Doctoral Thesis

Measuring risk beyond the cash-additive paradigm

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MEASURING RISK BEYOND THE CASH-ADDITIVE PARADIGM

A thesis submitted to attain the degree of

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Preface

The theory of risk measures has become a well-recognized research area since the publication of the landmark paper by Artzner, Delbaen, Eber and Heath in 1999. In the context of a one-period economy, the above authors called a risk measure any function assigning to the terminal position of a financial agent – be it the net capital position of a financial company, the profit-and-loss position of a single portfolio or the payoff of a given financial contract – the minimal amount of capital that has to be raised and invested in a pre-specified reference asset in order to take the agent’s position to a pre-specified level of acceptable risk.

The focus of the literature soon shifted towards the special class of cash-additive risk measures, for which the reference asset is a risk-free bond with zero interest rate. The central role assigned to cash-additivity was justified on the grounds of a suitable discounting argument. By virtue of this argument, it was claimed that the original theory of risk measures could be reduced to the cash-additive setting by means of discounting. Indeed, most of the subsequent theory builds, implicitly or explicitly, on this legitimation argument.

This work has three fundamental objectives. First, we aim to demonstrate that the above-mentioned discounting argument is only conclusive at a first sight and, unfortunately, hides a variety of financial and mathematical problems making the preceding reduction claim and the consequent exclusive focus on cash-additivity essentially illegitimate. Second, motivated by this failure, we aim to return to the original framework and develop a theory of risk measures with respect to a general reference asset without resorting to any change of numéraire. Third, we aim to lay the foundations of a general theory of risk measures beyond the case of a single reference asset. In particular, although the main explicit focus of our analysis
will be dedicated to univariate positions in the context of a one-period economy, we aim to provide a unifying perspective on the problem of measuring risk for general financial positions – univariate positions, multivariate positions, streams of univariate or multivariate positions – in the context of a general multi-period economy.

The distinguishing aspect of our approach is the articulation of a concept of a risk measure based on a prior notion of acceptable risk and on the specification of a class of admissible management actions that can be implemented in order to reach acceptability. From this perspective, a risk measure represents the minimal cost we have to pay to reach acceptability by means of admissible actions.
Prefazione

La teoria delle misure di rischio ha acquisito un contorno teorico autonomo in seguito alla pubblicazione dell’articolo di Artzner, Delbaen, Eber e Heath nel 1999. Nel contesto di una economia uniperiodale, gli autori di quell’articolo chiamarono misura di rischio ciascuna funzione che assegni alla posizione a scadenza di un agente finanziario – sia che questa rappresenti il bilancio netto di una compagnia finanziaria, il conto profitti-perdite di un singolo portafoglio o il valore a scadenza di un contratto finanziario – l’ammontare minimo di capitale che deve essere raccolto e investito in un dato titolo di riferimento al fine di garantire che la posizione dell’agente sia compatibile con un accettabile livello di rischio stabilito a priori.

La ricerca successiva si è di fatto concentrata su una sottoclasse particolare di misure di rischio, chiamate cash-additive, per cui il titolo di riferimento è un titolo obbligazionario privo di rischio a interesse nullo. La centralità di tali misure di rischio venne giustificata sulla base di una opportuna trasformazione matematica corrispondente a un cambiamento di valuta. In virtù di questa trasformazione, alcuni autori sostennero che la teoria originaria potesse essere ridotta allo studio delle misure di rischio cash-additive e, di fatto, la maggior parte della ricerca successiva si è da allora fondata, esplicitamente o implicitamente, su tale argomento.

La tesi ha tre obiettivi fondamentali. In primo luogo, intendiamo dimostrare che tale argomento è soltanto apparentemente conclusivo e presenta, in effetti, diversi problemi sia da un punto di vista finanziario che matematico, tali da rendere la corrispondente rivendicazione di centralità per le misure di rischio cash-additive di fatto illegittima. In secondo luogo, motivati da tali problemi, ci prefissiamo di (ri)prendere le mosse dalla definizione originaria al fine di costruire una teoria
delle misure di rischio rispetto a un generale titolo di riferimento, senza fare ricorso ad alcun procedimento di cambio di valuta. In terzo luogo, desideriamo porre le fondamenta di una teoria generale che spazi oltre il caso di un singolo titolo di riferimento. In particolare, nonostante la nostra analisi si concentrerà prevalentemente su posizioni espresse in un’unica valuta fissata nel contesto di un mercato uniperiodale, intendiamo offrire una prospettiva unificante sul problema della misurazione del rischio per posizioni finanziarie più generali – per esempio, vettori di posizioni espresse in valute diverse o flussi stocastici di posizioni nel tempo – nel contesto di un mercato multiperiodale.

La cifra caratteristica del nostro approccio consiste nell’articolazione di un concetto di misura di rischio fondato sulla definizione preliminare di rischio accettabile e sulla specificazione di una classe di azioni manageriali ammissibili a cui fare ricorso per garantire un’esposizione al rischio accettabile. In questa prospettiva, una misura di rischio rappresenta il costo minimale per garantire tale esposizione per mezzo di azioni ammissibili.
Acknowledgements

First of all, I wish to express my deep gratitude to Martin Schweizer and Walter Farkas for making it possible for me to pursue my doctoral studies at the Department of Mathematics of ETH Zurich and, at the same time, to have a privileged access to the Department of Banking and Finance of the University of Zurich. Sharing the research atmosphere of both institutions has been most inspiring for me. Moreover, I wish to thank Damir Filipović and Alexander Schied for readily accepting to act as my external examiners and for their valuable feedback on the thesis.

The germinal project behind this thesis can be traced back to a decisive afternoon session where I first met my future collaborator, mentor, and friend Pablo Koch-Medina. In the years to follow I had the joy and the fortune to be accompanied by Pablo through what has become for me a defining experience of exchange and collaboration. This thesis is dedicated to him.

During these years, I had the pleasure to exchange words on mathematics, finance and beyond with several (current and former) ETH and UZH people. In particular, I will remember with joy the discussions with Valeria Bignozzi, Paul Embrechts, Nicoletta Gabrielli, Martin Herdegen, Santiago Moreno-Bromberg, Ariel Neufeld, Michail Rassias, Oleg Reichmann, Mario Šikić, and Marcus Wunsch.

Finally, a note which is partially out of place. When it comes to emerge from the solipsistic spiral of the τέχνη, my favourite interlocutors are: Agostino, Antonella, Nicola, and Ambra. They accompany me through the διάλογος of my days. The thesis is secretly dedicated to them.

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Introduction

The theory of risk measures has become a well-recognized research area since the publication of the landmark paper by Artzner, Delbaen, Eber, Heath [9]. By establishing the foundations of a new theory of risk measurement beyond the classical mean-variance paradigm of Markowitz [80], that paper triggered a fertile research activity with a vast spectrum of applications, ranging from capital adequacy to portfolio selection, capital allocation and pricing and hedging in incomplete markets.\(^1\) In particular, the underlying theoretical framework has become the standard setting for the current debate on modern solvency regimes in both the banking and the insurance world.

The main contribution of [9] was the formulation of a concept of a risk measure endowed with a clear operational interpretation. In the context of a one-period economy where financial positions at maturity – e.g. net capital positions of a financial company, profit-and-loss positions of a single portfolio or payoffs of a given financial contract – are represented by random variables in a space \(\mathcal{X}\), the above four authors defined a risk measure to be a functional \(\rho_{\mathcal{A},S} : \mathcal{X} \rightarrow \mathbb{R}\) of the form

\[
\rho_{\mathcal{A},S}(X) = \inf \{ \lambda S_0 ; \ \lambda \in \mathbb{R}, \ X + \lambda S_T \in \mathcal{A} \}.
\]

Here, \(\mathcal{A}\) is a subset of \(\mathcal{X}\) consisting of those terminal positions that are deemed acceptable from a risk perspective, while \(S = (S_0, S_T)\) represents the price process of a liquidly traded asset that we use as the vehicle to reach acceptability. Hence, the quantity \(\rho_{\mathcal{A},S}(X)\) represents the “minimal” amount of capital we have to

\(^1\)The articulation of a theory of capital requirements was arguably the primitive motivation of [9]. For important contributions to capital allocation and pricing/hedging problems in the spirit of [9], we refer to Kalkbrener [67] and Acerbi [1] or Carr, Geman, Madan [20], respectively.
raise and invest, at inception, in the asset $S$ to meet the acceptability constraint specified by $\mathcal{A}$. In other words, the capital amount $\rho_{A,S}(X)$ can be interpreted as the minimal cost, measured in terms of the chosen reference asset, to ensure acceptability.

The focus soon shifted to the special class of *cash-additive* risk measures, for which the reference asset is a *risk-free bond with zero interest rate*. These risk measures correspond to functionals $\rho_{\mathcal{A}} : \mathcal{X} \to \mathbb{R}$ of the form

$$\rho_{\mathcal{A}}(X) = \inf\{\lambda \in \mathbb{R}, \ X + \lambda \in \mathcal{A}\}.$$ 

As pointed out by Delbaen [29], if we can *discount* future positions in $\mathcal{X}$ by the payoff $S_T$, then every risk measure of the form $\rho_{A,S}$ can be expressed in terms of a cash-additive risk measure. Indeed, setting

$$\hat{X} = \frac{X}{S_T}$$

and defining

$$\hat{\mathcal{A}} = \{\hat{X}; \ X \in \mathcal{A}\},$$

it is straightforward to verify that

$$\rho_{A,S}(X) = S_0 \rho_{\hat{\mathcal{A}}}(\hat{X})$$

for every position $X \in \mathcal{X}$. On the grounds of this observation, it was claimed that the theory of risk measures of the form $\rho_{A,S}$ could be *reduced* to the cash-additive case. In this sense, the preceding discounting argument provides the fundamental legitimation for the focus on cash-additive risk measures shared by most of the subsequent literature, hence representing the cornerstone of what one might call the *cash-additive paradigm.*

**Objective**

The thesis has three fundamental objectives. First, we aim to demonstrate that the above reduction claim is essentially illegitimate by unveiling a variety of financial pitfalls and of problematic mathematical subtleties hidden in the corresponding cash-additive reduction. In particular, we will argue that losing sight of the original mindset has led to abandon the constructive approach proposed by
Artzner, Delbaen, Eber, Heath [9] in favour of a less transparent approach where the role of the acceptance set has been subordinated and the critical problem of the choice of the reference asset has been essentially ignored.

Second, motivated by the failure of this reduction claim, we aim to go beyond the cash-additive paradigm and develop a theory of risk measures with respect to a general reference asset. In order to achieve full generality, we do not restrict the range of admissible reference assets to risk-free or default-free securities but allow for any nonnegative payoff profile. In particular, $S$ may represent a defaultable bond.

Third, by returning to the original constructive approach, we aim to lay the foundations of a general theory of risk measures based on the following primitive specifications:

1. a notion of acceptability for financial positions;
2. a class of admissible management actions.

From this perspective, a risk measure represents the “minimal” cost of reaching acceptability by means of some admissible management action. The admissible action associated with risk measures of the form $\rho_{A,S}$ corresponds to the simplest type of management decision: raising capital to invest in a single (liquid) asset. In the final part of the thesis, we drop the assumption of liquidity and broaden the scope of our research by allowing for multiple reference assets in the context of a general multi-period market.

**Structure**

In addition to this introductory part and to the following section, where we collect the basic notation and terminology we will be using in the sequel, mainly without explicit reference, the dissertation is divided into three chapters.

In Chapter 1 we start by recalling the original framework introduced by Artzner, Delbaen, Eber, Heath [9] and we discuss the subsequent reduction to cash-additive risk measures. After clarifying the corresponding discounting argument, we will proceed to unveil the problems hidden in this reduction procedure. We conclude with a brief overview of the literature.
In Chapter 2, after describing the mathematical model for financial positions, we introduce the central notion of acceptable positions. We discuss several properties of acceptance sets, with a special focus on dual representations. Along with more standard properties, we introduce the notion of numéraire- and surplus-invariance. In particular, we provide a representation result for the corresponding acceptance sets. Finally, we provide a comprehensive analysis of the acceptance sets commonly used in practice and encountered in the financial literature, namely acceptance sets based on Value-at-Risk, Expected Shortfall, Test Scenarios and Expected Utility.

In Chapter 3 we turn to investigate risk measures. We start by focusing on risk measures with respect to a single liquid reference asset. The emphasis is on finiteness, continuity properties and dual representations. We address the problem of the existence of an optimal reference asset minimizing the distance to acceptability. Moreover, we discuss how to extend risk measures on a larger domain if we want to preserve continuity properties and show that this problem is related with a certain form of statistical robustness. These results are then illustrated in the context of each of the acceptance sets introduced at the end of Chapter 2.

Finally, in Section 3.8, we broaden the scope of our research by allowing for multiple reference assets in the context of a general multi-period market. We provide a variety of reduction results showing how to express the corresponding risk measures in terms of risk measures with respect to a single liquid reference “asset”. In the setting of a single illiquid asset we focus on two properties that have been recently advocated capturing the presence of defaultable and illiquid reference assets, namely cash-subadditivity and quasiconvexity. Here, we provide new financial insights that partially challenge the standard interpretation, especially in the case of cash-subadditivity. Finally, we briefly focus on the class of set-valued risk measures and show how to exploit our approach to duality to obtain dual representations without resorting to set-valued analysis.

References

The dissertation is part of a broader project on risk measures conducted jointly with Pablo Koch-Medina. The core objective of the project is to develop a general
theory of risk measures beyond the cash-additive framework, with applications to capital adequacy, portfolio selection, capital allocation and pricing/hedging in incomplete markets.

The thesis comprises both published and original results. The former ones have been selected from a series of joint papers with Walter Farkas and Pablo Koch-Medina, namely [41], [42], [43], with Pablo Koch-Medina and Santiago Moreno-Bromberg, namely [68], and with Pablo Koch-Medina, namely [69].

The results in [41] are used in part of Section 3.7.1 and motivate the bulk of Section 3.4. The results in [42] constitute the bulk of Section 3.3 and Section 3.7. The results in [43] are used in Section 3.5.\(^2\)

The setting of [68] is recalled in Section 2.2.5 and the corresponding results are used in Section 2.3.4. The special dual representation for surplus-invariant acceptance sets, which is the main result of [68], is proved here without any separability assumption. Finally, the results in [69] are presented in Section 3.6. The corresponding applications to risk measures based on Expected Utility are extended here to the case of a general – not necessarily bounded from above – utility function.

The original contribution of the thesis is mainly contained in the following parts. First of all, with the partial exception of Section 1.3.1, the arguments presented in Chapter 1, which provide a detailed explanation of the problems hidden in the cash-additive reduction and thus constitute the very motivation for going beyond cash-additivity, are not found in the above papers.

In Chapter 2, the discussion on numéraire-invariance and the corresponding results are new and will constitute the subject of a joint paper in preparation; see [71]. Moreover, most of the duality results in Section 2.3 are also new.

In Chapter 3, the range of finiteness and continuity results in Section 3.3 has been expanded compared to [42]. In addition, most of the results on risk measures based on Test Scenarios and Expected Utility in Section 3.7 are new. The bulk of Section

\(^2\)In fact, the results in [42] and [43] are presented here under a more general mathematical architecture, which corresponds to assuming that the underlying model space is equipped with a quasiorder, i.e. a reflexive, transitive, binary relation, instead of an order, i.e. a quasiororder which is additionally antisymmetric. As argued later, this higher generality proves relevant when we work in the context of markets with frictions.
3.8 is new. In particular, the extension of risk measures to a general multi-period market described in the final part provides a considerable generalization of the one-period frictionless market studied in [43]. These general risk measures will be further pursued within the context of the above-mentioned research project. Finally, the results on set-valued risk measures in Section 3.8.7 are also new.
Chapter 1

Beyond the cash-additive paradigm

The objective of this chapter is to motivate the study of risk measures beyond the standard cash-additive setting. To this end, we start by recalling the original framework introduced in Artzner, Delbaen, Eber, Heath [9] and discuss the subsequent reduction to cash-additive risk measures. The prominent focus on cash-additivity was justified on the grounds of a suitable discounting argument. After discussing this argument in detail, we will proceed to unveil a variety of problematic aspects hidden in the corresponding discounting procedure, showing that the central role assigned to cash-additivity is essentially illegitimate.¹

1.1 The original framework

In this section we recall the original framework articulated in Artzner, Delbaen, Eber, Heath [9], with a special focus on the application of risk measures in a capital adequacy context.

¹The structure of this chapter is the result of innumerable discussions with Pablo Koch-Medina and may represent the manifesto of our common project. I’m extremely grateful for his suggestions on this fundamental part, which considerably improved the quality of a former version.
1.1.1 Capital adequacy and regulation

Liability holders of a financial institution – and regulators on their behalf – are concerned that the institution may fail to fully honor its future obligations. This will be the case if the institution’s capital position – the value of assets net of liabilities – becomes negative in some future state of the economy. To address this concern, financial institutions are required to hold an adequate amount of risk capital whose function is to absorb unexpected losses thereby ensuring an acceptable security level against insolvency. From a modelling perspective, the natural question in this respect is: how to articulate a mathematical theory of capital requirements providing these qualitative objectives with a clear quantitative formulation?

1.1.2 Acceptance sets and reference instruments

To address the preceding question, a clear program was proposed by Artzner, Delbaen, Eber, Heath in their seminal paper [9]:

Sets of acceptable future net worths are the primitive objects to be considered in order to describe acceptance or rejection of a risk. We present here how, given some “reference instrument”, there is a natural way to define a measure of risk by describing how close or how far from acceptance a position is.

More precisely,

We define the measure of risk of an unacceptable position [...] as the minimum extra capital which, invested in the reference instrument, makes the future value of the modified position become acceptable.

The program consists of two fundamental steps: the definition of acceptable positions and the specification of a reference investment instrument to use as the vehicle to reach acceptability.

In a capital adequacy context, the set of acceptable positions is to be specified by the regulator and establishes the fundamental threshold between companies that are adequately capitalized and companies that are not. The reference instrument is to be chosen by the management of the company, perhaps in agreement
with regulators. In the event that a position is not acceptable, the management will need to raise capital and invest in this pre-specified instrument in order to reach acceptability. The minimal amount of capital to raise for this purpose will determine the corresponding level of risk capital.

### 1.1.3 Measuring the distance to acceptability

Following [9], we formalize the concept of a risk measure described above in the setting of an economy with initial date \( t = 0 \) and terminal date \( t = T \).

Let \( \mathcal{X} \) be a real linear space consisting of random variables \( X : \Omega \to \mathbb{R} \) on a given probability triple \((\Omega, \mathcal{F}, \mathbb{P})\). The elements of \( \mathcal{X} \) represent capital positions (assets net of liabilities) of financial institutions at time \( T \). It is important to remark that financial positions are expressed in a fixed unit of account. For instance, we could take the reference domestic currency as the accounting unit.

In this setting, an acceptance set is any nontrivial subset \( \mathcal{A} \) of \( \mathcal{X} \) satisfying

\[
X \in \mathcal{A}, \; X \leq_{\mathbb{P}} Y \implies Y \in \mathcal{A}.
\]

The set \( \mathcal{A} \) consists of those final positions that are acceptable from a regulatory perspective. In this sense, the “monotonicity” requirement above captures the basic intuition according to which a company should be deemed adequately capitalized whenever its net profile is better than the net profile of a company that has already passed the capital adequacy test specified by \( \mathcal{A} \).

Finally, the reference instrument is represented by a couple \( S = (S_0, S_T) \) where \( S_0 > 0 \) is the initial value and \( S_T \in \mathcal{X} \) is the terminal (random) payoff of the instrument. The only assumption on the payoff \( S_T \) is that

\[
\mathbb{P}(S_T \geq 0) = 1.
\]

This assumption is general enough to accommodate a wide variety of choices, ranging from bonds and stocks to derivative instruments. In particular, the asset \( S \) need not be risk-free, i.e. it is not assumed to have a deterministic payoff. This is important since the existence and/or availability of such “riskless” assets can be questioned. In particular, \( S \) may represent a defaultable bond.\footnote{In the original paper, the set \( \Omega \) was assumed to be finite and \( S_T \) was allowed to take only strictly positive values.}

\footnote{We refer to Black [16] for an early study of capital markets without risk-free securities.}
Definition 1.1.1. The risk measure associated to $A$ and $S$ is the mapping $\rho_{A,S} : \mathcal{X} \to \mathbb{R}$ defined by

$$
\rho_{A,S}(X) := \inf\{\lambda S_0 ; \lambda \in \mathbb{R}, \ X + \lambda S_T \in A\}.
$$

The quantity $\rho_{A,S}(X)$ represents a capital amount expressed in the same accounting unit we have fixed at the beginning. Consistently with the above interpretation, it corresponds to the “minimal” amount of capital we have to raise and invest in the reference asset at inception in order to guarantee acceptability. More precisely, if positive it will define an injection of capital, if negative it will mean that capital can be extracted, by shorting $S$, without compromising acceptability.

From a capital adequacy perspective, the acceptance set consists of those positions that are deemed to provide a reasonable security against insolvency. In this sense, the quantity $\rho_{A,S}(X)$ can be interpreted as a measure of the risk of $X$ in terms of the “distance” of $X$ to acceptability. This “distance” is quantified by the capital amount we have to raise and invest in the asset $S$ – the chosen “yardstick” – to close the gap to acceptability.

A risk measure of the form $\rho_{A,S}$ is always decreasing, i.e.

$$
X \leq_{\mathbb{P}} Y \implies \rho_{A,S}(X) \geq \rho_{A,S}(Y).
$$

This property follows from the monotonicity of the underlying acceptance set and reflects the basic intuition according to which a company with a better capital position should need a lower capital buffer against insolvency. In particular, if a position $X$ is acceptable, i.e. $X \in A$, then $\rho_{A,S}(X) \leq 0$, in line with the interpretation of the quantity $\rho_{A,S}(X)$ as a “distance” to acceptability.

Moreover, $\rho_{A,S}$ is $S$-additive, i.e.

$$
\rho_{A,S}(X + \lambda S_T) = \rho_{A,S}(X) - \lambda S_0 \quad \text{for all } \lambda \in \mathbb{R}.
$$

This property shows that investing in the reference asset has an additive impact on the capital requirement figure. In particular, by raising capital and taking a long position in the asset $S$ we can always diminish the level of risk capital associated with a position $X$, provided that $\rho_{A,S}(X) < \infty$.

Remark 1.1.2. The property of $S$-additivity was called translation invariance in [9]. We prefer the term “$S$-additivity” for two reasons. First, because $\rho_{A,S}$
is not “invariant” with respect to translations along the direction $S_T$, but rather
displays an “additive” behavior. Second, because we prefer to maintain an explicit
(terminological) link with the chosen reference asset. This is particularly useful
when alternative choices of the reference asset need to be contemplated.

1.2 The cash-additive reduction

In this section we introduce the special class of cash-additive risk measures and
show how the general risk measures introduced in Artzner, Delbaen, Eber, Heath
[9] can be expressed, by means of a change of numéraire, in terms of these special
risk measures. The corresponding discounting argument constitutes the essential
legitimation for the prominent focus assigned to cash-additive risk measures in
the literature. Later, we will challenge the conclusive validity of this argument
and point out a variety of problems and pitfalls hidden in the corresponding cash-
additive reduction.

1.2.1 Cash-additive risk measures

We continue to denote by $\mathcal{X}$ the space of financial positions and additionally
assume that $1_\Omega \in \mathcal{X}$. The constant random variable equal to $\lambda \in \mathbb{R}$ will be
sometimes denoted simply by $\lambda$ instead of $\lambda 1_\Omega$.

**Definition 1.2.1.** A risk measure $\rho_{A,S} : \mathcal{X} \to \mathbb{R}$ is said to be cash-additive if
$S = (1, 1_\Omega)$. In this case, we simply write $\rho_{A,S} = \rho_A$, i.e.

$$\rho_A(X) := \inf\{\lambda \in \mathbb{R} ; \ X + \lambda \in A\}.$$ 

The special reference asset $S = (1, 1_\Omega)$ can be interpreted as a risk-free bond with
zero interest rate. When the underlying unit of account is some monetary unit,
this special asset is also referred to as cash, justifying the chosen terminology.
Hence, the quantity $\rho_A(X)$ represents the “minimal” amount of capital we have
to raise and invest in such a risk-free bond to ensure acceptability.

The corresponding cash-additivity property takes the form

$$\rho_A(X + \lambda) = \rho_A(X) - \lambda \quad \text{for all } \lambda \in \mathbb{R}.$$
Remark 1.2.2. In the literature, several alternative denominations have been used in place of cash-additivity, including translation property in [63], translation invariance in [48], translability in [51], translation equivariance in [85], translational covariance in [2], monetary property in [30], cash-invariance in [50].

1.2.2 The discounting argument

The cash-additive framework has become the standard axiomatic setting for risk measures. However, at a first sight, it seems less general than the original setting. Hence, the following question arises naturally: why did the focus shift from general reference assets to a riskless bond with zero interest rate?

An answer to the preceding question is provided by Delbaen [29]. In the remark after Definition 2.1 in that paper, the author refers to Artzner, Delbaen, Eber, Heath [9] for an interpretation and notes that

Here we are working in a model without interest rate, the general case can “easily” be reduced to this case by “discounting”.

No additional details were put forward in [29]. We also refer to the recent lecture notes by Delbaen [31]:

For simplicity, we also suppose that all (random) amounts of money available tomorrow have already been discounted. This practice [...] avoids a lot of notational problems. The discounting can take place with an arbitrary asset, provided the price is strictly positive. So we can use a “sure” bank account with known interest rate at time 0. But we could also use an asset with a return that is only known at date 1. After discounting, the interest rate disappears from the calculations and hence the discounting is equivalent to assume that the interest rate is zero.

The corresponding discounting argument can be formalized as follows. Assume we can discount future financial positions by $S_T$, i.e. we can express every position in terms of units of the reference asset.\footnote{The term “discounting” is sometimes used to refer to the special situation where the numéraire asset is a (zero-coupon) bond or a money market. In this thesis, following a rather}
to $X \in \mathcal{X}$ is then

$$\hat{X} := \frac{X}{S_T}.$$ 

In particular, note that $\hat{S}_T = 1_\Omega$. Define the space of discounted positions

$$\hat{\mathcal{X}} := \{\hat{X} \mid X \in \mathcal{X}\}$$

and the discounted acceptance set

$$\hat{\mathcal{A}} := \{\hat{X} \mid X \in \mathcal{A}\}.$$ 

Then, it is easy to see that

$$\rho_{\mathcal{A}, S}(X) = S_0 \rho_{\hat{\mathcal{A}}}((\hat{X})).$$ 

The preceding identity shows that we can express every risk measure of the form $\rho_{\mathcal{A}, S}$, originally defined on $\mathcal{X}$, in terms of a cash-additive risk measure defined on the discounted space $\hat{\mathcal{X}}$. This procedure will be therefore referred to as the cash-additive reduction. Setting $S_0 = 1$ for visual convenience, we can picture this reduction by means of the following commutative diagram

where the “hat” operator $\hat{\cdot} : \mathcal{X} \rightarrow \hat{\mathcal{X}}$ corresponds to discounting by $S_T$. This operator will be called the discounting operator. In other words, every risk measure $\rho_{\mathcal{A}, S}$ can be expressed as the composition between a discounting operator and a cash-additive risk measure.

It is important to stress that, when passing from $\mathcal{X}$ to $\hat{\mathcal{X}}$, we are changing the original unit of account: the positions in $\hat{\mathcal{X}}$ are expressed in units of the asset $S$. Hence, $\rho_{\hat{\mathcal{A}}}((\hat{X}))$ is also expressed in discounted terms. In particular, the “cash-additivity” of $\rho_{\hat{\mathcal{A}}}$ is nothing but the discounted form of $S$-additivity. In this light, the cash-additive theory emerges essentially as a theory for discounted positions.

standard terminology in mathematical finance, we use the term “discounting” as a synonym of “changing the numéraire”. In particular, one should not confuse a general reference asset with a bond or a money market. We refer to Vecer [98] for a detailed treatment of discounting techniques.
In conclusion, the cash-additive reduction would seem to show that the original mathematical problem can be simplified by a straightforward change of numéraire. This simplification mechanism constitutes the main motivation for investigating cash-additivity, as corroborated by the previous quotation from Delbaen [29] referring to the theory of risk measures of the form $\rho_{A,S}$ as “the general theory”.

**Remark 1.2.3.** The preceding discussion highlights the inadequacy of the term monetary property, see e.g. Delbaen [31], as an alternative to cash-additivity. Indeed, the quantity $\rho_A(\tilde{X})$ is generally not expressed in monetary units but in units of the asset $S$. Moreover, even if the underlying unit of account is a currency and $S_0 = 1$, then both $\rho_{A,S}$ and $\rho_A$ would be expressed in monetary terms.

### 1.3 Problems and pitfalls of discounting

The preceding discounting argument provides the fundamental legitimation to the prominent focus assigned to cash-additive risk measures in the literature. The objective of this section is to investigate whether this exclusive focus is justified or not.

Since the purpose of the cash-additive reduction is to translate the original mathematical problem into a more tractable one, the exclusive focus on cash-additivity will be justified only if we are able to show that the general theory can be reduced to the cash-additive theory without loss of generality. In this respect, the key financial question is:

*Can we ensure that every relevant financial problem in the context of the general theory can be still formulated in the cash-additive setting?*

The complementary key mathematical question is:

*Can we ensure that every relevant mathematical property in the context of the general theory can be still accounted for in the cash-additive setting?*

In the following sections we will provide abundant arguments showing that, unless the reference asset is a risk-free bond, neither question has a positive answer. In other words, the cash-additive theory fails to provide a conclusive perspective on the original theory of risk measures. This will motivate and justify the study of general risk measures beyond the cash-additive setting.
1.3 Problems and pitfalls of discounting

1.3.1 Discounting is not always possible

We start by noting that changing the numéraire is, sometimes, just not possible. Indeed, the discounting argument and the consequent cash-additive reduction cannot be applied whenever

\[ P(S_T = 0) > 0. \]

In this case the reference asset does not qualify to be a numéraire asset. From a financial perspective, this situation is far from being irrelevant. In addition to disallowing the choice of derivative instruments of option type, whose importance becomes manifest when risk measures are used in a pricing and/or hedging environment, the above “zero recovery” condition may have a natural interpretation also in the context of fixed-income instruments, as illustrated by the following example.

Example 1.3.1. Assume the asset \( S \) represents a defaultable bond maturing at time \( T \). In the event of default, the bond holder will only recover a portion of the entire face value. Hence, we can represent the payoff \( S_T \) in the form

\[ S_T = RF, \]

where the number \( F > 0 \) corresponds to the face value and the random variable \( R : \Omega \to [0,1] \) to the recovery rate. Since the bond is defaultable, we have \( P(R < 1) > 0 \). To be able to capture all realistic default “events”, we need to allow for the possibility that

\[ P(R = 0) = P(S_T = 0) > 0. \]

For instance, this zero recovery condition may describe a full default, in which case no payment will be made to the bond holder. Alternatively, in a practical context, it may describe a milder form of default where the contractual payment, or the recovery, is guaranteed but is delayed until after the modelling horizon \( T \). In this case, we are forced to model zero recovery at time \( T \), which is the horizon of the capital adequacy assessment.

The preceding example shows that, in important financial situations, the artifice of changing the numéraire and the consequent discounting reduction fail to work. Hence, not every relevant financial question arising in the general model can be articulated in the cash-additive language.
1.3.2 Losing control on the model space

Assume we can discount by $S_T$. Unless the payoff $S_T$ is bounded away from zero, i.e. $\mathbb{P}(S_T \geq \varepsilon) = 1$ for some $\varepsilon > 0$, when changing the numéraire we may abandon the original ambient space $\mathcal{X}$, i.e.

$$\hat{\mathcal{X}} \not\subseteq \mathcal{X}.$$  

The following simple example makes the point in the context of bounded positions.

**Example 1.3.2.** Take $\mathcal{X} = L^\infty$ and assume $S_T$ is not bounded away from zero. Then, the discounted counterpart of $1_\Omega$ is no longer bounded (sometimes not even integrable!).

The main consequence of the above observation is that, in general, we cannot equip the discounted space $\hat{\mathcal{X}}$ with the same structure of the original space $\mathcal{X}$. In particular, while the discounting operator allows to transfer the algebraic structure of $\mathcal{X}$ onto $\hat{\mathcal{X}}$, there is no natural way to transfer the corresponding topological structure. At a general level, we can only equip $\hat{\mathcal{X}}$ with the topology of convergence in probability, induced by the obvious embedding into $L^0$. However, this implies that some important questions for $\rho_{A,S}$ – for instance related to continuity or duality – may become difficult to address, or even meaningless, in the corresponding cash-additive model.\(^5\)

A natural way to proceed would be to specialize our analysis to the particular profile of the payoff $S_T$ and investigate, case by case, whether the corresponding discounted space $\hat{\mathcal{X}}$ could be embedded into one of the spaces commonly encountered in the cash-additive literature, namely spaces of integrable functions. This already suggests that passing to a cash-additive setting is at least inefficient. Moreover, the price to pay in this case is that the discounting operator and/or, more typically, its inverse, may fail to be continuous.

**Example 1.3.3.** Take $\mathcal{X} = L^\infty$ and assume $S_T$ is not bounded away from zero. In this case, the discounted space $\hat{\mathcal{X}}$ may contain nonintegrable positions. The

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\(^5\)It is well-known that, in a nonatomic context, the topological dual of $L^0$ reduces to zero. In particular, no cash-additive risk measure on $L^0$ can be finitely valued or continuous when convex, see also Remark 3.3.41 below.
1.3 Problems and pitfalls of discounting

The most natural way to recuperate integrability is to switch to a new (equivalent) probability $Q$ with density
\[ \frac{dQ}{dP} = \frac{S_T}{\mathbb{E}[S_T]} . \]
In this case, the space $\hat{X}$ can be naturally embedded into $L^1(Q)$. However, the inverse of the discounting operator fails to be continuous. To see this, consider a sequence $(A_n)$ of measurable sets with strictly positive probability shrinking to zero and define $X_n = 1_{A_n}$ for all $n \in \mathbb{N}$. Then $(\hat{X}_n)$ converges to 0 in $L^1(Q)$ but $(X_n)$ has no limit in the original space $L^\infty$.

The lack of (bi)continuity of the discounting operator has important consequences. The next example shows that a risk measure $\rho_{A,S}$ may be continuous even if $\rho_{\hat{A}}$ is not, highlighting that some important mathematical properties of the original risk measures may remain ultimately undetectable after the cash-additive reduction.

**Example 1.3.4.** Take $\mathcal{X} = L^\infty$ and assume $S_T$ is not bounded away from zero. Define $A = \{S_T \leq \varepsilon\}$ for some $0 < \varepsilon < \|S_T\|_\infty$ and consider the acceptance set
\[ \mathcal{A} = \{X \in L^\infty ; \mathbb{P}(X 1_A \geq 0) = 1\} . \]

Taking $X = 1_{\Omega}$, we claim that $\rho_{A,S}$ is continuous at $X$, under the usual topology of $L^\infty$, while $\rho_{\hat{A}}$ is not continuous at $\hat{X}$, under the topology of $L^1(Q)$ introduced in the example above. First, it is easy to verify that
\[ \rho_{A,S}(X) = -\frac{S_0}{\|S_T 1_A\|_\infty} . \]

Now, take $Y \in L^\infty$. If $\|Y - X\|_\infty \leq r$ for some $r \in (0,1)$, then
\[ -\frac{S_0(1 + r)}{\|S_T 1_A\|_\infty} = \rho_{A,S}(X + r) \leq \rho_{A,S}(Y) \leq \rho_{A,S}(X - r) = -\frac{S_0(1 - r)}{\|S_T 1_A\|_\infty} . \]

This immediately shows that $\rho_{A,S}$ is continuous at $X$. On the other side, set $X_n = 1_{A_n}$ where $A_n = \{S_T \geq \frac{1}{n}\}$ for $n \in \mathbb{N}$. Note that $(\hat{X}_n)$ converges to $\hat{X}$ in $L^1(Q)$ because $\mathbb{P}(A_n) \to 1$ as $n \to \infty$. Since the complement of $A_n$ is eventually contained in $A$, we have
\[ \mathbb{P}\left(\left(\hat{X}_n - \frac{1}{\|S_T\|_\infty}\right) 1_A \geq 0\right) \neq 1 \]
for $n$ large enough. As a result, we eventually obtain

$$\rho_{\hat{A}}(\hat{X}_n) \geq - \frac{1}{\|S_T\|_{\infty}} > - \frac{1}{\|S_T1_A\|_{\infty}} = \rho_{\hat{A}}(\hat{X}) ,$$

showing that $\rho_{\hat{A}}$ cannot be continuous at $\hat{X}$ under the topology of $L^1(Q)$.

A natural way to resolve this impasse is to equip the space $\hat{X}$ with the finest topology under which both the discounting operator and its inverse are continuous. In this way, the discounting operator would automatically establish, through the cash-additive reduction, a one-to-one correspondence of topological properties between $\rho_{A,S}$ and its cash-additive version $\rho_{\hat{A}}$. However, the relevant topological properties in the discounted model – for instance related to the topological role of the element $\hat{S}_T = 1_\Omega$, which is the payoff of the underlying reference “asset” in the cash-additive setting – would inextricably depend on the topological properties of the payoff $S_T$ in the original model. In particular, we would need to return to the “undiscounted” model $X$ to understand the associated cash-additive setting, showing that the cash-additive reduction would provide no simplification in this case.

In conclusion, the problem of finding a good topology for the discounted space has led us to unveil that, in most circumstances, we may lose important (topological) properties of risk measures of the form $\rho_{A,S}$ after the cash-additive reduction. To avoid this, we need to equip the discounted space with a topological structure that would force us to constantly refer to the topology of the original reference space, annihilating the benefit we could hope to gain by passing to cash-additivity.

### 1.3.3 Losing the structure of acceptability

Even if the discounted space can be naturally embedded into the original reference space, say if $S_T$ is bounded away from zero, when discounting we may alter considerably the structure of the original acceptance set, i.e.

$$\hat{A} \neq A .$$

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6This topology corresponds to the so-called final topology associated to the discounting operator and is such that a net $(\hat{X}_\alpha)$ converges to $\hat{X}$ if and only if the corresponding net $(X_\alpha)$ converges to $X$ in the original space.
The following example shows that acceptability based on Expected Shortfall is generally not preserved by discounting.

**Example 1.3.5.** Take $X = L^\infty$ and fix $\alpha \in (0, 1)$. We consider the acceptance set $\mathcal{A} = \mathcal{A}_{ES}(\alpha)$ based on Expected Shortfall at level $\alpha$, see Section 2.4.2, i.e.

$$\mathcal{A} = \{ X \in L^\infty ; \ ES_\alpha(X) \leq 0 \} .$$

Assume $\alpha < \frac{1}{2}$, which is the most relevant case for applications, and consider two disjoint measurable sets $A$ and $B$ satisfying $P(A) = P(B) = \frac{\alpha}{2}$. Moreover, define $C = (A \cup B)^c$ and note that $P(C) > \alpha$. The payoff of the reference asset is given by $S_T = 1_A + 31_B + 21_C$ so that $S_T$ is bounded away from zero. Finally, define the positions $X = -1_A + 21_B + 1_C$ and $Y = 21_A - 1_B + 1_C$. Then, it is easy to verify that

$$ES_\alpha(X) = 0 \quad \text{and} \quad ES_\alpha \left( \frac{X}{S_T} \right) = \frac{1}{4}$$

showing that $\hat{\mathcal{A}} \not\subseteq \mathcal{A}$, and

$$ES_\alpha(Y) = 0 \quad \text{and} \quad ES_\alpha(S_T Y) = \frac{1}{2}$$

showing that $\mathcal{A} \not\subseteq \hat{\mathcal{A}}$. Hence, in case of Expected Shortfall we may both lose and gain acceptability by passing to discounted positions.

**Remark 1.3.6.** The possible dependence of acceptability on the choice of the numéraire was already pointed out, even if not further investigated, by Artzner, Delbaen, Eber, Heath [9]: “acceptance sets allow us to address a question of importance to an international regulator and to the risk manager of a multinational firm, namely the invariance of acceptability of a position with respect to a change of currencies”. The same issue has been raised again in Artzner, Delbaen, Koch-Medina [10], where the numéraire dependence of acceptance sets based on Expected Shortfall was already highlighted.

Sometimes, the dependence on the unit of account is extremely strong and we systematically lose the structure of acceptability whenever the payoff of the reference asset is genuinely random. This is the case of acceptance sets based on Expected Shortfall.
Example 1.3.7. Assume $X = L^\infty$ and fix $\alpha \in (0, 1)$. We still consider the acceptance set $A = A_E^{\infty}(\alpha)$ introduced in the previous example. To avoid the trivial case $A = L^\infty_+$, we assume that $\mathbb{P}(E) \in (0, \alpha)$ for some $E \in \mathcal{F}$; see Proposition 2.4.11. Finally, take $S_T \in L^\infty_+$ and assume $\mathbb{P}(S_T \geq \varepsilon) = 1$ for a suitable $\varepsilon > 0$. Then, we claim that

$$\hat{A} = A \iff S_T \text{ is constant}.$$ 

Since $A$ is a cone, we only need to show the implication “$\implies$”. To this end, assume $S_T$ is not constant so that we find $\lambda > 0$ such that $\mathbb{P}(S_T \leq \lambda) > 0$ and $\mathbb{P}(S_T > \lambda) > 0$. We distinguish three cases.

First, assume $\mathbb{P}(S_T \leq \lambda) < \alpha$. In this case, set $A = \{S_T \leq \lambda\}$ and $B = \{S_T \geq \xi\}$ for a suitable $\xi > \lambda$ such that $\mathbb{P}(A) + \mathbb{P}(B) \geq \alpha$. Moreover, set $C = (A \cup B)^c$. Then, take strictly positive numbers $a, b, c \in \mathbb{R}$ satisfying

$$\frac{\lambda \alpha - \mathbb{P}(A)}{\xi} < \frac{a}{b} \leq \frac{\alpha - \mathbb{P}(A)}{\mathbb{P}(A)} \quad \text{and} \quad c > \frac{b\varepsilon}{\xi}.$$ 

Finally, consider the position given by $X = -a1_A + b1_B + c1_C$. It is not difficult to see that

$$\text{ES}_\alpha(X) = \frac{1}{\alpha}(a\mathbb{P}(A) - b(\alpha - \mathbb{P}(A))) \leq 0.$$ 

On the other side, since

$$\hat{X} \leq -\frac{a}{\lambda}1_A + \frac{b}{\xi}1_B + \frac{c}{\varepsilon}1_C,$$ 

it follows that

$$\text{ES}_\alpha(\hat{X}) \geq \frac{1}{\alpha} \left( \frac{a}{\lambda} \mathbb{P}(A) - \frac{b}{\xi} (\alpha - \mathbb{P}(A)) \right) > 0.$$ 

In conclusion, $\hat{X} \in \hat{A}$ but $\hat{X} \notin A$ showing that $\hat{A} \not\subseteq A$.

Second, assume that $\mathbb{P}(S_T \leq \lambda) \geq \alpha$ but $\mathbb{P}(S_T > \lambda) < \alpha$. In this case, we define $A = \{S_T \geq \xi\}$ for some $\xi > \lambda$ such that $\mathbb{P}(A) > 0$. Moreover, we consider $B = \{S_T \leq \lambda\}$ and $C = (A \cup B)^c$. Finally, we take strictly positive numbers $a, b, c \in \mathbb{R}$ satisfying

$$\frac{\alpha - \mathbb{P}(A)}{\mathbb{P}(A)} < \frac{a}{b} \leq \frac{\xi \alpha - \mathbb{P}(A)}{\lambda \mathbb{P}(A)} \quad \text{and} \quad c > \frac{b\|S_T\|_\infty}{\lambda}.$$
Now, consider the position \( X = -a1_A + b1_B + c1_C \). It easily follows that

\[
\text{ES}_\alpha(X) = \frac{1}{\alpha} \left( a \mathbb{P}(A) - b(\alpha - \mathbb{P}(A)) \right) > 0.
\]

On the other side, we have

\[
\hat{X} \geq -\frac{a}{\xi} 1_A + \frac{b}{\lambda} 1_B + \frac{c}{\|S_T\|_\infty} 1_C,
\]

implying that

\[
\text{ES}_\alpha(\hat{X}) \leq \frac{1}{\alpha} \left( \frac{a}{\xi} \mathbb{P}(A) - \frac{b}{\lambda}(\alpha - \mathbb{P}(A)) \right) \leq 0.
\]

In conclusion, \( \hat{X} \in A \) but \( \hat{X} \notin \hat{A} \) showing that \( A \not\subseteq \hat{A} \).

Third, assume that \( \mathbb{P}(S_T \leq \lambda) \geq \alpha \) and \( \mathbb{P}(S_T > \lambda) \geq \alpha \). In this situation we either have \( \mathbb{P}(E \cap \{S_T \leq \lambda\}) > 0 \), in which case we set \( A = E \cap \{S_T \leq \lambda\} \) and \( B = \{S_T \geq \xi\} \) for a suitable \( \xi > \lambda \) such that \( \mathbb{P}(A) + \mathbb{P}(B) \geq \alpha \) and we are thus back to the first case, or we have \( \mathbb{P}(E \cap \{S_T > \lambda\}) > 0 \), in which case we set \( A = E \cap \{S_T \geq \lambda\} \) for a suitable \( \xi > \lambda \) such that \( \mathbb{P}(A) > 0 \) and \( B = \{S_T \leq \lambda\} \) and we are back to the second case.

The preceding discussion has an important consequence: if acceptability depends on the numéraire, there is typically no connection between \( \rho_{A,S} \) and the cash-additive risk measure \( \rho_A \) applied to positions discounted by \( S_T \), i.e. we may have

\[
\rho_{A,S}(X) \neq S_0 \rho_A(\hat{X}).
\]

In other words, if we are interested in risk measures of the form \( \rho_{A,S} \) for some of the standard acceptance sets \( \mathcal{A} \), we can hope to exploit the cash-additive theory developed in the context of those standard acceptance sets only in the rare situation when \( \hat{A} = A \). Sometimes, for instance if \( \mathcal{A} \) is based on Expected Shortfall, this is never the case unless the payoff \( S_T \) is constant!

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7This argument unveils a critical misunderstanding implicit in much of the cash-additive literature: we cannot simply apply a standard cash-additive risk measure \( \rho_A \) to discounted positions and assume that \( \rho_A(\hat{X}) \) represents a number of units of the “numéraire” asset we have to purchase or sell in order to align \( X \) with the acceptability constraint specified by \( \mathcal{A} \). In fact, this would implicitly stipulate that \( \rho_{A,S}(X) = S_0 \rho_A(\hat{X}) \)!

8In Section 2.3.3 below we will characterize, in the convex case, the class of acceptance sets that fulfil this property.
Example 1.3.8. Take $\mathcal{X} = L^\infty$ and assume $S_T$ is bounded away from zero. In this case, $\hat{\mathcal{X}}$ is naturally embedded into $L^\infty$. We still consider the acceptance set $\mathcal{A} = A_{\text{ES}}(\alpha)$ introduced in the previous examples. Assume $\hat{\mathcal{A}} \neq \mathcal{A}$. We have seen in the preceding example that this is always the case whenever $S_T$ is not constant (and we are not in the trivial situation $\mathcal{A} = L^\infty_+$). Since $\mathcal{A}$ is well-known to be closed in $L^\infty$, the discounted acceptance set $\hat{\mathcal{A}}$ is also closed in $L^\infty$ and we have the standard relations

$$\mathcal{A} = \{ X \in L^\infty; \rho_\mathcal{A}(X) \leq 0 \} \quad \text{and} \quad \hat{\mathcal{A}} = \{ X \in L^\infty; \rho_{\hat{\mathcal{A}}}(X) \leq 0 \}.$$

As a consequence, $\rho_\mathcal{A}$ and $\rho_{\hat{\mathcal{A}}}$ cannot coincide, for otherwise $\hat{\mathcal{A}} = \mathcal{A}$. Hence, there must exist some $X \in L^\infty$ such that

$$\rho_{\mathcal{A},S}(X) = S_0 \rho_{\hat{\mathcal{A}}}(\hat{X}) \neq S_0 \rho_\mathcal{A}(\hat{X}),$$

where the first equality follows from the cash-additive reduction.

In conclusion, we have shown that the structure of acceptance sets is typically not preserved through the process of discounting. In particular, we may start with a risk measure associated to a standard acceptance set and, after discounting, end up with a cash-additive risk measure associated to a nonstandard acceptance set. The case of Expected Shortfall provides a cautionary example in this respect: acceptability is preserved if and only if the payoff of the reference asset is constant. In order to understand these nonstandard cash-additive risk measures we would invariably have to return to the original framework, highlighting once more, from a different angle, the inefficiency of the cash-additive reduction.

1.3.4 Losing properties of the acceptance set

Since the discounting process may alter the structure of acceptability, when changing the numéraire it is possible to lose some of the original properties of the acceptance set. Being a morphism of algebraic structures, the discounting operator will preserve every algebraic property. For instance, we have

$$\mathcal{A} \text{ convex/conic} \implies \hat{\mathcal{A}} \text{ convex/conic}.$$

However, properties that are related to the probabilistic or distributional structure of $\mathcal{A}$ get almost systematically lost whenever the payoff $S_T$ is genuinely random.
This is the case of law-invariance, i.e. the property for which acceptability only depends on the distribution of financial positions. Indeed, we may have

\[ \mathcal{A} \text{ law-invariant } \iff \hat{\mathcal{A}} \text{ law-invariant}. \]

**Example 1.3.9.** Consider the setting of Example 1.3.5. It is well-known that \( \mathcal{A} \) is law-invariant and, clearly, \( X \) and \( Y \) have the same distribution. However,

\[ ES_\alpha(S_T X) = -\frac{1}{2} \text{ while } ES_\alpha(S_T Y) = \frac{1}{2}, \]

implying that \( X \in \hat{\mathcal{A}} \) whereas \( Y \notin \hat{\mathcal{A}} \).

\[ \square \]

To exploit the “lost” properties of the original acceptance set we would need to undo the cash-additive reduction, providing additional evidence that changing the numéraire may considerably impoverish the structure of the original problem.

### 1.3.5 Losing sight of the operational dimension

The standard definition of a cash-additive risk measure commonly encountered in the literature is different from the one presented above. In fact, the cash-additive theory has been essentially developed without any explicit connection with the original motivating problem. In particular, the prominent role of the acceptance set and, especially, of the reference asset has been often neglected, if not ignored.\(^9\) The usual definition can be stated as follows. Here, we denote by \( \mathcal{X} \) a linear space of random variables on a given probability triple \( (\Omega, \mathcal{F}, \mathbb{P}) \) and assume \( 1_\Omega \in \mathcal{X} \).

**Definition 1.3.10.** A **cash-additive risk measure** is any nonconstant mapping \( \rho : \mathcal{X} \to \mathbb{R} \) which satisfies:

1. \( X \leq_P Y \) implies \( \rho(X) \geq \rho(Y) \);
2. \( \rho(X + \lambda) = \rho(X) - \lambda \) for all \( \lambda \in \mathbb{R} \).

The following questions, which are crucial in light of the cash-additive reduction, have been almost systematically evaded in most of the cash-additive literature:

*How to interpret the elements of \( \mathcal{X} \)?*

\(^9\)In this respect, the main exception is the paper by Artzner, Delbaen, Koch-Medina [10]. For a list of other relevant contributions we refer to Section 1.4.2 below.
Typically, the elements of $\mathcal{X}$ are assumed to represent financial positions expressed in a common, but unspecified, unit of account. However, they are rarely said to represent “discounted” positions and, as a rule, no explicit mention of the underlying numéraire is provided in this case.

How to interpret $\rho(X)$ in terms of management decisions?

The quantity $\rho(X)$ is typically interpreted as a capital amount to “add” to the position $X$ in order to become acceptable from a regulatory perspective. This interpretation is rarely accompanied by a more transparent operational description in terms of management actions and the “discounted nature” of the evaluation $\rho(X)$ is almost systematically left unnoticed.

What is the financial meaning of defining acceptability for positions expressed in units of the reference instrument?

