Limits of order types

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Abstract

The notion of limits of dense graphs was invented, among other reasons, to attack problems in extremal graph theory. It is straightforward to define limits of order types in analogy with limits of graphs, and this paper examines how to adapt to this setting two approaches developed to study limits of dense graphs.

We first consider *flag algebras*, which were used to open various questions on graphs to mechanical solving via semidefinite programming. We define flag algebras of order types, and use them to obtain, via the semidefinite method, new lower bounds on the density of 5- or 6-tuples in convex position in arbitrary point sets, as well as some inequalities expressing the difficulty of sampling order types uniformly.

We next consider *graphons*, a representation of limits of dense graphs that enable their study by continuous probabilistic or analytic methods. We investigate how planar measures fare as a candidate analogue of graphons for limits of order types. We show that the map sending a measure to its associated limit is continuous and, if restricted to uniform measures on compact convex sets, a homeomorphism. We prove, however, that this map is not surjective. Finally, we examine a limit of order types similar to classical constructions in combinatorial geometry (Erdős-Szekeres, Horton ...) and show that it cannot be represented by any somewhere regular measure; we analyze this example via an analogue of Sylvester's problem on the probability that k random points are in convex position.

1998 ACM Subject Classification G.2.m Discrete Mathematics, I.3.5 Computational Geometry and Object Modeling

Keywords and phrases order types, Limits of discrete structures, Flag algebras, Erdos-Szekeres, Sylvester's problem

Digital Object Identifier 10.4230/LIPIcs.SOCG.2015.300

1 Introduction

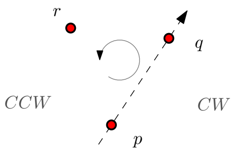
The order type of a point set is a combinatorial encoding of the respective positions of its elements that suffices to determine many of its properties. For instance, the order type determines the halving lines or more generally the k-sets of the point set, which graphs admit crossing-free straight line drawings with vertices supported on that point set, the structure of

* This work was partially supported by the ANR projects PRESAGE (ANR–11–BS02–003) and HEREDIA (ANR-10-JCJC-0204-01), by the Institut Universitaire de France, by the SNSF grant 200021-149111 and by the Advanced Grant of the European Research Council GUDHI (Geometric Understanding in Higher Dimensions).

its simplicial depth partition, etc. Order types have received continued attention in discrete and computational geometry since the 1980's and are known to be rather intricate objects, for instance difficult to axiomatise [13].

In this paper, we report on an effort to apply to order types ideas from the theories of *dense graph limits* developed by Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi and *flag algebras* developed by Razborov. While order types can be defined for points in d dimensions, in topological spaces, possibly with alignment, etc, all point sets considered in this paper are finite subsets of the euclidean plane, with no aligned triple.

Order types. Formally, order types are defined as follows. Define the *orientation* of a triangle pqr in the plane as *clockwise* (CW) if r lies to the right of the line pq oriented from p to q and *counter-clockwise* (CCW) if r lies to the left of that oriented line. (So the orientation of qpr is different from that of pqr.) We say that two planar point sets P and Q *have the same order type* if there exists a bijection $f : P \to Q$ that preserves orientations: for any triple of pairwise distinct points $p, q, r \in P$ the triangles pqr and $f(p)f(q)f(r)$ have the same orientation. The relation of having the same order type is easily checked to be an equivalence relation; the equivalence class, for this relation, of a finite point set P is called the *order type* of P. A point set P with order type ω is called a *realization* of ω.



When convenient, we extend to order types any notion that can be defined on a set of points and does not depend on a particular choice of realization. For instance we define the *size* of an order type ω as the cardinality $|\omega|$ of any of its realization. We adopt the convention that there is exactly one order type of each of the sizes 0, 1 and 2. We used the comprehensive list of all the order types of size up to 11, which was made available by Oswin Aichholzer [1] based on his work with Aurenhammer and Krasser [2] on the enumeration of order types. Throughout this paper, all non-trivial facts we use with reference on order types of small size can be traced back to that resource. We let $\mathcal{O}$ denote the set of order types and $\mathcal{O}_n$ the set of order types of size n.

Convergent sequences and limits of order types. We define the *density* $p(\omega, \omega')$ of an order type ω in another order type ω' as the probability that $|\omega|$ random points chosen uniformly from a point set realizing ω' have order type ω. (Observe that this probability depends solely on the order types and not on the choice of realization.) We say that a sequence $\{\omega_n\}_{n\in\mathbb{N}}$ of order types *converges* if the size $|\omega_n|$ goes to infinity as n goes to infinity, and if for any fixed order type ω the sequence of densities $p(\omega, \omega_n)$ converges. The *limit* of a convergent sequence of order types $\{\omega_n\}_{n\in\mathbb{N}}$ is the map

$$\begin{cases} \mathcal{O} & \to & [0,1] \\ \omega & \mapsto & \lim_{n\to\infty} p(\omega, \omega_n) \end{cases}$$

A standard compactness argument reveals that limits of order types abound. Indeed, for each element ω_n in a sequence of order types, the map $\omega \in \mathcal{O} \mapsto p(\omega, \omega_n)$ can be seen as a point in $[0,1]^{\mathbb{N}}$, which is compact by Tychonoff's theorem. Any sequence of order types whose size go to infinity therefore contains a convergent subsequence, and many extremal properties of point sets can be expressed in terms of limits of order types.

[1] `http://www.ist.tugraz.at/aichholzer/research/rp/triangulations/ordertypes/`

Problems and results. Let $\diamond_k$ denote the order type of k points in convex position, $\mathrm{conv}_k(n)$ the minimum number of convex k-gons in a set of n points in the plane, and $c_k = \lim_{n\to\infty} \mathrm{conv}_k(n)/\binom{n}{k}$ their minimum density. Determining $\mathrm{conv}_k(n)$ and c_k are classical problems in discrete geometry; see *eg* [6, Section 8.4, Problem 1]. Our first results are the following new lower bounds:

▶ **Proposition 1.** $c_5 \geq 0.0608516$ *and* $c_6 \geq 0.0018311$.

The best upper bounds we are aware of on these numbers are $c_5 \leq 0.0625$ and $c_6 \leq 0.005822$ and we are not aware of previously known lower bounds. We prove Proposition 1 by a reformulation of limits of order types as positive homomorphisms from a so-called flag algebra of order types into $\mathbb{R}$ (see Proposition 8); this point of view allows a semidefinite programming formulation of the search for inequalities satisfied by limits of order types. Specifically, we argue that for any limit of order type ℓ

$$\ell(\diamond_5) \geq 0.0608516 \qquad \text{and} \qquad \ell(\diamond_6) \geq 0.0018311.$$

The number c_4 corresponds to the celebrated rectilinear crossing number of the complete graph and has been extensively investigated; the best lower bound we could obtain on c_4 via flag algebra is $c_4 \geq 0.37843917$, which is inferior to the best known bound $c_4 \geq 277/729 \approx 0.3799$ (the best known upper bound being $83247328/218791125 \approx 0.3804$). We refer the interested reader to the survey of Abrego, Fernandez-Merchant and Salazar [1].

Probabilistic constructions are sometimes effective ways of finding extremal combinatorial structures, a textbook example being the lower bound on Ramsey numbers for graphs. It is of course easy to generate a random order type, for instance by sampling i.i.d. some measure over $\mathbb{R}^2$. It is not clear, however, how well such a method samples the space of order types, and hence how effective it would be to test conjectures and search for extremal examples (see *eg* [6, p 326]). Sampling order types of a given size *uniformly* looks difficult, as suggested by the lack of closed formulas for counting them, but we know of no formal justification of the hardness of this problem. As it turns out, limits of order types can also be defined as families of probability distributions on order types with certain internal consistencies (see Proposition 6) and our second result, also obtained by the semidefinite method of flag algebras, shows that a broad class of random generation method must exhibit some bias:

▶ **Proposition 2.** *For any limit of order types ℓ there exist two order types ω_1, ω_2 of size* 6 *such that* $\ell(\omega_1) > 1.8208\ \ell(\omega_2) > 0$.

