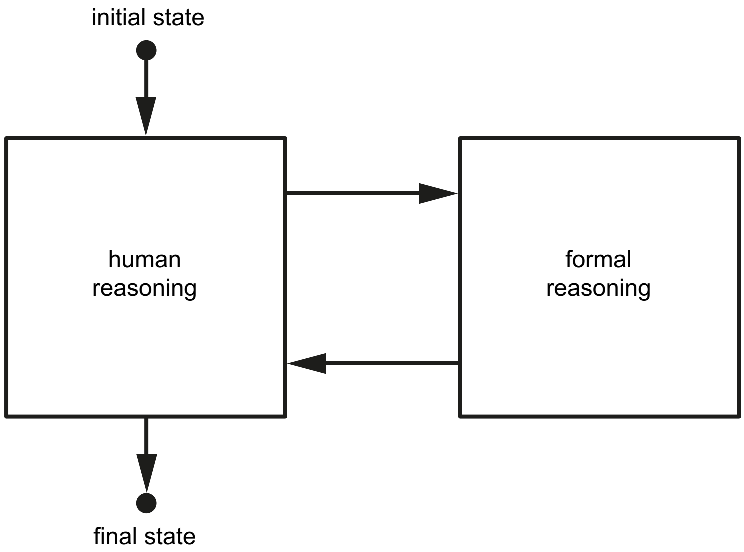

Figure 7.1: Computer-aided structural design as interaction between human reasoning and formal reasoning.

For the purpose of equilibrium modelling, formal processes are applied as 'directing' methods in the sense of Polónyi (see Section 3.1). These directing methods are defined here as form-finding processes, comparable to the self-forming process of a hanging model. Similarly to in the exemplary modelling approaches presented in Section 3.2.3, solely pin-jointed kinematic models are used. The designer first creates an initial kinematic model together with a set of external forces. Subsequently, the structure's geometry and inner forces are changed by the computational form-finding process towards an equilibrium state. Thereafter, the designer evaluates the form and inner force distribution of the equilibrium model. This generally includes the assessment of structural aspects (e.g. stiffening schemes, other loading cases, and construction) and the assessment of non-structural aspects (such as aesthetic, functional and economic design goals).

The model represents the geometry of the major structural axes and the equilibrium state for one dominant loading case. Thus, the designer has to bear in mind all further aspects of the design that are not represented explicitly in the model. If the model does not yet satisfy the design goals, it is changed and the modelling and form-finding process is repeated, until a satisfactory solution is obtained (Figure 7.2).

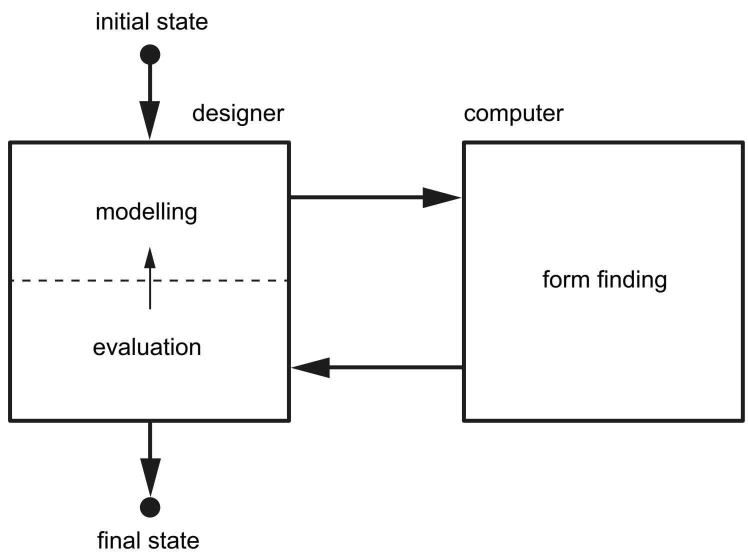

Figure 7.2: Computer-aided structural design as interaction between designer and computer: the designer conducts modelling and holistic design evaluation, the computer executes automatic form finding.

7.1.2 Search for equilibrium solutions

The modelling approach presented here is based on kinematic pin-jointed models, also referred to as *mechanisms*. The approach is defined as search within the *design world* in the sense of Mitchell (see Section 2.2.2). The design world $\boldsymbol{D}$ is constituted by the set of all possible *design states*. A single design state $\boldsymbol{a} \in \boldsymbol{D}$ consists of a kinematic pin-jointed model together with a set of external forces (Figure 7.3). The design world contains the subspace $\boldsymbol{D_e}$ of all design states in equilibrium.

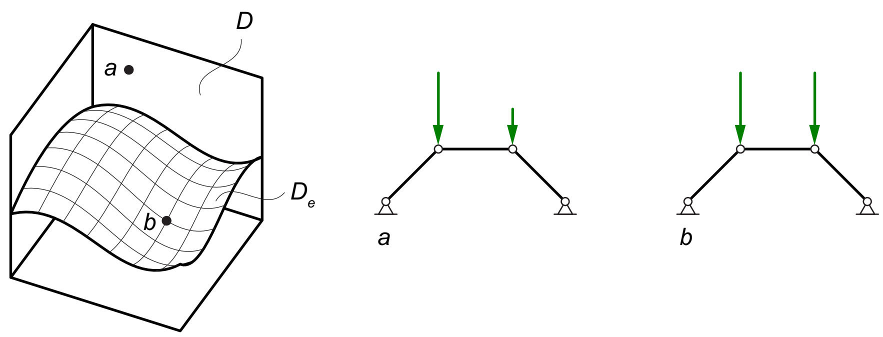

Figure 7.3: The design space $\boldsymbol{D}$ is schematically represented as a box, single design states are represented as points within the box, and the set $\boldsymbol{D_e}$ of all design states in equilibrium is represented by a surface. The design states $\boldsymbol{a}, \boldsymbol{b}$ are represented by structural diagrams; obviously $\boldsymbol{a} \notin \boldsymbol{D_e}$ is not in equilibrium, and $\boldsymbol{b} \in \boldsymbol{D_e}$ is in equilibrium.

The application and design operation by the designer generally maps one design state to another design state, by changing the model or the external forces or both. The form-finding process maps a design state to a design state in equilibrium, thus can be understood as 'projection' on to the equilibrium space. Figure 7.4 shows a schematic illustration of an iterative modelling process passing through six design states; the application of design operations is represented by dashed arrows, the execution of computational form finding is represented by continuous arrows. Starting with an initial design state $\notin \boldsymbol{D}_e$, the application of the form-finding method results in a design state $\boldsymbol{b} \in \boldsymbol{D}_e$. In the next step, a design operation is applied which maps $\boldsymbol{b}$ to a state $\boldsymbol{c}$. Design operations allow the model to be freely changed (by adding or deleting struts, by changing the geometry, constraints or support conditions), or the loading to be freely changed (by adding forces, removing forces or by changing magnitude and direction of existing forces). Since all models are kinematic, the application of design operations generally results in design states that are not in equilibrium, so one can assume $\boldsymbol{c} \notin \boldsymbol{D}_e$. This process, based on the alternating application of design operations and form-finding processes, is repeated until a final design state $\boldsymbol{f} \in \boldsymbol{D}_e$ is found.

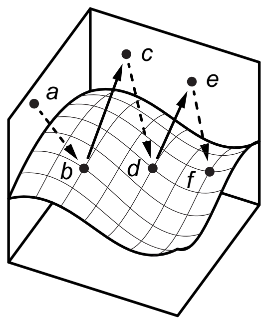

Figure 7.4: The iterative modelling process as search in the design space consisting of alternating application of the form-finding process (dashed arrows) and design operations (continuous arrows). The design states $\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{e}$ are not in equilibrium, the design states $\boldsymbol{b}, \boldsymbol{d}, \boldsymbol{f}$ are in equilibrium.

In order to enable the designer to deliberately control each execution of the form-finding process, the definition of additional constraints on geometry and inner forces is possible. The example presented in Figure 7.5 illustrates the effect of constraints in an exemplary manner: the unconstrained form-finding process generally maps a design state $\boldsymbol{a} \notin \boldsymbol{D}_e$ to a design state $\boldsymbol{b} \in \boldsymbol{D}_e$ close to the initial geometry. By defining additional geometric constraints, the form-finding processes result in the alternative solutions $\boldsymbol{c}, \boldsymbol{d} \in \boldsymbol{D}_e$. Within the schematic representation of the design space $\boldsymbol{D}$, all these three form-finding processes are interpreted as 'projections' of $\boldsymbol{a}$ to the equilibrium space $\boldsymbol{D}_e$ with different 'projection directions' defined by the constraints. Obviously it is possible to formulate unfeasible constraints, for instance by fixing the position of all free nodes of $\boldsymbol{a}$. This possibility to formulate unfeasible constraints is not a flaw in the approach, but rather a freedom for the designer to explore the very boundaries of kinematic equilibrium solutions.

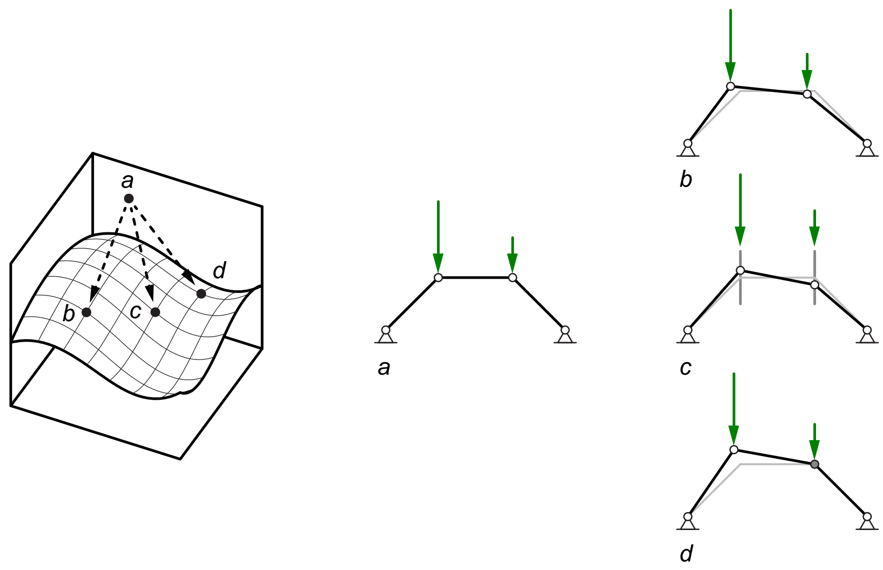

Figure 7.5: The form-finding process 'projects' a design state $\boldsymbol{a}$ which is not in equilibrium on to the equilibrium space $\boldsymbol{D_e}$. The formulation of additional constraints allows the direction of the 'projection' to be deliberately controlled, resulting in the different equilibrium solutions $\boldsymbol{b}$ (both nodes unconstrained), $\boldsymbol{c}$ (both nodes constrained to vertical axes), and $\boldsymbol{d}$ (left node unconstrained, right node constrained to fixed position).

7.2 Computer-aided form finding[43]

In this Section, the technical core of the dissertation – a new form-finding method – is presented. As described in Section 7.1.2, the novel modelling approach for pin-jointed structures is organised as a step-wise search through the design space consisting of design operations and form-finding steps. In Section 7.2.3, the new method for constrained form finding is presented, based on concepts of the Force Density Method. In Section 7.2.1, the problem of constrained form finding is stated in detail, and in Section 7.2.2, the Force Density Method is briefly summarised as background knowledge.

7.2.1 Problem of constrained form finding

Within the interactive search for equilibrium solutions, the problem of constrained form finding has to be solved repetitively (Section 7.1.2). The human designer changes the structural model, then the form-finding process 're-establishes' equilibrium after each design operation, with respect to given constraints on geometry and inner forces. The problem of constrained form finding constitutes the main technical challenge of equilibrium modelling. A suitable form-finding process has to be fast and stable, in order to enable a truly interactive modelling experience. Furthermore, the method should enable the designer to define the model and the constraints in a direct and visually intuitive way.

[43] This method has previously been published, with only minor changes, in the International Journal of Space Structures (Lachauer and Block, 2014)

The limitations of existing iterative modelling approaches are directly determined by the underlying form-finding methods. Experimental physical models enable the design of spatial compression dominated or tension dominated structures; the creation of models with mixed tension and compression structures is limited due to arising physical instabilities. Furthermore, the building of physical models is expensive and time-consuming, and sophisticated technical equipment is required for the precise measurement of the model geometry (see Section 3.3.1). Methods derived from graphic statics are powerful for design and modelling of planar structures, but the underlying concepts have only partially been generalised for three-dimensional cases (see Section 3.3.2). Efficient computational form- finding methods have been developed for specific structural typologies, e.g. for compression vaults, thin shells or tension membranes. The Statically-Geometrically Coupled Method (SGC) enables deliberately controlled form finding of general pin-jointed structures with mixed compression and tension forces (see Section 3.3.3). A detailed technical description of the SGC method is not disclosed, but it is stated that a mathematically unique description of the form-finding problem is required. This requirement undermines the goal of setting up a truly intuitive modelling approach based on visual feedback and direct interaction with the system.

None of the above-mentioned existing approaches offer a general and visually intuitive method of conducting form finding of mixed compression and tension structures in a flexible, yet highly controlled way.

7.2.2 Force Density Method

The Force Density Method was developed in the early 1970s (Linkwitz and Schek, 1971). In its basic form, the Force Density Method allows for form finding of equilibrium networks with defined topology for a given set of loads and support points (Schek, 1974). The key concept of the method is to replace the forces in the branches of the network by force-length ratios, called *force densities*, in order to obtain a linear system of equations for the unknown coordinates of the free nodes. For a detailed deduction of the method, the reader is referred to Schek (1974). Here, the concepts of the method are illustrated with one exemplary network.

For a given network with m branches and n_S nodes, the connectivity of the branches is described by a *branch-node matrix*[44] $\boldsymbol{C_S}$ with dimensions $m \times n_S$; if branch k connects nodes i and j, then the element (k, i) of $\boldsymbol{C_S}$ equals 1, and element (k, j) -1; all other elements are 0. The columns of the branch-node matrix are ordered such that one can define two sub-matrices $\boldsymbol{C_S} = [\boldsymbol{C}\ \boldsymbol{C_f}]$, $\boldsymbol{C}$ corresponding to the free nodes, and $\boldsymbol{C_f}$ corresponding to the support nodes. In Figure 7.6, an exemplary network with five branches and six nodes is shown, the nodes with the numbers 1 and 2 are considered as free nodes.

[44] The *branch-node matrix* is the transposed version of the *incidence matrix* used in graph theory.

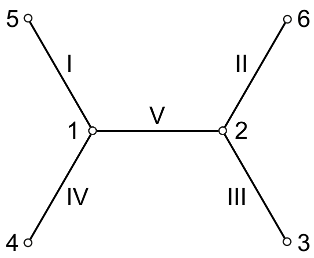

Figure 7.6: One example network with five branches and six nodes, the nodes 1 and 2 are free, the nodes 3, 4, 5, and 6 are support nodes.

The corresponding branch-node matrix $\boldsymbol{C_s}$ has the dimension of 5×6 and is composed as follows:

$$\boldsymbol{C_s} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \\ \mathrm{V} \end{array}$$

Each free node is interpreted as point i with the coordinates (x_i, y_i, z_i); the coordinates of the fixed support points are (x_{fi}, y_{fi}, z_{fi}). These coordinates constitute the free coordinate vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of length n and the support coordinate vectors $\boldsymbol{x_f}$, $\boldsymbol{y_f}$, $\boldsymbol{z_f}$ of length n_f, hence $n_s = n + n_f$. For the network shown in Figure 6.6, $n_s = n + n_f = 2 + 4 = 6$. The force densities $\boldsymbol{q}$ are defined as the force-length ratios: $q_i = \frac{f_i}{l_i}$, with f_i being the internal force, and l_i being the length of branch i. The force densities $\boldsymbol{q}$ form a vector of length m.

