# Accounting for Risk in Social Cost-Benefit Analysis
## Recursive Approaches

DISS. ETH NO. 22843

# Accounting for Risk in Social Cost-Benefit Analysis: Recursive Approaches

A thesis submitted to attain the degree of

DOCTOR OF SCIENCES of ETH ZÜRICH

(Dr. sc. ETH Zürich)

presented by

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2015

# Acknowledgements

Zurich, September 2015 *Svenja Hector*

# Abstract

In due consideration of attitudes towards risk, this thesis examines the consumption or social discount rate, one of the most controversially discussed concepts in the economic assessment of climate change policy.

**Chapter I** gives an overview of consumption discounting in the standard preference framework and goes into some of the related discussions. In particular, I describe the derivation of the consumption discount rate, elaborate on common parameter assumptions and some philosophical disputes, and dwell on the discounting debates that surround the Stern Review on the Economics of Climate Change (Stern 2007) and Nordhaus' (2008) integrated assessment model DICE. Eventually, I point to the problematic entanglement of risk and time preferences in the standard preference model, and emphasize recursive frameworks as an alternative.

**Chapter II** employs a two-period version of the recursive preference model to clarify the link between changes in risk aversion and the effect on the consumption discount rate. In a general framework that can cope with various forms of uncertainty, it is shown that the response of the consumption discount rate to a change in risk aversion depends on some fundamental properties of the considered uncertainties. The application of this general result to specific forms of uncertainty extends existing results to more general forms of risk and yields new findings related to preference uncertainty.

**Chapter III** revisits the consumption discount rate for a novel combination of standard assumptions. To disentangle risk and time preferences, I consider a decision maker with recursive preferences à la Kreps and Porteus (1978). Moreover, I assume that preferences are mutually utility independent in the sense of Koopmans (1960). In a first instance, I show that utility independence restricts Kreps-Porteus preferences to a specific parametric form, namely to the constant absolute risk aversion form of Hansen and Sargent's (1995) Risk-Sensitive preferences. Coming from a decision maker with Risk-Sensitive preferences, I analyze the consumption discount rate in an infinite horizon setting with independent growth risk and constant elasticity of substitution. I show that the discount rate is diminished by a previously unrecognized horizon effect, which may be significant if the rate of pure time preference is moderately small.

**Chapter IV** numerically examines the discount rate of a Risk-Sensitive social agent and thus yields insights on the magnitude of the horizon effect. In particular, I analyze how the size of the horizon effect relates to the rate of pure time preference and to the intertemporal elasticity of substitution. Taking a descriptive approach to discounting, a private agent's preference model is calibrated to the risk-free rate of return and the thus inferred risk preferences are employed as a descriptor of a social agent. Crucially, I postulate that the social agent's horizon extends that of the private agent.

# Zusammenfassung

Die vorliegende Dissertation beschäftigt sich, unter besonderer Berücksichtigung von Risikopräferenzen, mit der monetären Diskontrate, welche ein äusserst kontrovers diskutiertes Konzept in der ökonomischen Bewertung von Klimapolitik darstellt.

**Kapitel I** gibt einen Überblick über das Konzept der Diskontierung im Rahmen gängiger Präferenzmodelle und geht auf bestehende Debatten ein. Insbesondere beschreibe ich die Herleitung der monetären Diskontrate, erläutere geläufige Parameterwerte und philosophische Zerwürfnisse, und greife die Debatte bezüglich des Berichts zur Ökonomie des Klimawandels (Stern 2007) und dem Bewertungsmodell DICE (Nordhaus 2008) auf. Schliesslich verweise ich auf die problematische Verflechtung von Risiko- und Zeitpräferenzen im Standardmodell und hebe rekursive Modelle als Alternative hervor.

**Kapitel II** bedient sich einer zwei-Perioden Version des rekursiven Präferenzmodells, um den Einfluss von Veränderungen der Risikopräferenz auf die monetäre Diskontrate zu klären. In einem Modell, welches die Berücksichtigung verschiedener Unsicherheiten ermöglicht, wird gezeigt, dass der Effekt auf die Diskontrate von grundlegenden Eigenschaften der jeweiligen Unsicherheit abhängt. Die Anwendung dieses generellen Ergebnisses auf spezifische Unsicherheiten erweitert bestehende Resultate hinsichtlich der Risikospezifizierung und liefert neue Erkenntnisse bezüglich unsicherer Präferenzen.

**Kapitel III** analysiert die Diskontrate für eine neuartige Kombination gängiger Annahmen. Um Risiko- und Zeitpräferenzen zu entflechten, betrachte ich einen Entscheidungsträger mit rekursiven Präferenzen im Sinne von Kreps und Porteus (1978). Weiterhin unterstelle ich, dass diese Präferenzen eine Annahme von wechselseitiger Nutzenunabhängigkeit entsprechend Koopmans (1960) erfüllen. Zunächst zeige ich, dass Nutzenunabhängigkeit die spezifische Form von Kreps-Porteus Präferenzen einschränkt, so dass diese den Risiko-Sensitiven Präferenzen von Hansen und Sargent (1995) entsprechen. Ausgehend von einem Entscheidungsträger mit Risiko-Sensitiven Präferenzen analysiere ich die Diskontrate in einem Modell mit unendlichem Zeithorizont, unabhängigem Wachstumsrisiko und konstanter Substitutionselastizität. Es zeigt sich, dass die Diskontrate durch einen bisher unerkannten Zeithorizont-Effekt vermindert wird, welcher insbesondere dann von Belang ist, wenn die Rate der reinen Zeitpräferenz relativ gering ist.

**Kapitel IV** untersucht die Diskontrate eines Risiko-Sensitiven Agenten mittels numerischer Methoden. Insbesondere analysiere ich wie die Grössenordnung des Zeithorizont-Effekts von der Rate der reinen Zeitpräferenz und der Substitutionselastizität abhängt. Unter Annahme eines deskriptiven Ansatzes justiere ich die Präferenzen eines privaten Agenten anhand des risikolosen Zinssatzes, und verwende die dadurch abgeleiteten Risikopräferenzen zur Beschreibung eines sozialen Agenten. Entscheidend ist, dass der Zeithorizont des sozialen Agenten grösser ist als jener des privaten Agenten.

# Contents

# List of figures

# List of tables

# List of abbreviations

**CARA** constant absolute risk aversion

**CES** constant elasticity of substitution

**CRRA** constant relative risk aversion

**DM** decision maker

**EZ** Epstein-Zin

**KP** Kreps-Porteus

**MPI** mutual preference independence

**MUI** mutual utility independence

**PI** preference independence

**RS** Risk-Sensitive

**UI** utility independence

# Introduction

A benchmark of major importance in the evaluation of any public investment is a project's performance in cost-benefit analysis. To ensure that the funds entrusted to public decision makers are allocated in an efficient manner, the costs and the benefits of a given project must be assessed and weighed against each other. Only those projects that yield a positive net value, those that are desirable from a societal point of view, should be conducted. In assessing the net value of an investment project, cost-benefit analysis goes beyond the mere identification of financial in- and out-flows. Rather, it accounts for costs and benefits in various, often non-financial categories, converts them into monetary equivalents, and appraises them in accordance with presumed societal preferences.

A crucial aspect of this appraisal concerns the societal valuation of costs and benefits that accrue at different times, and under various economic circumstances. The standard assumption in this context is that costs and benefits are of smaller worth if they accrue in the future rather than in the present, i.e. future monetary values are discounted. Widely accepted rationales for such a devaluation of the future are a notion of empathic distance to future generations or the possibility that future generations may not exist (as reflected by the utility discount rate), and an aversion towards wealth inequality across generations (as reflected by the intertemporal elasticity of substitution). The *social or consumption discount rate*–the rate at which the societal valuation of monetary or consumption equivalents changes with time and circumstances–accounts for these rationales.[1]

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[1] Note the distinction of the social or consumption discount rate to the utility discount rate: The social/consumption discount rate is the rate at which future *monetary* values are discounted; the utility discount rate (also known as the rate of pure time preference, the rate of impatience, or the rate of empathic distance) is the rate at which future *felicity* (the lower case $u$) is discounted.

Disputes on the consumption discount rate have a long tradition in the economic assessment of public investment projects. A major point of disagreement concerns the question whether the consumption discount rate should be aligned along the opportunity costs of capital, and thus along society's revealed preferences for the allocation of wealth over time, or whether its choice should be guided by ethical principles. The former approach to discounting is the prevalent one in public project appraisal. As a result of the orientation along market rates of return and revealed preferences, this *descriptive* approach tends to imply rather high consumption discount rates. The European Commission's (2014) *Guide to Cost-benefit Analysis of Investment Projects*, for example, recommends the usage of a 5% rate as it reflects "the opportunity cost of capital from an inter-temporal perspective for society as a whole" (p. 289). Proponents of the latter, *prescriptive* approach to discounting typically advocate significantly lower rates. They object to the common practice of discounting the wellbeing of future generations for mere futurity, and thus propose a smaller discount on the societal value of future consumption.

Whether a high or a low discount rate is adopted for the conversion of future to present values has substantial implications in social cost-benefit analysis. These implications are particularly severe if the costs and the benefits of an investment are spread out over long time horizons. Under high consumption discount rates, projects with present costs and far-future benefits stand a high chance of failing the social cost-benefit test–the returns to investment are 'discounted away'. It therefore comes as no surprise that the choice of the consumption discount rate is a critical parameter in the economic assessment of climate change policies. Policies for the abatement of climate change entail large costs on present societies, e.g. in the form of higher energy prices or investments in clean technology, but deliver their full benefits, the avoided damages from severe climate change, only in the coming decades and centuries. Depending on the size of the consumption discount rate, these benefits to future generations may or may not justify the costs imposed on present generations.

