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Robust Stability of a Class of Interconnected Nonlinear Positive Systems

Marcello Colombino, Andreas B. Hempel and Roy S. Smith

***Abstract*— We present conditions for robust stability of a class of linear systems interconnected by uncertain nonlinear, norm-bounded functions. We show that such conditions can be reformulated as classical small gain like conditions for a related linear system. Under further assumptions that render the related linear system positive, we show that we can achieve sharp tractable conditions for robust stability of the original nonlinear system.**

I. Introduction

A dynamical system is said to be positive if, for every nonnegative input, the output trajectory remains nonnegative for all time and internally positive if this applies to the state as well, provided the initial condition is nonnegative [1]. Such systems arise naturally in several application areas including chemical reactions, population dynamics, job balancing in computer networks and consensus problems over graphs. Positive systems have received increasing attention over the last decade not only for their practical relevance but also for their system theoretic properties. It was shown recently that some generally intractable problems related to robust stability analysis become tractable for the class of positive linear systems. In [2] the authors show that the structured singular value, μ, is equal to its convex upper bound and thus easily computable. In [3] and [4], in the L_1 and L_∞ gain setting, the authors exploit linear copositive Lyapunov functions to provide tractable necessary and sufficient conditions for robustness analysis and controller synthesis. In this work we consider a class of linear systems coupled by uncertain nonlinear norm bounded functions introduced in [5]. The contributions of the paper are the following:

- Exploiting the S-Procedure we give sufficient conditions for robust stability with respect to the uncertain coupling functions.
- We draw the parallel of such conditions to more classical robustness conditions for a related linear system.
- We show that with some further assumptions we can exploit the properties of positive systems to obtain a convex characterization of robust stability of the original nonlinear interconnected system.

The paper is structured as follows: In Section II we present some basic definitions and preliminary results on positive systems. In Section III we review the robust stability problem with one uncertain nonlinear function and in section IV we discuss the more complex case with more than one uncertain function. In section V we illustrate the method by means of a numerical example. Appendix A contains the proof of our main result.

Notation

The set of real numbers is denoted by $\mathbb{R}$. By $\mathbb{R}_+$ ($\mathbb{R}_{++}$) we denote the set of nonnegative (positive) reals. With $\mathbb{Z}$ we denote the integers and with $\mathbb{Z}_{[a,b]}$ the set $[a,b] \cap \mathbb{Z}$. The set of $n \times n$ Metzler matrices (matrices with nonnegative off diagonal elements) is denoted by $\mathbb{M}^n$. The set of $n \times n$ nonnegative (positive) diagonal matrices is denoted by $\mathbb{D}^n_+$ ($\mathbb{D}^n_{++}$). Given a matrix A, $A^\top$ denotes its transpose. We use $\bar{\sigma}(A)$ to indicate the maximum singular value of A, $\mathrm{tr}(A)$ to denote its trace and $\mathrm{diag}(A)$ the vector whose elements are the diagonal elements of A. $A \succcurlyeq 0$ ($A \succ 0$) denotes that A is symmetric positive semidefinite (definite), all other relations $(>, <, \geq, \leq, =)$ between matrices are considered component wise. Given the matrices $A_1, \dots, A_n$, $\mathrm{blkdiag}(A_1, \dots, A_n)$ denotes the block diagonal matrix with $A_1, \dots, A_n$ on its diagonal. For a vector $x \in \mathbb{R}^n$, $\|x\| = \sqrt{x^\top x}$. Given a linear system M, $\|M\|_\infty$ denotes its $\mathcal{H}_\infty$ norm.

II. Preliminaries

A. Positive Systems

We will now give some background results on Positive systems. First we distinguish between positive and internally positive LTI systems:

Definition 1: A linear time invariant system is said to be *positive* if its impulse response is nonnegative.

Positivity is an input-output property, internal positivity however requires nonnegativity of the state, it is then a property of a specific realization, namely:

Definition 2: A realization (A, B, C, D) of a linear time invariant system described by the differential equation:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du,$$

is *internally positive* if and only if $A \in \mathbb{M}^n$ and $B, C, D \geq 0$ of appropriate dimensions.

Remark 1: Every LTI system that admits an internally positive realization is positive, the opposite is not true [6, Example 6].

It is well known that stability of a positive system can be proven with a diagonal quadratic Lyapunov function [1]. More recently it was shown that for such systems the KYP lemma also holds with a diagonal Lyapunov matrix [7], [8]. In this paper we show that this property is maintained even in the presence of a certain class of nonlinear norm-bounded uncertainty.