For a given branch-node matrix $\boldsymbol{C_s}$, a set of force densities $\boldsymbol{q}$, nodal loads $\boldsymbol{p_x}$, $\boldsymbol{p_y}$, $\boldsymbol{p_z}$, and the coordinates $\boldsymbol{x_f}$, $\boldsymbol{y_f}$, $\boldsymbol{z_f}$ of the supports, the coordinates $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of the free nodes are uniquely defined, and are calculated by solving one system of linear equations. In Figure 7.7 *(A),* one network together with one set of nodal loads is illustrated. Figures 7.7 *(B)–(D)* show three different equilibrium states resulting from three different sets of force densities.

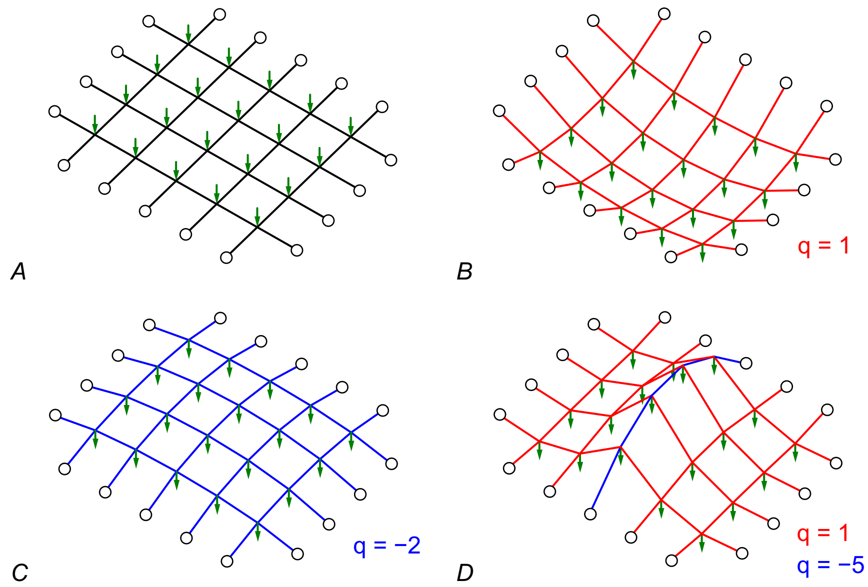

Figure 7.7: The network connectivity, support nodes, and a given set of nodal loads (*A*), the resulting equilibrium solution as tension structure based on uniform force densities with the value of 1 (*B*), the resulting equilibrium solution as compression structure based on uniform force densities with the value of -2 (*C*), and the resulting equilibrium solution as mixed compression and tension structure based on force densities with the value of 1 and -5 (*D*).

In this basic form of the Force Density Method, there is no way to define an 'initial' or 'starting' geometry of the structure. Only the coordinates of the supports can be determined, together with the network connectivity and the set of force densities and nodal loads. In simpler cases, such as e.g. shown in Figure 7.7, this is sufficient to describe an intended structure with some experience. But already if the structure contains edge arches or cables, the resulting equilibrium geometry becomes almost unpredictable, especially for structures with mixed compression and tension forces.

Several ways of extending the Force Density Method using iterative procedures in order to incorporate additional constraints, such as e.g. defined branch lengths or force values have been described (Linkwitz, Schek *et al.*, 1974). Two conceptually different iterative procedures for the optimisation of given network geometries are distinguished: the "Newtonverfahren" ("Newton Approach") and the "Ausgleichungsansatz" ("Variational Approach").

In the Newton Approach, the parameter space is constituted by the coordinates of the free nodes; the Force Density Method is used to compute the network geometry at iteration $t+1$ based on its previous state at iteration t:

$$(\Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{z}) = g(\boldsymbol{x}^t, \boldsymbol{y}^t, \boldsymbol{z}^t) \tag{7.1}$$

$$(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1}, \boldsymbol{z}^{t+1}) = (\boldsymbol{x}^t + \Delta\boldsymbol{x}, \boldsymbol{y}^t + \Delta\boldsymbol{y}, \boldsymbol{z}^t + \Delta\boldsymbol{z}) \tag{7.2}$$

In the Variational Approach, the parameter space is constituted by the force densities *and* the free node coordinates; both force densities and network geometry at iteration $t+1$ are calculated, based on their previous states at iteration t:

$$(\Delta\boldsymbol{q}, \Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{z}) = g(\boldsymbol{q}^t, \boldsymbol{x}^t, \boldsymbol{y}^t, \boldsymbol{z}^t) \tag{7.3}$$

$$(\boldsymbol{q}^{t+1}, \boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1}, \boldsymbol{z}^{t+1}) = (\boldsymbol{q}^t + \Delta\boldsymbol{q}, \boldsymbol{x}^t + \Delta\boldsymbol{x}, \boldsymbol{y}^t + \Delta\boldsymbol{y}, \boldsymbol{z}^t + \Delta\boldsymbol{z}) \tag{7.4}$$

Both methods work well for the optimisation of network geometries that are close to an equilibrium state purely in compression or tension. In the next Section, the latter is adopted for design purposes, allowing equilibrium networks close to a given input geometry to be found that consist of a combination of compression and tension branches, while imposing additional constraints on form and forces.

7.2.3 Computational form-finding method

Inspired by the concepts of graphic statics, user-defined constraints can be imposed both on force densities (hence implicitly on the forces), on the supports, and on the free nodes' coordinates. Force-density constraints are defined as lower and upper bounds $\boldsymbol{q}^{LB}$ and $\boldsymbol{q}^{UB}$. For a set of constrained force densities $\widehat{\boldsymbol{q}}$, the condition $\boldsymbol{q}^{LB} \leq \widehat{\boldsymbol{q}} \leq \boldsymbol{q}^{UB}$ is true. Support modes can either be fixed, movable along a line, or movable along a plane (movable support nodes are represented by free nodes with specific properties, see Section 7.2.3.3). Each node that is not defined as support node, can either be geometrically free, constrained to a line or a plane, or fixed in space. Initially, all geometric constraints have to be satisfied, meaning for instance that a free node constraint to a plane has to lie initially on this plane, the same rule applies for movable supports.

7.2.3.1 Overview

In each modelling step, the designer changes the current design state by modifications of geometry and/or topology of the model, or by altering the external forces; furthermore, the designer is able to impose the constraints on force densities, free nodes' coordinates, and supports (Figure 7.8). After this, the structure is generally no longer in equilibrium, or, alternatively, the imposed force constraints are not satisfied. The computational form-finding method iteratively minimises residual forces through redistribution of inner forces and through changes of the geometry, with respect to the constraints defined by the designer. In general, the form-finding method converges to a design state in equilibrium if the constraints are feasible. The set of resulting force densities allows for the calculation and visualisation of the internal forces of the design state. This alternating process of modelling and form finding is repeated until the designer decides that the equilibrium solution satisfies the overall design goals.

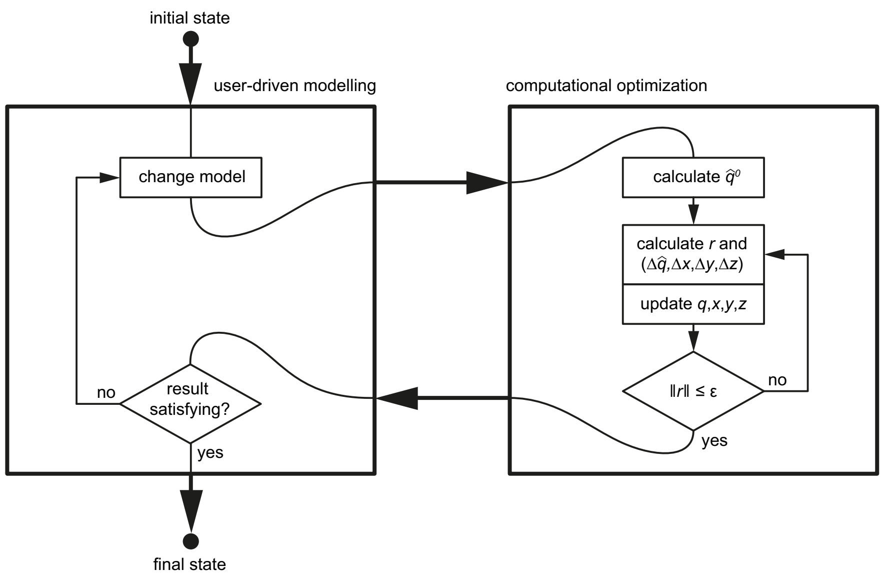

Figure 7.8: Flow chart of the iterative process consisting of alternating designer-driven modelling steps and computational form-finding steps.

For a given threshold value ε, the computational form-finding process works as follows:

I) Calculate a set of bounded force densities $\hat{\boldsymbol{q}}^0$ from the initial network geometry (Section 7.2.3.2);

II) Calculate the nodal residuals $\boldsymbol{r}$ and the gradient $(\Delta\hat{\boldsymbol{q}}, \Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{z})$ with respect to the constraints (Sections 7.2.3.3 and 7.2.3.4);

III) Update the force densities $\boldsymbol{q}$ and network geometry $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$; and

IV) If $\|\boldsymbol{r}\| > \varepsilon$ then return to step II).

A necessary condition is that the model forms a mechanism, which is the case if the following inequality is true (Timoshenko and Young, 1945: 188-194):

$$m < 3n. \tag{7.5}$$

For solving the iterative process, an explicit Runge-Kutta method with adaptive step size has been implemented (Kiusalaas, 2005: 275–282).

7.2.3.2 Initial step

First, the coordinate differences $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ per branch are calculated:

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{Cx} + \boldsymbol{C_f x_f} \\ \boldsymbol{v} &= \boldsymbol{Cy} + \boldsymbol{C_f y_f} \\ \boldsymbol{w} &= \boldsymbol{Cz} + \boldsymbol{C_f z_f} \end{aligned} \tag{7.6}$$

Based on the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, their diagonal matrices $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$ are constructed. This allows the equilibrium matrix $\boldsymbol{A}$ with dimensions $3n \times m$ to be defined:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{C}^T\boldsymbol{U} \\ \boldsymbol{C}^T\,\boldsymbol{V} \\ \boldsymbol{C}^T\boldsymbol{W} \end{bmatrix}. \tag{7.7}$$

Together with the vertically stacked load vector

$$\boldsymbol{p} = \begin{bmatrix} \boldsymbol{p}_x \\ \boldsymbol{p}_y \\ \boldsymbol{p}_z \end{bmatrix}, \tag{7.8}$$

the vector of nodal residuals $\boldsymbol{r}$ for a given set of *m* force densities $\boldsymbol{q}$ is then formulated as

$$\boldsymbol{r} = \boldsymbol{Aq} - \boldsymbol{p}. \tag{7.9}$$

The system of equations (7.9) has m unknowns and $3n$ equations, and because of (7.5), it is thus overdetermined. The goal is to find a set of initial force densities $\boldsymbol{q}^{\mathbf{0}}$ that minimises the initial nodal residuals $\boldsymbol{r}^{\mathbf{0}}$, which can be formulated as a linear least square problem:

$$\boldsymbol{q}^0 = \boldsymbol{min}\,\|\boldsymbol{r}^0\|^{\mathbf{2}} = \boldsymbol{min}\,\|\boldsymbol{Aq}^0 - \boldsymbol{p}\|^{\mathbf{2}}, \tag{7.10}$$

for which the solution can be written analytically using normal equations:

$$\boldsymbol{q}^0 = (\boldsymbol{A}^T\boldsymbol{A})^{-\mathbf{1}}\boldsymbol{A}^T\boldsymbol{p}. \tag{7.11}$$

7.2.3.3 Constraints

Inspired by an approach developed for geometric modelling, constraints are implemented using *projections* (Bouaziz, Deuss *et al.,* 2012).

Force-density constraints

In the first step of the iterative method, the constraints on the force densities, given as lower and upper boundaries $\boldsymbol{q}^{LB}$ and $\boldsymbol{q}^{UB}$, are imposed on $\boldsymbol{q}^{\mathbf{0}}$ by projecting them on the boundaries of the constrained set, if at least one constraint is violated (Figure 7.9).

Analytically, the constrained set of initial force densities $\widehat{\boldsymbol{q}}^{\mathbf{0}}$ is defined as:

$$\hat{q}_i^0 = \begin{cases} q_i^{LB}, & \text{if } q_i < q_i^{LB} \\ q_i\ \ , & \text{if } q_i^{LB} \le q_i \le q_i^{UB} \\ q_i^{UB}, & \text{if } q_i^{UB} < q_i \end{cases} \tag{7.12}$$

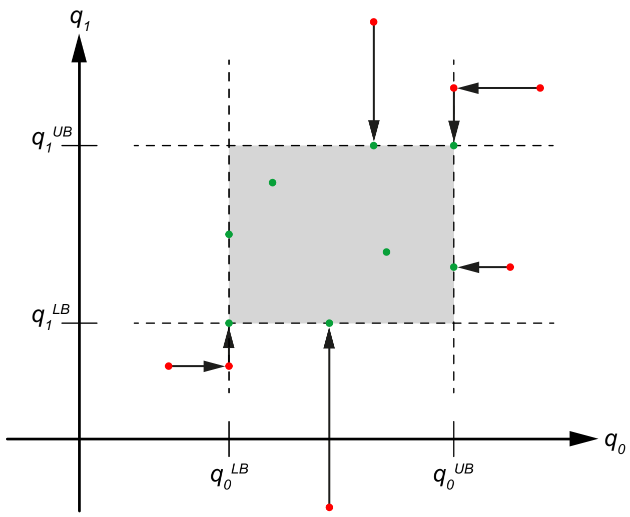

Figure 7.9: The diagram shows the imposing of the lower and upper bounds $\boldsymbol{q}^{LB}$ and $\boldsymbol{q}^{UB}$ on the force densities $\boldsymbol{q}$ for a two-dimensional case. The red dots represent states of $\boldsymbol{q}$ that are not within the bounds, and their projection bounded states $\widehat{\boldsymbol{q}}^0$ are represented as green dots.

Node constraints

Each node that is not a support node, can either be free, constrained to a line, constrained to a plane, or fixed in space. Initially, all geometric constraints have to be satisfied by the model, i.e. a node that is constrained to a plane has to lie on this plane before the form-finding process starts.

In each iteration step, the residual vector $\boldsymbol{r}$ is computed from $\boldsymbol{q}$, $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, i.e. Equations (7.6–7.9), and $\boldsymbol{r}$ is decomposed in 'unconstrained' components $\boldsymbol{r}^U$ and 'constrained' components $\boldsymbol{r}^C$, such that $\boldsymbol{r} = \boldsymbol{r}^U + \boldsymbol{r}^C$. The idea is to resolve the 'free' components $\boldsymbol{r}^U$ by changing the geometry according to $\Delta\boldsymbol{x}$, $\Delta\boldsymbol{y}$, $\Delta\boldsymbol{z}$, and to resolve the 'constrained' components $\boldsymbol{r}^C$ by altering the force densities according to $\Delta\boldsymbol{q}$. In this section, the decomposition $\boldsymbol{r} = \boldsymbol{r}^U + \boldsymbol{r}^C$ is explained; in Section 7.2.3.4, the computation of the gradient $(\Delta\widehat{\boldsymbol{q}}, \Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{z})$ from $\boldsymbol{r}^U$ and $\boldsymbol{r}^C$ is shown.