The recent debate which surrounds the two most well known economic assessments of climate policy, the *Stern Review on the Economics of Climate Change* (Stern 2007) and Nordhaus' (2008) integrated assessment model DICE, illustrates how critical the discount

rate is, and how unsettled the issue of social discounting remains to be. Both assessments set out to determine the socially optimal level and timing of climate change abatement, but they arrived at widely divergent policy recommendations. While Stern concludes by a call for *strong and early action*, Nordhaus suggests that a more conservative *climate policy ramp* is the optimal way to go. Several authors have attributed this discrepancy to the consumption discount rate.[2] Rather nonstandardly, Stern sets the preference parameters, in particular the utility discount rate and the intertemporal elasticity of substitution, in such a way that a very low consumption discount rate of 1.4% is implied. Nordhaus' preference parameters, in the contrary, are (descriptively) matched to an observed market rate of return, such that future monetary values are discounted at an annual rate of 5.5%.

While most critics expressed severe discontent with respect to the choice of preference parameters in the Stern Review, several authors have stressed that Stern's conclusions "may end up to be more right than wrong" (Weitzman 2007, p. 710) once risks and risk attitudes are comprehensively accounted for. In the Stern Review as well as in Nordhaus' original DICE model, stochasticities play a negligible role–the optimal extend of climate policy is hardly affected by the great many risks and uncertainties that loom in the future.

This insignificance of uncertainties in the social valuation of climate policies may be attributable to the presumed size of the considered stochasticities, but may as well result from the assumption that society is close to apathetic towards risk. In fact, the latter assumption is effectively built into the very preference framework that the Stern Review and the DICE model employ, namely the intertemporally additive expected utility model. In this model, attitudes with respect to consumption allocations over time and over states of the world are specified through a single parameter, such that it is impossible to select time and risk preferences independently of each other. In fixing the parameter that guides the allocation of consumption over time (the intertemporal elasticity of substitution), Stern as well as Nordhaus inevitably determine attitudes towards risk simultaneously, and end up with a degree of risk aversion that is too low in

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[2]See Tol and Yohe (2006), Nordhaus (2007, 2008), Weitzman (2007), Dasgupta (2007, 2008), Quiggin (2008) and the discussions in section 1.3.4, chapter 1.

view of empirical evidence.[3]

To reconcile the assumptions on the degree of risk aversion with empirical evidence while keeping those on the intertemporal elasticity of substitution constant, one must leave the familiar grounds of the additive expected utility model and turn to a more flexible preference framework. The recursive utility model developed by Kreps and Porteus (1978) and further advanced by Epstein and Zin (1989) offers the desired flexibility. In this model, the degree of risk aversion and the intertemporal elasticity of substitution are governed by different parameters, and can thus be independently aligned along empirical observations. Furthermore, it is possible in this model to vary the degree of risk aversion without affecting time preferences, such that analyses on the role of risk aversion in the context of consumption discounting can be conducted.

In the present thesis, I employ the Kreps-Porteus recursive utility model in order to examine the consumption discount rate in due consideration of risk attitudes. In doing so, I employ the *Risk-Sensitive preference* specification of Kreps-Porteus recursive preferences (Hansen and Sargent 1995). Thus, I deviate from the prevalent specification of the Kreps-Porteus model, namely Epstein-Zin preferences, which has recently been introduced into the discounting debate and into recursive versions of the DICE model.[4] Risk-Sensitive preferences, in contrast to Epstein-Zin preferences, represent constant absolute risk aversion and are 'well-ordered in terms of risk aversion'.[5] I enlarge upon the differences between the Risk-Sensitive and the Epstein-Zin preference specification at several instances throughout this thesis.

Chapter 1 of the thesis at hand gives an overview of the theoretical foundations of standard discounting and goes into some of the related discussions. In particular, I describe the derivation of the consumption discount rates that pertain to the discounted utility

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[3]See section 1.3.5, chapter 1, where I enlarge on the claim that the additive expected utility model is not substantiated empirically.

[4]Traeger (2009, 2011, 2014) disusses Epstein-Zin preferences in context of consumption discounting, Crost and Traeger (2011), Ackerman et al. (2013), Cai et al. (2013) and Jensen and Traeger (2013) analyze recursive versions of the DICE model with Epstein-Zin preferences. For more details, see section 1.4.2, chapter 1.

[5]The result that Risk-Sensitive preferences are well-ordered in terms of risk aversion goes back to Bommier and Le Grand (2014). Bommier et al. (2012) show that Epstein-Zin preferences are not well-ordered in terms of risk aversion. Already Chew and Epstein (1990) and Kimball and Weil (2009) commented on related issues with Epstein-Zin preferences. See section 1.4.3, chapter 1, and section 2.2.2, chapter 2.

model and to the additive expected utility model. Furthermore, I evolve on common parameter assumptions in the context of these models, point to some philosophical disputes regarding the general approach to discounting, and dwell on the debates that surround the Stern Review on the Economics of Climate Change and Nordhaus' integrated assessment model DICE. Lastly, I advocate the deviation from the standard models and emphasize recursive preference models as an alternative.

In chapter 2, I employ the Risk-Sensitive preference representation to study the link between changes in risk aversion and the effect on the consumption discount rate. A priori, it is unclear whether an increase in a decision maker's risk aversion would enhance his valuation of future consumption, and thus decrease the consumption discount rate, or whether it would have just the opposite effect. An increasingly risk averse decision maker may want to invest more for future generations in order to insure against the realization of bad states of the world, or he may want to invest less for an uncertain future to avoid putting resources at risk. To clarify the link between changes in risk aversion and the consumption discount rate, I develop a general framework that can cope with diverse forms of uncertainties. Within this framework, I show that the direction of the effect from a change in the decision maker's risk attitude on the discount rate depends on some fundamental properties of the uncertainty accounted for. These fundamental properties are then explored within a simple two-period endowment economy for three specific types of stochasticities, namely uncertainty on preferences, uncertainty on income, and uncertainty on an investment project.

In chapter 3, I examine the consumption discount rate of a decision maker whose preferences are Kreps-Porteus recursive and satisfy an assumption of (mutual) utility independence in the sense of Koopmans (1960). The Kreps-Porteus framework is chosen for its flexibility with respect to the disentanglement of risk aversion and the intertemporal elasticity of substitution. Utility independence is postulated as it is a broadly accepted assumption for intertemporal social welfare considerations, such as those underlying standard discounting frameworks. In a first instance, I show that utility independence restricts a Kreps-Porteus recursive decision maker's preference representation to a very specific parametric form, namely to the constant absolute risk aversion form of Risk-Sensitive preferences. Coming from a decision maker with Risk-Sensitive preferences, I

analyze the instantaneous consumption discount rate in an infinite horizon setting with independently distributed growth risk and constant elasticity of substitution. I find that the discount rate of the considered decision maker is subject to a previously unrecognized *horizon effect*, which is a function of the length of the time horizon *after* the period to which the discount rate applies. By means of an analytical solution for the consumption discount rate of a decision maker with Risk-Sensitive preferences, I illustrate that the horizon effect may diminish the discount rate to a significant extent, especially if the rate of pure time preference is small.

In chapter 4, I assess numerically how the horizon effect, and thus a Risk-Sensitive agent's consumption discount rate, reacts to changes in the utility discount rate and in the intertemporal elasticity of substitution. This analysis is conducted under a descriptive approach to social discounting. Assuming different parameter combinations of the utility discount rate and the intertemporal elasticity of substitution, I infer a Risk-Sensitive private agent's degree of temporal risk aversion from the risk-free rate of return, as observable in the financial markets. The private agent's preferences are then employed as a descriptor of the preferences of the social agent. However, I postulate an important distinction between the private and the social agent: The choices of the private agent are supposedly driven by a lifetime utility function with a relatively short horizon, whereas the choices of the social agent are driven by a preference model with a horizon that is considerably longer than the (remaining) lifetime of the private agent. The numerical results of this chapter shed further light on the significance of the horizon effect under various preference specifications.

# Chapter 1

# Discounting Basics
# and Related Discussions

## 1.1 Preface

This chapter gives an overview of the theoretical foundations of standard discounting
and dwells on some of the related discussions. In particular, I describe the derivation of
the consumption discount rates that pertain to the discounted utility model and to the
additive expected utility model (the standard models). Furthermore, I evolve on common parameter assumptions in the context of these models, point to some philosophical
disputes regarding the general approach to discounting, and dwell on the debates that
surround the Stern Review on the Economics of Climate Change (Stern 2007) and Nordhaus' (2008) integrated assessment model DICE. Lastly, I advocate the deviation from
the standard models and emphasize recursive preference models as an alternative.

Note that this not a comprehensive review of the discounting literature. Rather, I
restrict attention to those parts of the theory and the discussions that are conducive for
the understanding of the remainder of this thesis. For more comprehensive reviews, I
recommend Dasgupta's (2008) sharply written discussion *Discounting Climate Change*;
Gollier's (2011a) book *Pricing the Future*, which presents recent developments regarding
risks and uncertainties in relation to discounting in a very accessible manner; and Arrow

et al. (2012), the minutes of an expert panel on the question *How Should Benefits and Costs Be Discounted in an Intergenerational Context?*

Before going into the details, some clarifying remarks with respect to notation are in order to avoid misunderstandings. The object of interest in this thesis is the discount rate on future values that are expressed in monetary terms, such as consumption or consumption equivalents. I refer to this object as the *(instantaneous) consumption discount rate*. In the literature related to social cost-benefit analysis, this rate is also denoted as the *social discount rate* or as the *social rate of time preference*. Note that the consumption discount rate is distinct from the *utility discount rate*, which is the rate at which future felicity (the small $u$) is discounted. The utility discount rate appears under many names in the literature, e.g. as the *rate of pure time preference*, the *rate of impatience*, or the *rate of empathic distance*. Furthermore, regarding stochasticities, be aware that I make no distinction between *risk* and *uncertainty*. Both words are used interchangeably and refer to stochasticities in the form of Knightian risk. Knightian uncertainty, and thus ambiguity, is not considered in this thesis.