This inevitability of bias applies in particular to the random generation of order types by independent sampling of points from *any* measure over $\mathbb{R}^2$. Specifically, let μ be a finite measure over $\mathbb{R}^2$ and for any order type ω let $p(\omega,\mu)$ denote the probability that $|\omega|$ random points chosen independently from $\frac{1}{\mu(\mathbb{R}^2)}\mu$ realize ω; if every line is negligible for μ then $\ell_\mu : \omega \mapsto p(\omega,\mu)$ is a limit of order types (Lemma 7) and Proposition 2 applies.

Some hard problems in extremal graph theory were solved by representing limits of graphs by continuous functions, called *graphons*; a celebrated example is the application of large deviations principles to random Erdős-Renyi graphs $G(n,p)$ conditioned on the rare event of having triangle density q^3 for some $q > p$ [10], or on having a fixed degree sequence. At the heart of these results lies the fact that the relation between graphons and limits of graphs is not only a bijection, but an actual homeomorphism when both spaces are equipped with the adequate topologies. Since every finite measure μ over $\mathbb{R}^2$ (for which lines are negligible) defines a limit of order types ℓ_μ, it is natural to wonder if such measures can represent all limits of order types, and whether this representation can be made an homeomorphism.

Let $\mathcal{L}$ denote the space of limits of order types, endowed with the topology of the metric

$$d(\ell_1, \ell_2) := \sum_{i=1}^{\infty} 2^{-i} |\ell_1(\omega_i) - \ell_2(\omega_i)|, \tag{1}$$

where $\{\omega_1, \omega_2, \ldots\}$ is some arbitrary enumeration of the set of order types. We first show that the map $\mu \mapsto \ell_\mu$, from the space of finite measures over $\mathbb{R}^2$ for which every line is negligible, equipped with the topology of the weak convergence, into $\mathcal{L}$, is continuous (Proposition 10). We next consider the special case of restrictions of the Lebesgue measure (the area) to compact convex sets with non empty interior (*convex bodies*). Let $\mathcal{K}$ denote the quotient of the space of convex bodies by affine transforms: if K is a convex body, $[K] \in \mathcal{K}$ is the class of convex bodies affinely equivalent to K. We equip $\mathcal{K}$ with the Banach-Mazur distance[2] d_{BM}, and remark that if K is a convex body and μ_K is the uniform measure on K then the limit of order types ℓ_{μ_K} depends only on $[K]$. We prove:

▶ **Theorem 3.** *Let K and K' be two planar convex bodies.*

(i) *If for any $\omega \in \mathcal{O}$ we have $p(\omega, \mu_K) = p(\omega, \mu_{K'})$ then K and K' are affinely equivalent.*
(ii) *For any $\omega \in \mathcal{O}$ we have $|p(\omega, \mu_K) - p(\omega, \mu_{K'})| \leq 2|\omega| d_{BM}(K, K')$.*

As a consequence, the map $[K] \in \mathcal{K} \mapsto \ell_{\mu_K} \in \mathcal{L}$ is a homeomorphism to its image.

The type of rigidity expressed by Theorem 3 extends to a broader class of measures (see the journal version).

We next show that there exists a limit of order types that cannot be represented, in the sense defined above, by a measure. The gist of the construction is to consider a sequence of measures whose weak limit (in the measure sense) contains a Dirac mass. Specifically, for any real $t \in (0,1)$, let $\odot_t$ be a probability distribution over $\mathbb{R}^2$ supported on two concentric circles, with radii 1 and t, respectively. Each of the two circles has $\odot_t$-measure $1/2$, distributed proportionally to the length on that circle. We denote by $\ell_{\odot_t}$ the limit of order types associated to $\odot_t$ (*cf* Lemma 7) and we let $\ell_\odot$ be the limit of a convergent sub-sequence of $\{\ell_{\odot_{1/n}}\}_{n \in \mathbb{N}^*}$. Here we prove:

▶ **Proposition 4.** *If μ is a compactly supported measure over $\mathbb{R}^2$ then there exists $\omega \in \mathcal{O}$ such that $p(\omega, \mu) \neq \ell_\odot(\omega)$.*

The proof that the compactness assumption can be removed is postponed to the journal version.

We finally examine a variation on constructions of Erdős and Szekeres [7] and Horton [8] to construct a limit of order types that no measure that is somewhere regular can represent. We first define inductively a sequence $\{P_n\}_{n \in \mathbb{N}}$ of point sets. The set P_0 consists of a single point. Assuming P_n has been constructed, we let P_{n+1} to be the union of two congruent copies of P_n, P_n^0 and P_n^1, so that the following is true: any point in P_n^1 lies above every line spanned by two points from P_n^0, any point from P_n^0 lies below every line spanned by two points from P_n^1, and the least x coordinate of a point in P_n^1 is greater than the greatest x coordinate of a point in P_n^0. We then let ω_n denote the order type of P_n and let ℓ_H denote the limit of some convergent subsequence of $\{\omega_n\}$.

[2] Recall that d_{BM} is $d_{BM}([K],[K']) := \ln\left(\inf\{r : r \in \mathbb{R}^+, \exists A \in GA(2,\mathbb{R}) : K \subset AK' \subset rK\}\right)$ where rK denotes a scaling of K by a factor r; we abuse the terminology here as it is a distance only for symmetric convex sets.

▶ **Proposition 5.** *If μ is a measure over $\mathbb{R}^2$ that is, on an open set of positive μ-measure, absolutely continuous to either the Lebesgue measure or the length measure on a C^2 curve then there exists $k \geq 4$ such that $p(\diamond_k, \mu) > \ell_H(\diamond_k)$.*

Our proof hinges on the fact that when $k \to \infty$, $\ell_H(\diamond_k)$ decays faster than $p(\diamond_k, \mu)$ for any of the measures considered. For perspective, recall that it is known that the rectilinear crossing number equals the infimum, over all open sets $U \subset \mathbb{R}^2$ with finite Lebesgue measure, of $p(\diamond_4, \mu_U)$, where μ_U is the Lebesgue measure restricted to U [12].

2 Limits of order types

Order types can be understood as equivalence classes of chirotopes under the action of permutations (see below). As such, they are an example of *models* in the language of Razborov [11], and the theory of limits of order types is a special case of Razborov's work. In this section, we give a geometric presentation of the various faces of limits of order types. We intend the presentation to be as self-contained as possible, and refer to general results of Razborov when needed.

Limits as probability distributions on order types. The *split probability* $p(\omega', \omega''; \omega)$, where $\omega', \omega'', \omega$ are order types, is the probability that a random partition of a point set realizing ω into two classes of sizes $|\omega'|$ and $|\omega''|$, chosen uniformly among all such partitions, produces two sets with respective order types ω' and ω''. (In particular $p(\omega', \omega''; \omega) = 0$ if $|\omega| \neq |\omega_1| + |\omega_2|$.)