For a given node $\boldsymbol{i}$, depending on its degree of constraint, the corresponding residual $\boldsymbol{r}_i$ is decomposed as follows:

Point $\boldsymbol{i}$ is a

a) free node: a global weighting factor $0 > \alpha > 1$ is introduced to balance change of geometry and change of inner forces, so $\boldsymbol{r}_i^U = \alpha\boldsymbol{r}_i$, hence $\boldsymbol{r}_i^C = (1-\alpha)\boldsymbol{r}_i$ (Figure 7.10 *A*).

b) node constrained to a line Γ: $\boldsymbol{r}_i^U$ is the orthogonal projection of the residual $\boldsymbol{r}_i$ on to the line. Therefore, the projection matrix $\boldsymbol{P}^\Gamma$ is formed based on the unit vector $\boldsymbol{d}$ of the line's direction: $\boldsymbol{P}^\Gamma = \boldsymbol{d}\boldsymbol{d}^T$. Then, $\boldsymbol{r}_i^U = \boldsymbol{P}^\Gamma\boldsymbol{r}_i$, and $\boldsymbol{r}_i^C = \boldsymbol{r}_i - \boldsymbol{P}^\Gamma\boldsymbol{r}_i$ (Figure 7.10 *B*).

c) node constrained to a plane Ψ: $\boldsymbol{r}_i^U$ is obtained by an orthogonal projection of the residual $\boldsymbol{r}_i$ on to the plane. Therefore, the projection matrix $\boldsymbol{P}^\Psi$ is formed based on an orthonormal basis $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ (two orthogonal unit vectors lying within the plane Ψ): $\boldsymbol{P}^{\Psi} = [\boldsymbol{d}_1 \boldsymbol{d}_2][\boldsymbol{d}_1 \boldsymbol{d}_2]^{\mathrm{T}}$. Then, $\boldsymbol{r}_i^U = \boldsymbol{P}^{\Psi}\boldsymbol{r}_i$, and $\boldsymbol{r}_i^C = \boldsymbol{r}_i - \boldsymbol{P}^{\Psi}\boldsymbol{r}_i$ (Figure 7.10 *C*).

d) fixed node: the node should not move at all during the form-finding process, so the 'free' component vanishes, $\boldsymbol{r}_i^U = 0$, hence $\boldsymbol{r}_i^C = \boldsymbol{r}_i$ (Figure 7.10 *D*).

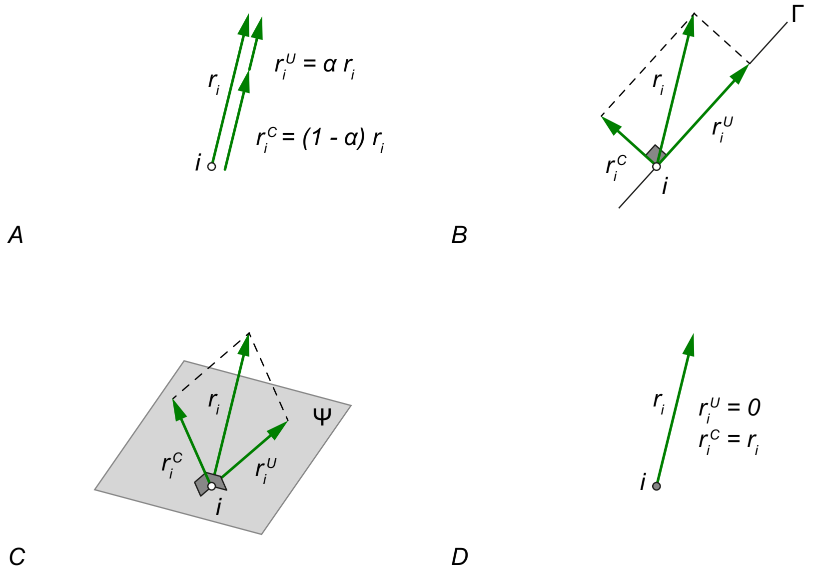

Figure 7.10: Decomposition of residual $\boldsymbol{r}_i$ for nodes. Node i is a free node (*A*), a node constrained to a line Γ (*B*), a node constraint to a plane Ψ (*C*), or a fixed node (*D*).

Support constraints

Each node that is a support node can either be fixed, constrained to a line, or constrained to a plane. Supports which are not fixed are moveable along the constraining line or plane, and can only resist forces that are perpendicular to the constraining geometry. Non-fixed supports are computed as free nodes with a specific residual decomposition (Figure 7.11). For a given movable support node i, the residual $\boldsymbol{r}_i$ is projected on to the constraining geometry, and decomposed as $\boldsymbol{r}_i = \boldsymbol{r}_i^* + \boldsymbol{r}_i^S$. The components $\boldsymbol{r}_i^S$, perpendicular to the constraining line or plane, are neglected in the calculation of $\|\boldsymbol{r}\|$ (see Section 7.2.3.1). This is valid, since by definition, the supports can resist these force components, perpendicular to the constraining geometry.

The 'unconstrained' and 'constrained' residual components are calculated as:
$\boldsymbol{r}_i^U = \alpha \boldsymbol{r}_i^*$ and
$\boldsymbol{r}_i^C = (1 - \alpha)\boldsymbol{r}_i^*$.

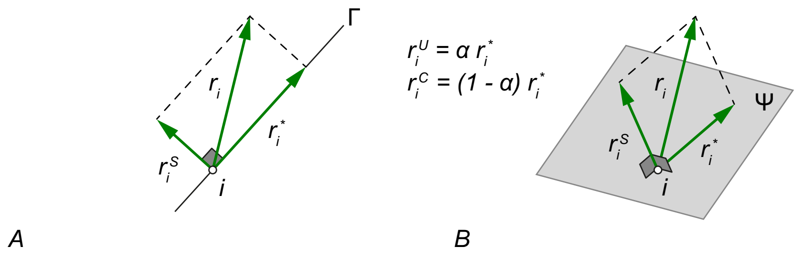

Figure 7.11: Decomposition of residual $\boldsymbol{r}_i$ for constrained supports. Node i is a support constrained to a line Γ (*A*) or a support constraint to a plane Ψ (*B*).

7.2.3.4 Gradient

Based on the decomposition $\boldsymbol{r} = \boldsymbol{r}^U + \boldsymbol{r}^C$ described in the previous Section, the computation of the constraint gradient $(\Delta\hat{\boldsymbol{q}}, \Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{z})$ is straightforward. The free components $\boldsymbol{r}^U$ are used to compute 'allowed' changes of the nodal positions within the geometric constraints using a stiffness $\boldsymbol{k}$:

$$\begin{aligned} \Delta\boldsymbol{x} &= \boldsymbol{k}\, \boldsymbol{r}_x^U \\ \Delta\boldsymbol{y} &= \boldsymbol{k}\, \boldsymbol{r}_y^U . \\ \Delta\boldsymbol{z} &= \boldsymbol{k}\, \boldsymbol{r}_z^U \end{aligned} \tag{7.13}$$

A variety of different definitions of stiffness matrices and scalars have been used in form finding (Veenendaal and Block, 2012); since this method does not count on any material properties, only 'geometric' stiffness is taken into account. In order to reduce the computational cost, the stiffness $\boldsymbol{k}$ is implemented as a lumped geometric stiffness vector, similar to that proposed by Barnes (Barnes, 1999). Here, the following definition for $\boldsymbol{k}$ as unitised, lumped geometric stiffness is introduced:

$$\boldsymbol{k} = \frac{|\boldsymbol{C}^T|\boldsymbol{L}\, \boldsymbol{q}}{|\boldsymbol{C}^T|\boldsymbol{L}\, |\boldsymbol{q}|}, \tag{7.14}$$

with element k_i having a value of 1 if the sum of forces in the adjacent branches of node i is greater than 0 ("tension-dominant" node), and -1 if the sum of forces in the adjacent branches of node i is smaller than 0 ("compression-dominant" node). Figure 7.12 illustrates the idea behind this definition for an unloaded node: for tension-dominant nodes, the residual points towards a close equilibrium position; for compression-dominant nodes, the residual points away from a close equilibrium position, so in this case the direction of the residual is flipped by the multiplication with -1.

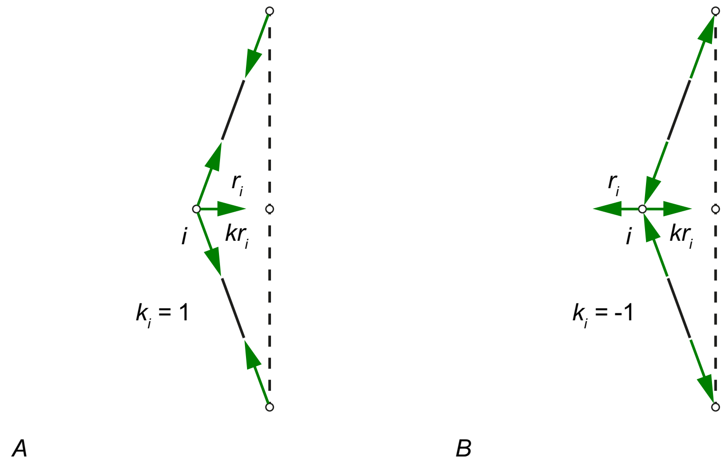

Figure 7.12: If node i is (*A*) "tension-dominant" or (*B*) "compression-dominant", residual $\boldsymbol{r}_i$ points towards, or away from, a close equilibrium state. In the latter case, the residual is flipped by the multiplication with -1.

The matrix $\boldsymbol{L}$ is constructed from the branch lengths along its diagonal, and is calculated as

$$\boldsymbol{L} = \sqrt{\boldsymbol{U}^T\boldsymbol{U}+\boldsymbol{V}^T\boldsymbol{V}+\boldsymbol{W}^T\boldsymbol{W}}. \tag{7.15}$$

The remaining 'constrained' components $\boldsymbol{r}^C$ of the residual vector $\boldsymbol{r}$ are resolved with $\Delta\boldsymbol{q}$, for which the following system of equations is formulated:

$$\boldsymbol{r}^C = \boldsymbol{A}\Delta\boldsymbol{q}. \tag{7.16}$$

Since $\boldsymbol{A}$ has dimensions $3n \times m$, and assumption (7.5) is true, the system of equations (7.16) is overdetermined, so $\Delta\boldsymbol{q}$ is computed as linear least-square approximation:

$$\Delta\boldsymbol{q} = \boldsymbol{min}\, \|\boldsymbol{A}\Delta\boldsymbol{q} - \boldsymbol{r}^C\|^2, \tag{7.17}$$

whose solution can be written analytically, using normal equations:

$$\Delta\boldsymbol{q} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{r}^C. \tag{7.18}$$

In order to impose the force-density bounds $\boldsymbol{q}^{LB}$ and $\boldsymbol{q}^{UB}$ during the iterative process, the bounded force-density differences $\Delta\widehat{\boldsymbol{q}}$ are calculated as follows:

$$\Delta\hat{q}_i = \begin{cases} q_i^{LB} - q_i, & \text{if } q_i < q_i^{LB} \\ \Delta q_i \quad\quad\;\, , & \text{if } q_i^{LB} \leq q_i \leq q_i^{UB} \\ q_i^{UB} - q_i, & \text{if } q_i^{UB} < q_i \end{cases} \tag{7.19}$$

7.3 Implemented prototype

In order to demonstrate the power of the new modelling approach, a prototype of the form-finding method has been implemented. The practical application of the modelling process using this prototype is shown in nine case studies (Section 8.3 and 8.4).

The prototype has been implemented within the commercial CAD software *Rhinoceros*[45], using the built-in scripting language. Rhinoceros is a tool specifically developed for 3D freeform design; all functions required by the designer to conduct design operation in a convenient and intuitive way are already implemented as standard CAD drawing operations. A pin-jointed model is represented by a set of lines in the three-dimensional drawing space; supports are represented by points. External forces are represented by directed green lines with arrowheads; the magnitude of a force is given by the line length (Figure 7.13). The connectivity of the structure is automatically detected by an algorithm which compares the coordinates of the line's end points. Identical coordinates of two different end points are considered as pin-jointed connection. The assignment of external forces to the nodes is done in a similar way.

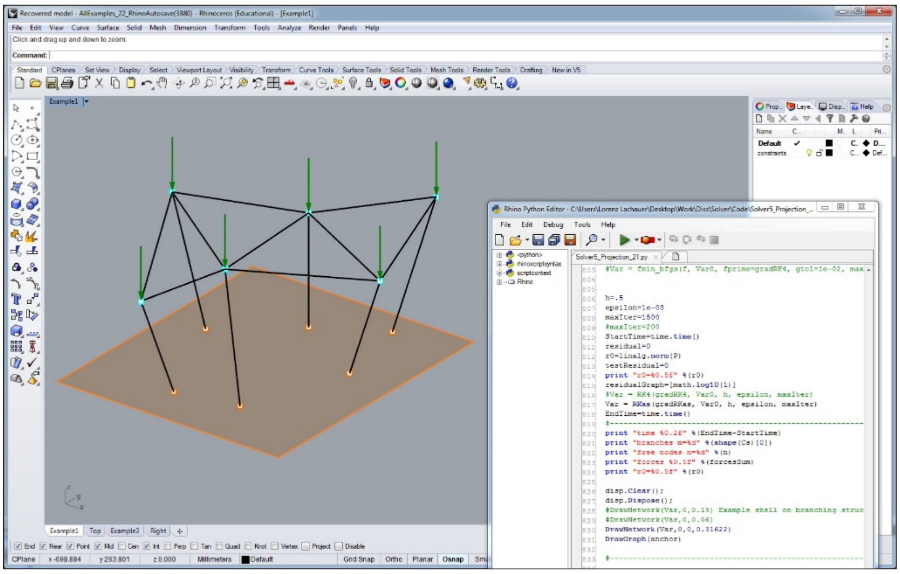

Figure 7.13: Screenshot of the prototypical modelling setup implemented within the commercial CAD software Rhinoceros, version 5; the built-in scripting editor is shown in the small window in front. The CAD modelling space shows the model of the example presented in Section 7.3.1: the branches are modelled as black lines, forces as green lines with arrowheads, the support nodes are modelled as orange points (constrained to the orange plane); the free nodes are modelled as cyan points (constrained to their position).

Furthermore, the designer is able to assign lower and upper force-density bounds $\boldsymbol{q}^{LB}$ and $\boldsymbol{q}^{UB}$ to the struts of the model by giving them specific colours; those colours relate to an editable table (in this prototype, as part of the Python source code) with the minimum and maximum allowed force densities.

45 McNeel – Rhinoceros: http://www.rhino3d.com/

Geometric constraints are modelled with points in specific colours; these colours correspond to the colours of constraining objects, e.g. lines or planes. While the points and lines representing the model are placed on a default layer, points, lines and planes representing geometric constraints are placed on a separate layer.

The form-finding algorithm has been implemented with the built-in *Python*[46] scripting language; the *NumPy*[47] open-source library has been used for the linear algebraic computations. Linear least-square problems, as Eq. (7.10) and (7.17), have been solved with the function *numpy.linalg.lstsq*. The orthonormal basis of planes representing node constraints is obtained by the built-in scripting command: *rhinoscriptsyntax.SurfaceFrame*. Furthermore, additional auxiliary functions have been implemented, for the generation of the matrix data structure from the CAD objects, and for the visualisation of resulting equilibrium solutions with internal forces.