## 1.2  The instantaneous consumption discount rate

The consumption discount rate (in an intergenerational context) is the rate at which the societal valuation of monetary values changes between two periods. As such, it is derived from an intertemporal social welfare or utility function $U(x)$, which represents the preferences of a social agent who acts in the interest of society as a whole. The ultimate goal of such a social agent is the efficient allocation of consumption $x_t$ across generations, i.e. to impose policies or conduct investments such that social welfare is maximized.

The concept of consumption discounting is intimately related to the marginal rate of substitution ($MRS$). The marginal rate of substitution measures an agent's valuation of marginal increases in future consumption relative to marginal increases in present consumption. In particular, the $MRS$ for the valuation of consumption in the instantaneous

future ($t = 2$) relative to consumption in the present ($t = 1$) is defined by

$$MRS_{1,2} = \frac{\partial U / \partial x_2}{\partial U / \partial x_1}.$$

The rate at which this valuation changes between periods 1 and 2 is the instantaneous consumption discount rate:[1,2]

$$DR_{1,2} = -\ln \frac{\partial U / \partial x_2}{\partial U / \partial x_1}. \tag{1.1}$$

In social cost-benefit analysis, this consumption or social discount rate is the rate at which differently dated monetary equivalents are converted to values of a common period in order to derive the net present value of an investment project. For a simple investment project with monetary costs $C_1$ in period 1 and monetary or consumption equivalent benefits $B_2$ in period 2, the net present value is derived from

$$NPV = \exp(-DR_{1,2}) \cdot B_2 - C_1. \tag{1.2}$$

A social investment project with $NPV > 0$ improves the wellbeing of society as a whole and is thus considered to be desirable or worth it.

## 1.3 Discounting with the standard models

Equation 1.1 indicates that the instantaneous consumption discount rate is very specific to the postulated form of the social welfare or utility function $U(x)$ and to the economic circumstances in period 2 relative to those in period 1. The prevalent functional forms of $U(x)$ are the discounted utility model for preferences over deterministic consumption (Ramsey 1928, Samuelson 1937, Koopmans 1960), and the intertemporally additive expected utility model (Von Neumann and Morgenstern 1944) for preferences over lotteries on consumption. Both models are detailed in the following sections.

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[1] An alternative but closely related way to derive the consumption discount rate is presented in chapter 2. See also Gollier (2011a), who presents three ways to determine the consumption discount rate.

[2] If future consumption is uncertain, the derivative of the utility function with respect to future consumption is defined as $\frac{\partial U}{\partial x_2} = \lim_{\varepsilon \to 0} \frac{U(\tilde{x}_2 - \varepsilon) - U(\tilde{x}_2)}{\varepsilon}$, such that the instantaneous consumption discount rate is $DR_{1,2} = -\ln \left( \lim_{\varepsilon \to 0} \frac{U(\tilde{x}_2 - \varepsilon) - U(\tilde{x}_2)}{U(x_1 - \varepsilon) - U(x_1)} \right).$

### 1.3.1 Discounting with the discounted utility model

Suppose that a social agent's preferences $\succeq^X$ are defined over the set of deterministic outcomes $X$, where an element $x \in X$ is a consumption stream $x = x_1, x_2, ...x_\infty$. Suppose furthermore that the agent's preferences are representable by an intertemporal utility function $U : X \to \mathbb{R}$, and in particular by the *additive discounted utility model*

$$U(x) = \sum_{t=1}^{\infty} \beta^{t-1} u(x_t),\tag{1.3}$$

where $\beta$ is a utility discount *factor*, $\delta = -\ln\beta$ is the respective utility discount *rate*, and $u(x_t)$ defines felicity from consumption in period $t$.

For an agent with preference representation (1.3), the instantaneous consumption discount rate is written as

$$DR_{1,2} = -\ln\beta - \ln\frac{u'(x_2)}{u'(x_1)}.$$

If felicity is given by the constant elasticity of substitution function (CES) $u(x_t) = \frac{x_t^\rho - 1}{\rho}$ with $\rho < 1$ we can further simplify the last equation such that

$$DR_{1,2} = \delta + (1 - \rho) g_2,\tag{1.4}$$

where, $IES = (1 - \rho)^{-1}$ defines the intertemporal elasticity of substitution and $g_2 = \frac{x_2}{x_1} - 1$ is the growth rate of consumption between periods 1 and 2.

In the literature related to social cost-benefit analysis under certainty, equation (1.4) is often referred to as a *Ramsey Equation*.[3] The Ramsey Equation (1.4) conveniently summarizes the factors that determine an agent's or society's valuation of consumption in period 2 relative to the valuation of consumption in period 1. These factors are the *utility discount rate* $\delta$ and the *wealth effect* $(1 - \rho) g_2$, both of which are detailed and discussed below.

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[3]Note that the implied reference to Ramsey (1928) is slightly misleading: In his seminal article on optimal savings behavior, Ramsey showed that the social rate of time preference that applies to monetary values (the consumption discount rate) must equal the productivity of capital along a full optimum path. Equation (1.4), however, does not necessarily evolve from an optimization but is merely a definition. Setting equation (1.4) equal to the rate of return to investment yields the 'true' Ramsey rule (see e.g. Arrow et al. 1996, Dasgupta 2008). It has, however, become customary in the discounting literature to refer to (1.4) as the Ramsey Equation even though the equivalence to the productivity of capital is not always presumed (see e.g. Arrow et al. 2012). Throughout this thesis, I follow this custom.

The discounted utility model (1.3) from which the Ramsey Equation emanates represents the societal valuation of deterministic consumption paths. Future consumption is hardly deterministic, however. Assuming deterministic economic developments is particularly inept in view of the valuation of far-future benefits from climate change abatement. The most simplistic way to account for the indeterminacy of future economic developments is to calculate expected consumption paths from probabilistic forecasts and use the expected consumption growth rate, rather than a deterministic one, in the Ramsey Equation (1.4).[4] Such a procedure, however, completely neglects (societal) attitudes towards future stochasticities. In view of uncertainty about the consumption levels of future generations, present societies may want to invest more in order to insure against the possibility that future generations end up in a world of very low consumption. To capture such preferences, the intertemporal utility function must account for risk aversion.

### 1.3.2 Discounting with the additive expected utility model

The prevalent approach to incorporate risk attitudes into intertemporal decision making is the additive expected utility framework. In this framework, a decision maker's preferences $\succeq^P$ are defined over the set of lotteries $P$ on $X$.[5] An *additive expected utility function* $U : P \to \mathbb{R}$ which represents $\succeq^P$ is written as

$$U\left(x\right) = u\left(x_1\right) + E_1 \left[\sum_{t=2}^{\infty} \beta^{t-1} u\left(\tilde{x}_t\right)\right], \tag{1.5}$$

where the tilde on the consumption level indicates uncertainty, and $E_1$ is an expectation operator based on information as of period 1. Note that the curvature of the felicity function $u\left(\tilde{x}_t\right)$ takes on two roles in the additive expected utility model. First, as in the discounted utility model above, it reflects preferences with respect to wealth inequality (the $IES$) over time. Second, it reflects preferences with respect to wealth inequalities across different states of the world (risk aversion). I enlarge upon this point and the implications for consumption discounting further below and throughout this thesis.

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[4]Nordhaus (2008) refers to such a procedure as a 'buest-guess' analysis.

[5]An element $p \in P$ is a lottery that assigns probabilities $l^{\omega}$ to outcomes $x^{\omega} \in X$. A state-indexed outcome $x^{\omega}$ defines the consumption stream in states $\omega = 1, ...N$. Since the $l^{\omega}$ are probabilities, we have $\sum_{\omega=1}^{N} l^{\omega} = 1$ and $l^{\omega} > 0 \; \forall \; \omega$.

The instantaneous consumption discount rate of an agent with additive expected utility preferences as represented by (1.5) can be written as

$$DR_{1,2} = -\ln \beta - \ln \frac{E_1 \left[ u' \left( \tilde{x}_2 \right) \right]}{u' \left( x_1 \right)}. \tag{1.6}$$

Given the CES function $u\left(x_t\right) = \frac{x_t^\rho - 1}{\rho}$ ($\rho < 1$)–which, in the present setting, represents constant relative risk aversion (CRRA) in addition to constant elasticity of substitution– and normally distributed consumption growth $\tilde{g}_t \sim N\left(\mu_t, \sigma_t^2\right)$, (1.6) can be stated in a more convenient analytical form:[6]

$$DR_{1,2} = \delta + \left(1 - \rho\right)\mu_2 - \frac{\sigma_2^2}{2}\left(1 - \rho\right)^2. \tag{1.7}$$

Equation (1.7) is sometimes referred to as the *Extended Ramsey Equation* since it extends the framework for discounting in a deterministic world (the Ramsey Equation) to a world with risk on consumption growth and non-neutral risk attitudes.[7] The additional third term in the Extended Ramsey Equation, the *precautionary effect* as detailed below, discloses how the presence of normally distributed growth risk affects the instantaneous consumption discount rate in an additive expected utility framework with CES/CRRA felicity.[8]

### 1.3.3 Parameterization

Debates on the 'correct' consumption discount rate in a societal context date back as far as Ramsey (1928).[9] The major disagreements concern the general approach to discounting (the descriptive vs. the prescriptive approach), and, on a related note, the

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[6] To derive (1.7), use the functional form of the felicity function and the definition of consumption growth in (1.6) and rewrite it as $DR_{1,2} = -\ln\beta - \ln E_1 \left[\exp\left((\rho - 1)\ln\left(\tilde{g}_2 + 1\right)\right)\right]$. For $\tilde{g}_2$ small, we can substitute the logarithmic expression such that $\ln\left(\tilde{g}_2 + 1\right) \approx \tilde{g}_2$. Since $\tilde{g}_2$ is a random and normally distributed variable, we can employ a moment generating function such that $E_1 \left[\exp\left((\rho - 1)\tilde{g}_2\right)\right] = \exp\left((\rho - 1)\mu_2 + \frac{\sigma_2^2}{2}(\rho - 1)^2\right)$. This finally yields equation (1.7).