The risk measure $\rho$ does not explicitly build on any acceptance set. However, the following notion of acceptability is implicit in the interpretation of $\rho$ as a capital requirement functional: a position is acceptable if it belongs to the set

$$\mathcal{A}(\rho) = \{ X \in \mathcal{X}; \: \rho(X) \leq 0 \}.$$

In particular, it is easy to verify that $\mathcal{A}(\rho)$ is an acceptance set and

$$\rho(X) = \rho_{\mathcal{A}(\rho)}(X).$$

Since the natural domain of a cash-additive risk measure is a space of positions expressed in units of the reference instrument, if we model acceptability in the above way we are actually specifying a notion of acceptability in terms of units of the reference instrument. However, the possible dependence of acceptability on the unit of account leads to a critical reservation about the financial meaning of this procedure. Assuming that a concept of acceptability which is not invariant across changes of numéraire is meaningful, the financial intuition about the notion of acceptability might depend on the chosen unit of account. For instance, we may have a sharper intuition for positions expressed in monetary terms than for positions expressed in units of a given stock.

\(^{10}\text{Sometimes, this monetary interpretation is not mentioned and } \rho(X) \text{ is vaguely defined to represent the “risk” of the position } X.\)
1.3 Problems and pitfalls of discounting

In conclusion, the exclusive focus on “discounted” positions and the subordination of the notion of acceptability described above have contributed, in a decisive way, to abandon the original constructive and operational mindset in favour of a less transparent approach. In particular, the problem of the choice of the reference asset has been essentially lost in the literature.11

1.3.6 The unfulfilled promise of cash-additivity

The discussion in the preceding section reveals a critical aspect of the approach common to the vast majority of the cash-additive literature, according to which the cash-additive theory is not understood as the reduced form of a more general theory, but rather as a primitive modelling environment. This theoretical mistake has led to neglect the following fundamental question:

What are the implicit restrictions on the general theory imposed by stipulating properties in the cash-additive setting?

Indeed, we have provided abundant examples showing that most properties of acceptance sets and risk measures get lost after the change of numéraire. Retroactively, this shows that imposing properties in the cash-additive setting may have a dramatic impact in terms of the range of reference assets that are implicitly compatible with them. The case of Expected Shortfall is paradigmatic in this respect: the standard cash-additive results for Expected Shortfall can be exploited to obtain corresponding results for risk measures based on Expected Shortfall in the general setting only if the reference instrument is a risk-free bond!

In conclusion, it follows from all the preceding arguments and examples that, unless we restrict our attention to risk-free reference assets, the cash-additive theory cannot offer a correct mathematical perspective to explore the general theory of risk measures as first articulated by Artzner, Delbaen, Eber, Heath [9].

11 As pointed out by Artzner, Delbaen, Koch-Medina [10], the germinal instance of a “discounting attitude” can already be found in the original paper by Artzner, Delbaen, Eber, Heath [9]. There, in the proof of Proposition 4.1, namely the representation result for coherent risk measures by means of “generalized scenarios”, the authors pass from \( \rho_{A,S} \) to the “cash-additive” functional defined by \( E^*(X) = \rho_{A,S}(-S_T X) \) in order to apply a representation result from (the first edition of) Huber, Ronchetti [61]. From that point on, the authors rely systematically on discounted positions; see the definition of the Tail Conditional Expectation and the Worst Conditional Expectation.
1.4 Beyond the cash-additive setting

In this section we draw the conclusions of the preceding discussion and provide a brief bibliographical note, setting the scene for the remainder of the dissertation.

1.4.1 Returning to the original risk measures

The above discussion has proved the structural inadequacy of the cash-additive reduction as a means to simplify the study of the original general theory of risk measures. To properly understand and investigate these general risk measures we are therefore forced to abandon the usual cash-additive setting and return to the original framework.

From this perspective, the main objective of the dissertation is to develop a theory of risk measures with respect to a general reference asset. In order to achieve full generality, we do not restrict the range of admissible reference assets to risk-free or default-free securities but allow for any nonnegative payoff profile. In particular, the reference asset may represent a defaultable bond. The study of risk measures of the form $\rho_{A,S}$ – corresponding to the most elementary management action, namely raising capital and investing in a single (liquid) asset – constitutes the natural first step towards the formulation of a general theory of risk measurement, which will be outlined in the last part of the thesis.

In many financial applications, most notably in a regulatory framework, it is of cardinal importance to rely on a concept of a risk measure with a clear operational interpretation. For this reason, we will follow the constructive approach of [9] and base the notion of a risk measure on the following prior specifications:

1. a notion of acceptability for financial positions;
2. a class of management actions to reach acceptability.

From this perspective, a risk measure represents the “minimal” cost of reaching acceptability by means of some admissible management strategy. By doing so, we can unambiguously interpret the output of a risk measure in terms of management decisions and clearly visualize the relative interplay between the acceptance set and the class of admissible management actions, which represent the main vehicles for our financial interpretation. In particular, every property stipulated directly
for a risk measure should be capable of being traced back to this fundamental level. This will be our guiding principle throughout the entire dissertation.

1.4.2 General reference assets in the literature

In addition to the seminal contribution by Artzner, Delbaen, Eber, Heath [9], risk measures with respect to a general – not necessarily risk-free – reference asset had been considered by few other authors before. The most relevant contributions are, in chronological order, those by Jaschke, Küchler [63], by Scandolo [89], partially merged into the paper by Frittelli, Scandolo [52], by Filipović, Kupper [45], and by Artzner, Delbaen, Koch-Medina [10]. We also refer to the paper by Konstantinides, Kountzakis [72]. In particular, the authors of [10] had already warned about the critical gap between the original framework and the posterior theory, with a prominent focus on the interpretational problems arising when risk measures are disconnected from their primitive operational dimension.

The above papers present a variety of results about finiteness, continuity and dual representations for risk measures of the form $\rho_{A,S}$. With the exception of the dual representations in [63] and [52], the main results are obtained, implicitly or explicitly, under the assumption that the payoff of the reference asset is an interior point of the positive cone, induced by a suitable (quasi)order on the model space. This critically restricts the range of admissible reference assets and limits the applicability of the corresponding results to spaces of bounded random variables, since the positive cone of most of the other spaces commonly encountered in the literature has empty interior.\(^\text{12}\)

Finally, we mention that “risk measures” of the form $\rho_{A,S}$ have been considered, since the publication by Gerstewitz (Tammer), Iwanow [53], in some branches of pure mathematics, especially nonlinear analysis – where functionals of the form $\rho_{A,S}$ are used to obtain separation results for nonconvex sets – and optimization – where they are typically used to scalarize vector optimization problems. For a recent overview, we refer to Tammer, Zălinescu [97] and the references therein.\(^\text{13}\)

The common assumption in this literature is, once again, the requirement that

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\(^{12}\)For instance, the standard positive cone in $L^p$, $p \in [0, \infty)$, has empty interior whenever the underlying $\sigma$-algebra is not finite.

\(^{13}\)Many thanks to Andreas Hamel for unveiling this “prehistorical” theory of risk measures and to Constantin Zălinescu for pointing out his recent contribution.
the “payoff” $S_T$ be an interior point of the underlying positive cone. Hence, our mathematical contribution can be appreciated also from this perspective.
Chapter 2

Financial positions and acceptance sets

In this chapter we describe the mathematical model for financial positions and introduce the notion of acceptable positions. We present a variety of results on acceptance sets, with a special focus on dual representations. Finally, we examine the most important examples of acceptance sets used in a risk measurement context.

2.1 The space of financial positions

We start by introducing the notion of financial positions and their underlying mathematical model.

2.1.1 Introducing financial positions

We consider a one-period financial market where agents trade at a fixed initial date \( t = 0 \), called \textit{inception}, and where uncertainty resolves at a fixed terminal date \( t = T \), called \textit{maturity}. The agent’s variable of interest at maturity will be referred to as the agent’s \textit{(financial) position}. Depending on the underlying problem, e.g. capital adequacy, portfolio selection, pricing or hedging, a terminal
Financial positions and acceptance sets

position may describe:

- the *capital position* (assets net of liabilities) of a financial institution;
- the *profit-and-loss profile* (gains net of losses) of a portfolio of assets;
- the *payoff* of a financial contract; or
- the *exposure* of a financial intermediary.

In the sequel, we will always assume that financial positions are expressed in a fixed unit of account, e.g. a fixed currency.

### 2.1.2 The mathematical model

We assume that financial positions are represented by the elements of a nonempty set $\mathcal{X}$. For instance, we could assume that the elements of $\mathcal{X}$ correspond to functions of the form

$$X : \Omega \to \mathbb{R},$$

where $\Omega$ is a nonempty set representing the possible *states*, or *scenarios*, of the market at maturity, and $X(\omega)$ is a real number describing the terminal value of the position should the market turn out to be in the state $\omega \in \Omega$. This is the standard choice in the mathematical finance literature, where $\mathcal{X}$ is assumed to consist of measurable functions, provided $\Omega$ is equipped with a convenient $\sigma$-algebra $\mathcal{F}$ and $\mathbb{R}$ is equipped with the canonical Borel $\sigma$-algebra. In this respect, we can distinguish between three fundamental modelling approaches in the literature\textsuperscript{1}:

- the *historical probability* approach, according to which $(\Omega, \mathcal{F})$ is equipped with a fixed probability measure describing the historical frequency of market scenarios;

- the *model uncertainty* approach, according to which $(\Omega, \mathcal{F})$ is equipped with a set of possibly nondominated probability measures representing subjective beliefs on the frequency of market scenarios;

\textsuperscript{1}In the risk measure literature, the first model approach is standard. We refer to Föllmer, Schied [49] and to Bion-Nadal, Kervarec [15] for a study of risk measures in different model uncertainty settings, and to Föllmer, Schied [48] for a study in a model-free setting.
• the *model-free* approach, according to which no probability measure is specified on the measurable space \((\Omega, \mathcal{F})\).

Depending on the chosen approach, the space \(\mathcal{X}\) is equipped with a convenient topological linear structure and with a convenient (quasi)order reflecting the underlying system of preferences towards future positions.\(^2\)

In this thesis we would like to keep a unified perspective on the above tripartition. For this reason, we follow a more abstract approach and work in the context of a general quasioordered topological linear space. This setting is general enough to accommodate the full range of market models that are commonly encountered in the literature. Moreover, it gives the opportunity to capture the key underlying structure surfacing from the single particular models, at the same time preparing the ground for new applications.\(^3\)

The following structural assumptions on the set \(\mathcal{X}\) will be maintained throughout the entire dissertation.\(^4\)

**Assumption.** We assume \(\mathcal{X}\) is a real topological linear space equipped with the compatible quasiorder \(\leq\) induced by a fixed convex cone containing zero, denoted by \(\mathcal{X}_+\). We will always assume that \(\mathcal{X}_+\) is not reduced to the zero element. Note that we do *not* postulate that the interior of the positive cone \(\mathcal{X}_+\) is nonempty.

Before proceeding, we examine the main assumptions stipulated for the space \(\mathcal{X}\), providing additional arguments for the choice of our abstract framework.

**Algebra.** Since financial positions essentially result from the aggregation of different financial contracts, the linear structure is the natural setting to model the aggregation process accounting for both sides of each contract. In particular, we can model both *long* and *short* positions in any asset.

**Order.** The quasiorder \(\leq\) establishes a way to compare different positions. In this sense, being flexible in the choice of the (quasi)order is important to allow

\(^2\)For a thorough study of preference relations in a risk measurement context we refer to Drapeau [33].

\(^3\)In particular, the corresponding “abstract” notion of a financial position is well-suited to model both univariate and multivariate positions as well as streams of positions in a multi-period setting. The advantage of this approach is illustrated in Section 3.8.

\(^4\)This setting was already adopted, in a risk measure context, by Jaschke, Küchler [63] and by Filipović, Kupper [45]. We also refer to Drapeau [33].
for different types of preferences. Note that, since \( \leq \) need not be a total order, there may exist uncomparable positions. Moreover, since \( \leq \) is not assumed to be antisymmetric, we generally have

\[
X \leq Y, \quad Y \leq X \implies X = Y.
\]

In other words, it is possible to remain indifferent between two alternative positions. This generality is meant to capture situations where, due to market frictions such as transaction costs, taxes or trading constraints, the comparison across positions might be problematic and/or lead to decisional loops.

**Topology.** The existence of a topological structure allows to analyze the continuity behaviour of functionals defined on \( \mathcal{X} \) and to apply convergence and approximation schemes. In particular, the continuity of maps \( \rho : \mathcal{X} \to \mathbb{R} \) can be interpreted as a form of robustness with respect to slight perturbations or misspecifications of the corresponding positions. The flexibility in the choice of the underlying topology is then important to test different forms of robustness.

**Order and topology.** Note that, sometimes, the same underlying model space may display very different properties depending on the interplay between order and topology. A relevant example is that of Fréchet lattices, which are ubiquitous in the literature on risk measures. Indeed, whenever we consider a coarser topology on a Fréchet lattice we are immediately taken outside the Fréchet lattice domain.\(^5\) However, changing the underlying topology is sometimes needed to obtain better continuity properties, e.g. more tractable dual representations.\(^6\) Incidentally, this shows the importance of going beyond the Fréchet lattice setting.

**Positive cone.** The elements of the positive cone \( \mathcal{X}_+ \) can be interpreted as those positions that are deemed desirable according to our fixed preference system. Since the positive cone of many of the model spaces commonly encountered in the literature has empty interior, it is important to develop a general theory without ruling out this possibility a priori. In spaces where the positive cone has nonempty interior, e.g. spaces of bounded random variables, the link between

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\(^5\)This is because, by Corollary 9.9 in Aliprantis, Border [4], the topology turning a real linear space equipped with a lattice order into a Fréchet lattice is unique.

\(^6\)The well-known Fatou property for risk measures on \( L^\infty \) is nothing but lower semicontinuity with respect to the topology \( \sigma(L^\infty; L^1) \), see Theorem 3.2 by Delbaen [29].
order and topology is very strong, typically leading to statements that cannot be exported outside this special setting.

2.2 The set of acceptable positions

In this section we formalize the concept of acceptable positions and discuss the most important properties of acceptance sets.

2.2.1 Introducing acceptance sets

Definition 2.2.1. A nonempty, strict subset $A \subset X$ is called an acceptance set if it is monotone, i.e.

$$X \in A, \ Y \geq X \implies Y \in A.$$ 

The defining properties of an acceptance set constitute the minimal requirements to represent a sound concept of acceptability: the first conditions allow to discriminate between “good” and “bad” positions, and monotonicity captures the intuition that any financial position dominating, with respect to the given quasiorder, an already accepted position should also be deemed acceptable.

2.2.2 The monotonicity property

We start by presenting few basic properties of monotone sets. The first result shows a useful equivalent way to check for monotonicity, whose elementary proof is left to the reader. We will use this basic result without further reference.

Proposition 2.2.2. A set $A \subseteq X$ is monotone if and only if $A + X_+ \subseteq A$.

The property of monotonicity is preserved under several set and algebraic “operations”. The most relevant cases are listed below for easy reference.

Proposition 2.2.3. Assume $A_1$ and $A_2$ are monotone sets in $X$. Then $A_1 \cap A_2$ and $A_1 \cup A_2$ are also monotone.

Proposition 2.2.4. Assume $A \subseteq X$ is monotone and take a set $S \subseteq X$. Then $A + S$ is also monotone.
Proposition 2.2.5. Assume \( A \subseteq X \) is monotone. Then \( \text{co} A \) and \( \text{cone} A \) are also monotone.

Proof. Take arbitrary \( X \in \text{co} A \) and \( U \in X_+ \). We need to show that \( X + U \in \text{co} A \). Indeed, we can write
\[
X = \sum_{i=1}^{n} \lambda_i X_i
\]
for suitable \( X_i \in A \) and \( \lambda_i \in [0,1] \) such that \( \sum_{i=1}^{n} \lambda_i = 1 \), where \( n \in \mathbb{N} \). Since \( X_i + U \in A \) for any \( i = 1, \ldots, n \) by monotonicity, we immediately get
\[
X + U = \sum_{i=1}^{n} \lambda_i (X_i + U) \in \text{co} A,
\]
showing that \( \text{co} A \) is monotone. The proof for the conic hull is similar.

We turn to the interplay between monotonicity and topology.

Proposition 2.2.6. Let \( A \subseteq X \). If \( A \) is monotone, then \( \text{int} A \) and \( \text{cl} A \) are also monotone.

Proof. First, assume \( \text{int} A \) is nonempty. Since \( \text{int} A + X_+ \) is open, we must have
\[
\text{int} A + X_+ \subseteq \text{int}(A + X_+) = \text{int} A,
\]
showing that \( \text{int} A \) is monotone. To prove the second assertion, note that
\[
\text{cl} A + X_+ \subseteq \text{cl}(A + X_+) = \text{cl} A,
\]
implying that \( \text{cl} A \) is a monotone set and concluding the proof.

The interior of any monotone set in \( X \) is automatically nonempty whenever the positive cone of \( X \) has nonempty interior.

Proposition 2.2.7. Assume \( X_+ \) has nonempty interior and let \( A \subseteq X \) be a monotone set. Then, \( A \) has nonempty interior. Moreover, if \( A \) is an acceptance set, then \( \text{int} A \) and \( \text{cl} A \) are also acceptance sets.

Proof. The first assertion follows immediately from Proposition 2.2.2. Now, assume \( A \) is an acceptance set. By Proposition 2.2.6 we know that \( \text{int} A \) and \( \text{cl} A \) are both monotone. Hence, it is enough to show that \( \text{int} A \) is nonempty and \( \text{cl} A \) is a strict subset of \( X \). The former property is a consequence of the preceding proposition. To establish the latter one, we prove that \( \text{int}(\text{cl} A) \subseteq A \). Indeed,
take $X \in \text{int}(\text{cl} \ A)$ and let $U$ be an interior point of $\mathcal{X}_+$. Note that $X - \varepsilon U \in \text{cl} \ A$ for a suitably small $\varepsilon > 0$. Then, for $\delta \in (0, \varepsilon)$ we find $Y \in A$ satisfying

$$-\delta U \leq Y - (X - \varepsilon U) \leq \delta U$$

by Theorem 9.40 in Aliprantis, Border [4]. This implies that $X \geq Y + (\varepsilon - \delta)U$, hence $X \in A$ by monotonicity. In conclusion, $\text{int}(\text{cl} \ A) \subseteq A$ holds showing that $\text{cl} \ A$ is strictly contained in $\mathcal{X}$.

For a general subset $A \subseteq \mathcal{X}$ it is typically easier to show that an element $X \in A$ belongs to the core of $A$ than to show it belongs to the interior of $A$. Hence, it is of practical relevance to know whether these two sets coincide. The next result shows that, whenever the underlying reference space is a Fréchet lattice, the core and the interior of any monotone set coincide.

**Proposition 2.2.8.** Assume $\mathcal{X}$ is a Fréchet lattice and let $A \subseteq \mathcal{X}$ be a monotone set. Then, $\text{core} \ A = \text{int} \ A$.

**Proof.** Clearly, it is enough to show that $\text{core} \ A \subseteq \text{int} \ A$. Take $X \in \text{core} \ A$ and assume that $X \notin \text{int} \ A$. It follows that there exists a sequence $(Y_n)$ converging to 0 such that $X + Y_n \notin A$ for all $n \in \mathbb{N}$. Since the lattice operations are continuous by Theorem 8.41 in Aliprantis, Border [4], we also have $|Y_n| \to 0$ as $n \to \infty$. Without loss of generality we may assume that $d(|Y_n|, 0) \leq 4^{-n}$ for all $n \in \mathbb{N}$, where $d$ denotes the distance on $\mathcal{X}$. Recall that $d$ is translation invariant by Theorem 5.10 in [4]. Hence, setting $\lambda_n = 2^n$ for any $n \in \mathbb{N}$ we see that

$$Y = \sum_{n \in \mathbb{N}} \lambda_n |Y_n| \in \mathcal{X}.$$ 

Since $X \in \text{core} \ A$, it follows that $X - \delta Y \in A$ for a suitable $\delta > 0$. For a sufficiently large $m \in \mathbb{N}$ we have $\lambda_m \geq \delta^{-1}$, hence

$$X - \delta Y \leq X - \delta \lambda_m |Y_m| \leq X + Y_m.$$ 

The monotonicity of $A$ thus implies $X + Y_m \in A$, contradicting the fact that $X + Y_n \notin A$ holds for all $n \in \mathbb{N}$. In conclusion, $X$ must belong to $\text{int} \ A$, proving the initial claim. \qed
Remark 2.2.9. The proof of the preceding proposition is inspired by the proof of Lemma 4.1 in Cheridito, Li [23], obtained in the context of monotone functionals on Banach lattices. Note that, in contrast to corresponding standard results in functional analysis, the presence of monotonicity requires no convexity assumption here.

We conclude with a simple characterization of core points of a monotone set in a lattice setting.

Proposition 2.2.10. Assume $X$ is a topological Riesz space and let $A \subseteq X$ be a monotone set. For $X \in X$, we have $X \in \operatorname{core} A$ if and only if for any $U \in X^+$ there exists $\varepsilon > 0$ such that $X - \lambda U \in A$ for all $\lambda \in [0, \varepsilon)$.

Proof. We only need to prove the “if” part. Fix $Y \in X$ and assume we can find $\varepsilon > 0$ such that $X - \lambda Y^- \in A$ for any $\lambda \in [0, \varepsilon)$. Since

$$X + \lambda Y = X + \lambda Y^+ - \lambda Y^- \geq X - \lambda Y^-,$$

the monotonicity of $A$ implies $X + \lambda Y \in A$ for all $\lambda \in [0, \varepsilon)$.

2.2.3 Closedness, convexity, conicity

In this section we briefly discuss some of the properties that acceptance sets are typically required to satisfy and highlight their underlying financial motivation. The most important examples of acceptance sets are given by closed sets. In the literature, this property is commonly assumed to be a mere technical requirement, since it simplifies the study of dual representations for the corresponding risk measures (as documented below in Section 3.5). However, the assumption of closedness carries also a financial meaning since it corresponds to requiring that an unacceptable position cannot be made acceptable by arbitrarily small perturbations.

We will pay special attention to convex acceptance sets. Indeed, the assumption of convexity corresponds to the diversification principle for which portfolios of acceptable positions are also to be deemed acceptable. In other words, under convexity, acceptability is preserved when passing to aggregated portfolios.

Remark 2.2.11. Convex acceptance sets (associated to convex, cash-additive risk measures) were first studied in Föllmer, Schied [48].
2.2 The set of acceptable positions

Some of the most important acceptance sets are conic sets. The assumption of conicity is equivalent to assuming that acceptability is independent of the size of financial positions and, together with convexity, implies that adding up, or merging, acceptable positions does not compromise acceptability. However, positions with arbitrarily large loss exposures may be deemed acceptable.

**Remark 2.2.12.** As it is well-known, acceptance sets that are convex cones were introduced in Artzner, Delbaen, Eber, Heath [9] under the name of *coherent* acceptance sets. Even if the terminology has become well-established in the risk measure literature, we will refrain from adopting it and rather opt for a more transparent mathematical specification.

2.2.4 Numéraire-invariance

Assume $\mathcal{X}$ consists of random variables over a given probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. As already discussed in Section 1.3.4, the acceptability of a position may depend on the underlying unit of account. The objective of this section is to elaborate upon this issue, with a special focus on capital adequacy, and articulate a notion of numéraire-invariant acceptability. Incidentally, the following discussion will raise a severe concern about the economical legitimation of capital adequacy tests based on Expected Shortfall, which fail to be “numéraire-invariant” as shown in Example 1.3.5 and further discussed in Example 1.3.7.

Along with other fundamental economical properties\(^7\), we argue that acceptability should not depend on the aggregation currency of a financial institution, but rather reflect a structural aspect of the corresponding balance sheet: the ability to cover liabilities. In other words, the outcome of a capital adequacy test should not depend on the particular unit of account in which capital positions are expressed since, following Artzner, Delbaen, Koch-Medina [10], “changing the unit of account is essentially a different way to quote prices and should by itself have no impact on whether a financial position is accepted or not”. Moreover, once acceptability depends on the aggregation currency, the capital adequacy test might lead to ambiguous capital adequacy assessments: an unacceptable company might become acceptable by a mere change of the currency in which risks are aggregated.

\(^7\)We refer to the recent work by Herdegen [60] for a careful presentation of a numéraire-independent approach to financial modelling.
so that otherwise identical companies might receive an opposite assessment only because they have selected a different reporting currency. If unregulated, this situation would create a clear incentive, from the management’s perspective, to choose the aggregation currency minimizing the level of required capital, making the comparison across institutions belonging to the same jurisdiction fundamentally problematic.

Beside this fundamental issue, the possible dependence of acceptability on the aggregation currency is problematic also from the perspective of the global financial system. We illustrate this in the case of acceptance sets based on Expected Shortfall. Consider a capital position $X \in \mathcal{X}$ and recall that, if we model the process of changing the numéraire by way of a multiplication with a suitable rescaling factor $D$ satisfying $\mathbb{P}(D > 0) = 1$, we may end up having

$$ES_\alpha(X) > 0 \text{ but } ES_\alpha(DX) \leq 0.$$ 

This situation has been encountered in Example 1.3.5. Now, consider a financial company subject to a regulatory regime based on Expected Shortfall and assume the corresponding capital adequacy test is performed in the domestic currency. After a change of ownership, the head office may be transferred to a different jurisdiction and, hence, the company might become subject to a different regulatory regime. Assume the new capital adequacy test is still defined by the same formal rule, i.e. Expected Shortfall, but is performed in the new domestic currency. Since the result of the capital adequacy assessment depends on the reference currency, the same company could pass or fail the same capital adequacy test depending on the underlying jurisdiction. As a result, this situation would create a form of regulatory arbitrage in terms of a conflict between different jurisdictions: the management would have a strong incentive to transfer the legal residence of the company in order to minimize the level of required capital, or even to “transform” an unacceptable company into an acceptable one.\(^8\)

Motivated by the above discussion, we proceed to introduce a notion of numéraire-invariance that seems most natural in the context of the model space $\mathcal{X}$. Here,

\(^8\)The same type of regulatory arbitrage would arise if, instead of moving to a new jurisdiction, a company would transfer risk to a sister company in another jurisdiction. This situation is not uncommon in the insurance world, where risks are often transferred using internal reinsurance contracts within a group.
we denote by $D(\mathcal{X})$ the set of all $D \in \mathcal{X}$ such that $\mathbb{P}(D > 0) = 1$ and $DX \in \mathcal{X}$ for every $X \in \mathcal{X}$. A random variable $D \in D(\mathcal{X})$ may represent a (stochastic) exchange rate between two currencies or a discount factor obtained by considering the inverse of the price of some reference instrument.

**Definition 2.2.13.** An acceptance set $A \subset \mathcal{X}$ is said to be *numéraire-invariant* whenever

$$X \in A, \ D \in D(\mathcal{X}) \implies DX \in A.$$

**Remark 2.2.14.** As mentioned in Section 1.3.4, the critical aspect of the possible dependence of acceptability on the underlying numéraire was already pointed out in Artzner, Delbaen, Eber, Heath [9]. Later, this problem was discussed by Artzner, Delbaen, Koch-Medina [10], where a notion of numéraire-invariance is implicitly captured by the notion of compatibility of risk measures in various currencies. 

### 2.2.5 Surplus-invariance

Assume $\mathcal{X}$ consists of random variables on a given probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. The positive and the negative part of a position $X \in \mathcal{X}$ have a natural financial interpretation in the case that $X$ represents the net capital position of a company with limited liability. In this case, the positive part $X^+$ represents the surplus of the company and corresponds to the excess of available funds over the amount needed to meet liabilities. On the other side, the negative part $X^-$ represents the amount over which available funds fall short of meeting liabilities. Since $X^-$ reflects the limited liability of the owners of the institution, it is often referred to as the (owners’) default option. Equivalently, $X^-$ represents the difference between the contractual and the actual payment received by liability holders.

The property of surplus-invariance is aligned with the primary mandate of regulation, i.e. the protection of the interests of liability holders: whenever a company fails to pass the capital adequacy test, any other company with a higher default option should also fail the test. In other words, the result of the capital adequacy assessment should only depend on the default profile, which directly affects liability holders, and should not be driven by the surplus of a financial institution, which benefits exclusively the shareholders after liability holders have been paid.
Definition 2.2.15. An acceptance set \( \mathcal{A} \subset \mathcal{X} \) is said to be **surplus-invariant** if

\[
X \in \mathcal{A}, \ X^- \geq_{\mathbb{P}} Y^- \implies Y \in \mathcal{A}.
\]

The next result provides an elementary characterization of surplus-invariant acceptance sets emphasizing that, under surplus-invariance, acceptability is essentially driven by the default option.

**Proposition 2.2.16.** Let \( \mathcal{A} \subset \mathcal{X} \) be an acceptance set. The following statements are equivalent:

(i) \( \mathcal{A} \) is surplus invariant;

(ii) if \( X \in \mathcal{A} \) and \( Y^- = X^- \), then \( Y \in \mathcal{A} \);

(iii) if \( X \in \mathcal{A} \), then \( -X^- \in \mathcal{A} \);

(iv) if \( X \in \mathcal{A} \), then \( X 1_A \in \mathcal{A} \) for any \( A \in \mathcal{F} \).

**Proof.** It is clear that (i) implies (ii), which in turn implies (iii). Now, assume (iii) holds and take \( X \in \mathcal{A} \) and \( A \in \mathcal{F} \). Since \( X 1_A \geq_{\mathbb{P}} -X^- \), we immediately conclude that (iv) is satisfied. Finally, assume (iv) holds and take \( X \in \mathcal{A} \). If \( Y^- \leq_{\mathbb{P}} X^- \), then \( Y \geq_{\mathbb{P}} X 1_{\{X < 0\}} \) implying that \( Y \in \mathcal{A} \) and showing that \( \mathcal{A} \) is surplus invariant.

**Remark 2.2.17.** The concept of surplus-invariant acceptance sets has been recently discussed in Koch-Medina, Moreno-Bromberg, Munari [68] and further investigated in Koch-Medina, Munari, Šikić [71]. We also refer to the related works by Cont, Deguest, He [27] and by Staum [94].

2.2.6 **Law-invariance**

Assume \( \mathcal{X} \) consists of random variables on a given probability triple \((\Omega, \mathcal{F}, \mathbb{P})\). We will write \( X \sim_{\mathbb{P}} Y \) whenever \( X \) and \( Y \) have the same probability law, i.e. \( \mathbb{P}(X \in A) = \mathbb{P}(Y \in A) \) for any Borel set \( A \subseteq \mathbb{R} \).

Acceptance sets for which acceptability is only driven by the probability distribution of financial positions are called law-invariant.
Definition 2.2.18. An acceptance set $\mathcal{A} \subset \mathcal{X}$ is said to be law-invariant if
\[ X \in \mathcal{A}, \ Y \sim_{\mathbb{P}} X \implies Y \in \mathcal{A}. \]

Remark 2.2.19. The property of law-invariance has been thoroughly investigated in the context of (cash-additive) risk measures since the early contribution by Kusuoka [77].

2.2.7 Sensitivity

Assume $\mathcal{X}$ consists of random variables on a given probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. An acceptance set which does not contain any nonzero negative position is called sensitive.

Definition 2.2.20. An acceptance set $\mathcal{A} \subset \mathcal{X}$ is said to be sensitive whenever
\[ X \in \mathcal{A}, \ X \in \mathcal{X}_{-} \implies X = 0. \]

Remark 2.2.21. This property was already listed among the axioms for acceptance sets in Artzner, Delbaen, Eber, Heath [9].

2.3 Representing acceptance sets

In this section we provide a dual representation of closed, convex acceptance sets which will later constitute the key ingredient to obtain dual representations for the corresponding risk measures. In order to apply separation techniques, we assume that $\mathcal{X}$ is locally convex and maintain this assumption throughout the whole section.

2.3.1 Support functions and barrier cones

Consider a nonempty subset $\mathcal{A} \subseteq \mathcal{X}$. The support function of $\mathcal{A}$ is the mapping $\sigma_{\mathcal{A}} : \mathcal{X}' \to \mathbb{R} \cup \{-\infty\}$ defined by
\[ \sigma_{\mathcal{A}}(\psi) := \inf_{X \in \mathcal{A}} \psi(X). \]

The domain of $\sigma_{\mathcal{A}}$ is called the barrier cone of $\mathcal{A}$ and is denoted by $\mathcal{B}(\mathcal{A})$, i.e.
\[ \mathcal{B}(\mathcal{A}) := \{ \psi \in \mathcal{X}' ; \ \sigma_{\mathcal{A}}(\psi) > -\infty \}. \]
If the dual $\mathcal{X}'$ can be identified with a space $\mathcal{Y}$, we will write $\sigma_A(Y)$ to denote the action of the support function on the functional corresponding to the element $Y$. Moreover, when convenient, we will also understand $\mathcal{B}(\mathcal{A})$ as a subset of $\mathcal{Y}$.

**Remark 2.3.1.** Support functions are sometimes defined as suprema rather than infima, see e.g. Aubin, Ekeland [11]. We have deviated from this convention because using the infimum seems more natural in the context of acceptance sets, as shown by the dual representations below.

The salient properties of support functions are listed in the next proposition and will be used without further reference.

**Proposition 2.3.2.** For a nonempty set $\mathcal{A} \subseteq \mathcal{X}$, the following statements hold:

(i) $\sigma_A$ is concave and conic;

(ii) $\mathcal{B}(A)$ is a convex cone;

(iii) $\sigma_A = \sigma_{cl.A}$ and $\mathcal{B}(A) = \mathcal{B}(cl.A)$;

(iv) if $\mathcal{A}$ is a cone, then

$$\mathcal{B}(A) = \{\psi \in \mathcal{X}' \mid \sigma_A(\psi) = 0\} = \{\psi \in \mathcal{X}' \mid \psi(X) \geq 0, \forall X \in A\}.$$  

**Remark 2.3.3.** If $\mathcal{A}$ is a cone, $\mathcal{B}(A)$ is nothing but the (one-sided) polar of $\mathcal{A}$. We refer to Section 5.16 in Aliprantis, Border [4] for more details. In particular, note that $\mathcal{B}(\mathcal{X}_+) = \mathcal{X}_+^\prime$.

The next important proposition shows that every functional belonging to the barrier cone of a monotone set is automatically positive. This result will constitute the key ingredient to provide dual representations of acceptance sets.

**Proposition 2.3.4.** Let $\mathcal{A} \subseteq \mathcal{X}$ be a nonempty monotone set. Then, $\mathcal{B}(\mathcal{A}) \subseteq \mathcal{X}_+^\prime$.

**Proof.** Let $\psi \in \mathcal{B}(\mathcal{A})$ and take $X \in \mathcal{A}$ and $U \in \mathcal{X}_+$. Then, by monotonicity, we have $X + \lambda U \in \mathcal{A}$ for all $\lambda \geq 0$. Hence,

$$\psi(X) + \lambda \psi(U) \geq \sigma_A(\psi) > -\infty$$

for all $\lambda \geq 0$, which is only possible if $\psi(U) \geq 0$. Since $U$ was arbitrary, we conclude that $\psi \in \mathcal{X}_+^\prime$. 

$\square$
Remark 2.3.5. The above result can be generalized in the following way: every (not necessarily continuous) linear functional \( \psi : \mathcal{X} \to \mathbb{R} \) that is bounded from below on some nonempty monotone subset of \( \mathcal{X} \) is automatically positive. \( \square \)

2.3.2 Dual representations

The following result provides the fundamental dual representation of closed, convex monotone sets.

**Theorem 2.3.6.** Assume \( \mathcal{A} \subseteq \mathcal{X} \) is a nonempty monotone set. If \( \mathcal{A} \) is closed and convex, then

\[
\mathcal{A} = \bigcap_{\psi \in B(\mathcal{A})} \{ X \in \mathcal{X} ; \psi(X) \geq \sigma_\mathcal{A}(\psi) \} = \bigcap_{\psi \in \mathcal{X}'_+} \{ X \in \mathcal{X} ; \psi(X) \geq \sigma_\mathcal{A}(\psi) \}.
\]

Moreover, if \( \mathcal{A} \) is also a cone we have

\[
\mathcal{A} = \bigcap_{\psi \in B(\mathcal{A})} \{ X \in \mathcal{X} ; \psi(X) \geq 0 \}.
\]

**Proof.** In light of Proposition 2.3.4, the representation of \( \mathcal{A} \) in the convex case follows as a direct consequence of the Hahn-Banach separation theorem in the version stated in Aubin, Ekeland [11]. In particular, since \( B(\mathcal{A}) \subseteq \mathcal{X}'_+ \), we can replace \( B(\mathcal{A}) \) with the larger set \( \mathcal{X}'_+ \) without altering the intersection. The representation in the conic case is an immediate corollary of Proposition 2.3.2. \( \square \)

In the above dual representation we can sometimes reduce the range of functionals over which intersections are taken by a suitable “normalization”. We present two typical situations. The corresponding statements under the additional assumption of conicity are straightforward and left to the reader.

**Proposition 2.3.7.** Let \( \mathcal{A} \subseteq \mathcal{X} \) be a closed, convex acceptance set. Take \( U \in \mathcal{X}_+ \) and assume \( \varphi(U) > 0 \) for some functional \( \varphi \in B(\mathcal{A}) \). Then, we have

\[
\mathcal{A} = \bigcap_{\psi \in B(\mathcal{A}) \atop \psi(U) = 1} \{ X \in \mathcal{X} ; \psi(X) \geq \sigma_\mathcal{A}(\psi) \}.
\]
Proof. For simplicity, define
\[ B_U(A) = \{ \psi \in B(A) ; \; \psi(U) = 1 \} . \]

First, note that we can normalize \( \varphi \) to get \( \varphi(U) = 1 \). Since \( B(A) \) is a cone, we see that the set \( B_U(A) \) is nonempty. Now take \( X \in \mathcal{X} \) such that \( \phi(X) \geq \sigma_A(\phi) \) for all \( \phi \in B_U(A) \), and fix an arbitrary \( \psi \in B(A) \). Since \( B_U(A) \subseteq B(A) \), Theorem 2.3.6 implies that, to prove the claim, we only need to show that
\[ \psi(X) \geq \sigma_A(\psi) . \] (2.1)

Assume first that \( \psi(U) > 0 \). In this case we can normalize \( \psi \) to get \( \psi \in B_U(A) \), hence (2.1) is satisfied by the assumption on \( X \). As a result, assume \( \psi(U) = 0 \) and set \( \varphi_n = \varphi + n\psi \) for all \( n \in \mathbb{N} \). Since \( B(A) \) is a convex cone, the functional \( \varphi_n \) also belongs to \( B_U(A) \). Hence, we obtain
\[ \frac{1}{n} \varphi(X) + \psi(X) = \frac{1}{n} \varphi_n(X) \geq \frac{1}{n} \sigma_A(\varphi_n) \geq \frac{1}{n} \sigma_A(\varphi) + \sigma_A(\psi) . \]

Letting \( n \to \infty \) we finally get (2.1), concluding the proof.

The next “normalization” follows immediately from the conicity of the support function.

**Proposition 2.3.8.** Assume \( \mathcal{X} \) is a normed space and let \( A \subset \mathcal{X} \) be a closed, convex acceptance set. Then
\[ A = \bigcap_{\psi \in B(A) ; \; \| \psi \| \leq 1} \{ X \in \mathcal{X} ; \; \psi(X) \geq \sigma_A(\psi) \} = \bigcap_{\psi \in B(A) ; \; \| \psi \| = 1} \{ X \in \mathcal{X} ; \; \psi(X) \geq \sigma_A(\psi) \} . \]

### 2.3.3 Duality and numéraire-invariance

Using the above duality results, we provide a representation of numéraire-invariant acceptance sets that are closed, convex subsets of \( L^p \). In this case, acceptability reduces to requiring solvency on a pre-specified set of scenarios. This shows that, if we insist on convexity, then numéraire-invariance is compatible with a very narrow range of acceptability criteria. In light of the problems discussed in Section 2.2.4, arising when acceptability depends on the numéraire, the following result has therefore important implications for policy making since it makes manifest
that the choice of a regulatory risk measure is not straightforward and tradeoffs
between possibly incompatible requirements – such as convexity and numéraire-
invariance – need to be made.

In the proof, we will make use of the following existence result for measurable
sets. The corresponding proof follows a rather standard exhaustion argument,
which essentially underpins the proof of Lemma 1.60 in Föllmer, Schied [50].

Lemma 2.3.9. Consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Assume \(\mathcal{G}\) is a nonempty
subset of \(\mathcal{F}\) and \(\mathbb{P}(A) > 0\) for every \(A \in \mathcal{G}\). Moreover, assume \(\mathcal{G}\) is closed under
countable unions. Then, there exists \(M \in \mathcal{G}\) such that:

(i) \(\mathbb{P}(M) \geq \mathbb{P}(A)\) for all \(A \in \mathcal{G}\);

(ii) \(\mathbb{P}(A \cap M^c) = 0\) for all \(A \in \mathcal{G}\).

Proof. We can always find a sequence \((A_n)\) in \(\mathcal{G}\) converging to
\[
m = \sup_{A \in \mathcal{G}} \mathbb{P}(A).
\]
Without loss of generality, we can assume that \(A_n \subseteq A_{n+1}\) for every \(n \in \mathbb{N}\). Since
\(\mathcal{G}\) is closed under countable unions, setting
\[
M = \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{G}
\]
we obtain that \(\mathbb{P}(M) = m\) so that (i) holds. If (ii) is not satisfied, we would find
\(A \in \mathcal{G}\) with \(\mathbb{P}(A \cap M^c) > 0\). However, \(A \cup M\) would belong to \(\mathcal{G}\) and
\[
\mathbb{P}(A \cup M) = \mathbb{P}(M) + \mathbb{P}(A \cap M^c) > \mathbb{P}(M),
\]
contradicting (i). Hence, (ii) also holds.

Theorem 2.3.10. Fix \(p \in [1, \infty]\) and assume \(\mathcal{A} \subset L^p\) is a closed \((\sigma(L^\infty, L^1)\)-
closed if \(p = \infty\), convex, numéraire-invariant acceptance set. Then, there exists
\(A \in \mathcal{F}\) such that
\[
\mathcal{A} = \{X \in L^p ; X1_A \geq 0\}.
\]

Proof. The set \(\mathcal{D}(L^p)\) of all \(D \in L^p\) such that \(\mathbb{P}(D > 0) = 1\) and \(DX \in L^p\) for all
\(X \in L^p\) is easily seen to consist of all bounded random variables that are strictly
positive almost surely. Since $\lambda 1_\Omega \in \mathcal{D}(L^p)$ for every $\lambda > 0$, the acceptance set $\mathcal{A}$ is a cone by numéraire-invariance. Hence, Theorem 2.3.6 implies that

$$\mathcal{A} = \bigcap_{Z \in \mathcal{B}(\mathcal{A})} \{ X \in L^p ; \mathbb{E}[ZX] \geq 0 \}.$$  \hspace{1cm} (2.2)

Now, consider the collection

$$\mathcal{G} = \{ A \in \mathcal{F} ; \mathbb{P}(A) > 0, X^{-1}A = 0, \forall X \in \mathcal{A} \}.$$  

Note that $\{ Z > 0 \} \in \mathcal{G}$ for every nonzero $Z \in \mathcal{B}(\mathcal{A})$. To see this, assume by contrast that $X^{-1}\{Z > 0\}$ is nonzero for some $X \in \mathcal{A}$ and $Z \in \mathcal{B}(\mathcal{A})$. Moreover, set $A = \{ X < 0 \} \cap \{ Z > 0 \}$ and consider $D_n = n1_A + 1_{A^c}$ for $n \in \mathbb{N}$. Since $D_n \in \mathcal{D}(L^p)$ for all $n \in \mathbb{N}$, we have $D_nX \in \mathcal{A}$ for every $n \in \mathbb{N}$ by numéraire-invariance. Thus, (2.2) would imply

$$n\mathbb{E}[ZX1_A] + \mathbb{E}[ZX1_{A^c}] = \mathbb{E}[ZD_nX] \geq 0$$

for every $n \in \mathbb{N}$, which is clearly impossible as $\mathbb{E}[ZX1_A] < 0$. Hence, $\{ Z > 0 \} \in \mathcal{G}$ for all nonzero $Z \in \mathcal{G}$. In particular, $\mathcal{G}$ is nonempty. It is immediate to see that $\mathcal{G}$ is closed under countable unions. Then, as a consequence of Lemma 2.3.9, we find a set $M \in \mathcal{G}$ satisfying $\mathbb{P}(A \cap M^c) = 0$ for all $A \in \mathcal{G}$. We claim that

$$\mathcal{A} = \{ X \in L^p ; X1_M \geq_\mathbb{P} 0 \}.$$  

To prove the inclusion “⊆”, assume $X \in \mathcal{A}$. Then, since $M \in \mathcal{G}$, we must have $X^{-1}M = 0$ and this implies $X1_M \geq_\mathbb{P} 0$. To prove the converse inclusion, take $X \in L^p$ with $X1_M \geq_\mathbb{P} 0$ and fix an arbitrary nonzero $Z \in \mathcal{B}(\mathcal{A})$. Recall that $\{ Z > 0 \} \in \mathcal{G}$. Since $\mathbb{P}(\{ Z > 0 \} \cap M^c) = 0$, it follows that $\mathbb{E}[ZX^-] = 0$ and, thus, $\mathbb{E}[ZX] \geq 0$. Hence, the representation (2.2) shows that $X \in \mathcal{A}$, concluding the proof. \hfill \Box

**Remark 2.3.11.** As proved in Koch-Medina, Munari, Šikić [71], the above representation result can be extended to any $L^p$ space, including the range $p < 1$. \hfill \Box

### 2.3.4 Duality and surplus-invariance

In this section we focus on surplus-invariant acceptance sets in the context of $L^p$ spaces. We start by providing a representation of the corresponding support function. The immediate proof is left to the reader.
Lemma 2.3.12. Fix \( p \in [1, \infty] \) and let \( q = \frac{p}{p-1} \). Assume \( \mathcal{A} \subset L^p \) is a closed (\( \sigma(L^\infty, L^1) \)-closed if \( p = \infty \)), convex, surplus-invariant acceptance set. Then, for any \( Z \in L^q_+ \) we have

\[
\sigma_{\mathcal{A}}(Z) = \inf_{X \in \mathcal{A}} \mathbb{E}[-ZX].
\]

The next “localization” lemma will be useful to understanding the structure of surplus-invariant acceptance sets in the context of \( L^p \) spaces.

Lemma 2.3.13. Let \( p \in [1, \infty] \) and assume \( \mathcal{A} \subset L^p \) is a closed (\( \sigma(L^\infty, L^1) \)-closed if \( p = \infty \)), convex, surplus-invariant acceptance set. Then, there exists a set \( C \in \mathcal{F} \) such that

(i) \( Z^* > 0 \) almost surely on \( C \) for some \( Z^* \in \mathcal{B}(\mathcal{A}) \);

(ii) \( Z = 0 \) almost surely on \( C^c \) for every \( Z \in \mathcal{B}(\mathcal{A}) \).

Proof. Define \( q = \frac{p}{p-1} \). We first prove that the class

\[ \mathcal{G} := \{ \{ Z > 0 \}; \; Z \in \mathcal{B}(\mathcal{A}), \; \mathbb{P}(Z > 0) > 0 \} \]

is closed under countable unions. Indeed, consider a sequence \((Z_n)\) in \( \mathcal{B}(\mathcal{A}) \) and take a sequence \((\lambda_n)\) of positive real numbers such that both series

\[ \sum_{n \in \mathbb{N}} \lambda_n \|Z_n\|_q \quad \text{and} \quad \sum_{n \in \mathbb{N}} \lambda_n \sigma_{\mathcal{A}}(Z_n) \]

converge. As a result, setting

\[ Z = \sum_{n \in \mathbb{N}} \lambda_n Z_n \in L^q_+ \]

we easily see that

\[ \sigma_{\mathcal{A}}(Z) \geq \sum_{n \in \mathbb{N}} \lambda_n \sigma_{\mathcal{A}}(Z_n) > -\infty \]

so that \( Z \in \mathcal{B}(\mathcal{A}) \). Moreover, we clearly have

\[ \{ Z > 0 \} = \bigcup_{n \in \mathbb{N}} \{ Z_n > 0 \} . \]

It follows that \( \mathcal{G} \) is closed under countable unions. As a result, Lemma 2.3.9 implies that we can find a suitable \( M \in \mathcal{G} \) satisfying \( \mathbb{P}(A \cap M^c) = 0 \) for all \( A \in \mathcal{G} \). Assume \( Z^* \in \mathcal{B}(\mathcal{A}) \) is such that

\[ M = \{ Z^* > 0 \} . \]
We claim that \( M \) has the desired properties. In fact, we only need to show that \( Z = 0 \) almost surely on \( M^c \) for every \( Z \in \mathcal{B}(A) \). To prove this, suppose that \( Z > 0 \) on a measurable subset \( E \subseteq M^c \) with nonzero probability for some \( Z \in \mathcal{B}(A) \). Then, \( Z + Z^* \) would be an element of \( B(A) \) satisfying

\[
P(Z + Z^* > 0) \geq P(E) + P(M) > P(M),
\]

in contrast with the maximality of \( M \). Hence, the lemma is proved.

We focus on surplus-invariant acceptance sets that are closed, convex subsets of \( L^p \). In this case, acceptability can be described by a set of requirements on the behaviour of financial positions on each of the “atoms” of a measurable partition \( \{A, B, C\} \) of \( \Omega \): no defaults are allowed on \( A \), a “controlled” form of default is allowed on \( B \), while no requirements are imposed on \( C \). The second condition will be expressed in terms of first order stochastic dominance. More precisely, the negative part of each position restricted to \( B \) must be bounded, in this preference sense, by a target “worst” loss. Here, we write \( Y \geq_{\text{FSD}} X \) whenever \( Y \) is preferable than \( X \) in the first order stochastic dominance sense, i.e. whenever the corresponding cumulative distribution functions satisfy \( F_Y \leq F_X \).

**Theorem 2.3.14.** Fix \( p \in [1, \infty] \) and assume \( A \subset L^p \) is a a closed \((\sigma(L^\infty, L^1)\)-closed if \( p = \infty \)), convex, surplus-invariant acceptance set. Then, there exist disjoint sets \( A, B \in \mathcal{F} \) and a random variable \( L \in L^0_- \) satisfying

\[
A = \{X \in L^p; \; X 1_A \geq_p 0, \; -X^-1_B \in \mathcal{L}\}
\]

where

\[
\mathcal{L} = \{X \in A; \; X \geq_{\text{FSD}} L\}.
\]

**Proof.** Consider a set \( C \in \mathcal{F} \) and a random variable \( Z^* \in \mathcal{B}(A) \) satisfying the conditions stated in the preceding lemma. We claim there exists \( L \in L^p_- \) such that \( -X^-1_C \geq_{\text{FSD}} L \) for all \( X \in A \). To prove this, consider the function \( F : \mathbb{R} \to [0, 1] \) defined by

\[
F(x) = \sup_{X \in A} F_{-X^-1_C}(x).
\]

Note that \( F \) is increasing and identically 1 on \([0, \infty)\). First, assume \( F(a) = 0 \) for some \( a < 0 \) so that \( -X^-1_C \geq_p a 1_\Omega \) for every \( X \in A \). In this case, the claim is
easily satisfied by $L = a_1\Omega$. Otherwise, assume $F(x) > 0$ for any $x < 0$. We aim to show that

$$\lim_{x \to -\infty} F(x) = 0. \quad (2.3)$$

On the contrary, we would find $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ we would find $X_n \in \mathcal{A}$ with $\mathbb{P}(X_n^{-1}C \geq n) > \varepsilon$. Taking $\delta > 0$ small enough to ensure $\mathbb{P}(\mathcal{C} \cap \{Z^* > \delta\}) > \mathbb{P}(\mathcal{C}) - \frac{\varepsilon}{2}$, we would then obtain

$$\mathbb{P}(X_n^{-1}C \geq n, Z^* > \delta) > \frac{\varepsilon}{2}.$$  

However, since $-X_n^{-1}C \in \mathcal{A}$ for any $n \in \mathbb{N}$ by surplus-invariance, this would imply that

$$-n\delta\frac{\varepsilon}{2} \geq \mathbb{E}[Z^*(-X_n^{-1}C)] \geq \sigma_{\mathcal{A}}(Z^*) > -\infty$$

for every $n \in \mathbb{N}$, which is impossible. Hence, (2.3) must hold. In this case, there always exists a measurable partition $(A_n)$ of $\Omega$ such that $\mathbb{P}(A_n) > 0$ for all $n \in \mathbb{N}$. By (2.3), we can find a decreasing sequence $(a_n)$ of strictly negative numbers satisfying

$$F(a_n) \leq 1 - \sum_{i=1}^{n} \mathbb{P}(A_i)$$

for every $n \in \mathbb{N}$ and such that

$$L = \sum_{n \in \mathbb{N}} a_n 1_{A_n} \in L^p_-. $$

We claim that $-X^{-1}C \geq_{FSD} L$ for all $X \in \mathcal{A}$. To show this, take $x < 0$. If $x \geq a_1$ we easily see that

$$F(x) \leq 1 = \mathbb{P}(L \leq x).$$

Otherwise, assume $a_{k+1} \leq x < a_k$ for some $k \in \mathbb{N}$ and note that

$$F(x) \leq F(a_k) < 1 - \sum_{i=1}^{k} \mathbb{P}(A_i) = \mathbb{P}(L \leq x).$$

Hence, the claim is true. To conclude the proof, it is now sufficient to define

$$A = \mathcal{C} \cap \left\{\text{essinf}_{X \in \mathcal{A}} X^- = 0\right\} \quad \text{and} \quad B = \mathcal{C} \setminus A.$$  

Indeed, the inclusion “$\subseteq$” follows from what we proved above while the converse inclusion is an immediate consequence of Theorem 2.3.6. \qed
In the conic case, surplus-invariance is equivalent to numéraire-invariance so that acceptability amounts to requiring solvency over a pre-specified set of scenarios.