[46] Python scripting for Rhinoceros: http://wiki.mcneel.com/developer/python

[47] NumPy – scientific computing with Python: http://www.numpy.org/

8 Case studies

In this Section, applications of the general equilibrium modelling methodology will be presented as cases studies.

8.1 Goals

The case studies are divided into two groups. The goal of the first group of examples is to demonstrate the power of the method for creating formal design variations of existing structural typologies (Section 8.3). The goal of the second group is to demonstrate the exploration of new equilibrium structures through the recombination of existing types (Section 8.4). Each case study is divided into three steps. All steps are presented both as initial input model, and as resulting equilibrium structure. For all examples, the threshold value is defined as $\varepsilon = \|\boldsymbol{r}^0\|/1000$. The global weighting factor α is set to 0.5, changes of this value did not affect solving time significantly. In the Appendix (Section 10), solving times, iteration numbers and convergence graphs are illustrated for each case study.

8.2 Graphical legend

For the illustration of the cases studies, a specific graphical convention is used for the representation of the structural model with its supports, external forces and constraints (Table 8.1). All case studies are organised in two columns: the left column represents the input models (e.g. Figure 8.3 *A*, *C*, *E*), and the right column represents the models in equilibrium, resulting from the form-finding process (e.g. Figure 8.3 *B*, *D*, *F*).

Input model with constraints and external loads		Resulting constrained model in equilibrium, with support forces	
	branch without force density bounds		compession force, line weight proportional to the square root of inner force
q ϵ [0;1]	branch with force density bounds		tension force, line weight proportional to the square root of inner force
			geometry of the input model
	external force, magnitude equals arrow length		external force, magnitude equals arrow length
		$\varphi = 0.1$	external force, magnitude equals arrow length multiplied by φ
	fixed support node		fixed support node
	movable support node constrained to a plane		movable support node
	unsupported node with fixed position		
	unsupported movable node constrained to an axis		
	unsupported movable node constrained to a plane		

Table 8.1 Graphic nomenclature used in the illustration of the case studies

8.3 Variation of structural models

In this Section, six case studies are presented, which focus on the creation of new structural models through the geometrical and topological variation of simpler, mostly well-known typologies.

8.3.1 Inclined columns

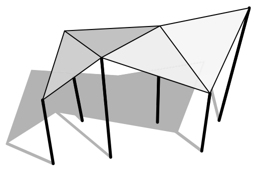

Figure 8.2: A folded roof supported by inclined columns.

This example demonstrates the step-wise modelling of a folded roof supported by inclined columns (Figure 8.2). Initially, an intended roof geometry is given. The goal is to find a constellation of inclined columns balancing the roof's self-weight. A model with an arbitrary constellation of inclined columns is generally not in equilibrium, since both a translational force and a rotational moment is induced on the roof (Lachauer and Block, 2012).

The edges of the triangular faces are considered as structural members, and the roof's self-weight is approximated by vertical forces of equal magnitude (Figure 8.3 *A*). The roof is modelled with geometrically fixed vertices to deliberately control the shape. The supports of the inclined columns are movable along the ground plane, to provide geometrical freedom for form finding. This setting leads to a structure with vertical columns, since the movable supports are not able to balance horizontal forces (Figure 8.3 *B*). By adding struts between the movable supports (Figure 8.3 *C*), an equilibrium state with non-vertical columns becomes possible (Figure 8.3 *D*).

Thereafter, the shape of the roof is formally refined by altering the vertex heights (Figure 8.3 *E*), and a final equilibrium state is found (Figure 8.3 *F*).

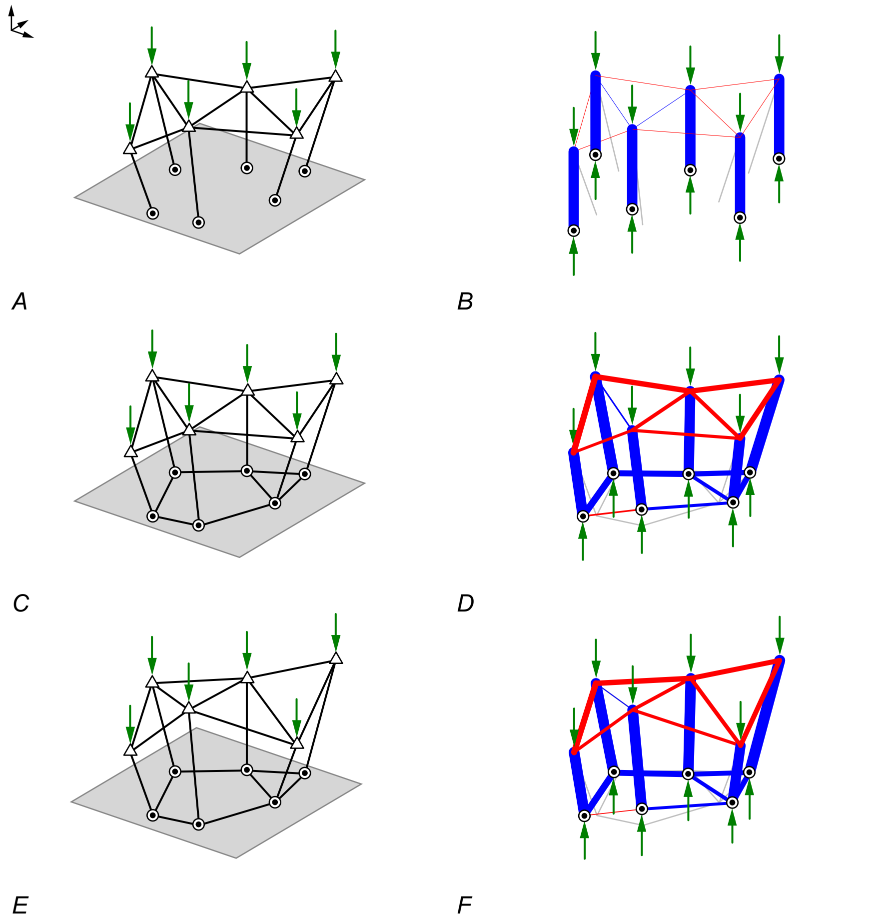

Figure 8.3: Iterative modelling of a folded roof on inclined columns. Initially, the roof is resting on inclined columns with movable supports; this results in an equilibrium solution with vertical columns (*A*, *B*). Then, structural elements between the supports are added (*C*, *D*). Finally the roof geometry is refined by changing the height of the vertices (*E*, *F*).

8.3.2 Branching columns

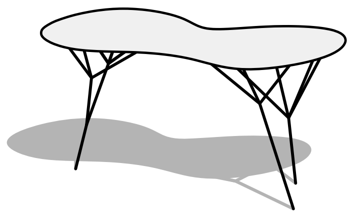

Figure 8.4: A flat roof supported by branching columns.

This case study demonstrates the step-wise modelling of a planar horizontal roof with irregular outline, supported by branching, or 'tree', columns (Figure 8.4). Initially, the shape of the planar roof is given. The goal is to find a constellation of branching columns, which balance the roof's self-weight. Similarly to the previous example, an arbitrary geometry of the branching columns is generally not in equilibrium, since both a translational force and a rotational moment is induced on the roof. Furthermore, the branching nodes of the columns are generally not in equilibrium (Lachauer and Block, 2012).

A horizontal truss structure is inscribed into the roof plate, the roof's self-weight is approximated by vertical forces of equal magnitude (Figure 8.5 *A*). The roof itself is modelled with geometrically fixed vertices. The supports of the branching columns are movable along the ground plane, to provide geometrical freedom for form finding. In addition, the branching nodes are constrained to horizontal planes, to control the geometry of the support structure. This setting leads to a branching structure with vertical columns, since the movable supports are not able to balance horizontal forces (Figure 8.5 *B*). By adding struts between the movable supports (Figure 8.5 *C*), an equilibrium state with non-vertical branching columns becomes possible (Figure 8.5 *D*).

Thereafter, the height of the roof is increased, and two branching columns are joined together, to refine the formal expression of the design (Figure 8.5 *E*). Based on this model, a final equilibrium state is found (Figure 8.5 *F*).

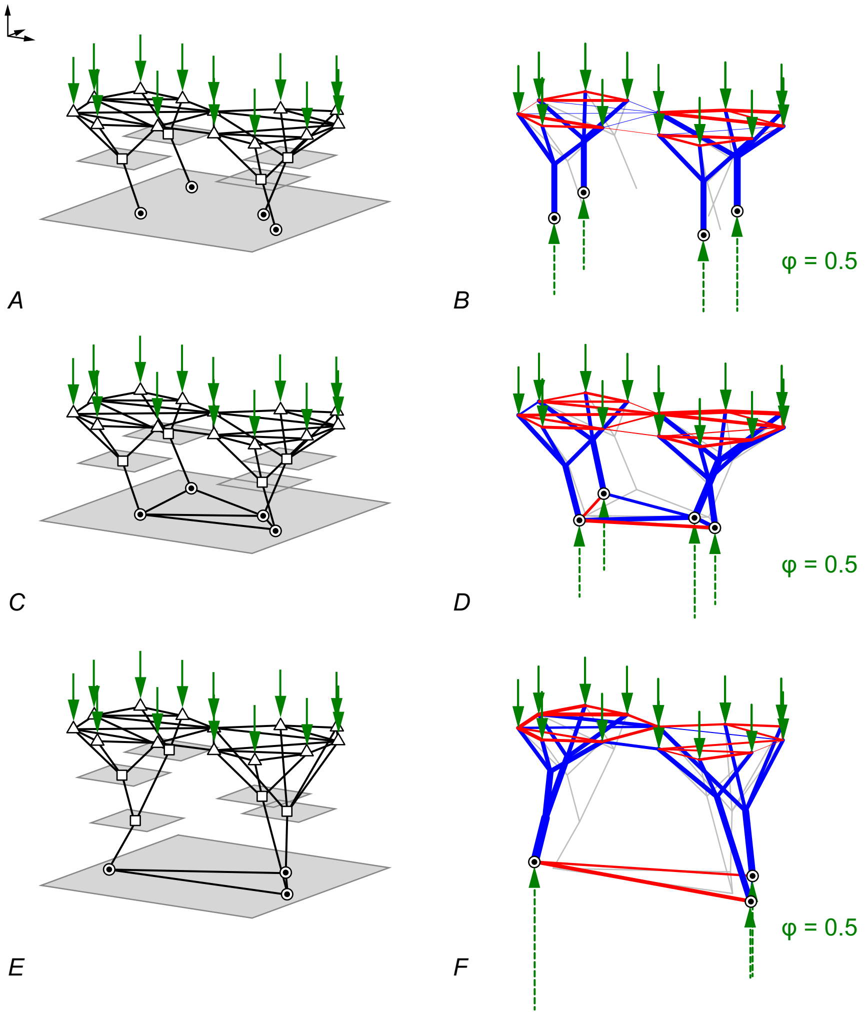

Figure 8.5: Iterative modelling of a planar roof on branching columns. Initially, the roof is resting on branching columns with movable supports; this results in an equilibrium solution with vertical columns (*A*, *B*). Then, structural elements between the supports are added (*C*, *D*). Finally two branching columns are joined together and the height of the roof is changed (*E*, *F*).

8.3.3 Shell structure

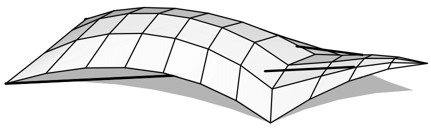

Figure 8.6: A compression shell with a cantilevering part.

This example demonstrates the formal exploration of a compression shell under self-weight in three variations, resulting in a cantilevering shell with tension ties (Figure 8.6). The initial geometry is based on a discretised freeform NURBS surface, geometrically close to a compression shell. Inspired by *Thrust Network Analysis* (Block and Ochsendorf, 2007), different types of constraints are applied.

In a first step, all structural members are modelled as compression elements (orange), therefore the upper bound $\boldsymbol{q} \leq \mathbf{0}$ is enforced on all force densities (Figure 8.7 *A*). This results in a compression structure close to the input geometry, with higher forces in the edge arches, and lower forces within the shell surface (Figure 8.7 *B*). In the second step (Figure 8.7 *C*), the force flow within the structure is adjusted, by enforcing the force density bound $\boldsymbol{q} \leq -\mathbf{5}$ for one row of struts (dark red). The resulting equilibrium model is a shell with a necking (Figure 8.7 *D*). In the third step, the goal is to add a cantilevering 'nose' to the shell. For this purpose, the surface itself is extended by a triangular patch, and two additional members between supports and the tip of the patch are introduced.

In order to maintain this specific proportion between the span of the shell, and the span of the cantilever, the tip of the cantilever is constrained to a vertical line (Figure 8.7 *E*). The form-finding process results in a cantilevering equilibrium solution, the tip of the patch is pulled back by two tension ties (Figure 8.7 *F*).

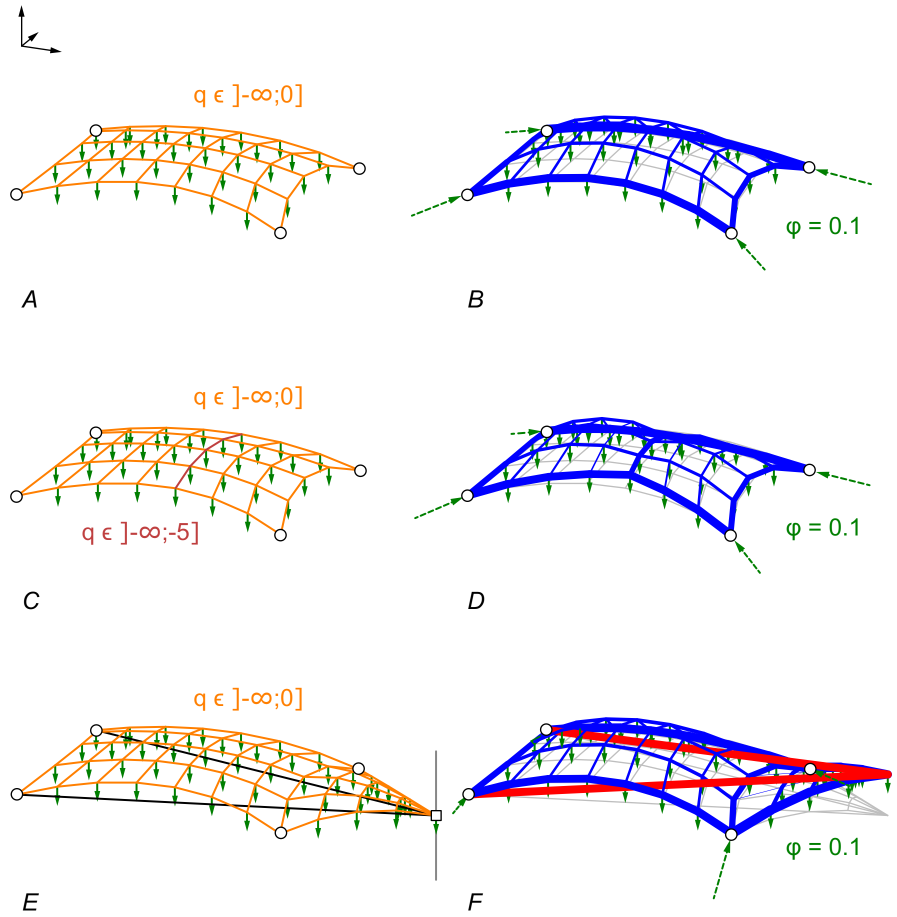

Figure 8.7: Exploration of formal variations of a compression shell. Firstly, the shell is modelled as 'freeform' network (*A*, *B*). Secondly, the forces in the shell are redistributed; this results in a necking (*C*, *D*). Lastly, a cantilevering triangular patch is added, tied back with tension elements (*E*, *F*).