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III. ROBUST STABILITY OF SYSTEMS WITH ONE NONLINEAR UNCERTAINTY

In this section we review some known necessary and sufficient conditions for the robust stability of a class of uncertain nonlinear systems presented in [5]. We show the equivalence of these conditions with the classical small gain conditions for a related LTI system. Then we make some further assumptions to be able to refine these robust stability conditions using the theory of positive systems.

A. General systems

Consider a system of the form:

$$\dot{x} = Ax + h(x,t), \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $h : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a potentially nonlinear, piecewise-continuous time-dependent function.

Assumption 1: For given $H \in \mathbb{R}^{n\times n}$, the function $h(\cdot)$ satisfies the following inequality in its domain of continuity:

$$h^\top(x,t)h(x,t) \le \varphi^2 x^\top H^\top Hx, \forall\, (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

We wish to find conditions for system (1) to be stable for all functions $h(\cdot,\cdot)$ satisfying Assumption 1. More formally:

Definition 3: A system in the form of (1) is *quadratically robustly stable* with degree φ if there exists $P \succ 0$ such that $V(x) = x^\top Px$ is a Lyapunov function for all $h(\cdot,\cdot)$ satisfying Assumption 1.

Following the approach in [5] we seek such a quadratic Lyapunov function. Then the following must hold:

$$\begin{aligned} &x^\top P(Ax + h(x,t)) + (Ax + h(x,t))^\top Px < 0,\\ &\forall\, x,t \text{ s.t. } x \in \mathbb{R}^n,\;\; h^\top(x,t)h(x,t) \le \varphi^2 x^\top H^\top Hx. \end{aligned} \tag{2}$$

Condition (2) is equivalent to the following:
$\forall\, x \in \mathbb{R}^n \setminus \{0\},\; h \in \mathbb{R}^n \setminus \{0\}$ such that

$$\begin{bmatrix} x \\ h \end{bmatrix}^\top \begin{bmatrix} \varphi^2 H^\top H & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x \\ h \end{bmatrix} \ge 0 \tag{3a}$$

it must hold that

$$\begin{bmatrix} x \\ h \end{bmatrix}^\top \begin{bmatrix} A^\top P + PA & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x \\ h \end{bmatrix} < 0. \tag{3b}$$

We can now apply the S-Procedure [9] and conclude that (3) is equivalent to the existence of P and $\theta > 0$ such that the following matrix inequality is feasible:

$$\begin{gathered} P \succ 0 \\ \theta \begin{bmatrix} \varphi^2 H^\top H & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} A^\top P + PA & P \\ P & 0 \end{bmatrix} \prec 0. \end{gathered} \tag{4}$$

Theorem 1 ([9], Theorem 2.16): System (1) is quadratically robustly stable with degree $1/\sqrt{\gamma^\star}$ if and only if the following Semidefinite Program (SDP) is feasible:

$$\begin{aligned} \gamma^\star = \inf_{\gamma,Y} \quad & \gamma \\ \text{s.t.} \quad & Y \succ 0 \\ & \begin{bmatrix} AY + YA^\top & I & YH^\top \\ I & -I & 0 \\ HY & 0 & -\gamma I \end{bmatrix} \prec 0. \end{aligned} \tag{5}$$

Sketch of the proof: pre- and post-multiply (4) by $\text{blkdiag}\left(\theta^{1/2}P^{-1}, \theta^{-1/2}I\right)$, substitute $Y = \theta P^{-1}$ and $\gamma = 1/\varphi^2$, then apply the Schur complement. ■

Remark 2: We can interpret the SDP in (5) as a small gain condition. If we consider the system M:

$$M = \begin{cases} \dot{x} = Ax + w \\ z = Hx, \end{cases} \tag{6}$$

we notice that $\gamma^\star$ in (5) can be found equivalently as $\gamma^\star = \|M\|_\infty^2$.

By the small gain theorem this implies that the robust stability of the nonlinear system in (1) with respect to the function $h(\cdot)$ satisfying Assumption 1 is equivalent to the robust stability of the interconnection in Figure 1 with respect to Δ, where Δ is a LTI system and $\|\Delta\|_\infty < 1$.

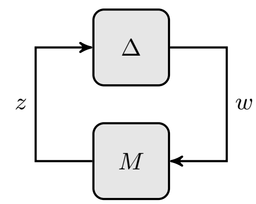

Fig. 1: The interconnection of M and Δ.