[7] The term 'Extended Ramsey Equation' is, e.g., used in Gollier (2011a).

[8] See Gollier (2011a), Arrow et al. (2012), and in particular Gollier (2002b), but note that the precautionary savings effect has also been examined intensely in the consumption/savings literature (Leland 1968, Drèze and Modigliani 1972, Kimball 1990).

[9] Or even further: The two arguments for positive consumption discounting in the Ramsey Equation have already been discussed as two (out of three) rationales for positive interest rates in Von Böhm-Bawerk's (1890) theory of interest.

parameterization of the variables on the right-hand side of (1.4) and (1.7). This parametrization also constitutes one of the most controversially discussed issues in the economic assessment of climate change policies, as particularly evident from the debate surrounding the Stern Review on the Economics of Climate Change (Stern 2007) and Nordhaus' (2008) integrated assessment model DICE.

The disagreement with respect to the general approach to discounting regards the question whether equations (1.4) and (1.7) should reflect how individual agents *actually* value future consumption, or whether it should reflect how the present generation *ought to* handle consumption tradeoffs between different generations. Arrow et al. (1996) coined the terms of a *descriptive* (positive) and a *prescriptive* (normative) approach to discounting in this context.

The descriptive approach holds that the wellbeing of society is maximized if the social welfare function, and thus the Ramsey Equation, draws on the actual preferences of the individual parts of society, i.e. the preferences of private agents. Private agents' preferences with respect to the allocation of consumption over time are, presumably, revealed through their choices, e.g. through the choices which agents make in the financial markets. Since equilibrium prices in financial markets reflect these private choices, they can be employed to elicit the preferences of private agents. Thus, the descriptive approach to social discounting generally supports the calibration of the discounting function along observable market rates of return.[10,11]

An additional but related argument of the descriptionist school of thought involves the opportunity costs of capital. To attain economic efficiency, the cost-effectiveness of a (social) investment project should be measured against the rate of return on alternative investment projects. Employing a market rate of return to discount the benefits from an investment insures that society invests its limited resources in the project that yields the

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[10] See e.g. Arrow et al. (1996) or Arrow et al. (2012).

[11] There is an issue regarding the question *which* of the many rates of return in financial markets should constitute this benchmark. In context of the evaluation of climate policies, Nordhaus (2008) remarks that he chooses a (risky) rate of return to capital, rather than a substantially lower risk-free rate of return, since the risk properties of climate investments are assumed to be equivalent to those of other capital investments. Whether or not this is a reasonable assumption has been adressed in several contributions, see e.g. Weitzman (2007). Furthermore, it is not clear whether market rates of return are suitable to reflect intragenerational preferences given the relatively short maturities of the underlying assets. Arrow et al. (2012) put foward that market rates of return are likely to reflect intra- rather than intergenerational preferences.

highest present value. Thus, economic resources are maximized and future generations are free to use them as they see fit.[12]

The prescriptive school of thought holds that assumptions on the parameters in the social welfare function (social preference parameters) should be based on ethical principles rather than observable private preferences. However, some advocates of the prescriptive approach support the view that–as a substitute for indeterminate ethical principles– social preferences could be observed from (democratically founded) public policy decisions (e.g. from the progressivity of the income tax structure), or could be elicited through public debate and stated preference models (Arrow et al. 2012). While such 'revealed social preferences' seem to be a more ethical foundation for intergenerational decision making than revealed private preferences, it remains questionable whether they actually conform with some yet-to-be-determined ethical principles.

**The utility discount rate**

The divide between the descriptionist and the prescriptionist school of thought is most obvious when it comes to the parameterization of the utility discount rate–the first term on the left-hand side of equations (1.4) and (1.7). Proponents of the descriptive approach to social discounting refer to the observation that individuals as well as societies *do* discount future felicity, and thus advocate a utility discount rate $\delta > 0$. The traditional interpretation of a positive rate of pure time preference is a notion of impatience: Individuals prefer to consume now rather than later simply because now is closer in time. Empirical evidence for such a characterization of individuals' preferences is given by experiments (see e.g. Frederick et al. 2002) and is inferable from market rates of return, i.e. from prices that result through the choices of individuals in financial markets. The practise of drawing on the observably impatient behavior of individuals as an argument for positive utility discounting in a societal context has been subject to much criticism. Schelling (1995), e.g., reasons that individuals' preferences over their own life-time felicity should not guide assumptions on preferences with respect to the wellbeing of future generations. As an alternative, Schelling advocates a notion of *empathic distance* as an

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[12]See e.g. Arrow et al. (1996). They also point out that prescriptionists tend to put the opportunity cost of capital argument into question as they doubt the transferability of economic resources to future generations.

argument for positive utility discounting.[13]

Proponents of the prescriptive approach are generally opposed to discounting the felicity of future generations for impatience or empathic distance. They consider the practise of ascribing a lower weight to the wellbeing of future generations as unethical, and thus typically advocate $\delta = 0$. Popular and oft-cited supporters of this stance are Ramsey (1928), who puts forward that "[positive utility discounting is] ethically indefensible and arises merely from the weakness of the imagination" (p. 543); Pigou (1932), who ascribes private agents' positive utility discount rate to a defect in their "telescopic faculty" (p. 25); and Harrod (1948), who dismisses utility discounting as it constitutes "a polite expression for rapacity and the conquest of reason by passion" (p. 40). Those who are critical to zero utility discounting sometimes refer to Koopmans (1965) and Arrow (1999) to counter that $\delta = 0$ implies the impoverishment of present generations in an infinite horizon setting, or at least unacceptably high savings rates in a finite horizon setting. An elegant way to bypass this criticism while staying within the prescriptive school of thought is to accept a very small utility discount rate for the chance of human extinction.[14,15]

The choice of the utility discount rate in some of the most popular economic assessments of climate change differs considerably.[16] Nordhaus (1994) employs $\delta = 3\%$ in an early version of the DICE model, but changes this value to $\delta = 1.5\%$ in the updated version (Nordhaus 2008).[17] Cline (1992) advocates zero-utility discounting, since he considers $\delta > 0$ to be unethical. Stern's (2007) stance on utility discounting mirrors that of Cline, yet Stern acknowledges the possibility of human extinction and thus sets $\delta = 0.1\%$.

Note that the utility discount rate in the discounted utility model, and thus the utility discount rate in the Ramsey Equation (1.4), is presumed to be constant. With constant rate utility discounting, the weight attached to the felicity of future generations

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[13]In particular, Schelling (1995) points out that individuals as well as societies discount the wellbeing of people that are distant in geography or culture, and argues that a similar notion of empathic distance also holds with respect to the temporal distance between generations.

[14]Dasgupta (2008) refers to Yaari (1965) as the source of this risk-of-human-extinction argument.

[15]Bommier and Zuber (2008) show that the risk-of-human-extinction argument can imply substantial discounting of future welfare if the decision maker is characterized by temporal risk aversion, a concept which I discuss in section 1.4 below.

[16]See also the extended discussion on the Stern Review (Stern 2007) and on Nordhaus' DICE model (Nordhaus 2008) below.

[17]Note that the value of the $IES$ is changed simultaneously, see the next section on the wealth effect.

declines exponentially over time. A general dissatisfaction with this exponential decline, especially in context of intergenerational decision making, helped the popularization of declining utility discount rates. A descriptive argument for declining utility discount rates is that individuals are typically characterized by hyperbolic discounting, i.e. the decay in the valuation of felicity is slowed down as more distant time periods are considered.[18],[19] Prescriptive arguments for a declining time path of the utility discount rate are offered by Chichilnisky (1996, 1997) and Li and Löfgren (2000), who derive their arguments through different sustainability axioms.[20] While arguments for declining utility discount rates are attractive for constituting a compromise between rather high descriptive and very low prescriptive rates, they have to stand up to the critique of time inconsistency of optimal consumption plans. Time inconsistency is implied whenever a consumption plan that is deemed optimal in $t$ is not optimal anymore in $\tau > t$, given that no new information is revealed between $t$ and $\tau$. Strotz (1956) formalizes the notion of time inconsistency and proves that time consistency is only provided in a discounted utility framework if the discounting function has an exponential form, i.e. if the utility discount rate is constant.[21]

**The wealth effect**

Future consumption may not only be discounted for reasons of mere futurity (utility discounting), but also because the circumstances that pertain to future generations may differ from those that pertain to the present generation. In particular, with positive economic growth, future generations will be richer than present generations. With the standard assumption of decreasing marginal felicity ($u'(x_t) > 0$, $u''(x_t) < 0$), an additional unit of consumption has a higher societal value if given to the poor present rather than to a rich future generation. The curvature of the felicity function, and thus the

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[18] Naturally, Schelling's (1995) critique to employing individual preferences in a societal context also applies in context of hyperbolic discounting.

[19] Frederick et al. (2002) review the experimental literature on this issue.

[20] In Chichilnisky (1996, 1997), this emphasis appears in the form of axioms that require the non-dictatorship of present or future generations; in Li and Löfgren (2000), the introduction of two agents, namely a representative of the present who discounts felicity at a positive rate and a representative of the future who does not discount felicity, implies declining utility discount rates.

[21] Optimal consumption paths that follow from frameworks with a hyperbolic utility discount rate, as those implied by Chichilnisky (1996,1997), Li and Löfgren (2000), and Fredrick et al. (2002), are thus subject to time inconsistency.

intertemporal elasticity of substitution, determines how much the societal valuation of consumption differs at different wealth levels. The second term in equations (1.4) and (1.7)–the wealth effect–captures this motive for consumption discounting.