Fix two order types $\omega', \omega'' \in \mathcal{O}$, consider a converging sequence $\{\omega_n\}_{n\in\mathbb{N}}$ of order types, and let n_0 be such that $|\omega_n| \geq |\omega'| + |\omega''|$ for any $n \geq n_0$. For any $n \geq n_0$ let

$$\alpha_n = p(\omega', \omega_n)p(\omega'', \omega_n) \qquad \text{and} \qquad \beta_n = \sum_{\omega\in\mathcal{O}_{|\omega'|+|\omega''|}} p(\omega', \omega''; \omega)p(\omega, \omega_n).$$

Now, fix some point set P with order type ω_n. On the one hand, α_n equals the probability that two independent events both happens: (i) that a set P' of $|\omega'|$ random points chosen uniformly from P have order type ω', and (ii) that another set P'' of $|\omega''|$ random points chosen uniformly from P have order type ω''. On the other hand, observe that β_n equals the probability that (i) and (ii) happen *and* that P' and P'' are disjoint. The difference $|\alpha_n - \beta_n|$ is therefore bounded from above by the probability that P' and P'' intersect. Bounding from above the probability that P' and P'' have an intersection of one or more elements by the expected size of $P' \cap P''$, we have

$$\left|p(\omega', \omega_n)p(\omega'', \omega_n) - \sum_{\omega\in\mathcal{O}_{|\omega'|+|\omega''|}} p(\omega', \omega''; \omega)p(\omega, \omega_n)\right| \leq \mathbb{E}\left(|P' \cap P''|\right) = \frac{|\omega'||\omega''|}{|\omega_n|}. \tag{2}$$

Taking $n \to \infty$ in (2) we see that every limit of order types ℓ satisfies

$$\forall \omega', \omega'' \in \mathcal{O}, \qquad \ell(\omega')\ell(\omega'') = \sum_{\omega\in\mathcal{O}_{|\omega'|+|\omega''|}} p(\omega', \omega''; \omega)\ell(\omega). \tag{3}$$

These internal consistency relations provide the following alternative characterization of limits as families of distributions on order types:

▶ **Proposition 6** (Lovasz and Szegedy [9, Theorem 2.2], Razborov [11, Theorem 3.3])**.** *A function $\ell : \mathcal{O} \to \mathbb{R}$ is a limit of order types if and only if it satisfies Condition* (3) *and for every $n \in \mathbb{N}$ the restriction $\ell_{|\mathcal{O}_n}$ is a probability distribution on $\mathcal{O}_n$.*

Limits from measures over $\mathbb{R}^2$. As spelled out in the paragraph following Proposition 2, measures over $\mathbb{R}^2$ provide examples of limits of order types.

▶ **Lemma 7.** *The map $\ell_\mu : \omega \in \mathcal{O} \mapsto p(\omega, \mu)$ is a limit of order types if and only if μ is a measure for which every line is negligible.*

Proof. Assume that ℓ_μ is a limit of order types and let $\{\omega_n\}_{n\in\mathbb{N}}$ be a sequence converging to μ. Let $\therefore$ denote the order type of size 3. We have

$$p(\therefore, \mu) = \ell_\mu(\therefore) = \lim_{n\to\infty} p(\therefore, \omega_n) = 1$$

so three random points chosen independently from $\frac{1}{\mu(\mathbb{R}^2)}\mu$ are aligned with probability 0, and every line is negligible for μ.

Conversely, assume that μ is a measure for which every line is negligible. For every $n \geq 3$ the restriction of ℓ_μ to $\mathcal{O}_n$ is a probability distribution. Moreover, for any order types $\omega', \omega'' \in \mathcal{O}$ we have

$$Pr_\mu(\omega')Pr_\mu(\omega'') = \sum_{\omega\in\mathcal{O}_{|\omega'|+|\omega''|}} Pr_\mu(\omega)p(\omega', \omega''; \omega)$$

since the union of two independent random sets of sizes $|\omega_1|$ and $|\omega_2|$ has size $|\omega_1| + |\omega_2|$ almost surely. Proposition 6 implies that ℓ_μ is a limit of order types. ◀

Limits as positive algebra homomorphisms. Let $\{\omega_n\}_{n\in\mathbb{N}}$ be a sequence of order types converging to a limit ℓ. Let $\omega \in \mathcal{O}$, let $k \geq |\omega|$ and let n_0 be large enough so that $|\omega_n| \geq k$ for $n \geq n_0$. A simple conditioning argument yields that for any $n \geq n_0$,

$$p(\omega, \omega_n) = \sum_{\omega'\in\mathcal{O}_k} p(\omega, \omega')p(\omega', \omega_n).$$

Indeed, the probability that a random sample realizes ω is the same if we sample uniformly $|\omega|$ points from a realization of ω_n, and if we sample k points uniformly from that realization, then select a subset of $|\omega|$ of these k points uniformly. It follows that any limit ℓ of order types satisfies:

$$\forall\omega \in \mathcal{O}, \forall k \geq |\omega|, \quad \ell(\omega) = \sum_{\omega'\in\mathcal{O}_k} p(\omega, \omega')\ell(\omega'). \tag{4}$$

Now, let $\mathbb{R}\mathcal{O}$ be the set of all finite formal linear combinations of elements of $\mathcal{O}$ with real coefficients and consider the quotient vector space

$$\mathcal{A} = \mathbb{R}\mathcal{O}/\mathfrak{K} \qquad \text{where} \qquad \mathfrak{K} = \operatorname{vect}\Big\{\omega - \sum_{\omega'\in\mathcal{O}_{|\omega|+1}} p(\omega, \omega')\omega' : \omega \in \mathcal{O}\Big\}.$$

We define a product on $\mathcal{O}$ by

$$\forall\omega_1, \omega_2 \in \mathcal{O}, \qquad \omega_1 \times \omega_2 = \sum_{\omega\in\mathcal{O}_{|\omega_1|+|\omega_2|}} p(\omega_1, \omega_2; \omega)\omega \tag{5}$$

and extend it linearly to $\mathbb{R}\mathcal{O}$. This extension is compatible with the quotient by $\mathfrak{K}$ [11, Lemma 2.4] and therefore turns $\mathcal{A}$ into an algebra.

We call an algebra homomorphism from $\mathcal{A}$ to $\mathbb{R}$ *positive* if it maps every element of $\mathcal{O}$ to a non-negative real, and denote by $\mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ the set of positive algebra homomorphism from $\mathcal{A}$ to $\mathbb{R}$. (Note that any algebra homomorphism sends $\cdot$, the order-type of size one, to the real 1 as it is the neutral element for the product on order types.)

▶ **Proposition 8** ([11, Theorem 3.3b]). *A map $f : \mathcal{O} \to \mathbb{R}$ is a limit of order types if and only if its linear extension is compatible with the quotient by $\mathfrak{K}$ and defines a positive homomorphism from $\mathcal{A}$ to $\mathbb{R}$.*

We write that an element of $\mathcal{A}$ is non-negative when its image under any positive homomorphisms is non-negative. The algebra $\mathcal{A}$ allows us to compute effectively with density relations that hold for *every* limit ℓ.

▶ **Example 9.** Let us denote by $\cdot$ the order type on one point, by and the two order types of size four and by , , and by the three order types of size five, seen as elements of $\mathcal{A}$. From Identity (4) we get

$$= + \frac{3}{5} + \frac{1}{5} \quad \text{and} \quad + + = \cdot \tag{6}$$

Since for any limit of order types ℓ we have $\ell(\cdot) = 1$, the above easily implies that $\ell(\diamond_4) \geq 1/5$. Using again Identity (4), and the non-negativity of we then obtain:

$$\frac{2}{5} \geq - \frac{3}{5} (+ +) = - \frac{3}{5} \cdot$$

and $\ell(\diamond_5) \geq \frac{5}{2}\ell(\diamond_4) - \frac{3}{2}$ for any limit of order types ℓ.

3 The semidefinite method for order types

Let us give an intuition of how the *semidefinite method* works on an example. A simple (mechanical) examination of 6405 order types reveals that $p(\diamond_4, \omega) \geq 19/70$ for any $\omega \in \mathcal{O}_8$. With Identity (4) this implies $\geq 19/70 \cdot$ or equivalently $c_4 \geq 19/70 > 0.2714$. Observe that for any $C \in \mathcal{A}$ and any (linear extension of a) limit of order types ℓ we have $\ell(C \times C) = \ell(C)^2 \geq 0$ by Proposition 8. We thus have at our command an infinite source of inequalities to consider to try and improve the above bounds. For instance, a tedious but elementary computation yields that

$$\left(\frac{6}{25} - \frac{11}{125} \right)^2 + \frac{298819}{1093750} \sum_{\omega \in \mathcal{O}_8} \omega = \sum_{\omega \in \mathcal{O}_8} a_\omega \omega,$$

where $a_\omega \leq p(\diamond_4, \omega)$ for every $\omega \in \mathcal{O}_8$. This implies that $\ell(\diamond_4) \geq 298819/1093750 > 0.2732$ for any limit of order types ℓ. The search for interesting combinations of such inequalities can be done by semidefinite programming.