B. Positive systems

We now make assumptions on A and H so that we can use the theory of positive systems to refine the conditions of Theorem 1. We consider a system in of the form

$$\dot{x} = Ax + h(x,t), \tag{7}$$

where $A \in \mathbb{M}^n$ is a Metzler matrix. We also restrict H in Assumption 1 to be nonnegative. Note that these further assumptions do not make the system in (7) necessarily positive as we do not impose that the function $h(\cdot)$ preserves positivity. We can however refine the conditions of Theorem 1 exploiting the theory of positive systems as follows.

Theorem 2: If $A \in \mathbb{M}^n$ and $H \in \mathbb{R}_+^{n\times n}$, system (7) is quadratically robustly stable with degree $1/\sqrt{\gamma^\star}$ if and only if the following SDP is feasible:

$$\begin{aligned} \gamma^\star = \inf_{\gamma,Y} \quad & \gamma \\ \text{s.t.} \quad & Y \in \mathbb{D}_{++}^n \\ & \begin{bmatrix} AY + YA^\top & I & YH^\top \\ I & -I & 0 \\ HY & 0 & -\gamma I \end{bmatrix} \prec 0. \end{aligned} \tag{8}$$

Proof: For fixed θ and φ, we define:

$$\theta \begin{bmatrix} \varphi^2 H^\top H & 0 \\ 0 & -I \end{bmatrix} =: Q_{\theta,\varphi}$$

Using the KYP Lemma [10], the LMI in (4) is feasible *if and only if* for all $\omega \in \mathbb{R} \cup \{\infty\}$:

$$\begin{bmatrix} (j\omega I - A)^{-1} \\ I \end{bmatrix}^* Q_{\theta,\varphi} \begin{bmatrix} (j\omega I - A)^{-1} \\ I \end{bmatrix} \prec 0. \tag{9}$$

Note that $(j\omega I - A)$ is always invertible as A must be Hurwitz. Under the assumption that $A \in \mathbb{M}^n$ and $H \in \mathbb{R}_+^{n\times n}$, $\theta, \varphi > 0$, all the elements in $Q_{\theta,\varphi}$ are nonnegative with the exception of the last n diagonal elements. It follows from [8, Theorem 1] that condition (9) holds *if and only if* there exists a D, such that:

$$D \in \mathbb{D}^n_{++}$$
$$Q_{\theta,\varphi} + \begin{bmatrix} A^\top D + DA & D \\ D & 0 \end{bmatrix} \prec 0. \tag{10}$$

We can then continue as for Theorem 1 and the proof is complete. ■

It is not surprising that under the assumption that A is Metzler and H is nonnegative we can restrict to a diagonal Lyapunov function, especially in view of Remark 2: with the new assumptions the system M in (6) is an internally positive system and it is well known that its contractiveness can be proven with a diagonal Lyapunov function [7].

IV. Robust stability of systems coupled by several nonlinear uncertainties

We will now extend the results in Section III to address the robust stability of a network of N systems with uncertain coupling [11]. Again we first consider the case of general systems and then we make further assumptions to exploit powerful results on positive systems.

A. General systems

Consider a set of N subsystems indexed by $\mathbb{Z}_{[1,N]}$. We partition the state vector as $x = [x_1, \ldots, x_N]^\top$, where $x_i \in \mathbb{R}^{n_i}$, $\forall i \in \mathbb{Z}_{[1,N]}$. We define a set of neighbors $\mathcal{N}_i \subseteq \mathbb{Z}_{[1,N]}$ for each subsystem i. Each subsystem has its local dynamics described by:

$$\dot{x}_i = \sum_{j\in\mathcal{N}_i} A_{ij}x_j + h_i(x,t), \tag{11}$$

where $A_{ij} \in \mathbb{R}^{n_i\times n_j}$ and $h_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ are potentially nonlinear, piecewise-continuous time-dependent functions. Since the $h_i(\cdot,\cdot)$ are functions of the whole state x, they potentially couple all subsystems.

Assumption 2: For given $H_i \in \mathbb{R}^{n_i\times n}$, the functions $h_i(\cdot,\cdot)$ are piecewise continuous and satisfy the following inequality in their domain of continuity:

$$h_i^\top(x,t)h_i(x,t) \le \varphi^2 x^\top H_i^\top H_i x,\ \forall\ (x,t) \in \mathbb{R}^n_+ \times \mathbb{R}.$$

Remark 3: The matrices H_i are assumed to be in $\mathbb{R}^{n_i\times n}$ to simplify the notation later. All results still hold if $H_i \in \mathbb{R}^{q_i\times n}$ for any $q_i \in \mathbb{Z}_{[1,\infty]}$.