To illustrate the effect of the intertemporal elasticity of substitution on the consumption discount rate, consider a social welfare function with $IES = 1$, $\delta = 0$ and $g_2 = 0.01$.[22] Plugging these values into the Ramsey Equation (or into the Extended Ramsey Equation with $\sigma_2^2 = 0$) shows that the 1% difference in consumption levels translates into a discount rate of $DR_{1,2} = 1\%$. An additional unit of consumption in the (rich) second period is thus 1% less valuable than an additional unit of consumption in the (poor) first period. Given $IES < 1$ ($IES > 1$), $g_2 = 0.01$ translates into $DR_{1,2} > 1\%$ ($DR_{1,2} < 1\%$). These effects on the consumption discount rate reflect that the more inelastic intertemporal substitution is (the smaller the $IES$), the more costly it is in terms of social welfare to give a marginal unit of consumption to the rich (future) rather than to the poor (present). Put differently, the more inelastic substitution is, the higher is the decision maker's aversion towards (intergenerational or intertemporal) consumption inequality.[23]

In some of the most well-known assessments of climate policy, the intertemporal elasticity of substitution ranges between 0.5 and 1. Nordhaus (1994) calibrates the early DICE model such that $IES = 1$, but employs $IES = 0.5$ in the updated version of DICE (Nordhaus 2008). Cline (1992) employs $IES = 2/3$ and Stern (2007) uses $IES = 1$. Dasgupta (2008), who discusses these parameter values (except that of Nordhaus 2008), argues that the $IES$ should reside in the range $[0.33, 0.5]$. He motivates these values by an example on the (rather low) inequality aversion that is implied by the (rather high) $IES$ in Nordhaus (1994), Cline and Stern. With Dasgupta's low values for the $IES$, consumption increases that pertain to the rich future generation are discounted more strongly.[24,25]

The wealth effect generally justifies positive consumption discounting since most econo-

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[22] Note that the felicity function is logarithmic if $IES = 1$ ($\rho = 0$): $\lim_{\rho \to 0} \frac{x_t^\rho - 1}{\rho} = \ln x_t$.

[23] See Dasgupta (2008), who formalizes the interrelation between the $IES$ and aversion towards consumption inequality in a societal context.

[24] Dasgupta (2008) also points out that $IES \geq 1$ (in combination with $\delta \approx 0$) implies absurdly high savings rates which are at odds with empirical observations. Quiggin (2008), however, brings forward that the savings rate would not be so high after all if technological progress is taken into account.

[25] See also Gollier (2011a), who describes how the $IES$ can be estimated and enlarges on further parameter estimates from the literature.

mists act on the assumption of sustained economic growth. Beyond the mainstream and outside economics, however, sustained growth is not always treated as a matter of fact. Assumptions of zero or negative growth can equally justify non-positive consumption discounting. Dasgupta et al. (1999) and Sterner (1994), for example, emphasize that ecological boundaries may limit growth in the long run, which would be reflected in very low or even negative long-run consumption discount rates.

**The precautionary effect**

The precautionary effect in the Extended Ramsey Equation (1.7) extends the discount rate under certainty by a third term which accounts for risk on consumption growth and the societal (the social decision maker's) attitude towards this risk. Intuitively, this effect is explicable by a precautionary savings motive. Risk on future growth ($\sigma_2^2 > 0$) goes along with uncertainty on future consumption levels. Such uncertainty induces an agent with discounting function (1.7) to value additional consumption in period 2 more (discount it less), and thus save/invest more for the future to insure against the realization of bad states of the world.[26]

However, given a normal distribution with moderate variance and an $IES$ within a standard range, the precautionary savings motive decreases the consumption discount rate by an almost negligible amount.[27] With $\delta = 0$, $IES = 0.5$ and with Kocherlakota's (1996) estimates for the moments of stochastic consumption growth, $\mu = 0.018$ and $\sigma^2 = 0.00127$ (estimated from historic US data, 1889-1978), the precautionary effect is of magnitude 0.25%.[28] The precautionary effect thus reduces the consumption discount rate under certainty, $DR_{1,2} = 0.5^{-1} \cdot 0.018 = 3.6\%$, to a level of $DR_{1,2} = 3.35\%$ only.[29]

The presentation of the Extended Ramsey Equation in this chapter pertains to the in-

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[26] It is well known that this depends on the third derivative of the felicity function, which is said to represent prudence if $u''' (\cdot) > 0$, see e.g. Gollier (2002a). The constant elasticity of substitution/constant relative risk aversion felicity function $u(x_t) = \frac{x_t^\rho - 1}{\rho}$ on which the Extended Ramsey Equation is based satisfies this condition.

[27] This has been noted by several authors, see e.g. Traeger (2009), Gollier (2011a) Arrow et al. (2012).

[28] The values $\delta = 0$, $IES = 0.5$ are suggested by Dasgupta (2008) and Gollier (2008), as mentioned in Arrow et al. (2012).

[29] Gollier (2011b) shows that this precautionary effect can be more sizeable if the variance on consumption growth in a broad selection of countries (190 countries) is considered. Hence, Gollier highlights that the exclusive focus on developed countries in most empirical investigations of growth uncertainty implies an underestimation of the precautionary effect.

stantaneous consumption discount rate, i.e. the rate at which consumption in period 2 is valued relative to consumption in period 1. Such a restricted view into the instantaneous future is sufficient if the consumption discount rate is constant over time, that is if consumption in period 2 is discounted at the same rate as consumption in later periods. With such constant rate discounting, the value of consumption declines exponentially over time, i.e. very quickly. However, the consumption discount rate need not be constant. In fact, arguments why the discount rate's term structure should be declining flourished over the past 15 years. Some of these arguments have obtained wide approval, especially in context of climate change abatement, since they imply a slowdown in the 'discounting-away' of future generations.

One particularly interesting argument in view of the precautionary effect is that of Gollier (2008). Gollier shows that the consumption discount rate that evolves from the Extended Ramsey Equation is constant only if the risk on growth is independently and identically distributed. If the risk on growth is positively correlated instead, the consumption discount rate must have a declining term structure.[30] Intuitively, given correlated growth shocks, future risks are amplified and the precautionary savings motive is enhanced as more distant time periods are considered. More distant consumption is therefore discounted at a smaller rate.[31,32]

### 1.3.4 Stern, Nordhaus and the debates

In 2006, Nicholas Stern and his team prominently released the Stern Review on the Economics of Climate Change (later published as Stern 2007)–a voluminous report on

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[30] Gollier (2002a, 2002b) also motivates a declining time path for the consumption discount rate, yet without assuming correlated growth risk. Instead, an assumption of decreasing relative risk aversion (rather than constant relative risk aversion as in the Extended Ramsey Equation) and the absence of a risk of recession are required to justify declining consumption discount rates.

[31] See also Arrow et al. (2012), who discuss how such correlated growth risk could be parameterized, and Gollier (2011a), who explains the effect from non-*iid* growth risk in textbook manner.

[32] Another popular argument for declining consumption discount rates is that of Weitzman (1998, 2001). In contrast to that of Gollier (2008), Weitzman's rationale is independent of the constituent parts of the Extended Ramsey Equation. Weitzman adresses the key function of social cost-benefit analysis, namely the net present value equation, directly. He introduces uncertainty on the consumption discount rate into this equation, states it in terms of an *expected* net present value (in which expectations are taken of the consumption discount *factor*), and shows that the respective *certainty equivalent consumption discount rate* must be declinig over time. Weitzman (2001) draws on the diverse domain of expert opinions with respect to the 'correct' consumption discount rate in order to motivate the presence of uncertainty on this rate, and in order to specify the distribution which governs the uncertainty on the consumption discount rate.

the costs and benefits of climate change abatement which was officially commissioned by the British government. The Review's oft-cited conclusion is that "the benefits of strong and early action far outweigh the economic costs of not acting" (Stern 2007, Summary of Conclusions, p. vi). More specifically, Stern and his team put forward that the costs of inaction range between annual losses of $5 - 20\%$ of GDP, depending on the range of the risks and impacts accounted for. The costs from significant reductions in greenhouse gas emissions, in contrast, are estimated to be only $1\%$ of annual GDP.

Stern's (2007) grave call for action stands in stark contrast to the more conservative recommendations of several other authors, most notably Nordhaus (2008). Nordhaus' analyses in the integrated assessment model DICE constitute the benchmark against which economic assessments of climate change are typically measured. This model, which has been developed and updated over the past three decades, enjoys wide recognition, and thus has served as the starting point for many extensions (some of which are mentioned in section 1.4.2). The main conclusion from a recent update of the DICE model is that "efficient emissions reductions follow a 'policy ramp' in which policies involve modest rates of emissions reductions in the near term, followed by sharp reductions in the medium and long terms" (Nordhaus 2008, p. 14).

Unsurprisingly, Stern's (2007) clear deviation from such more common policy recommendations implied that the Review became subject to severe scrutiny–the conclusions of which were mostly not favorable. While the criticism touched on many issues in Stern's analysis,[33] the one point most, if not all examiners objected to, was the treatment of the consumption discount rate and the lack of a sensitivity analysis with respect to the selection of preference parameters. Tol and Yohe (2006), Nordhaus (2007, 2008), Weitzman (2007), Dasgupta (2007, 2008) and Quiggin (2008) suggest that Stern's (2007) main conclusion–that the costs of strong climate change abatement are outweighed by the immense benefits–are largely driven by the use of a very low consumption discount rate of $1.4\%$. This discount rate results from two crucial assumptions. First, in setting the utility discount rate to $\delta = 0.1\%$ and thus close to zero, Stern follows the prescriptive school of thought. He advocates the ethical indefensibility of utility discounting but accounts for a small risk of human extinction. The exact choice of $\delta = 0.1\%$ is arguably arbitrary,

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[33]Tol and Yohe (2006) criticize a double-counting of risk, an assumption of constant vulnerability, an underestimation of the costs of climate change policy and inconsistencies in results and conclusions.

however. With $\delta = 0.1\%$, the probability that humanity still exists in 100 years is only 90%–a very pessimistic stance indeed.[34] Second, the $IES$ is assumed to be rather high ($IES = 1$), i.e. the social decision maker does not care much about intertemporal consumption inequalities.[35] The specific choice of the $IES$ is substantiated by reference to a single article without discussing and, more importantly analyzing, deviations from this value. Stern simply notes that this choice is "in line with recent empirical estimates" (Stern 2007, Part II, p. 161).[36] The previous discussion on the wealth suggests that Stern's choice of the $IES$ is at the upper end of a commonly accepted range, and, in particular, is significantly higher than the range advocated by Dasgupta (2008).[37]

Nordhaus (2007, 2008) confirms the hypothesis that Stern's (2007) controversial conclusions result from a very low consumption discount rate rather than from advancements in modeling or from new insights on the economics or science of climate change. In his integrated assessment model DICE, Nordhaus (2007, 2008) conduct's a sensitivity analysis with respect to the preference parameters $\delta$ and $IES$. With Nordhaus' preferred parameters, $\delta = 1.5\%$ and $IES = 0.5$, DICE yields an optimal carbon tax of 35$ per ton of carbon, whereas with Stern's parameter values, the optimal carbon tax suggested by the model increases by an order of magnitude to 360$. In setting $\delta = 1.5\%$ and $IES = 0.5$, Nordhaus takes a purely descriptive approach to social discounting. His choice of preference parameters is aligned along a market rate of return of 5.5%.[38] Nordhaus draws on the opportunity costs of capital argument to justify this approach: "When countries weigh their self-interest in international bargains about emissions reductions [...], they look at the actual gains from bargains, and the returns on these relative to other investments [...]" (Nordhaus 2008, p. 175).