3.1 Improving the semidefinite method via rooting and averaging

The effectiveness of the semidefinite method for limits of graphs was greatly enhanced by considering partially labelled graphs. We unfold here a similar machinery, using some blend of order types and chirotopes.

Partially labelled point sets, flags, σ-flags and $\mathcal{A}^\sigma$. A point set *partially labelled* by a finite set $\mathcal{Z}$ (the *labels*) is a finite point set P together with some injective map $L : \mathcal{Z} \to P$; we will write this $(P, \mathcal{Z}, L)$ when we need to make explicit the set of labels and the label map. We say that two partially labelled point sets $(P, \mathcal{Z}, L)$ and $(P', \mathcal{Z}, L')$ have the same *flag* if there exists a bijection $\phi : P \to P'$ that preserves both the orientation and the labelling, the latter meaning that $\phi(L(i)) = L'(i)$ for any $i \in \mathcal{Z}$. The relation of having the same flag is an equivalence relation, and a *flag* is an equivalence class for this relation. Again, we call any partially labelled point set a realization of its equivalence class, and the size $|\tau|$ of a flag τ is the cardinality of any of its realizations.

We call a flag where all the points are labelled, *ie* where $|P| = |\mathcal{Z}|$ in some realization $(P, \mathcal{Z}, L)$, a $\mathcal{Z}$*-chirotope*. (When $\mathcal{Z} = [k] = \{1, 2, \ldots, k\}$ a $\mathcal{Z}$-chirotope coincides with the classical notion of chirotope.) Discarding the unlabelled part of a flag τ with label set $\mathcal{Z}$ yields some $\mathcal{Z}$-chirotope σ called the *root* of τ. We call a flag with root σ a σ*-flag* and we denote by $\mathcal{X}^\sigma$ the set of σ-flags. The *unlabelling* $\tau^\emptyset$ of a flag τ with realization $(P, \mathcal{Z}, L)$ is the order type of P.

Let $\mathcal{Z}$ be a set of labels and σ a $\mathcal{Z}$-chirotope. We define densities and split probabilities for σ-flags like for order types. Namely, let τ, τ' and τ'' be σ-flags realized, respectively, by $(P, \mathcal{Z}, L)$ and $(P', \mathcal{Z}, L')$ and $(P'', \mathcal{Z}, L'')$. The *density* of τ in τ' is the probability that for a random subset S of size $|P| - |\mathcal{Z}|$, chosen uniformly in $P' \setminus L'(\mathcal{Z})$, the partially labelled set $(S \cup L'(\mathcal{Z}), \mathcal{Z}, L')$ has flag τ. The *split probability* $p(\tau, \tau'; \tau'')$ is the probability that for a random subset S of size $|P| - |\mathcal{Z}|$, chosen uniformly in $P'' \setminus L''(\mathcal{Z})$, the partially labelled set $(S \cup L''(\mathcal{Z}), \mathcal{Z}, L'')$ and $(P'' \setminus S, \mathcal{Z}, L'')$ have, respectively, flags τ and τ'.

We can finally define an algebra of σ-flags as for order types. We equip the quotient vector space

$$\mathcal{A}^\sigma = \mathbb{R}\mathcal{X}^\sigma / \mathfrak{K}^\sigma \qquad \text{where} \qquad \mathfrak{K}^\sigma = \mathrm{vect}\Big\{\omega - \sum_{\omega' \in \mathcal{X}^\sigma_{|\omega|+1}} p(\omega, \omega')\omega' : \omega \in \mathcal{X}^\sigma\Big\}$$

by the linear extension of the product defined on $\mathcal{X}^\sigma$ by $\tau \times \tau' = \sum_{\tau'' \in \mathcal{X}^\sigma_{|\tau|+|\tau'|-|\sigma|}} p(\tau, \tau'; \tau'')\tau''$.

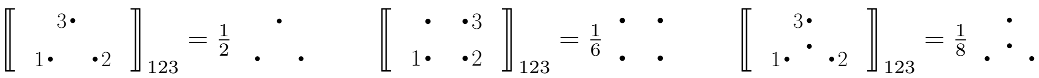
Rooted homomorphisms and averaging. The use of the $\mathcal{A}^\sigma$'s to study $\mathcal{A}$ relies on three tools which we now introduce. We first define an *embedding* of a $\mathcal{Z}$-chirotope in an order type ω as a σ-flag with root σ and unlabelling ω. We use *random embeddings* with the following distribution in mind: fix some point set realizing ω, consider the set I of injections $f : \mathcal{Z} \to P$ such that $(P, \mathcal{Z}, f)$ is a σ-flag, choose some injection f_r from I uniformly at random, and consider the flag of $(P, \mathcal{Z}, f_r)$. We call this the *labelling* distribution on embeddings of σ in ω.

Next, we associate to any convergent sequence of order types $\{\omega_n\}_{n\in\mathbb{N}}$, and for any $\mathcal{Z}$-chirotope σ, a probability distribution on $\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$. For any $n \in \mathbb{N}$, the labelling distribution on embeddings of σ in ω_n defines a probability distribution $\mathbf{P}^\sigma_\mathbf{n}$ on mappings from $\mathcal{A}^\sigma$ to $\mathbb{R}$; specifically, for each embedding θ_n of σ in ω_n we consider the map

$$f_{\theta_n} : \left\{ \begin{array}{ccc} \mathcal{A}^\sigma & \to & \mathbb{R} \\ \tau & \mapsto & p(\tau, \theta_n) \end{array} \right.$$

and assign to it the same probability, under $\mathbf{P}^\sigma_\mathbf{n}$, as the probability of θ_n under the labelling distribution. Since $p(\omega, \omega_n)$ converges as $n \to \infty$ for every $\omega \in \mathcal{O}$, the sequence $\{\mathbf{P}^\sigma_\mathbf{n}\}_{n\in\mathbb{N}}$ weakly converges to a Borel probability measure on $\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ [11, Theorems 3.12 and 3.13] which we denote by $\mathbf{P}^\sigma_\ell$. Moreover, if $\ell(\sigma^\emptyset) > 0$ then the homomorphism induced by ℓ determines the probability distribution $\mathbf{P}^\sigma_\ell$ [11, Theorem 3.5].