8.3.4 Lookout tower

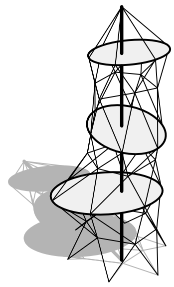

Figure 8.8: A pre-stressed lookout tower with irregular geometry.

This case study demonstrates the modelling of a pre-stressed lookout tower in three different variations, resulting in a proposal with irregular geometry (Figure 8.8). All variations of the structural model consist of an outer tension net, a vertical compression pylon in the centre, and three horizontal platforms. The case study is inspired by the "Panoramic Tower Killesberg" in Stuttgart, Germany, designed by schlaich bergermann und partner. For modelling, the following structural preconditions are assumed: the tower is heavily pre-stressed and the self-weight of the platforms is neglected. Furthermore, the platforms are considered as horizontally stiff plates; the pylon and the platforms are structurally not connected.

In the first step, a model with three regular, trussed hexagons of the same size, representing the platforms, is created (Figure 8.9 *A*). The vertices of the platforms and both ends of the pylon are modelled as geometrically fixed nodes. For defining a pre-stressing force, one vertical load is applied to the lower end of the pylon. Currently, the computational setup of the form-finding method does not allow the defining of models without external force at all, so pre-stressing has to be introduced as external force. The network (orange) is constrained to tension forces only, by defining the lower bound $\boldsymbol{q} \geq \mathbf{0}$ **for** the force densities. The form-finding process results in an equilibrium state very close to the input geometry (Figure 8.9 *B*).

In the next step, the geometry of the tower is stretched in one direction, increasingly towards the bottom. This results in a "tent-shaped" tower. The connectivity, support conditions and constraints of the model remain unchanged (Figure 8.9 *C*). Again, an equilibrium state very close to the input geometry is found (Figure 8.9 *D*). Finally, an asymmetric model is created, by stretching the platforms to different degrees in different directions (Figure 8.9 *E*). The goal of these operations is to spatially differentiate the three platforms.

This model exhibits an unsteady solving behaviour (Figure 10.4 *F*); the author interprets this as a sign that the defined constraints are close to an unfeasible constellation. Nevertheless, a final equilibrium state is found (Figure 8.9 *F*).

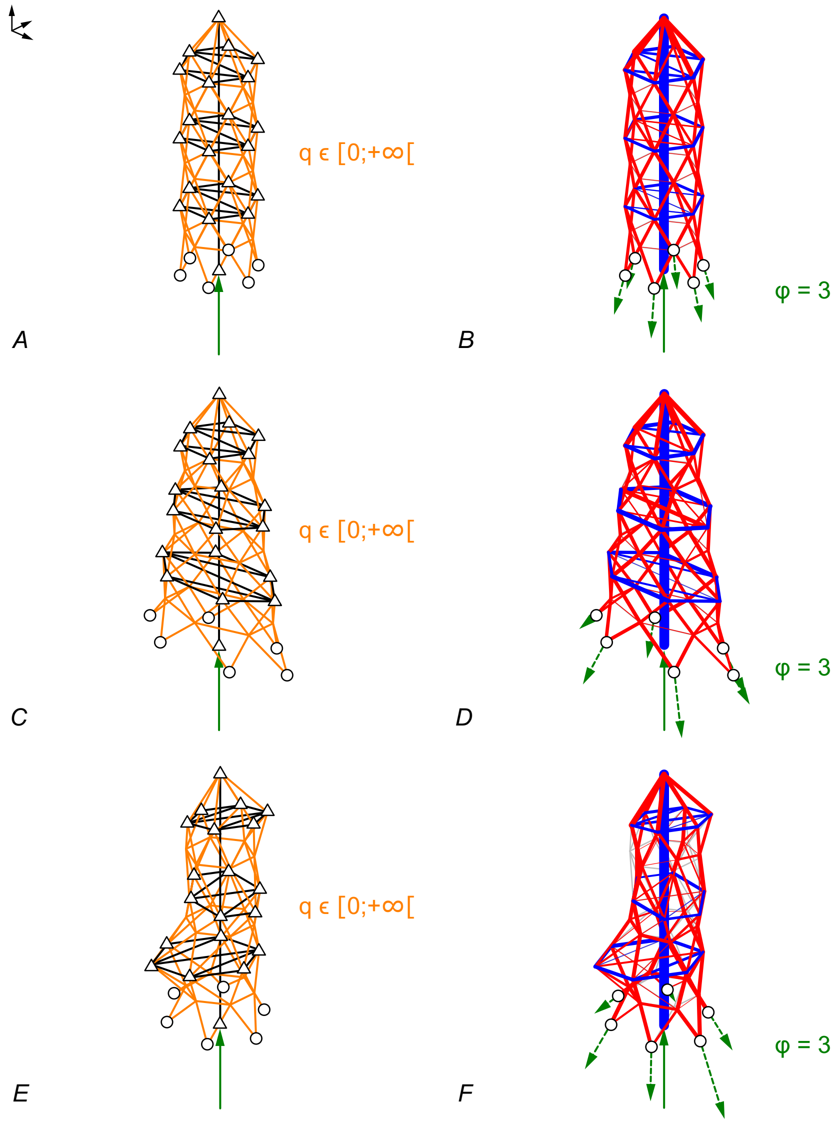

Figure 8.9: Exploration of formal variations of a pre-stressed lockout tower. Firstly, the tower is modelled with hexagonal platforms of the same size (*A*, *B*). Secondly, the platforms are stretched in one direction, increasingly towards the bottom (*C*, *D*). Lastly, the platforms are stretched and shifted irregularly (*E*, *F*).

8.3.5 Stadium roof

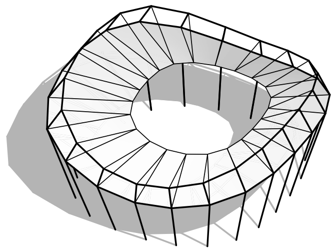

Figure 8.10: A pre-stressed stadium roof.

This example demonstrates the iterative modelling of a pre-stressed stadium roof with complex geometry (Figure 8.10). It is inspired by the constructive principle of the "spoked wheel" for stadium roofs, developed in the office of schlaich bergermann und partner (Göppert and Stein, 2007). As additional design challenge, the goal is to create a roof that is asymmetrical along the longitudinal axis, which is sometimes required for covering a large VIP tribune on one of the longer sides. The Camp Nou soccer stadium in Barcelona, Spain, for instance, has such an asymmetrical layout. For form finding, the pre-stress of the roof is considered as being dominant, self-weight is neglected.

The geometry of the initial model consists of a discretised spatial NURBS curve, representing the compression ring resting on inclined columns. The inner tension hoop is connected to the outer ring with radial cables (Figure 8.11 *A*). The geometry of the compression ring is fixed, the nodes of the tension hoops are freely movable. The current set-up of the form-finding method does not allow models to be defined without external forces. Therefore, one strut of the compression ring is replaced by a force pair to create a self-stressed structure. As result of the form-finding step, an equilibrium state is found, with low compression and tension forces in the columns, and an adjusted geometry of the tension hoop (Figure 8.11 *B*). The spatial geometry of the pre-stressed compression ring creates force components perpendicular to the roof surface; these residual forces are carried by the columns.

In the next modelling step, a second pre-stressed compression ring is introduced, with a second layer of radial tension cables (Figure 8.11 *C*). The form-finding process results in a similar hoop geometry to that in the first model (Figure 8.11 *D*). Thereafter, the geometry of the compression ring is transformed by a scaling operation (Figure 8.11 *E*), leading to a final equilibrium state (Figure 8.11 *F*).

The asymmetric setting of the roof layout is less explicit than in the input model, and has potential for further refinement. In Lachauer and Block (2014), the modelling of a similar roof is shown, with additional vertical tension elements between the two layers of radial cables.

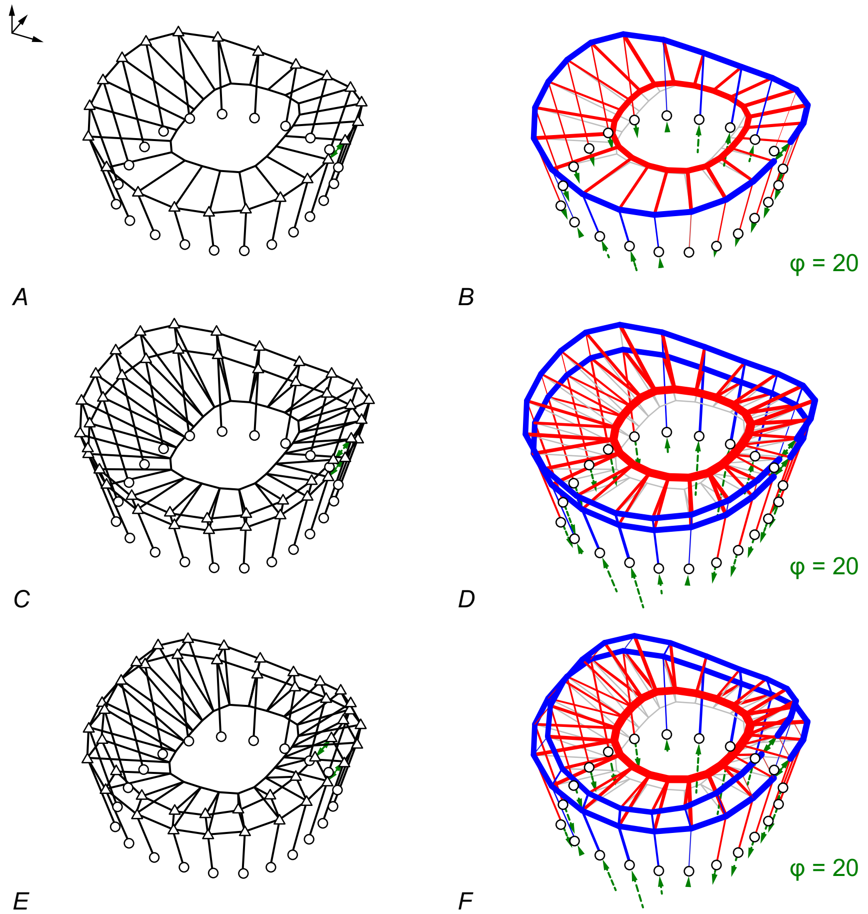

Figure 8.11: Iterative modelling of a pre-stressed stadium roof on inclined columns. Initially, the roof is modelled based on an asymmetric set-up of tension cables and spatial compression ring (*A*, *B*). Then a second layer of cables and a second ring are added (*C*, *D*). Finally, the geometry of the second ring is transformed by a scaling operation (*E*, *F*)

8.3.6 Curved Bridge

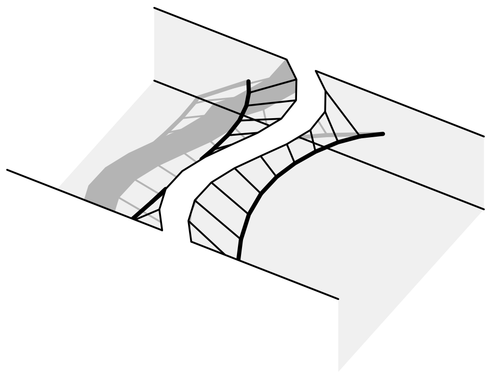

Figure 8.12: A curved bridge supported by spatial arches.

This case study demonstrates the iterative modelling of a curved bridge supported by two arches (Figure 8.12). Initially, the axis of the deck is given. The goal is to find a support structure which balances the deck's self-weight. A similar design problem was addressed by the author earlier, with a less general, tailored form-finding method (Lachauer and Kotnik, 2011).

In the first, naive approach, the horizontally planar bridge deck is modelled as a discretised curved axis, with geometrically fixed nodes, and a linear support structure with unconstrained nodes. The self-weight of the deck is represented by vertical forces of equal magnitude (Figure 8.13 *A*). Based on this input model, the form-finding process generates an inclined, supporting arch, which is itself in equilibrium, but the deck is not in equilibrium, and residual forces remain at the nodes (Figure 8.13 *B*). The form-finding process does not converge: after a few iterations, the sum of residuals remains constant, hence the model remains in a non-equilibrium state (Figure 10.6 *B*).

In order to create a structural model with feasible constraints, a second support structure is added in the same horizontal plane as the deck curve, to bear the residual forces at the deck nodes. Furthermore, vertical planes are added as constraints for the supporting structure (Figure 8.13 *C*). This model converges and creates a deck in pure tension, supported by two compression arches: one arch is exactly horizontal, and the other is inclined, below the deck (Figure 8.13 *D*).

Thereafter, in the third step, the model is refined: additionally to the deck axis, the two edges are modelled, and two linear supporting structures are placed below the deck (Figure 8.13 *E*). In this model, the edges are geometrically fixed, and the axis is modelled using unconstrained nodes. This input model results in an equilibrium solution with a support structure consisting of two arches and a structural deck, with both compression and tension forces (Figure 8.13 *F*).

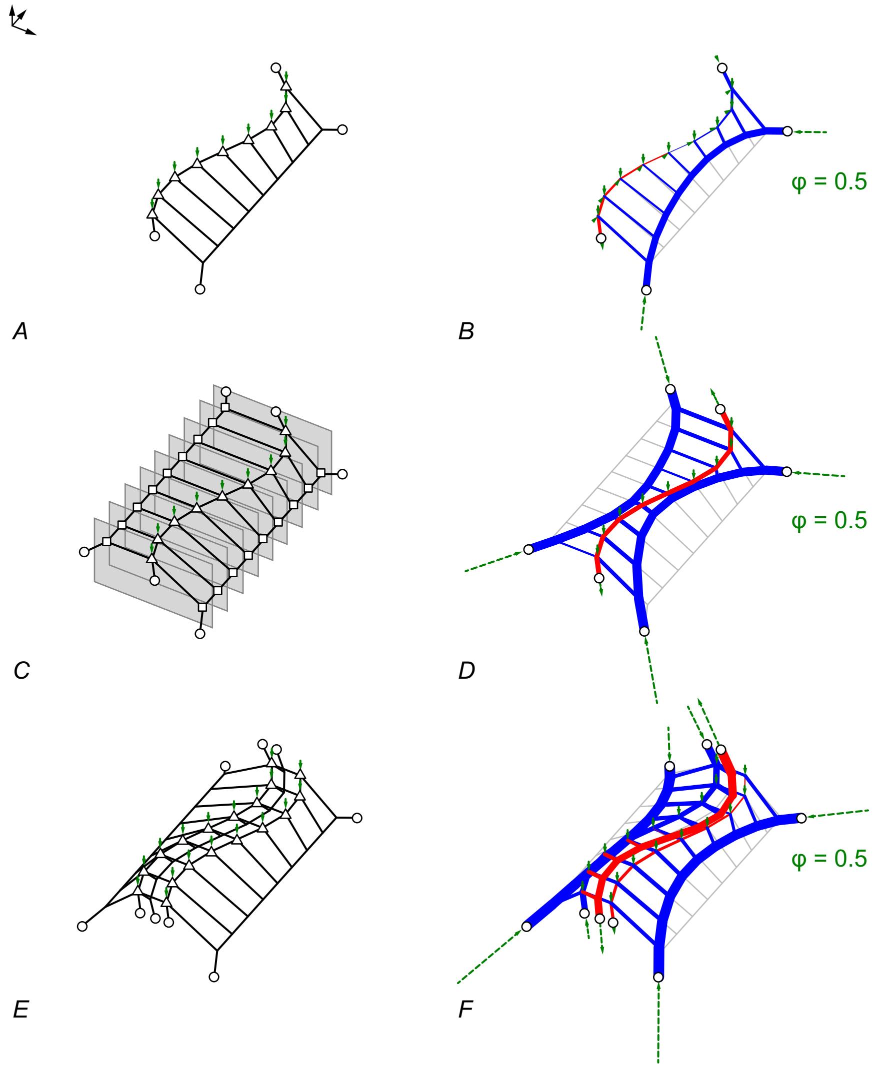

Figure 8.13: Iterative modelling of curved bridge supported by spatial arches. Initially, the bridge is modelled as 'freeform' deck with one linear support structure; this model does not converge, residual forces remain at the deck nodes (*A*, *B*). Then a second linear support structure is added; this results in a tension deck supported by two compression arches (*C*, *D*). Finally, the structural model is refined, consisting of elements representing the deck edges, the deck axis and two arches (*E*, *F*).