**Lemma 2.3.15.** Fix $p \in [1, \infty]$ and assume $A \subset L^p$ is a a closed ($\sigma(L^\infty, L^1)$-closed if $p = \infty$), convex, conic, acceptance set. Then, the following statements are equivalent:

(i) $A$ is surplus-invariant;

(ii) $A$ is numéraire-invariant.

**Proof.** First, assume $A$ is surplus-invariant. Take $X \in A$ and $D \in D(L^p)$. Recalling that $D \in L^\infty$, we clearly have $DX \geq P - \|D\|\infty X^-$. Since $-X^- \in A$ by surplus-invariance, we conclude that $DX \in A$ by virtue of the conicity of $A$. Hence, $A$ is numéraire-invariant. Conversely, assume $A$ is numéraire-invariant. Take $X \in A$ and set $D_n = 1_{\{X < 0\}} + \frac{1}{n}1_{\{X \geq 0\}}$ for every $n \in \mathbb{N}$. Since $D_nX \in A$ for all $n \in \mathbb{N}$ by numéraire-invariance and $D_nX \to -X^-$ as $n \to \infty$, it follows that $-X^- \in A$. Hence, $A$ is surplus-invariant. \qed

**Theorem 2.3.16.** Fix $p \in [1, \infty]$ and assume $A \subset L^p$ is a a closed ($\sigma(L^\infty, L^1)$-closed if $p = \infty$), convex, conic, surplus-invariant acceptance set. Then, there exists $A \in \mathcal{F}$ such that

$$A = \{X \in L^p ; \ X1_A \geq_0 0\}.$$ 

**Remark 2.3.17.** As proved in Koch-Medina, Munari, Šikić [71], the above representation results can be extended to any $L^p$ space, including the range $p < 1$. \qed

### 2.3.5 Duality and law-invariance

In this section we focus on the representation of law-invariant acceptance sets in the context of $L^p$ spaces. First, we need to recall the notion of quantiles.

Fix $\alpha \in (0, 1)$. An $\alpha$-**quantile** of $X \in L^0$ is any number $q \in \mathbb{R}$ with the property

$$\mathbb{P}(X < q) \leq \alpha \leq \mathbb{P}(X \leq q).$$

The set of all $\alpha$-quantiles of $X$ is the closed interval $[q^-_\alpha(X), q^+_\alpha(X)]$ where the upper $\alpha$-quantile is given by

$$q^+_\alpha(X) := \inf\{\lambda \in \mathbb{R} ; \ \mathbb{P}(X \leq \lambda) > \alpha\} = \sup\{\lambda \in \mathbb{R} ; \ \mathbb{P}(X < \lambda) \leq \alpha\}$$
and the lower $\alpha$-quantile by
\[
q^{-\alpha}(X) := \sup\{\lambda \in \mathbb{R}; \ P(X < \lambda) < \alpha\} = \inf\{\lambda \in \mathbb{R}; \ P(X \leq \lambda) \geq \alpha\}.
\]

The following important result was first shown in the context of standard probability spaces, see Remark 4.4 by Jouini, Schachermayer, Touzi [65], and was later extended to general nonatomic spaces in Proposition 1.2 by Svindland [96].

**Lemma 2.3.18.** Assume $(\Omega, \mathcal{F}, P)$ is nonatomic. Let $A \subset L^\infty$ be a closed, convex, law-invariant acceptance set. Then $A$ is $\sigma(L^\infty, L^p)$-closed for all $p \in [1, \infty]$.

**Theorem 2.3.19.** Assume $(\Omega, \mathcal{F}, P)$ is nonatomic. Fix $p \in [1, \infty]$ and let $q = \frac{p}{p-1}$. If $A \subset L^p$ is a closed, convex, law-invariant acceptance set, then
\[
A = \bigcap_{Z \in L^q_+} \{X \in L^p; \ E[ZX] \geq \sigma_A(Z)\}.
\]

In this case, for any $Z \in L^q_+$ we have
\[
\sigma_A(Z) = \inf_{X \in A} \int_0^1 q^+_\alpha(X)q^+_{1-\alpha}(Z)d\alpha.
\]

**Proof.** The above representation of $A$ follows directly (or via the preceding lemma when $p = \infty$) from Theorem 2.3.6. Now, take any $Z \in L^q_+$ and note that, by law-invariance, we have
\[
\sigma_A(Z) = \inf_{X \in A} \inf_{Y \sim P_X} E[ZY].
\]

The representation of $\sigma_A$ follows from Theorem 13.4 in Chong, Rice [25].

### 2.3.6 Duality and sensitivity

In this section, we provide a dual characterization of sensitive acceptance sets that are convex cones in $L^p$ in terms of the existence of some strictly positive element in the corresponding barrier cone.

**Theorem 2.3.20.** Fix $p \in [1, \infty]$ and assume $A \subset L^p$ is a closed ($\sigma(L^\infty, L^1)$-closed if $p = \infty$), convex, conic, sensitive acceptance set. Then, there exists $Z \in B(A)$ such that $P(Z > 0) = 1$ and
\[
A = \bigcap_{Z \in B(A) \atop P(Z > 0) = 1} \{X \in L^p; \ E[ZX] \geq 0\}.
\]
Proof. Set $q = \frac{p}{p-1}$. It follows from Proposition 2.3.4 and Theorem 2.3.6 that $\mathcal{B}(A) \subseteq L^q_+$ and

$$\mathcal{A} = \bigcap_{Z \in \mathcal{B}(A)} \{X \in L^p ; \ E[ZX] \geq 0\}.$$ 

First, we claim that

$$\mathbb{P}(Z > 0) = 1 \text{ for some } Z \in \mathcal{B}(A). \quad (2.4)$$

We start by noting that, by sensitivity, for every nonzero $X \in \mathcal{X}_-$ we find some $Z \in \mathcal{B}(A)$ such that $E[ZX] < 0$. Now, consider the collection

$$\mathcal{G} = \{A \in \mathcal{F}; \ \mathbb{P}(Z1_A > 0) = 1 \text{ for some } Z \in \mathcal{B}(A)\}.$$ 

Note that $\mathcal{G}$ contains every set of the form $\{Z > 0\}$ for $Z \in \mathcal{B}(A)$. We claim that $\mathcal{G}$ is closed under countable unions. Indeed, take a sequence $(A_n)$ in $\mathcal{G}$. Take $Z_n \in \mathcal{B}(A)$ satisfying $\mathbb{P}(Z_n 1_{A_n} > 0) = 1$ and set $\lambda_n = (2^n \|Z_n\|_q)^{-1}$ for all $n \in \mathbb{N}$. Then, it follows that

$$Z = \sum_{n \in \mathbb{N}} \lambda_n Z_n \in L^q_+.$$ 

Moreover, it is not difficult to verify that

$$\sigma_A(Z) \geq \sum_{n \in \mathbb{N}} \lambda_n \sigma_A(Z_n) = 0,$$

hence $Z \in \mathcal{B}(A)$. If we denote by $\bar{A}$ the union of the sets $A_n$, we see that $\mathbb{P}(Z1_{\bar{A}} > 0) = 1$, proving the claim. As a consequence, it follows from Lemma 2.3.9 that $\mathcal{G}$ admits an element $M$ such that $\mathbb{P}(A \cap M^c) = 0$ for all $A \in \mathcal{G}$. In particular, $\mathbb{P}(\{Z > 0\} \cap M^c) = 0$ for every $Z \in \mathcal{B}(A)$. As a result, we must have $\mathbb{P}(M) = 1$, for otherwise $1_{M^c}$ would not be zero and, by sensitivity, we would find $Z \in \mathcal{B}(A)$ satisfying $E[Z(-1_{M^c})] < 0$. Then, (2.4) follows immediately from $\mathbb{P}(M) = 1$ once we recall that $M \in \mathcal{G}$. To conclude, it is enough to note that every element in $\mathcal{B}(A)$ can be approximated, through convex combinations, by elements in $\mathcal{B}(A)$ that are strictly positive with full probability. $\Box$

Remark 2.3.21. Note that, if $p = 1$, the assertion in (2.4) is nothing but a different formulation of the well-known Kreps-Yan theorem which, in our language, can be then reformulated as a statement on sensitive acceptance sets. We refer to Theorem 1.62 in Föllmer, Schied [50] for a precise statement. $\Box$
2.4 Examples of acceptance sets

In this section we discuss the most important examples of acceptance sets used in risk management. Throughout we fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We will mainly focus on the cases which are most relevant for applications, namely the finite and the nonatomic case. The underlying order structure is always understood with respect to the canonical almost-sure order.

2.4.1 Acceptability based on Value-at-Risk

We start by introducing the framework of Value-at-Risk.\(^9\) The underlying model space is \(\mathcal{X} = L^p\) for fixed \(p \in [0, \infty]\). Given a probability level \(\alpha \in (0, 1)\), we define the \textit{Value-at-Risk} (VaR) of a position \(X \in L^p\) at the level \(\alpha\) as

\[
\text{VaR}_\alpha(X) := \inf \{ m \in \mathbb{R} ; \mathbb{P}(X + m < 0) \leq \alpha \}.
\]

Note that, up to a sign, \(\text{VaR}_\alpha(X)\) coincides with the upper \(\alpha\)-quantile of the random variable \(X\). The corresponding acceptance set is defined by

\[
\mathcal{A}_\text{VaR}^p(\alpha) := \{ X \in L^p ; \text{VaR}_\alpha(X) \leq 0 \}.
\]

We start by summarizing, for easy reference, some elementary properties of VaR and provide an equivalent formulation of VaR-acceptability. This formulation has a more transparent financial interpretation: the set \(\mathcal{A}_\text{VaR}^p(\alpha)\) consists of those positions whose \textit{default probability} is not exceeding the threshold \(\alpha\). As such, VaR-acceptability depends only on the \textit{frequency} of losses and not on their \textit{magnitude}. This is perhaps one of the most controversial aspects of VaR together with the lack of convexity discussed below. Recall that a map \(\rho : L^p \to \mathbb{R}\) is called \textit{law-invariant} if \(\rho(X) = \rho(Y)\) whenever \(X \sim \mathbb{P} Y\).

\textbf{Lemma 2.4.1.} For any \(\alpha \in (0, 1)\) the map \(\text{VaR}_\alpha\) is cash-additive, decreasing, conic and law-invariant. Moreover,

\[
\mathcal{A}_\text{VaR}^p(\alpha) = \{ X \in L^p ; \mathbb{P}(X < 0) \leq \alpha \}.
\]

\(^9\)This framework was developed by J.P. Morgan and publicly released in the RiskMetrics Technical Document of 1994; see Guldimann [54] for more information.
Proof. We only prove the last assertion. The inclusion “⊇” is immediate. Conversely, take $X \in L^p$ and assume $\text{VaR}_\alpha(X) \leq 0$ so that $\mathbb{P}(X < -\frac{1}{n}) \leq \alpha$ for all $n \in \mathbb{N}$. We conclude by noting that $\mathbb{P}(X < 0) = \lim_{n \to \infty} \mathbb{P}(X < -\frac{1}{n}) \leq \alpha$. 

**Proposition 2.4.2.** For any $\alpha \in (0, 1)$ the set $A_{\text{VaR}}^p(\alpha)$ is a conic, numéraire-, surplus-, law-invariant acceptance set.

It is well-known that $A_{\text{VaR}}^p(\alpha)$ fails to be convex in general. Here, we provide a simple characterization of this fact.

**Proposition 2.4.3.** Let $\alpha \in (0, 1)$. The following statements are equivalent:

(i) $A_{\text{VaR}}^p(\alpha)$ is convex;

(ii) for any $A, B \in \mathcal{F}$ we have $\mathbb{P}(A) \leq \alpha, \mathbb{P}(B) \leq \alpha \implies \mathbb{P}(A \cup B) \leq \alpha$.

**Proof.** First, assume $A_{\text{VaR}}^p(\alpha)$ is convex but we find $A, B \in \mathcal{F}$ such that $\mathbb{P}(A) \leq \alpha, \mathbb{P}(B) \leq \alpha$ and $\mathbb{P}(A \cup B) > \alpha$. Setting $X = -1_A$ and $Y = -1_B$, it is immediate to see that $X$ and $Y$ both belong to $A_{\text{VaR}}^p(\alpha)$ but $X + Y$ does not. Since $A_{\text{VaR}}^p(\alpha)$ is a cone, this implies that it cannot be convex, contradicting the assumption. Hence, (i) must imply (ii). Conversely, assume (ii) holds but $A_{\text{VaR}}^p(\alpha)$ is not convex so that we find two elements $X$ and $Y$ in $A_{\text{VaR}}^p(\alpha)$ whose sum lies outside $A_{\text{VaR}}^p(\alpha)$. Set $A = \{X < 0\}$ and $B = \{Y < 0\}$ and note that $\{X + Y < 0\} \subseteq A \cup B$. Since this contradicts (ii), we conclude that $A_{\text{VaR}}^p(\alpha)$ must be convex. 

The following immediate result shows that $A_{\text{VaR}}^p(\alpha)$ is sensitive only in trivial cases.

**Proposition 2.4.4.** Let $\alpha \in (0, 1)$. The following statements are equivalent:

(i) $A_{\text{VaR}}^p(\alpha)$ is sensitive;

(ii) $\mathbb{P}(A) > \alpha$ for every nonempty $A \in \mathcal{F}$;

(iii) $A_{\text{VaR}}^p(\alpha) = L_+^p$.

We turn to investigate the topological properties of $A_{\text{VaR}}^p(\alpha)$. A different proof of the following closedness result can be found in Theorem 3 by Chambers [22].
Proposition 2.4.5. For any $\alpha \in (0, 1)$ the set $A_{VaR}^p(\alpha)$ is closed in $L^p$.

Proof. It is enough to show that $A_{VaR}^p(\alpha)$ is closed with respect to the convergence in probability. Take a sequence $(X_n)$ in $A_{VaR}^p(\alpha)$ converging in probability to $X \in L^p$. We claim that $X \in A_{VaR}^p(\alpha)$. Fix an arbitrary $m > 0$ and take $\varepsilon \in (0, m)$. Then, set $A_n = \{|X_n - X| > \varepsilon\}$ for any $n \in \mathbb{N}$. We easily see that

$$P(X + m < 0) \leq P(A_n) + P(X_n + m - \varepsilon < 0) \leq P(A_n) + \alpha$$

for every $n \in \mathbb{N}$. Since $P(A_n) \to 0$ as $n \to \infty$, it follows that $P(X + m < 0) \leq \alpha$ for all $m > 0$. Hence, we conclude that $X \in A_{VaR}^p(\alpha)$, proving the claim.

Proposition 2.4.6. Assume $p = \infty$. For any $\alpha \in (0, 1)$ the set $A_{VaR}^\infty(\alpha)$ has nonempty interior in $L^\infty$. Moreover,

$$\text{int } A_{VaR}^\infty(\alpha) = \{X \in L^\infty; \text{Var}_\alpha(X) < 0\}.$$

Proof. The interior of $A_{VaR}^\infty(\alpha)$ is nonempty by Proposition 2.2.7. The characterization of the corresponding interior points follows immediately by cash-additivity.

Proposition 2.4.7. Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic and take $p \in [0, \infty)$. For any $\alpha \in (0, 1)$ the set $A_{VaR}^p(\alpha)$ has nonempty interior in $L^p$. Moreover,

$$\text{int } A_{VaR}^p(\alpha) = \{X \in L^p; \mathbb{P}(X \leq 0) < \alpha\}.$$

In particular, for $U \in L^p_+$ we have $U \in \text{int } A_{VaR}^p(\alpha)$ if and only if $\mathbb{P}(U = 0) < \alpha$.

Proof. First recall that, by Proposition 2.2.8, the core and the interior of any acceptance set in $L^p$ coincide. Take now $X \in L^p$ with $\mathbb{P}(X \leq 0) < \alpha$. If $X \notin \text{core } A_{VaR}^p(\alpha)$, then we can find a positive $Z \in L^p_+$ and a sequence of real numbers $(\lambda_n)$ converging to zero from above such that $\mathbb{P}(X < \lambda_n Z) > \alpha$ for any $n \in \mathbb{N}$. This follows from Proposition 2.2.10. However, this would imply $\mathbb{P}(X \leq 0) \geq \alpha$, contradicting what assumed above. In conclusion, $X$ must belong to $\text{core } A_{VaR}^p(\alpha)$, showing that $A_{VaR}^p(\alpha)$ has nonempty interior and proving the inclusion “$\supset$”.

To prove the converse inclusion take $X \in \text{core } A_{VaR}^p(\alpha)$ and assume $\mathbb{P}(X \leq 0) \geq \alpha$. Note that $\mathbb{P}(X > 0) > 0$, hence $\mathbb{P}(0 < X < \varepsilon) > 0$ for some $\varepsilon > 0$, because
$X \in \mathcal{A}_{\text{VaR}}^p(\alpha)$. Since $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic, we find a sequence $(A_n)$ of pairwise disjoint measurable subsets of $\{0 < X < \varepsilon\}$ satisfying $0 < \mathbb{P}(A_n) < n^{-p-2}$ for any $n \in \mathbb{N}$. Setting

$$U = 1_{\{X \leq 0\}} + \sum_{n=1}^{\infty} n1_{A_n} \in L^p_+$$

it is easy to see that for every $\lambda > 0$ there exists $n \in \mathbb{N}$ satisfying

$$\mathbb{P}(X < \lambda U) \geq \mathbb{P}(X \leq 0) + \mathbb{P}(A_n) > \alpha.$$ 

However, this contradicts $X \in \text{core} \mathcal{A}_{\text{VaR}}^p(\alpha)$. Hence the inclusion "$\subseteq$" must also hold, concluding the proof.

2.4.2 Acceptability based on Expected Shortfall

In this section we discuss acceptability based on Expected Shortfall. Throughout, we set $\mathcal{X} = L^p$ for a fixed $p \in [1, \infty]$. Given a probability level $\alpha \in (0, 1)$, we define the Expected Shortfall (ES) of a position $X \in L^p$ at the level $\alpha$ as

$$\text{ES}_\alpha(X) := \frac{1}{\alpha} \int_0^{\alpha} \text{VaR}_\beta(X)d\beta.$$ 

The corresponding acceptance set is defined by

$$\mathcal{A}_{\text{ES}}^p(\alpha) := \{X \in L^p ; \text{ES}_\alpha(X) \leq 0\}.$$ 

The following proposition summarizes some well-known facts about ES, which will be used without further reference.

**Lemma 2.4.8.** For any $\alpha \in (0, 1)$ the map $\text{ES}_\alpha$ is cash-additive, decreasing, convex, conic and law-invariant. Moreover, $\text{ES}_\alpha$ is continuous on $L^p$ and $\text{ES}_\alpha(X) \geq \text{VaR}_\alpha(X)$ for all $X \in L^p$.

The corresponding properties of $\mathcal{A}_{\text{ES}}^p(\alpha)$ are recorded in the next result. In particular, note that ES-acceptability is more stringent than VaR-acceptability.

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10The notion of Expected Shortfall was introduced, under different names and guises, by several authors at the beginning of the 2000s. We refer to Acerbi, Tasche [3] for a general overview.
Proposition 2.4.9. For any $\alpha \in (0,1)$ the set $A^p_{ES}(\alpha)$ is a convex, conic, law-invariant, sensitive acceptance set. Moreover, $A^p_{ES}(\alpha) \subseteq A^p_{VaR}(\alpha)$.

The following lemma will help clarifying when acceptance sets based on ES are numéraire- and surplus-invariant.

Lemma 2.4.10. Fix $\alpha \in (0,1)$ and take $X \notin A^p_{ES}(\alpha)$. Then, the following statements are equivalent:

(i) $\mathbb{P}(X < 0) < \alpha$;

(ii) $Y^- = X^-$ for some $Y \in A^p_{ES}(\alpha)$;

(iii) $Y^- \geq_{\mathbb{P}} X^-$ for some $Y \in A^p_{ES}(\alpha)$.

Proof. First, assume that $\mathbb{P}(X < 0) < \alpha$ and consider $Y = X + \lambda 1_{\{X \geq 0\}}$ for $\lambda > 0$. Note that $X^- = Y^-$. Moreover, set $\gamma = \mathbb{P}(X < 0)$. Note that we always have $\text{VaR}_\beta(Y) \leq \text{VaR}_\beta(-Y^-) = \text{VaR}_\beta(-X^-)$.

In addition, if $\beta \in (\gamma, \alpha)$, we obtain

$$\mathbb{P}(Y - \lambda < 0) = \mathbb{P}(X < 0) = \gamma < \beta,$$

showing that $\text{VaR}_\beta(Y) \leq -\lambda$. As a consequence, it follows that

$$\text{ES}_\alpha(Y) = \frac{1}{\alpha} \int_0^\gamma \text{VaR}_\beta(Y)d\beta + \frac{1}{\alpha} \int_{\gamma}^\alpha \text{VaR}_\beta(Y)d\beta \leq \frac{\gamma}{\alpha} \text{ES}_\gamma(-X^-) - \frac{\lambda \alpha - \gamma}{\alpha}.$$

Since $\alpha > \gamma$, we see that $Y \in A^p_{ES}(\alpha)$ if we take $\lambda$ sufficiently large. Hence, (i) implies (ii). To conclude, it is enough to show that (iii) implies (i). To this end, assume that $Y^- \geq_{\mathbb{P}} X^-$ for some $Y \in A^p_{ES}(\alpha)$ but $\mathbb{P}(X < 0) \geq \alpha$. Then $\mathbb{P}(Y < 0) \geq \alpha$, implying that $\text{VaR}_\beta(Y) > 0$ for every $\beta \in (0,\alpha)$. As a result, we get $Y \notin A^p_{ES}(\alpha)$, contradicting the initial assumption. Hence, we must have $\mathbb{P}(X < 0) < \alpha$.

Using the preceding lemma we can show that, unless we consider trivial situations, the acceptance set based on ES fails to be either numéraire- or surplus-invariant.

Proposition 2.4.11. For any $\alpha \in (0,1)$, the following statements are equivalent:
(i) $\mathcal{A}_{ES}^p(\alpha)$ is numéraire-invariant;

(ii) $\mathcal{A}_{ES}^p(\alpha)$ is surplus-invariant;

(iii) $\mathbb{P}(A) \geq \alpha$ for every $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$;

(iv) $\mathcal{A}_{ES}^p(\alpha) = L_p^+$.

**Proof.** Assume that (i) holds. Note that $\mathcal{A}_{ES}^p(\alpha)$ is a closed, convex cone. In particular, the continuity of $ES_\alpha$ ensures that $\mathcal{A}_{ES}^p(\alpha)$ is closed. Moreover, $\mathcal{A}_{ES}^\infty(\alpha)$ is also $\sigma(L^\infty, L^1)$-closed as a consequence of Theorem 4.52 in Föllmer, Schied [50]. Hence, (ii) follows from Theorem 2.3.10.

Then, assume (ii) holds and take $A \in \mathcal{F}$ satisfying $\mathbb{P}(A) > 0$. If $\mathbb{P}(A) < \alpha$, setting $X = -1_A$ we would find, by the preceding lemma, some $Y \in \mathcal{A}_{ES}^p(\alpha)$ such that $Y^- \geq \mathbb{P} X^-$. However, we clearly have $X \notin \mathcal{A}_{ES}^p(\alpha)$ by sensitivity, contradicting surplus-invariance. Hence, (iii) must hold.

Now, assume (iii) holds and take $X \notin L_p^+$ so that $\mathbb{P}(X < 0) > 0$. Then, we must have $\mathbb{P}(X < 0) > \beta$, hence $VaR_\beta(X) > 0$, for every $\beta \in (0, \alpha)$. This yields $ES_\alpha(X) > 0$, proving that $\mathcal{A}_{ES}^p(\alpha) \subseteq L_p^+$. Since the converse inclusion is trivial, it follows that (iv) holds. We conclude by noting that $L_p^+$ is clearly numéraire-invariant.

The lack of numéraire-invariance and surplus-invariance has important financial implications. First, we show that for any position that is acceptable under VaR below the level $\alpha$, we find a position with the same negative part that is acceptable under ES at the level $\alpha$. Hence, there is virtually no difference in terms of the range of tail behaviours allowed by VaR and ES.

**Proposition 2.4.12.** Take $\alpha \in (0, 1)$ and assume $\beta \in (0, \alpha)$. Then, for any $X \in \mathcal{A}_{VaR}^p(\beta)$ there exists $Y \in \mathcal{A}_{ES}^p(\alpha)$ such that $Y^- = X^-$.  

**Proof.** Take $X \in \mathcal{A}_{VaR}^p(\beta)$ and assume $X \notin \mathcal{A}_{ES}^p(\alpha)$, otherwise the claim is trivial. In this case, the assertion follows immediately from Lemma 2.4.10. 

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[11] A detailed analysis of the following “unexpected shortfalls” of Expected Shortfall can be found in Koch-Medina, Munari [70].
The next result shows that, for any position say $X$ that is rejected by ES at a level $\alpha > P(X < 0)$, we can find an acceptable position with a larger default probability than that of $X$ (in fact, as close to $\alpha$ as we like) and such that, when it defaults, it can default by strictly larger amounts than $X$ (in fact, as large as we choose). This implies, from a different angle, that the tail behaviour allowed by capital adequacy tests based on ES is arguably not too different to that allowed by VaR tests.

**Proposition 2.4.13.** Assume $(\Omega, \mathcal{F}, P)$ is nonatomic and fix $\alpha \in (0, 1)$. If $X \notin A_{ES}^p(\alpha)$ and $P(X < 0) < \alpha$, then for any $\gamma \in (P(X < 0), \alpha)$ and $\lambda > 0$ there exists $Y \in A_{ES}^p(\alpha)$ satisfying:

(i) $\{X < 0\} \subset \{Y < 0\}$;

(ii) $P(Y < 0) = \gamma$;

(iii) $Y^- \geq \max\{X^-, \lambda\}$ on $\{Y < 0\}$.

**Proof.** Set $A = \{X < 0\}$ and $\beta = P(A)$. Moreover, take any $\gamma \in (\beta, \alpha)$. Since $(\Omega, \mathcal{F}, P)$ is nonatomic, we find a measurable set $B \supseteq A$ such that $P(B) = \gamma$. Then, take $\lambda > 0$ and set

$$Z = X - \lambda 1_B - X 1_{B \setminus A} \in L^p.$$ 

Clearly, we have $\{Z < 0\} = B$ and

$$Z^- = (-X + \lambda) 1_A + \lambda 1_{B \setminus A} = X^- + \lambda 1_B.$$ 

In particular, we have $P(Z < 0) = \gamma$. Since $\gamma < \alpha$, Lemma 2.4.10 tells us that we find $Y \in A_{ES}^p(\alpha)$ such that $Y^- = Z^-$. The random variable $Y$ satisfies the above conditions. \qed 

We turn to focus on the main topological properties of $A_{ES}^p(\alpha)$. We denote by $\text{cl}_p S$ the closure in $L^p$ of any subset $S$ of $L^p$.

**Proposition 2.4.14.** For any $\alpha \in (0, 1)$ the set $A_{ES}^p(\alpha)$ is closed in $L^p$. In particular, $A_{ES}^p(\alpha) = \text{cl}_p A_{ES}^\infty(\alpha)$. 

Proof. The closedness of $A^p_{ES}(\alpha)$ follows directly from the continuity of $ES_\alpha$. To conclude, we only need to prove that any $X \in A^p_{ES}(\alpha)$ is the limit in $L^p$ of a suitable sequence of elements in $A^\infty_{ES}(\alpha)$. First, set $X_\lambda = \lambda + (1 - \lambda)X$ for $\lambda \in (0, 1)$ and note that $X_\lambda \rightarrow X$ in $L^p$ as $\lambda \rightarrow 0$. Since

$$ES_\alpha(X_\lambda) \leq \lambda ES_\alpha(1) + (1 - \lambda) ES_\alpha(X) < 0,$$

we can assume without loss of generality that $ES_\alpha(X) < 0$. In this case, for any sequence $(X_n) \subset L^\infty$ converging to $X$ in $L^p$, we must eventually have $ES_\alpha(X_n) < 0$ by continuity, concluding the proof.

**Proposition 2.4.15.** For any $\alpha \in (0, 1)$ the set $A^p_{ES}(\alpha)$ has nonempty interior in $L^p$. Moreover, $\text{Int} A^p_{ES}(\alpha) = \{X \in L^p ; ES_\alpha(X) < 0\} \subseteq \{X \in L^p ; \mathbb{P}(X \leq 0) < \alpha\}$.

If $U \in L^p_+$, we have $U \in \text{Int} A^p_{ES}(\alpha)$ if and only if $\mathbb{P}(U = 0) < \alpha$.

**Proof.** The equality follows directly by the cash-additivity and continuity of $ES_\alpha$. Moreover, take $X \in L^p$ satisfying $ES_\alpha(X) < 0$. Then we must have $\text{VaR}_\beta(X) < 0$ for some $\beta \in (0, \alpha)$. Hence, we find $m < 0$ such that

$$\mathbb{P}(X \leq 0) \leq \mathbb{P}(X + m < 0) \leq \beta < \alpha,$$

showing the inclusion in the above assertion. To conclude the proof, we only need to prove the “if” implication. Take $U \in L^p_+$ and note, first, that $\text{VaR}_\beta(U) \leq 0$ for every $\beta \in (0, \alpha)$. Hence, $ES_\alpha(U) \leq 0$ as well. Now, assume that $\mathbb{P}(U = 0) < \alpha$. Then we must find $\lambda > 0$ such that $\gamma = \mathbb{P}(U < \lambda) < \alpha$. As a result, $\text{VaR}_\beta(U) < 0$ for all $\beta \in (\gamma, \alpha)$, implying that $ES_\alpha(U) < 0$.

Finally, we provide an explicit dual representation of $A^p_{ES}(\alpha)$.

**Proposition 2.4.16.** Let $q = \frac{p}{p-1}$. For any $\alpha \in (0, 1)$ we have

$$A^p_{ES}(\alpha) = \bigcap_{Z \in \mathcal{B}(A^p_{ES}(\alpha))} \{X \in L^p ; \mathbb{E}[ZX] \geq 0\}$$

where

$$\mathcal{B}(A^p_{ES}(\alpha)) = \left\{Z \in L^q_+ ; Z \leq_{\mathbb{P}} \frac{1}{\alpha}\right\}.$$
Proof. The above representation of $A^p_{ES}(\alpha)$ follows from Theorem 2.3.6. If $p = \infty$, the explicit form of $B(A^\infty_{ES}(\alpha))$ follows from Theorem 4.52 in Föllmer, Schied [50]. Now, take any $p \in [1, \infty)$. In this case, the claim follows once we observe that $B(A^p_{ES}(\alpha)) = B(A^\infty_{ES}(\alpha)) \cap L^q$ by virtue of Proposition 2.4.14. 

2.4.3 Acceptability based on Test Scenarios

In this section, we focus on acceptability based on the SPAN methodology. Throughout, we set $X = L^p$ for a fixed $p \in [1, \infty]$. For any set of scenarios $A \in \mathcal{F}$ we define the set

$$\text{SPAN}^p(A) := \{X \in L^p ; X1_A \geq \mathbb{P} 0\}.$$ 

A financial position $X$ is acceptable with respect to $\text{SPAN}^p(A)$ if the default probability of $X$ conditional to the scenario set $A$ is null or, equivalently, if a company with position $X$ is solvent in the scenarios specified by $A$ with full probability.

We will always assume that $\mathbb{P}(A) > 0$ so that $\text{SPAN}^p(A)$ is a strict subset of $L^p$. In this case, we say that the scenario set $A$ is admissible. The proof of the next proposition is straightforward and left to the reader.

Proposition 2.4.17. For any admissible $A \in \mathcal{F}$, the set $\text{SPAN}^p(A)$ is a convex, conic, numéraire-, surplus-invariant acceptance set.

If $\Omega$ is finite, the acceptance set $\text{SPAN}^p(A)$ may be either law-invariant or not. In the nonatomic case, the following result holds.

Proposition 2.4.18. Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic and take an admissible $A \in \mathcal{F}$. Then, $\text{SPAN}^p(A)$ is law-invariant if and only if $\mathbb{P}(A) = 1$.

Proof. If $\mathbb{P}(A) = 1$ it is immediate to see that $\text{SPAN}^p(A) = L^p_+$, hence $\text{SPAN}^p(A)$ is law-invariant. Conversely, assume $\mathbb{P}(A) < 1$ and take measurable sets $B \subseteq A$ and $C \subseteq A^c$ satisfying $\mathbb{P}(B) = \mathbb{P}(C)$. Setting $X = 1_B - 1_C$ and $Y = 1_C - 1_B$, it is easy to see that $X \sim \mathbb{P} Y$ and $X \in \text{SPAN}^p(A)$, while $Y \notin \text{SPAN}^p(A)$. This proves the “only if” implication.

The acronym SPAN stands for Standard Portfolio Analysis of Risk and refers to a methodology used to compute margin requirements developed and implemented, since 1988, by the Chicago Mercantile Exchange. For more detailed information, we refer to the documentation at http://www.cmegroup.com/clearing/span-methodology.html.
The following result on sensitivity is also immediate.

**Proposition 2.4.19.** For any admissible \( A \in \mathcal{F} \), the set \( \text{SPAN}^p(A) \) is sensitive if and only if \( \mathbb{P}(A) = 1 \).

Next, we focus on the topological properties of acceptance sets of \( \text{SPAN} \) type.

**Proposition 2.4.20.** For any admissible \( A \in \mathcal{F} \) the acceptance set \( \text{SPAN}^p(A) \) is closed in \( L^p \). Moreover, we have \( \text{SPAN}^p(A) = \text{cl}_p \text{SPAN}^\infty(A) \).

**Proof.** It suffices to prove that \( \text{SPAN}^p(A) \) is closed with respect to convergence in probability. But this is clear since any sequence converging in probability admits a subsequence which converges almost surely.

We know from Proposition 2.2.6 that \( \text{SPAN}^\infty(A) \) has nonempty interior in \( L^\infty \). The characterization of the corresponding interior is straightforward.

**Proposition 2.4.21.** Assume \( p = \infty \). For any admissible \( A \in \mathcal{F} \) the acceptance set \( \text{SPAN}^\infty(A) \) has nonempty interior in \( L^\infty \). Moreover,

\[
\text{int SPAN}^\infty(A) = \{ X \in L^\infty ; X1_A \geq \mathbb{P} \varepsilon 1_A \text{ for some } \varepsilon > 0 \}.
\]

**Proposition 2.4.22.** Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is nonatomic and take \( p \in [1, \infty) \). For any admissible \( A \in \mathcal{F} \) the acceptance set \( \text{SPAN}^p(A) \) has empty interior in \( L^p \).

**Proof.** Take an admissible \( A \in \mathcal{F} \) and recall that \( \text{SPAN}^p(A) \) is convex. We show that any open neighborhood of \( X \in \text{SPAN}^p(A) \cap L^\infty \) contains elements that do not belong to \( \text{SPAN}^p(A) \). The statement will then follow by the density of \( L^\infty \) in \( L^p \).

Take a bounded \( X \in \text{SPAN}^p(A) \) and fix \( r > 0 \). Moreover, take \( \lambda > \|X\|_{\infty} \). Since \((\Omega, \mathcal{F}, \mathbb{P})\) is nonatomic, we find a measurable set \( B \subseteq A \) with \( 0 < \mathbb{P}(B) < r^p \lambda^{-p} \).

If we define \( Y = (X - \lambda)1_B + X1_{B^c} \in L^p \), we easily see that \( \|X - Y\|_p < r \) but \( Y \notin \text{SPAN}^p(A) \). This concludes the proof.

We conclude by providing a dual representation of \( \text{SPAN} \) acceptance sets.

**Proposition 2.4.23.** Let \( q = \frac{p}{p-1} \). For any admissible \( A \in \mathcal{F} \) we have

\[
\text{SPAN}^p(A) = \bigcap_{Z \in \mathcal{B}(\text{SPAN}^p(A))} \{ X \in L^p ; \mathbb{E}[ZX] \geq 0 \}
\]

where

\[
\mathcal{B}(\text{SPAN}^p(A)) = \{ Z \in L^q_+ ; Z1_{A^c} = 0 \}.
\]
Proof. First of all, we claim that $\text{SPAN}^\infty(A)$ is $\sigma(L^\infty,L^1)$-closed. Indeed, take a net $(X_\alpha) \subset \text{SPAN}^\infty(A)$ converging to some $X \in L^\infty$ under $\sigma(L^\infty,L^1)$. Assume $X \notin \text{SPAN}^\infty(A)$. Setting $B = A \cap \{X < 0\}$, we see that $\mathbb{E}[X_\alpha 1_B] \to \mathbb{E}[X 1_B]$. However, $\mathbb{E}[X_\alpha 1_B] \geq 0$ for all $\alpha$ while $\mathbb{E}[X 1_B] < 0$, showing that $\text{SPAN}^\infty(A)$ must be $\sigma(L^\infty,L^1)$-closed.

Since $\text{SPAN}^p(A)$ is a closed ($\sigma(L^\infty,L^1)$-closed if $p = \infty$) convex cone, the above representation follows immediately from Theorem 2.3.6. Moreover, Lemma 2.3.12 implies that

$$B(\text{SPAN}^p(A)) = \{Z \in L^q_+; \ Z X^- = 0, \ \forall X \in \text{SPAN}^p(A)\},$$

yielding the explicit representation for the barrier cone. \hfill \qed

### 2.4.4 Acceptability based on Expected Utility

In this section we define a concept of acceptability based on expected utility over a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A nonconstant function $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is called a utility function if it is right-continuous, increasing and concave. Being concave, $u$ can be discontinuous at most at one point, where it is assumed to be right-continuous. Moreover, we have $u(-\infty) = -\infty$. These properties will be used without further reference.

Before proceeding, we want to highlight the following simple result about expected utility. It can be regarded as a Jensen-type result for functions that may take infinite values.

**Lemma 2.4.24.** For any $X \in L^1$, the function $u(X) : \Omega \to \mathbb{R} \cup \{-\infty\}$ is measurable and $\mathbb{E}[u(X)]$ exists. Moreover, $\mathbb{E}[u(X)] < \infty$.

**Proof.** The measurability is an immediate consequence of the monotonicity of $u$. We conclude by showing that $\mathbb{E}[u(X)^+] < \infty$. Set $x_0 = \inf\{x \in \mathbb{R}; \ u(x) \geq 0\}$ and note that the claim is trivially satisfied if $x_0 = \infty$. Otherwise, since $u$ is concave, we find $x_1 > x_0$ and $\alpha > 0$ such that $u(x) \leq \alpha x$ whenever $x \geq x_1$. In this case, we have

$$\mathbb{E}[u(X)^+] \leq u(x_1)\mathbb{P}(x_0 \leq X < x_1) + \alpha\mathbb{E}[X 1_{\{X \geq x_1\}}] < \infty. \hfill \qed$$
For every $p \in [1, \infty]$ and every expected utility level $\alpha \in \mathbb{R}$ we define the set
\[
\mathcal{A}_u^p(\alpha) := \{ X \in L^p ; \mathbb{E}[u(X)] \geq \alpha \}.
\]
We say that the couple $(u, \alpha)$ is admissible if $u(x_0) \geq \alpha$ for some $x_0 \in \mathbb{R}$. In this case $\mathcal{A}_u^p(\alpha)$ is easily seen to be nonempty.

The first result is an immediate consequence of the monotonicity and concavity of $u$. Note that, in general, acceptance sets based on expected utility fail to be conic, numéraire- and surplus-invariant, and sensitive.

**Proposition 2.4.25.** Assume $(u, \alpha)$ is admissible. Then, $\mathcal{A}_u^p(\alpha)$ is a convex, law-invariant acceptance set.

We turn to the topological properties of acceptance sets of expected-utility type.

**Proposition 2.4.26.** Assume $(u, \alpha)$ is admissible. Then, $\mathcal{A}_u^p(\alpha)$ is closed in $L^p$. Moreover, we have $\mathcal{A}_u^p(\alpha) = \text{cl}_p \mathcal{A}_u^\infty(\alpha)$.

**Proof.** Let $(X_n)$ be a sequence of elements in $\mathcal{A}_u^p(\alpha)$ converging to some $X \in L^p$. Without loss of generality, we can assume $\|X_n - X\|_p \leq 2^{-n}$ for all $n \in \mathbb{N}$. Define
\[
Y_n = \sum_{k=n}^{\infty} |X_k - X| \in L^1
\]
and set $Z_n = X + Y_n \in L^1$ for all $n \in \mathbb{N}$. Note that $Z_n \geq_p X_n$ and, thus,
\[
\mathbb{E}[u(Z_n)] \geq \mathbb{E}[u(X_n)] \geq \alpha
\]
for any $n \in \mathbb{N}$. In addition, $(Z_n)$ is a decreasing sequence converging to $X$ in the $L^1$ topology, hence we can assume that $Z_n \rightarrow X$ almost surely from above as $n \rightarrow \infty$. This can always be ensured by passing to a suitable subsequence. Since $u$ is right-continuous, it follows that $u(Z_n) \rightarrow u(X)$ almost surely from above as $n \rightarrow \infty$. Since $\mathbb{E}[u(Z_1)] < \infty$ due to Lemma 2.4.24, we finally find that $\mathbb{E}[u(Z)] \rightarrow \mathbb{E}[u(X)]$ by monotone convergence, showing that $\mathbb{E}[u(X)] \geq \alpha$ and proving closedness.

Now, take $p < \infty$. To conclude, we only need to prove that any element $X \in \mathcal{A}_u^p(\alpha)$ is the limit in $L^p$ of a suitable sequence $(X_n)$ of elements in $\mathcal{A}_u^\infty(\alpha)$. To this end, fix $X \in \mathcal{A}_u^p(\alpha)$. If $X$ is bounded from above almost surely, define
Moreover, \( X_n = X1_{\{X > -n\}} \in L^\infty \) for all \( n \in \mathbb{N} \). Clearly, \( X_n \to X \) in \( L^p \) as \( n \to \infty \). Moreover, \( X_n \geq_P X \) implies that \( \mathbb{E}[u(X_n)] \geq \alpha \) for any \( n \in \mathbb{N} \), proving the claim. To prove the statement in the general case it is then enough to find a sequence \( (X_n) \) of elements in \( A^p_u(\alpha) \) that are bounded from above almost surely and converge to \( X \).

First, assume \( u \) is bounded from above by \( \alpha \) and set \( a = \inf \{ x \in \mathbb{R} ; u(x) = \alpha \} \). Note that \( u(a) = \alpha \) by right-continuity. Since \( \mathbb{E}[u(X)] \geq \alpha \), we must have \( X \geq_P a \). Setting

\[
X_n = X1_{\{X \leq n\}} + a1_{\{X > n\}} \in L^\infty
\]

for all \( n \in \mathbb{N} \), it is easy to see that \( X_n \to X \) in \( L^p \) as \( n \to \infty \) and \( X_n \geq_P a \), hence \( X_n \in A^p_u(\alpha) \), for any \( n \in \mathbb{N} \), proving the claim.

Next, assume \( u(x_0) > \alpha \) for some \( x_0 \in \mathbb{R} \) and define \( X_\lambda = \lambda X + (1 - \lambda)x_0 \) for \( \lambda \in (0, 1) \). Then, the concavity of \( u \) implies

\[
\mathbb{E}[u(X_\lambda)] \geq \lambda \mathbb{E}[u(X)] + (1 - \lambda)u(x_0) > \alpha.
\]

Since \( X_\lambda \to X \) in \( L^p \) as \( \lambda \to 1 \), this shows we may assume \( \mathbb{E}[u(X)] > \alpha \) without loss of generality.

If \( u \) is continuous on \( \mathbb{R} \), define \( X_n = X1_{\{X \leq n\}} \in L^p \) and note that \( (X_n) \) converges to \( X \) in \( L^p \). By continuity, we can assume that \( u(X_n) \to u(X) \) almost surely as \( n \to \infty \). This is always true by passing to a suitable subsequence. Moreover, note that \( u(X_n) \) is increasingly convergent to \( u(X) \) almost surely. Since \( \mathbb{E}[u(X)] > -\infty \), we must have \( u(X) \in L^1 \) due to Lemma 2.4.24. Hence, \( \mathbb{E}[u(X_1)] > -\infty \) and we can apply monotone convergence to obtain \( \mathbb{E}[u(X_n)] \to \mathbb{E}[u(X)] \) as \( n \to \infty \).

But \( \mathbb{E}[u(X)] > \alpha \) implies that we eventually have \( X_n \in A^p_u(\alpha) \), proving the claim.

Finally, assume \( u \) is discontinuous at \( a \in \mathbb{R} \). Note that \( u(a) > -\infty \). Since \( \mathbb{E}[u(X)] > -\infty \), we must have \( X \geq_P a \). Now, define

\[
X_n = X1_{\{X \leq n\}} + a1_{\{X > n\}} \in L^\infty
\]

for any \( n \in \mathbb{N} \) so that \( (X_n) \) converges to \( X \) in \( L^p \). Note that \( X_n \geq_P a \) for all \( n \in \mathbb{N} \). Hence, by passing to a convenient subsequence, we can always assume that \( u(X_n) \to u(X) \) almost surely for \( n \to \infty \). Indeed, \( u(X_n) \) is increasingly convergent to \( u(X) \) almost surely. Since \( \mathbb{E}[u(X_1)] \geq u(a) > -\infty \), we conclude
that $\mathbb{E}[u(X_n)] \to \mathbb{E}[u(X)]$ as $n \to \infty$ by monotone convergence. As above, this eventually implies that $X_n \in \mathcal{A}_u^p(\alpha)$, proving the claim also in this case and concluding the proof.

Next we characterize when the interior of $\mathcal{A}_u^p(\alpha)$ is nonvoid. Note that $\mathcal{A}_u^\infty(\alpha)$ has nonempty interior in $L^\infty$ due to Proposition 2.2.6.

**Proposition 2.4.27.** Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic and take $p \in [1, \infty)$. Moreover, assume $(u, \alpha)$ is admissible. The following statements are equivalent:

(i) $\mathcal{A}_u^p(\alpha)$ has nonempty interior in $L^p$;

(ii) $u(x_0) > \alpha$ for some $x_0 \in \mathbb{R}$ and

$$\lim_{x \to \infty} \frac{x^p}{u(-x)} < 0.$$  

(2.5)

**Proof.** To prove that (i) implies (ii), assume first that $u(x) \leq \alpha$ for all $x \in \mathbb{R}$ and take $X \in \mathcal{A}_u^p(\alpha)$ and $r > 0$. Choose $\gamma > 0$ with $\mathbb{P}(|X| < \gamma) > 0$ and then $\xi > 0$ such that $u(\gamma - \xi) < \alpha$. Since $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic we find a measurable set $A \subset \{|X| < \gamma\}$ with $\mathbb{P}(A) < r^p \xi^{-p}$. Set now $Y = (X - \xi)1_A + X1_{A^c}$ and note that $\|X - Y\|_p^p = \xi^p \mathbb{P}(A) < r^p$. Moreover,

$$\mathbb{E}[u(Y)] = \mathbb{E}[u(X - \xi)1_A] + \mathbb{E}[u(X)1_{A^c}] \leq u(\gamma - \xi) \mathbb{P}(A) + \alpha \mathbb{P}(A^c) < \alpha.$$  

Hence, in every neighborhood of $X$ there exists some element which does not belong to $\mathcal{A}_u^p(\alpha)$. Since $X$ was arbitrary, this implies that $\mathcal{A}_u^p(\alpha)$ has empty interior.

Second, assume that (2.5) does not hold so that the limit is zero. Take an element $X \in \mathcal{A}_u^p(\alpha) \cap L^\infty$ satisfying $u(\|X\|_\infty) \geq \mathbb{E}[u(X)] \geq \alpha$. For any fixed $r > 0$ it is easy to see that we can find $\xi > 0$ sufficiently large to yield

$$0 \leq \frac{u(\|X\|_\infty) - \alpha}{u(\|X\|_\infty) - u(\|X\|_\infty - \xi)} < \frac{r^p}{\xi^p} < 1.$$  

As a consequence, taking $\lambda \in (0, 1)$ with

$$\frac{u(\|X\|_\infty) - \alpha}{u(\|X\|_\infty) - u(\|X\|_\infty - \xi)} < \lambda < \frac{r^p}{\xi^p},$$
we obtain $\xi^p \lambda < r^p$, hence
\[
\lambda u(\|X\|_\infty - \xi) + (1 - \lambda)u(\|X\|_\infty) < \alpha.
\]
Since $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic, $\mathbb{P}(A) = \lambda$ for a suitable $A \in \mathcal{F}$. Now, consider the random variable $Y = (X - \xi)1_A + X1_{A^c}$. Clearly, $\|X - Y\|^p_p = \xi^p \mathbb{P}(A) < r^p$. Moreover, as a consequence of the above inequality we have
\[
\mathbb{E}[u(Y)] \leq \mathbb{P}(A) u(\|X\|_\infty - \xi) + \mathbb{P}(A^c) u(\|X\|_\infty) < \alpha.
\]
This implies that $X$ is not an interior point of $\mathcal{A}_u^p(\alpha)$. As a result, by the density of $L^\infty$ in $L^p$ we conclude that $\mathcal{A}_u^p(\alpha)$ has empty interior. In conclusion, (i) must imply both conditions in (ii).

Now, assume that (ii) holds. In particular, by (2.5) we find $\varepsilon > 0$ and a suitable $\xi > 0$ such that $|x|^p \geq -\varepsilon u(-x)$ whenever $x > \xi$. As a consequence, we obtain
\[
\mathbb{E}[u(X)] \geq u(-\xi)\mathbb{P}(X \geq -\xi) - \frac{1}{\varepsilon} \|X\|^p_p > -\infty
\]
for any $X \in L^p$. Together with Lemma 2.4.24, this implies that the map assigning to every $X \in L^p$ the quantity $\mathbb{E}[u(X)]$ is finitely valued. Being increasing and concave, this map is also continuous on $L^p$ by Theorem 0.51. Since $u(x_0) > \alpha$ for some $x_0 \in \mathbb{R}$, we can conclude that $\mathcal{A}_u^p(\alpha)$ has nonempty interior.