We finally define, for any $\mathcal{Z}$-chirotope σ, an *averaging* (or downward) operator $\llbracket\cdot\rrbracket_\sigma : \mathcal{A}^\sigma \to \mathcal{A}$ as the linear operator defined on the elements of $\tau \in \mathcal{X}^\sigma$ by $\llbracket\tau\rrbracket_\sigma := p^\sigma_\tau \cdot \tau^\emptyset$, where p^σ_τ is the probability that a random embedding of σ to $\tau^\emptyset$ (for the labelling distribution) equals τ. Here are a few examples of σ-flags, where $\sigma = 123$ is the CCW chirotope of size 3:



For any given $\mathcal{Z}$-chirotope σ and a limit of order types ℓ, we have the following important identity [11, Lemma 3.11]:

$$\forall \tau \in \mathcal{A}^\sigma, \quad \ell\left(\llbracket \tau \rrbracket_\sigma\right) = \ell\left(\llbracket \sigma \rrbracket_\sigma\right) \int_{\phi^\sigma \in \mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})} \phi^\sigma(\tau) d\mathbf{P}_\ell^\sigma. \tag{7}$$

In particular, $\ell\left(\llbracket C^\sigma \rrbracket_\sigma\right) \geq 0$ for any $C^\sigma \in \mathcal{A}^\sigma$ such that $\phi^\sigma(C^\sigma) \geq 0$ almost surely for $\phi^\sigma \in \mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$, relatively to $\mathbf{P}_\ell^\sigma$; for any limit of order types ℓ and any $\mathcal{Z}$-chirotope σ we therefore have

$$\forall C^\sigma \in \mathcal{A}^\sigma, \quad \ell\left(\left\llbracket (C^\sigma)^2 \right\rrbracket_\sigma\right) \geq 0. \tag{8}$$

3.2 The semidefinite method for order types

The operator $\llbracket \cdot \rrbracket_\sigma$ is linear, so for every $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$, any $A_1^\sigma, A_2^\sigma, \ldots, A_I^\sigma \in \mathcal{A}^\sigma$, and any non-negative reals $z_1, z_2, \ldots, z_I$, we have

$$\phi\left(\left\llbracket \sum_{i \in [I]} z_i \cdot (A_i^\sigma)^2 \right\rrbracket_\sigma\right) \geq 0.$$

For any finite set of flags $S \subseteq \mathcal{O}^\sigma$ and for any real, symmetric, positive semidefinite matrix M of size $|S| \times |S|$, we have $\phi\left(\llbracket v_S^T M v_S \rrbracket_\sigma\right) \geq 0$, where v_S is the vector in $(\mathcal{A}^\sigma)^{|S|}$ whose ith coordinate equals the ith element of S (for some given order). This recasts the search for a good "positive" quadratic combination as a semidefinite programming problem.

Let N be an integer, $f = \sum_{\omega \in \mathcal{O}_N} f_\omega \omega$ some target function, and $\sigma_1, \ldots, \sigma_k$ a finite list of chirotopes so that $|\sigma_i| \equiv N \mod 2$. For each $i \in [k]$, let v_i be the $|\mathcal{X}^{\sigma_i}_{(N+|\sigma_i|)/2}|$-dimensional vector with ith coordinate equal to the ith element of $\mathcal{X}^{\sigma_i}_{(N+|\sigma_i|)/2}$. We look for a real b as large as possible subject to the constraint that there exists k real, symmetric, positive semidefinite matrices $M_1, M_2, \ldots, M_k$, where M_i has size $|v_i| \times |v_i|$, so that

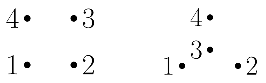
$$\forall \omega \in \mathcal{O}_N, \quad f_\omega \geq a_\omega \qquad \text{where} \qquad \sum_{\omega \in \mathcal{O}_N} a_\omega \omega = \sum_{i \in [k]} \llbracket v_i^T M_i v_i \rrbracket_{\sigma_i} + b \sum_{\omega \in \mathcal{O}_N} \omega. \tag{9}$$

The values of the a_ω's are determined by b, the entries of the matrices $M_1, M_2, \ldots, M_k$, the splitting probabilities $p(\tau', \tau''; \tau)$, where $\tau', \tau'' \in \mathcal{X}^{\sigma_i}_{(N+|\sigma_i|)/2}$ and $\tau \in \mathcal{X}^{\sigma_i}_N$, and the probabilities $p_\tau^{\sigma_i}$, where $\tau \in \mathcal{O}^{\sigma_i}_N$. Moreover, finding the maximum value of b and the entries of the matrices M_i can be formulated as a semidefinite program.

Effective semidefinite programming for flags of order types. In order to use a semidefinite programming software for finding a solution of programs in the form of (9), it is enough to generate the sets $\mathcal{O}_N$ and $\mathcal{X}^{\sigma_i}_N$, the split probabilities $p(\tau', \tau''; \tau)$, where $\tau', \tau'' \in \mathcal{X}^{\sigma_i}_{(N+|\sigma_i|)/2}$ and $\tau \in \mathcal{X}^{\sigma_i}_N$, and the probabilities $p_\tau^{\sigma_i}$, where $\tau \in \mathcal{O}^{\sigma_i}_N$.

We generated the sets and the values by brute force up to $N = 8$. The only non-trivial algorithmic step is deciding whether two order types, represented by point sets, are equivalent. This can be done by computing some canonical ordering of the points that turn two point sets with the same order type into point sequences with the same chirotope. Aloupis et al. [4] recently proposed an algorithm performing that in time $O(n^2)$; the method we implemented takes time $O(n^2 \log n)$ and seems to be folklore (we learned it from Pocchiola and Pilaud). For solving the semidefinite program itself, we used a library called CSDP [5]. The input data for CSDP was generated using a mathematical software SAGE [14].

Setting up the semidefinite programs. In the rest of this section we work with $N = 8$ and use chirotopes labelled $\sigma_1, \sigma_2, \ldots, \sigma_{24}$ where σ_1 the empty chirotope, σ_2 the only chirotope of size two, σ_3 and σ_4 the two chirotopes of size 4 depicted on the left, and $\sigma_5, \ldots, \sigma_{24}$ a fixed set of 20 chirotopes of size 6 so that $\mathcal{O}_6 = \{\sigma_5^\emptyset, \ldots, \sigma_{24}^\emptyset\}$; note that since $|\mathcal{O}_6| = 20$, what follows will not depend on the choices made in labelling $\sigma_5, \ldots, \sigma_{24}$. The vectors $v_1, v_2, \ldots, v_{24}$ described in the previous paragraph for this choice of N and σ_i's have lengths 2, 44, 468, 393, 122, 112, 114, 101, 101, 103, 106, 103, 103, 120, 102, 108, 94, 90, 91, 91, 95, 95, 92, 104, respectively.



Computations proving Propositions 1 and 2. We solved two semidefinite programs with the above choice of parameters for $f = \sum_{\omega \in \mathcal{O}_8} p(\diamond_5, \omega)$ and $f = \sum_{\omega \in \mathcal{O}_8} p(\diamond_6, \omega)$ and obtained real symmetric positive semidefinite matrices $M_1, \ldots, M_{24}$ and $M'_1, \ldots, M'_{24}$ with rational entries so that

$$\sum_{\omega \in \mathcal{O}_8} p(\diamond_5, \omega)\omega \geq \sum_{i \in [24]} [\![v_i^T M_i v_i]\!]_{\sigma_i} + \frac{15715211616602583691}{258254417031933722624} \sum_{\omega \in \mathcal{O}_8} \omega,$$

and

$$\sum_{\omega \in \mathcal{O}_8} p(\diamond_6, \omega)\omega \geq \sum_{i \in [24]} [\![v_i^T M'_i v_i]\!]_{\sigma_i} + \frac{67557324685725989}{368934881474191032320} \sum_{\omega \in \mathcal{O}_8} \omega.$$

The lower bounds on c_5 and c_6 then follow from Identity (4).

Assume (without loss of generality) that $\mathcal{O}_6 = \{\omega_{6,1}, \omega_{6,2}, \ldots, \omega_{6,20}\}$. Solving two semidefinite programs, we obtained real symmetric positive semidefinite matrices $M_1, \ldots, M_{24}$ and $M'_1, \ldots, M'_{24}$ as well as non-negative rational values $d_1, \ldots, d_{20}$ and $d'_1, \ldots, d'_{20}$ so that

$$\sum_{j \in [20]} d_j \left(\omega_{6,j} - \frac{1}{32} \sum_{\omega \in \mathcal{O}_8} \omega \right) + \sum_{i \in [24]} [\![v_i^T M_i v_i]\!]_{\sigma_i} < 0$$

and

$$\sum_{j \in [20]} d'_j \left(-\omega_{6,j} + \frac{1}{18} \sum_{\omega \in \mathcal{O}_8} \omega \right) + \sum_{i \in [24]} [\![v_i^T M'_i v_i]\!]_{\sigma_i} < 0.$$

They imply that there is no $\ell \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ such that, respectively $\ell(\omega) \geq 1/32$ for every $\omega \in \mathcal{O}_6$, or such that $\ell(\omega) \leq 1/18$ for every $\omega \in \mathcal{O}_6$. Together this proves Proposition 2 with an imbalance bound of $32/18 > 1.77$. The better bound of Proposition 2 is obtained by a refinement of this approach where the order types with minimum and maximum probability are prescribed; this requires solving over 700 semidefinite programs.