We can write the system in (11) in compact form:

$$\dot{x} = Ax + h(x,t), \tag{12}$$

where

$$A \in \mathbb{R}^{n\times n} = \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{bmatrix}$$

and $h(x,t) = [h_1^\top(x,t), \ldots, h_N^\top(x,t)]^\top$. Note that if $j \notin \mathcal{N}_i$ $A_{ij} = 0$. We define the matrices $E_i \in \mathbb{R}^{n_i\times n}$ such that $h_i(x,t) = E_i h(x,t)$ and the matrices $\Pi_i = E_i^\top E_i$. Notice that the matrices Π_i form a partition of the identity and $\sum_{i=1}^N \Pi_i = I_n$. Assumption 2 can then be summarized as follows:

$$h^\top(x,t)\Pi_i h(x,t) \le \varphi^2 x^\top H_i^\top H_i x,$$
$$\forall\ (x,t) \in \mathbb{R}^n \times \mathbb{R},\ \forall i \in \mathbb{Z}_{[1,N]} \tag{13}$$

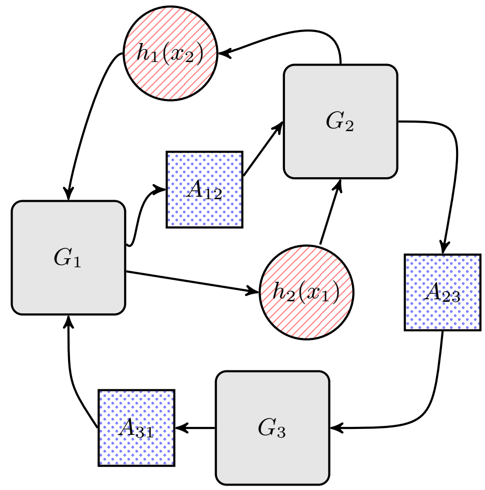

Fig. 2: Block diagram of an uncertain interconnected system in the form of (11). The striped red blocks represent the uncertain nonlinear couplings, the dotted blue blocks represent the known linear couplings.

As in Section III, we seek a quadratic Lyapunov function that certifies the robust stability of system (12) with respect to $h(\cdot,\cdot)$ satisfying (13). Then we need to find a matrix $P \succ 0$ such that:

$$x^\top P(Ax + h(x,t)) + (Ax + h(x,t))^\top Px < 0,$$
$$\forall\, x, t \ \text{ s.t. }\ x \in \mathbb{R}^n,\ \ h^\top(x,t)\Pi_i h(x,t) \le \varphi^2 x^\top H_i^\top H_i x$$
$$\forall\ i \in \mathbb{Z}_{[1,N]}. \tag{14}$$

Remark 4: Note that the compact reformulation of (13) given in [5, Equation 5.9] and [11, Equation 4] is incorrect and produces extremely conservative conditions. It is not possible to reduce (13) to a single quadratic inequality, the analysis is therefore more complex than what was previously reported in the literature.

Condition (14) is equivalent to:
$\forall x, h \in \mathbb{R}^n_+ \setminus \{0\}$ such that

$$\begin{bmatrix} x \\ h \end{bmatrix}^\top \begin{bmatrix} \varphi^2 H_i^\top H_i & 0 \\ 0 & -\Pi_i \end{bmatrix} \begin{bmatrix} x \\ h \end{bmatrix} \ge 0 \tag{15a}$$

holds $\forall\ i \in \mathbb{Z}_{[1,N]}$, it must hold that

$$\begin{bmatrix} x \\ h \end{bmatrix}^\top \begin{bmatrix} A^\top P + PA & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x \\ h \end{bmatrix} < 0. \tag{15b}$$

We can now apply the S-Procedure and we get a *sufficient* condition for robust stability.

Proposition 3: System (11) is quadratically robustly stable with respect to all functions $h_1(\cdot), \ldots, h_N(\cdot)$ that respect Assumption 2 if there exist multipliers $\theta_1, \ldots, \theta_N \in \mathbb{R}_{++}$ and a matrix $P \succ 0$ such that

$$\begin{bmatrix} A^\top P + PA & P \\ P & 0 \end{bmatrix} + \theta_1 \begin{bmatrix} \varphi^2 H_1^\top H_1 & 0 \\ 0 & -\Pi_1 \end{bmatrix} + \cdots$$
$$\cdots + \theta_N \begin{bmatrix} \varphi^2 H_N^\top H_N & 0 \\ 0 & -\Pi_N \end{bmatrix} \prec 0. \quad (16)$$