Following the approach proposed by Biagini, Frittelli [13], we consider acceptance sets based on expected utility also within the context of Orlicz spaces. Indeed, the authors showed there how to associate a suitable Orlicz space with any finitely valued utility $u$ by passing to the Young function corresponding to $u$, i.e. the function $\hat{u} : \mathbb{R} \to \mathbb{R}$ defined by
\[
\hat{u}(x) := u(0) - u(-|x|).
\]
It is easy to show that $\hat{u}$ is an Orlicz function. In particular, the right-continuity of $u$ implies that $\hat{u}$ is lower semicontinuous. As a result, we can consider the associated Orlicz space $L\hat{u}$ and Morse space $M\hat{u}$. In the following we will mainly focus on $M\hat{u}$. Unless otherwise stated, the results below will be also valid in the context of $L\hat{u}$. 
Remark 2.4.28. As shown in [13], if $u$ takes the value $-\infty$ one can still define Orlicz spaces associated to the Orlicz function $\tilde{u}_a : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$

$$\tilde{u}_a(x) := u(a) - u(a - |x|),$$

where $a$ is an interior point of $\text{dom}(u)$. However, in this case we have $L^{\tilde{u}_a} = L^\infty$ and $M^{\tilde{u}_a} = \{0\}$. \hfill $\Box$

Assume $u$ is finitely valued. For any expected utility level $\alpha \in \mathbb{R}$ we define the set

$$A_u(\alpha) := \{X \in M^{\tilde{u}}; \mathbb{E}[u(X)] \geq \alpha\}.$$

Again, note that $A_u(\alpha)$ is nonempty only when the couple $(u, \alpha)$ is admissible. The first result is straightforward. Once again, note that $A_u(\alpha)$ is generally neither conic, nor numéraire- or surplus-invariant, nor sensitive.

Proposition 2.4.29. Assume $u$ is finitely valued and $(u, \alpha)$ is admissible. Then, $A_u(\alpha)$ is a convex, law-invariant acceptance set.

Since $M^{\tilde{u}}$ is continuously embedded in $L^1$, the first assertion in next result follows directly from Proposition 2.4.26. The second assertion is a consequence of the density of $L^\infty$ in $M^{\tilde{u}}$ and does not hold in the context of $L^{\tilde{u}}$. The proof mimicks that of Proposition 2.4.26. The closure of a set $A \subseteq M^{\tilde{u}}$ in the norm topology of $M^{\tilde{u}}$ will be denoted by $\text{cl}_{\tilde{u}} A$.

Proposition 2.4.30. Assume $u$ is finitely valued and $(u, \alpha)$ is admissible. Then, $A_u(\alpha)$ is closed in $M^{\tilde{u}}$. Moreover, $A_u(\alpha) = \text{cl}_{\tilde{u}} A_u^\infty(\alpha)$.

Proposition 2.4.31. Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic. Moreover, assume $u$ is finitely valued and $(u, \alpha)$ is admissible. The following statements are equivalent:

(i) $A_u(\alpha)$ has nonempty interior in $M^{\tilde{u}}$;

(ii) $u(x_0) > \alpha$ for some $x_0 > 0$.

Proof. To prove that (i) implies (ii), assume $u(x) \leq \alpha$ for all $x \in \mathbb{R}$. Fix $X \in A_u(\alpha)$ and $r > 0$. We claim that $Y \notin A_u(\alpha)$ for some $Y \in M^{\tilde{u}}$ with $\|Y - X\|_{\tilde{u}} \leq r$. To this end, take $\xi > 0$ such that $\mathbb{P}(|X| \leq \xi) > 0$ and choose some $\lambda > 0$ for which $u(\xi - \lambda) < \alpha$. Since $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic, we can find a measurable set
A \subset \{ |X| \leq \xi \}$ such that $\hat{u}(\frac{1}{r})\mathbb{P}(A) \leq 1$. Setting $Y = (X - \lambda)1_A + X1_{A^c} \in M^{\hat{u}}$, it follows that $\|Y - X\|_{\hat{u}} \leq r$. Moreover, since $u(\xi - \lambda) < \alpha$, we obtain

$$\mathbb{E}[u(Y)] \leq u(\xi - \lambda)\mathbb{P}(A) + \alpha \mathbb{P}(A^c) < \alpha,$$

showing that $Y \notin \mathcal{A}_u(\alpha)$ and proving the claim. Hence, (i) must imply (ii).

To prove the converse implication, assume $u(x_0) > \alpha$ for some $x_0 > 0$. We claim that $X = x_0$ is an interior point of $\mathcal{A}_u(\alpha)$. Indeed, choose $\lambda \in (0, 1)$ in such a way that

$$\alpha - \lambda u(x_0) + (1 - \lambda)(1 - u(0)) \leq 0.$$

Note that for every $Y \in L^{\hat{u}}$ with $\|Y\|_{\hat{u}} < 1 - \lambda$ we have

$$\mathbb{E}\left[\frac{1}{1 - \lambda} Y\right] \leq 1,$$

yielding

$$\mathbb{E}[u(X + Y)] \geq \lambda \mathbb{E}\left[u\left(\frac{1}{\lambda} X\right)\right] + (1 - \lambda)\mathbb{E}\left[u\left(\frac{1}{1 - \lambda} Y\right)\right]$$

$$\geq \lambda u\left(\frac{x_0}{\lambda}\right) - (1 - \lambda)\mathbb{E}\left[\hat{u}\left(\frac{1}{1 - \lambda} Y\right)\right] + (1 - \lambda)u(0)$$

$$\geq \alpha.$$

As a result, $X + Y \in \mathcal{A}_u(\alpha)$ whenever $\|Y\|_{\hat{u}} < 1 - \lambda$, showing that $X$ does belong to the interior of $\mathcal{A}_u(\alpha)$. Hence, (ii) implies (i).

We conclude by providing a dual representation of acceptance sets based on expected utility on $L^p$ spaces. The corresponding representation on the Morse space $M^{\hat{u}}$ is similar and left to the reader. Recall that $u_*$ denotes the concave conjugate function of $u$, i.e. for any $z \in \mathbb{R}$ we have

$$u_*(z) = \inf_{x \in \mathbb{R}} \{zx - u(x)\}.$$

**Proposition 2.4.32.** Let $q = \frac{p}{p-1}$. For any admissible $(u, \alpha)$ we have

$$\mathcal{A}_u^p(\alpha) = \bigcap_{Z \in L^q_+} \{X \in L^p \mid \mathbb{E}[ZX] \geq \sigma_{\mathcal{A}_u^p(\alpha)}(Z)\}$$
where for any $Z \in L^q_+$

$$
\sigma_{A^p_u(\alpha)}(Z) = \sup_{\lambda > 0} \frac{1}{\lambda} \left( \alpha + \mathbb{E}[u_*(\lambda Z)] \right).
$$

Proof. Since $A^p_u(\alpha)$ is a closed ($\sigma(L^{\infty}, L^1)$-closed if $p = \infty$ by Proposition 4.113 in Föllmer, Schied [50]) convex set, the above representation follows from Theorem 2.3.6. If $p = \infty$, the explicit form of the support function can be derived from Theorem 4.115 in [50]. Now, let $p \in [1, \infty)$ and note that Proposition 2.4.26 implies that $\sigma_{A^p_u(\alpha)}(Z) = \sigma_{A^\infty_u(\alpha)}(Z)$ for all $Z \in L^q_+$. This concludes the proof of the proposition. \qed
Chapter 3

Management actions and risk measures

In this chapter we turn to the study of risk measures. Once the underlying notion of acceptability has been specified, to define a risk measure we need to describe a class of admissible management actions, or strategies, that are eligible to modify the acceptability profile of financial positions. The associated risk measure will determine the “minimal” cost to ensure acceptability by implementing some admissible action. We start by focusing on the simplest form of management action, namely raising capital and investing in a single pre-specified asset in the context of a one-period liquid market. At a later stage, we will extend the scope of our research by allowing for more general types of management strategies, including raising capital and investing in multiple reference assets traded in a general multi-period market.

3.1 Investing in a single liquid asset

The acceptability profile of a financial position can be altered by implementing a pre-specified set of admissible management actions. We start by considering a simple class of admissible actions, namely raising capital and investing in a pre-specified liquid asset. We maintain the assumption that financial positions are
represented by the elements of the space $\mathcal{X}$.

### 3.1.1 The class of admissible actions

We assume that acceptability can be achieved by investing in a pre-specified liquid asset, namely by purchasing or selling the asset. The corresponding class of admissible management actions, or strategies, can be therefore identified with a subset $\mathcal{M} \subseteq \mathbb{R}$. Every strategy $\lambda \in \mathcal{M}$ corresponds to the number of units that are purchased, if positive, or sold, if negative. Since the reference asset is assumed to be liquid, we set $\mathcal{M} = \mathbb{R}$. In other words, the reference asset is not subject to any trading constraint, including limitations in short-selling.

The corresponding implementation cost at inception is described by a functional

$$\pi : \mathcal{M} \rightarrow \mathbb{R}.$$  

We interpret $\pi$ as a pricing functional from a buyer’s perspective. In other words, the quantity $\pi(\lambda)$ represents the buying price, the so-called ask price, of $\lambda$ units of the reference asset. Conversely, the quantity $-\pi(-\lambda)$ represents the corresponding selling price, the so-called bid price. We follow the usual convention to model an injection, respectively an extraction, of capital by means of a positive, respectively negative, number. Since the reference asset is liquidly traded, we assume that $\pi$ is linear so that trading at inception is frictionless, i.e. ask and bid prices coincide. In particular, if we set $S_0 = \pi(1)$ it follows that

$$\pi(\lambda) = \lambda S_0 \quad \text{for all } \lambda \in \mathcal{M}.$$  

In other words, $S_0$ represents the unitary (bid and ask) price of the reference asset at inception. We will always assume that $S_0 > 0$.

Finally, the terminal payoff of a strategy at maturity is represented by a map

$$Z : \mathcal{M} \rightarrow \mathcal{X}.$$  

Contrary to $\pi$, we interpret $Z$ as a “pricing functional” from a seller’s perspective. In particular, the position $Z(\lambda)$ corresponds to the payoff we receive if we are long $\lambda$ units of the underlying asset. Similarly, the position $-Z(-\lambda)$ has to be interpreted as the payout we have to deliver in case we are short $\lambda$ units. In case
the asset does not entitle to a contractual payment at maturity, the quantity $Z(\lambda)$ can be interpreted as a terminal liquidation value, while $-Z(-\lambda)$ corresponds to a final acquisition value. Since the reference asset is liquid, we assume that $Z$ is linear. In particular, the liquidation value and the acquisition value at maturity are equal. Note that, setting $S_T = Z(1)$ we obtain

$$Z(\lambda) = \lambda S_T \quad \text{for all } \lambda \in \mathcal{M}.$$ 

Hence, $S_T$ represents the unitary (seller’s and buyer’s) payoff of the reference asset at maturity. We will always assume that $S_T$ is a nonzero element in $\mathcal{X}_+$. 

### 3.1.2 Introducing risk measures

Once we have specified a notion of acceptability and a class of admissible management actions to use in order to reach acceptability, we can finally define a risk measure as the mapping assigning to each financial position the “minimal” cost we have to pay to implement some admissible action moving the position into the acceptance set. Before we formalize this concept, it is useful to highlight the primitive role of the acceptance set and of the class of management actions in the following definition.

**Definition 3.1.1.** Consider an acceptance set $A \subset \mathcal{X}$, a subset $\mathcal{M} \subseteq \mathbb{R}$, a functional $\pi : \mathcal{M} \to \mathbb{R}$ and a map $Z : \mathcal{M} \to \mathcal{X}$. Then, the quadruple

$$S = (A, \mathcal{M}, \pi, Z)$$

is called a risk measurement system (associated to a single reference asset). The reference asset is said to be liquid if $\mathcal{M} = \mathbb{R}$ and both $\pi$ and $Z$ are linear. In this case, we identify $S$ with the couple

$$S = (A, S) \quad \text{for } S = (S_0, S_T),$$

where $S_0 = \pi(1)$ and $S_T = Z(1)$. Moreover, we assume that $S_0 > 0$ and $S_T$ is a nonzero element of $\mathcal{X}_+$. 

From now on, we fix a risk measurement system $S = (A, S)$ associated to a single liquid asset. The corresponding risk measure can be formally defined in the following way.
Definition 3.1.2. The risk measure associated with $S = (A, S)$ is the mapping $\rho_{A,S}: \mathcal{X} \to \mathbb{R}$ defined by

$$\rho_{A,S}(X) := \inf\{\lambda S_0; \lambda \in \mathbb{R}, X + \lambda S_T \in A\}.$$ 

In line with the notion of a risk measure discussed in Chapter 1, the quantity $\rho_{A,S}(X)$ represents the “minimal” amount of capital we have to raise and invest in the reference asset at inception to ensure the acceptability of $X$.

Remark 3.1.3. Without loss of generality, we could assume that $S_0 = 1$. However, to allow for more transparent financial interpretations we will keep explicit track of the unitary price $S_0$. [\qed]

3.2 Basic properties of risk measures

In this section we discuss some basic properties of risk measures of the form $\rho_{A,S}$. In particular, we focus on the properties of $S$-additivity and monotonicity, which will play a fundamental role in the sequel.

3.2.1 Additivity and monotonicity

We start by showing that the infimum defining $\rho_{A,S}$ is taken over an interval of the real line. Before proving this simple but crucial lemma, define for $X \in \mathcal{X}$

$$\mathcal{R}_{A,S}(X) := \{\lambda \in \mathbb{R}; X + \lambda S_T \in A\}.$$ 

Lemma 3.2.1. For any $X \in \mathcal{X}$ the set $\mathcal{R}_{A,S}(X)$ is either empty or it is an interval in $\mathbb{R}$ unbounded to the right.

Proof. Fix $X \in \mathcal{X}$ and assume $\mathcal{R}_{A,S}(X)$ is nonempty. Take $\lambda \in \mathcal{R}_{A,S}(X)$ and $\xi > \lambda$. Since $S_T \geq 0$, we have $X + \xi S_T \geq X + \lambda S_T$. Hence, $\xi \in \mathcal{R}_{A,S}(X)$ by the monotonicity of $A$. [\qed]

Every risk measure of the form $\rho_{A,S}$ is nonconstant, decreasing and satisfies the following “additivity” property. These basic properties will be freely used without reference.
Definition 3.2.2. We say that a map \( \rho : \mathcal{X} \to \mathbb{R} \) is \( S \)-additive if for any \( X \in \mathcal{X} \)
\[
\rho(X + \lambda S_T) = \rho(X) - \lambda S_0 \quad \text{for all } \lambda \in \mathbb{R}.
\]

Proposition 3.2.3. The map \( \rho_{A,S} \) is nonconstant, \( S \)-additive and decreasing.

Proof. First, take \( X \in \mathcal{X} \) and \( \lambda \in \mathbb{R} \). Then
\[
\rho_{A,S}(X + \lambda S_T) = \inf \{ (\xi - \lambda)S_0 \ ; \ \xi \in \mathbb{R}, \ X + \xi S_T \in A \} = \rho_{A,S}(X) - \lambda S_0.
\]
Since \( A \) is an acceptance set, we find \( X \in A \) and \( Y \in A^c \). In particular, the previous lemma implies that \( \rho_{A,S}(X) \leq 0 \) and \( \rho_{A,S}(Y) \geq 0 \). As a consequence of \( S \)-additivity, it follows that \( \rho_{A,S} \) is nonconstant. Finally, take \( X, Y \in \mathcal{X} \) and assume \( X \leq Y \). Then, it is immediate to see that \( \mathcal{R}_{A,S}(X) \subseteq \mathcal{R}_{A,S}(Y) \) by the monotonicity of \( A \). As a result, we obtain
\[
\rho_{A,S}(X) = S_0 \inf \mathcal{R}_{A,S}(X) \geq S_0 \inf \mathcal{R}_{A,S}(Y) = \rho_{A,S}(Y).
\]

Every nonconstant map which is \( S \)-additive and decreasing is, in fact, a risk measure of the form \( \rho_{A,S} \). In the sequel, for any \( \rho : \mathcal{X} \to \mathbb{R} \) we define
\[
\mathcal{A}(\rho) := \{ X \in \mathcal{X} \ ; \ \rho(X) \leq 0 \}.
\]

Proposition 3.2.4. Assume \( \rho : \mathcal{X} \to \mathbb{R} \) is nonconstant, \( S \)-additive and decreasing. Then \( \mathcal{A}(\rho) \) is an acceptance set and \( \rho = \rho_{A(\rho),S} \).

Proof. Since \( \rho \) is nonconstant and decreasing, it is immediate to see that \( \mathcal{A}(\rho) \) is an acceptance set. Moreover, since \( \rho \) is \( S \)-additive, for any \( X \in \mathcal{X} \) we have
\[
\rho_{A(\rho),S}(X) = S_0 \inf \{ \lambda \in \mathbb{R} \ ; \ \rho(X) \leq \lambda S_0 \} = \rho(X).
\]

In light of the previous result, it is natural to ask what is the link between a given acceptance set \( A \) and the acceptance set induced by \( \rho_{A,S} \). Here, for a position \( Z \in \mathcal{X} \), we say that a set \( A \subseteq \mathcal{X} \) is directionally closed along \( Z \) whenever
\[
X + \lambda_n Z \in A, \ \lambda_n \to 0 \implies X \in A.
\]
In particular, a closed set is directionally closed along any direction.

Proposition 3.2.5. We have \( A = \mathcal{A}(\rho_{A,S}) \) if and only if \( A \) is directionally closed along \( S_T \).
Proof. To prove the “if” implication, assume $\mathcal{A}$ is directionally closed along $S_T$. Since every $X \in \mathcal{A}$ is such that $\rho_{A,S}(X) \leq 0$, we focus on the inclusion “$\supseteq$”. Assume $X \in \mathcal{A}(\rho_{A,S})$. Then, we find a sequence $(\lambda_n)$ of strictly positive real numbers converging to zero and such that $X + \lambda_n S_T \in \mathcal{A}$ for all $n \in \mathbb{N}$. By the closedness property of $\mathcal{A}$, we immediately conclude that $X \in \mathcal{A}$. To prove the “only if” implication, assume $\mathcal{A} = \mathcal{A}(\rho_{A,S})$ and take a sequence $(\lambda_n)$ of real numbers converging to zero and satisfying $X + \lambda_n S_T \in \mathcal{A}$ for every $n \in \mathbb{N}$. In particular, we have $\rho_{A,S}(X) \leq \lambda_n S_0$ for all $n \in \mathbb{N}$ so that $\rho_{A,S}(X) \leq 0$. By assumption this implies that $X \in \mathcal{A}$, concluding the proof.

Remark 3.2.6. We can visualize the preceding results by means of the following schemes: on one side, we always have

$$\rho \rightarrow \mathcal{A}(\rho) \rightarrow \rho_{A(\rho),S} = \rho;$$

on the other side, we generally have

$$\mathcal{A} \rightarrow \rho_{A,S} \rightarrow \mathcal{A}(\rho_{A,S}) \neq \mathcal{A}.$$

In particular, we cannot always recuperate the initial acceptance set $\mathcal{A}$ if we consider the acceptance set induced by $\rho_{A,S}$. This has the following important consequence. Assume $\mathcal{A}$ is an acceptance set and $\mathcal{A} = \mathcal{A}(\rho)$ for some nonconstant, $S$-additive and decreasing map $\rho$. Then, we always have $\rho = \rho_{A,S}$. This shows that the notion of a risk measure compatible with a given acceptability criterion is well-posed. However, assume $\rho$ is nonconstant, $S$-additive and decreasing and $\rho = \rho_{A,S}$ for some acceptance set $\mathcal{A}$. Then, we generally have $\mathcal{A} \neq \mathcal{A}(\rho)$. Hence, the notion of an acceptance set compatible with a given risk measure is not well-posed. This highlights the conceptual prominence of the notion of acceptability in contrast to that of a risk measure.

The next important result shows what is the relative position of the acceptance set with respect to the level sets of the corresponding risk measure. We will use the inclusions below without further reference.

Proposition 3.2.7. The following inclusions hold:

$$\text{int} \mathcal{A} \subseteq \{\rho_{A,S} < 0\} \subseteq \mathcal{A} \subseteq \{\rho_{A,S} \leq 0\} \subseteq \text{cl} \mathcal{A}.$$

In particular, we have $\{\rho_{A,S} = 0\} \subseteq \text{bd} \mathcal{A}$. 
Proof. The last assertion follows directly from the above chain of inclusions. Now, take \( X \in \mathcal{X} \). If \( X \in \text{int} \, \mathcal{A} \), then \( X - \varepsilon S_T \in \mathcal{A} \) for some \( \varepsilon > 0 \), which is equivalent to \( \rho_{A,S}(X) < 0 \). In particular, \( \rho_{A,S}(X) < 0 \) implies \( X \in \mathcal{A} \) by monotonicity. Hence, the first two inclusions hold. If \( X \in \mathcal{A} \), then clearly \( \rho_{A,S}(X) \leq 0 \) so that the third inclusion holds. Finally, assume \( \rho_{A,S}(X) \leq 0 \) and set \( X_n = X + \frac{1}{n} S_T \) for any \( n \in \mathbb{N} \). Note that \( \rho_{A,S}(X_n) = \rho_{A,S}(X) - \frac{S_0}{n} < 0 \), implying \( X_n \in \mathcal{A} \) for any \( n \in \mathbb{N} \) by virtue of the second inclusion above. Since \( X_n \to X \) as \( n \to \infty \), we conclude that \( X \in \text{cl} \, \mathcal{A} \), proving the last inclusion.

### 3.2.2 Convexity and conicity

We provide sufficient conditions for \( \rho_{A,S} \) to be convex and conic, showing how to transfer properties of \( \mathcal{A} \) to properties of the corresponding risk measure. The following result will be sometimes used without reference.

**Proposition 3.2.8.** Assume that \( \mathcal{A} \) is convex, respectively conic. Then, \( \rho_{A,S} \) is convex, respectively conic.

**Proof.** First, assume \( \mathcal{A} \) is convex. Take arbitrary \( X, Y \in \mathcal{X} \) and assume that \( \rho_{A,S}(X) \leq m_X \) and \( \rho_{A,S}(Y) \leq m_Y \) for suitable \( m_X, m_Y \in \mathbb{R} \). Since \( \mathcal{A} \) is convex, for arbitrary \( \lambda \in (0, 1) \) we have

\[
\lambda X + (1 - \lambda) Y + \left( \frac{\lambda m_X + (1 - \lambda)m_Y + \varepsilon}{S_0} \right) S_T \in \mathcal{A}
\]

for every \( \varepsilon > 0 \), implying that \( \rho_{A,S}(\lambda X + (1 - \lambda)Y) \leq \lambda m_X + (1 - \lambda)m_Y \). This shows that the epigraph of \( \rho_{A,S} \) is convex and proves the first claim.

Similarly, assume \( \mathcal{A} \) is a cone. Take \( X \in \mathcal{X} \) and assume \( \rho_{A,S}(X) \leq m \) for \( m \in \mathbb{R} \). Since \( \mathcal{A} \) is a cone, for arbitrary \( \lambda > 0 \) we have

\[
\lambda X + \left( \frac{\lambda m + \lambda \varepsilon}{S_0} \right) S_T \in \mathcal{A}
\]

for all \( \varepsilon > 0 \). As a result, \( \rho_{A,S}(\lambda X) \leq \lambda m \). This shows that the epigraph of \( \rho_{A,S} \) is a cone and concludes the proof.

The next result shows that the convex and the conic hull of a risk measure \( \rho_{A,S} \) are still risk measures of the same form.
Proposition 3.2.9. The following statements hold:

(i) \( \text{co} \rho_{A,S} = \rho_{\text{co}A,S} \);

(ii) \( \text{cone} \rho_{A,S} = \rho_{\text{cone}A,S} \).

Proof. (i) Note that \( \rho_{\text{co}A,S} \) is convex by Proposition 3.2.8. Moreover, we always have \( \rho_{\text{co}A,S} \leq \rho_{A,S} \) since \( A \subseteq \text{co}A \). Now, take a convex map \( f : \mathcal{X} \rightarrow \mathbb{R} \) such that \( f \leq \rho_{A,S} \). To conclude the proof we only need to show that \( f \leq \rho_{\text{co}A,S} \). Fix \( X \in \mathcal{X} \) and assume \( X + \xi S_T \in \text{co}A \) for some \( \xi \in \mathbb{R} \). Then we can write \( X + \xi S_T \) as a convex combination

\[
X + \xi S_T = \sum_{i=1}^{n} \lambda_i Y_i
\]

with \( Y_1, \ldots, Y_n \in \mathcal{A} \). Note that

\[
f(Y_i - \xi S_T) \leq \rho_{A,S}(Y_i - \xi S_T) \leq \xi S_0
\]

for any \( i = 1, \ldots, n \). Then, by convexity, we have

\[
f(X) = f\left( \sum_{i=1}^{n} \lambda_i(Y_i - \xi S_T) \right) \leq \sum_{i=1}^{n} \lambda_i \xi S_0 = \xi S_0.
\]

This shows that \( f(X) \leq \rho_{\text{co}A,S}(X) \) and proves (i).

(ii) By Proposition 3.2.8, the map \( \rho_{\text{cone}A,S} \) is conic. Moreover, since \( A \subseteq \text{cone}A \), we have \( \rho_{\text{cone}A,S} \leq \rho_{A,S} \). Take a conic map \( f : \mathcal{X} \rightarrow \mathbb{R} \) satisfying \( f \leq \rho_{A,S} \). As above, we only need to show that \( f \leq \rho_{\text{cone}A,S} \). To this end, fix \( X \in \mathcal{X} \) and assume \( X + \xi S_T \in \text{cone}A \) for some \( \xi \in \mathbb{R} \). Hence, there exists \( \lambda > 0 \) such that

\[
X + \xi S_T = \lambda Y
\]

for a suitable \( Y \in \mathcal{A} \). Thus, the conicity of \( f \) implies

\[
f(X) = \lambda f\left( Y - \frac{\xi}{\lambda} S_T \right) \leq \lambda \rho_{A,S}(Y - \frac{\xi}{\lambda} S_T) \leq \xi S_0.
\]

This yields \( f(X) \leq \rho_{\text{cone}A,S}(X) \) and concludes the proof of (ii). \( \Box \)
3.2.3 Maxima and minima

In this section we prove that the class of risk measures of the form \( \rho_{A,S} \) is stable when taking maxima and minima of risk measures with respect to different acceptance sets. The “maximum” case corresponds to taking a worst-case approach prescribing to reach acceptability under every acceptance set. Similarly, the “minimum” case corresponds to a best-case approach, for which overall acceptability amounts to reaching acceptability under some of the pre-specified acceptance sets.

**Proposition 3.2.10.** Assume \( A_1 \) and \( A_2 \) are acceptance sets in \( \mathcal{X} \). The following statements hold:

(i) \( \max\{\rho_{A_1,S}, \rho_{A_2,S}\} = \rho_{A_1 \cap A_2,S} \);

(ii) \( \min\{\rho_{A_1,S}, \rho_{A_2,S}\} = \rho_{A_1 \cup A_2,S} \).

**Proof.** We only focus on the first claim, since the proof of the second one proceeds along similar lines. Since \( A_1 \cap A_2 \) is contained in both \( A_1 \) and \( A_2 \), the inequality “\( \leq \)” is clear. Now, assume this inequality is strict for some \( X \in \mathcal{X} \). Then we would find \( \lambda \in \mathbb{R} \) such that \( X + \lambda S_T \notin A_1 \cap A_2 \) but, at the same time, \( \rho_{A_1,S}(X) < \lambda S_0 \) and \( \rho_{A_2,S}(X) < \lambda S_0 \). Since this is not possible, the inequality “\( \geq \)” must also hold.

**Remark 3.2.11.** The extension to arbitrary intersections and unions is straightforward, provided we replace the maximum, respectively the minimum, by a supremum, respectively an infimum.

3.2.4 Infimal convolutions

In this brief section we show that the class of risk measures of the form \( \rho_{A,S} \) is closed with respect to infimal convolutions. First, recall that the infimal convolution of \( f_1 : \mathcal{X} \to \mathbb{R} \cup \{\infty\} \) and \( f_2 : \mathcal{X} \to \mathbb{R} \cup \{\infty\} \) is the map \( f_1 \square f_2 : \mathcal{X} \to \mathbb{R} \) defined by

\[
 f_1 \square f_2(X) := \inf\{f_1(Y) + f_2(X - Y) \mid Y \in \mathcal{X}\}.
\]

For a fixed position \( X \in \mathcal{X} \), any couple \( (Y, X - Y) \) with \( Y \in \mathcal{X} \) can be interpreted as an allocation of \( X \) across two “business lines”. If \( f_1 \) and \( f_2 \) represent cost functionals corresponding to a certain capital allocation scheme, the quantity
$f_1 \Box f_2(X)$ is the “minimal” total required capital across all possible allocations of the aggregated position $X$. As such, infimal convolutions play a central role in the context of optimal risk sharing across different agents or business lines. For more details on infimal convolutions and their applications to the theory of (cash-additive) risk measures we refer to Barrieu, El Karoui [12] and Filipović, Svindland [46].

Here, we fix two acceptance sets $A_1$ and $A_2$ in $\mathcal{X}$ and consider risk measures associated to $S^1 = (S^1_0, S^1_T)$ and $S^2 = (S^2_0, S^2_T)$. Moreover, we define

$$\mathcal{P}_0(S^1, S^2) := \left\{ \frac{\lambda}{S^1_0} S^1_T - \frac{\lambda}{S^2_0} S^2_T ; \lambda \in \mathbb{R} \right\}.$$  

This set consists of the payoffs of all “portfolios” we can form at zero cost by combining the assets $S^1$ and $S^2$.

**Proposition 3.2.12.** Assume $\rho_{A_1, S^1}$ and $\rho_{A_2, S^2}$ do not take the value $-\infty$. Then

$$\rho_{A_1, S^1} \Box \rho_{A_2, S^2} = \rho_{A_1 + A_2 + \mathcal{P}_0(S^1, S^2), S^1}.$$  

**Proof.** Fix $X \in \mathcal{X}$. To show the inequality “$\leq$”, assume $X + \lambda S^1_T$ belongs to $A_1 + A_2 + \mathcal{P}_0(S^1, S^2)$ for some $\lambda \in \mathbb{R}$. Take $Y \in A_1$, $Z \in A_2$ and $\xi \in \mathbb{R}$ in such a way that

$$X + \lambda S^1_T = Y + Z + \frac{\xi}{S^1_0} S^1_T - \frac{\xi}{S^2_0} S^2_T.$$  

It follows that

$$\rho_{A_1, S^1} \Box \rho_{A_2, S^2}(X) \leq \rho_{A_1, S^1}(Y - \lambda S^1_T + \frac{\xi}{S^1_0} S^1_T) + \rho_{A_2, S^2}(Z - \frac{\xi}{S^2_0} S^2_T) \leq \lambda S^1_0.$$  

The inequality follows by taking the infimum over $\lambda$. To show the converse inequality, assume $Y + \lambda S^1_T \in A_1$ and $X - Y + \xi S^2_T \in A_2$ for some $Y \in \mathcal{X}$ and suitable $\lambda, \xi \in \mathbb{R}$. Since the position

$$X + \left( \lambda + \frac{\xi S^2_0}{S^1_0} \right) S^1_T = Y + \lambda S^1_T + X - Y + \xi S^2_T + \frac{\xi S^2_0}{S^1_0} S^1_T - \frac{\xi S^2_0}{S^2_0} S^2_T,$$

lies in $A_1 + A_2 + \mathcal{P}_0(S^1, S^2)$, it follows that $\rho_{A_1 + A_2 + \mathcal{P}_0(S^1, S^2), S^1}(X) \leq \lambda S^1_0 + \xi S^2_0$. Taking the infimum over $\lambda$ and $\xi$ and, in a second step, over $Y$, we obtain the desired inequality. $\square$
Remark 3.2.13. We have focused on two business lines for ease of notation. The extension to a general number of business lines is straightforward and left to the reader.

3.3 Finiteness and continuity of risk measures

In this section we investigate finiteness and continuity properties of risk measures of the form $\rho_{A, S}$, highlighting the interplay between the acceptance set and the reference asset.

3.3.1 The geometry of finiteness

We start by showing that a risk measure of the form $\rho_{A, S}$ may only take infinite values. In this case, we will say that $\rho_{A, S}$ is degenerate.

Example 3.3.1. Consider an infinite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and take $\mathcal{X} = L^\infty$. Assume $S_T$ is not bounded away from zero, i.e. $\mathbb{P}(S_T \leq \varepsilon) > 0$ for all $\varepsilon > 0$. This is equivalent to requiring that $S_T$ is not an interior point of $L^\infty_+$. Finally, consider the acceptance set

$$A = \{X \in L^\infty; \ X \geq_{\mathbb{P}} \lambda S_T \text{ for some } \lambda \in \mathbb{R}\}.$$

In particular, note that $A$ is strictly contained in $L^\infty$ because of our assumption on $S_T$. Then it is immediate to see that $X \in A$ if and only if $X + \lambda S_T \in A$ for any $\lambda \in \mathbb{R}$. Hence, $\rho_{A, S}(X) = -\infty$ whenever $X \in A$ and $\rho_{A, S}(X) = \infty$ whenever $X \notin A$, showing that $\rho_{A, S}$ is degenerate.

The first result provides equivalent conditions for a risk measure $\rho_{A, S}$ to be finitely valued, whose easy verification is left to the reader. This basic characterization will be used without further reference. We only note that the “if” part in the second assertion is a consequence of Lemma 3.2.1.

Proposition 3.3.2. Take $X \in \mathcal{X}$. The following statements hold:

(i) $\rho_{A, S}(X) < \infty$ if and only if $X + \lambda S_T \in A$ for some $\lambda \in \mathbb{R}$;

(ii) $\rho_{A, S}(X) > -\infty$ if and only if $X + \lambda S_T \notin A$ for some $\lambda \in \mathbb{R}$.
The next result, which is less useful from an operational point of view, provides a geometrical interpretation of finiteness: the risk measure $\rho_{A,S}$ is finite at $X \in \mathcal{X}$ if the curve $\gamma_X : \mathbb{R} \to \mathcal{X}$ defined by
\[
\gamma_X(\lambda) = X + \lambda S_T
\]
lies outside the acceptance set $A$ till it enters $A$ and then remains there. We can picture this situation in the following way:

\[
\begin{array}{c|c|c}
X + \lambda S_T \notin A & X + \lambda S_T \in A \\
\rho_{A,S}(X) > -\infty & \lambda_X \\
\rho_{A,S}(X) < \infty & \lambda
\end{array}
\]

The formal statement is recorded in the next proposition.

**Proposition 3.3.3.** The risk measure $\rho_{A,S}$ is finitely valued if and only if for any $X \in \mathcal{X}$ there exists $\lambda_X \in \mathbb{R}$ such that:

(i) $X + \lambda S_T \notin A$ whenever $\lambda < \lambda_X$;

(ii) $X + \lambda S_T \in A$ whenever $\lambda > \lambda_X$.

In this case, we have $\rho_{A,S}(X) = \lambda_X S_0$.

**Proof.** The “if” part follows immediately from Proposition 3.3.2. To prove the “only if” part, assume $\rho_{A,S}$ is finitely valued and set $\lambda_X = \rho_{A,S}(X)/S_0$ for any $X \in \mathcal{X}$. Clearly, if $\lambda < \lambda_X$ we must have $X + \lambda S_T \notin A$, since otherwise
\[
\lambda_X S_0 = \rho_{A,S}(X) \leq \lambda S_0 < \lambda_X S_0.
\]

Take now $\lambda > \lambda_X$. By definition of $\rho_{A,S}(X)$, we find a sequence $(\lambda_n)$ of real numbers satisfying $X + \lambda_n S_T \in A$ for all $n \in \mathbb{N}$ and such that, as $n \to \infty$,
\[
\lambda_n \to \frac{\rho_{A,S}(X)}{S_0} = \lambda_X.
\]

As a result, we must have $\lambda_k < \lambda$ for some $k \in \mathbb{N}$. Then $X + \lambda S_T \in A$ by monotonicity and this concludes the proof of the proposition. \qed
3.3.2 The geometry of continuity

We start this section on continuity by showing that a risk measure of the form \( \rho_{A,S} \) may be discontinuous at every point.

**Example 3.3.4.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a nonatomic probability space and take \( \mathcal{X} = L^1 \).

Consider the cash-additive risk measure associated to the acceptance set

\[
A = \{ X \in L^1 ; \mathbb{E}[X] \geq 0, X \geq_{\mathbb{P}} \alpha \text{ for some } \alpha \in \mathbb{R} \}.
\]

It is easy to see that \( \rho_{A}(X) = -\mathbb{E}[X] \) whenever \( X \) is bounded from below and \( \rho_{A}(X) = \infty \) otherwise. First, assume \( X \) is bounded from below and assume \( Y \) is not bounded from below. Setting \( X_n = X + Y 1_{\{Y \leq -n\}} \) for all \( n \in \mathbb{N} \), it is easy to see that \( X_n \to X \) as \( n \to \infty \). However, \( \rho_{A}(X_n) = \infty \) for all \( n \in \mathbb{N} \) while \( \rho_{A}(X) < \infty \), showing that \( \rho_{A} \) is not continuous at \( X \). Then, assume \( X \) is unbounded from below so that \( \rho_{A}(X) = \infty \). Set \( X_n = X 1_{\{X \geq -n\}} \) for any \( n \in \mathbb{N} \). Since \( X_n \to X \) as \( n \to \infty \) and \( |\rho_{A}(X_n)| \leq \|X\|_1 \) for all \( n \in \mathbb{N} \), it follows that \( \rho_{A} \) cannot be continuous at \( X \).

The following results establish equivalent conditions for risk measures of the form \( \rho_{A,S} \) to be semicontinuous. We start with a pointwise characterization.

**Proposition 3.3.5.** Take \( X \in \mathcal{X} \). The following assertions hold:

(a) The following statements are equivalent:

(i) \( \rho_{A,S} \) is lower semicontinuous at \( X \);

(ii) \( X + \lambda S_T \notin \text{cl} \mathcal{A} \) whenever \( \lambda S_0 < \rho_{A,S}(X) \);

(iii) \( \rho_{A,S}(X) = \rho_{\text{cl} \mathcal{A},S}(X) \).

(b) The following statements are equivalent:

(i) \( \rho_{A,S} \) is upper semicontinuous at \( X \);

(ii) \( X + \lambda S_T \in \text{int} \mathcal{A} \) whenever \( \lambda S_0 > \rho_{A,S}(X) \);

(iii) \( \rho_{A,S}(X) = \rho_{\text{int} \mathcal{A},S}(X) \).

**Proof.** We only prove part (a), since the proof of the other part proceeds along similar lines. Assume \( \rho_{A,S} \) is lower semicontinuous at \( X \) and take \( \lambda \in \mathbb{R} \) satisfying
$\lambda S_0 < \rho_{A,S}(X)$. By lower semicontinuity, we find a neighborhood $U$ of $X$ such that $\rho_{A,S}(Y) > \lambda S_0$ for all $Y \in U$. This implies $Y + \lambda S_T \notin \mathcal{A}$ for any element $Y \in U$ and, therefore, $X + \lambda S_T$ must lie outside $\text{cl} \mathcal{A}$. Hence, (i) implies (ii).

Now, assume (ii) holds. Since $\mathcal{A} \subseteq \text{cl} \mathcal{A}$, we always have $\rho_{\text{cl}A,S}(X) \leq \rho_{A,S}(X)$. The converse inequality follows immediately from (ii), showing that (iii) holds.

Finally, assume $\rho_{\text{cl}A,S}(X) = \rho_{A,S}(X)$ and let $\rho_{A,S}(X) > m$ for some $m \in \mathbb{R}$. Moreover, take $\varepsilon > 0$ such that $m + \varepsilon < \rho_{A,S}(X))$. Since $\rho_{\text{cl}A,S}(X) > m + \varepsilon$, it follows that we must have $X + \frac{m+\varepsilon}{S_0} S_T \notin \text{cl} \mathcal{A}$ or, equivalently, $U + \frac{m+\varepsilon}{S_0} S_T \subseteq (\text{cl} \mathcal{A})^c$ for some neighborhood $U$ of $X$. Since $\rho_{A,S}(Y) \geq m + \varepsilon$ for all $Y \in U$, we conclude that $\rho_{A,S}$ is lower semicontinuous at $X$, proving that (iii) implies (i).

Remark 3.3.6. Note that $\rho_{A,S}$ cannot be upper semicontinuous at any point $X$ such that $\rho_{A,S}(X) < \infty$ if $\text{int} \mathcal{A}$ is empty. This follows directly from the above proposition. In particular, if the topological dual of $\mathcal{X}$ is trivial, i.e. $\mathcal{X}' = \{0\}$, and if $\mathcal{A}$ is convex, then $\rho_{A,S}$ cannot be continuous at any point of its effective domain.

A global version of the preceding result is recorded in the next proposition.

**Proposition 3.3.7.** The following assertions hold:

(a) The following statements are equivalent:

(i) $\rho_{A,S}$ is lower semicontinuous;

(ii) $\{\rho_{A,S} \leq 0\}$ is closed;

(iii) $\text{cl} \mathcal{A} = \{\rho_{A,S} \leq 0\}$;

(iv) $\rho_{A,S} = \rho_{\text{cl}A,S}$.

(b) The following statements are equivalent:

(i) $\rho_{A,S}$ is upper semicontinuous;

(ii) $\{\rho_{A,S} < 0\}$ is open;

(iii) $\text{int} \mathcal{A} = \{\rho_{A,S} < 0\}$;

(iv) $\rho_{A,S} = \rho_{\text{int}A,S}$.
Proof. We only prove part (a), since the proof of the other part proceeds along similar lines. By Proposition 3.0.52, we immediately see that (i) implies (ii). If (ii) holds, then \( \text{cl } \mathcal{A} \subseteq \{ \rho_{A,S} \leq 0 \} \subseteq \text{cl } \mathcal{A} \) by virtue of Proposition 3.2.7, showing that (iii) holds as well. Now, assume (iii) holds and take \( m \in \mathbb{R} \). Since

\[
\{ \rho_{A,S} \leq m \} = \{ \rho_{A,S} \leq 0 \} - \frac{m}{S_0} S_T
\]

by \( S \)-additivity, we conclude that \( \{ \rho_{A,S} \leq m \} \) is also closed and, thus, \( \rho_{A,S} \) is lower semicontinuous by Proposition 3.0.52. Hence, (iii) implies (i) which is equivalent, in light of the proposition above, to (iv).

As a corollary, we immediately obtain the following sufficient condition for \( \rho_{A,S} \) to be lower, respectively upper, semicontinuous. This result will be sometimes used without reference.

**Corollary 3.3.8.** Assume that \( \mathcal{A} \) is closed, respectively open. Then \( \rho_{A,S} \) is lower, respectively upper, semicontinuous.

Next, we show that the lower and upper semicontinuous hull of a risk measure of the form \( \rho_{A,S} \) is a risk measure of the same type.

**Proposition 3.3.9.** The following statements hold:

(i) \( \text{cl } \rho_{A,S} = \rho_{\text{cl } \mathcal{A},S} \);

(ii) \( \text{int } \rho_{A,S} = \rho_{\text{int } \mathcal{A},S} \).

Proof. We only prove (i) since the proof of (ii) proceeds along similar lines. By the preceding corollary we know that \( \rho_{\text{cl } \mathcal{A},S} \) is lower semicontinuous. Moreover, since \( \mathcal{A} \subseteq \text{cl } \mathcal{A} \) we always have \( \rho_{\text{cl } \mathcal{A},S} \leq \rho_{A,S} \). Now, fix a lower semicontinuous map \( f : \mathcal{X} \to \mathbb{R} \) such that \( f \leq \rho_{A,S} \). To prove (i) we only need to show that \( f \leq \rho_{\text{cl } \mathcal{A},S} \). Assume, by contrast, that \( f(X) > m > \rho_{\text{cl } \mathcal{A},S}(X) \) for some \( X \in \mathcal{X} \) and \( m \in \mathbb{R} \). Since \( f \) is lower semicontinuous, we find a neighborhood \( \mathcal{U} \) of \( X \) so that \( f(Y) > m \) for all \( Y \in \mathcal{U} \). Note that \( X + \frac{m}{S_0} S_T \) belongs to \( \text{cl } \mathcal{A} \), hence it is the limit of a suitable net \( (Y_\alpha) \subseteq \mathcal{A} \). As a result, it follows that

\[
m < f \left( Y_\alpha - \frac{m}{S_0} S_T \right) \leq \rho_{A,S} \left( Y_\alpha - \frac{m}{S_0} S_T \right) \leq m
\]

for some index \( \alpha \), which is clearly impossible. In conclusion, \( f \leq \rho_{\text{cl } \mathcal{A},S} \) must hold and the proof is complete.
Combining the above results on lower and upper semicontinuity, it is possible to obtain corresponding statements about the continuity of risk measures $\rho_{A,S}$. Here, we present a statement highlighting the underlying geometrical interpretation of continuity: the risk measure $\rho_{A,S}$ is continuous at $X \in \mathcal{X}$ if the curve $\gamma_X : \mathbb{R} \rightarrow \mathcal{X}$ defined by

$$\gamma_X(\lambda) = X + \lambda S_T$$

lies outside the closure $\text{cl} \ A$ till it penetrates the boundary $\text{bd} \ A$ and immediately enters the interior $\text{int} \ A$. In other words, the line $\gamma_X$ cannot adhere to the boundary of $A$ at more than one point for $\rho_{A,S}$ to be continuous at $X$. This situation can be pictured in the following way:

\[ X + \lambda S_T \notin \text{cl} \ A \quad \text{if} \quad \lambda < \lambda_X \]

\[ X + \lambda S_T \in \text{int} \ A \quad \text{if} \quad \lambda > \lambda_X \]

\[ \rho_{A,S} \text{ lower semicontinuous} \quad \lambda_X \quad \rho_{A,S} \text{ upper semicontinuous} \]

The formal proof of this “transversality” condition is provided in the following result.

**Proposition 3.3.10.** The risk measure $\rho_{A,S}$ is continuous if and only if for any $X \in \mathcal{X}$ there exists $\lambda_X \in \mathbb{R}$ such that

(i) $X + \lambda S_T \notin \text{cl} \ A$ whenever $\lambda < \lambda_X$;

(ii) $X + \lambda S_T \in \text{int} \ A$ whenever $\lambda > \lambda_X$.

In this case, we have $\rho_{A,S}(X) = \lambda_X S_0$.

**Proof.** By virtue of Proposition 3.3.5, we only need to prove the “if” part. To this end, assume we find $\lambda_X \in \mathbb{R}$ satisfying both statements. If we can show that $\rho_{A,S}(X) = \lambda_X S_0$, then the claim would follow by Proposition 3.3.5. First, note that (ii) implies that $\rho_{A,S}(X) \leq \lambda_X S_0$. On the other side, if $\rho_{A,S}(X) < \lambda_X S_0$ we would find $\lambda < \lambda_X$ such that $X + \lambda S_T \in A$, in contrast to (i). Hence, we must have $\rho_{A,S}(X) = \lambda_X S_0$, concluding the proof. \[\square\]

We conclude this section with a necessary condition for a risk measure of the form $\rho_{A,S}$ to be continuous on its effective domain, which follows directly from the above proposition (or Remark 3.3.6).
Corollary 3.3.11. Assume \( \rho_{A,S} \) is continuous at some point \( X \in \mathcal{X} \) such that \( \rho_{A,S}(X) < \infty \). Then \( \text{int } A \) is nonempty.

Remark 3.3.12. Consider the space \( \mathcal{X} = L^p, p \in [1, \infty) \), in a nonatomic setting. By the previous corollary, Theorem 2.9 in Kaina, Rüschendorf [66] cannot be true in the stated generality, namely that any lower semicontinuous, coherent cash-additive risk measure \( \rho : L^p \rightarrow \mathbb{R} \cup \{\infty\} \) must automatically be finitely valued and continuous. To see this, consider the closed, convex cone \( A = L^p_+ \). The corresponding cash-additive risk measure \( \rho_A \) is coherent and lower semicontinuous by Proposition 3.2.8 and Corollary 3.3.8, respectively. However, \( \rho_A \) cannot be continuous at any point of finiteness since \( L^p_+ \) has empty interior. Moreover, note that \( \rho_A \) is not even finitely valued, since \( \rho_A(X) = \infty \) whenever \( X \) is not bounded from below. The problem in [66] originates with the proof of Proposition 2.8 in that paper which only works for finitely-valued functions.

3.3.3 General acceptance sets

It is well-known that every cash-additive risk measure on the space \( L^\infty \) is finitely valued and (even Lipschitz) continuous. Our first result can be seen as a generalization of this fact and unveils the key property in that context: the random variable \( 1_{\Omega} \) is an interior point of the positive cone in \( L^\infty \).

Theorem 3.3.13. The following statements hold:

(i) If \( S_T \in \text{core } \mathcal{X}_+ \), then \( \rho_{A,S} \) is finitely valued;

(ii) If \( S_T \in \text{int } \mathcal{X}_+ \), then \( \rho_{A,S} \) is finitely valued and continuous.

Proof. (i) Fix \( X \in \mathcal{X} \) and take \( Y \in A \) and \( Z \in A^c \). Since \( S_T \) is a core point of \( \mathcal{X}_+ \), there exists \( \lambda > 0 \) such that \( Y - X \leq \lambda S_T \). As a result, we have \( X + \lambda S_T \in A \), implying \( \rho_{A,S}(X) < \infty \). On the other hand, we can also find \( \xi > 0 \) so that \( X - Z \leq \xi S_T \). Thus, \( X - \xi S_T \notin A \) by monotonicity, showing that \( \rho_{A,S}(X) > -\infty \).

(ii) Clearly, \( \rho_{A,S} \) is finite by the preceding point. To prove continuity, take an arbitrary \( X \in \mathcal{X} \) and assume it is the limit of a net \( (X_\alpha) \). Then, for every \( \varepsilon > 0 \) there exists \( \alpha_\varepsilon \) such that \( -\varepsilon S_T \leq X_\alpha - X \leq \varepsilon S_T \) whenever \( \alpha \geq \alpha_\varepsilon \) by Theorem 9.40 in Aliprantis, Border [4]. As a consequence, monotonicity and \( S \)-additivity
implies
\[ |\rho_{A,S}(X_\alpha) - \rho_{A,S}(X)| \leq \varepsilon S_0 \]
for any \( \alpha \geq \alpha_\varepsilon \), showing that \( \rho_{A,S} \) is continuous at \( X \).

**Corollary 3.3.14.** Assume \( \mathcal{X} \) is a normed space and \( S_T \in \text{int} \mathcal{X}_+ \). Then, \( \rho_{A,S} \) is finitely valued and Lipschitz continuous.

**Proof.** Given the previous result, we only need to prove Lipschitz continuity. Since \( S_T \) is an interior point of \( \mathcal{X}_+ \), we find \( \lambda > 0 \) such that \( X \leq \lambda \|X\| S_T \) for every \( X \in \mathcal{X} \). Hence, taking \( X, Y \in \mathcal{X} \) we obtain
\[
Y \leq X + \lambda \|X - Y\| S_T
\]
and \( S \)-additivity, together with monotonicity, implies
\[
\rho_{A,S}(X) - \rho_{A,S}(Y) \leq \lambda S_0 \|X - Y\|.
\]
Exchanging \( X \) and \( Y \), we conclude the proof.

**Remark 3.3.15.** The preceding result implies Lemma 3.5 by Filipović, Kupper [45], which is in fact a continuity criterion on a quasiordered normed space whose positive cone has nonempty interior. Indeed, the condition on the payoff of their “numéraire” asset amounts to assuming that it is an interior point of the corresponding positive cone.

### 3.3.4 Convex acceptance sets

In this section we focus on convex acceptance sets. Convexity allows to obtain finiteness and continuity results for a wider range of payoffs \( S_T \) without requiring that the positive cone has nonempty core or nonempty interior. In particular, all results in this section apply to \( L^p \), \( p \in [1, \infty] \), and Orlicz spaces.

We start by focusing on a salient aspect of convexity, i.e. the possibility to derive automatic continuity results. It is known from Borwein [17] that, under suitable assumptions\(^1\) on the underlying reference space \( \mathcal{X} \), every map \( f : \mathcal{X} \to \mathbb{R} \cup \{\infty\} \) which is convex and decreasing is automatically continuous on the interior of

\(^1\)As recorded in Theorem 0.51, the positive cone \( \mathcal{X}_+ \) must have nonempty interior or, alternatively, the space \( \mathcal{X} \) has to be completely metrizable and \( \mathcal{X}_+ \) must be closed and generate
its effective domain. In particular, finiteness implies continuity for this type of mappings.