8.4 Combination of structural models

In this Section, three case studies are presented that focus on the creation of new structural models through the combination of existing typologies.

8.4.1 Twin stadium roof

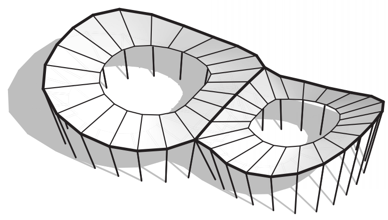

Figure 8.14: A "twin stadium" with pre-stressed tension roof.

This example demonstrates the iterative modelling of a roof for a "twin stadium", covering both a soccer arena and an ice-hockey arena (Figure 8.14). Such a twin stadium project has recently been proposed for Zurich, Switzerland (Aeschlimann, 2013). The goal is to create one merged structure from two distinct roofs. Therefore, one of the two has to be adjusted in size, since ice-hockey arenas are smaller than soccer arenas.

The initial model consists of two mirrored stadium roofs, similar to the one presented in Section 8.3.5 (Figure 8.11 *A*), merged together. The model is composed of inclined columns, two pre-stressed, spatial compression rings with fixed geometry, and a pair of inner tension hoops connected to the rings with radial cables (Figure 8.15 *A*). Both models share one part of the compression ring, and four columns. In this example, in addition to the pre-stressing force, the self-weight of the compression ring is also considered. The equilibrium model resulting from the form-finding process adapts the geometry of the tension hoop and cables (Figure 8.15 *B*).

In the second modelling step, one of the two roof structures is scaled (two-dimensionally, in plan, not in height) to fit the smaller size of the ice-hockey arena (Figure 8.15 *C*). This input model results in a corresponding equilibrium model (Figure 8.15 *D*).

In the third step, the goal is to create a structurally more open connection between the two arenas, hence to replace the four shared columns by a truss. For this purpose, the shared columns are shortened, and additional horizontal struts are introduced (Figure 8.15 *E*). This finally results in an equilibrium model without columns between the two arenas. At the connection of the two roofs, a funicular suspended girder carries the loads (Figure 8.15 *F*).

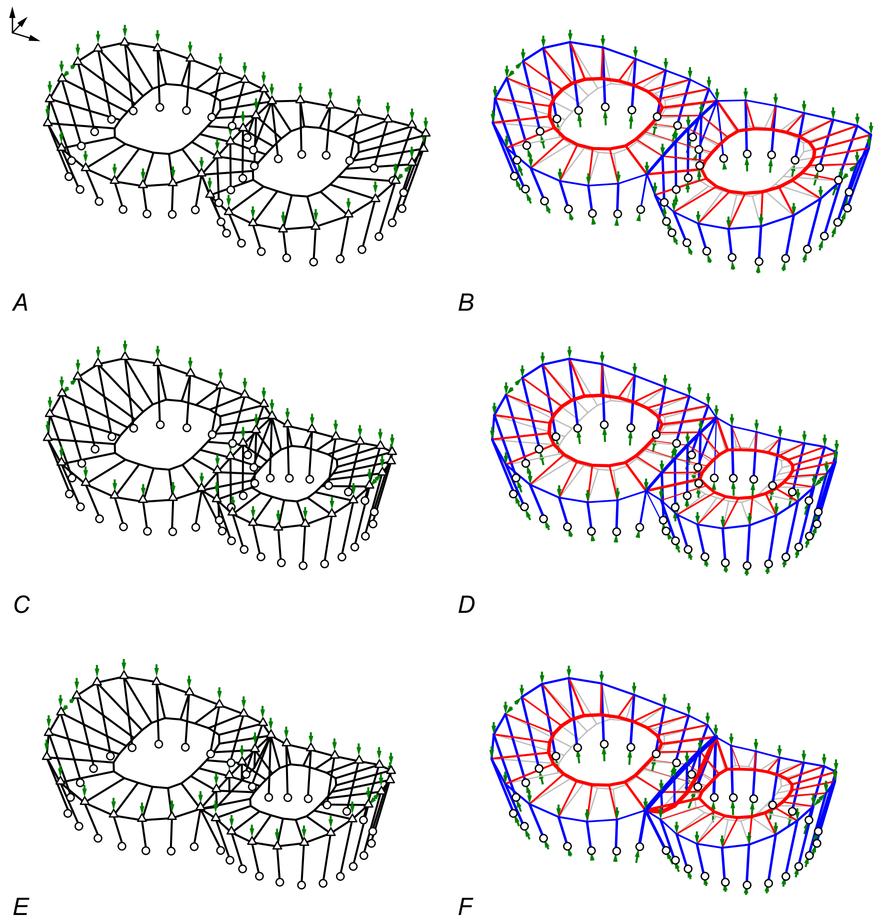

Figure 8.15: Iterative modelling of double stadium with pre-stressed tension roof. Initially, two symmetric stadium models are merged together (*A*, *B*). Then one of the stadiums is reduced in size by a two-dimensional scaling operation (*C*, *D*). Finally, the columns between the two models are replaced by a suspended girder (*E*, *F*).

8.4.2 Vault on columns

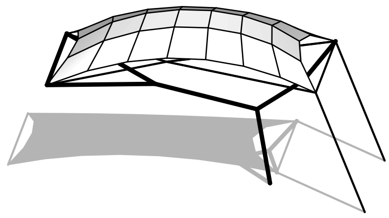

Figure 8.16: A shell supported by a cantilevering structure.

This case study demonstrates the iterative modelling of a combined roof construction, consisting of a compression shell supported by a cantilevering structure (Figure 8.16). The goal is furthermore to illustrate how to decompose a complex structural system, and then to solve the subsystems independently. For this reason, the shell structure is modelled first, and the forces acting on the supports of the shell are used as loading for the support structure.

The first modelling and form-finding steps are equal to the models already presented in Section 8.3.3 (Figure 8.7 *A* and *B*): a discretised NURBS surface is used as input, with struts constrained to compression forces only (in orange). To enforce compression forces, the upper bound $\boldsymbol{q} \leq \mathbf{0}$ is set for all force densities. As self-weight, vertical forces of equal magnitude are assigned to the free nodes (Figure 8.17 *A*). The resulting equilibrium model is balanced by four support forces (Figure 8.17 *B*). A set of forces inverse to the support forces is applied as loading in the next modelling step.

As initial naive approach, an asymmetric branching structure with horizontal bracings, carried by one movable support, is modelled. The upper vertices of the structure are geometrically fixed and loaded with the forces induced by the shell (Figure 8.17 *C*). The movable support results in an equilibrium model with one vertical column (Figure 8.17 *D*). The design goal is to create an asymmetric cantilevering structure. For this purpose, two more supports are added which are connected to the movable support and to two of the corner vertices (Figure 8.17 *E*). This input model results in an asymmetric cantilevering structure.

Finally, the two substructures in equilibrium (Figure 8.17 *B* and *F*) are combined, forming the resulting design proposal (Figure 8.16). This operation is maintaining equilibrium, since the reaction forces of the shell are cancelling out the forces acting on the support structure.

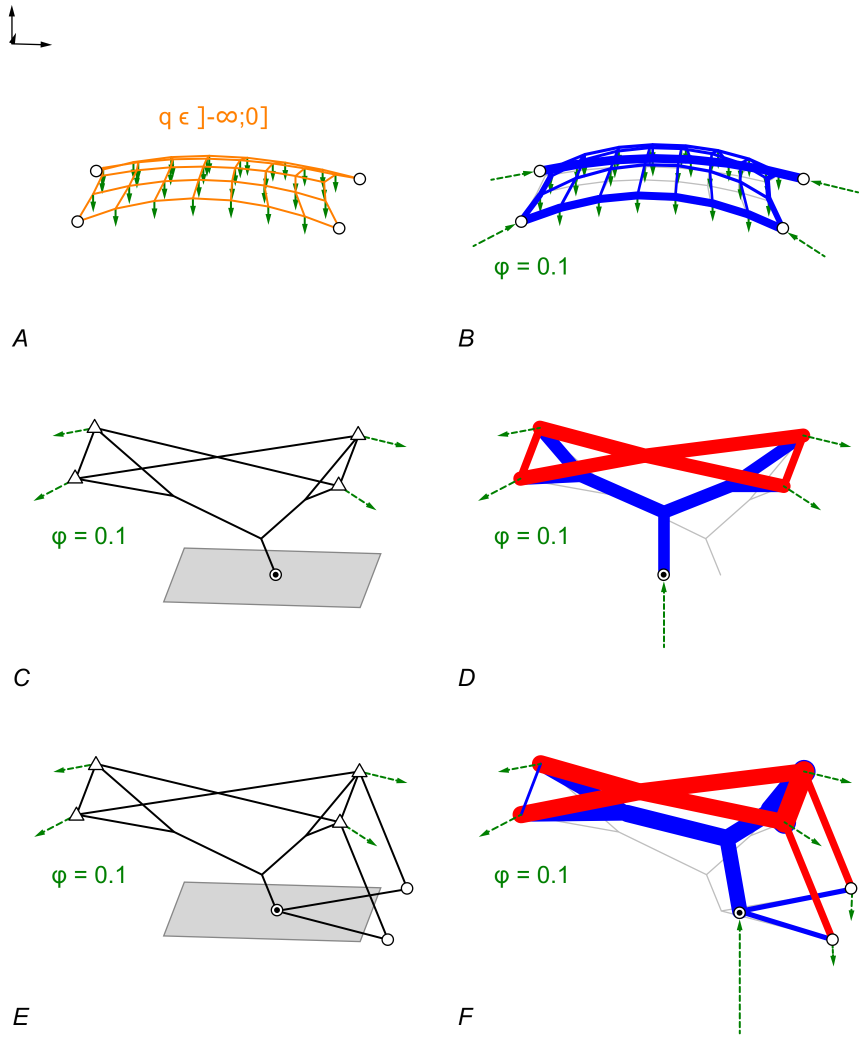

Figure 8.17: Iterative modelling of a cantilevering support structure combined with a compression shell. The structure is decomposed in two subsystems, which are modelled sequentially: the shell *(A, B*), and the support structure *(C, D*, *E*, *F).*

8.4.3 Shell structure as bridge

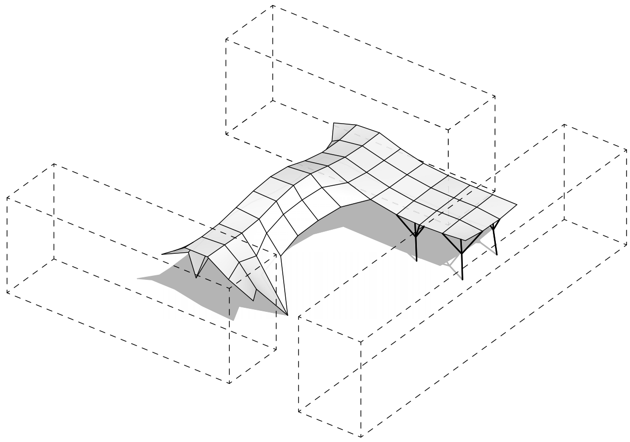

Figure 8.18: A freeform bridge connecting three existing buildings.

This case study demonstrates the iterative modelling of a freeform bridge, connecting three existing buildings above ground level (Figure 8.18). Furthermore, the goal is to demonstrate the ability of the equilibrium modelling method to create structures with complex, hybrid structural behaviour. Initially, the modelling process is started with a freeform network derived from a NURBS surface. Thereafter, columns are added and, finally, additional geometric constraints and diagonals are introduced, in order to achieve a smooth gradient for the bridge.

The first input model consists of a discretised freeform surface patch in an L-shape, supported at specific edge nodes. The longer leg of the L will be used as the bridge deck, the shorter leg will be used as the exterior terrace. This spatial layout leads to the specific loading pattern applied in the model (Figure 8.19 *A*). The form-finding process results in a rather steep compression shell, supported at some edges by compression arches and at other edges by tension elements (Figure 8.19 *B*).

In the next modelling step, the structure is refined: branching columns are introduced below the planed terrace, and three supports of the shell are lowered to the ground level (Figure 8.19 *C*). These operations improve the geometry of the planed terrace, but the gradient of the bridge deck is still too steep (Figure 8.19 *D*). Finally, supports at both ends of the deck are added, and diagonals are inserted in the region of the terrace and the bridge deck. In addition, the loaded nodes of the terrace and of the bridge deck are geometrically fixed (Figure 8.19 *E*).

This highly constrained input model leads to a structure with the desired geometric properties and a hybrid structural behaviour: a gently sloped bridge deck supported by a surface structure, blending into the planar terrace, which is supported by branching columns (Figure 8.19 *F*).

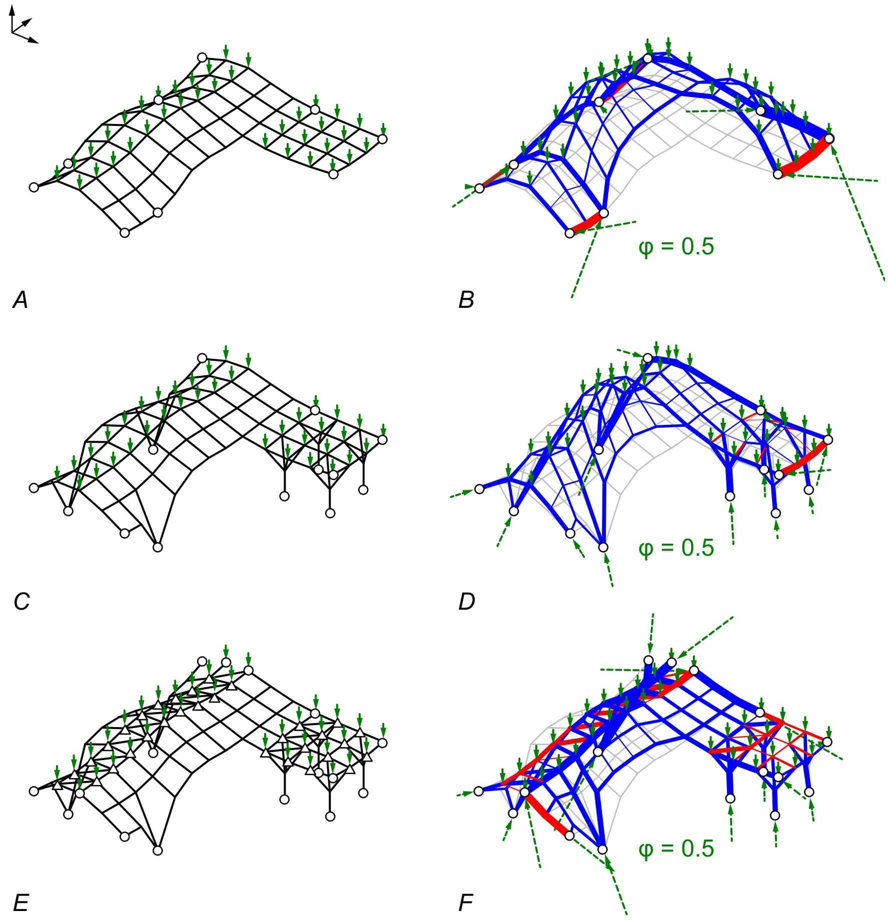

Figure 8.19: Iterative modelling of a combined bridge and platform as hybrid structure. Initially, the structure is modelled as freeform network supported at the edges (*A*, *B*). Then branching columns are added and certain supports are lowered in height (*C*, *D*). Finally, geometric constraints and diagonals are introduced (*E*, *F*).