The numerical values of the entries of all the matrices $M_1, \ldots, M_{24}$ and coefficients $d_1, \ldots, d_{20}$ mentioned above can be downloaded from the web page `http://honza.ucw.cz/proj/ordertypes/`. In fact, the matrices $M_1, \ldots, M_{24}$ are not stored directly, but as an appropriate non-negative sum of squares, which makes the verification of positive semidefiniteness trivial. To make an independent verification of our computations easier, we created sage scripts called "verify_prop*.sage", available from the same web page.

4 Representation of limits by measures

Let $\mathcal{L}$ denote the space of limits of order types endowed with the topology of the distance given by Equation (1). Let $\mathcal{M}$ denote the space of finite measures over $\mathbb{R}^2$ for which every line is negligible, equipped with the topology of the weak convergence[3].

▶ **Proposition 10.** *The map $\mu \in \mathcal{M} \mapsto \ell_\mu \in \mathcal{L}$ is continuous.*

Proof. For $k \geq 1$ and any measure μ over $\mathbb{R}^2$ we let μ^k denote the k-fold product measure over $\mathbb{R}^{2k}$. For any order type ω we let $\mathcal{R}_\omega \subset \mathbb{R}^{2|\omega|}$ denote the space of all realizations of ω, that is $\mathcal{R}_\omega$ contains all $2|\omega|$-tuples $(x_1, y_1, x_2, y_2, \ldots, x_{|\omega|}, y_{|\omega|})$ such that the points $(x_1, y_1), (x_2, y_2), \ldots, (x_{|\omega|}, y_{|\omega|})$ realize ω. Observe that $p(\omega, \mu) = \mu^k(\mathcal{R}_\omega)$.

Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of measures in $\mathcal{M}$ weakly converging to a measure $\mu \in \mathcal{M}$. For any k, the k-fold product measures μ_n^k converge weakly to μ^k. Moreover, for every order type ω the boundary $\partial\mathcal{R}_\omega$ consists solely of planar point sets with at least one aligned triple. The measure $\mu^k(\partial\mathcal{R}_\omega)$ is therefore bounded from above by the probability that $|\omega|$ random points sampled from μ contains at least three aligned points. Since every line is negligible for μ, this ensures that $\mu^k(\partial\mathcal{R}_\omega) = 0$ and therefore for any ω, $\ell_{\mu_n}(\omega) = \mu_n^k(\mathcal{R}_\omega) \to \mu^k(\mathcal{R}_\omega) = \ell_\mu(\omega)$. ◀

In the rest of this section we prove Theorem 3, which strengthens Proposition 10 for uniform measures on convex bodies, and prove Proposition 4 and 5.

4.1 Proof of Theorem 3

The gist of our proof is to relate a convex set K to the limit of order types ℓ_K induced by the measure μ_K through a family of positive algebra homomorphism $\phi_{P,\mu_K,P'}(\tau) \in \mathrm{Hom}^+(\mathcal{A}^{\sigma'}, \mathbb{R})$ defined for any point sequences P and P'.

For two chirotopes σ, σ' we write $\sigma' \rhd \sigma$ and say that σ' extends σ if there exists sequences of points $P = \{p_1, p_2, \ldots, p_n\}$ and $P' = \{p'_1, p'_2, \ldots, p'_{n'}\}$ so that P has chirotope σ and the sequence $P \cup P' := \{q_1, q_2, \ldots, q_{n+n'} : q_i = p_i \text{ for } i \leq n \text{ and } q_i = p'_{i-n} \text{ for } i > n\}$ has chirotope σ'. Let μ be a measure over $\mathbb{R}^2$ for which lines are negligible. For any σ'-flag τ we let $\phi_{P,\mu,P'}(\tau)$ denote the probability that $|\tau| - |\sigma'|$ random unlabeled points chosen independently from μ define, together with $P \cup P'$, a partially labelled sequence realizing τ. The map $\tau \in \mathcal{A}^{\sigma'} \mapsto \phi_{P,\mu,P'}(\tau)$ is easily seen to be a positive algebra homomorphism from $\mathcal{A}^{\sigma'}$ to $\mathbb{R}$. For a fixed P and varying P' such that $n' = |P'|$, we define a map

$$\phi_{P,\mu,\cdot} : \begin{cases} (K)^{n'} & \to \quad \bigcup_{\sigma' \rhd \sigma; |\sigma'| = |\sigma| + n'} \mathrm{Hom}^+(\mathcal{A}^{\sigma'}, \mathbb{R}) \\ \quad P' & \mapsto \quad \{\tau \mapsto \phi_{P,\mu,P'}(\tau)\} \end{cases}$$

where we assume that τ is a σ'-flag and $P \cup P'$ have chirotope σ'. (For the sake of the presentation, we write $\phi_{P,\mu,t}$ in place of $\phi_{P,\mu,\{t\}}$ when applying $\phi_{P,\mu,\cdot}$ to a singleton.) The key fact about this map is that if we push forward $\mu^{n'}$ through $\phi_{P,\mu,\cdot}$ it induces a probability distribution on $\cup_{|\sigma'|=|\sigma|+n', \sigma' \rhd \sigma} \mathrm{Hom}^+(\mathcal{A}^{\sigma'}, \mathbb{R})$ that turns out, due to a theorem of Razborov, to be essentially determined by ℓ_μ. We will denote by $\mathbf{Q}$ a set of n' random points chosen independently from μ, and by $\phi_{P,\mu,\mathbf{Q}}$ the random homomorphism corresponding to the push forward of $\mu^{n'}$.

[3] A sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of measures weakly converges to a measure μ if $\mu_n(A) \to \mu(A)$ for every measurable set A such that $\mu(\partial A) = 0$, where ∂ stands for the topological boundary.

We first argue that the geometry of K, up to affine transformation, can be retrieved from these homomorphisms since they encode ratios of triangle areas that determine certain barycentric coordinates.

▶ **Lemma 11.** *Let K be a convex body, $\{t_1, t_2, t_3, t\} \subset K$. For any triangle T' supported in $\{t_1, t_2, t_3, t\}$, the ratio of the area of T' to the area of $t_1t_2t_3$ is determined by the values of $\phi_{\{t_1,t_2,t_3\},\mu_K,t}$ on σ-flags of size 5, where σ is the chirotope of $\{t_1, t_2, t_3, t\}$.*

Proof. The relative area of a triangle T' with respect to a triangle T is the quotient $\frac{\text{area}(T')}{\text{area}(T)}$. Let us begin with the case in which $t \in \text{conv}(T)$ with $T = \{t_1, t_2, t_3\}$. The point t subdivides T into 3 triangles. Without loss of generality, let τ be the σ-flag corresponding to appending a point t' inside the triangle $\{t, t_2, t_3\}$. By definition $\phi_{T,\mu_k,t}(\tau) = \frac{\text{area}(t,t_2,t_3)}{\text{area}(t_1,t_2,t_3)}$. When t belongs to any of the six remaining regions defined by the lines spanned by $\{t_1, t_2, t_3\}$, a triangle of the form $\{t, t_2, t_3\}$ is divided into two triangles by T, and as before we can determine the relative area of each of these triangles and their sum provides the relative area of $\{t, t_2, t_3\}$. ◀

We next show that measures that induce the same limit give rise to equivalent families of homomorphisms (due to lack of space we defer the proof to the journal version).