For risk measures of the form $\rho_{A,S}$ we can prove an automatic continuity result without any assumption on the reference space $\mathcal{X}$. We will only assume that the underlying acceptance set has nonempty interior, which appeared to be a necessary condition for continuity in Corollary 3.3.11.

**Theorem 3.3.16.** Assume $A$ is convex and $\text{int} \ A$ is nonempty. Moreover, assume $\rho_{A,S}$ does not take the value $-\infty$. Then, $\rho_{A,S}$ is continuous on the interior of $\text{dom}(\rho_{A,S})$.

**Proof.** Note first that the domain of $\rho_{A,S}$ has nonempty interior because it contains $A$. Since $A \subseteq \{\rho_{A,S} \leq 0\}$, the risk measure $\rho_{A,S}$ is bounded above on some open set, namely $\text{int} \ A$. As a result, we can apply Theorem 0.50 to conclude that $\rho_{A,S}$ is continuous on the interior of $\text{dom}(\rho_{A,S})$.

**Corollary 3.3.17.** Assume $A$ is convex and $\text{int} \ A$ is nonempty. If $\rho_{A,S}$ is finitely valued, then it is continuous.

**Remark 3.3.18.** The requirement that $\text{int} \ A$ is nonempty is necessary to obtain the automatic continuity result above. In fact, the following interesting example shows that, if $\text{int} \ A$ is empty, then a convex risk measure of the form $\rho_{A,S}$ may be finitely valued but discontinuous at any point. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a nonatomic space and equip $L^\infty$ with the topology $\sigma(L^\infty, L^1)$. Assume $S_T \in L^\infty_+$ is an order unit. Then $\rho_{L^\infty_+,S}$ is finitely valued by Theorem 3.3.13. However, $\rho_{L^\infty_+,S}$ cannot be continuous at any point due to Corollary 3.3.11, since the interior of $L^\infty_+$ is empty.

Motivated by the preceding results, we will now focus on conditions ensuring finiteness. We start by showing that a general necessary condition for a convex risk measure $\rho_{A,S}$ to be finite is that the underlying acceptance set has nonempty core.

the whole space $\mathcal{X}$. An additional result in the context of bornological spaces is also proved in [17]. These specifications correspond to the classical conditions ensuring the continuity of positive linear functionals, see Section 5.5 in Schaefer [90]. The results in [17] encompass both the “extended Namioka-Klee theorem” by Biagini, Frittelli [14] and the automatic continuity criterion obtained by Ruszczyński, Shapiro [88].
Lemma 3.3.19. Assume $\mathcal{A}$ is convex and $\rho_{A,S}$ does not attain the value $-\infty$. Then, $\text{dom}(\rho_{A,S})$ has nonempty core if and only if $\mathcal{A}$ has nonempty core.

Proof. Since $\mathcal{A} \subseteq \text{dom}(\rho_{A,S})$, it is enough to prove the “only if” part. To this end, let $X$ be a core point of $\text{dom}(\rho_{A,S})$. By $S$-additivity, we can assume without loss of generality that $\rho_{A,S}(X) < 0$. Take a nonzero $Y \in \mathcal{X}$. Then, we find $\varepsilon > 0$ such that $X + \lambda Y \in \text{dom}(\rho_{A,S})$ whenever $|\lambda| < \varepsilon$. As a result, $f(\lambda) = \rho_{A,S}(X + \lambda Y)$ defines a real-valued, convex function on the interval $(-\varepsilon, \varepsilon)$, which must be continuous by a standard result in convex analysis. Since $f(0) = \rho_{A,S}(X) < 0$, it follows from continuity that there exists $\delta > 0$ such that $\rho_{A,S}(X + \lambda Y) < 0$, hence $X + \lambda Y \in \mathcal{A}$, for all $|\lambda| < \delta$. This proves that $X \in \text{core } \mathcal{A}$. □

Corollary 3.3.20. Assume $\mathcal{A}$ is convex. If $\rho_{A,S}$ is finitely valued, then $\text{core } \mathcal{A}$ is nonempty.

We now turn to sufficient conditions for finiteness. First, we show that if $\rho_{A,S}$ is finitely valued in the direction of some strictly positive element, then it is finitely valued on $\mathcal{X}$. This provides a simple criterion for finiteness and continuity which will be useful in the sequel. Note that we do not require any explicit assumption on the payoff $S_T$.

Theorem 3.3.21. Let $U$ be a strictly positive element of $\mathcal{X}$. Assume $\mathcal{A}$ is convex and $\text{int } \mathcal{A}$ is nonempty. Moreover, assume $\rho_{A,S}$ does not attain the value $-\infty$. The following statements are equivalent:

(i) $\rho_{A,S}$ is finitely valued;

(ii) $\rho_{A,S}(-\lambda U) < \infty$ for all $\lambda > 0$.

In this case, $\rho_{A,S}$ is also continuous.

Proof. We only need to prove that (ii) implies (i). Assume $X \notin \text{dom}(\rho_{A,S})$. Since $\text{dom}(\rho_{A,S})$ is convex and has nonempty interior, by separation we find a nonzero functional $\psi \in \mathcal{X}'$ satisfying

$$
\psi(X) \leq \sigma_{\text{dom}(\rho_{A,S})}(\psi) \leq \psi(-\lambda U)
$$

for all $\lambda > 0$. Now, being monotone, $\text{dom}(\rho_{A,S})$ is itself an acceptance set. Since $\psi$ belongs to its barrier cone, $\psi$ must be positive by virtue of Proposition 2.3.4.
As a result, the above inequality can only hold if $\psi(U) = 0$, contradicting the strict positivity of $U$. In conclusion, $\rho_{A,S}$ must be finite, hence continuous by Corollary 3.3.17, on the whole of $X$.

\begin{remark}
The above result is particularly useful in the context of $L^p$ spaces, $p \in [1, \infty]$, or (nontrivial) Morse spaces, since $1_\Omega$ is a strictly positive element in these spaces.
\end{remark}

From Theorem 3.3.13 we know that, for a general acceptance set, we always have finiteness if the payoff $S_T$ is an order unit. If the acceptance set is convex and has nonempty interior, it suffices to require that $S_T$ be strictly positive. Note that, in contrast to other results in this section, the assumption that $\rho_{A,S}$ does not attain the value $-\infty$ is not needed here.

\begin{theorem}
Assume $A$ is convex and int $A$ is nonempty. If $S_T$ is strictly positive, then $\rho_{A,S}$ is finitely valued and continuous.
\end{theorem}

\begin{proof}
First, we show that $\rho_{A,S}$ never attains the value $-\infty$. Indeed, assume to the contrary that $\rho_{A,S}(X) = -\infty$ for some $X \in X$. In particular, $X + \lambda S_T \in A$ for every $\lambda \in \mathbb{R}$. Now take $Y \notin A$. By separation, there exists a nonzero $\psi \in X'$ such that $\psi(Y) \leq \psi(Z)$ for all $Z \in A$. In particular,

$$\psi(Y) \leq \psi(X + \lambda S_T)$$

for any $\lambda \in \mathbb{R}$, so that we must have $\psi(S_T) = 0$. However, $\psi$ is positive due to Proposition 2.3.4 and, consequently, $\psi(S_T) = 0$ cannot hold because $S_T$ is strictly positive. We conclude that $\rho_{A,S}$ cannot attain the value $-\infty$.

By Theorem 3.3.21, the claim will follow once we show that $\rho_{A,S}(-\lambda S_T) < \infty$ for all $\lambda > 0$ or, equivalently by $S$-additivity, that $\rho_{A,S}(0) < \infty$. Assume this is not the case. Then, we have span$(S_T) \cap A = \emptyset$. As a result, we can find a nonzero separating functional $\psi \in X'$ such that

$$\lambda \psi(S_T) \leq \psi(X)$$

for every $X \in A$ and $\lambda \in \mathbb{R}$. This implies $\psi(S_T) = 0$, which is again in contrast to the positivity of $\psi$ ensured by Proposition 2.3.4. Hence $\rho_{A,S}(0) < \infty$, concluding the proof.
\end{proof}
We conclude this section by showing that, when the underlying acceptance set has nonempty interior, a convex risk measure which is finitely valued on a dense subspace is automatically finitely valued on the whole space. This is particularly useful when dealing with risk measures defined on $L^p$, $p \in [1, \infty)$, or on (nontrivial) Morse spaces since, typically, it is not difficult to establish finiteness on the dense subspace $L^\infty$.

**Proposition 3.3.24.** Assume $\mathcal{A}$ is convex and $\text{int} \mathcal{A}$ is nonempty. Moreover, assume $\rho_{\mathcal{A},S}$ does not attain the value $-\infty$. If $\rho_{\mathcal{A},S}$ is finitely valued on a dense linear subspace $\mathcal{M}$ of $\mathcal{X}$, then $\rho_{\mathcal{A},S}$ is finitely valued and continuous on $\mathcal{X}$.

**Proof.** Assume $X \notin \text{dom}(\rho_{\mathcal{A},S})$. Since the domain of $\rho_{\mathcal{A},S}$ is convex and contains $\mathcal{A}$, by separation we find a nonzero $\psi \in \mathcal{X}'$ such that

$$\psi(X) \leq \psi(Y)$$

for all $Y \in \mathcal{M}$. But this implies that $\psi$ must annihilate $\mathcal{M}$ and hence, by density, the whole space $\mathcal{X}$. As a result, the risk measure $\rho_{\mathcal{A},S}$ must be finitely valued, hence continuous by Corollary 3.3.17, on the whole of $\mathcal{X}$. $\square$

### 3.3.5 Conic acceptance sets

In this section, we focus our analysis on conic acceptance sets. The following results will be of practical importance for risk measures based on VaR-acceptability in Section 3.7.

**Theorem 3.3.25.** Assume $\mathcal{A}$ is a cone. The following statements hold:

(i) $\rho_{\mathcal{A},S}(X) < \infty$ for all $X \in \mathcal{X}$ if and only if $S_T \in \text{core} \mathcal{A}$;

(ii) $\rho_{\mathcal{A},S}(X) > -\infty$ for all $X \in \mathcal{X}$ if and only if $-S_T \in \text{core} \mathcal{A}^c$.

**Proof.** We only prove part (i). The proof of part (ii) proceeds along similar lines. Assume first that $S_T$ is a core point of $\mathcal{A}$ so that for every $X \in \mathcal{X}$ there exists $\lambda > 0$ with $S_T + \lambda X \in \mathcal{A}$. Since $\mathcal{A}$ is a cone, this implies $X + \frac{1}{\lambda} S_T \in \mathcal{A}$ showing that $\rho_{\mathcal{A},S}(X) < \infty$. To prove the converse implication, assume $S_T \notin \text{core} \mathcal{A}$. Then, we can find $X \in \mathcal{X}$ such that $S_T + \lambda_n X \notin \mathcal{A}$ for a suitable sequence $(\lambda_n)$ of strictly positive numbers converging to zero. Equivalently, $X + \frac{1}{\lambda_n} S_T \notin \mathcal{A}$ for every $n \in \mathbb{N}$, implying $\rho_{\mathcal{A},S}(X) = \infty$. $\square$
Hence, as in the context of convex acceptance sets, a risk measure \( \rho_{\mathcal{A},S} \) based on a conic acceptance set can be finitely valued only if \( \mathcal{A} \) has nonempty core.

**Corollary 3.3.26.** Assume \( \mathcal{A} \) is a cone. If \( \rho_{\mathcal{A},S} \) is finitely valued, then core \( \mathcal{A} \) is nonempty.

The next result shows that finiteness is a necessary condition for continuity in case of a conic acceptance set. The converse implication is, however, not true, as exemplified by risk measures based on VaR-acceptability in Section 3.7.

**Proposition 3.3.27.** Assume \( \mathcal{A} \) is a cone. If \( \rho_{\mathcal{A},S} \) is continuous at 0, then \( S_T \in \text{int} \mathcal{A} \) and \( -S_T \in \text{int} \mathcal{A}^c \).

*Proof.* Since \( \rho_{\mathcal{A},S} \) is conic, we must have \( \rho_{\mathcal{A},S}(0) = 0 \) by continuity. Indeed, observe that, taking \( X \in \mathcal{A} \) and \( Y \in \mathcal{A}^c \),

\[
\rho_{\mathcal{A},S}(0) = \lim_{n \to \infty} \rho_{\mathcal{A},S}\left(\frac{1}{n}X\right) = \lim_{n \to \infty} \rho_{\mathcal{A},S}\left(\frac{1}{n}Y\right).
\]

Therefore, since \( \frac{1}{n}X \in \mathcal{A} \) and \( \frac{1}{n}Y \in \mathcal{A}^c \) for all \( n \in \mathbb{N} \), it follows that \( \rho_{\mathcal{A},S}(0) = 0 \) must hold. For the next implications we rely on Proposition 3.3.5. First, since \( \rho_{\mathcal{A},S} \) is lower semicontinuous at 0, we must have \( -S_T \notin \text{cl} \mathcal{A} \) or, equivalently, \( -S_T \in \text{int} \mathcal{A}^c \). On the other side, the upper semicontinuity of \( \rho_{\mathcal{A},S} \) at 0 immediately implies that \( S_T \in \text{int} \mathcal{A} \). \( \square \)

**Corollary 3.3.28.** Assume \( \mathcal{A} \) is a cone. If \( \rho_{\mathcal{A},S} \) is continuous, then it is finitely valued.

### 3.3.6 Convex and conic acceptance sets

In this section we characterize the range of payoffs \( S_T \) for which a risk measure of the form \( \rho_{\mathcal{A},S} \) is finitely valued, respectively continuous, when \( \mathcal{A} \) is a convex cone. We will apply these results to risk measures based on ES-acceptability in Section 3.7. We start with a general lemma about core points.

**Lemma 3.3.29.** Assume \( \mathcal{A} \) is convex and core \( \mathcal{A} \) is nonempty. Then, for every \( \lambda \in (0, 1) \) we have

\[
\lambda \text{core} \mathcal{A} + (1 - \lambda)(\text{core} \mathcal{A}^c)^c \subseteq \text{core} \mathcal{A}.
\]
Proof. Take arbitrary \( X \in \text{core} \mathcal{A} \) and \( Y \notin \text{core} \mathcal{A}^c \) and fix \( \lambda \in (0, 1) \). Assume that \( Z = \lambda X + (1 - \lambda)Y \notin \text{core} \mathcal{A} \). By the separation criterion recorded in Theorem 0.45, we find a nonzero linear functional \( \psi : \mathcal{X} \rightarrow \mathbb{R} \) such that

\[
\psi(A) \leq \psi(Z)
\]

for all \( A \in \mathcal{A} \). Moreover, \( \psi(X) < \psi(Z) \). Since \( Y \notin \text{core} \mathcal{A}^c \), we find \( U \in \mathcal{X} \) and a suitable sequence \((\lambda_n)\) of real numbers converging to zero such that \( Y + \lambda_n U \in \mathcal{A} \) for all \( n \in \mathbb{N} \). As a result, we get \( \psi(Y) \leq \psi(Z) \). However, this implies

\[
\psi(Z) = \lambda \psi(X) + (1 - \lambda)\psi(Y) < \lambda \psi(Z) + (1 - \lambda)\psi(Z) = \psi(Z)
\]

which is clearly not possible. In conclusion, we must have \( Z \in \text{core} \mathcal{A} \). \( \square \)

**Theorem 3.3.30.** Assume \( \mathcal{A} \) is a convex cone. The following statements are equivalent:

(i) \( \rho_{\mathcal{A}, S} \) is finitely valued;

(ii) \( S_T \in \text{core} \mathcal{A} \).

Proof. By Theorem 3.3.25, we only need to prove that \( S_T \in \text{core} \mathcal{A} \) implies that \( -S_T \in \text{core} \mathcal{A}^c \). Assume to the contrary that \( -S_T \notin \text{core} \mathcal{A}^c \). In this case, any nontrivial convex combination of \( -S_T \) and \( S_T \) would lie in the core of \( \mathcal{A} \) by the preceding lemma, hence in particular \( 0 \in \text{core} \mathcal{A} \). Since \( \mathcal{A} \) is a cone, this would imply \( \mathcal{A} = \mathcal{X} \), which is not possible because \( \mathcal{A} \) is a proper subset of \( \mathcal{X} \). \( \square \)

**Theorem 3.3.31.** Assume \( \mathcal{A} \) is a convex cone. The following statements are equivalent:

(i) \( \rho_{\mathcal{A}, S} \) is continuous;

(ii) \( \rho_{\mathcal{A}, S} \) is continuous at 0;

(iii) \( S_T \in \text{int} \mathcal{A} \).

Proof. Clearly (i) implies (ii), which in turn implies (iii) by Proposition 3.3.27. To prove that (iii) implies (i), assume that \( S_T \in \text{int} \mathcal{A} \). Then \( \rho_{\mathcal{A}, S} \) is finitely valued by the preceding theorem. Moreover, since \( \text{int} \mathcal{A} \) is nonempty, we can apply Corollary 3.3.17 to conclude that \( \rho_{\mathcal{A}, S} \) is also continuous on the full \( \mathcal{X} \). \( \square \)
Corollary 3.3.32. Assume $\mathcal{X}$ is a normed space and $\mathcal{A}$ is a convex cone. If $S_T \in \text{int } \mathcal{A}$, then $\rho_{\mathcal{A},S}$ is Lipschitz continuous.

Proof. Since $\rho_{\mathcal{A},S}$ is finitely valued and continuous by the preceding results, we have Lipschitz continuity on a suitable neighborhood of zero, say $U$, by Theorem 5.44 in Aliprantis, Border [4]. Note that $\rho_{\mathcal{A},S}(0) = 0$, for otherwise $\rho_{\mathcal{A},S}(0)$ would be nonfinite by conicity. Hence, we find a constant $\alpha > 0$ such that $|\rho_{\mathcal{A},S}(X)| \leq \alpha \|X\|$ for any $X \in U$. By conicity, the same inequality holds for any $X \in \mathcal{X}$. Now, for arbitrary $X, Y \in \mathcal{X}$ we have $\rho_{\mathcal{A},S}(X) \leq \rho_{\mathcal{A},S}(X - Y) + \rho_{\mathcal{A},S}(Y)$ by subadditivity, and therefore

$$\rho_{\mathcal{A},S}(X) - \rho_{\mathcal{A},S}(Y) \leq \alpha \|X - Y\|.$$  

Exchanging $X$ and $Y$ we see that $\rho_{\mathcal{A},S}$ is indeed Lipschitz continuous on $\mathcal{X}$.

Remark 3.3.33. Note that, if $\mathcal{A}$ is a convex cone containing $0$, the binary relation $X \leq_{\mathcal{A}} Y$ defined by $Y - X \in \mathcal{A}$ is a quasiorder on $\mathcal{X}$ with positive cone $\mathcal{A}$. This gives an alternative way to prove the implications “(ii) $\implies$ (i)” in Theorem 3.3.30 and “(iv) $\implies$ (i)” in Theorem 3.3.31 and the claim in Corollary 3.3.32. Indeed, $\mathcal{A}$ is an acceptance set with respect to the quasiorder induced by $\mathcal{A}$ itself, hence we can rely on the results in Section 3.3.3.

We conclude this section by specifying Theorem 3.3.21 to acceptance sets that are convex cones. The result follows immediately using the conicity of the corresponding risk measures.

Corollary 3.3.34. Let $U$ be a strictly positive element of $\mathcal{X}$, and assume $\mathcal{A}$ is a convex cone and $\text{int } \mathcal{A}$ is nonempty. Moreover, assume $\rho_{\mathcal{A},S}$ does not attain the value $-\infty$. The following statements are equivalent:

(i) $\rho_{\mathcal{A},S}$ is finitely valued;

(ii) $\rho_{\mathcal{A},S}(-U) < \infty$.

In this case, $\rho_{\mathcal{A},S}$ is also continuous.

Remark 3.3.35. The previous result is, again, particularly useful when $\mathcal{X}$ is an $L^p$ space, $p \in [1, \infty]$, or an Morse space since $1_\Omega$ is strictly positive in these spaces.
3.3.7 Acceptance sets in a lattice setting

We now turn to general acceptance sets in topological Riesz spaces. In this context, recall that an element \( U \in \mathcal{X}_+ \) is called a weak topological unit if, for every \( X \in \mathcal{X}_+ \), we have \( X \wedge nU \rightarrow X \) as \( n \rightarrow \infty \).

The first result provides a sufficient condition for \( \rho_{A,S} \) to be finitely valued.

**Theorem 3.3.36.** Let \( \mathcal{X} \) be a topological Riesz space. Assume \( \text{int} A \) is nonempty and \( \rho_{A,S} \) does not attain the value \(-\infty\). If \( S_T \) is a weak topological unit, then \( \rho_{A,S} \) is finitely valued.

**Proof.** Take \( Z \in \text{int} A \) and choose a neighborhood of zero \( U \) with \( Z + U \subseteq A \). Fix \( Y \in \mathcal{X}_+ \) and note that
\[
Y = Y \wedge (nS_T) + (Y - nS_T)^+
\]
for any \( n \in \mathbb{N} \). Since \( S_T \) is a weak topological unit, it follows that \( (Y - nS_T)^+ \rightarrow 0 \) as \( n \rightarrow \infty \), so that \( -(Y - mS_T)^+ \in U \) for a sufficiently large \( m \in \mathbb{N} \). In particular, note that \( Z - (Y - mS_T)^+ \in A \). Moreover, it is easy to show that
\[
Z - (Y - mS_T)^+ - mS_T \leq Z - Y.
\]
Hence, by monotonicity and \( S \)-additivity, we conclude that
\[
\rho_{A,S}(Z - Y) \leq mS_0 < \infty.
\]
Now, take an arbitrary \( X \in \mathcal{X} \). Setting \( Y = (Z - X)^+ \), it follows that
\[
\rho_{A,S}(X) \leq \rho_{A,S}(Z - Y) < \infty.
\]

**Remark 3.3.37.** Note that the acceptance set \( A \) in the previous theorem is not required to be convex. Hence, our result contains as a special case nonconvex extensions of two well-known finiteness results for convex risk measures in a cash-additive setting: Theorem 2.3 by Svindland [95] on \( L^p \) spaces, \( p \in [1, \infty] \), and Theorem 4.6 by Cheridito, Li [23] on Morse spaces. The corresponding proofs rely on separation arguments which cannot be reproduced in the nonconvex setting above. In fact, our approach is simpler and depends solely on the lattice structure. It is closer in spirit to the proof of Proposition 6.7 in Shapiro, Dentcheva, Ruszczyński [92] who, however, make use of a category argument that only works if lower semicontinuity is additionally assumed.
If $\mathcal{X}$ is additionally assumed to be a Fréchet lattice, we can restate the preceding theorem in the following way by virtue of Proposition 2.2.8.

**Corollary 3.3.38.** Let $\mathcal{X}$ be a Fréchet lattice, and assume $\text{core } \mathcal{A}$ is nonempty and $\rho_{\mathcal{A},S}$ does not attain the value $-\infty$. If $S_T$ is a weak topological unit, then $\rho_{\mathcal{A},S}$ is finitely valued.

We conclude this section by focusing on automatic continuity results in the context of Fréchet lattices, following a different approach from the one adopted in Biagini, Frittelli [14].

**Proposition 3.3.39.** Let $\mathcal{X}$ be a Fréchet lattice, and assume $\mathcal{A}$ is convex and $\rho_{\mathcal{A},S}$ does not attain the value $-\infty$. Then, $\rho_{\mathcal{A},S}$ is continuous on the interior of $\text{dom}(\rho_{\mathcal{A},S})$.

**Proof.** Assume $\text{dom}(\rho_{\mathcal{A},S})$ does possess nonempty interior, otherwise the claim is trivially satisfied. In this case, since $\mathcal{A}$ is convex, Lemma 3.3.19 implies that $\text{core } \mathcal{A}$ is nonempty. Then, $\text{int } \mathcal{A}$ is also nonempty by Proposition 2.2.8. In conclusion, it follows from Theorem 3.3.16 that $\rho_{\mathcal{A},S}$ is indeed continuous on the interior of its domain of finiteness.

**Corollary 3.3.40.** Let $\mathcal{X}$ be a Fréchet lattice and assume $\mathcal{A}$ is convex. If $\rho_{\mathcal{A},S}$ is finitely valued, then it is continuous.

**Remark 3.3.41.** The finiteness result on Fréchet lattices stated in Corollary 3.3.38 immediately provides a continuity criterion in case of convex acceptance sets in light of the preceding corollary. Moreover, note that, if $\mathcal{A}$ is convex, no risk measure $\rho_{\mathcal{A},S}$ can be finitely valued whenever the topological dual of $\mathcal{X}$ is trivial, i.e. $\mathcal{X}' = \{0\}$. See also Remark 3.3.6 above.

**Remark 3.3.42.** The proof of Proposition 3.3.39 provides an alternative approach to the extended Namioka-Klee theorem by Biagini, Frittelli [14], stating that every convex, decreasing function $f : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ on a Fréchet lattice $\mathcal{X}$ is continuous on the interior of its domain. Indeed, assume $f$ is such a map and let $X$ be an interior point of its domain. Moreover, set $\mathcal{A} = \{f < \lambda\}$ for some $\lambda > f(X)$. As in the proof of Lemma 3.3.19 above, it is not difficult to show that $X$ belongs to the core of $\mathcal{A}$. Since $\mathcal{A}$ is monotone, $X$ is actually an interior point of $\mathcal{A}$ due to Proposition 2.2.8. Hence, $f$ turns out to be bounded above on a neighborhood of $X$ and the claim follows from Theorem .0.50.
3.4 Comparing risk measures

In this section we address the problem of the optimal choice of the reference asset when alternative choices are possible. The optimal choice correspond to the reference asset minimizing the value of a risk measure associated with a pre-specified acceptance set.

3.4.1 Equality of risk measures

We start by discussing when two risk measures of the form $\rho_{A,S}$ coincide. To this end, assume $A_1$ and $A_2$ are acceptance sets in $\mathcal{X}$ and consider the reference assets specified by $S^1 = (S_0, S_T^1)$ and $S^2 = (S_0, S_T^2)$. For later use, we define for any $m \in \mathbb{R}$

$$P_m(S^1, S^2) := \left\{ \frac{m}{S_0} S_T^1 + \lambda (S_T^2 - S_T^1) ; \lambda \in \mathbb{R} \right\}.$$ 

Every element in $P_m(S^1, S^2)$ represents the payoff of a possible “portfolio” we can set up by implementing a combination of $S^1$ and $S^2$ with initial cost $m$.

**Proposition 3.4.1.** Assume that $\rho_{A_1,S^1}$ and $\rho_{A_2,S^2}$ are lower semicontinuous. Then, $\rho_{A_1,S^1} = \rho_{A_2,S^2}$ if and only if the following conditions are satisfied:

(i) $\text{cl} \, A_1 = \text{cl} \, A_2$;

(ii) $\text{cl} \, A_1 = \text{cl} \, A_1 + P_0(S^1, S^2)$.

**Proof.** Without loss of generality we may assume that $A_1$ and $A_2$ are both closed. This follows from Proposition 3.3.5. To prove the “if” part, assume both conditions hold and take $X \in \mathcal{X}$. In particular, we have $A_1 = A_2$. Since

$$X + \lambda S_T^2 = X + \lambda S_T^1 + \lambda (S_T^2 - S_T^1)$$

for any $\lambda \in \mathbb{R}$, we see that $X + \lambda S_T^1 \in A_1$ implies $X + \lambda S_T^2 \in A_2$ by means of (ii). Hence, it follows that $\rho_{A_2,S^2}(X) \leq \rho_{A_1,S^1}(X)$. By exchanging the roles of $S^1$ and $S^2$ we obtain the reverse inequality. To prove the “only if” part, assume that $\rho_{A_1,S^1}$ and $\rho_{A_2,S^2}$ coincide. As a consequence of Proposition 3.3.5 we immediately obtain (i). It remains to show that $X + \lambda (S_T^2 - S_T^1) \in A_1$ whenever $X \in A_1$ and $\lambda \in \mathbb{R}$. But this follows again from Proposition 3.3.5 since, in this case,

$$\rho_{A_1,S^1}(X + \lambda (S_T^2 - S_T^1)) = \rho_{A_1,S^1}(X) + \lambda (S_0 - S_0) \leq 0.$$
Under additional assumptions we can obtain a sharper characterization of the above result. In particular, we will assume that \( \rho_{A_1,S_1}(0) \in \mathbb{R} \) and

\[
P_m(S^1, S^2) \cap \text{cl } A_1 = \left\{ \frac{m}{S_0} S^1_T \right\} \quad \text{for } m = \rho_{A_1,S_1}(0).
\]

To best interpret this condition, assume \( m = 0 \). In this case, the elements in \( P_0(S^1, S^2) \) correspond to fully-leveraged positions where either a long position in \( S^1 \) is financed by “borrowed money” obtained by entering a corresponding short position in \( S^2 \), or vice versa. Thus, the above condition is equivalent to requiring that no fully-leveraged combination of \( S^1 \) and \( S^2 \) is acceptable.

**Corollary 3.4.2.** Assume that \( \rho_{A_1,S_1} \) and \( \rho_{A_2,S_2} \) are lower semicontinuous. Furthermore, assume \( \rho_{A_1,S_1}(0) \) is finite and

\[
P_{\rho_{A_1,S_1}(0)}(S^1, S^2) \cap \text{cl } A_1 = \left\{ \frac{\rho_{A_1,S_1}(0)}{S_0} S^1_T \right\}.
\]

Then \( \rho_{A_1,S_1} = \rho_{A_2,S_2} \) if and only if \( \text{cl } A_1 = \text{cl } A_2 \) and \( S^1_T = S^2_T \).

**Proof.** By Proposition 3.3.5 we can assume that \( A_1 \) and \( A_2 \) are both closed. The “if” part is then obvious. To prove the “only if” part, assume that \( \rho_{A_1,S_1} \) and \( \rho_{A_2,S_2} \) coincide. Then \( A_1 = A_2 \) and \( A_1 + P_0(S^1, S^2) = A_1 \) by Proposition 3.4.1. Since \( \frac{\rho_{A_1,S_1}(0)}{S_0} S^1_T \in A_1 \), for all \( \lambda \in \mathbb{R} \) we obtain

\[
\frac{\rho_{A_1,S_1}(0)}{S_0} S^1_T + \lambda(S^2_T - S^1_T) \in P_{\rho_{A_1,S_1}(0)} \cap A_1
\]

which, under our assumption, can only hold if \( S^1_T = S^2_T \).

### 3.4.2 Optimal reference asset

The following result shows that, in the common cases, the optimality problem cannot be solved at a global level but only makes sense at the level of individual positions. This has important consequences in a capital adequacy framework since it implies that, if a regulator allows financial institutions to make a position acceptable by raising capital and, irrespective of their individual balance sheets, investing this capital amount in the same way – for instance in a pre-specified bond – then some institutions may be forced to reach acceptability at a higher cost than would have been possible by choosing an alternative reference asset.
Theorem 3.4.3. Assume that \( \rho_{A,S^1} \) and \( \rho_{A,S^2} \) are lower semicontinuous. If \( \rho_{A,S^1}(X) \leq \rho_{A,S^2}(X) \) for every \( X \in \mathcal{X} \), then \( \rho_{A,S^1} = \rho_{A,S^2} \).

Proof. As a consequence of Proposition 3.3.5, we can assume that \( A \) is closed without losing generality. Then, by Proposition 3.4.1, it suffices to show that \( X + \lambda(S^2_T - S^1_T) \in A \) for every \( X \in A \) and \( \lambda \in \mathbb{R} \). In this case, our assumption implies that

\[
\rho_{A,S^1}(X + \lambda(S^2_T - S^1_T)) \leq \rho_{A,S^2}(X + \lambda S^2_T) + \lambda S_0 = \rho_{A,S^2}(X) \leq 0
\]

and the claim follows from Proposition 3.3.5. \( \square \)

If we require that fully-leveraged combinations of \( S^1 \) and \( S^2 \) are not acceptable, the following stronger statement can be made. In light of the preceding theorem, the result follows immediately from Corollary 3.4.2.

Corollary 3.4.4. Assume that \( \rho_{A,S^1} \) and \( \rho_{A,S^2} \) are lower semicontinuous. Furthermore, assume that \( \rho_{A,S^1}(0) \) is finite and

\[
\mathcal{P}_{\rho_{A,S^1}(0)} \cap \text{cl}A = \left\{ \frac{\rho_{A,S^1}(0)}{S_0} \frac{S^1_T}{S^1_T} \right\}.
\]

If \( \rho_{A,S^1}(X) \leq \rho_{A,S^2}(X) \) for every \( X \in \mathcal{X} \), then \( S^1_T = S^2_T \).

Remark 3.4.5. In Filipović [44] a different sort of optimality is addressed. That paper deals with a fixed convex, cash-additive risk measure \( \rho : L^\infty \to \mathbb{R} \) and shows that, under a sensitivity assumption, there is no “numéraire” \( U \in L^\infty \) which minimizes \( \rho(X/U) \) for \( X \in L^\infty \). Note that, in this setting, \( \rho \) is kept fixed and the numéraire changes. Hence, with every choice of the numéraire the set of acceptable positions may change. Moreover, we can only consider positions \( X \in L^\infty \) such that \( X/U \) is still bounded. By contrast, in this section we have investigated what happens if we keep the acceptability criterion fixed and vary the chosen reference asset, without imposing any constraint on the range of “feasible” positions and without any sensitivity assumption. Moreover, the interpretation of \( \rho(X/U) \) may be problematic whenever the induced acceptance set \( A(\rho) \) is not numéraire-invariant, as shown in Section 1.3.3 in the context of Expected Shortfall. \( \square \)
3.5 Representing risk measures

In this section we focus on dual representations of risk measures of the form $\rho_{A,S}$. In order to apply separation techniques, we assume throughout that our reference space $\mathcal{X}$ is \textit{locally convex}. Instead of resorting to conjugate duality, we opt for an alternative approach that exploits the particular structure of risk measures and is based on a dual representation of the underlying acceptance set.

3.5.1 Nondegeneracy conditions

In the sequel we will focus on risk measures that are convex and lower semicontinuous. The next example shows that a risk measure of the form $\rho_{A,S}$ may satisfy these properties but be degenerate, i.e. may only take nonfinite values.

Example 3.5.1. Consider the setting of Example 3.3.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an infinite probability space and take $\mathcal{X} = L^\infty$. Assume $S_T$ is not an interior point of $L^\infty_+$ and consider the convex acceptance set

$$ A = \{ X \in L^\infty ; \ X \geq_\mathbb{P} \lambda S_T \text{ for some } \lambda \in \mathbb{R} \}. $$

Then $\text{cl} A$ is also a convex acceptance set by Proposition 2.2.7 and the risk measure $\rho_{\text{cl} A,S}$ is convex and lower semicontinuous by virtue of Proposition 3.2.8 and Corollary 3.3.8, respectively. Now, it is easy to see that

$$ \rho_{\text{cl} A,S}(X) \leq \rho_{A,S}(X) = -\infty $$

whenever $X \in A$. As a result, $\rho_{\text{cl} A,S}$ cannot take any finite value due to Proposition 2.4 in Ekeland, Témam [39].

Motivated by the preceding example, we provide a characterization of nondegeneracy for convex, lower semicontinuous risk measures. Before stating this result, it is useful to introduce a notation for the set of all continuous linear functionals on $\mathcal{X}$ assigning to $S_T$ the value $S_0$, i.e.

$$ \mathcal{E}(S) := \{ \psi \in \mathcal{X}^\prime ; \ \psi(S_T) = S_0 \}. $$

The elements in $\mathcal{E}(S)$ can be interpreted as those “pricing” functionals on $\mathcal{X}$ that are consistent with the asset $S$. 

Lemma 3.5.2. Assume $A$ is convex and $\rho_{A,S}$ is lower semicontinuous at $X \in X$. If $B(A) \cap E(S) = \emptyset$, then $\rho_{A,S}(X) \notin \mathbb{R}$.

Proof. First of all, recall that $\rho_{A,S}(X) = \rho_{\text{cl} A,S}(X)$ by Proposition 3.3.5. Hence, Theorem 2.3.6 implies that

$$\rho_{A,S}(X) = S_0 \inf \{ \lambda \in \mathbb{R} ; \psi(X) + \lambda \psi(S_T) \geq \sigma_A(\psi), \forall \psi \in B(A) \} . \quad (3.1)$$

Since $\psi(S_T) = 0$ for all $\psi \in B(A)$, it is immediate to see that $\rho_{A,S}(X) = \infty$ if $X \notin \text{cl} A$ and $\rho_{A,S}(X) = -\infty$ otherwise. \hfill \Box

The previous result highlights the importance of the condition

$$B(A) \cap E(S) \neq \emptyset$$

for a convex, lower semicontinuous risk measure of the form $\rho_{A,S}$ to be nondegenerate. In line with the above interpretation of $E(S)$, the intersection $B(A) \cap E(S)$ consists of all “pricing” functionals $\psi$ that are consistent with the asset $S$ and that support the underlying acceptance set $A$. Note that, if $A$ is a convex cone, then $\psi$ must assign a nonnegative “price” to each position in $A$.

In the next proposition, we prove that the above condition is indeed equivalent to nondegeneracy.

Proposition 3.5.3. Assume $A$ is closed and convex. Then, the following statements are equivalent:

(i) $\rho_{A,S}$ is nondegenerate;

(ii) $\rho_{A,S}(X) > -\infty$ for all $X \in X$;

(iii) $B(A) \cap E(S) \neq \emptyset$.

Proof. First of all, note that $\rho_{A,S}$ is convex and lower semicontinuous. If $\rho_{A,S}$ takes the value $-\infty$, then it must be degenerate by Proposition 2.4 in Ekeland, Téman [39]. Hence, (i) implies (ii). In turn, (ii) must imply (iii) by virtue of the preceding lemma. Indeed, as shown in the corresponding proof, $\rho_{A,S}(X) = -\infty$ for any $X \in A$ whenever $B(A) \cap E(S) = \emptyset$. Finally, assume there exists $\psi \in B(A)$ satisfying $\psi(S_T) = S_0$. Then, for every $X \in A$ we have

$$0 \geq \rho_{A,S}(X) \geq \sigma_A(\psi) - \psi(X) > -\infty ,$$
the central inequality being a consequence of the representation in (3.1). This shows that \((iii)\) implies \((i)\), concluding the proof of the equivalence. 

**Remark 3.5.4.** Clearly, the previous result still holds if \(A\) is not closed, provided that \(\rho_{A,S}\) be lower semicontinuous. 

We conclude this section with an additional characterization of nondegeneracy.

**Corollary 3.5.5.** Assume \(A\) is closed and convex. Moreover, assume \(\lambda S_T \in A\) for some \(\lambda \in \mathbb{R}\). Then, the following statements are equivalent:

1. \(\rho_{A,S}\) is nondegenerate
2. \(\rho_{A,S}(0) > -\infty\);
3. \(\xi S_T \notin A\) for some \(\xi \in \mathbb{R}\).

**Proof.** By the preceding proposition, we see that \((i)\) implies \((ii)\), which is clearly equivalent to \((iii)\). Now, assume that \(\xi S_T \notin A\) for some \(\xi \in \mathbb{R}\). By separation, we find a nonzero functional \(\psi \in X'\) such that

\[
\xi \psi(S_T) < \sigma_A(\psi) \leq \lambda \psi(S_T).
\]

In particular, we have \(\psi(S_T) \neq 0\). Since \(\psi\) must be positive by Proposition 2.3.4, we conclude that, up to rescaling, \(\psi\) belongs to \(B(A) \cap E(S)\). Hence, it follows from the previous proposition that \((iii)\) implies \((i)\). 

**Remark 3.5.6.** In the result above, we can again replace the assumption of closedness for \(A\) with the more general assumption of lower semicontinuity for \(\rho_{A,S}\) provided we replace \(A\) with \(\text{cl}\ A\) in the third assertion.

### 3.5.2 Dual representations

In this section, we focus on dual representations for risk measures of the form \(\rho_{A,S}\). We start with a pointwise result.

**Lemma 3.5.7.** Assume \(A\) is convex and \(\rho_{A,S}\) is lower semicontinuous at \(X \in X\). If \(B(A) \cap E(S) \neq \emptyset\), then

\[
\rho_{A,S}(X) = \sup_{\psi \in B(A) \cap E(S)} \{\sigma_A(\psi) - \psi(X)\}.
\]
Moreover, if $A$ is also a cone, we have

$$\rho_{A, S}(X) = \sup_{\psi \in B(A) \cap E(S)} \psi(-X).$$

**Proof.** First, recall that $\rho_{A, S}(X) = \rho_{\text{cl}\, A, S}(X)$ by Proposition 3.3.5. Since $\rho_{A, S}$ is nonconstant, $\text{cl}\, A$ must be strictly contained in $\mathcal{X}$. In particular, $\text{cl}\, A$ is itself an acceptance set by Proposition 2.2.6. As a consequence, Proposition 2.3.7 implies that

$$\text{cl}\, A = \bigcap_{\psi \in B(A) \cap E(S)} \{X \in \mathcal{X}; \psi(X) \geq \sigma_{A}(\psi)\}.$$  

Hence, we immediately obtain

$$\rho_{A, S}(X) = S_0 \inf\{\lambda \in \mathbb{R}; \psi(X) + \lambda \psi(S_T) \geq \sigma_{A}(\psi), \forall \psi \in B(A) \cap E(S)\}.$$  

Since $\psi(S_T) = S_0$ for all $\psi \in B(A) \cap E(S)$, the first representation easily follows. The representation in the conic case is a direct consequence of the fact that the corresponding support function is null on $B(A)$.

**Remark 3.5.8.** Dual representations are typically obtained for convex maps that are *globally* lower semicontinuous. Note that the representation above only requires *pointwise* lower semicontinuity. Indeed, a risk measure of the form $\rho_{A, S}$ may be lower semicontinuous at some point but not on the whole of $\mathcal{X}$. For instance, consider the setting of Example 3.3.4 and take $\mathcal{X} = L^1$ over a nonatomic probability space. The cash-additive risk measure associated to the acceptance set

$$\mathcal{A} = \{X \in L^1; \mathbb{E}[X] \geq 0, X \geq \alpha \text{ for some } \alpha \in \mathbb{R}\}$$

is lower semicontinuous at 0 but cannot be globally lower semicontinuous. First, note that

$$\text{cl}\, \mathcal{A} = \{X \in L^1; \mathbb{E}[X] \geq 0\}$$

and $\rho_{\mathcal{A}}(0) = -\mathbb{E}[0] = 0$. Since for any $\lambda < 0$ we have $\lambda \notin \text{cl}\, \mathcal{A}$, Proposition 3.3.5 implies that $\rho_{\mathcal{A}}$ is lower semicontinuous at 0. If $\rho_{\mathcal{A}}$ were lower semicontinuous on $L^1$, then

$$\rho_{\mathcal{A}}(X) = \rho_{\text{cl}\, \mathcal{A}}(X) = -\mathbb{E}[X]$$

for any $X \in L^1$ as a consequence of Proposition 3.3.5. But this is impossible since $\rho_{\mathcal{A}}$ is not finitely valued. The lack of global lower semicontinuity for the above risk measure was also remarked by Filipović, Svindland [47].
The corresponding global representation result is recorded in the next theorem and easily follows by combining the above lemma with Proposition 3.5.3.

**Theorem 3.5.9.** Assume \( A \) is closed and convex. Moreover, assume that \( \rho_{A,S} \) is nondegenerate. Then, for every \( X \in \mathcal{X} \)

\[
\rho_{A,S}(X) = \sup_{\psi \in \mathcal{B}(A) \cap \mathcal{E}(S)} \left\{ \sigma_A(\psi) - \psi(X) \right\}.
\]

Moreover, if \( A \) is also a cone, we have for every \( X \in \mathcal{X} \)

\[
\rho_{A,S}(X) = \sup_{\psi \in \mathcal{B}(A) \cap \mathcal{E}(S)} \psi(-X).
\]

**Remark 3.5.10.** Clearly, the above result continues to hold if we replace the assumption of closedness for \( A \) with the more general assumption of lower semi-continuity for \( \rho_{A,S} \).

**Remark 3.5.11.** Differently from the standard approach to duality via conjugation, the above “geometrical” approach – first pursued in Farkas, Koch-Medina, Munari [43] – exploits the very structure of risk measures of the form \( \rho_{A,S} \), above all the geometry of the underlying acceptance sets. By doing so it naturally leads to a version of the dual representation where the relative impact of \( A \) and \( S \) is made immediately visible. In the conic case, the above representation should be compared with the dual representation obtained by Jaschke, Küchler [63]. For the general convex case, we refer to Staum [93] and to Frittelli, Scandolo [52].

A natural and important question to ask is when the supremum in the above representation formula is attained. The next result provides a sufficient condition in this respect.

**Proposition 3.5.12.** Assume \( A \) is convex and \( \rho_{A,S} \) is finite and continuous at \( X \in \mathcal{X} \). Then, \( \mathcal{B}(A) \cap \mathcal{E}(S) \neq \emptyset \) and

\[
\rho_{A,S}(X) = \max_{\psi \in \mathcal{B}(A) \cap \mathcal{E}(S)} \left\{ \sigma_A(\psi) - \psi(X) \right\}.
\]

Moreover, if \( A \) is also a cone, we have

\[
\rho_{A,S}(X) = \max_{\psi \in \mathcal{B}(A) \cap \mathcal{E}(S)} \psi(-X).
\]
Proof. Let $X \in \mathcal{X}$ be a point of finiteness and continuity for $\rho_{\mathcal{A},S}$. Then, Proposition 3.3.5 implies that $\mathcal{A}$ has nonempty interior and $X + \lambda S_T \in \text{int} \mathcal{A}$ whenever $\rho_{\mathcal{A},S}(X) < \lambda S_0$. Fix $\lambda \in \mathbb{R}$ satisfying this inequality. Since $X + \frac{\rho_{\mathcal{A},S}(X)}{S_0} S_T$ is a boundary point for $\mathcal{A}$, it is also a support point in the terminology of Lemma 7.7 in Aliprantis, Border [4], i.e. there exists a nonzero functional $\psi \in \mathcal{X}'$ such that

$$\psi\left(X + \frac{\rho_{\mathcal{A},S}(X)}{S_0} S_T\right) = \sigma_{\mathcal{A}}(\psi).$$

In particular, we see that $\psi \in \mathcal{B}(\mathcal{A})$ and thus $\psi$ must be positive by Proposition 2.3.4. Moreover, note that we must have $\psi(S_T) > 0$, as otherwise

$$\psi(X + \lambda S_T) = \psi\left(X + \frac{\rho_{\mathcal{A},S}(X)}{S_0} S_T\right) = \sigma_{\mathcal{A}}(\psi)$$

which is not possible since $X + \lambda S_T \in \text{int} \mathcal{A}$. Normalizing, we can then assume $\psi(S_T) = S_0$ so that $\psi \in \mathcal{B}(\mathcal{A}) \cap \mathcal{E}(S)$ and

$$\psi(X) + \rho_{\mathcal{A},S}(X) = \sigma_{\mathcal{A}}(\psi),$$

concluding the proof of the proposition. \qed

Remark 3.5.13. The above statement is in line with known results, see e.g. Theorem 7.12 in Aliprantis, Border [4]. Here, we have provided an alternative proof based on the particular structure of our risk measures. \qed

3.6 Extending risk measures

Assume $\rho_{\mathcal{A},S}$ is finitely valued and continuous on a given reference space. In this section we characterize when $\rho_{\mathcal{A},S}$ can be extended to a risk measure of the same type defined on a larger domain, preserving finiteness and continuity.

This extension problem was inspired by Filipović, Svindland [47]. The main result there is that every law-invariant map $f : \ell^\infty \to \mathbb{R} \cup \{\infty\}$ which is convex and lower semicontinuous can be uniquely extended to a map on $\ell^1$ satisfying the same properties. In this sense, $\ell^1$ can be viewed as the canonical space for this type of maps. The results in that paper are presented in the context of a standard probability space but can be extended to a nonatomic setting as shown in Svindland [96].
Remark 3.6.1. A different extension problem for law-invariant risk measures in a cash-additive setting has been recently addressed by Pichler [86].

In this spirit, we investigate which is the canonical model space for risk measures of the form $\rho_{A,S}$ if we want to preserve finiteness and continuity, instead of lower semicontinuity only. Before starting, an important observation is in order. First, we prove the following result on the interplay between cash-additivity and law-invariance.

Proposition 3.6.2. Let $A \subset L^\infty$ be a law-invariant acceptance set. Then, $\rho_A$ is law-invariant.

Proof. Take $X \sim P$ $Y$ and assume $\rho_A(X) < \rho_A(Y)$. By cash-additivity we find a suitable $\lambda \in \mathbb{R}$ such that $\rho_A(X + \lambda) < 0 < \rho_A(Y + \lambda)$. Clearly, we still have $X + \lambda \sim P Y + \lambda$. However, the previous inequality implies that $X + \lambda \in A$ while $Y + \lambda \notin A$, contradicting law-invariance.

In contrast to the cash-additive setting, a risk measure of the form $\rho_{A,S}$ may fail to be law-invariant even if $A$ is law-invariant, as shown by the next example.

Example 3.6.3. Assume $(\Omega, \mathcal{F}, P)$ is nonatomic and take $\mathcal{X} = L^\infty$. Moreover, set $A = A_{\infty}^{\alpha}$ and assume $S_T \in L^\infty$. We claim that $\rho_{A,S}$ is never law-invariant unless $S_T$ is deterministic, in which case $\rho_{A,S}$ is just a multiple of $E_S \alpha$.

To prove this, assume $S_T$ is not deterministic so that there exist $\xi_2 > \xi_1 > 0$ for which $P(S_T \leq \xi_1) > 0$ and $P(S_T \geq \xi_2) > 0$. Since $(\Omega, \mathcal{F}, P)$ is nonatomic, we can find measurable sets $A \subseteq \{S_T \leq \xi_1\}$ and $B \subseteq \{S_T \geq \xi_2\}$ satisfying $P(A) = P(B) = p$ with $p \in (0, 1 - \alpha)$. Set now $C = (A \cup B)^c$ and note that $P(C) = 1 - 2p$. Moreover, for $\gamma \in (-\xi_2, -\xi_1)$ we define

$$X = \gamma 1_A - S_T 1_C \quad \text{and} \quad Y = \gamma 1_B - S_T 1_C.$$

Then $X \sim P Y$ but

$$\rho_{A,S}(X) > S_0 \geq \rho_{A,S}(Y),$$

implying that $\rho_{A,S}$ is not law-invariant. Indeed, note first that $Y + S_T \succeq P 0$ so that $\rho_{A,S}(Y) \leq S_0$. Since

$$P(X + S_T + \lambda < 0) \geq P(A) + P(C) = 1 - p > \alpha$$
for all $\lambda < 0$, we must have $\text{ES}_\alpha(X + S_T) \geq 0$. Moreover, taking $\lambda \in [0, -\gamma - \xi_1)$ we obtain
\[ P(X + S_T + \lambda < 0) = P(A) = p \]
and therefore $\text{VaR}_\beta(X + S_T) \geq -\gamma - \xi_1 > 0$ whenever $\beta \in (0, p)$. By and large, we must have $\text{ES}_\alpha(X + S_T) > 0$. Since $\mathcal{A}$ is closed, we conclude that $\rho_{\mathcal{A}, S}(X) > S_0$ holds, proving (3.2).

However, $\rho_{\mathcal{A}, S}$ is sometimes law-invariant even if $S_T$ is not deterministic.

**Example 3.6.4.** Take $\mathcal{X} = L^\infty$ and consider the law-invariant acceptance set
\[ \mathcal{A} = \{ X \in L^\infty ; \mathbb{E}[X] \geq \alpha \} \]
for a given $\alpha \in \mathbb{R}$. Moreover, assume $S_T \in L^\infty_+$. Since
\[ \rho_{\mathcal{A}, S}(X) = \frac{S_0}{\mathbb{E}[S_T]} (\alpha - \mathbb{E}[X]) \]
for any $X \in L^\infty$, the risk measure $\rho_{\mathcal{A}, S}$ is law-invariant regardless of the choice of the payoff $S_T$.