□

Again we can draw a parallel from condition (16) to a more classical condition regarding the robust stability of linear systems. In particular if we define $H := [H_1^\top, \ldots, H_N^\top]^\top$, and the system

$$M = \begin{cases} \dot{x} = Ax + w \\ z = Hx, \end{cases} \quad (17)$$

condition (16) is equivalent to saying that $\exists\, P \succ 0,\ \ \Theta \in \mathbf{\Theta}$ such that

$$\begin{bmatrix} A^\top P + PA + \varphi^2 H^\top \Theta H & P \\ P & -\Theta \end{bmatrix} \prec 0, \quad (18)$$

where

$$\mathbf{\Theta} := \left\{ \text{blkdiag}(\theta_1 I_{n_1}, \ldots, \theta_N I_{n_N}),\, \theta_i > 0 \right\}, \quad (19)$$

or equivalently the frequency domain condition on the transfer matrix $M(j\omega)$

$$\sup_{\omega \in \mathbb{R}} \inf_{\Theta \in \mathbf{\Theta}} \bar{\sigma} \left(\Theta^{\frac{1}{2}} M(j\omega) \Theta^{-\frac{1}{2}} \right) < \frac{1}{\varphi},$$

which is the standard μ upper bound [12, Proposition 8.6] for the system M.

B. Positive systems

Similarly to Section III-B we will now make some extra assumptions in order to exploit the powerful conditions on robust stability for positive systems [2]. Each subsystem now has its local dynamics described by:

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} A_{ij} x_j + h_i(x, t), \quad (20)$$

where $A_{ii} \in \mathbb{M}^{n_i}$ and $A_{ij} \in \mathbb{R}_+^{n_i \times n_j}$ for all $i \in \mathbb{Z}_{1,N}$ and $j \neq i$. The functions $h_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ still satisfy Assumption 2, however now the matrices $H_i \in \mathbb{R}_+^{n_i \times n}$. Again we can write the system in (20) in compact form

$$\dot{x} = Ax + h(x, t), \quad (21)$$

but now $A \in \mathbb{M}^n$. We now show that, under the new assumptions, the conditions derived with the S-Procedure hold for a diagonal Lyapunov Matrix and are now both necessary and sufficient for robust stability.

Theorem 4: If $A \in \mathbb{M}^n$ and $H_i \in \mathbb{R}_+^{n_i \times n}$, system (20) is robustly stable with respect to all $h_1(\cdot), \ldots, h_N(\cdot)$ satisfying Assumption 2 *if and only if* $\exists\ D \in \mathbb{D}_{++}^n,\ \theta_1, \ldots, \theta_N \in \mathbb{R}_{++}$ such that

$$\begin{bmatrix} A^\top D + DA & D \\ D & 0 \end{bmatrix} + \theta_1 \begin{bmatrix} \varphi^2 H_1^\top H_1 & 0 \\ 0 & -\Pi_1 \end{bmatrix} + \cdots$$
$$\cdots + \theta_N \begin{bmatrix} \varphi^2 H_N^\top H_N & 0 \\ 0 & -\Pi_N \end{bmatrix} \prec 0. \quad (22)$$

□

In other words, the S-Procedure is lossless. The proof of Theorem 4 is presented in Appendix A.

Again we can draw a parallel to more classical robustness conditions. As already stated above the Linear Matrix Inequality (LMI) condition of Theorem 4 is equivalent to the standard μ upper bound for system (17). Since with the new assumptions the system (17) is a positive system, we know from [2, Theorem 10] that μ is equal to the upper bound. We can therefore conclude that for such systems the robust stability with respect to nonlinear static perturbation is equivalent to the robust stability with respect to linear time invariant dynamical perturbations.

V. Numerical Example

We illustrate the result by means of a numerical example. Consider a system in the form (20) where the state vector is partitioned as $x = [x_1^\top, x_2^\top, x_3^\top]^\top$ and:

$$A_{11} = \begin{bmatrix} -1 & 0.4 \\ 0.1 & -1 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0 & I & 0 \end{bmatrix},$$
$$A_{22} = \begin{bmatrix} -1 & 0 \\ 0.5 & -1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0 & I \end{bmatrix},$$
$$A_{33} = \begin{bmatrix} -1 & 0.2 \\ 0.7 & -1 \end{bmatrix}, \quad H_3 = \begin{bmatrix} I & 0 & 0 \end{bmatrix},$$