▶ **Lemma 12.** *Let μ and μ' be two measures in $\mathbb{R}^2$ for which lines are negligible. Let $\mathbf{Q}$ be a set of m random points chosen independently from μ, and $\mathbf{Q}'$ be a set of m random points chosen independently from μ'. If $\ell_\mu = \ell_{\mu'} = \ell$ then for every chirotope σ such that $\ell(\llbracket\sigma\rrbracket_\sigma) > 0$, there exist sequences of points P and P' with chirotope σ such that $\phi_{P,\mu,\mathbf{Q}} = \phi_{P',\mu',\mathbf{Q}'}$.*

We now have all the ingredients to prove Theorem 3.

Proof of Theorem 3. We begin by proving the consequence of (i) and (ii). The space $(\mathcal{K}, d_{BM})$ is a compact Hausdorff space, so (ii) implies that $\mathcal{L}_\mathcal{K}$ is compact and (i) implies that the map is a bijection with its image. Any continuous bijection from a Hausdorff space to a compact space is a homemorphism.

We now prove (ii). Let $d_{TV}(\mu_1, \mu_2) := \sup_A |\mu_1(A) - \mu_2(A)|$, where the supremum is taken among all measurable sets A, denote the total variation distance between two probability measures μ_1 and μ_2. It is classical that $d_{TV}(\mu_1^k, \mu_2^k) \le k d_{TV}(\mu_1, \mu_2)$ so in particular $|p(\omega, \mu_K) - p(\omega, \mu_{K'})| \le |\omega| d_{TV}(\mu_K, \mu'_K)$. Hence it is enough to show that $d_{TV}(\mu_K, \mu_{gK'}) \le 2d_{BM}(K, K')$ for some nondegenerate affine transformation g. Without loss of generality we can assume that $K \subset K' \subset rK$ where $r = e^{d_{BM}(K,K')}$. Since $K \subset K'$ the supremum $\sup_A |\mu_K(A) - \mu_{K'}(A)|$ is attained by $A = K$. Indeed, for every measurable set A, the signed measure $\mu_K(A) - \mu_{K'}(A) = \frac{\text{area}(A\cap K)}{\text{area } K} - \frac{\text{area}(A\cap K')}{\text{area } K'}$ does not decrease by substituting A by $A' = A \cap K$, and among subsets of K this signed measure does not decrease by substituting A by a superset. Hence $d_{TV}(\mu_K, \mu_{K'}) = 1 - \frac{\text{area}(K)}{\text{area}(K')} \le 1 - \frac{\text{area}(K)}{\text{area}(rK)} = 1 - \frac{1}{r^2} \le 2 \ln r$. The last inequality is true provided $r \le 1$, which is the case.

Finally we prove item (i). Let K and K' be two convex bodies such that $\ell_K = \ell_{K'}$. By Lemma 12, there exists triangles T and T' such that $\phi_{T,\mu_K,\mathbf{t}} = \phi_{T',\mu_{K'},\mathbf{t}'}$, where $\mathbf{t}$ and $\mathbf{t}'$ are points chosen uniformly at random from K and K' respectively. Define the signed area of an ordered triangle as its area multiplied by its orientation (i.e. it is positive if the triangle is CCW oriented and negative otherwise) and denote it by area*. Remark that the relative signs of the triangles depend only on the chirotope σ' of $\{t_1, t_2, t_3, t\}$. By Lemma 11, for every $t \in K$ the homomorphism $\phi_{T,\mu_K,t} \in \cup_{|\sigma'|=4} \text{Hom}^+(\mathcal{A}^{\sigma'}, \mathbb{R})$ is enough to reconstruct the relative area* with respect to T of each triangle in t_1, t_2, t_3, t. Using barycentric coordinates and T as an affine basis:

$$t = \frac{\text{area}^*(t, t_2, t_3)}{\text{area}^*(t_1, t_2, t_3)} t_1 + \frac{\text{area}^*(t_1, t, t_3)}{\text{area}^*(t_1, t_2, t_3)} t_2 + \frac{\text{area}^*(t_1, t_2, t)}{\text{area}^*(t_1, t_2, t_3)} t_3,$$

we recover t from $\phi_{T,\mu_K,t}$. Writing t in this way for every homomorphism in the support of $\phi_{T,\mu_K,\mathbf{t}}$ we reconstruct the convex body K. Analogously, writing t' using T' as an affine basis and $\phi_{T',\mu_{K'},t'}$ to compute the relative areas for every homomorphism in the support of $\phi_{T',\mu_{K'},\mathbf{t}'}$, we reconstruct K'. Since $\phi_{T',\mu_{K'},\mathbf{t}'}$ and $\phi_{T,\mu_K,\mathbf{t}}$ are identical, K' is the image of K under the affine map taking T to T'. ◀

4.2 Proof of Proposition 4

It is perhaps tempting, when searching for a measure representing a given limit ℓ, to take a sequence of random order types $\mathbf{r_n}$ from ℓ, with $\lim_{n\to\infty} |\omega_n| = \infty$, take for each n a realization P_n of ω_n and expect that the empirical measure $\mu_{P_n} := \frac{1}{|P_n|}\sum_{s\in P_n}\delta_s$ converges to a measure representing ℓ. The next lemma gives necessary and sufficient conditions for this approach to work (due to space constraint we defer the proof to the journal version):

▶ **Lemma 13.** *Let ℓ be a limit of order types. There exists a measure μ for which lines are negligible and such that $P(\omega,\mu) = \ell(\omega)$ for all $\omega \in \mathcal{O}$ if and only if there exists a sequence of point sets $\{P_n\}_{n\in\mathbb{N}}$ whose order types converge to ℓ and such that for any $\epsilon > 0$ the following two conditions hold:*

(i) *there exists $R > 0$ such that for n large enough, all but at most a fraction ϵ of P_n lies within distance R from the origin.*

(ii) *for any line $h \subset \mathbb{R}^2$, there exists $\delta > 0$ such that for n large enough, the fraction of points from P_n within distance δ from h is at most ϵ,*

The condition of Lemma 13 is both necessary and sufficient, and allows us to prove that $\ell_\odot$ cannot be represented by a compactly supported measure.

Proof of Proposition 4. Let $\mathbf{R_n^t}$ be a point set of size $N = n^2$ sampled according to $\odot_t$. Order the points of $\mathbf{R_n^t}$ on the boundary of $\partial(\mathrm{conv}(\mathbf{R_n^t}))$ following the counterclockwise orientation. Denote this set by $\mathrm{out}(\mathbf{R_n^t}) := \{\mathbf{s_1}, \mathbf{s_2}, \ldots, \mathbf{s_m}\}$ and order its complement in some arbitrary fashion and denote it by $\mathrm{in}(\mathbf{R_n^t}) := \{\mathbf{t_1}, \mathbf{t_2}, \ldots, \mathbf{t_{N-m}}\}$.