### 3.6.1 The general extension result

Throughout this section we assume $\mathcal{M}$ is a dense linear subspace of $\mathcal{X}$ endowed with the relative topology induced by $\mathcal{X}$. This is the reference topological structure we fix on $\mathcal{M}$. The positive cone of $\mathcal{M}$ is denoted by $\mathcal{M}_+$. Since every functional on $\mathcal{X}$ can be restricted to a functional on $\mathcal{M}$, we may also consider the topology $\sigma(\mathcal{M}, \mathcal{X}')$ on $\mathcal{M}$ where, by abuse of notation, we do not distinguish between functionals on $\mathcal{X}$ and their restrictions to $\mathcal{M}$. In the next section we will take $\mathcal{X} = L^p$, for some $p \in [1, \infty)$, and $\mathcal{M} = L^\infty$.

The following result shows when we can ensure that a risk measure $\rho_{\mathcal{A}, S}$ on $\mathcal{M}$ can be extended to the whole of $\mathcal{X}$ preserving finiteness and continuity. To avoid confusion, we denote by $\text{cl}_X(\mathcal{A})$ the closure of a subset $\mathcal{A} \subset \mathcal{X}$ with respect to the topology on $\mathcal{X}$.

**Theorem 3.6.5.** Let $\mathcal{A} \subset \mathcal{M}$ be a convex, $\sigma(\mathcal{M}, \mathcal{X}')$-closed acceptance set and let $S_T \in \mathcal{M}_+$. Assume $\rho_{\mathcal{A}, S}$ is finitely valued and continuous on $\mathcal{M}$. The following statements are equivalent:
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(i) \( \rho_{A,S} \) can be extended to a finitely-valued, continuous, risk measure of the same type defined on \( \mathcal{X} \);

(ii) \( \rho_{A,S} \) is continuous at 0 with respect to the relative topology on \( \mathcal{M} \);

(iii) \( A \) has nonempty interior with respect to the relative topology on \( \mathcal{M} \);

(iv) \( \text{cl}_X(A) \) has nonempty interior.

In this case, the extension is unique and is given by \( \rho_{\text{cl}_X(A),S} \).

Proof. Assume (i) holds so that \( \rho_{A,S} \) admits an extension to a finitely-valued, continuous risk measure on \( \mathcal{X} \). Then, clearly, \( \rho_{A,S} \) must also be continuous with respect to the relative topology on \( \mathcal{M} \). It follows that (ii) holds.

Now assume \( \rho_{A,S} \) is continuous at 0 with respect to the relative topology on \( \mathcal{M} \). Since \( \rho_{A,S} \) is finite at \( 0 \in \mathcal{M} \), the set \( A \) must have nonempty interior with respect to the relative topology on \( \mathcal{M} \) by Corollary 3.3.11. Hence, (ii) implies (iii).

Next, assume that (iii) holds so that \( A \) has nonempty interior with respect to the relative topology on \( \mathcal{M} \). As a result, \( \mathcal{U} \cap \mathcal{M} \subseteq A \) for a suitable open subset \( \mathcal{U} \subseteq \mathcal{X} \). Since \( \mathcal{M} \) is dense in \( \mathcal{X} \), it follows that

\[
\mathcal{U} \subset \text{cl}_X(\mathcal{U}) = \text{cl}_X(\mathcal{U} \cap \mathcal{M}) \subseteq \text{cl}_X(A),
\]

proving that \( \text{cl}_X(A) \) has nonempty interior and, hence, that (iv) holds.

Finally, assume that \( \text{cl}_X(A) \) has nonempty interior. Since \( A \) is \( \sigma(\mathcal{M},\mathcal{X}') \)-closed, the risk measure \( \rho_{A,S} \) is lower semicontinuous with respect to \( \sigma(\mathcal{M},\mathcal{X}') \) by Corollary 3.3.8. Moreover, since \( A \) is convex and \( \rho_{A,S} \) is nondegenerate, we can apply Proposition 3.5.3 to find that \( \mathcal{B}(A) \cap \mathcal{E}(S) \) is nonempty. Here, \( \mathcal{B}(A) \) and \( \mathcal{E}(S) \) are both understood as subsets of \( \mathcal{X}' \). As a result, Lemma 3.5.7 implies that

\[
\rho_{A,S}(X) = \sup_{\psi \in \mathcal{B}(A) \cap \mathcal{E}(S)} \{ \sigma_A(\psi) - \psi(X) \}
\]

holds for any \( X \in \mathcal{M} \). With a similar argument we also obtain

\[
\rho_{\text{cl}_X(A),S}(X) = \sup_{\psi \in \mathcal{B}(A) \cap \mathcal{E}(S)} \{ \sigma_A(\psi) - \psi(X) \}
\]

for all \( X \in \mathcal{X} \). This shows that \( \rho_{\text{cl}_X(A),S} \) extends \( \rho_{A,S} \) to the whole of \( \mathcal{X} \). We claim that \( \rho_{\text{cl}_X(A),S} \) is finitely valued and continuous. To prove this, note that, by
Proposition 3.5.3, the risk measure $\rho_{\text{cl}_X(A),S}$ cannot take the value $-\infty$, otherwise it would be degenerate on $\mathcal{M}$. Then, Proposition 3.3.24 immediately implies the claim. It follows that (iv) implies (i).

We conclude the proof of the theorem by observing that $\rho_{A,S}$ can have a unique continuous extension because $\mathcal{M}$ is dense in $\mathcal{X}$. \qed

### 3.6.2 Law-invariance and extensions

We now apply the previous extension theorem to risk measures of the form $\rho_{A,S}$ in the context of $L^p$ spaces under the assumption that the reference probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic. Recall that every finitely-valued risk measure of the form $\rho_{A,S}$ defined on $L^p$ is automatically continuous whenever $A$ is convex by virtue of Corollary 3.3.40. Moreover, recall that, for any subset $A \subset L^\infty$, we denote by $\text{cl}_p(A)$ the closure of $A$ in $L^p$.

**Theorem 3.6.6.** Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic. Let $A \subset L^\infty$ be a convex, law-invariant acceptance set and take $S_T \in L^\infty_+$. Moreover, assume $\rho_{A,S}$ is finitely valued and, hence, continuous on $L^\infty$. For every $p \in [1, \infty)$, the following statements are equivalent:

(a) $\rho_{A,S}$ can be extended to a finitely-valued and, hence, continuous risk measure of the same type defined on $L^p$;

(b) if $(X_n) \subset L^\infty$ and $X_n \to 0$ in $L^p$, then $\rho_{A,S}(X_n) \to \rho_{A,S}(0)$;

(c) $\text{cl}_\infty(A)$ has nonempty interior with respect to the relative topology on $L^\infty$ induced by $L^p$;

(d) $\text{cl}_p(A)$ has nonempty interior in $L^p$.

In this case, the extension is unique and is given by $\rho_{\text{cl}_p(A),S}$.

**Proof.** Since $\rho_{A,S}$ is continuous on $L^\infty$, we have $\rho_{A,S} = \rho_{\text{cl}_\infty(A),S}$ by Proposition 3.3.5. Note that $\text{cl}_\infty(A)$ is still convex and law-invariant. In particular, the law-invariance of $\rho_A$ proved in Proposition 3.6.2 implies that $\text{cl}_\infty(A)$ is law-invariant because $\text{cl}_\infty(A)$ consists of all $X \in L^\infty$ such that $\rho_A(X) \leq 0$. As a consequence, $\text{cl}_\infty(A)$ is also $\sigma(L^\infty, L^p)$-closed by virtue of Lemma 2.3.18. The claim is now a direct corollary of Theorem 3.6.5. \qed
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3.6.3 The index of continuity

As a consequence of Theorem 3.6.6, it is natural to define the index of continuity of a risk measure \( \rho_{A,S} \) defined on \( L^\infty \) as follows. We keep our assumption that the underlying probability space is nonatomic.

**Definition 3.6.7.** Let \( A \subset L^\infty \) be a convex, law-invariant acceptance set and assume \( S_T \in L^\infty_+ \). If \( \rho_{A,S} \) is finitely valued on \( L^\infty \), the *index of continuity* of \( \rho_{A,S} \) is defined as

\[
ic(\rho_{A,S}) := \inf\{p \in [1, \infty) ; \cl_p(A) \text{ has nonempty interior in } L^p\}.
\]

If the infimum above is attained and \( p = \nic(\rho_{A,S}) \), then \( L^p \) is the largest space supporting a finitely-valued and, hence, continuous extension of \( \rho_{A,S} \). Therefore, if we are interested in preserving finiteness and continuity properties of a risk measure, the space \( L^p \) is to be considered the canonical model space for \( \rho_{A,S} \). In this respect, we will show that for any \( p \in [1, \infty] \) we can find a risk measure of the form \( \rho_{A,S} \) whose index of continuity is exactly \( p \). In particular, there are risk measures of this type which cannot be extended beyond \( L^\infty \) preserving finiteness and continuity.

Note that the existence of a finitely-valued, continuous extension of a risk measure \( \rho_{A,S} \) satisfying the assumptions of Theorem 3.6.6 does not depend on the properties of the payoff \( S_T \), but only on the topological properties of the acceptance set \( A \). This observation has the following important consequence.

**Proposition 3.6.8.** Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is nonatomic. Let \( A \subset L^\infty \) be a convex, law-invariant acceptance set and assume \( S_T \in L^\infty_+ \). If \( \rho_{A,S} \) is finite on \( L^\infty \), then \( \nic(\rho_{A,S}) = \nic(\rho_A) \).

The preceding result shows that, whenever we are able to assess that \( \rho_{A,S} \) is finitely valued on \( L^\infty \), the index of continuity of \( \rho_{A,S} \) can be determined by looking at the cash-additive risk measure \( \rho_A \). Hence, in this case, the presence of a general payoff \( S_T \) has no impact and we could assume that \( S \) is a risk-free asset. However, the finiteness of \( \rho_{A,S} \) on \( L^\infty \) does depend on the interplay between \( A \) and \( S \), as illustrated by our results in Section 3.7.
### 3.6.4 The index of qualitative robustness

In this section we recall the notion of qualitative robustness introduced in the recent paper by Krätschmer, Schied, Zähle [73] and we discuss the link with our index of continuity.

Consider a law-invariant acceptance set $A \subset L^\infty$ and its associated cash-additive risk measure $\rho_A$ which is, then, also law-invariant. If we denote by $\mathbb{P}_X$ the law of $X$, i.e. $\mathbb{P}_X(A) := \mathbb{P}(X \in A)$ for all Borel sets $A \subset \mathbb{R}$, and set

$$
\mathcal{M}_\infty := \{\mathbb{P}_X; X \in L^\infty\},
$$

we can define a functional $\mathcal{R}_A : \mathcal{M}_\infty \to \mathbb{R}$ by

$$
\mathcal{R}_A(\mathbb{P}_X) := \rho_A(X).
$$

The capital position $X$ of a financial institution is often estimated through a sequence of historical observations $x_1, \ldots, x_N \in \mathbb{R}$, and the quantity $\mathcal{R}_A(m)$, where $m$ denotes the empirical distribution of these observations, is used as a natural proxy for $\rho_A(X)$.

The importance of the robustness properties of the operator $\mathcal{R}_A$ was discussed in detail by Cont, Deguest, Scandolo [28]. Based on that paper, a refined notion of qualitative robustness has been recently proposed in Krätschmer, Schied, Zähle [73] and further studied by the same authors in [74].

Let $\mathcal{M}$ denote the set of (Borel) probability measures over $\mathbb{R}$. To any $\mu \in \mathcal{M}$ which is not a Dirac measure, we can associate a nonatomic probability space $(\Omega^\mu, \mathcal{F}^\mu, \mathbb{P}^\mu)$ supporting a sequence $(X_n)$ of i.i.d. random variables having $\mu$ as their common law; see e.g. Section 11.4 in Dudley [35]. For $n \in \mathbb{N}$, the empirical distribution of $X_1, \ldots, X_n$ is the map $m_n^\mu : \Omega^\mu \to \mathcal{M}_\infty$ defined by

$$
m_n^\mu(\omega) := \frac{1}{n} \sum_{i=1}^n \delta_{X_i(\omega)},
$$

where $\delta_{X_i(\omega)}$ denotes the standard Dirac measure associated to the singleton $\{X_i(\omega)\}$. Moreover, we can consider the random variable $\mathcal{R}_A(m_n^\mu)$ given by

$$
\mathcal{R}_A(m_n^\mu)(\omega) := \mathcal{R}_A(m_n^\mu(\omega)).
$$
for $\omega \in \Omega^\mu$. The following notion of qualitative robustness is a generalization of the classical notion introduced by Hampel [58]. For $p \in [1, \infty)$ define $\psi_p(x) = \frac{1}{p} |x|^p$, $x \in \mathbb{R}$, and recall from [74] that a set $N \subset M$ is said to be uniformly $p$-integrating if

$$
\lim_{c \to \infty} \sup_{\mu \in N} \int_{\{\psi_p \geq c\}} \psi_p(x) d\mu(x) = 0.
$$

**Definition 3.6.9.** Fix $p \in [1, \infty)$. The functional $R_A$ is said to be $p$-robust on $M_\infty$ if for any uniformly $p$-integrating set $N \subset M_\infty$, $\mu \in N$ and $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$
d_P(\mu, \nu) + \left| \int_{\mathbb{R}} \psi_p(x) d\mu(x) - \int_{\mathbb{R}} \psi_p(x) d\nu(x) \right| \leq \delta \tag{3.3}
$$

implies

$$
d_P(\mathbb{P}_{R_A(m_n^\mu)}, \mathbb{P}_{R_A(m_n^\nu)}) \leq \varepsilon
$$

for $\nu \in N$ and $n \geq n_0$, where $d_P$ denotes the usual Prohorov metric over $M$.

If $R_A$ is $p$-robust on $M_\infty$, then a suitable small change in the law of the data entails an arbitrarily small change in the law of the corresponding estimators.

**Remark 3.6.10.** As discussed in [73] and [74], the choice to add an additional term to the Prohorov metric in (3.3), as opposed to the classical framework developed by Hampel [58], has the main advantage of making $R_A(\mu)$ sensitive to the tail behavior of $\mu$. Indeed, under the Prohorov metric, or equivalently under any metric inducing the weak topology on $M$, like the Lévy metric, two distributions $\mu$ and $\nu$ may possess a different tail behavior but be rather close in metric terms.

In this case, qualitative robustness would essentially prevent $R_A$ from discriminating across different tail profiles. For more details about statistical robustness and topological properties related to measures we refer to Dudley [35] or Huber, Ronchetti [61].

Based on Krätschmer, Schied, Zähle [73], the same authors introduced in [74] the so-called index of qualitative robustness for convex, law-invariant, cash-additive risk measures.

**Definition 3.6.11.** Let $A \subset L^\infty$ be a convex, law-invariant acceptance set. The index of qualitative robustness of $\rho_A$ is defined as

$$
iqr(\rho_A) := (\inf\{p \in [1, \infty) : R_A \text{ is } p\text{-robust on } M_\infty\})^{-1}.
$$
By combining Theorem 2.16 in [74] and our Theorem 3.6.6 we obtain the following result relating the qualitative robustness of the operator $R_A$ to the topological properties of the acceptance set $A$.

**Theorem 3.6.12.** Assume $A \subset L^\infty$ is a convex, law-invariant acceptance set, and fix $p \in [1, \infty)$. The following statements are equivalent:

(i) $R_A$ is $p$-robust on $\mathcal{M}_\infty$;

(ii) $\text{cl}_p(A)$ has nonempty interior in $L^p$.

Moreover, we have

$$\text{iqr}(\rho_A) = \frac{1}{\text{ic}(\rho_A)}.$$

**Remark 3.6.13.** In light of the previous result, the index of continuity turns out to be important also from the perspective of qualitative robustness. In the following section, we will characterize the index of continuity of risk measures based on Expected Shortfall, Test Scenarios and Expected Utility. We refer to Koch-Medina, Munari [69] for corresponding results in the context of max-correlation risk measures and risk measures based on distortion functions.

### 3.7 Illustration of the main results

In this section we apply our results to risk measures associated with a variety of explicit acceptance sets. The main focus will be on finiteness and continuity results, including the analysis of the index of continuity.

#### 3.7.1 Risk measures based on Value-at-Risk

Throughout the entire section we set $\mathcal{X} = L^p$ for some $p \in [0, \infty]$. We start by discussing finiteness.

**Proposition 3.7.1.** Assume $p = \infty$. For any $\alpha \in (0, 1)$ the following assertions hold:

(a) The following statements are equivalent:

(i) $\rho_{\mathcal{A}_{\text{VaR}}(\alpha), S}(X) < \infty$ for every $X \in L^\infty$;
(ii) \( \text{VaR}_\alpha(S_T) < 0; \)
(iii) \( \mathbb{P}(S_T < \lambda) \leq \alpha \) for some \( \lambda > 0. \)

(b) The following statements are equivalent:

(i) \( \rho_{A^\infty_{\text{Var}}(\alpha), S}(X) > -\infty \) for every \( X \in L^\infty; \)
(ii) \( \text{VaR}_\alpha(-S_T) > 0; \)
(iii) \( \mathbb{P}(S_T = 0) < 1 - \alpha. \)

Proof. (a) Note that \( \text{VaR}_\alpha(S_T) < 0 \) is equivalent to \( S_T \in \text{int } A^\infty_{\text{Var}}(\alpha) \) by Proposition 2.4.7. Since the interior and the core of \( A^\infty_{\text{Var}}(\alpha) \) coincide due to Proposition 2.2.8, the equivalence of (i) and (ii) follows from Theorem 3.3.25. The equivalence of (ii) and (iii) is a direct consequence of the definition of \( \text{VaR}. \)

(b) Condition (i) is equivalent to \( -S_T \) being a core point of the complement of \( A^\infty_{\text{Var}}(\alpha) \) by Theorem 3.3.25. Note that \( -(A^\infty_{\text{Var}}(\alpha))^c \) is monotone, hence the core and the interior of the complement of \( A^\infty_{\text{Var}}(\alpha) \) coincide by virtue of Proposition 2.2.8. Since the complement of \( A^\infty_{\text{Var}}(\alpha) \) is open as a consequence of Proposition 2.4.5, we conclude that (i) and (ii) are equivalent. Finally, the equivalence of (ii) and (iii) follows from the definition of \( \text{VaR}. \)

Proposition 3.7.2. Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is nonatomic, and let \( p \in [0, \infty). \) For any \( \alpha \in (0, 1) \) the following statements are equivalent:

(i) \( \rho_{A^p_{\text{Var}}(\alpha), S} \) is finitely valued;
(ii) \( \mathbb{P}(S_T = 0) < \min\{\alpha, 1 - \alpha\}. \)

Proof. By Proposition 2.4.7 and Theorem 3.3.25, we only need to show that \( \rho_{A^p_{\text{Var}}(\alpha), S} \) never attains the value \( -\infty \) if and only if \( \mathbb{P}(S_T = 0) < 1 - \alpha. \) First, assume \( \mathbb{P}(S_T = 0) \geq 1 - \alpha. \) Then clearly \( \rho_{A^p_{\text{Var}}(\alpha), S}(0) = -\infty. \) On the other side, assume \( \mathbb{P}(S_T = 0) < 1 - \alpha \) and fix \( X \in L^p. \) Note that

\[
\mathbb{P}(\{X < nS_T\} \cap \{S_T > 0\}) \rightarrow \mathbb{P}(S_T > 0)
\]
as \( n \to \infty, \) implying that \( \mathbb{P}(X < nS_T) > \alpha \) for \( n \in \mathbb{N} \) large enough. Hence it follows that \( \rho_{A^p_{\text{Var}}(\alpha), S}(X) > -\infty. \) \[ \square \]
We now turn to continuity results for risk measures based on VaR-acceptability. Recall that, in light of Corollary 3.3.28 above, finiteness is always a necessary condition for continuity. First, we remark that such risk measures are always lower semicontinuous. This follows directly from Proposition 2.4.5.

**Proposition 3.7.3.** Assume $p \in [0, \infty]$. For any $\alpha \in (0, 1)$ the risk measure $\rho_{A^{p}_{\text{Var}}(\alpha), S}$ is lower semicontinuous on $L^p$.

We start by discussing the case of bounded positions on a finite probability space. The following lemma is the key result in this setting.

**Lemma 3.7.4.** Assume $\Omega$ is finite and $\alpha \in (0, 1)$. Moreover, assume $\rho_{A^{\infty}_{\text{Var}}(\alpha), S}$ is finitely valued. For $X \in L^{\infty}$, the following statements are equivalent:

(i) $\rho_{A^{\infty}_{\text{Var}}(\alpha), S}$ is continuous at $X$;

(ii) $\mathbb{P}(\tilde{X} < 0) + \mathbb{P}(X = 0, S_T = 0) \leq \alpha$, where

\[
\tilde{X} = X + \frac{\rho_{A^{\infty}_{\text{Var}}(\alpha), S}(X)}{S_0} S_T.
\]

In particular, the preceding conditions hold whenever $\mathbb{P}(X = 0, S_T = 0) = 0$.

**Proof.** Since $\mathbb{P}(\tilde{X} < 0) \leq \alpha$, the last assertion follows immediately from (ii). Hence, we focus on the equivalence above. Throughout the whole proof, set

\[
A = \{\tilde{X} < 0\} \cup (\{X = 0\} \cap \{S_T = 0\}).
\]

First, assume (i) holds but (ii) is not satisfied. Set $B = \{\tilde{X} < 0\} \cap \{S_T > 0\}$ and define

\[
m = \begin{cases} 
\min \left\{ -\frac{S_0 \tilde{x}(\omega)}{S_T(\omega)} ; \ \omega \in B \right\} & \text{if } B \neq \emptyset, \\
1 & \text{otherwise.}
\end{cases}
\]

Note that $m > 0$. As a result, $B \subset \{\tilde{X} + \frac{m}{S_0} S_T < \frac{1}{n}\}$ for every $n \in \mathbb{N}$. Since (ii) does not hold, we must have

\[
\mathbb{P}\left(\tilde{X} + \frac{m}{S_0} S_T < \frac{1}{n}\right) > \alpha,
\]
implying that

\[
\rho_{A^{\infty}_{\text{Var}}(\alpha), S}\left(X - \frac{1}{n}\right) \geq \rho_{A^{\infty}_{\text{Var}}(\alpha), S}(X) + m.
\]
for all \( n \in \mathbb{N} \). However, this means that \( \rho_{A^\infty_{\text{var}}(\alpha),S} \) is not continuous at \( X \), contradicting the initial assumption. Hence, (i) implies (ii).

To prove the converse implication, assume (ii) holds but \( \rho_{A^\infty_{\text{var}}(\alpha),S} \) is not continuous at \( X \). By Proposition 3.7.3, this implies that \( \rho_{A^\infty_{\text{var}}(\alpha),S} \) is not upper semicontinuous at \( X \) or, equivalently by \( S \)-additivity, at \( \tilde{X} \). It follows from Proposition 2.4.6 and Proposition 3.3.5 that we can find \( m > 0 \) such that \( \text{VaR}_\alpha(\tilde{X} + \frac{m}{S_0}S_T) \geq 0 \), hence

\[
\mathbb{P}(\tilde{X} + \frac{m}{S_0}S_T < \lambda) > \alpha
\]

for all \( \lambda > 0 \). Now, take

\[
\xi = \min \left\{ \tilde{X}(\omega) + \frac{m}{S_0}S_T(\omega) ; \ \omega \in A^c \right\} > 0.
\]

Since \( \{ \tilde{X} + \frac{m}{S_0}S_T < \xi \} \subset A \), we obtain that (ii) fails to hold, contradicting the initial assumption. In conclusion, (ii) implies (i).

\[\square\]

**Proposition 3.7.5.** Assume \( \Omega \) is finite and let \( \alpha \in (0,1) \). The following conditions are equivalent:

(i) \( \rho_{A^\infty_{\text{var}}(\alpha),S} \) is continuous on \( L^\infty \);

(ii) \( \mathbb{P}(A) \leq \alpha - \mathbb{P}(S_T = 0) \) for every \( A \in \mathcal{F} \) with \( A \subseteq \{ S_T > 0 \} \) and \( \mathbb{P}(A) \leq \alpha \).

**Proof.** For any \( X \in L^\infty \) define \( \tilde{X} \) as in the preceding lemma and recall that \( \mathbb{P}(\tilde{X} < 0) \leq \alpha \). To prove that (i) implies (ii), take a measurable set \( A \subseteq \{ S_T > 0 \} \) with \( \mathbb{P}(A) \leq \alpha \), and set \( X = -1_A \). Note that \( \mathbb{P}(X < 0) = \mathbb{P}[A] \leq \alpha \), implying \( \rho_{A^\infty_{\text{var}}(\alpha),S}(X) \leq 0 \). Moreover, since \( X \leq 0 \), we actually have \( \rho_{A^\infty_{\text{var}}(\alpha),S}(X) = 0 \), so that \( \tilde{X} = X \). Since \( \rho_{A^\infty_{\text{var}}(\alpha),S} \) is continuous at \( X \) by assumption, Lemma 3.7.4 implies that

\[
\mathbb{P}(A) + \mathbb{P}(S_T = 0) = \mathbb{P}(\tilde{X} < 0) + \mathbb{P}(X = 0, S_T = 0) \leq \alpha,
\]

showing that (ii) holds.

Conversely, to prove that (ii) implies (i), take \( X \in L^\infty \) and define

\[
A = \{ \tilde{X} < 0 \} \cap \{ S_T > 0 \}.
\]

Note that

\[
\{ \tilde{X} < 0 \} \cup (\{ X = 0 \} \cap \{ S_T = 0 \}) \subset A \cup \{ S_T = 0 \}. \quad (3.4)
\]
Since $P(A) \leq P(\tilde{X} < 0) \leq \alpha$, we obtain by assumption that $P(A) + P(S_T = 0) \leq \alpha$. Hence it follows from (3.4) that

$$P(\tilde{X} < 0) + P(X = 0, S_T = 0) \leq \alpha$$

and therefore $\rho_{A_{\text{Var}}(\alpha), S}$ is seen to be continuous at $X$ by Lemma 3.7.4, concluding the proof.

The situation is quite different in a nonatomic setting: continuity is only possible if the payoff of the reference asset is bounded away from zero.

**Proposition 3.7.6.** Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic and let $\alpha \in (0, 1)$. Moreover, assume $\rho_{A_{\text{Var}}(\alpha), S}$ is finitely valued. Then, the following statements are equivalent:

(i) $\rho_{A_{\text{Var}}(\alpha), S}$ is continuous on $L^\infty$;

(ii) $P(S_T \geq \varepsilon) = 1$ for some $\varepsilon > 0$.

**Proof.** By Theorem 3.3.13 we only need to prove that (i) implies (ii). Assume to the contrary that $P(S_T < \lambda) > 0$ for every $\lambda > 0$. Note that, by Proposition 3.7.1, we have $P(S_T = 0) < 1 - \alpha$. Hence, there exists $\xi > 0$ such that

$$0 < P(S_T < \xi) < 1 - \alpha.$$ 

Since $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic, we can find a measurable set $A \subset \{S_T \geq \xi\}$ with $P(A) = \alpha$. Now, define

$$X = -\|S_T\|_{\infty}1_A.$$ 

Taking $\lambda \in (0, \xi]$ we see that

$$\{X + S_T < \lambda\} \cap A = A, \quad \{X + S_T < \lambda\} \cap A^c = \{S_T < \lambda\},$$

so that

$$P(X + S_T < \lambda) = P(A) + P(S_T < \lambda) > \alpha.$$ 

As a result, we have $\text{VaR}_\alpha(X + S_T) \geq 0$, hence $X + S_T \not\in \text{int} A_{\text{Var}}(\alpha)$ by Proposition 2.4.6. Since $X \in A_{\text{Var}}(\alpha)$ and thus $\rho_{A_{\text{Var}}(\alpha), S}(X) \leq 0$, Theorem 3.3.5 implies that $\rho_{A_{\text{Var}}(\alpha), S}$ is not (upper semi)continuous at $X$. It follows that (i) must imply (ii), concluding the proof. \qed
3.7 Illustration of the main results

We now turn to spaces of unbounded positions. In this case risk measures based on VaR are never continuous on the whole space.

**Proposition 3.7.7.** Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is nonatomic and \(p \in [0, \infty)\). Then, for any \(\alpha \in (0, 1)\) the risk measure \(\rho_{\mathcal{A}_\text{VaR}(\alpha), S}\) is not continuous on \(L^p\).

**Proof.** Take \(\varepsilon > 0\) and \(A \in \mathcal{F}\) with \(\mathbb{P}(A) = \alpha\), and set 
\[ X = -(S_T + \varepsilon)1_A. \]
Note that \(\mathbb{P}(X < 0) = \alpha\) and \(\mathbb{P}(X + S_T \leq 0) \geq \alpha\). As a result, we obtain 
\[ \rho_{\mathcal{A}_\text{VaR}(\alpha), S}(X) \leq 0 < S_0 \leq \rho_{\text{int} \mathcal{A}_\text{VaR}(\alpha), S}(X), \]
where the last inequality is a consequence of Proposition 2.4.7. In conclusion, Proposition 3.3.5 shows that \(\rho_{\mathcal{A}_\text{VaR}(\alpha), S}\) is not (upper semi)continuous at \(X\). \(\square\)

In Proposition 3.3.5 we provided equivalent conditions for a risk measure of the form \(\rho_{\mathcal{A}, S}\) to be (semi)continuous. Here, we sharpen that result in case of VaR itself. The lack of continuity of \(\text{VaR}_\alpha\) at a point \(X \in L^p\) depends on the flatness of the cumulative distribution function \(F_X\) at the level \(\alpha\). In particular, \(\text{VaR}_\alpha\) is continuous at \(X\) except for at most countably many levels.

**Proposition 3.7.8.** Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is nonatomic and \(p \in [0, \infty)\). Then, for any \(\alpha \in (0, 1)\) and \(X \in L^p\) the following statements are equivalent:

(i) \(\text{VaR}_\alpha\) is continuous at \(X\);

(ii) \(q^-_\alpha(X) = q^+_\alpha(X)\).

**Proof.** Recall that \(q^+_\alpha(X) = -\text{VaR}_\alpha(X)\) for all \(X \in L^p\), so that \(q^+_\alpha\) is upper semicontinuous on \(L^p\) by Proposition 3.7.3. Since \(q^-_\alpha(X) = -q^+_1(-X)\) for every \(X \in L^p\), it follows that \(q^-_\alpha\) is lower semicontinuous on \(L^p\). Now, consider the acceptance set 
\[ \mathcal{A} = \{X \in L^p ; \mathbb{P}(X < 0) < \alpha\} \]
and note that, by definition, \(\rho_{\mathcal{A}} = -q^-_\alpha\). In particular, \(\rho_{\mathcal{A}}\) is upper semicontinuous on the whole \(L^p\). Moreover, Proposition 2.4.7 implies that 
\[ \text{int} \mathcal{A}_\text{VaR}(\alpha) \subseteq \mathcal{A} \subseteq \mathcal{A}_\text{VaR}(\alpha), \]
hence \( \text{int} \mathcal{A}_{\text{VaR}}^p(\alpha) = \text{int} \mathcal{A} \). As a result, it follows from Proposition 3.3.5 that

\[
\rho_{\text{int}} \mathcal{A}_{\text{VaR}}^p(\alpha) = \rho_{\text{int}} \mathcal{A} = \rho \mathcal{A} = -q_\alpha^-.
\]

In addition, we have

\[
\rho_{\text{cl}} \mathcal{A}_{\text{VaR}}^p(\alpha) = \rho \mathcal{A}_{\text{VaR}}^p(\alpha) = \text{VaR}_\alpha = -q_\alpha^+,
\]

where the first equality is a consequence of \( \mathcal{A}_{\text{VaR}}^p(\alpha) \) being closed in \( L^p \) due to Proposition 2.4.5. The statement now follows from Proposition 3.3.5.

**Corollary 3.7.9.** Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is nonatomic and \( p \in [0, \infty) \). Then, \( \text{VaR}_\alpha \) is continuous at \( X \in L^p \) for every \( \alpha \in (0, 1) \) if and only if \( F_X \) is strictly increasing on the interval \( \{x \in \mathbb{R}; 0 < F_X(x) < 1\} \).

**Remark 3.7.10.** Under the assumptions of Proposition 3.7.7 we see that \( \text{VaR}_\alpha \) cannot be continuous on the whole space \( L^p \) unless \( p = \infty \). In particular, \( \text{VaR}_\alpha \) cannot be continuous with respect to convergence in distribution, since otherwise it would be continuous on \( L^0 \). This shows that part (b) of Proposition 7.4 in Rüschendorf [87] is not correct.

### 3.7.2 Risk measures based on Expected Shortfall

In this section we set \( \mathcal{X} = L^p \) for some \( p \in [1, \infty] \). The following result on lower semicontinuity is an immediate consequence of Proposition 2.4.14.

**Proposition 3.7.11.** For any \( \alpha \in (0, 1) \) the risk measure \( \rho_{\mathcal{A}_{\text{ES}}^p(\alpha), \mathcal{S}} \) is lower semicontinuous on \( L^p \).

**Proposition 3.7.12.** For any \( \alpha \in (0, 1) \), the following statements are equivalent:

(i) \( \rho_{\mathcal{A}_{\text{ES}}^p(\alpha), \mathcal{S}} \) is finitely valued on \( L^p \);

(ii) \( \rho_{\mathcal{A}_{\text{ES}}^p(\alpha), \mathcal{S}} \) is continuous on \( L^p \);

(iii) \( \text{ES}_\alpha(S_T) < 0 \);

(iv) \( \mathbb{P}(S_T = 0) < \alpha \).

In this case, \( \rho_{\mathcal{A}_{\text{ES}}^p(\alpha), \mathcal{S}} \) is Lipschitz continuous on \( L^p \).
Proof. Recall that the core and the interior of $\mathcal{A}_{ES}^p(\alpha)$ coincide by Proposition 2.2.8. Hence the first two assertions are equivalent to $S_T$ being an interior point of $\mathcal{A}_{ES}^p(\alpha)$ by Theorem 3.3.30 and Theorem 3.3.31, respectively. The equivalence with the remaining assertions is a direct consequence of Proposition 2.4.15. 

We conclude by highlighting the corresponding index of continuity. The result, which is in line with Example 2.23 in [74], follows directly from Theorem 3.6.6 in light of Proposition 2.4.14 and Proposition 2.4.15.

**Proposition 3.7.13.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be nonatomic and take $\alpha \in (0, 1)$. Moreover, take $S_T \in L^\infty_+$ and assume $\rho_{\mathcal{A}_{ES}^\infty(\alpha), S}$ is finitely valued. Then $\text{ic}(\rho_{\mathcal{A}_{ES}^\infty(\alpha), S}) = 1$ and the index is attained.

### 3.7.3 Risk measures based on Test Scenarios

In this section we set $\mathcal{X} = L^p$ for some $p \in [1, \infty]$. As in the previous section, the following result on lower semicontinuity is a direct consequence of Proposition 2.4.14.

**Proposition 3.7.14.** Assume $A \in \mathcal{F}$ is admissible. Then, $\rho_{\text{SPAN}^p(A), S}$ is lower semicontinuous on $L^p$.

However, differently from the case of Expected Shortfall, the finiteness and continuity properties of risk measures based on SPAN acceptability strongly depend upon the choice of the underlying model space.

**Proposition 3.7.15.** Take $p = \infty$ and assume $A \in \mathcal{F}$ is admissible. The following statements are equivalent:

1. $\rho_{\text{SPAN}^\infty(A), S}$ is finitely valued on $L^\infty$;
2. $\rho_{\text{SPAN}^\infty(A), S}$ is continuous on $L^\infty$;
3. $\mathbb{P}(A \cap \{S_T \geq \varepsilon\}) = 1$ for some $\varepsilon > 0$.

In this case, $\rho_{\text{SPAN}^\infty(A), S}$ is Lipschitz continuous on $L^\infty$.

Proof. The core and the interior of $\text{SPAN}^\infty(A)$ coincide by Proposition 2.2.8. Hence the first two assertions are equivalent to $S_T$ being an interior point of $\text{SPAN}^\infty(A)$ by Theorem 3.3.30 and Theorem 3.3.31, respectively. The assertion follows from Proposition 2.4.21.
The result in the unbounded case follows immediately from Proposition 2.4.22 by mimicking the above proof.

**Proposition 3.7.16.** Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is nonatomic. Take \(p \in [1, \infty)\) and assume \(A \in \mathcal{F}\) is admissible. Then, \(\rho_{\text{SPAN}^p(A),S}\) is neither finitely valued nor continuous on \(L^p\).

We conclude by highlighting that the corresponding index of continuity is infinite.

**Proposition 3.7.17.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be nonatomic and assume \(A \in \mathcal{F}\) is admissible. Moreover, take \(S_T \in L^\infty_+\) and assume \(\rho_{\text{SPAN}^\infty(A),S}\) is finitely valued. Then, we have \(\text{ic}(\rho_{\text{SPAN}^\infty(A),S}) = \infty\).

**Proof.** By reproducing the first part of the proof of Proposition 2.4.23, it is easy to see that \(\text{SPAN}^\infty(A)\) is \(\sigma(L^\infty, L^p)\)-closed for any \(p \in [1, \infty]\). As a result, the assertion follows by applying Theorem 3.6.5 in combination with Proposition 2.4.20 and Proposition 2.4.22. \(\square\)

### 3.7.4 Risk measures based on Expected Utility

In this section we start by setting \(\mathcal{X} = L^p\) for \(p \in [1, \infty]\). The first result is an immediate consequence of Proposition 2.4.26.

**Proposition 3.7.18.** Let \((u, \alpha)\) be admissible. For any \(p \in [1, \infty]\) the risk measure \(\rho_{A^p_u(\alpha),S}\) is lower semicontinuous on \(L^p\).

We turn to establishing finiteness and continuity results for risk measures based on expected utility. First, we focus on the case of bounded positions.

**Proposition 3.7.19.** Assume \(p = \infty\) and let \((u, \alpha)\) be admissible. The following statements hold:

(a) Assume \(u\) is finite and unbounded from above. Then, \(\rho_{A^\infty_u(\alpha),S}\) is finitely valued and continuous on \(L^\infty\).

(b) Assume \(u\) is finite and bounded from above. Moreover, assume \(u(x) > \alpha\) for some \(x \in \mathbb{R}\). Then, the following assertions are equivalent:

(i) \(\rho_{A^\infty_u(\alpha),S}\) is finitely valued and continuous on \(L^\infty\);
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(ii) \( \mathbb{P}(S_T = 0) = 0. \)

(c) Assume that either \( u \) attains the value \(-\infty\) or \( u(x) \leq \alpha \) for all \( x \in \mathbb{R} \). Then, the following assertions are equivalent:

(i) \( \rho_{A^\infty_u}(\alpha), S \) is finitely valued and continuous on \( L^\infty; \)

(ii) \( \mathbb{P}(S_T \geq \varepsilon) = 1 \) for some \( \varepsilon > 0. \)

Proof. First, recall that finiteness implies continuity for \( \rho_{A^\infty_u}(\alpha), S \) by Corollary 3.3.40. Moreover, \( \rho_{A^\infty_u}(\alpha), S \) never takes the value \(-\infty\). To show this, fix \( X \in L^\infty \) and \( \xi > 0 \) such that \( \mathbb{P}(S_T \geq \xi) > 0. \) Then, since \( u \) is unbounded from below, we can always find \( \lambda > 0 \) sufficiently large to yield

\[
\mathbb{E}[u(X - \lambda S_T)] \leq u(||X||_\infty - \lambda \xi) \mathbb{P}(S_T \geq \xi) + u(||X||_\infty) \mathbb{P}(S_T < \xi) < \alpha.
\]

This implies \( X - \lambda S_T \notin A^\infty_u(\alpha) \) and, hence, \( \rho_{A^\infty_u(\alpha), S}(X) > -\infty. \)

(a) Take \( \xi > 0 \) such that \( \mathbb{P}(S_T \geq \xi) > 0. \) Since \( u \) is unbounded from above, for any \( X \in L^\infty \) we can find \( \lambda > 0 \) large enough to get

\[
\mathbb{E}[u(X + \lambda S_T)] \geq u(-||X||_\infty + \lambda \xi) \mathbb{P}(S_T \geq \xi) + u(-||X||_\infty) \mathbb{P}(S_T < \xi) \geq \alpha.
\]

This implies that \( \rho_{A^\infty_u(\alpha), S} \) is finitely valued.

(b) Assume first that (i) holds so that \( \rho_{A^\infty_u(\alpha), S}(-\xi) < \infty \) for any \( \xi > 0. \) As a result, for every \( \xi > 0 \) there exists \( \lambda > 0 \) such that

\[
u(-\xi) \mathbb{P}(S_T = 0) + \sup_{x \in \mathbb{R}} u(x) \mathbb{P}(S_T > 0) \geq \mathbb{E}[u(-\xi + \lambda S_T)] \geq \alpha.
\]

Since \( u \) is unbounded from below, this is only possible if \( \mathbb{P}(S_T = 0) = 0, \) proving (ii).

Conversely, assume (ii) holds and take \( X \in L^\infty. \) Since \( u(x) > \alpha \) for some \( x \in \mathbb{R} \) and \( \mathbb{P}(S_T = 0) = 0, \) we can find \( \varepsilon > 0 \) sufficiently small to ensure that

\[
\mathbb{E}
\left[
u\left(X + \frac{1}{\varepsilon^2} S_T\right)\right] \geq u\left(\frac{1}{\varepsilon} - ||X||_\infty\right) \mathbb{P}(S_T \geq \varepsilon) + u(-||X||_\infty) \mathbb{P}(S_T < \varepsilon) \geq \alpha.
\]

As a result, \( \rho_{A^\infty_u(\alpha), S} \) is never equal to \( \infty \) and (i) follows.

(c) First note that (ii) always implies (i) by Theorem 3.3.13, since \( S_T \) is an interior point of \( L^\infty \) under (ii). Conversely, assume that (i) holds and \( u \) takes the
value $-\infty$. By contradiction, let $\mathbb{P}(S_T < \varepsilon) > 0$ for all $\varepsilon > 0$. Take $\xi > 0$ with $u(-\xi) = -\infty$ and set $X = -\xi - 1$. Then for every $\lambda > 0$ there exists $\varepsilon > 0$ such that $u(-\xi - 1 + \lambda \varepsilon) = -\infty$, implying that

$$
\mathbb{E}[u(X + \lambda S_T)] \leq u(-\xi - 1 + \lambda \varepsilon) \mathbb{P}(S_T < \varepsilon) + u(-\xi - 1 + \lambda \|S_T\|_\infty) \mathbb{P}(S_T \geq \varepsilon) < \alpha .
$$

As a result we obtain $\rho_{\mathcal{A}_u(\alpha),S}(X) = \infty$, contradicting (i). Hence (ii) must hold. Otherwise, assume (i) holds and $u$ is finitely valued but bounded from above by $\alpha$, and set $x_0 = \inf\{x \in \mathbb{R} ; u(x) = \alpha\}$. Moreover, take $\xi > -x_0$. Since $\rho_{\mathcal{A}_u(\alpha),S}(-\xi) < \infty$, there exists $\lambda > 0$ such that $\mathbb{E}[u(-\xi + \lambda S_T)] \geq \alpha$. But this is only possible if $-\xi + \lambda S_T \geq x_0$, showing that (ii) holds with $\varepsilon = (\xi + x_0)\lambda^{-1} > 0$. This concludes the proof of the proposition.

**Proposition 3.7.20.** Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic and take $p \in [1, \infty)$. Moreover, let $(u, \alpha)$ be admissible. The following statements hold:

(a) If $u$ is bounded from above, the following statements are equivalent:

(i) $\rho_{\mathcal{A}_u(\alpha),S}$ is finitely valued and continuous on $L^p$;

(ii) $u(x_0) > \alpha$ for some $x_0 > 0$, $\lim_{x \to \infty} \frac{x^p}{u(-x)} < 0$ and $\mathbb{P}(S_T = 0) = 0$.

(b) If $u$ is not bounded from above, the following statements are equivalent:

(i) $\rho_{\mathcal{A}_u(\alpha),S}$ is finitely valued and continuous on $L^p$;

(ii) $\lim_{x \to \infty} \frac{x^p}{u(-x)} < 0$.

**Proof.** Since $\mathcal{A}^p_u(\alpha)$ is convex, recall that finiteness implies continuity for $\rho_{\mathcal{A}^p_u(\alpha),S}$ by virtue of Corollary 3.3.40.

(a) First, assume (i) holds. Then, $\mathcal{A}^p_u(\alpha)$ must have nonempty core by Corollary 3.3.20, hence nonempty interior by Proposition 2.2.8. As a result, Proposition 2.4.27 implies that $u(x_0) > \alpha$ for some $x_0 > 0$ and $\lim_{x \to \infty} \frac{x^p}{u(-x)} < 0$. Assume now that $\mathbb{P}(S_T = 0) > 0$. Since $u(-\infty) = -\infty$, taking $\xi > 0$ large enough we obtain

$$
\mathbb{E}[u(-\xi + \lambda S_T)] \leq u(-\xi) \mathbb{P}(S_T = 0) + \sup_{x \in \mathbb{R}} u(x) \mathbb{P}(S_T > 0) < \alpha
$$

for all $\lambda \in \mathbb{R}$. As a result $\rho_{\mathcal{A}^p_u(\alpha),S}(-\xi) = \infty$, contradicting (i). In conclusion, we see that (i) implies (ii).
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To prove the converse implication, assume (ii) holds. Since \( A^p_u(\alpha) \) has nonempty interior in \( L^p \) by Proposition 2.4.27 and \( S_T \) is strictly positive, assertion (i) is an immediate consequence of Theorem 3.3.23.

(b) If (i) holds, then \( A^p_u(\alpha) \) must have nonempty interior as argued above, hence (ii) follows from Proposition 2.4.27. Conversely, assume (ii) holds. Since \( u \) is unbounded from above, we always have \( u(x_0) > \alpha \) for some \( x_0 > 0 \), hence the interior of \( A^p_u(\alpha) \) is nonempty again by Proposition 2.4.27.

We claim that \( \rho_{A^p_u(\alpha),S} \) does not attain the value \(-\infty\). Indeed, assume to the contrary that \( \rho_{A^p_u(\alpha),S}(X) = -\infty \) for some \( X \in L^p \) and take \( \xi > 0 \) such that

\[ E[u(X)1_{\{u(X) > 0\}}] - \xi \mathbb{P}(S_T > 0) < \alpha. \tag{3.5} \]

Define the function \( v : \mathbb{R} \to \mathbb{R} \) by \( v(x) = \max\{u(x), -\xi\} \). Then

\[ E[v(X - nS_T)] \geq E[u(X - nS_T)] \geq \alpha \]

for every \( n \in \mathbb{N} \). Hence, using dominated convergence it is not difficult to show that

\[ E[u(X)1_{\{u(X) > 0\}}] - \xi \mathbb{P}(S_T > 0) \geq E[-\xi 1_{\{S_T > 0\}} + v(X)1_{\{S_T = 0\}}] \geq \alpha. \]

But this contradicts (3.5) showing that \( \rho_{A^p_u(\alpha),S} \) cannot attain the value \(-\infty\).

Finally, take \( \gamma > 0 \) so that \( \mathbb{P}(S_T \geq \gamma) > 0 \). Since \( u \) in unbounded from above, for any \( \xi > 0 \) we find \( \lambda > 0 \) such that

\[ E[u(-\xi + \lambda S_T)] \geq u(-\xi + \lambda \gamma) \mathbb{P}(S_T \geq \gamma) + u(-\xi) \mathbb{P}(S_T < \gamma) \geq \alpha, \]

showing that \( \rho_{A^p_u(\alpha),S}(-\xi) < \infty \). Since \( 1_{\Omega} \) is a strictly positive element in \( L^p \), we can apply Theorem 3.3.21 to find that \( \rho_{A^p_u(\alpha),S} \) is finitely valued, showing that (i) holds and concluding the proof.

Next we consider the case \( \mathcal{X} = M^\hat{u} \) over a nonatomic probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

Unless otherwise stated, the following results are also valid in the context of the Orlicz space \( L^\hat{u} \). The first result is a direct application of Proposition 2.4.30.

**Proposition 3.7.21.** Assume \( u \) is finitely valued and let \((u, \alpha)\) be admissible. Then, \( \rho_{A^p_u(\alpha),S} \) is lower semicontinuous on \( M^\hat{u} \).
In light of Proposition 2.4.31, the statement of the Proposition 3.7.20 takes the following form. The straightforward adjustment of the proof is left to the reader.

**Proposition 3.7.22.** Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is nonatomic. Moreover, assume \(u\) is finitely valued and let \((u, \alpha)\) be admissible.

(a) If \(u\) is bounded from above, the following statements are equivalent:

1. \(\rho_{A_u(\alpha), S}\) is finitely valued and continuous on \(M^{\hat{u}}\);
2. \(u(x_0) > \alpha\) for some \(x_0 > 0\) and \(\mathbb{P}(S_T = 0) = 0\).

(b) If \(u\) is not bounded from above, then \(\rho_{A_u(\alpha), S}\) is finitely valued and continuous on \(M^{\hat{u}}\).

If the underlying reference space is taken to be the Orlicz space \(L^{\hat{u}}\), the previous result does not hold any longer. In fact, the following example shows that, when \(u\) is the exponential utility function, the associated risk measure is never finitely valued on \(L^{\hat{u}}\).

**Example 3.7.23.** Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is nonatomic. Let \(u(x) = 1 - e^{-x}\) for any \(x \in \mathbb{R}\) and assume \((u, \alpha)\) is admissible. Then, \(\rho_{A_u(\alpha), S}\) is not finitely valued on \(L^{\hat{u}}\). Indeed, since \((\Omega, \mathcal{F}, \mathbb{P})\) is nonatomic, we can always find \(Y \in L^{1/2} \setminus L^1\) such that \(Y \geq 1\) almost surely and such that \(Y\) is independent of \(S_T\). Setting \(X = -\log(Y)\), it is easy to see that \(X \in L^{\hat{u}}\) and \(\mathbb{E}[e^{-X}] = \infty\). As a consequence, for any \(\lambda \in \mathbb{R}\) we have

\[
\mathbb{E}[u(X + \lambda S_T)] = 1 - \mathbb{E}[e^{-X}] \mathbb{E}[e^{-\lambda S_T}] = -\infty < \alpha
\]

showing that \(\rho_{A_u(\alpha), S}(X) = \infty\).

**Remark 3.7.24.** The previous example has an interesting consequence. Under the setting above, the acceptance set \(A_u(\alpha)\) is convex and has nonempty interior in \(L^{\hat{u}}\) by a suitable version of Proposition 2.4.31. Hence, Theorem 3.3.23 implies that the Orlicz space \(L^{\hat{u}}\) has no strictly positive elements! We do not know whether, more generally, whenever a nontrivial Morse space and the corresponding Orlicz space do not coincide, as is the case in the presence of exponential utility, the Orlicz space cannot possess strictly positive elements. The lack of strictly positive elements contributes to better explaining the cash-additive example provided at the end of Section 5.1 in Biagini, Frittelli [14], which was used to highlight that
corresponding results in Cheridito, Li [23], obtained for Morse spaces, are not valid in the context of general Orlicz spaces: the random variable $1_{\Omega}$ is not strictly positive in $L^{\hat{w}}$.

In the final part of this section we determine the index of continuity of risk measures based on expected utility. Note that, even though (cash-additive) risk measures of this type were treated in [74], no result concerning their statistical robustness is proved there.

In light of Proposition 2.4.27 we can refine Theorem 3.6.6 as follows.

**Proposition 3.7.25.** Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic and $(u, \alpha)$ is admissible. Take $S_T \in L_{+}^{\infty}$ and assume $\rho_{A_{u}^{\infty}(\alpha), S}$ is finitely valued and, hence, continuous on $L^{\infty}$. For any $p \in [1, \infty)$ the following statements are equivalent:

(i) $\rho_{A_{u}^{\infty}(\alpha), S}$ can be extended to a finitely-valued and, hence, continuous risk measure of the same type on $L^{p}$;

(ii) $u(x_0) > \alpha$ for some $x_0 \in \mathbb{R}$ and $\lim_{x \to -\infty} \frac{x^p}{u(-x)} < 0$.