Note that both authors, Stern (2007) as well as Nordhaus (2008), pay attention to

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[34] Stern (2007) himself points out that such a high chance of human extinction may be unrealistic.

[35] Note that, since Stern (2007) eventually conducts an expected utility analysis, the choice of $IES = 1$ implies that he also sets the parameter of relative risk aversion to 1.

[36] The reference which Stern refers to is Pearce and Ulph (1999) (as cited in Stern, 2007), which is a meta-review with some in-depth discussions on the $IES$.

[37] Dasgupta (2007, 2008) also points to the inconsistency in the choice of preference parameters in the Review. Stern (2007) takes a prescriptive stance on the utility discount rate, but refers (descriptively) to consumer behavior when it comes to the intertemporal elasticity of substitution. Dasgupta opposes this approach for constituting "neither good economics nor good philosophy" (Dasgupta 2008, p. 159).

[38] This market rate of return is obtained for the endogenously determined growth rate over the first century in the DICE model. Nordhaus (2008) remarks that alternative values of $\delta$ and the $IES$ can be chosen to obtain the same market rate of return.

stochasticities and the evaluation thereof. Stern's main analysis is an expected utility analysis: He computes social welfare (discounted utility) for 1000 runs of GDP, taking into account damages from climate change and adaptation measures, and finally takes the expectation of probabilistic social welfare, i.e. he eventually derives the expected utility.[39] The Review, at several instances, emphasizes the importance of accounting for risk,[40] and conveys the impression that the graveness of the main results is ascribable to the incorporation of several stochasticities: "The monetary cost of climate change is now expected to be higher than many earlier studies suggested, because these studies tended not to include some of the most uncertain but potentially most damaging impacts." (Stern 2007, Part II, p. 143). Quite in the contrary, Nordhaus, who conducts an expected utility analysis as well as a best-guess analysis[41], finds that "the best-guess policy is a good approximation to the expected-value policy. There appears to be no empirical ground for paying a major risk premium for future uncertainties beyond what would be justified by the averages" (Nordhaus 2008, p. 28).

In view of Nordhaus' (2007, 2008) sensitivity analyses on $(\delta, IES)$ and on 'best-guess' versus 'expected utility', it seems apparent that Stern's (2007) results are not driven by a comprehensive consideration of uncertainties, but rather by the very low consumption discount rate. One may thus be drawn to the impression that Stern's choice of the discount rate emanates from a desire to incorporate high precaution and the possibility of severe climate change through a back door, i.e. through setting the discount rate to an artificially low level rather than explicitly modelling climate catastrophes or a significant aversion towards stochasticities.[42]

Does all this imply that Nordhaus' (2008) policy ramp is the proper way to go whereas Stern's (2007) call for action should be put down as uncalled-for alarmism? Not nec-

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[39] See Stern (2007), Part II, p. 161.

[40] See e.g. Stern (2007), Executive Summary, p. ix: "[..] the modelling framework used by this Review had to be built around the economics of risk. Averaging across possibilities conceals risks. The risks of outcomes much worse than expected are very real and they could be catastrophic. Policy on climate change is in large measure about reducing these risks.", "[t]he Review uses the results from one particular model, PAGE2002, to illustrate how the estimates derived from these integrated assessment models change in response to updated scientific evidence on the probabilities attached to degrees of temperature rise. The choice of model was guided by our desire to analyse risks explicitly - this is one of the very few models that would allow that exercise."

[41] Nordhaus (2008) explains that by a buest-guess analysis, he means an analysis in which one takes expectations of uncertain variables and then treats these means as determinstic.

[42] A similar point is made in Weitzman (2007).

essarily. The previous discussions merely suggest that the parameter assumptions in the Stern Review do not pass the majority vote amongst economists, whereas Nordhaus' values for $\delta$ and the $IES$ find broader acceptance. However, both authors fail to account for climate related disasters and work within a framework–the additive expected utility model–in which risk aversion cannot be accounted for properly. I address the latter point in the next sections and throughout this thesis. Once severe risks and risk aversion are modeled explicitly, Stern's conclusions may well *end up to be more right than wrong,* even with Nordhaus' descriptively chosen preference parameters.[43]

### 1.3.5 The problem of preference entanglement

The additive expected utility model, the model on which Stern's (2007) and Nordhaus (2007, 2008) analyses are build, postulates an entanglement of attitudes towards risk and attitudes towards intertemporal inequality. In particular, if the social decision maker's preferences are represented by the additive expected utility function (1.5) and if felicity has the constant elasticity of substitution/constant relative risk aversion property–as is the case in Stern's and Nordhaus' analyses–then the degree of relative risk aversion ($RRA$) is the inverse of the intertemporal elasticity of substitution: $RRA = IES^{-1}$. Thus, assumptions on the value of the $IES$ determine the degree of relative risk aversion at the same time. As remarked above, this entanglement is due to the twofold role of the felicity function $u(\cdot)$. The concavity of the felicity function governs not only the desire to smooth consumption between different periods (reflected by the $IES$), but also the desire to smooth consumption between different states of the world (reflected by the degree of risk aversion).

This preference entanglement is problematic for (at least) two reasons. First, in view of the relationship $RRA = IES^{-1}$, it is clear that an increase in the degree of risk aversion must go hand in hand with a decrease of the intertemporal elasticity of substitution. Therefore, the additive expected utility model is not fit to analyze the implications of changes in risk aversion (or in the $IES$). This defect is specifically severe in contexts

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[43]I am paraphrasing Weitzman (2007) here, who points out that Stern's (2007) choice of discount rate "may end up to be more right than wrong" (p. 710) if uncertainty on the discount rate itself is accounted for. Weitzman also emphasizes the need to incorporate risks in the very left tail of the growth rate's probability distribution, i.e. the need to account for climate catastrophes.

in which risks are sizeable and the effects from changes in risk aversion therefore pronounced. Such a context is given by almost any kind of long run investment decision, and in particular by investment decisions related to climate change policy.

Second, the entanglement, and thus the additive expected utility model itself, is not substantiated empirically. This widely accepted (and yet often neglected) fact has been commented on extensively, most prominently in context of asset pricing. In their seminal article, Mehra and Prescott (1985) show that the additive expected utility model fails in explaining the high premiums on financial assets with risky returns, unless absurdly high degrees of risk aversion are postulated. Weil (1989) reiterates the puzzle as he shows that, with the high degrees of risk aversion necessary to explain the equity premium, the risk-free rate would have to be much bigger than what is observable in the financial markets.[44] To match observably low risk-free rates, the degree of risk aversion must be lowered (the *IES* must be increased) which in turn implies that the equity premium cannot be explained. These are the notorious *equity premium puzzle* and its little sister, the *risk-free rate puzzle*.

To understand the risk-free rate puzzle and the relation to the additive expected utility model, note that the risk-free rate $r_f$ is simply the instantaneous consumption discount rate for the valuation of riskless benefits/costs. Hence, given the additive expected utility model with CES felicity and normally distributed growth risk, $r_f$ is defined by equation (1.7). Now consider what happens in this equation if the degree of relative risk aversion $RRA = (1 - \rho)$ is increased in order to match some equity premium: Increasing $(1 - \rho)$ amplifies the precautionary effect and thus lowers the risk-free rate (equivalently $DR$), but increasing $(1 - \rho)$ also enhances the wealth effect and thus increases $r_f$. With standard assumptions on the mean and variance of economic growth, the latter effect will dominate the former, and what is intended as an increase in risk aversion will eventually increases the risk-free rate through the wealth effect.[45]

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[44] The risk-free rate of return is the return of an asset with price 1 and certain payoff (or with payoffs that are not correlated with the riskiness of the economy); the equity premium is added to the risk-free rate of return if the payoffs of the respective asset parallel those of the economy, i.e. if an asset's payoff positively correlates with macroeconomic risk.

[45] All this can be found in any standard textbook on asset pricing, e.g. in Cochrane (2005). Weitzman (2007), Gollier (2011a) and many others, make similar comments.

## 1.4 Recursive preference models

To study the role of risk aversion in intertemporal contexts, such as e.g. in the assessment of climate policies, we must leave the familiar grounds of the additive expected utility model and turn to a more flexible preference representation. A model that has proven to be particularly useful to represent preferences in a more flexible manner is the recursive utility model developed by Kreps and Porteus (1978) and further advanced by Epstein and Zin (1989). This model generalizes the standard model in a crucial aspect: It disentangles risk aversion from the $IES$.