9 Conclusions

In this Section, the unique contributions of this dissertation are resumed, together with a discussion on the limitations of the developed approaches. In addition, suggestions for future work are presented, and final conclusions are drawn.

9.1 Overview

As defined in Section 4, the problem of structural design modelling is to develop methods which enable interactive design processes as an iterative search within the space of efficient structural systems. Within the conceptual framework of equilibrium solutions (Section 5), the following main contributions have been made:

- Type-specific equilibrium modelling methods have been developed, based on customized form-finding techniques tailored to selected structural typologies (Section 6).
- A general equilibrium modelling method has been developed, based on a new form-finding method which allows to define boundary conditions on form and forces (Section 7).
- Case studies have been presented, demonstrating the flexibility of the general modelling approach (Section 8).

A general method for constrained form finding of structures with both compression and tension forces was earlier developed and is being used internally in the office of schlaich bergermann und partner for the design of footbridges and stadium roofs (Kemmler, 2012). This method requires the formulation of a mathematically unique form-finding problem, hence lacks the possibility for a direct and interactive design exploration. However, form-finding methods which enable an interactive exploration with respect to additional constraints have been developed for specific structural typologies, for instance for compression-only structures (Kilian and Ochsendorf, 2005; Block and Ochsendorf, 2007) and for tensegrity structures (Tachi, 2012). In the main, such methods enable the designer to conduct an iterative search within the space of equilibrium solutions, consisting of alternating phases of computational form-finding and designer-driven modelling. Structural design has generally been framed as an iterative search process consisting of both intuitive and analytical reasoning (Hossdorf, 2003: 121-142); the need for systematic "form-directing" methods has been pointed out (Polónyi, 1987). In this sense, computer-aided structural modelling is understood as interaction between designer-driven modelling processes and computational processes (Kotnik, 2010; Schumacher, 2012: 284-294), hence structural design is conceptualised as a search process (Simon, 1996: 111-138). This notion of structural design gives the designer the possibility of addressing additional goals beyond the physical necessities of statics. Overall, the need for a holistic understanding of structural design, besides mere verification of usability and safety, also including architectural goals, such as aesthetic, spatial, social, economic and ecological aspects has recently been emphasised (Schlaich, 2006; Schwartz, 2012). Such an autonomous, creative practice rooted in a tradition of structural design has been described as "structural art" (Billington, 1983).

The problem of structural design modelling addresses the challenge to give a maximum of formal flexibility and control to the designer, while ensuring efficient structural behaviour through computation. This is furthermore triggered by the goal of reducing the gap between engineers and architects by reaching out from structurally efficient and simple "engineering solutions" towards novel design proposals with high architectural quality and potentially complex form.

In Section 9.2, the unique contributions of this dissertation to the body of knowledge are discussed. In Section 9.3, the limitations of the presented modelling approaches and form-finding methods are illustrated, and directions for future research are proposed. In Section 9.4, a final conclusion is drawn.

9.2 Contributions

This dissertation contributes to the field of structural design, based on the conceptual framework of equilibrium solutions and form-finding methods. Through interactive form finding, pin-jointed structures which are in equilibrium for the given loading case are generated. Different modelling approaches enable the modification and iterative refinement of these initial equilibrium models. In this context, equilibrium models are the starting point for later phases of the design process dealing with questions of dimensioning, constructive detailing and erection. All presented modelling methods are limited to structures that form mechanisms, thus stiffening and bracing schemes have to be developed. Subsequently, member dimensions are assigned, and analysis regarding structural safety for live loads, stability and serviceability has to be performed. These later phases of the design process are not within the scope of this dissertation, exclusively modelling approaches and form-finding methods are discussed.

9.2.1 Tailored modelling approaches

In early stages of the structural design process, namely in the "conceiving phase" (Section 3.1), a potpourri of different approaches, physical models, geometric routines based on graphic statics, and computational methods are used in practice. In Section 6, exemplary computer-aided modelling approaches for specific typologies of structures based on the rigorous conceptual framework of equilibrium solutions are presented. These structural typologies are defined by underlying structural concepts. The boundary conditions and constraints of these concepts are inscribed in the modelling process through custom adaptation of standard form finding methods.

As unique contribution, the presented approaches enable the interactive modelling of

- freeform roof girders,
- curved arch or suspension bridges,
- plates supported by branching structures, and
- compression-only vaults.

The method for modelling freeform roof girders (Lachauer and Kotnik, 2010) is based on a graphical design routine for efficient trusses (Zalewski and Allen, 1998: 275–300). The modelling method for curved bridges (Lachauer and Kotnik, 2011) combines structural concepts developed in the office of schlaich bergermann und partner (Keil, 2004) with computational form-finding techniques. The modelling method for branching structures (Lachauer and Block, 2012) is a topologically flexible extension of the method for curved bridges. The method for modelling compression vaults (Rippmann, Lachauer *et al.*, 2012) extends Thrust Network Analysis (Block and Ochsendorf, 2007).

Earlier approaches to the geometric modelling of curved bridges (Laffranchi, 1999: 23-30) and branching structures (Hunt, Haase *et al.,* 2009) do not allow for the iterative refinement of the model, previously existing approaches to interactive modelling of compression-only vaults provides less control of the vault form during the design process (Kilian and Ochsendorf, 2005; Kilian, 2006).

9.2.2 Extension of the Force Density Method

The modelling approaches for specific typologies of structures discussed above are based on custom adaptations of standard form-finding methods. These tailored methods are addressing the boundary conditions and constraints of the type-specific structural concepts. The general equilibrium modelling approach (Section 7) enables the designer to define and modify boundary condition freely; this approach is based *one* general form-finding method.

As described in Section 7.1, the general equilibrium modelling method is conceptualised as an alternating, step-wise search process. The designer constructs a loaded, pin-jointed kinematic model in the drawing space. Such a model is generally not in equilibrium, the application of the developed form-finding process establishes equilibrium by automatic changes in the model's geometry and inner force distribution. In the next step, the designer alters the model, and repeats the form-finding process. These two steps are repeated, until a solution is found that satisfies the designer's goals.

In cases of mixed compression and tension members, the form-finding process often causes large deformations to the initial model geometry. By enabling the designer to impose constraints on geometry, inner forces, and support conditions, a deliberate 'steering' of the search process becomes possible.

This new equilibrium modelling method combines the advantages of graphic statics with the advantages of physical modelling. On the one hand, the modelling method enables the designer to precisely control the geometry during the form-finding process, similarly as in graphic statics. On the other hand, the method gives the designer the freedom to directly interact with the spatial pin-jointed model. After each form-finding operation, the loading, the connectivity and the geometry of the model can be changed. In this sense, the method is comparable to the design process using physical models.

The new method is based on early descriptions of non-linear extensions of the Force Density Method for compression and tension structures (Linkwitz and Schek, 1971; Linkwitz, Schek *et al.*, 1974) and inspired by recent developments for the form finding of structures with mixed compression and tension forces (Miki and Kawaguchi, 2010; Tachi, 2012). Furthermore, the method is influenced by concepts from computer graphics, such as the implementation of constraints using projections (Bouaziz, Deuss *et al.*, 2012). A precedent version of the method presented in this dissertation has been published recently (Lachauer and Block 2014). Lately, an extension of the Force Density Method has been presented, that allows form-finding of mixed compression and tension forces with respect to geometric constraints on the free nodes and additional force-density bounds (Tamai 2013). Miki and Kawaguchi (2010) and Tamai (2013) solve the constrained form-finding problem using numerical optimisation methods with an objective function. The approaches by Tachi (2012), by Lachauer and Block (2014), and the method presented in this dissertation are based on numerical integration with explicit calculation of a constrained gradient. In this context, numerical integration offers greater flexibility in the definition of constraints, such as for instance the possibility to define movable supports.

In this dissertation, a new non-linear extension of the Force Density Method is presented, which generates equilibrium solutions close to the geometry of a given pin-jointed, kinematic model. This method is especially suited for the modelling of structures with mixed compression and tension forces. As unique contribution, this new form-finding method allows the following constraints and boundary conditions to be defined:

- Geometric constraints for support nodes, enabling movable supports which can resist forces in specific directions only to be defined
- Upper and lower bounds on the force densities, enabling bounds on the inner forces of the structure to be implicitly defined
- Geometric constraints for free nodes, enabling nodes which move along lines, within planes, and nodes that are fixed in space to be defined

9.2.3 Equilibrium modelling case studies

The general equilibrium modelling approach has been prototypically implemented within a commercial CAD system, using a built-in scripting language and an additional numerical mathematical library. Constraints are represented through standard geometric objects and colour codes (Section 7.3). This prototypical modelling setup has been used for the interactive design of the nine exemplary case studies presented in Sections 8.3 and 8.4.

In earlier publications by the author, the computational modelling of the following typologies of structures has been addressed: modelling of compression shells (Rippmann, Lachauer *et al.*, 2012), curved bridges (Lachauer and Kotnik, 2011), branching columns and inclined columns (Lachauer and Block, 2012), and the modelling of a stadium roof (Lachauer and Block, 2014). Other relevant approaches in the field of structural modelling have been published for branching structures (Hunt, Haase *et al.,* 2009), compression shells (Kilian and Ochsendorf, 2005; Block and Ochsendorf, 2007), compression shells with tension ties (Rippmann and Block, 2013), and compression shells, pre-stressed tower, and curved bridges (Tamai, 2013).

All these approaches are based on different and often highly customised modelling methods. In this dissertation, the following contributions regarding the modelling of equilibrium structures are made:

- Unification of the modelling method for a wide variety of different equilibrium structures
- Demonstration of the design of new structures through the combination of structural models
- Demonstration of the design of new structures through geometrical and topological variation of structural models

9.3 Limitations and future work

Type-specific modelling methods are limited to certain structural typologies 'by definition', through their tailored form-finding processes. The great flexibility of the general equilibrium modelling method regarding the diversity of possible modelling results has been demonstrated. Nonetheless also the presented general modelling method has limitations, and offers various possibilities for improvement. The general method enables the designer to define infeasible conditions, hence background knowledge in the field of equilibrium modelling is needed. In this context, a heuristic technique that identifies infeasible constraints would be helpful. But even with such a technique, equilibrium modelling still requires deep understanding of the equilibrium principles, and experience in structural design.

The improvement of the form-finding method itself will lead to higher solving stability. Furthermore, the definition of additional types of constraints and support conditions will allow for greater control on the structural form. Finally, the improvement of the implementation of the form-finding method will lead to faster convergence, and will enable the modelling of larger equilibrium models.

9.3.1 Improvement of the form-finding method

The form-finding method in its current definition is flexible and powerful, nevertheless it has some limitations. The described types of geometric constraints for free nodes are restricted to lines and planes. Furthermore, supports are either unconstrained and fixed, or movable along lines or planes. At this point, it is impossible to define constrained, fixed supports (e.g. 'roller supports'). In its current formulation, the method does not allow form finding on pre-stressed models without external loads to be performed. In such cases, at least one of the struts has to be replaced by a pair of external pre-stressing forces. Highly indeterminate structures, such as large triangular networks, lead to slow and sometimes unstable convergence. Topological degeneration during the form-finding process (struts with vanishing length) leads to abnormal termination of the method.

Future work will focus as a first step on the extension of the existing types of geometric constraints beyond lines and planes. The concept of coordinate decomposition by projection can easily be extended to other geometric objects such as curves, curved surfaces or meshes. In this case, the residuals are decomposed based on projections on to tangential lines or planes of the constraining object. In a similar manner, supports movable along curves, surfaces or meshes will be implemented. Furthermore, also constrained and fixed, hence roller supports with different degrees of freedom, will be realised. The use of other stiffness matrices might generally improve solving stability. Besides geometric constraints on free nodes, additional constraints on force densities will also be included in future work, e.g. the possibility of defining struts with the same, but unknown, force density. Moreover, the approach offers the opportunity of also including geometric constraints on strut lengths. Finally, the most challenging remaining aspect is the formulation of mathematical convergence criteria and the definition of a process that identifies unfeasible sets of constraints.

9.3.2 Application of the method to real design cases

The presented examples demonstrate the flexibility of the modelling approach for a wide variety of cases, but most of the case studies are still in a conceptual state, with a small number of structural elements. One essential future step would be to substantially increase the computational performance of the implemented prototype. This would be a precondition for providing an interactive modelling experience in the design of large equilibrium models, such as that required, for instance, for a real stadium roof.

Future work includes the implementation of the form-finding method using sparse matrix libraries and highly optimised numerical solvers, such as the LAPACK open-source software package. The current prototype is developed, based on the IronPython scripting language which does not support sparse matrices. The implementation of the same method with, for instance, CPython will most likely increase solving speed by at least one order of magnitude. Furthermore, the simple iterative structure of the solving method allows for parallel computing, and might potentially be executed on the graphics processor (GPU). This offers the possibility of real-time form finding, and the integration of the method in node-based parametric modelling systems, e.g. Grasshopper[48].

9.4 Conclusions

This dissertation contributes to the body of knowledge in the field of structural design based on computer-aided modelling approaches and equilibrium solutions. Tailored form-finding techniques allowed for interactive equilibrium modelling of specific types of structures. Furthermore, a general modelling method has been presented. The implemented prototype of this general method has allowed for the development of several case studies, demonstrating the iterative modelling of various types of advanced spatial equilibrium structures such as curved bridges, branching columns and stadium roofs. The two disparate topics of geometric modelling, usually associated with Architecture, and structural form finding, usually associated with Structural Engineering, are pulled together. By enabling the interactive exploration of structural forms within the physical necessities of static equilibrium, structural design modelling is located in the overlap of these two domains. This dissertation is triggered by the hope of expanding the design culture rooted at the very intersection of Structural Design and Architecture, and to foster the design of buildings with both freedom in architectural form *and* efficiency in structural performance.