In this case, the extension is unique and given by $\rho_{A_{u}^{p}(\alpha), S}$. Moreover,

$$ic(\rho_{A_{u}^{\infty}(\alpha), S}) = \inf \left\{ p \in [1, \infty) : \lim_{x \to -\infty} \frac{x^p}{u(-x)} < 0 \right\}.$$  

**Non-finite utilities**

As a first example, we consider the case of a utility $u$ taking the value $-\infty$. Some of the most familiar utility functions present this feature, like the logarithmic utility

$$u(x) = \begin{cases} 
\log(x) & \text{if } x > 0, \\
-\infty & \text{if } x \leq 0, 
\end{cases}$$

or the power utility with parameter $\gamma \in (0, 1)$

$$u(x) = \begin{cases} 
x^\gamma & \text{if } x \geq 0, \\
-\infty & \text{if } x < 0. 
\end{cases}$$

As an immediate consequence of Proposition 3.7.25 above, it follows that risk measures associated with a non-finite utility can never be extended beyond $L^{\infty}$ preserving finiteness and continuity.
Proposition 3.7.26. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be nonatomic. Assume $u$ takes the value $-\infty$ and let $(u, \alpha)$ be admissible. Moreover, take $S_T \in L_{\infty}^+$ and assume $\rho_{A_{u}^{-\infty}(\alpha), S}$ is finitely valued. Then, we have $\text{ic}(\rho_{A_{u}^{-\infty}(\alpha), S}) = \infty$.

**Exponential utility**

The index of continuity may be infinite even if $u$ does not attain the value $-\infty$. To see this we consider the exponential utility with parameter $\gamma > 0$

$$u(x) = 1 - e^{-\gamma x}.$$  

The following result shows that risk measures on $L_{\infty}$ based on expected exponential utility do not admit finitely-valued, hence continuous, extensions to any other $L^p$ space.

Proposition 3.7.27. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be nonatomic. Assume $u$ is the exponential utility with parameter $\gamma > 0$ and let $(u, \alpha)$ be admissible. Moreover, take $S_T \in L_{\infty}^+$ and assume $\rho_{A_{u}^{-\infty}(\alpha), S}$ is finitely valued. Then, we have $\text{ic}(\rho_{A_{u}^{-\infty}(\alpha), S}) = \infty$.

Proof. Since

$$\lim_{x \to \infty} \frac{x^p}{u(-x)} = \lim_{x \to \infty} \frac{x^p}{1 - e^{\gamma x}} = 0$$

for any $p \in [1, \infty)$, the assertion follows immediately from Proposition 3.7.25.

**Flat power utility**

We now show that there exist convex risk measures on $L_{\infty}$ whose index of continuity is equal to any prescribed number in $[1, \infty)$. We consider the flat power utility with parameter $\gamma \in [1, \infty)$

$$u(x) = \begin{cases} 
-|x|^\gamma & \text{if } x < 0, \\
0 & \text{if } x \geq 0.
\end{cases}$$

Note that, in order to have $u(x_0) > \alpha$ for some $x_0 \in \mathbb{R}$, we must impose $\alpha < 0$ in the present case.

Proposition 3.7.28. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be nonatomic. Assume $u$ is the flat power utility with parameter $\gamma \in [1, \infty)$ and let $\alpha < 0$. Moreover, take $S_T \in L_{\infty}^+$ and assume $\rho_{A_{u}^{-\infty}(\alpha), S}$ is finitely valued. Then, $\text{ic}(\rho_{A_{u}^{-\infty}(\alpha), S}) = \gamma$ and the index is attained.
3.7 Illustration of the main results

Proof. Take \( p \in [1, \infty) \) and note that

\[
\lim_{x \to \infty} \frac{x^p}{u(-x)} = - \lim_{x \to \infty} x^{p-\gamma} < 0
\]

if and only if \( p \geq \gamma \). Then, the claim follows directly from Proposition 3.7.25. \( \square \)

A power-like utility alternative to exponential utility

In this section we focus on the power-like utility with parameter \( \gamma > 0 \)

\[
u(x) = \frac{1}{\gamma}(1 + \gamma x - \sqrt{1 + \gamma^2 x^2}).
\]

This utility was discussed in Henderson, Hobson [59] as a tractable alternative to exponential utility if one wants to penalize negative wealth less severely.

Recall that \( \sqrt{1+x} - 1 \approx \frac{1}{2}x \) when \( x \to 0 \), hence

\[
\lim_{x \to \infty} u(x) = 1 - \lim_{x \to \infty} \gamma x \left( \sqrt{1 + \frac{1}{\gamma^2 x^2}} - 1 \right) = 1.
\]

Since \( u \) is bounded from above by 1, we must impose \( \alpha < 1 \) in order to have \( u(x_0) > \alpha \) for some \( x_0 \in \mathbb{R} \). The following result shows that the corresponding risk measures can always be extended to \( L^1 \). This is in strong contrast to the exponential case, reinforcing the role of power-like utility as a valid alternative to exponential utility.

**Proposition 3.7.29.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be nonatomic. Assume \( u \) is the power-like utility with parameter \( \gamma > 0 \) and let \( \alpha < 1 \). Moreover, take \( S_T \in L^\infty_+ \) and assume \( \rho_{A^\infty_+(\alpha), S} \) is finitely valued. Then, \( \text{ic}(\rho_{A^\infty_+(\alpha), S}) = 1 \) and the index is attained.

Proof. Note that

\[
\lim_{x \to \infty} \frac{x}{u(-x)} = - \lim_{x \to \infty} \left( \sqrt{1 + \frac{1}{\gamma^2 x^2}} - 1 \right)^{-1} = -\infty.
\]

As a result, the assertion follows from Proposition 3.7.25. \( \square \)
3.8 Beyond a single liquid asset

In this final section we aim to prepare the ground for future research by extending the range of admissible actions to account for the possibility of investing, for instance, in multiple reference assets in the context of a general multi-period market. After discussing several examples of financial relevance, we provide a variety of reduction results showing how to exploit the theory developed so far to investigate risk measures in this more general framework. Moreover, we focus on two properties, namely cash-subadditivity and quasiconvexity, that have been recently investigated, in a risk measure context, in connection with the presence of defaultable and/or illiquid assets. Since we have a natural way to incorporate defaultability and illiquidity by means of the payoff map \( Z \) and the pricing functional \( \pi \), respectively, we compare the standard interpretation of these properties in light of our constructive approach. Especially in the case of cash-subadditivity, this analysis will challenge the commonly accepted interpretation. Finally, we briefly focus on the class of set-valued risk measures, which has been introduced as a natural model setting when dealing with markets with frictions, and show how to exploit our approach to duality to obtain dual representations without resorting to set-valued analysis.

3.8.1 Beyond univariate positions

Consider a general multi-period market. So far we have interpreted the elements of the underlying space \( \mathcal{X} \) as financial positions at maturity – e.g. capital positions, profit-and-loss profiles, payoffs – expressed in a fixed unit of account. However, in the context of a general multi-period economy, it is important to extend the range of admissible positions to account for two fundamental situations.

First, we may want to follow the evolution of a position across time and define a notion of a risk measure for a stream of positions across time. In this case, the elements of \( \mathcal{X} \) can be represented by stochastic processes of the form

\[
X : \Omega \times T \rightarrow \mathbb{R},
\]

where \( T \) is a suitable set of trading or reporting dates. For example, \( X \) may describe a flow of capital positions, profit-and-loss profiles, payoffs or credit exposures, expressed in a given unit of account.
Second, especially in illiquid markets, the possibility to aggregate in a common unit of account a basket of positions originally expressed in different currencies may be subject to restrictions. Hence, we may want to define a notion of a risk measure for a vector of positions across time. In this case, the elements of $\mathcal{X}$ can be represented by multi-dimensional stochastic processes of the form

$$X : \Omega \times T \rightarrow \mathbb{R}^d,$$

where $T$ is a suitable set of trading or reporting dates and $d$ corresponds to the number of basic currencies or securities traded in the market. For instance, $X$ may describe a sequence of capital positions, profit-and-loss profiles, payoffs or credit exposures, denominated in different currencies, or a sequence of portfolio components expressed in “physical” units.

In the sequel we will therefore extend the notion of a financial position to capture all the above multi-period specifications, both in the univariate and in the multivariate case.

**Remark 3.8.1.** If $d = 1$ and $T$ consists of a single date, we are back to the context of univariate positions in a one-period economy we have been conceptually dealing with so far.

### 3.8.2 The class of admissible actions

If we analyze the concept of a risk measure articulated in the original paper by Artzner, Delbaen, Eber, Heath [8], the following primitive components can be distilled out in the context of the general ambient space $\mathcal{X}$:

- a target set $\mathcal{A} \subset \mathcal{X}$ of acceptable or desirable positions;
- a class $\mathcal{M}$ of admissible management actions to use in order to move a position into the target set;
- a functional $\pi : \mathcal{M} \rightarrow \mathbb{R}$ assigning to each admissible action the corresponding implementation cost;
- a map $\zeta : \mathcal{X} \times \mathcal{M} \rightarrow \mathcal{X}$ describing how to move financial positions by means of admissible actions.
In the sequel, we will always represent the map \( \zeta \) by means of

- a map \( Z : \mathcal{M} \to \mathcal{X} \) describing the positions obtained by implementing admissible actions, so that

\[
\zeta(X, m) := X + Z(m) \quad \text{for all } m \in \mathcal{M}.
\]

In line with Section 3.1.2, we collect these fundamental building blocks under the concept of a risk measurement system.

**Definition 3.8.2.** Assume \( \mathcal{A} \subset \mathcal{X} \) is an acceptance set and \( \mathcal{M} \) is a subset of a suitable real linear space. Moreover, consider a functional \( \pi : \mathcal{M} \to \mathbb{R} \) and a map \( Z : \mathcal{M} \to \mathcal{X} \). Then, we call risk measurement system the quadruple

\[
S = (\mathcal{A}, \mathcal{M}, \pi, Z).
\]

The following example provides the key model specifications to interpret the notion of a risk measurement system, especially the role of the payoff map \( Z \).

**Example 3.8.3.** Consider a multi-period market where \( d \) assets are traded. If we equip a given probability triple \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration structure, the class \( \mathcal{M} \) may be taken to consist of suitable \( d \)-dimensional stochastic processes, which are typically interpreted as admissible trading strategies. In this setting, we may consider the following basic specifications.

(i) The space \( \mathcal{X} \) may consist of random variables

\[
X : \Omega \to \mathbb{R}
\]

representing univariate positions, e.g. capital positions, profit-and-loss profiles, payoffs, credit exposures, expressed in a given unit of account. In this context, the random variable \( Z(m) \) may describe the liquidation value of the strategy \( m \in \mathcal{M} \).

(ii) The space \( \mathcal{X} \) may consist of random vectors

\[
X : \Omega \to \mathbb{R}^d
\]

representing multivariate positions, e.g. vectors of univariate positions. In this context, the random vector \( Z(m) \) may describe the terminal multivariate position or portfolio vector associated to the process \( m \in \mathcal{M} \).
(iii) The space $X$ may consist of stochastic processes

$$X : \Omega \times T \rightarrow \mathbb{R}$$

representing sequences of univariate positions across time. In this context, the one-dimensional process $Z(m)$ may describe the stream of proceedings we receive by rebalancing the portfolio according to the strategy $m \in \mathcal{M}$.

(iv) The space $X$ may consist of stochastic processes

$$X : \Omega \times T \rightarrow \mathbb{R}^d$$

representing sequences of multivariate positions across time. In this context, the multi-dimensional process $Z(m)$ may describe the stream of proceedings we receive, for each univariate component, by rebalancing the portfolio according to the strategy $m \in \mathcal{M}$, or the stream of portfolio weights corresponding to $m \in \mathcal{M}$.

### 3.8.3 Introducing risk measures

To each risk measurement system we can associate a corresponding risk measure, following the constructive approach we have adopted to define risk measures of the form $\rho_{A,S}$.

**Definition 3.8.4.** The risk measure associated with $S = (A, \mathcal{M}, \pi, Z)$ is the map $\rho_S : X \rightarrow \mathbb{R}$ defined by

$$\rho_S(X) = \inf \{ \pi(m) ; \ m \in \mathcal{M}, \ X + Z(m) \in A \}.$$  

In line with our interpretation, the quantity $\rho_S(X)$ represents the “minimal” amount of capital we have to raise and invest in some admissible strategy at inception to ensure that $X$ satisfies the given acceptability constraint.

**Remark 3.8.5.** Note that risk measures of the form $\rho_{A,S}$ correspond to the system $S = (A, \mathbb{R}, \pi, Z)$ with linear $\pi$ and $Z$ satisfying $\pi(1) = S_0$ and $Z(1) = S_T$. In this sense, risk measures of the form $\rho_{A,S}$ constitute the basic instance of these general risk measures.
We proceed to listing a variety of examples illustrating the range of financial problems where risk measures of type $\rho_S$ find natural applications. We start with capital adequacy, even though functionals of the form $\rho_S$ are better known in a pricing/hedging environment.\footnote{Note that some of the following examples already fit the framework of risk measures of the form $\rho_{A,S}$, once we use the elements of the underlying space $\mathcal{X}$ to model financial positions beyond the univariate setting.}

**Example 3.8.6 (Capital Adequacy).** The risk measures studied by Artzner, Delbaen, Koch-Medina [10] are functionals of type $\rho_S$. In the context of a one-period market, let $\mathcal{X}$ be a space of random variables representing capital positions at maturity. We denote by $\mathcal{M}$ the linear space generated by the payoffs of $d$ traded assets and assume $\mathcal{M}$ is equipped with a linear pricing functional $\pi$, so that $\pi(M)$ is the market price of each payoff $M \in \mathcal{M}$. Then, the risk measure with multiple eligible assets is defined by

$$\rho_{A,\mathcal{M},\pi}(X) = \inf \{ \pi(M); \; M \in \mathcal{M}, \; X + M \in A \}.$$ 

The quantity $\rho_{A,\mathcal{M},\pi}(X)$ represents the minimal amount of capital we have to raise and invest is some marketed payoff making the position $X$ acceptable. In particular, $\rho_{A,\mathcal{M},\pi}$ can be seen as the basic multi-asset generalization of risk measures of the form $\rho_{A,S}$. Clearly, we have

$$\rho_{A,\mathcal{M},\pi}(X) = \rho_S(X)$$

once we set $S = (A, \mathcal{M}, \pi, Z)$ where $Z$ is the identity function on $\mathcal{M}$. \qed

**Example 3.8.7 (Capital Adequacy).** The problem of defining a notion of a risk measure for multivariate positions has been first systematically addressed by Jouini, Meddeb, Touzi [64]. There, the authors describe two fundamental ways to define a risk measure for random vectors: the first by means of a set-valued mapping, the second by means of a scalarization procedure. Here we focus on the scalar approach, which has been notably pursued by Burgert, Rüschendorf [19] and by Ekeland, Schachermayer [38]. We also refer to the scalarized set-valued risk measures in Hamel, Heyde [55]. The above contributions can be embraced into the setting of [55]. In the context of a one-period model, let $\mathcal{X}$ be a space of
3.8 Beyond a single liquid asset

$d$-dimensional random vectors whose components represent suitable capital positions, e.g. expressed in different currencies. Assume $\mathcal{P}$ is a linear subspace of $\mathcal{X}$ consisting of admissible deterministic portfolios to be set up at inception. In [19] and [38] the space $\mathcal{P}$ is assumed to be generated by the unitary vector in $\mathbb{R}^d$. Moreover, assume $\mathcal{P}$ is equipped with a linear functional $\pi$ assigning to each portfolio $u \in \mathcal{P}$ a certain “value” or “weight”. The functional $\pi$ is assumed to represent a suitable aggregation procedure, e.g. an internal price system chosen by the regulator; see [55]. Then, the risk measure for multivariate positions can be defined as

$$\rho_{A,\mathcal{P},\pi}(X) = \inf \{ \pi(u); u \in \mathcal{P}, X + u \in A \}.$$ 

In other words, the quantity $\rho_{A,\mathcal{P},\pi}(X)$ helps selecting, by means of a suitable aggregation scheme, the “optimal” initial portfolios making the multivariate position $X$ acceptable. It is easy to see that

$$\rho_{A,\mathcal{P},\pi}(X) = \rho_S(X)$$

for $S = (A, \mathcal{P}, \pi, Z)$, where $Z$ is the identity function on $\mathcal{P}$.

Example 3.8.8 (Hedging). The various versions of the superhedging price fit the framework of risk measures of the form $\rho_S$. We refer to the survey article by Pennanen [82] for a detailed account of the literature, especially for the case of illiquid markets. Here, we describe the problem of hedging a single-payment univariate claim. Let $\mathcal{X}$ be a space of random variables representing future claims to hedge at maturity. We denote by $\Theta$ a class of suitable $d$-dimensional processes representing admissible trading strategies. Moreover, we denote by $\alpha_0(\vartheta)$ the implementation cost (ask price) at inception and by $\beta_T(\vartheta)$ a random variable describing the liquidation value (bid price) at maturity of a strategy $\vartheta \in \Theta$. The superhedging price for a claim $X \in \mathcal{X}$ is then defined as

$$\pi_A(X) = \inf \{ \alpha_0(\vartheta); \vartheta \in \Theta, \beta_T(\vartheta) - X \in A \}.$$ 

The quantity $\pi_A(X)$ represents the minimal amount of capital we have to raise to implement some admissible strategy at inception so that we can cover the claim $X$ at maturity according to the hedging tolerance specified by $A$. If $A$ is the usual positive cone, $\pi_A(X)$ is the classical superhedging price. If we enlarge the acceptance set so that we allow for an acceptable hedging error, we are in
the framework of good deal prices. The pricing theory under acceptable risk
initiated by Carr, Geman, Madan [20] and developed by Madan, Cherny in [79] is
a prominent example of this approach. In particular, $\mathcal{A}$ can be defined in terms
of a given acceptability index; see Cherny, Madan [24]. We also refer to the good
deal valuations recently studied in Arai, Fukasawa [7]. In any case, it is easy to
see that
$$\pi_\mathcal{A}(X) = \rho_S(-X)$$
once we set $S = (\mathcal{A}, \Theta, \alpha_0, \beta_T)$.
The usual formulation of these “no-arbitrage” pricing bounds is, in fact, slightly
different from the one presented above. Indeed, one typically assumes the exis-
tence of a risk-free asset and considers a set $\Xi$ of admissible trading strategies for
the other $d - 1$ assets. Moreover, one denotes by $\beta_T(\xi)$ a random variable rep-
resenting the bid price at maturity of a strategy $\xi \in \Xi$ (the map $\beta_T$ is typically
assumed to be linear). In this setting, the superhedging price for a claim $X \in \mathcal{X}$
is given by
$$\pi_\mathcal{A}(X) = \inf \{\lambda \in \mathbb{R} : \exists \xi \in \Xi : \lambda + \beta_T(\xi) - X \in \mathcal{A}\}.$$ 
In this case, we have
$$\pi_\mathcal{A}(X) = \rho_S(-X)$$
provided we set $S = (\mathcal{A} - \beta_T(\Xi), \mathbb{R}, \pi, Z)$ with linear $\pi$ and $Z$ satisfying $\pi(1) = 1$
and $Z(1) = 1_\Omega$, respectively. In other words, $\pi_\mathcal{A}$ is nothing but a cash-additive
risk measure. \hfill \Box

**Example 3.8.9** (Hedging). The hedging functionals recently studied by Penna-
nen in [81] and in [83] are examples of risk measures of the form $\rho_S$. In the context
of a multi-period economy, let $\mathcal{X}$ be a space of stochastic processes representing
sequences of payments across time and consider a process $P \in \mathcal{X}$, the so-called
premium process, describing the “cash-flow that a seller receives in exchange for
delivering a claim” in $\mathcal{X}$. The associated superhedging price, or reservation value,
is defined by
$$\pi_{\mathcal{A},P}(X) = \inf \{\lambda \in \mathbb{R} : \lambda P - X \in \mathcal{A}\}.$$ 
The quantity $\pi_{\mathcal{A},P}(X)$ gives the “least contribution rate”, expressed in terms of
the premium process, that is sufficient to hedge the claim payments up to the
hedging tolerance specified by $\mathcal{A}$. Clearly, we have

$$\pi_{\mathcal{A}, P}(X) = \rho_S(-X)$$

where $S = (\mathcal{A}, \mathbb{R}, \pi, Z)$ for linear $\pi$ and $Z$ satisfying $\pi(1) = 1$ and $Z(1) = P$. □

Example 3.8.10 (Pricing). The liquidation value studied in Bouchard, Kabanov, Touzi [18] is, up to a sign, a risk measure of the form $\rho_S$. Assume $\mathcal{X} = \mathbb{R}^d$ is a space of portfolio vectors equipped with a suitable quasiorder. The corresponding positive cone, called solvency region in [18], is assumed to consist of those portfolios that can be exchanged into the null portfolio with no injection of capital. In this setting, the liquidation value for a portfolio $x \in \mathcal{X}$ was defined by

$$\ell(x) = \sup\{\lambda \in \mathbb{R} ; x \geq \lambda e_1\}.$$ 

If we assume the first component of each vector in $\mathcal{X}$ represents a cash amount, the quantity $\ell(x)$ is the “maximal cash endowment that one can get from $x$ by clearing all the positions in the risky assets”. It is easy to verify that

$$\ell(x) = -\rho_S(x)$$

once we set $S = (\mathcal{X}_+, \mathbb{R}, \pi, Z)$ for linear $\pi$ and $Z$ satisfying $\pi(1) = 1$ and $Z(1) = e_1$, respectively. The above notion of a liquidation value seems to be the first instance where a “risk measure” approach was implicitly adopted to define pricing functionals for portfolios of assets outside the standard frictionless theory. □

Example 3.8.11 (Pricing). The various versions of the indifference price provide abundant examples of risk measures of the form $\rho_S$. Here, we describe the problem of pricing a single-payment contract. We refer to the survey article by Pennanen [82] for additional information about indifference pricing. Let $\mathcal{X}$ be a space of random variables representing the terminal payoff of contracts to price. Assume $\mathcal{M} \subseteq \mathbb{R}$ and consider a pricing functional $\pi : \mathcal{M} \to \mathbb{R}$ and a payoff function $Z : \mathcal{M} \to \mathcal{X}$. Moreover, consider a utility mapping $U : \mathcal{X} \to \mathbb{R}$ assigning to each payoff a certain utility coefficient. Given a pre-specified position $Y \in \mathcal{X}$, we define the (seller’s) indifference price of a contract with payoff $X \in \mathcal{X}$ by setting

$$I_U(X; Y) = \inf\{\pi(\lambda) ; \lambda \in \mathcal{M}, U(Z(\lambda) - X + Y) \geq U(Y)\}.$$
The quantity $I_U(X; Y)$ represents the natural seller’s pricing bound making the agent indifferent, from the point of view of the utility mapping $U$, between selling $X$ and doing nothing. If we define

$$\mathcal{A} = \{X \in \mathcal{X} ; U(X + Y) \geq U(Y)\},$$

it is immediate to see that

$$I_U(X; Y) = \rho_S(-X)$$

once we set $S = (\mathcal{A}, \mathcal{M}, \pi, Z)$. Typically, the reference asset is taken to be liquid and one assumes that $\pi(1) = 1$ and $Z(1) = 1_\Omega$ hold. For instance, this is the case for the risk indifference price studied in Arai, Fukasawa [7].

**Example 3.8.12** (Pricing). The indifference swap rates recently studied by Pennanen [83] are examples of risk measures of type $\rho_S$. In the context of a multi-period economy, let $\mathcal{X}$ be a space of stochastic processes representing sequences of payments across time and consider a premium process $P \in \mathcal{X}$. Moreover, consider a suitable utility mapping $U : \mathcal{X} \to \mathbb{R}$. Given a liability process $Y \in \mathcal{X}$, the *indifference swap rate* for $X \in \mathcal{X}$ is defined by setting

$$\pi_{U,P}(X; Y) = \inf\{\lambda \in \mathbb{R} ; U(\lambda P - X - Y) \geq U(-Y)\}.$$

The quantity $\pi_{U,P}(X; Y)$ represents the “lowest swap rate (premium rate) that would allow the agent to enter the swap contract without worsening his risk-return profile”. Setting

$$\mathcal{A} = \{X \in \mathcal{X} ; U(X - Y) \geq U(-Y)\}$$

we clearly have

$$\pi_{U,P}(X; Y) = \rho_S(-X)$$

where $S = (\mathcal{A}, \mathbb{R}, \pi, Z)$ for linear $\pi$ and $Z$ satisfying $\pi(1) = 1$ and $Z(1) = P$.

**Example 3.8.13** (Risk Assessment). The liquidity-adjusted risk measures by Weber, Anderson, Hamm, Knispel, Liese, Salfeld [99] are functionals of type $\rho_S$. In the context of a one-period economy, let $\mathcal{X} = \mathbb{R}^d$ be a space of portfolio vectors whose first component corresponds to cash. The *liquidity-adjusted risk measure* is given by

$$\rho^L(x) = \inf\{\lambda \in \mathbb{R} ; x + \lambda e_1 \in \mathcal{A}\},$$
where \( A \) incorporates the relevant information about liquidity and portfolio constraints. The quantity \( \rho^L(x) \) is used to compare the riskiness of different portfolio vectors. It is immediate to see that
\[
\rho^L(x) = \rho_S(x)
\]
provided \( S = (A, \mathbb{R}, \pi, Z) \) for linear \( \pi \) and \( Z \) satisfying \( \pi(1) = 1 \) and \( Z(1) = e_1 \), respectively.

**Example 3.8.14 (Risk Assessment).** The risk measures for processes by Frittelli, Scandolo [52] are risk measures of the form \( \rho_S \). In the context of a multi-period market, let \( \mathcal{X} \) be a space of stochastic processes representing payoff streams. Denote by \( C \) a linear subspace of \( \mathcal{X} \) representing suitable loan processes and assume \( C \) is equipped with a linear pricing rule \( \pi \). The *risk measure for processes* is then defined by
\[
\rho_{A,C,\pi}(X) = \inf\{\pi(C) ; C \in C, X + C \in A\}.
\]
The quantity \( \rho_{A,C,\pi}(X) \) is supposed to be used to compare the riskiness of different payoff streams. Clearly, we have
\[
\rho_{A,C,\pi}(X) = \rho_S(X)
\]
for \( S = (A, C, \pi, Z) \), where \( Z \) is the identity function on \( C \).

**Example 3.8.15 (Utility Assessment).** The benefit functions introduced by Luenberger [78] are risk measures of the form \( \rho_S \). Let \( \mathcal{X} = \mathbb{R}^d \) be the space of commodity bundles and consider a consumption possibility set \( C \subset \mathbb{R}^d \), a utility function \( u : \mathbb{R}^d \to \mathbb{R} \) and a nonzero reference bundle \( g \in \mathbb{R}^d_+ \). The corresponding *benefit function* is defined by
\[
b(x; g, \alpha) = \sup\{\lambda \in \mathbb{R} ; x - \lambda g \in C, u(x - \lambda g) \geq \alpha\},
\]
where \( \alpha \in \mathbb{R} \) is a pre-specified utility level. The quantity \( b(x; g, \alpha) \) is interpreted as the “amount of commodity bundle \( g \) that the consumer would be willing to trade for the possibility of moving from utility level \( \alpha \) to the bundle \( x \)”.

\[3\] The paper [78] appeared few years before the publication by Artzner, Delbaen, Eber, Heath [9]. From a conceptual perspective, the points of contact between the two papers are surprisingly strong. The author of [78] even uses the term “measure” to refer to his reference bundle: “\( g \) is a reference vector defining the measure by which alternative bundles are compared”!
define the set
\[ \mathcal{A} = \{ x \in \mathbb{R}^d ; \ x \in \mathcal{C}, \ u(x) \geq \alpha \} , \]
it is clear that
\[ b(x; g, \alpha) = -\rho_S(x) \]
once we consider \( S = (\mathcal{A}, \mathbb{R}, \pi, Z) \) for linear \( \pi \) and \( Z \) satisfying \( \pi(1) = 1 \) and \( Z(1) = g \).

### 3.8.4 Reduction theorems

In this section we provide a variety of reduction results showing how to exploit the theory of single-asset risk measures to investigate the general risk measures of the form \( \rho_S \). We divide our analysis into two steps.

We start by considering the situation where the market allows for a “direction of liquidity”. This condition is assumed, implicitly or explicitly, in the vast majority of market models encountered in the literature.

**Definition 3.8.16.** We say that a strategy \( s \in \mathcal{M} \) is **liquid** if \( \mathcal{M} + \text{span}(s) \subseteq \mathcal{M} \) and if both \( \pi \) and \( Z \) are linear along \( s \).

**Remark 3.8.17.** Consider a market where \( d \) assets are traded and assume \( s \in \mathcal{M} \) corresponds to buy-and-holding one unit of the \( i \)-th asset. If \( s \) is liquid, we can also say that asset \( i \) is **liquid**. This terminology is consistent with the notion of a liquid reference asset introduced in Section 3.1. In that case we had \( \mathcal{M} = \mathbb{R} \) and the strategy \( s = 1 \), corresponding to buy-and-holding one unit of the reference asset, was indeed a liquid strategy.

The following result shows that, in the presence of a liquid strategy, the study of risk measures of the form \( \rho_S \) can be reduced to the study of risk measures with respect to a single “asset”. Here, we denote by \( \ker(\pi) \) the **kernel** of \( \pi \), i.e.
\[ \ker(\pi) := \{ m \in \mathcal{M} ; \ \pi(m) = 0 \} . \]

**Theorem 3.8.18.** Consider a liquid strategy \( s \in \mathcal{M} \) and assume \( \pi(s) > 0 \) and \( Z(s) \) is a nonzero element of \( \mathcal{X}_+ \). Moreover, set \( S = (\pi(s), Z(s)) \). Then, for every \( X \in \mathcal{X} \) we have
\[ \rho_S(X) = \rho_{\mathcal{A} - Z(\ker(\pi)), S}(X) . \]
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Proof. Since \( \pi \) is linear along \( s \) and \( \pi(s) > 0 \), we have \( \mathcal{M} = \ker(\pi) + \text{span}(s) \). As a result, using the linearity of \( Z \) along \( s \) we obtain

\[
\rho_S(X) = \inf\{\lambda \pi(s) ; \lambda \in \mathbb{R}, X + \lambda Z(s) \in \mathcal{A} - Z(\ker(\pi))\}
\]

for every \( X \in \mathcal{X} \), proving the claim. \( \square \)

Remark 3.8.19. The monotone set \( \mathcal{A} - Z(\ker(\pi)) \) consists of those positions that can be made acceptable by means of zero-cost admissible strategies. Note that \( \mathcal{A} - Z(\ker(\pi)) \) is an acceptance set if and only if it is a proper subset of \( \mathcal{X} \). \( \square \)

The preceding result can only be applied if the market allows for the existence of a liquid strategy. In the fully illiquid case we can still obtain a suitable reduction to risk measures of the form \( \rho_{\mathcal{A},S} \) provided we enlarge the underlying space. The new ambient space is the product space \( \mathbb{R} \times \mathcal{X} \). Under the natural topological linear structure, we equip this space with the compatible quasiorder induced by the convex cone \( \mathbb{R}_+ \times \mathcal{X}_+ \). As usual, for a map \( f : \mathbb{R} \times \mathcal{X} \to \mathbb{R} \) we will write \( f(\lambda, X) \) instead of \( f((\lambda, X)) \).

Theorem 3.8.20. Consider the subset of \( \mathbb{R} \times \mathcal{X} \) defined by

\[
\mathcal{C} = \mathbb{R}_+ \times \mathcal{A} - \{(-\pi(m), Z(m)) ; m \in \mathcal{M}\}.
\]

Moreover, set \( S = (1, (1, 0)) \) where \( (1, 0) \in \mathbb{R} \times \mathcal{X} \). Then, for every \( X \in \mathcal{X} \) we have

\[
\rho_S(X) = \rho_{\mathcal{C},S}(0, X).
\]

Proof. Take \( X \in \mathcal{X} \) and note that, by definition of \( \mathcal{C} \),

\[
\rho_{\mathcal{C},S}(0, X) = \inf\{\lambda \in \mathbb{R} ; \exists m \in \mathcal{M} : (\lambda - \pi(m), X + Z(m)) \in \mathbb{R}_+ \times \mathcal{A}\}.
\]

To prove the inequality \( \leq \), take \( \lambda \in \mathbb{R} \) such that \( \lambda - \pi(m) \in \mathbb{R}_+ \) and \( X + Z(m) \in \mathcal{A} \) for some \( m \in \mathcal{M} \). Then, it follows that

\[
\rho_S(X) \leq \pi(m) \leq \lambda,
\]

showing the desired inequality. To prove the converse inequality, take \( m \in \mathcal{M} \) such that \( X + Z(m) \in \mathcal{A} \). Then, we immediately see that

\[
\rho_{\mathcal{C},S}(0, X) \leq \pi(m),
\]
proving the other inequality and concluding the proof.

\[\square\]

**Remark 3.8.21.** The monotone set \(C\) introduced above is such that a couple \((\lambda, X)\) belongs to \(C\) if and only if \(X\) can be made acceptable by means of some admissible strategy whose implementation cost is not higher than \(\lambda\).

\[\square\]

The next example unveils a surprising connection between the preceding reduction theorem and the recent work by Pennanen [83].

**Example 3.8.22.** Consider a risk measurement system \(S\) in the context of a fully illiquid market. The previous result shows that we can express \(\rho_S\) in terms of the superhedging functionals studied in Pennanen [83]; see Example 3.8.9. Indeed, using the above reduction it is not difficult to see that

\[
\rho_S(X) = \rho_{C,S}(0, X) = \pi_{C,P}(0, -X)
\]

provided the premium process \(P = (1, 0) \in \mathbb{R} \times \mathcal{X}\) is the “payoff” of \(S\). In particular, note that \(\pi_{C,P}\) is defined on the product space \(\mathbb{R} \times \mathcal{X}\). To best exemplify this link, we focus on the classical superhedging problem in the context of the market model adopted in [83].

Consider a multi-period discrete market where \(d\) assets are traded. The set of relevant dates is \(T = \{0, 1, \ldots, T\}\). In particular, trading is possible at each date prior to maturity. We denote by \(\mathcal{X}\) a space of stochastic processes of the form

\[
X = (X_1, \ldots, X_T),
\]

representing multi-payment claims. To hedge these claims, we can implement a suitable trading strategy modelled by a \(d\)-dimensional process belonging to a set \(\Theta\). For each strategy \(\vartheta \in \Theta\) we denote by \(\alpha_0(\vartheta)\) the corresponding implementation cost at inception and by

\[
\beta(\vartheta) = (\beta_1(\vartheta), \ldots, \beta_T(\vartheta))
\]

the cash-flow of proceedings we receive across time by rebalancing the portfolio accordingly. The superhedging price of a claim process \(X \in \mathcal{X}\) is then given by

\[
\pi_{\mathcal{X}_+}(X) = \inf\{\alpha_0(\vartheta); \vartheta \in \Theta, \beta(\vartheta) - X \in \mathcal{X}_+\},
\]
where $\mathcal{X}_+$ is the standard positive cone in $\mathcal{X}$. Note that, similarly to what discussed in Example 3.8.8, we have

$$\pi_{\mathcal{X}_+}(X) = \rho_S(-X)$$

for $S = (\mathcal{X}_+, \Theta, \alpha_0, \beta)$. Now, consider the space $\mathbb{R} \times \mathcal{X}$ and take $P = (1, 0) \in \mathbb{R} \times \mathcal{X}$. Moreover, define

$$C = \mathbb{R}_+ \times \mathcal{X}_+ - \{(-\alpha_0(\vartheta), \beta(\vartheta)) ; \vartheta \in \Theta\}.$$

As a consequence of the previous theorem, it follows that

$$\pi_{\mathcal{X}_+}(X) = \rho_{C,S}(0, -X)$$

where $S = (1, P)$. To highlight the link with [83], it remains to note that the focus in that paper is precisely devoted to claim processes of the form

$$(0, X_1, \ldots, X_T) = (0, X) \in \mathbb{R} \times \mathcal{X}$$

and the corresponding superhedging price, i.e. $\pi_{C,P}(0, X)$, is defined directly in the form $\rho_{C,S}(0, -X)$ rather than in the standard form $\pi_{\mathcal{X}_+}(X)$. In particular, $P$ is the basic instance of a premium process.

We conclude this section by focusing on the case of a single illiquid reference asset in the context of a one-period market. In this case, the class $\mathcal{M}$ of admissible actions can be represented by a subset of $\mathbb{R}$. We will show that, by imposing more assumptions on the system $\mathcal{S}$, we can provide a sharper reduction which does not require to enlarge the original ambient space $\mathcal{X}$. Before proving the corresponding reduction result, we set

$$m := \inf \mathcal{M} \quad \text{and} \quad \overline{m} := \sup \mathcal{M}$$

and we introduce the modified pricing functional $\pi_{\mathcal{M}} : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\pi_{\mathcal{M}}(\lambda) := \begin{cases} \pi(m) & \text{if } \lambda \leq m, \\ \pi(\lambda) & \text{if } m < \lambda \leq \overline{m}, \\ \infty & \text{if } \lambda > \overline{m}. \end{cases}$$

Note that, in case $\mathcal{M} = \mathbb{R}$, the modified pricing functional $\pi_{\mathcal{M}}$ coincides with $\pi$. \qed
**Theorem 3.8.23.** Assume $\mathcal{A}$ is closed, $\mathcal{M}$ is closed and convex, $\pi(\infty) = \infty$ and $Z$ is linear. Moreover, define $S = (\pi(1), Z(1))$. Then, for every $X \in \mathcal{X}$ we have

$$\rho_S(X) = \pi(\mathcal{M}\left(\frac{\rho_{A,S}(X)}{\pi(1)}\right)).$$

**Proof.** Fix $X \in \mathcal{X}$. Assume first that $\rho_{A,S}(X) \leq m\pi(1)$. If $m = -\infty$, we must have $\rho_{A,S}(X) = -\infty$ and, thus,

$$\rho_S(X) = \pi(-\infty) = \pi(\mathcal{M}(-\infty)),$$

proving the assertion. Otherwise, let $m > -\infty$ so that the right-hand side is equal to $\pi(m)$. We claim that $\rho_S(X) = \pi(m)$. Since we clearly have $\rho_S(X) \geq \pi(m)$, we focus on the converse inequality. To this end, take a sequence $(\lambda_n)$ in $\mathcal{M}$ converging to $m$ from above and such that $X + \lambda_nZ(1) \in \mathcal{A}$. Since $\mathcal{A}$ is closed, we must have $X + mZ(1) \in \mathcal{A}$. Moreover, since $\mathcal{M}$ is closed, we also have $m \in \mathcal{M}$, implying that $\rho_S(X) \leq \pi(m)$.

Second, assume that $m\pi(1) < \rho_{A,S}(X) \leq m\pi(1)$. If $\rho_{A,S}(X)$ is finite, we have to prove that

$$\rho_S(X) = \pi\left(\frac{\rho_{A,S}(X)}{\pi(1)}\right).$$

(3.6)

Note that, by the convexity of $\mathcal{M}$, the quantity $\frac{\rho_{A,S}(X)}{\pi(1)}$ belongs to $\mathcal{M}$. Moreover, since $\mathcal{A}$ is closed, we have

$$X + \frac{\rho_{A,S}(X)}{\pi(1)}Z(1) \in \mathcal{A}.$$

This shows the inequality “$\leq$” in (3.6). The converse inequality follows immediately from the monotonicity of $\pi$. Otherwise, assume $\rho_{A,S}(X) = \infty$, which is only possible if $m = \infty$. In this case, it is easy to see that

$$\rho_S(X) = \infty = \pi(\infty) = \pi(\mathcal{M}(\infty)).$$

Finally, assume $\rho_{A,S}(X) > m\pi(1)$. In this case, the assertion is trivial since both sides are equal to $\infty$. 

**Remark 3.8.24.** It is not difficult to verify that the preceding statement holds even if $\mathcal{A}$ is not closed, provided $\pi$ is assumed to be right-continuous.
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3.8.5 A note on cash-subadditivity

In this section we focus on the property of cash-subadditivity, which has been recently introduced in the influential paper by El Karoui, Ravanelli [40] to challenge the “axiom” of cash-additivity.

The motivation for a new axiomatic setting provided in [40] is that “while regulators and financial institutions determine and collect today the reserve amounts to cover future risky positions, the cash additivity requires that risky positions and reserve amounts are expressed in the same numéraire”. Based on this argument, the main contribution of that paper is “to propose a new class of risk measures [...] that are directly defined on the future risky positions and provide the reserve amounts in terms of the current numéraire”. Moreover, “replacing the cash additive axiom with the cash subadditive axiom is sufficient to characterize risk measures that can be used also when cash-additive risk measures cannot, for instance, under stochastic and/or ambiguous interest rates or assessing (sic) the risk of defaultable contingent claims”.

In light of the analysis of the discounting reduction undertaken in Chapter 1, the above motivation to abandon cash-additivity in favour of cash-subadditivity seems to result from a misinterpretation of the cash-additive property. First, risk measures of the form $\rho_{A,S}$ determine, by construction, capital requirements today to “cover the risk” of future positions. Moreover, the cash-additive reduction is not postulating the existence and availability of any “risk-free” asset but is only reflecting, in discounted terms, the simplified mathematical formulation of a more original problem. By no means is the original reference asset required to be “risk-free”.

To further confront the above interpretation, we will focus on risk measures with respect to a single liquid reference bond $S = (S_0, S_T)$. The following results will show that, contrary to what argued in [40], the corresponding risk measures $\rho_{A,S}$ typically fails to be cash-subadditive precisely when the bond is defaultable and the invested capital is at risk, i.e. when

$$\mathbb{P}(S_T < S_0) > 0.$$

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume $\mathcal{X}$ is a topological linear space of random variables containing $1_\Omega$. The model space $\mathcal{X}$ is ordered by the canonical
almost-sure order $\leq_P$.

**Definition 3.8.25.** A map $f : \mathcal{X} \rightarrow \mathbb{R}$ is called *cash-subadditive* whenever

$$f(X + m) \geq f(X) - m \text{ for all } m > 0.$$ 

We start by showing that the prototype of a cash-subadditive risk measure presented in [40] is, in fact, a risk measure of the form $\rho_{A,S}$ where $S$ is a bond which can never default on the invested capital!

**Example 3.8.26.** To provide “the intuition for introducing cash-subadditive risk measures”, the authors in [40] considered maps $\rho : \mathcal{X} \rightarrow \mathbb{R}$ of the form

$$\rho(X) = \rho_0(D_T X),$$

where $\rho_0$ is a cash-additive risk measure defined on the convenient corresponding space and $D_T$ is a random variable taking values in $(0, 1]$ and representing a (possibly stochastic) discount factor (the value 0 is also allowed in [40]). It is easy to see that $\rho$ is cash-subadditive. However, even if not cash-additive, $\rho$ is in fact additive with respect to the reference asset $S = (1, 1/D_T)$, provided that $1/D_T$ belongs to $\mathcal{X}$. In particular, we may think of $S$ as the reference bond used to discount future values. The bond may default on the interest payment but the invested capital is never at risk since $\mathbb{P}(S_T < S_0) = \mathbb{P}(D_T > 1) = 0$.

We provide a general sufficient condition for cash-subadditivity, which covers the situation of the previous example: if the invested capital is not at risk, i.e. if the bond can only default on the interest payment, then $\rho_{A,S}$ is always cash-subadditive.

**Proposition 3.8.27.** Assume $\mathbb{P}(S_T < S_0) = 0$. Then, $\rho_{A,S}$ is cash-subadditive.

**Proof.** Take $X \in \mathcal{X}$ and $m > 0$. Setting $\lambda = m/S_0$ we easily obtain that

$$\rho_{A,S}(X + m) = \rho_{A,S}(X + \lambda S_0) \geq \rho_{A,S}(X + \lambda S_T) = \rho_{A,S}(X) - m.$$ 

Next we show a general necessary condition for cash-subadditivity.

**Proposition 3.8.28.** Assume that $0 \in \mathcal{A}$. If $\rho_{A,S}$ is cash-subadditive, then we have $S_T - S_0 \in \text{cl} \mathcal{A}$. 

Proof. The claim follows immediately from Proposition 3.2.7 since
\[ \rho_{A,S}(S_T - S_0) = \rho_{A,S}(-S_0) - S_0 \leq \rho_{A,S}(S_0 - S_0) = \rho_{A,S}(0) \leq 0. \]
The preceding condition becomes equivalent to cash-subadditivity if the underlying acceptance set is a closed, convex cone, as implied by the next result.

**Proposition 3.8.29.** Assume \( A \) is a convex cone. If \( S_T - S_0 \in A \), then \( \rho_{A,S} \) is cash-subadditive.

**Proof.** Assume \( S_T - S_0 \in A \) but \( \rho_{A,S} \) is not cash-subadditive. Then, without loss of generality, we find \( X \in \mathcal{X} \) and \( m > 0 \) satisfying
\[ \rho_{A,S}(X + m) < 0 < \rho_{A,S}(X) - m = \rho_{A,S}\left(X + \frac{m}{S_0} S_T\right). \]
In particular, \( X + m \in A \) while \( X + \frac{m}{S_0} S_T \notin A \) by Proposition 3.2.7. Since \( A \) is closed under addition and
\[ X + \frac{m}{S_0} S_T = X + m + \frac{m}{S_0} (S_T - S_0), \]
we immediately conclude that \( S_T - S_0 \notin A \), contradicting the initial assumption. Hence, \( \rho_{A,S} \) must be cash-subadditive.

**Corollary 3.8.30.** Assume that \( A \) is a closed, convex cone. Then, \( \rho_{A,S} \) is cash-subadditive if and only if \( S_T - S_0 \in A \).

We can apply the preceding result to risk measures based on ES-acceptability and SPAN-acceptability. The easy verification is left to the reader. For acceptability based on ES at level \( \alpha \), it turns out that the corresponding risk measure is never cash-subadditive whenever the probability that the invested capital is at risk exceeds the threshold \( \alpha \).

**Proposition 3.8.31.** Take \( p \in [1, \infty] \) and fix \( \alpha \in (0, 1) \). Then, \( \rho_{A_{ES(\alpha)},S} \) is cash-subadditive if and only if \( ES_\alpha(S_T) \leq -S_0 \). In this case, \( \mathbb{P}(S_T < S_0) \leq \alpha \).

For acceptability based on the test scenario \( A \), cash-subadditivity is equivalent to the bond never defaulting on the invested capital in the event \( A \).

**Proposition 3.8.32.** Take \( p \in [1, \infty] \) and assume \( A \in \mathcal{F} \) is admissible. Then, \( \rho_{SPAN^p(A),S} \) is cash-subadditive if and only if \( \mathbb{P}(A \cap \{S_T < S_0\}) = 0 \).
We conclude by focusing on acceptability based on VaR. In this case, cash-subadditivity fails whenever the probability that the invested capital is at risk exceeds the threshold $\alpha$. The claim follows directly from Proposition 3.8.28.

**Proposition 3.8.33.** Take $p \in [0, \infty]$ and fix $\alpha \in (0, 1)$. If $\rho_{A_{\text{VaR}}(\alpha), S}$ is cash-subadditive, then $\mathbb{P}(S_T < S_0) \leq \alpha$.

In fact, if the underlying probability space is nonatomic, then cash-subadditivity always fails as soon as the invested capital is at risk.

**Proposition 3.8.34.** Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic. Take $p \in [0, \infty]$ and fix $\alpha \in (0, 1)$. Then, $\rho_{A_{\text{VaR}}(\alpha), S}$ is cash-subadditive if and only if $\mathbb{P}(S_T < S_0) = 0$.

**Proof.** By Proposition 3.8.27 we only need to prove the “only if” implication. To this end, assume $\rho_{A_{\text{VaR}}(\alpha), S}$ is cash-subadditive but $\mathbb{P}(S_T < S_0) > 0$. Take $\varepsilon \in (0, 1)$ such that $\mathbb{P}(S_T \leq \varepsilon S_0) > 0$. Since $\mathbb{P}(S_T < S_0) \leq \alpha$ by virtue of Proposition 3.8.33, we can find $0 < \delta < \mathbb{P}(S_T \leq \varepsilon S_0)$ satisfying

$$\mathbb{P}(S_T \geq S_0) > \alpha + \delta - \mathbb{P}(S_T < S_0) > 0.$$  

Moreover, since $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic we have $\mathbb{P}(A) = \alpha + \delta - \mathbb{P}(S_T < S_0)$ for some measurable subset $A$ of $\{S_T \geq S_0\}$. Now, take $m > 0$ such that $(1+m)\varepsilon < 1$ and set

$$X = \begin{cases} 
-1 & \text{on } B \\
-\frac{2+m}{S_0}S_T & \text{on } B^c
\end{cases}$$

where

$$B = \{S_T \leq \varepsilon S_0\} \cup (\{S_T \geq S_0\} \setminus A).$$

Then $\mathbb{P}(X + 1 < 0) \leq \mathbb{P}(B^c) < \alpha$, implying $\rho_{A_{\text{VaR}}(\alpha), S}(X + 1) \leq 0$. Moreover,

$$X + \frac{1}{S_0}S_T + \frac{m}{S_0}S_T \leq -1 + (1+m)\varepsilon < 0 \quad \text{on } \{S_T \leq \varepsilon S_0\}$$

and

$$X + \frac{1}{S_0}S_T + \frac{m}{S_0}S_T = -\frac{1}{S_0}S_T < 0 \quad \text{on } B^c.$$ 

Hence, it follows that

$$\mathbb{P}\left(X + \frac{1}{S_0}S_T + \frac{m}{S_0}S_T < 0\right) \geq \mathbb{P}(S_T < S_0) + \mathbb{P}(A) = \alpha + \delta > \alpha.$$
whence
\[ \rho_{\text{Var}(\alpha),S}(X) - 1 = \rho_{\text{Var}(\alpha),S}(X + \frac{1}{S_0} S_T) \geq m > 0 \geq \rho_{\text{Var}(\alpha),S}(X + 1). \]
This contradicts cash-subadditivity, showing that \( \mathbb{P}(S_T < S_0) = 0 \) must hold. \( \square \)

**Remark 3.8.35.** The above result shows that Corollary 3.8.30 fails if we drop the assumption of convexity. In fact, the assumption of conicity cannot be dropped either. Indeed, consider the acceptance set in \( L^1 \)
\[ \mathcal{A} = \{ X \in L^1 : \mathbb{E}[X] \geq \alpha \} \]
for a fixed \( \alpha \in \mathbb{R} \). Then, it is not difficult to check that \( \rho_{\mathcal{A},S} \) is cash-subadditive if and only if \( \mathbb{E}[S_T] \geq S_0 \). \( \square \)

In Cerreia-Vioglio, Maccheroni, Marinacci, Montrucchio [21], it is suggested that cash-subadditivity may arise when the reference bonds are illiquid. Hence, we conclude this section by briefly investigating the role of cash-subadditivity when the pricing rule \( \pi \) is not assumed to be linear.

The following result shows that, in the common situations, \( \rho_S \) can only be cash-subadditive if \( \pi \) is continuous, thus ruling out examples of illiquid markets where the pricing rule may present jumps at inception.

**Proposition 3.8.36.** Consider \( \mathcal{X} = L^p \) for \( p \in [1, \infty] \). Assume \( \mathcal{A} \) is closed, convex and has nonempty interior. Take \( \mathcal{M} = \mathbb{R} \) and assume \( \pi(\infty) = \infty \). Moreover, assume \( Z \) is linear and \( Z(1) \) is strictly positive. If \( \rho_S \) is cash-subadditive, then \( \pi \) is continuous.