For each point $s_i \in \mathrm{out}(\mathbf{R_n^t})$ consider the total order on $\mathbf{R_n^t} \setminus \{\mathbf{s_i}\}$ induced by rotating a semiline about s_i, starting with the semiline at s_{i+1}. This order is called *the local sequence* of s_i. It is well known and not hard to show that it is a chirotope invariant. In this case, the local sequence of $s_i \in \mathrm{conv}(\mathbf{r_n})$, is $(s_{i+1}, s_{i+2}, \ldots, s_j, t_{k_1}t_{k_2}, \ldots, t_{k_{\mathrm{in}(\mathbf{r_n})}} s_{j+1}s_{j+2}, \ldots, s_{i-1})$, where the order of the points in $\mathrm{in}(r_n)$ depends on i, but this will be irrelevant. Denote by $j\colon \mathrm{out}(\mathbf{R_n^t}) \to \mathrm{out}(\mathbf{R_n^t})$ a function that assigns to s_i the last element of $\mathrm{out}(\mathbf{R_n^t})$ in its local sequence before it reaches the elements of $\mathrm{in}(\mathbf{R_n^t})$. Since the number of points in $\mathrm{in}(\mathbf{R_n^t})$ is distributed like a binomial with N trials and probability $\frac{1}{2}$, for each i the triangle $\mathrm{conv}(s_i, s_{j(i)}, s_{j(i)+1})$ contains the points of $\mathrm{in}(\mathbf{R_n^t})$ with probability at least $1 - \frac{1}{2^{N+2}}f(t)$, where $f(t)$ is a continuous function that approaches 1 as t approaches 0 . By the union bound this happens for all i with probability at least $1 - \frac{N}{2^{N+2}}f(t)$. Let $|j(i) - i|$ be the number of vertices on $\mathrm{out}(\mathbf{R_n^t})$ on a counterclockwise walk on $\partial(\mathrm{conv}(\mathbf{R_n^t}))$. For each i, the random variable $|j(i) - i|$ is distributed like a binomial with N trials and probability $\frac{1}{4}$. Hoeffding inequality implies that there exists an absolute constant $C > 0$ such that,

$$Pr\Big[||j(i)-i| - \frac{N}{4}| \geq C\sqrt{N\log N})\Big] = O\left(\frac{1}{N^2}\right).$$

By the union bound,

$$Pr\Big[\forall i : ||j(i)-i| - \frac{N}{4}| > C\sqrt{N\log N}\Big] = O\left(\frac{1}{N}\right).$$

We can conclude that with high probability the image of j contains more than $\Omega(\sqrt{\frac{N}{\log N}})$ points and that each triangle of the form $\mathrm{conv}(s_i, s_{j(i)}, s_{j(i)+1})$ contains $\mathrm{in}(\mathbf{R_n^t})$.

Now assume for contradiction that μ is a compactly supported measure representing $\ell_\odot$. Let $\mathbf{r_n}$ be a random order type of size $N = n^2$ chosen according to $\ell_\odot$. Let $\mathbf{R_n}$ be a set of N points sampled uniformly and independently from μ. Since μ represents $\ell_\odot$ the order type of $\mathbf{R_n}$ is distributed like $\mathbf{r_n}$. Let $\mathbf{r_n^t}$ be the random order type of $\mathbf{R_n^t}$. Define $\mathrm{out}(\mathbf{R_n}) := \{\mathbf{s'_1}, \mathbf{s'_2}, \dots, \mathbf{s'_{m'}}\}$ and $\mathrm{in}(\mathbf{R_n}) := \{\mathbf{t'_1}, \mathbf{t'_2}, \dots, \mathbf{t'_{N-m'}}\}$, analogously as we did for of $\mathrm{out}(\mathbf{R_n^t})$ and $\mathrm{in}(\mathbf{R_n^t})$ for $\mathbf{R_n}$. Since the distributions of order types $\mathbf{r_n^t}$ and $\mathbf{r_n}$ can be made arbitrarily close in total variation distance by making t small enough, we can conclude that with high probability, for each i, $\mathrm{conv}[s'_i, s'_{j(i)}, s'_{j(i)+1}]$ contains $\mathrm{in}(\mathbf{R_n})$.

On the other hand, if the support of μ has finite perimeter, then the sum of the lengths of the edges of $\mathrm{out}(\mathbf{R_n})$ is also finite, hence the infimal length among such edges approaches zero as n approaches infinity. Let i_0 be such that the edge $s_{j(i_0)}, s_{j(i_0)+1}$ realizes the infimal length. Let h be the line spanned by s_{i_0} and $s_{j(i_0)}$. Given $\epsilon < \frac{1}{8}$, there exists $\delta(\epsilon) > 0$ such that $\mu(h + B(\delta)) < \epsilon/2$ and hence, by the law of large numbers $\mu_{\mathbf{R_n}}(h + B(\delta)) < \epsilon$ almost surely. But we showed that $\mathrm{conv}[s'_{i_0}, s'_{j(i_0)}, s_{j(i_0)'+1}]$ contains $\mathrm{in}(\mathbf{R_n})$ with high probability, which implies that $\mu_{\mathbf{R_n}}(h + B(\delta)) > \frac{1}{2} - \epsilon$ with high probability, which is a contradiction. ◀

4.3 Proof of Proposition 5

Recall that $\diamond_k$ is the order type of k points in convex position. It is folklore that any set of n points contains at least $\frac{k^{3k/2}}{4^{k^2}}\binom{n}{k}$ subsets of k points in convex position, so for any limit of order types ℓ we must have $\ell(\diamond_k) \geq \frac{k^{3k/2}}{4^{k^2}}$ (due to space constraint we defer the proof to the journal version). We first show that this bound is essentially attained by ℓ_H:

▶ **Lemma 14.** $\ell_H(\diamond_k) \leq 2^{-\frac{k^2}{2}+k\log k}$.

Proof. Define a k-cup to be a sequence of points lying on the graph of a convex function, and a k-cap to be a sequence of points lying on the graph of a concave function. Let $q_+(k, P_n)$ be the fraction of k-tuples of P_n forming a k-cup, and $q_-(k, P_n)$ be the fraction of k-tuples of P_n forming a k-cap. Since a k-tuple in convex position contains either a $\frac{k}{2}$-cup or a $\frac{k}{2}$-cap the union bound gives $p(\diamond_k, \omega_n) \leq q_+(\frac{k}{2}, P_n) + q_-(\frac{k}{2}, P_n)$. By symmetry is enough to bound $q_+(k, P_n)$. Denote by $Q_+(k, P_n)$ the number of k-cups in P_n. Since every k-cup containing points from P_n^0 and P_n^1 contains at most one point from P_n^1,

$$Q_+(k, P_{n+1}) \leq Q_+(k-1, P_n^0)|P_n^1| + Q_+(k, P_n^0) + Q_+(k, P_n^1).$$

Note that $Q_+(3, P_n) \leq \binom{2^n}{3}$ and $Q_+(k, P_0) \leq 1$. By induction on $n + k$ we get that $Q_+(k, P_n) \leq 2^{nk-\frac{k^2}{2}}$. With Stirling's formula, we thus have $q_+(k, P_n) \leq 2^{-\frac{k^2}{2}+k\log k}$ for n large enough. ◀

We next bound from below $p(\diamond_k, \mu)$ under some regularity assumptions on μ. These bounds are up to an undetermined constant; the fact that the rate of decay of $p(\diamond_k, \mu)$ is by an order of magnitude slower than that of $\ell_H(\diamond_k)$ is enough, however, to ensure that for any such μ there exists some n such that $p(\diamond_k, \mu) \neq \ell_H(\diamond_k)$, thus proving Proposition 5.

▶ **Lemma 15.** *Let μ be a measure over $\mathbb{R}^2$ for which lines are negligible.*

(i) *If there exists an open set of positive μ-measure on which μ is absolutely continuous to the Lebesgue measure then $p(\diamond_k, \mu) \geq 2^{-2k\log k + O(k)}$.*

(ii) *If there exists an open set of positive μ-measure on which μ is absolutely continuous to the length measure on a C^2 curve then $p(\diamond_k, \mu) \geq 2^{-O(k)}$.*

The number of different order types in the plane is $2^{4n\log n}$, up to multiplicative factors of order $2^{o(n\log n)}$ [3, Section 4]. Notice that the asymptotic bounds presented on $p(\diamond_k,\mu)$ both for smooth curves and for the Lebesgue measure, imply that there exists a sequence of order types ω_k such that $\frac{\ell_\mu(\omega_k)}{\ell_\mu(\diamond_k)}$ approaches zero as k approaches infinity. On the other hand, the bounds for $\ell_H(\diamond_k)$ imply that there exists an order type such that $\frac{\ell_H(\omega_k)}{\ell_H(\diamond_k)}$ approaches infinity.

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