$A_{ij} = 0$ for $i \neq j$, and $\varphi = 0.7$. Using MOSEK [13] we find that

$$D = \text{diag}(0.7410, 0.9876, 0.8954, 0.5627, 0.9779, 0.6076)$$

and $(\theta_1, \theta_2, \theta_3) = (1.1248, 0.9070, 1.3251)$ solve (22) which, by Theorem 4, is equivalent to the system being robustly stable with respect to all perturbations $h_1(\cdot), h_2(\cdot), h_3(\cdot)$ such that $h_i(x,t)^\top h_i(x,t) \leq \varphi^2 x^\top H_i^\top H_i x$. Since it is not possible to test all perturbations that respect this bound, for illustration we choose

$$h_1(x) = \varphi \frac{\sin(2x_2) \circ (\sin(x_2) + 3x_2)}{4},$$
$$h_2(x) = \varphi \frac{\sin(2x_3) \circ (\sin(x_3) + 3x_3)}{4},$$
$$h_3(x) = \varphi \frac{\sin(2x_1) \circ (\sin(x_1) + 3x_1)}{4}, \quad (23)$$

where $\sin(\cdot)$ is taken element-wise and $\circ$ denotes the Hadamard product. As expected the origin is stable for the interconnected system, in Figure 3 we see the state evolution from the initial condition $x_0 = [5, 1, 1, 5, 10, 5]^\top$. In Figure 4 we see the evolution of $x(t)^\top D x(t)$ that, as expected, decreases exponentially to zero.

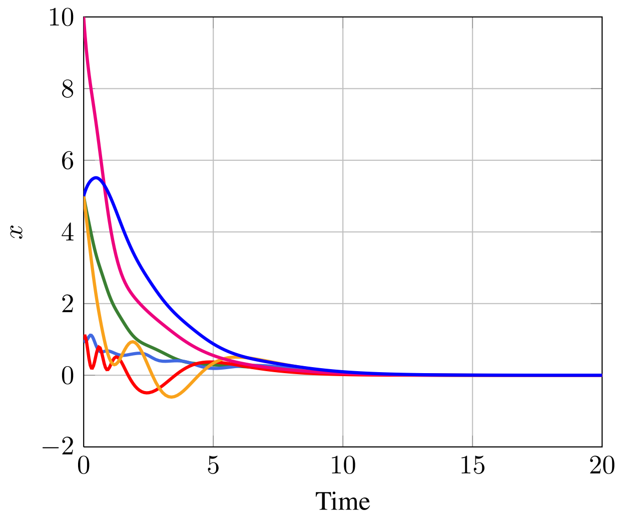

Fig. 3: As expected, the states of the nonlinear system converge to the origin.

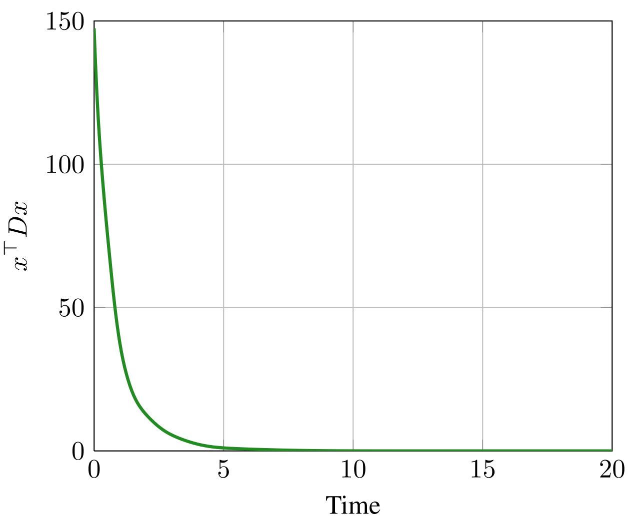

Fig. 4: $V(x) = x^\top Dx$ is a Lyapunov function for the nonlinear system.

If we increase φ to 0.9, equation (22) has no solution. By Theorem 4 we know there exist functions $h_1(\cdot), h_2(\cdot), h_3(\cdot)$ satisfying the norm bounds $h_i(x,t)^\top h_i(x,t) \leq \varphi^2 x^\top H_i^\top H_i x$ such that the nonlinear interconnected system is not asymptotically stable. If we use the functions in (23) we can see from Figure 5 that the trajectories no longer converge to the origin.