**Proof.** Set \( S = (\pi(1), Z(1)) \) and note that, by Theorem 3.8.23, we have
\[ \rho_S(X) = \pi\left( \rho_{\mathcal{A},S}(X) \right) \]
for every \( X \in \mathcal{X} \). As a consequence of Theorem 3.3.23, the map \( \rho_{\mathcal{A},S} \) is finitely valued and continuous. Now, assume \( \rho_S \) is cash-subadditive but \( \pi \) is not left-continuous at \( \lambda_0 \in \mathbb{R} \). In this case, we find \( m > 0 \) such that
\[ m < \pi(\lambda_0) - \lim_{\lambda \uparrow \lambda_0} \pi(\lambda). \]
By $S$-additivity we always find $X \in \mathcal{X}$ satisfying $\rho_{A,S}(X) = \lambda_0 \pi(1)$. We claim that $\rho_{A,S}(X + m) < \lambda_0 \pi(1)$. Indeed, since $X + \lambda_0 S_T \in \text{bd} \ A$ we must have $X + \lambda_0 S_T + m \in \text{cl} \ A$ by the monotonicity of $\text{cl} \ A$. If $X + \lambda_0 S_T + m$ were a boundary point of $A$, we could find a nonzero functional $\psi \in \mathcal{X}'$ such that $\psi(Y) \geq \psi(X + \lambda_0 S_T + m)$ for all $Y \in \text{cl} \ A$ by Lemma 7.7 in Aliprantis, Border [4]. In particular, $\psi(X + \lambda_0 S_T) \geq \psi(X + \lambda_0 S_T + m)$ implying that $\psi(1_\Omega) \leq 0$. However, this would contradict the positivity of $\psi$ ensured by Proposition 2.3.4. As a result, $X + \lambda_0 S_T + m \in \text{int} \ A$ and therefore $\rho_{A,S}(X + m) < \lambda_0 \pi(1)$ by virtue of Proposition 3.2.7, proving the claim. However, this inequality contradicts the assumption of cash-subadditivity since

$$\rho_S(X) - \rho_S(X + m) = \pi(\lambda_0) - \pi\left(\frac{\rho_{A,S}(X + m)}{\pi(1)}\right) > m.$$ 

Similarly, we can prove that cash-subadditivity forces $\pi$ to be right-continuous, concluding the proof. 

The continuity requirement implicitly imposed by cash-subadditivity is sometimes quite strong, as described by the next example. This example shows that the sufficient condition stated in Example 2.2 in Cerreia-Vioglio, Maccheroni, Marinacci, Montrucchio [21] is, in fact, also necessary.

**Example 3.8.37.** In addition to the assumptions of the preceding result, suppose that $Z(1) = 1_\Omega$. In this case, $\rho_S$ is cash-subadditive only if $\pi$ is Lipschitz continuous (with constant 1), i.e. only if for every $\lambda, \xi \in \mathbb{R}$ we have

$$|\pi(\lambda) - \pi(\xi)| \leq |\lambda - \xi|.$$ 

Indeed, take $\lambda > \xi$. By cash-additivity we have $\rho_A(X) = \lambda \pi(1)$ for some $X \in \mathcal{X}$. As a result, $\rho_A(X + \lambda - \xi) = \xi \pi(1)$. Hence, cash-subadditivity implies that

$$\pi(\lambda) - \pi(\xi) = \rho_S(X) - \rho_S(X + \lambda - \xi) \leq \lambda - \xi,$$

where we used that

$$\rho_S(X) = \pi\left(\frac{\rho_A(X)}{\pi(1)}\right)$$

for every $X \in \mathcal{X}$ by Theorem 3.8.23. 

\[ \square \]
3.8 Beyond a single liquid asset

3.8.6 A note on quasiconvexity

In this section we briefly focus on another property of risk measures that has been recently investigated by several authors, namely quasiconvexity.

This property plays a cardinal role in the context of preference theory and their numerical representations as the minimal condition to require if one wants to model risk aversion and diversification benefits. For more details, we refer to the extensive study by Drapeau [33] and to the papers by Cerreia-Vioglio, Maccheroni, Marinacci, Montrucchio [21] and by Drapeau, Kupper [34]. In particular, in [21] the role of quasiconvexity has been justified in connection to cash-subadditivity and to the potential illiquidity of the reference bonds. Here, we provide an independent justification and show that quasiconvexity naturally appears, even in the absence of cash-subadditivity, when the reference asset is not liquidly traded.

We start by recalling the definition of quasiconvexity. The reference space \( \mathcal{X} \) is assumed to be a real topological linear space equipped with a convenient quasiorder structure.

**Definition 3.8.38.** A map \( f : \mathcal{X} \to \mathbb{R} \) is called **quasiconvex** whenever

\[
  f(X) \leq f(Y) \implies f(\lambda X + (1 - \lambda)Y) \leq f(Y) \quad \text{for all } \lambda \in (0, 1).
\]

It is easy to see that quasiconvexity is equivalent to \( \{ f \leq \lambda \} \) being convex for any \( \lambda \in \mathbb{R} \). Note that any convex map is automatically quasiconvex.

If some admissible strategy is liquid, there is no difference between quasiconvexity and convexity. Hence, quasiconvexity can be genuinely observed only in case of full illiquidity.

**Proposition 3.8.39.** Assume \( s \in \mathcal{M} \) is liquid. Then, \( \rho_S \) is quasiconvex if and only if \( \rho_S \) is convex.

**Proof.** Clearly, we only need to prove the “only if” statement. To this end, assume \( \rho_S \) is quasiconvex. Then, the monotone set \( A(\rho_S) = \{ \rho_S \leq 0 \} \) is convex. It follows from Theorem 3.8.18 that \( \rho_S \) is \( S \)-additive with respect to \( S = (\pi(s), Z(s)) \).

Hence, we can write \( \rho_S = \rho_{A(\rho_S), S} \), showing that \( \rho_S \) is convex. \( \square \)

The next proposition provides a general sufficient condition for quasiconvexity in the case of a single illiquid reference asset. Here, we understand that \( Z \) is **concave**
whenever
\[ Z(\alpha \lambda + (1 - \alpha)\xi) \geq \alpha Z(\lambda) + (1 - \alpha)Z(\xi) \quad \text{for all } \alpha \in (0, 1). \]

Moreover, we say that \( Z \) is increasing whenever
\[ \lambda < \xi \implies Z(\lambda) \leq Z(\xi). \]

**Proposition 3.8.40.** Assume \( \mathcal{X} \) is locally convex. Moreover, assume \( \mathcal{A} \) is closed and convex, \( \mathcal{M} \subseteq \mathbb{R} \) is closed and convex, \( \pi \) is increasing and \( \pi(\infty) = \infty \), and \( Z \) is concave and increasing. Then, \( \rho_S \) is quasiconvex.

**Proof.** In the setting of the product space \( \mathbb{R} \times \mathcal{X} \) equipped with its canonical induced structure, consider the monotone set
\[ \mathcal{C} = \{(\lambda, X) \in \mathbb{R} \times \mathcal{X}; \ X + Z(\lambda) \in \mathcal{A}\}. \]

Then, for every \( X \in \mathcal{X} \), it is easy to see that
\[ \rho_S(X) = \inf\{\pi(\lambda); \ \lambda \in \mathcal{M}, \ (X, 0) + \lambda(0, 1) \in \mathcal{C}\}. \quad (3.7) \]

The expression on the right-hand side defines a general risk measure on the space \( \mathcal{X} \times \mathbb{R} \). Note that \( \mathcal{C} \) is convex. Now, observe that, for any \( \psi \in \mathcal{X}' \), the composition of \( Z \) and \( \psi \) is a concave map from \( \mathbb{R} \) to itself, which is then continuous by a standard result in convex analysis. This implies that \( Z \) is continuous with respect to the topology \( \sigma(\mathcal{X}, \mathcal{X}') \) on the corresponding image space \( \mathcal{X} \). Since \( \mathcal{A} \) is closed and convex, it is also \( \sigma(\mathcal{X}, \mathcal{X}') \)-closed. Hence, \( \mathcal{C} \) is closed with respect to the corresponding weak product topology. In light of the representation (3.7) and by Theorem 3.8.23, it is not difficult to see that \( \rho_S \) can be expressed as the composition between the convex risk measure \( \rho_{\mathcal{C}, S} \), where \( S = (\pi(1), (1, 0)) \), and the modified pricing functional \( \pi_{\mathcal{M}} \). Since \( \pi_{\mathcal{M}} \) is increasing, it follows that \( \rho_S \) is quasiconvex.

### 3.8.7 A note on set-valued risk measures

In this final section we briefly focus on the class of risk measures introduced, under the name of *vector-valued* risk measures, by Jouini, Meddeb, Touzi [64] and later extended, under the name of *multidimensional* risk measures, by Kulikov.
[76]. Recently, starting with the publication by Hamel, Heyde [55] and by Hamel, Heyde, Rudloff [56], this type of risk measures has become the subject of active research under the label of set-valued risk measures. The objective of this section is to show how to exploit our approach to duality in order to trace a different route towards dual representations of set-valued risk measures, which arguably constitute the main mathematical results in the above literature. Notably, we will not rely on set-valued tools.

As articulated in [64], the purpose of set-valued risk measures is to extend the framework developed in Artzner, Delbaen, Eber, Heath [9] to multivariate positions in a context where “investors are in general not able to aggregate their portfolio because of liquidity problems and/or transaction costs between the different security markets”. The usual setting is the following. We fix a probability triple \((\Omega, \mathcal{F}, \mathbb{P})\) and denote by \(L^p_d\) the space of all \(d\)-dimensional random vectors with components in \(L^p\), for \(p \in [1, \infty]\). The space \(L^p_d\) becomes a topological linear space when endowed with the usual norm topology. Moreover, we assume \(L^p_d\) is equipped with the usual compatible quasiorder induced by

\[
(L^p_d)_+ = \{ X \in L^p_d ; \ X_i \geq_{\mathbb{P}} 0, \ \forall i = 1, \ldots, d \}.
\]

The elements of \(L^p_d\) are taken to represent random portfolios of \(d\) pre-specified assets, e.g. currencies, at maturity. In particular, each component of \(X \in L^p_d\) represent the random number of “physical” units of the corresponding asset.

We fix an acceptance set \(A \subset L^p_d\). The acceptability profile of a position in \(L^p_d\) can be altered by investing in deterministic portfolios represented by the elements of a linear space \(\mathcal{M} \subseteq \mathbb{R}^d\). The set \(\mathcal{M}\) is endowed with the relative topology and we set

\[
\mathcal{M}_+ = \mathcal{M} \cap (L^p_d)_+.
\]

We will always work under the standard assumption \(\mathcal{M}_+ \neq \{0\}\). Following [56], the definition below provides a unifying formulation for all set-valued risk measures considered in the above-mentioned literature. In the sequel, we denote by \(u\) the random vector in \(L^p_d\) which is constantly equal to \(u \in \mathbb{R}^d\).

**Definition 3.8.41.** The set-valued risk measure associated to \((A, \mathcal{M})\) is the correspondence \(\mathcal{R}_{A, \mathcal{M}} : \mathcal{X} \rightrightarrows \mathcal{M}\) defined by setting

\[
\mathcal{R}_{A, \mathcal{M}}(X) := \{ u \in \mathcal{M} ; \ X + u \in A \}.
\]
Hence, $\mathcal{R}_{\mathcal{A},\mathcal{M}}$ associates to any position those admissible deterministic portfolios that, when added to it, align the position with the pre-specified acceptable risk tolerance. The link with our general risk measures is clear. In particular, the space $\mathcal{M}$ plays the role of the set of admissible strategies. However, differently from our framework, the original idea in [64] was to define risk measures before any pricing rule, or cost functional, is applied.\footnote{The problem of how to choose some “optimal” element from the set $\mathcal{R}_{\mathcal{A},\mathcal{M}}(X)$ – for instance by means of a pricing scalarization – was discussed, but not further pursued, in [64]. The authors of [55] proposed a linear scalarization. We also refer to Example 3.8.7.}

We turn to dual representations of set-valued risk measures under the assumption that $\mathcal{A}$ is closed and convex. This is the standing assumption in the above literature. In the sequel, the \textit{kernel} of a functional $\pi \in \mathcal{M}'$ will be denoted by $\ker \pi$, i.e.

$$\ker \pi := \{ u \in \mathcal{M} ; \pi(u) = 0 \}.$$ 

Furthermore, the \textit{annihilator} of a subset $\mathcal{C} \subseteq L^p_d$ will be denoted by $\mathcal{C}^\perp$, i.e.

$$\mathcal{C}^\perp := \{ \psi \in (L^p_d)' ; \psi(X) = 0, \ \forall X \in \mathcal{C} \}.$$ 

The following lemma records the key observation we will exploit to deriving dual representations in the set-valued framework.

\textbf{Lemma 3.8.42.} Assume $\mathcal{A}$ is closed and convex. Then, for any $\pi \in \mathcal{M}'$ we have

$$\mathcal{B}(\mathcal{A} + \ker \pi) = \mathcal{B}(\mathcal{A}) \cap (\ker \pi)^\perp$$

and

$$\sigma_{\mathcal{A} + \ker \pi}(\psi) = \begin{cases} \sigma_{\mathcal{A}}(\psi) & \text{if } \psi \in (\ker \pi)^\perp \\ -\infty & \text{otherwise} \end{cases}$$

for all $\psi \in (L^p_d)'$. Moreover,

$$\mathcal{A} = \bigcap_{\pi \in \mathcal{M}'_+} \text{cl}(\mathcal{A} + \ker \pi).$$

\textbf{Proof.} We only prove the last assertion since the remaining ones easily follow from the fact that $\ker \pi$ is a linear subspace of $L^p_d$. The inclusion “$\subseteq$” is clear, hence we focus on the converse inclusion. To this end, assume $X \in \text{cl}(\mathcal{A} + \ker \pi)$ for any
\( \pi \in M'_+ \) and take \( \psi \in B(A) \). In light of Theorem 2.3.6, to conclude the proof we have to show that \( \psi(X) \geq \sigma_A(\psi) \). Now, consider the functional \( \pi_\psi \in M' \) defined by

\[
\pi_\psi(u) = \psi(u).
\]

Since \( \psi \) is positive by Proposition 2.3.4, it follows that \( \pi_\psi \in M'_+ \). Moreover, note that \( \psi \in (\ker \pi_\psi)^\perp \). As a result, \( \psi \) belongs to the barrier cone of \( \text{cl}(A + \ker \pi_\psi) \).

Since \( X \in \text{cl}(A + \ker \pi_\psi) \), we conclude that

\[
\psi(X) \geq \sigma_{A + \ker \pi_\psi} = \sigma_A(\psi)
\]

by Theorem 2.3.6, where we have used the explicit expression of the support function stated above.

The next theorem provides a general formulation for all dual representations of set-valued risk measures considered in the above literature. To this end, for any functional \( \pi \in M' \) it is useful to introduce the set

\[
E_M(\pi) := \{ \psi \in (L^p_d)' \mid \psi(u) = \pi(u), \forall u \in M \}
\]

consisting of all linear, continuous extensions of \( \pi \) on the whole space \( L^p_d \).

**Theorem 3.8.43.** Assume \( A \) is closed and convex and \( R_{A,M} \) never attains the value \( M \). Then, for every \( X \in L^p_d \)

\[
R_{A,M}(X) = \bigcap_{\pi \in M'_+} \bigcap_{\psi \in B(A) \cap E_M(\pi)} \{ u \in M ; \pi(u) \geq \sigma_A(\psi) - \psi(X) \}.
\]

**Proof.** By combining Theorem 2.3.6 with the preceding lemma, it is straightforward to verify that

\[
R_{A,M}(X) = \bigcap_{\pi \in M'_+} \bigcap_{\psi \in B(A) \cap (\ker \pi)^\perp} \{ u \in M ; \psi(u) \geq \sigma_A(\psi) - \psi(X) \} \quad (3.8)
\]

for every \( X \in L^p_d \). Thus, the intersection “\( \subseteq \)” is immediately seen to hold. To prove the converse inclusion, note first that we can always find a nonzero functional \( \pi^* \in M'_+ \) satisfying

\[
B(A) \cap E_M(\pi^*) \neq \emptyset,
\]
for otherwise $\mathcal{B}(A) \cap (\ker \pi)^\perp = \mathcal{B}(A) \cap \mathcal{M}^\perp$ would hold for all $\pi \in \mathcal{M}_1'$ and (3.8) would imply that $\mathcal{R}_{A,M}(X) = \mathcal{M}$ for any $X \in A$ by Theorem 2.3.6. Now, fix $X \in L^p_d$ and assume $u \in \mathcal{M}$ is such that

$$\pi(u) \geq \sigma_A(\varphi) - \varphi(X)$$

for all $\pi \in \mathcal{M}_1'$ and $\varphi \in \mathcal{B}(A) \cap \mathcal{E}_M(\pi)$. In particular, this is valid for $\pi = \pi^*$. To conclude the proof, we need to prove that $u$ belongs to the right-hand side of (3.8). To this end, take $\pi \in \mathcal{M}_1'$ and $\psi \in \mathcal{B}(A) \cap (\ker \pi)^\perp$. Clearly, the only interesting case to verify is $\psi \notin \mathcal{E}_M(\pi)$ or, equivalently, $\psi \in \mathcal{M}^\perp$. In this case, the following two situations are possible: either $\psi(X) \geq \sigma_A(\psi)$, in which case we can discard $\psi$ from the intersection in (3.8), or

$$\sigma_A(\psi) > \psi(X), \tag{3.9}$$

in which case $\mathcal{R}_{A,M}(X) = \emptyset$. To conclude, it is therefore enough to show that, in the latter circumstance, we have

$$\bigcap_{\varphi \in \mathcal{B}(A) \cap \mathcal{E}_M(\pi^*)} \{u \in \mathcal{M}; \pi^*(u) \geq \sigma_A(\varphi) - \varphi(X)\} = \emptyset. \tag{3.10}$$

To prove this, take $\varphi \in \mathcal{B}(A) \cap \mathcal{E}_M(\pi^*)$ and set $\varphi_n = \varphi + n\psi$ for each $n \in \mathbb{N}$. Since $\psi \in \mathcal{B}(A) \cap \mathcal{M}^\perp$, it is not difficult to verify that every functional $\varphi_n$ also belongs to $\mathcal{B}(A) \cap \mathcal{E}_M(\pi^*)$. Then, using that $\sigma_A(\psi) > \psi(X)$ by (3.9), the final claim (3.10) immediately follows from

$$\sup_{n \in \mathbb{N}} \{\sigma_A(\varphi_n) - \varphi_n(X)\} \geq \sigma_A(\varphi) - \varphi(X) + \sup_{n \in \mathbb{N}} \{n(\sigma_A(\psi) - \psi(X))\} = \infty.$$ 

Remark 3.8.44. In the preceding theorem we have assumed that $\mathcal{R}_{A,M}$ cannot take the value $\mathcal{M}$. Indeed, under the above assumptions, it is easy to prove that $\mathcal{R}_{A,M}$ could only take the values $\emptyset$ and $\mathcal{M}$ whenever $\mathcal{R}_{A,M}(X) = \mathcal{M}$ for some $X \in \mathcal{X}$. To see this, take $Y \in L^p_d$ and assume that $\mathcal{R}_{A,M}(Y)$ is nonempty so that $Y + u \in A$ for a given $u \in \mathcal{M}$. Note that, for any $v \in \mathcal{M}$ and $\lambda \in (0,1)$, we have

$$\lambda \left(X + \frac{1}{\lambda}(v - u)\right) + (1 - \lambda)(Y + u) \in A.$$

This follows, in particular, since $\mathcal{R}_{A,M}(X) = \mathcal{M}$. Hence, letting $\lambda \to 0$, we obtain $Y + v \in A$ and this implies $\mathcal{R}_{A,M}(Y) = \mathcal{M}$.
Appendix

The purpose of this chapter is to introduce basic notation and terminology and collect some general background results, mostly from the theory of ordered topological linear spaces. We assume the reader is familiar with the basic concepts of this theory and refer to the book by Aliprantis, Border [4] for the corresponding details. Other useful references are the monographs by Peressini [84], by Jameson [62] and by Schaefer [90]. We also refer to Dunford, Schwartz [36] for more details about the function spaces considered in the last part.

Sets

First of all, we set our axiomatic horizon. We assume to work under the standard axioms of Zermelo-Fraenkel, including the Axiom of Choice. For more details about these fundamental assumptions, we refer to the book by Devlin [32]. The empty set will be denoted by $\emptyset$. Consider a nonempty set $\mathcal{X}$. For given subsets $A$ and $B$ of $\mathcal{X}$, we write $A \subseteq B$ if $A$ is contained in $B$ and $A \subset B$ if $A$ is strictly contained in $B$, i.e. if $A$ is contained in $B$ but does not coincide with it. The difference between $A$ and $B$ is the set

$$A \setminus B := \{ X \in A ; X \notin B \}.$$ 

The complement of a subset $A$ in $\mathcal{X}$ is denoted by $A^c$, i.e.

$$A^c := \mathcal{X} \setminus A.$$ 

The set of natural numbers – excluding zero – is denoted by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$. For $d \in \mathbb{N}$, the set of all $d$-tuples of real numbers is denoted by $\mathbb{R}^d$. 

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In particular, we identify \( \mathbb{R}^1 \) with \( \mathbb{R} \). We will always consider \( \mathbb{R}^d \) equipped with its canonical (order, linear, topological, measurable) structure. For any \( i = 1, \ldots, d \), we denote by \( e_i \) the element of \( \mathbb{R}^d \) having the \( i \)th component equal to 1 and all others equal to zero.

**The extended real line**

In the sequel, we will often encounter real-valued functions possibly taking “non-finite” values. For this reason, it is important to specify the structure of the corresponding “extended” real line. We refer to Hamel, Schrage [57] for more details about the following axiomatics.

We denote by \( \overline{\mathbb{R}} \) the set \( \mathbb{R} \) enlarged with two additional distinct elements denoted by \( \infty \) and \( -\infty \). We extend the canonical order on \( \mathbb{R} \) to a total order satisfying

\[-\infty < \lambda < \infty \quad \text{for all } \lambda \in \mathbb{R}.
\]

In particular, we have

\[
\inf \overline{\mathbb{R}} = \inf \mathbb{R} = -\infty \quad \text{and} \quad \sup \overline{\mathbb{R}} = \sup \mathbb{R} = \infty.
\]

Moreover, we set \( \inf \emptyset := \infty \) and \( \sup \emptyset := -\infty \). We extend the usual addition by setting for every \( \alpha, \beta \in \overline{\mathbb{R}} \)

\[
\alpha + \beta := \inf \{ \lambda + \xi ; \lambda, \xi \in \mathbb{R}, \lambda \geq \alpha, \xi \geq \beta \}.
\]

In particular, we have

\[
\infty + (-\infty) = \infty \quad \text{and} \quad -\infty + \infty = \infty.
\]

It is easy to verify that, under this extended addition, \( \overline{\mathbb{R}} \) becomes a commutative semigroup with neutral element 0. However, \( \overline{\mathbb{R}} \) is clearly not a group. Finally, we also extend the usual multiplication by setting

\[
\lambda \cdot \infty := \begin{cases} 
\infty & \text{if } \lambda > 0 \\
-\infty & \text{if } \lambda < 0
\end{cases}
\quad \text{and} \quad \lambda \cdot (-\infty) := \begin{cases}
-\infty & \text{if } \lambda > 0 \\
\infty & \text{if } \lambda < 0.
\end{cases}
\]

Moreover, we set \( 0 \cdot \infty := 0 \) and \( 0 \cdot (-\infty) := 0 \). In particular, note that

\[-(\infty + (-\infty)) \neq -\infty + \infty \quad \text{and} \quad -(\infty + \infty) \neq \infty + (-\infty).
\]
We extend the usual topological structure by defining a neighborhood of $\infty$, respectively $-\infty$, to be any set of the form
\[ \{ \alpha \in \mathbb{R} ; \: \alpha \geq \lambda \} \quad \text{respectively} \quad \{ \alpha \in \mathbb{R} ; \: \alpha \leq \lambda \}, \]
for $\lambda \in \mathbb{R}$. The measurable structure of $\overline{\mathbb{R}}$ is understood with respect to the corresponding Borel $\sigma$-algebra.

**Algebra**

Let $\mathcal{X}$ be a (nontrivial) linear space over $\mathbb{R}$. If $\mathcal{A}$ and $\mathcal{B}$ are subsets of $\mathcal{X}$ and $\lambda \in \mathbb{R}$, we define
\[ \mathcal{A} + \mathcal{B} := \{ X + Y ; \: X \in \mathcal{A}, \: Y \in \mathcal{B} \}, \]
\[ \lambda \mathcal{A} := \{ \lambda X ; \: X \in \mathcal{A} \}. \]
Moreover, we set $\mathcal{A} - \mathcal{B} := \mathcal{A} + (-\mathcal{B})$. Finally, for $X \in \mathcal{X}$ we will simply write $\mathcal{A} + X := \mathcal{A} + \{ X \}$ and $\mathcal{A} - X := \mathcal{A} - \{ X \}$. The linear space generated by a set $\mathcal{A} \subseteq \mathcal{X}$ is denoted by $\text{span}(\mathcal{A})$. For any $X \in \mathcal{X}$, we simply write $\text{span}(X)$ instead of $\text{span}(%{X})$.

A set $\mathcal{A} \subseteq \mathcal{X}$ is said to be **convex** if
\[ \lambda \mathcal{A} + (1 - \lambda) \mathcal{A} \subseteq \mathcal{A} \quad \text{for any} \: \lambda \in (0, 1). \]
Moreover, $\mathcal{A}$ is said to be a **cone**, or a **conic** set, if
\[ \lambda \mathcal{A} \subseteq \mathcal{A} \quad \text{for any} \: \lambda > 0. \]

Note that a convex cone $\mathcal{A}$ is **closed under addition**, i.e. $\mathcal{A} + \mathcal{A} \subseteq \mathcal{A}$. The **convex hull** of $\mathcal{A}$, denoted $\text{co} \mathcal{A}$, is the smallest convex set containing $\mathcal{A}$ or, equivalently, the intersection of all convex sets containing $\mathcal{A}$. It is easy to show that
\[ \text{co} \mathcal{A} = \left\{ \sum_{i=1}^{n} \lambda_i X_i ; \: n \in \mathbb{N}, \: \lambda_i \in [0, 1], \: \sum_{i=1}^{n} \lambda_i = 1, \: X_i \in \mathcal{A} \right\}. \]

Similarly, the **conic hull** of $\mathcal{A}$, denoted $\text{cone} \mathcal{A}$, is the smallest cone containing $\mathcal{A}$ or, equivalently, the intersection of all conic sets containing $\mathcal{A}$. Note that
\[ \text{cone} \mathcal{A} = \{ \lambda X ; \: \lambda > 0, \: X \in \mathcal{A} \}. \]
An element \( U \in \mathcal{A} \) is called a **core point** of \( \mathcal{A} \) if for any \( X \in \mathcal{X} \) there exists \( \varepsilon > 0 \) such that \( U + \lambda X \in \mathcal{A} \) whenever \( \lambda \in (-\varepsilon, \varepsilon) \). Note that, since \( X \) is arbitrary, it is sufficient to check the last condition for any \( \lambda \in (0, \varepsilon) \). The set of all core points of \( \mathcal{A} \) is called the **core** of \( \mathcal{A} \) and is denoted by \( \text{core} \mathcal{A} \), i.e.

\[
\text{core} \mathcal{A} := \{ U \in \mathcal{A} ; \forall X \in \mathcal{X}, \exists \varepsilon > 0 : U + \lambda X \in \mathcal{A}, \forall \lambda \in (-\varepsilon, \varepsilon) \}.
\]

If \( \mathcal{A} \) is convex, then it is easy to show that

\[
\text{core} \mathcal{A} = \{ U \in \mathcal{A} ; \forall X \in \mathcal{X}, \exists \lambda > 0 : U + \lambda X \in \mathcal{A} \}.
\]

The following result constitutes the basic separation criterion for convex sets and is a refinement of Theorem 5.61 in Aliprantis, Border [4].

**Theorem 0.45.** Assume \( \mathcal{A} \) and \( \mathcal{B} \) are disjoint, convex subsets of \( \mathcal{X} \) and assume \( \text{core} \mathcal{A} \) is nonempty. Then, there exists a nonzero linear functional \( \psi : \mathcal{X} \to \mathbb{R} \) such that \( \psi(X) < \psi(Y) \) for all \( X \in \text{core} \mathcal{A} \) and \( Y \in \mathcal{B} \).

**Order**

Assume \( \mathcal{X} \) is a (nontrivial) linear space over the field of real numbers \( \mathbb{R} \). A binary relation \( \leq \) on \( \mathcal{X} \) is called a **quasiorder** if it is reflexive, i.e. \( X \leq X \) for all \( X \in \mathcal{X} \), and transitive, i.e.

\[
X \leq Y, \ Y \leq Z \implies X \leq Z.
\]

A quasiorder is said to be an **order** if it is antisymmetric, i.e.

\[
X \leq Y, \ Y \leq X \implies X = Y.
\]

In this case, we say that \( \mathcal{X} \) is a **quasiordered**, respectively an **ordered**, space. A quasiorder \( \leq \) is called **compatible** (with the linear structure of \( \mathcal{X} \)) if

\[
X \leq Y \implies X + Z \leq Y + Z \quad \text{for any} \ Z \in \mathcal{X}
\]

and

\[
X \leq Y \implies \lambda X \leq \lambda Y \quad \text{for any} \ \lambda > 0.
\]

The **positive cone** induced by a quasiorder \( \leq \) is defined as

\[
\mathcal{X}_+ := \{ X \in \mathcal{X} ; \ X \geq 0 \}.
\]
Note that $\mathcal{X}_+$ is a convex cone containing 0. Every element in $\mathcal{X}_+$ is said to be positive. The negative cone is just the negative of $\mathcal{X}_+$, i.e.

$$\mathcal{X}_- := \{X \in \mathcal{X}; X \leq 0\}.$$ 

Any element in $\mathcal{X}_-$ is said to be negative. If $\leq$ is additionally assumed to be antisymmetric, then $\mathcal{X}_+$ is pointed, i.e.

$$\mathcal{X}_+ \cap \mathcal{X}_- = \{0\}.$$ 

Conversely, if $\mathcal{K} \subseteq \mathcal{X}$ is a convex cone containing 0, the binary relation $\leq_{\mathcal{K}}$ defined by

$$X \leq_{\mathcal{K}} Y \iff Y - X \in \mathcal{K}$$

is a compatible quasiorder on $\mathcal{X}$ with positive cone $\mathcal{K}$. If $\mathcal{K}$ is additionally assumed to be pointed, then $\leq_{\mathcal{K}}$ is also an order.

Let $\leq$ be a compatible quasiorder on $\mathcal{X}$. An element $U \in \mathcal{X}$ is called an order unit if, for any $X \in \mathcal{X}$, there exists $\lambda > 0$ such that $\lambda U \geq X$. It is easy to show that $U$ is an order unit if and only if $U \in \text{core}\ X_+$. 

Let $\leq$ be a compatible order on $\mathcal{X}$. The infimum of two elements $X$ and $Y$ in $\mathcal{X}$ is the unique element $X \land Y$ in $\mathcal{X}$ such that $X \land Y \leq X$ and $X \land Y \leq Y$, and

$$Z \leq X, Z \leq Y \implies Z \leq X \land Y.$$ 

Similarly, the supremum of $X$ and $Y$ is the unique element $X \lor Y$ in $\mathcal{X}$ such that $X \lor Y \geq X$ and $X \lor Y \geq Y$, and

$$Z \geq X, Z \geq Y \implies Z \geq X \lor Y.$$ 

Note that $(-X) \land (-Y) = -(X \lor Y)$ for all $X, Y \in \mathcal{X}$. If $X \land Y$ (hence $X \lor Y$) exists for any choice of $X$ and $Y$ in $\mathcal{X}$, we say that $\mathcal{X}$ is a Riesz space, or a linear lattice. In this case, the positive part, the negative part, and the absolute part of $X \in \mathcal{X}$ are defined as

$$X^+ := X \lor 0, \quad X^- := -(X \land 0), \quad |X| := X^+ + X^-,$$

respectively. Note that $X = X^+ - X^-$ for any $X \in \mathcal{X}$. 
Topology

Assume $\mathcal{X}$ is a (nontrivial) topological linear space over $\mathbb{R}$ with topology $\tau$. The interior and the closure of a set $A \subseteq \mathcal{X}$ are denoted by $\text{int} A$ and $\text{cl} A$, respectively. Moreover, the boundary of $A$ is denoted by $\text{bd} A$.

Consider another topology $\sigma$ on $\mathcal{X}$. We say that $\sigma$ is coarser than $\tau$ whenever $\sigma \subseteq \tau$. In this case, we also say that $\tau$ is finer than $\sigma$. Given a family $\mathcal{F}$ of maps from $\mathcal{X}$ to $\mathbb{R}$, we denote by $\sigma(\mathcal{X}, \mathcal{F})$ the coarsest topology on $\mathcal{X}$ making every map in $\mathcal{F}$ continuous.

We denote by $\mathcal{X}'$ the topological dual of $\mathcal{X}$, i.e., the set of all linear continuous functionals $\psi : \mathcal{X} \to \mathbb{R}$. Note that $\mathcal{X}'$ is a real linear space with respect to the canonical pointwise operations. The topology $\sigma(\mathcal{X}, \mathcal{X}')$ is sometimes called the weak topology on $\mathcal{X}$. If $\mathcal{X}'$ is canonically identified with a set $\mathcal{Y}$, we will also write $\sigma(\mathcal{X}, \mathcal{Y})$ in place of $\sigma(\mathcal{X}, \mathcal{X}')$.

The next theorem, sometimes referred to as the Hahn-Banach separation, records the basic topological separation result we will need in the sequel. We will typically use it without explicit reference. The reader may consult, respectively, Theorem 5.67 and Theorem 5.79 in [4].

**Theorem 0.46.** Assume $A$ and $B$ are disjoint, convex subsets of $\mathcal{X}$. The following statements hold:

$(i)$ If $\text{int} A$ is nonempty, there exists a nonzero $\psi \in \mathcal{X}'$ such that $\psi(X) < \psi(Y)$ for all $X \in \text{int} A$ and $Y \in B$;

$(ii)$ Assume $\mathcal{X}$ is locally convex. If $A$ is closed and $B$ is compact, there exists a nonzero $\psi \in \mathcal{X}'$ such that

$$\sup_{X \in A} \psi(X) < \inf_{Y \in B} \psi(Y).$$

Assume $\mathcal{X}$ is equipped with a compatible quasiorder $\leq$ with positive cone $\mathcal{X}_+$. The dual $\mathcal{X}'$ can be naturally equipped with the compatible quasiorder induced by the convex cone

$$\mathcal{X}'_+ := \{ \psi \in \mathcal{X}' ; \; \psi(X) \geq 0, \; \forall X \in \mathcal{X}_+ \}.$$
Appendix

Any functional in $\mathcal{X}'_+$ is said to be positive. We say that an element $U \in \mathcal{X}_+$ is strictly positive if

$$\psi \in \mathcal{X}'_+ \setminus \{0\} \implies \psi(U) > 0.$$ 

Similarly, we say that a functional $\psi \in \mathcal{X}'_+$ is strictly positive if

$$X \in \mathcal{X}_+ \setminus \{0\} \implies \psi(X) > 0.$$ 

We say that $\mathcal{X}$ is a topological Riesz space if it is a Riesz space with respect to a given order $\leq$. In this case, $\mathcal{X}$ is said to be locally solid when there exists a neighborhood system consisting of solid sets, i.e. neighborhoods $U$ such that

$$|X| \leq |Y|, \ Y \in U \implies X \in U.$$ 

A topological Riesz space $\mathcal{X}$ is called a Fréchet lattice if it is locally solid and completely metrizable. A Riesz space is called a Banach lattice if it is equipped with a complete norm $\|\cdot\|$ such that

$$|X| \leq |Y| \implies \|X\| \leq \|Y\|.$$ 

Clearly, every Banach lattice is also a Fréchet lattice.

Assume $\mathcal{X}$ is a topological Riesz space. An element $U \in \mathcal{X}_+$ is called a weak topological unit if for every $X \in \mathcal{X}_+$ we have $X \wedge nU \to X$ as $n \to \infty$. The next result extends Theorem 6.3 in [91] beyond the normed space setting and establishes the link between weak topological units and strictly positive elements. The proof can be found in Farkas, Koch-Medina, Munari [42].

**Proposition 0.47.** Let $\mathcal{X}$ be a topological Riesz space. Then, every weak topological unit is strictly positive. If $\mathcal{X}$ is also locally convex and locally solid, the converse is true as well.

**Mappings**

Assume $\mathcal{X}$ is a (nontrivial) real topological linear space over $\mathbb{R}$ equipped with a compatible quasiorder with positive cone $\mathcal{X}_+$. Consider a map $f : \mathcal{X} \to \mathbb{R}$. If $f$ takes only values in $\mathbb{R}$, we say that $f$ is finitely valued, or simply finite. Otherwise, we say that $f$ takes nonfinite values. The domain (of finiteness) of $f$ is defined as

$$\text{dom}(f) := \{X \in \mathcal{X} ; f(X) \in \mathbb{R}\}.$$
If \( f \) does not attain the value \(-\infty\), the set \( \text{dom}(f) \) coincides with the usual effective domain from convex analysis. The epigraph of \( f \) is the set

\[
\text{epi}(f) := \{(X, \lambda) \in \mathcal{X} \times \mathbb{R} ; \ f(X) \leq \lambda \}.
\]

When no confusion on the underlying space is possible, we write for \( \lambda \in \mathbb{R} \)

\[
\{f \leq \lambda\} := \{X \in \mathcal{X} ; \ f(X) \leq \lambda\}.
\]

In a similar way we define other types of level sets. In case of two maps \( f_1 : \mathcal{X} \rightarrow \overline{\mathbb{R}} \) and \( f_2 : \mathcal{X} \rightarrow \overline{\mathbb{R}} \), we write \( f_1 \leq f_2 \) whenever \( f_1(X) \leq f_2(X) \) for all \( X \in \mathcal{X} \), and similarly for other types of relations on the image space.

The map \( f \) is called convex, respectively conic or positively homogeneous, if \( \text{epi}(f) \) is convex, respectively conic. The negative of a convex map is called concave. The next result provides a standard characterization of convex and conic maps.

**Proposition 0.48.** For a map \( f : \mathcal{X} \rightarrow \overline{\mathbb{R}} \) the following statements hold:

(i) \( f \) is convex if and only if for every \( X, Y \in \mathcal{X} \)

\[
f(\lambda X + (1 - \lambda)Y) \leq \lambda f(X) + (1 - \lambda)f(Y) \quad \text{for any } \lambda \in (0, 1);
\]

(ii) \( f \) is conic if and only if for every \( X \in \mathcal{X} \)

\[
f(\lambda X) = \lambda f(X) \quad \text{for any } \lambda > 0.
\]

The convex hull of \( f \) is the unique convex map \( \text{co} f : \mathcal{X} \rightarrow \overline{\mathbb{R}} \) such that \( \text{co} f \leq \rho \) and, for any convex map \( g : \mathcal{X} \rightarrow \overline{\mathbb{R}} \),

\[
g \leq f \quad \implies \quad g \leq \text{co} f.
\]

Similarly, the conic hull of \( f \) is the unique conic map \( \text{cone} f : \mathcal{X} \rightarrow \overline{\mathbb{R}} \) such that \( \text{cone} f \leq f \) and, for any conic map \( g : \mathcal{X} \rightarrow \overline{\mathbb{R}} \),

\[
g \leq f \quad \implies \quad g \leq \text{cone} f.
\]

Moreover, \( f \) is said to be subadditive if \( \text{epi}(f) \) is closed under addition. Clearly, every map \( f \) which is convex and conic is also subadditive. The negative of a subadditive map is called superadditive.
Proposition 0.49. A map $f : \mathcal{X} \to \mathbb{R}$ is subadditive if and only if for every $X,Y \in \mathcal{X}$

$$f(X + Y) \leq f(X) + f(Y).$$

The convex conjugate of $f$ is the convex map $f^* : \mathcal{X}' \to \mathbb{R}$ defined by

$$f^*(\psi) := \sup_{X \in \mathcal{X}} \{\psi(X) - f(X)\}.$$ 

Similarly, the concave conjugate of $f$ is the concave map $f_* : \mathcal{X}' \to \mathbb{R}$ defined by

$$f_*(\psi) := \inf_{X \in \mathcal{X}} \{\psi(X) - f(X)\}.$$ 

In the sequel, we will use the following continuity criteria for convex maps. The first result is part of Theorem 5.43 in [4].

Theorem 0.50. Assume $f : \mathcal{X} \to \mathbb{R}$ is convex and finite on some open, convex subset $A \subseteq \mathcal{X}$. Then, $f$ is continuous on $A$ if and only if it is bounded from above on a subset of $A$ with nonempty interior.

The second criterion is a general automatic continuity result obtained by Borwein [17]. Here, we say that $f$ is increasing if

$$X \leq Y \implies f(X) \leq f(Y).$$

The negative of an increasing function is called decreasing.

Theorem 0.51. Assume $f : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ is convex and increasing (or decreasing) and one of the following conditions is satisfied:

(i) $\mathcal{X}_+$ has nonempty interior;

(ii) $\mathcal{X}$ is completely metrizable, $\mathcal{X}_+$ is closed and $\mathcal{X} = \mathcal{X}_+ - \mathcal{X}_+$.

Then, $f$ is continuous on the core of $\text{dom}(f)$.

We say that $f$ is lower semicontinuous if $\text{epi}(f)$ is closed. The negative of a lower semicontinuous map is called upper semicontinuous. The following result is also standard.

Proposition 0.52. A map $f : \mathcal{X} \to \mathbb{R}$ is lower semicontinuous if and only if $\{f \leq \lambda\}$ is closed for any $\lambda \in \mathbb{R}$. 
Now take $X \in \mathcal{X}$. We say that $f$ is lower semicontinuous at $X$ if for every $\varepsilon > 0$ there exists a neighborhood $U$ of $X$ such that

$$Y \in U \implies f(Y) \geq f(X) - \varepsilon.$$ 

Similarly, $f$ is upper semicontinuous at $X$ if $-f$ is lower semicontinuous at $X$. It is easy to prove that lower semicontinuity on $\mathcal{X}$ is equivalent to lower semicontinuity at each point in $\mathcal{X}$. Moreover, $f$ is continuous at $X$ if and only if it is both lower and upper semicontinuous at $X$.

The lower semicontinuous hull of $f$ is the unique lower semicontinuous map $\text{cl} \ f : \mathcal{X} \to \overline{\mathbb{R}}$ such that $\text{cl} \ f \leq f$ and, for any lower semicontinuous map $g : \mathcal{X} \to \overline{\mathbb{R}}$,

$$g \leq f \implies g \leq \text{cl} \ f.$$

A similar definition, with a reverse inequality, holds for the upper semicontinuous hull of $f$, denoted by $\text{int} \ f$.

**Standard spaces**

In this section we briefly present the standard examples of function spaces we will encounter in the sequel.

**Spaces of bounded functions**

Let $(\Omega, \mathcal{F})$ be a given measurable space. We denote by $\mathcal{L}^0$ the real linear space of all measurable functions $X : \Omega \to \mathbb{R}$. The indicator function of a set $A \in \mathcal{F}$ is the measurable function $1_A$ defined by

$$1_A(\omega) := \begin{cases} 
1 & \text{if } \omega \in A, \\
0 & \text{if } \omega \notin A.
\end{cases}$$

In particular, $1_\Omega$ is the function constantly equal to 1. Sometimes, for any $\lambda \in \mathbb{R}$, we simply write $\lambda$ instead of $\lambda 1_\Omega$. The subset of all bounded, measurable functions is denoted by $\mathcal{L}^\infty$. This set becomes a Banach lattice when equipped with the norm

$$\|X\|_\infty := \sup_{\omega \in \Omega} |X(\omega)|.$$
and with the order induced by the positive cone

\[ \mathcal{L}_+^\infty := \{ X \in \mathcal{L}^\infty ; \, X(\omega) \geq 0, \, \forall \omega \in \Omega \}. \]

The dual of \( \mathcal{L}^\infty \) can be identified with the set \( ba \) of all finitely additive set functions \( \mu : \mathcal{F} \to \mathbb{R} \) that have bounded variation. For any \( X \in \mathcal{L}^\infty \) and \( \mu \in ba \), the corresponding pairing is given by

\[ \langle \mu, X \rangle = \int_{\Omega} X d\mu. \]

**Spaces of integrable functions**

Let \( (\Omega, \mathcal{F}, P) \) be a given probability space. The two cases which are most relevant for applications are those of a finite probability space and of a nonatomic probability space. Recall that \( (\Omega, \mathcal{F}, P) \) is said to be **nonatomic** if it contains no set \( A \in \mathcal{F} \) with \( P(A) > 0 \) such that either \( P(B) = 0 \) or \( P(B) = P(A) \) holds whenever \( B \) is a measurable subset of \( A \). In this case, the space supports random variables that are uniformly distributed, hence random variables having any prescribed distribution.

The quotient space of \( \mathcal{L}^0 \) with respect to the equivalence relation identifying any two elements of \( \mathcal{L}^0 \) that coincide almost surely under \( P \) is denoted by \( L^0 \). As usual, we will not distinguish between equivalence classes in \( L^0 \) and their representative elements in \( \mathcal{L}^0 \). The standard **almost-sure order** under \( P \) is denoted by \( \leq_P \), i.e.

\[ X \leq_P Y \iff P(X \leq Y) = 1. \]

The space \( L^0 \) becomes a Fréchet lattice when equipped with the metric

\[ d_0(X, Y) := \mathbb{E} \left[ \frac{|X - Y|}{1 + |X - Y|} \right] \]

and the order induced by the positive cone

\[ L^0_+ := \{ X \in L^0 ; \, X \geq_P 0 \}. \]

Here, we have denoted by \( \mathbb{E}[\cdot] \) the expectation with respect to \( P \). We always suppress the reference to the underlying probability measure when there is no ambiguity. Recall that convergence with respect to \( d_0 \) is equivalent to convergence in probability.
We denote by $L^\infty$ the linear subspace of $L^0$ consisting of all functions that are bounded almost surely. The space $L^\infty$ is a Banach lattice under the norm

$$\|X\|_\infty := \inf\{\lambda > 0 ; \, \mathbb{P}(|X| \leq \lambda) = 1\}.$$ 

The dual of $L^\infty$ can be identified with the set $ba(\mathbb{P})$ of all finitely additive set functions $\mu : \mathcal{F} \to \mathbb{R}$ that have bounded variation and that are absolutely continuous with respect to $\mathbb{P}$. For any $X \in L^\infty$ and $\mu \in ba(\mathbb{P})$, the pairing is given by

$$\langle \mu, X \rangle = \int_\Omega X \, d\mu.$$ 

Moreover, for any $p \in (0, \infty)$ we denote by $L^p$ the linear subspace of $L^0$ consisting of all functions satisfying $\mathbb{E}[|X|^p] < \infty$. If $p \in [1, \infty)$, the space $L^p$ is a Banach lattice under the norm

$$\|X\|_p := \mathbb{E}[|X|^p]^{\frac{1}{p}}$$

while, if $p \in (0, 1)$, the space $L^p$ is a Fréchet lattice with respect to the metric

$$d_p(X, Y) := \mathbb{E}[|X - Y|^p]^{\frac{1}{p}}.$$ 

Recall that $L^\infty$ is dense in any space $L^p$, $p \in [0, \infty)$. If $p \in [0, 1)$, the dual of $L^p$ is trivial, i.e. $(L^p)' = \{0\}$. In particular, this implies that the only nonempty, open, convex subset of $L^p$ is $L^p$ itself. If $p \in [1, \infty)$, the dual of $L^p$ can be identified with the space $L^q$ for $q = \frac{p}{1-p}$. The index $q$ is sometimes called the conjugate of $p$. For any $X \in L^p$ and $Z \in L^q$, the pairing is given by

$$\langle Z, X \rangle = \mathbb{E}[ZX].$$

Orlicz spaces

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A map $\Phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is called an Orlicz function if it is convex, lower semicontinuous, symmetric, i.e. $\Phi(-x) = \Phi(x)$ for all $x \in \mathbb{R}$, and satisfies $\Phi(0) = 0$ and $\Phi(x) > 0$ for some $x > 0$. The Orlicz space associated to $\Phi$, denoted $L^\Phi$, is the linear subspace of $L^0$ consisting of all functions such that

$$\mathbb{E}[\Phi(\lambda X)] < \infty \quad \text{for some } \lambda > 0.$$ 

$^5$We use the convention $\frac{1}{0} := \infty.$
The space $L^\Phi$ is a Banach lattice under the norm

$$\|X\|_\Phi := \inf \left\{ \lambda > 0 ; \ E \left[ \Phi \left( \frac{1}{\lambda} X \right) \right] \leq 1 \right\}.$$  

The Morse space associated to $\Phi$, denoted $M^\Phi$, is the Banach lattice consisting of all functions in $L^\Phi$ satisfying

$$E[\Phi(\lambda X)] < \infty \quad \text{for all } \lambda > 0.$$ 

The space $L^\infty$ is continuously embedded into $L^\Phi$ which, in turn, is continuously embedded into $L^1$. In particular, if $\Phi$ is not finitely valued, then $L^\Phi = L^\infty$ and $M^\Phi = \{0\}$. Recall that $L^\Phi = M^\Phi$ whenever $\Phi$ is finite and there exist $x_0 > 0$ and $\alpha > 0$ such that

$$x > x_0 \implies \Phi(2x) \leq \alpha \Phi(x).$$ 

If $\Phi$ is finitely valued, the space $L^\infty$ is dense in $M^\Phi$. Moreover, the dual of $M^\Phi$ can be identified with $L^{\Phi^*}$. For a discussion on the dual of $L^\Phi$ we refer to Biagini, Frittelli [14] and the references therein. For more details about Orlicz spaces and, especially, Morse spaces, we refer to Edgar, Sucheston [37].

**Order units and strictly positive elements**

If the positive cone of a given reference space $X$ has empty interior, we may consider a variety of surrogates for interior points. In the sequel, we will mainly focus on order units and strictly positive elements. Clearly, every interior point of the positive cone is an order unit, which in turn is strictly positive. The converse implications may be strict, but they hold whenever the positive cone has nonempty interior. Here, we clarify the situation in the case of the standard spaces described above.

**Nonempty interior.** The positive cone of $L^\infty$ has nonempty interior and we have $X \in \text{int} L^\infty_+$ if and only if $X$ is bounded away from zero, i.e. $X \geq_p \varepsilon$ for some $\varepsilon > 0$. In particular one should not confuse strictly positive elements with functions that are strictly positive almost surely. The same holds for the space $L^\infty$. 

**Empty interior, nonempty core.** Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is nonatomic. If we endow $L^\infty$ with the topology $\sigma(L^\infty, L^1)$, then it is not difficult to see that the interior of the positive cone is empty. Note that, as above, any positive element in $L^\infty$ which is
bounded away from zero is an order unit. Moreover, the strictly positive elements are precisely those \( X \in L^\infty \) such that \( \mathbb{P}(X > 0) = 1 \). As a result, there exist strictly positive elements that are not order units, even if the positive cone has nonempty core.

**Empty core, strictly positive elements.** Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is nonatomic. The positive cone of \( L^p, p \in [1, \infty) \), has empty core. However, the elements \( X \in L^p \) such that \( \mathbb{P}(X > 0) = 1 \) correspond to the strictly positive elements. The same is true for any (nontrivial) Morse space \( M^\Phi \).

**No strictly positive elements.** It is known that strictly positive elements may not exist, see Section 2.2 in Aliprantis, Tourky [5], where the space \( X \) is a nonstandard function space. In Remark 3.7.24, we provide a more interesting example in the context of financial mathematics. We show that the Orlicz space corresponding to exponential utility has no strictly positive elements whenever \((\Omega, \mathcal{F}, \mathbb{P})\) is nonatomic.

In the context of a topological Riesz space we can also consider weak topological units. By Proposition .0.47, weak topological units in \( L^p \) spaces, \( p \in [1, \infty) \), or in (nontrivial) Morse spaces are precisely those positive elements \( X \) for which \( \mathbb{P}(X > 0) = 1 \). In \( L^\infty \) they correspond to elements that are bounded away from zero.
Bibliography


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5.  Capital adequacy tests and limited liability of financial institutions
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    *Journal of Banking & Finance*, 2015
4.  Measuring risk with multiple eligible assets
    (with W. Farkas and P. Koch-Medina)
3.  Law-invariant risk measures: Extension properties and qualitative robustness
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    *Statistics & Risk Modeling*, 2014
Publications (continued)

2. Beyond cash-additive risk measures: when changing the numéraire fails  
   (with W. Farkas and P. Koch-Medina)  
   *Finance and Stochastics*, 2014

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Talks

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          Set Optimization Meets Finance, Brunico

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          Bachelier Finance Society, World Congress, Bruxelles

2014.05  What is a risk measure?  
          Zurich Graduate Colloquium, Zürich

2013.02  Beyond cash-additive risk measures: capital efficiency and default risk  
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          Risk Day, Zürich

2012.08  Multi-asset capital requirements  
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