In this section, I briefly present the preference representation of Kreps-Porteus recursive preferences, show how it enables a disentanglement of risk aversion and the $IES$, and enlarge on two specific forms of the Kreps-Porteus model, namely Epstein-Zin and Risk-Sensitive preferences. Kreps-Porteus recursive preferences, in particular the Risk-Sensitive specification, are discussed at more length in chapters 2 and 3. In this section, I focus on pointing to some of the recent literature that introduces recursive preferences, in particular the Epstein-Zin form, into the discounting debate.

### 1.4.1 Kreps-Porteus recursive preferences

Kreps-Porteus (KP) recursive preferences are defined over the set of temporal lotteries $D$. A temporal lottery $d \in D$ can be understood as the information that pertains to a given node in a multiperiod probabilistic consumption tree. Any node in such a tree is specified by the consumption level in the given node and by a probability measure over the subsequent nodes.[46] Given this specification, a temporal lottery, in contrast to a normal (atemporal) lottery $p \in P$, contains information with respect to the timing of the resolution of uncertainty. Thus, Kreps-Porteus recursive preferences which are defined over $D$, in contrast to additive expected utility preferences which are defined over $P$, enable the characterization of an agent who has a preference for the early or late resolution of uncertainty. Such a specification of preferences is not possible in the additive expected utility model. An agent with additive expected utility preferences is

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[46] More generally, a temporal lottery can in addition contain information about the consumption path that lead up to the given node.

neutral towards the timing of the resolution of uncertainty. In appendix 1.A, I present an example to illustrate temporal lotteries and show how they incorporate the timing of the resolution of uncertainty.

The preferences $\succeq^D$ of a Kreps-Porteus recursive agent can be represented by a recursive utility function $U : D \rightarrow \mathbb{R}$ of the form

$$U_t (x_t, m) = u (x_t) + \beta \phi^{-1} (E_m [\phi (U (x_{t+1}, m))]) , \tag{1.8}$$

where $x_t$ specifies consumption in $t$ (the consumption level at a given node in a consumption tree) and $m$ is a probability measure over the subsequent temporal lotteries. A recursive utility function $U_t$ aggregates certain felicity $u (x_t)$ and the certainty equivalent of continuation utility $U_{t+1}$, i.e. $\phi^{-1} (E_m [\phi (U (x_{t+1}, m))])$. In this way, starting with the nodes in the penultimate period $T-1$,[47] the subtrees of a consumption tree are recursively collapsed into (continuation) utility measures until the intertemporal utility in the first period is derived.

To see how this preference representation enables the disentanglement of risk aversion and the $IES$, consider the two-period special case such that we can write

$$U_1 (x_1, m) = u (x_1) + \beta \phi^{-1} (E_m [\phi (u (x_2))]) , \tag{1.9}$$

and define $v (\cdot) = \phi (u (\cdot))$ as an atemporal expected utility function. With this formulation, an increase in the agents risk aversion, i.e. an increase in the concavity of $v (\cdot)$, can be achieved by solely increasing the concavity of $\phi (\cdot)$ and thus without affecting the curvature of $u (\cdot)$ which defines the $IES$.

Note that in the special case in which $\phi (\cdot)$ is linear, i.e. in the case in which $v (\cdot) = u (\cdot)$, equation (1.8) reduces to the additive expected utility model (1.5) in which the curvature of $u (\cdot)$ governs risk aversion as well as the $IES$. Thus, the KP recursive utility function nests the additive expected utility function as a special case. Note furthermore that if continuation utility is deterministic, equation (1.8) reduces to the form of the discounted utility model (1.3) in which the curvature of $u (\cdot)$ solely governs the intertemporal elasticity of substitution.

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[47] In the terminal period $T$, the recursive utility function is given by $U_T (x_T) = u (x_T)$.

### 1.4.2 Epstein-Zin preferences

Epstein and Zin (1989) inquire the circumstances under which preferences over the space of infinite temporal lotteries can be represented recursively. In particular, they prove the existence of a recursive utility function that has the constant elasticity of substitution property and is combined with a broad class of continuous mean value functionals (the integration over stochastic continuation utility). For illustrative purposes, they examine three (homogenous) subclasses of the mean value functional. A class that gained considerable popularity in the subsequent literature pertains to a constant relative risk aversion (CRRA) specification of the function $\phi(\cdot)$ in the Kreps-Porteus preference representation above.[48],[49] Throughout this thesis, I refer to this CRRA specification as *Epstein-Zin (EZ) preferences*.

In various contributions, Traeger (e.g. 2009, 2011, 2014) examines Epstein-Zin preferences in context of consumption discounting and optimal climate policy. He employs the EZ preference representation[50],[51]

$$U_t = \frac{x_t^\rho}{\rho} + \beta \left( E_t U_{t+1}^{\frac{\alpha}{\rho}} \right)^{\frac{\rho}{\alpha}}, \tag{1.10}$$

where $IES = (1 - \rho)^{-1}$ and $RRA = (1 - \alpha)$ is a measure for Arrow-Pratt relative risk aversion. If $\alpha = \rho$ (if $RRA = IES^{-1}$), expression (1.10) nests the additive expected utility function. However if $\alpha < \rho$, then an agent with preference representation (1.10) will be more risk averse than an agent with additive expected utility preferences. Traeger (2014) derives an analytical solution for the instantaneous consumption discount rate

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[48] Epstein-Zin preferences have been employed intensely in the asset pricing literature, notably by Bansal and Yaron (2004), who show that such preferences can be conducive in resolving the equity premium puzzle. For a review on the role of Epstein-Zin preferences in relation to the equity premium puzzle, see e.g. Donaldson and Mehra (2008).

[49] The prevalence of Epstein-Zin preferences is, to a big part, attributable to the fact that they are homothetic/scale invariant. The homotheticity of the EZ utility function implies that all arguments can be scaled by a common constant without affecting preference orderings. Homotheticity entails considerable analytical convenience in many settings and is an essential component in some frameworks, as e.g. in balanced growth models (Skiadas 2013).

[50] Note that we can get from the general formulation of KP recursive preferences to this EZ form by using $u(x_t) = \frac{x_t^\rho}{\rho}$ and $\phi(z) = (\rho z)^{\frac{\alpha}{\rho}}$ in (1.8).

[51] Traeger (2014, footnote 17) shows how this representation of EZ preferences can be renormalized to

$$U(x_t, m) = \left( (1 - \beta) x_t^\rho + \beta \left( E_m [U(x_t, m)]^\alpha \right)^{\frac{\rho}{\alpha}} \right)^{\frac{1}{\rho}},$$

which is the homothetic form orginally examined in Epstein and Zin (1989).

of an agent with preference representation (1.10). Employing parameter estimates from the asset pricing literature, he illustrates how the possibility to disentangle risk aversion and the $IES$ with EZ preferences can imply significant reductions in the consumption discount rate relative to the value that pertains to the additive expected utility model.[52] I present and discuss this analytical solution of the consumption discount rate for Epstein-Zin preferences in chapter 3.

In order to gain insights on the effects of risks and uncertainties on optimal climate policy, Crost and Traeger (2011), Ackerman et al. (2013), Cai et al. (2013) and Jensen and Traeger (2013) examine Epstein-Zin versions of Nordhaus' (2008) DICE model. They construct different recursive versions of DICE, employ the Epstein-Zin preference specification, and examine the implications for climate policy in presence of different uncertainties. Crost and Traeger examine the case of damage uncertainty, Ackerman et al. are interested in uncertainty on climate sensitivity, Cai et al. investigate the effect of uncertainty on climate tipping events, and Jensen and Traeger explore the presence of long-term growth uncertainty. In contrast to Nordhaus' additive expected utility version of the DICE model, these EZ-DICE models can be equipped with risk and time preferences that are independent of each other. Thus, sensitivity analyses with respect to the $IES$ and the degree of risk aversion can be conducted, and the model can be equipped with independent parameter values that are aligned along empirical observations. Most EZ-DICE models employ an $IES$ that lies slightly above that of Nordhaus, and a degree of risk aversion that considerably exceeds the inverse of the $IES$, and thus the degree of risk aversion in the original DICE model.[53] A general insight from all of these EZ-DICE models is that the optimal price on carbon (the social cost of carbon) increases relative to that derived in the original DICE model, and the optimal pace of emissions reductions is faster.[54]

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[52] In particular, Traeger (2014) employs Vissing-Jørgensen and Attanasio' (2003) parameter estimates for the EZ preference model, $IES = \frac{3}{2}$ and $RRA = 9.5$, and sets the economic parameters (the moments of stochastic consumption growth) to $\mu = 2\%$ and $\sigma = 4\%$.

[53] Nordhaus (2008) employs $IES = 0.5$ which, since his framework is that of an additive expected utility model, implies $RRA = 2$. Crost and Traeger (2011), Ackerman et al. (2013), Cai et al. (2013) and Jensen and Traeger (2013) refer to Vissing-Jørgensen and Attanasio's (2003) and Bansal and Yaron's (2004) parameter estimates for the EZ model and select $IES = \frac{3}{2}$ and $RRA = [9.5, 10]$. All authors also examine deviations form these parameter values.

[54] Some authors (Crost and Traeger 2011, Ackerman et al. 2013) remark that the results from the EZ-DICE models are very sensitive to changes in the $IES$, but do not change much with large variations in the degree of risk aversion. The difference in the policy recommendations relative to those of Nordhaus (2008) thus seem to stem from the higher $IES$ rather than the higher degree of risk aversion. Cai et al.

### 1.4.3   Risk-Sensitive preferences

*Risk-Sensitive (RS) preferences* constitute an interesting alternative to the Epstein-Zin preferences specification of the Kreps-Porteus recursive model. Using the constant absolute risk aversion (CARA) functional $\phi(z) = -\exp(-kz)$ in the general formulation (1.8) of KP recursive preferences, yields

$$U_t(x_t, m) = u(x_t) - \frac{\beta}{k} \ln(E_m \exp(-kU(x_{t+1}, m))). \qquad (1.11)$$

Here, $k$ measures the degree of *temporal risk aversion*. In the special case in which $k = 0$, expression (1.11) is equivalent to the additive expected utility model and the agent is temporally risk neutral. If however $k > 0$ ($k < 0$), then an agent with preference representation (1.11) is more (less) risk averse than an additive expected utility agent. In this case, the agent is said to be (strictly) temporally risk averse (loving).