VI. Conclusion

In this paper we analyze the robust stability of systems interconnected by nonlinear norm bounded uncertain functions. The conditions derived resemble more classical small gain type conditions. Using a duality approach introduced in [14] and [15], we can show that such conditions are tight under the positivity assumption. The same duality approach

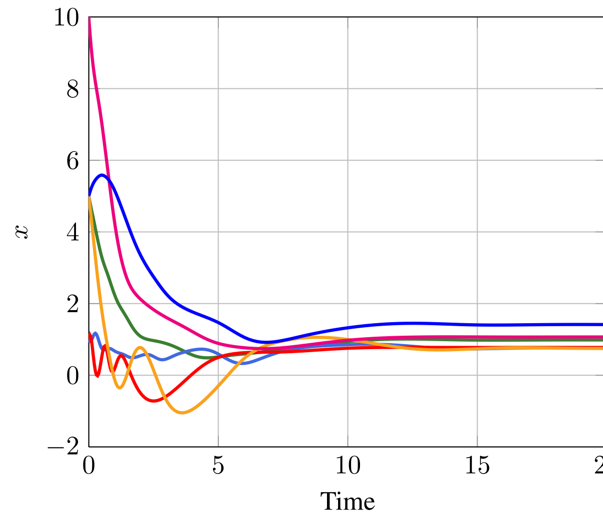

Fig. 5: For $\varphi = 0.9$ the origin is no longer globally asymptotically stable. This is possible since (22) is infeasible.

could be adapted to give an alternative proof of the result of [2] for robust control of linear positive systems.

Appendix A

A. Proof of Theorem 4

The technique for the proof is inspired by [15]. We first state two results that will be needed for the proof:

Lemma 5 ([15] Proposition 3 + Lemma 3): Let $A \in \mathbb{M}^n$, then the following are equivalent:

a) There exists $z \in \mathbb{R}_+ \backslash \{0\}$ such that $\mathrm{diag}(zz^\top A) \in \mathbb{R}_+^n$.
b) A is not Hurwitz. □

Lemma 6 ([15] Lemma 2): Let $A \in \mathbb{M}^n$ and $Z \succcurlyeq 0$, $Z \neq 0$ with elements $[z_{ij}]$, then the following statements are true:

a) $\mathrm{diag}(ZA) \in \mathbb{R}_+^n \implies \mathrm{diag}(\bar{z}\bar{z}^\top A) \in \mathbb{R}_+^n$ and
b) $\mathrm{tr}(ZA) \geq 0 \implies \mathrm{tr}(\bar{z}\bar{z}^\top A) \geq 0$,

where $\bar{z} := [\sqrt{z_{11}}, \ldots, \sqrt{z_{nn}}]^\top$. □

We can now procede to the main proof. The fact that (22) implies robust stability is known and easy to prove, we now focus on proving that robust stability implies (22).

Condition (22) can be rewritten as $\exists\ D \in \mathbb{D}_{++}^n,\ \ \Theta \in \boldsymbol{\Theta}$ such that

$$\begin{bmatrix} A^\top D + DA + \varphi^2 H^\top \Theta H & D \\ D & -\Theta \end{bmatrix} \prec 0, \qquad (24)$$

where $H := [H_1^\top \ldots H_N^\top]^\top$ and the set $\boldsymbol{\Theta}$ is defined in (19). For the sake of contradiction we assume that the system is robustly stable, but there exist no $(D, \Theta) \in \mathbb{D}_{++} \times \boldsymbol{\Theta}$ that satisfy (24). We will then show that, under these assumptions, we can construct a function $h(\cdot)$ that satisfies the bounds of Assumption 2 and makes (20) unstable thus reaching a contradiction. If condition (24) is not satisfied by any $(D, \Theta) \in \mathbb{D}_{++} \times \boldsymbol{\Theta}$, by the Separating Hyperplane Theorem [12, Theorem 1.5] there exists

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^\top & Z_{22} \end{bmatrix} \succcurlyeq 0, \quad Z \neq 0,$$

such that

$$\mathrm{tr}\left(Z\begin{bmatrix} A^\top D + DA + \varphi^2 H^\top \Theta H & D \\ D & -\Theta \end{bmatrix}\right) \geq 0 \quad \forall (D,\Theta) \in \mathbb{D}_{++} \times \mathbf{\Theta}, \tag{25}$$

or equivalently

$$2\,\mathrm{tr}\left(\left(Z_{11}A^\top + Z_{12}\right)D\right) + + \mathrm{tr}\left(Z\begin{bmatrix} \varphi^2 H^\top \Theta H & 0 \\ 0 & -\Theta \end{bmatrix}\right) \geq 0 \quad \forall (D,\Theta) \in \mathbb{D}_{++} \times \mathbf{\Theta}. \tag{26}$$