[48] McNeel – Grasshopper for Rhinoceros: http://www.grasshopper3d.com/

10 Appendix

10.1 Convergence graphs

In this appendix, solving times and convergence behavior of the presented case studies are listed.

	branches m	free nodes n	iterations t	time [s]
Fig. 8.3 *(B)*	15	12	20	1.8
Fig. 8.3 *(D)*	22	12	19	2.0
Fig. 8.3 *(F)*	22	12	19	1.8
Fig. 8.5 *(B)*	40	21	129	12.2
Fig. 8.5 *(D)*	45	21	148	16.2
Fig. 8.5 *(F)*	44	21	297	33.1
Fig. 8.7 *(B)*	67	36	86	9.2
Fig. 8.7 *(D)*	67	36	83	14.7
Fig. 8.7 *(F)*	101	52	107	24.0
Fig. 8.9 *(B)*	124	38	11	2.7
Fig. 8.9 *(D)*	124	38	11	2.3
Fig. 8.9 *(F)*	124	38	106	21.6
Fig. 8.11 *(B)*	87	44	136	26.6
Fig. 8.11 *(D)*	152	66	232	94.5
Fig. 8.11 *(F)*	152	66	202	83.2
Fig. 8.13 *(B)*	29	18	–	–
Fig. 8.13 *(D)*	48	27	376	52.3
Fig. 8.13 *(F)*	85	45	336	51.4
Fig. 8.15 *(B)*	165	82	210	105.6
Fig. 8.15 *(D)*	165	82	196	102.4
Fig. 8.15 *(F)*	170	86	221	119.4
Fig. 8.17 *(B)*	67	36	86	9.2
Fig. 8.17 *(D)*	13	9	226	29.7
Fig. 8.17 *(F)*	17	9	284	37.6
Fig. 8.19 *(B)*	122	62	220	55.1
Fig. 8.19 *(D)*	139	66	103	33.0
Fig. 8.19 *(F)*	150	63	127	42.8

Table 10.1: Number of branches, free nodes, iterations, and solving times for the case studies presented in Sections 6.3 and 6.4. The computation has been executed on a standard PC (Intel Core Duo, 2.8 GHz).

The following convergence graphs are plotting residuals $\rho = \|\boldsymbol{r}^t\|/\|\boldsymbol{r}^0\|$ against iterations t for the case studies presented in Section 8.3 and 8.4.

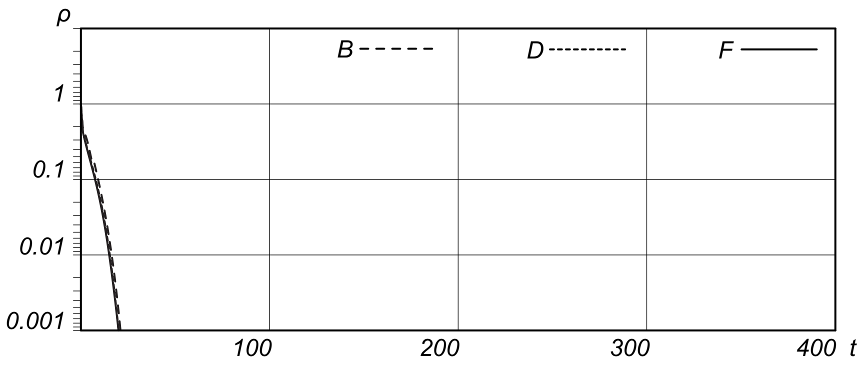

Figure 10.1: Convergence graph for case study 8.3.1 (see Figure 8.3).

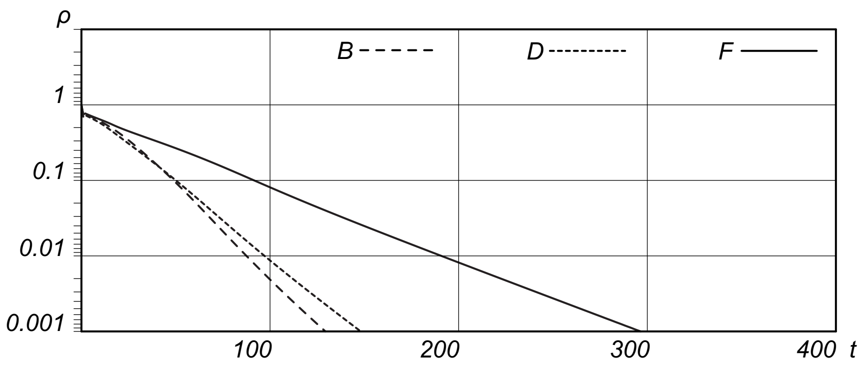

Figure 10.2: Convergence graph for case study 8.3.2 (see Figure 8.5).

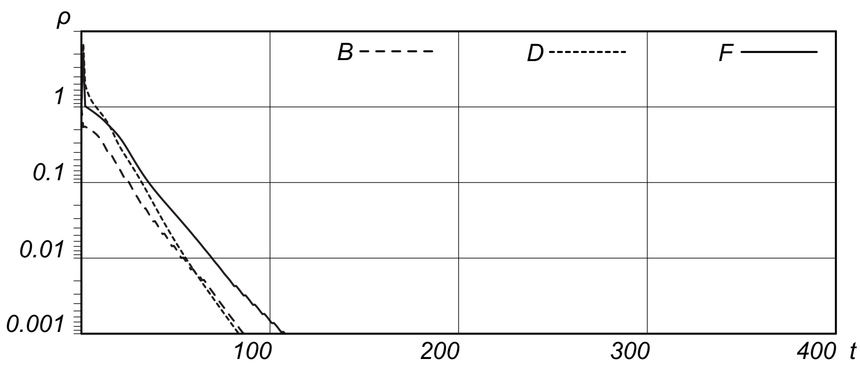

Figure 10.3: Convergence graph for case study 8.3.3 (see Figure 8.7).

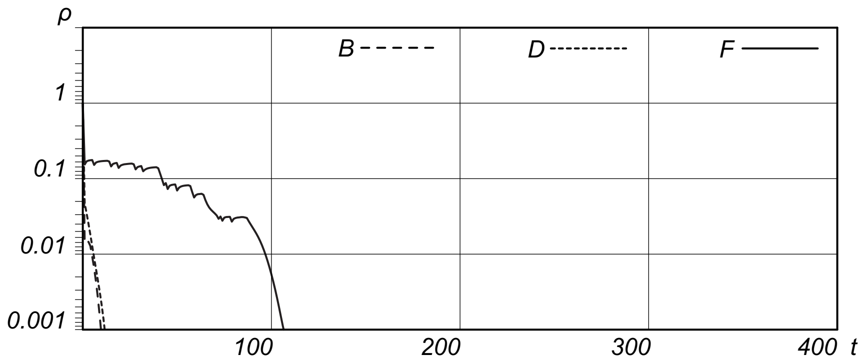

Figure 10.4: Convergence graph for case study 8.3.4 (see Figure 8.9)

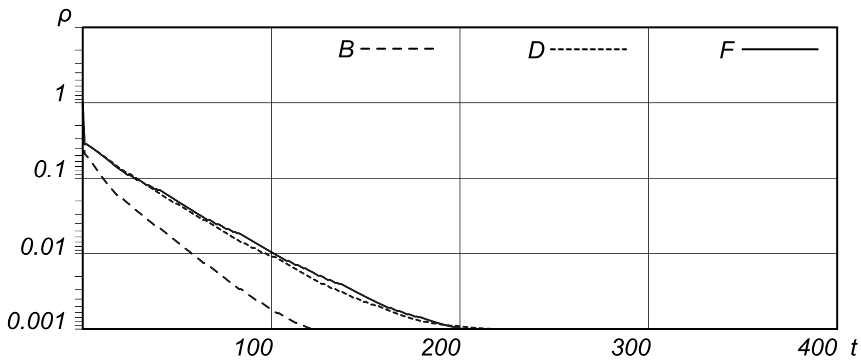

Figure 10.5: Convergence graph for case study 8.3.5 (see Figure 8.11).

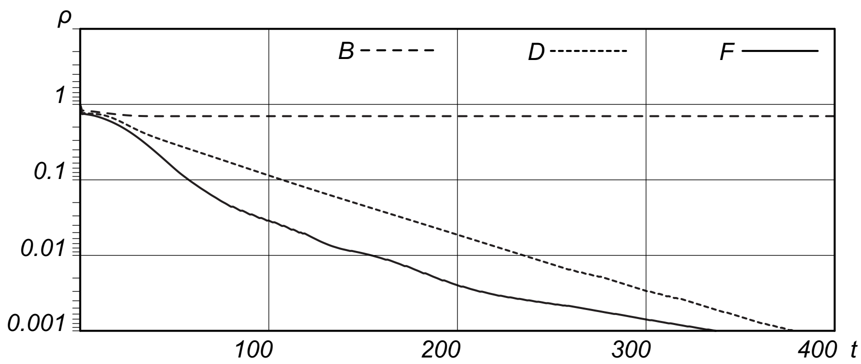

Figure 10.6: Convergence graph for case study 8.3.6 (see Figure 8.13)

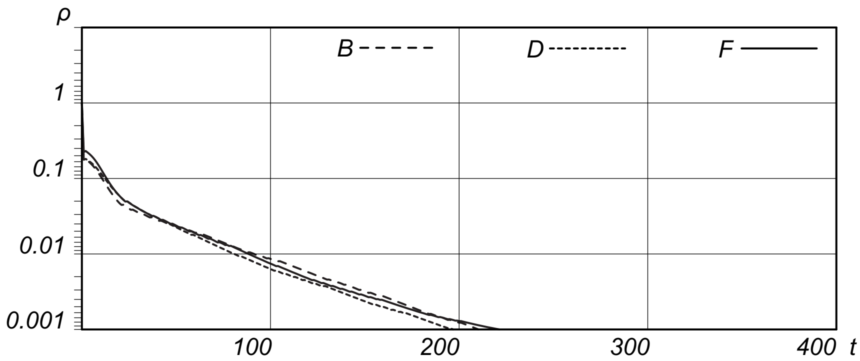

Figure 10.7: Convergence graph for case study 8.4.1 (see Figure 8.15)

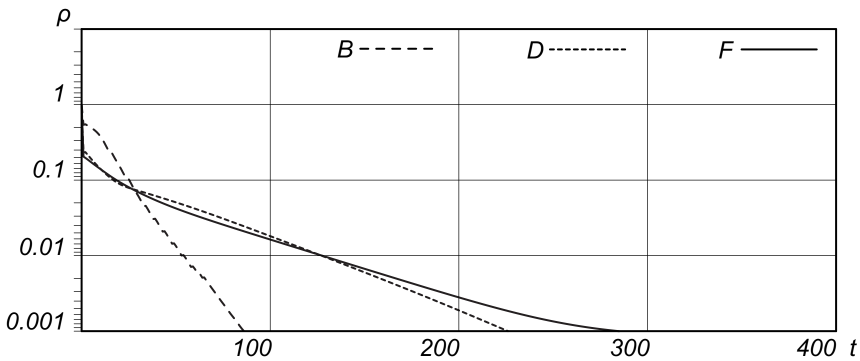

Figure 10.8: Convergence graph for case study 8.4.2 (see Figure 8.17)

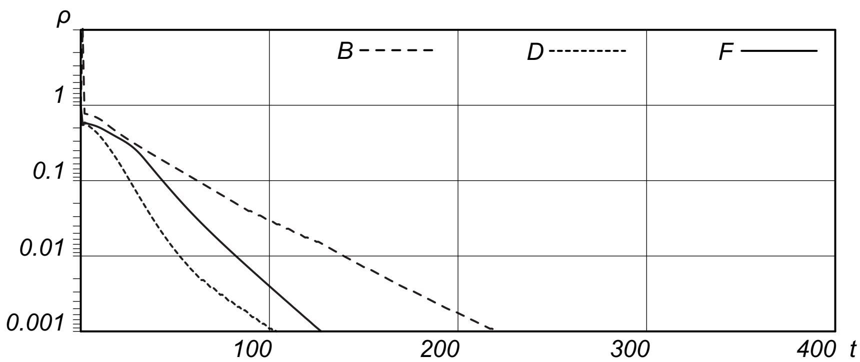

Figure 10.9: Convergence graph for case study 8.4.3 (see Figure 8.19)

10.2 Additional diagrams

For better readability, the following diagrams are printed in larger size.

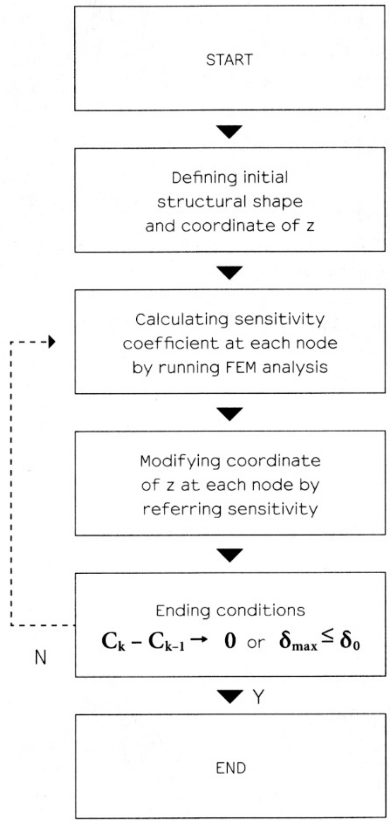

Figure 10.10: Flow diagram of Sensitivity Analysis method (see Figure 3.10 *A*)
(Sasaki, 2007: 105)

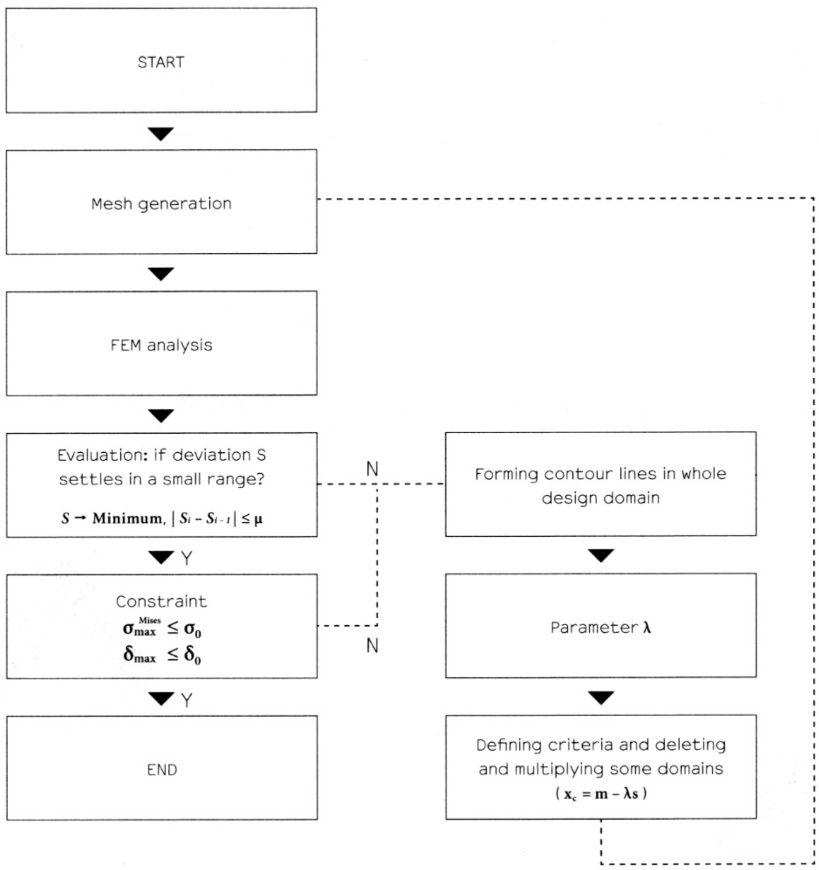

Figure 10.11: Flow diagram of 3D Extended ESO method (see Figure 3.11 *A*) (Sasaki, 2007: 108)

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