Condition (26) has two main implications:

a) $\mathrm{diag}\left(Z_{11}A^\top + Z_{12}\right) \in \mathbb{R}^n_+$ and (27)

b) $\mathrm{tr}\left(Z\begin{bmatrix} \varphi^2 H^\top \Theta H & 0 \\ 0 & -\Theta \end{bmatrix}\right) \geq 0, \quad \forall \Theta \in \mathbf{\Theta}.$ (28)

To see this, fix $\Theta \in \mathbf{\Theta}$. If either (27) or (28) did not hold then one can always find a $D \in \mathbb{D}_{++}$ that invalidates the inequality in (26). We can write (27) as

$$\mathrm{diag}\left(Z\begin{bmatrix} A^\top & 0 \\ I & 0 \end{bmatrix}\right) \in \mathbb{R}^{2n}_+.$$

Since both

$$\begin{bmatrix} \varphi^2 H^\top \Theta H & 0 \\ 0 & -\Theta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A^\top & 0 \\ I & 0 \end{bmatrix}$$

are Metzler, we can apply Lemma 6 and conclude that:

a) $\mathrm{diag}\left(\bar{z}\bar{z}^\top \begin{bmatrix} A^\top & 0 \\ I & 0 \end{bmatrix}\right) \in \mathbb{R}^{2n}_+$ and

b) $\mathrm{tr}\left(\bar{z}\bar{z}^\top \begin{bmatrix} \varphi^2 H^\top \Theta H & 0 \\ 0 & -\Theta \end{bmatrix}\right) \geq 0, \quad \forall \Theta \in \mathbf{\Theta},$

where $\bar{z} := [\sqrt{z_{11}}, \ldots, \sqrt{z_{nn}}]^\top \in \mathbb{R}^{2n}_+$. If we partition $\bar{z} = [\bar{z}_1^\top, \bar{z}_2^\top]^\top$ we can rewrite the conditions above as

a) $\mathrm{diag}\left(\bar{z}_1\bar{z}_1^\top A^\top + \bar{z}_1\bar{z}_2^\top\right) \in \mathbb{R}^n_+$ and

b) $\bar{z}_1^\top \varphi^2 H^\top \Theta H \bar{z}_1 \geq \bar{z}_2^\top \Theta \bar{z}_2, \quad \forall \Theta \in \mathbf{\Theta},$

or equivalently,

a) $\mathrm{diag}\left(\bar{z}_1\bar{z}_1^\top A^\top + \bar{z}_1\bar{z}_2^\top\right) \in \mathbb{R}^n_+$ (29)

b) $\varphi\|\Theta^{\frac{1}{2}} H\bar{z}_1\| \geq \|\Theta^{\frac{1}{2}}\bar{z}_2\|, \quad \forall \Theta \in \mathbf{\Theta}.$ (30)

Note that $\bar{z}_2 \neq 0$, otherwise from (29) and Lemma 5 the matrix A would not be Hurwitz. From (30) we deduce that for every block component corresponding to H_i it must hold that

$$\varphi\|H_i\bar{z}_1\| \geq \|(\bar{z}_2)_i\|. \tag{31}$$

To see that, suppose (31) did not hold for component j then one could let $\theta_i \to 0$ for all $i \neq j$ and inequality (30) would fail for some $\Theta \in \mathbf{\Theta}$. By [2, Lemma 6] we notice that (31) is equivalent to the existence of $\Delta_i \in \mathbb{R}^{ni}_+$, with $\bar{\sigma}(\Delta_i) \leq 1$ such that $\Delta_i\varphi H_i\bar{z}_1 = (\bar{z}_2)_i$, or equivalently

$$\Delta\varphi H\bar{z}_1 = \bar{z}_2, \tag{32}$$

where $\Delta = \mathrm{blkdiag}(\Delta_1 \ldots \Delta_N) \in \mathbb{R}^{n\times n}_+$. If we insert (32) into (29) we get

$$\mathrm{diag}\left(\bar{z}_1\bar{z}_1^\top\left(A^\top + \varphi H^\top\Delta^\top\right)\right) \in \mathbb{R}^n_+. \tag{33}$$

Since $\Delta \in \mathbb{R}^{n\times n}_+$, $A + \varphi\Delta H \in \mathbb{M}^n$. By Lemma 5, (33) implies that $A + \varphi\Delta H$ is not Hurwitz. We now consider system (20) where $h_i(x,t) = \varphi\Delta_i H_i x$. We know that such $h_i(\cdot)$ satisfiy the norm bounds because of (31) but as we have shown, the system is unstable. We have then reached a contradiction and the proof is complete. ■

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