The denotation as Risk-Sensitive preferences relates expression (1.11) to Hansen and Sargent (1995), and thus to robust and risk-sensitive control theory, both of which are concepts that originated in engineering and applied mathematics. Robust control theory, on the one hand, seeks to facilitate the design of robust decision or control rules when the potential misspecification of models is taken into account. This branch of control theory thus incorporates a sense of Knightian uncertainty on, or a decision maker's distrust in the 'approximating model'. Risk-sensitive control theory, on the other hand, is concerned with the amplification of the effect of risk in decision rules. Here, a decision maker's underlying model is understood to be true, but his preferences are adjusted such that his objective function is more sensitive to risky outcomes. Jacobson (1973) and Whittle (1981) established a link between robust and risk-sensitive control theory, which was then connected to economic problems by Hansen, Sargent, and several co-authors. Hansen et al. (2006) link robust control theory to the economic concept of ambiguity aversion; Hansen and Sargent (1995) establish the connection between risk-sensitive control theory and the economic concept of risk aversion in an intertemporal context. In particular, Hansen and Sargent combine the risk-sensitivity notion from control theory with the economic concept of utility discounting, and establish a Risk-Sensitive preference repre-

(2013), however, find that changes in risk aversion may have a significant effect on the optimal price of carbon.

sentation which constitutes a special case of the recursive preference representation of Kreps and Porteus (1978) and Epstein and Zin (1989) [55,56]

In this thesis, I pay special attention to the consumption discount rate that pertains to a (social) decision maker with the Risk-Sensitive preference representation (1.11). Thus, I deviate from preference frameworks that are more common in the consumption discounting literature. On the one hand, I deviate from the prevalent additive expected utility model, since this model is not flexible enough to examine the role of risk aversion in an intertemporal context. On the other hand, I deviate from Epstein-Zin preferences for several reasons. First, Risk-Sensitive preferences offer the same flexibility regarding the disentanglement of risk aversion and the $IES$ as Epstein-Zin preferences, but are connected to constant absolute rather than constant relative risk aversion. The examination of this constant absolute risk aversion specification of Kreps-Porteus recursive preferences yields an interesting extension within the discounting literature.[57] Second, as I show in chapter 3, Risk-Sensitive preferences are implied if a condition of mutual utility independence in the sense of Koopmans' (1960) is imposed on Kreps-Porteus recursive preferences. Lastly, as annotated by Chew and Epstein (1990), highlighted by Kimball and Weil (2009) and more formally examined by Bommier et al. (2012), Epstein-Zin preferences are not monotonic with respect to first-order stochastic dominance, which may bring about perverted results when studying the role of risk aversion.[58] Bommier and Le Grand (2014) point out that Risk-Sensitive preferences, in contrast, are not subject to such problems.

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[55] Hansen, Sargent and co-authors point out that the risk-sensitivity parameter in a Risk-Sensitive preference framework can be interpreted as a preference for robustness, see e.g. Hansen et al. (2006).

[56] See Hansen et al. (1999), Hansen et al. (2006), Williams (2008).

[57] To the best of my knowledge, Risk-Sensitive preferences have not been examined in context of the consumption discounting/optimal climate policy literature. Some authors have, however, linked environmental policy problems to the ambiguity aversion interpretation of robust control theory, see e.g. Athanassoglou and Xepapadeas (2012) and Funke and Paetz (2011).

[58] Bommier et al. (2012) point out that Epstein-Zin preferences are not 'well ordered in terms of risk aversion'.

# Appendices for chapter 1

## 1.A    Temporal lotteries

A major difference between the standard additive expected utility model and the Kreps-Porteus recursive utility model lies in the domain over which the underlying preferences are defined. The standard additive expected utility function represents preferences that are defined over the set of (atemporal) lotteries $P$ on outcomes $X$. A KP-recursive utility function, in the contrary, represents preferences that are defined over the set $D$ of temporal lotteries. The key feature by which temporal lotteries are distinct from atemporal lotteries is the time-dating of the resolution of uncertainty.

I illustrate this characteristic feature of temporal lotteries by a classic example of Kreps and Porteus (1978).[59] Consider the two probability trees depicted in figure 1.1. The two trees have equivalent payoff structures: both yield payoff 0 in period 1, payoff 5 in periods 2 and 4 and, with equal probabilities, payoffs 0 or 10 in period 3. If we think of these trees in terms of atemporal lotteries, we find that both trees are equally likely to produce the consumption paths $(0, 5, 10, 5)$ or $(0, 5, 0, 5)$. Thus, if viewed through the lens of atemporal lotteries, the two trees appear to be equivalent although their structure is clearly different.

Temporal lotteries, in the contrary explicitly account for the timing of the resolution of uncertainty. A temporal lottery $d \in D$ is identified by a pair $(x, m)$, where $x$ denotes immediate consumption at a given node in a probability tree and $m$ is a probability

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[59]More comprehensive expositions of temporal lotteries can be found in Kreps and Porteus' (1978) original contribution and in Epstein and Zin (1989), who extend Kreps and Porteus' finite space of temporal lotteries to an infinite horizon.
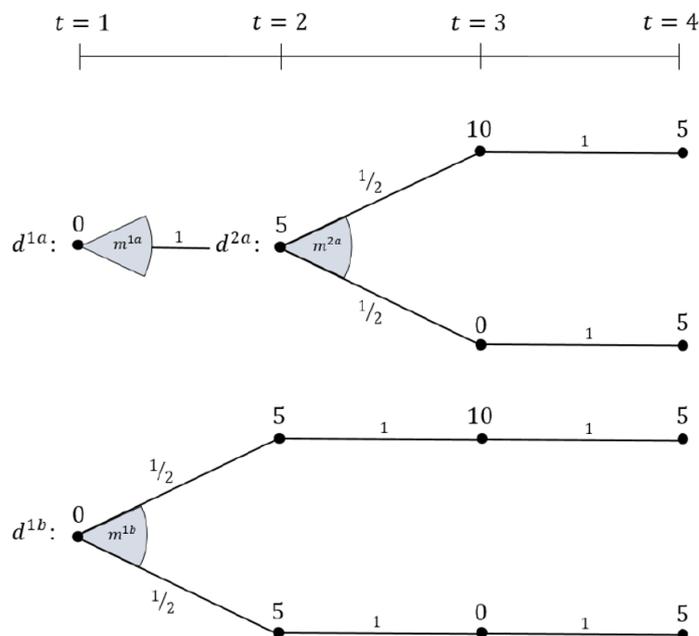
Figure 1.1: Temporal lotteries.

measure over the remaining nodes.[60] Consider the upper tree in figure 1.1. The temporal lottery $d^{2a}$ is characterized by as the tuple $(5, m^{2a})$, where $m^{2a}$ is a probability measure over the nodes in period 3. Likewise, $d^{1a}$ is identified as $(0, m^{1a})$, where $m^{1a}$ is a degenerate probability distribution over the node in period 2, i.e. over the temporal lottery $d^{2a}$. In the lower tree of figure1.1, the temporal lottery $d^{1b}$ is characterized by the pair $(0, m^{1b})$. The two temporal lotteries $d^{1a}$ and $d^{1b}$, i.e. the upper and the lower tree in figure 1.1, are distinct since the probability measures $m^{1a}$ and $m^{1b}$ dictate a different uncertainty structure. In $d^{1a}$, all uncertainty is resolved in period 3, whereas in $d^{1b}$, all uncertainty is resolved as of period 2.

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[60]More generally, a temporal lottery can be identified by a pair $(x, m)$ where $x$ denotes the consumption vector (the history of consumption) up until the respective node.

# Chapter 2

# Accounting for Different Uncertainties: Implications for Climate Investments?

## 2.1 Introduction

Assessing optimal measures to tackle climate change is a prominent example of the evaluation of long-term investments. The consumption discount rate plays a crucial role in this context, since long time horizons amplify the significance of the discount rate in determining an investment's desirability. Seemingly small differences in the discount rate have a major effect on suggested climate policy, as exemplified by the debate surrounding the Stern Review on the Economics of Climate Change and William Nordhaus' integrated assessment model DICE. Stern (2007), whose consumption discount rate of 1.4% is rather small, calls for *strong and early action* to tackle climate change. Nordhaus (2008) matches his discount rate to a comparatively high market rate of return of 5.5% and consequently argues for a more conservative *climate policy ramp*. This discord arises from differences in the rate of pure time preference and the intertemporal elasticity of substitution, which, together with the economic growth rate, determine the consumption discount rate in a deterministic setting.

Both the Stern Review on the Economics of Climate Change and Nordhaus' DICE model lack an in-depth treatment of future stochasticities and a comprehensive treatment of risk aversion. Uncertainty, however, looms large in the context of climate change. The natural sciences as well as the economic side of the evaluation contribute a considerable number of uncertainties, such as those on climate sensitivity, the damage function, economic growth, investment payoffs and future preferences. How these uncertainties affect the assessment of climate policy depends not only on their number and magnitude but also on the attitudes of policy makers towards stochasticity. A priori, it is unclear whether increases in risk aversion influence investments positively, out of a precautionary or insurance motive, or whether a more risk averse policy maker invests less for future generations to avoid putting resources at risk. The effect of a change in the decision maker's degree of risk aversion on the discount rate may thus be positive or negative.

The present chapter clarifies the link between changes in risk aversion and the effect on the consumption discount rate. For this purpose, I develop a general framework that can cope with diverse forms of uncertainties.[1] Within this framework, I show that the direction of the effect from a change in the decision maker's risk attitude on the discount rate depends on some fundamental properties of the uncertainty accounted for. These fundamental properties are then explored within a simple two-period endowment economy for three specific types of stochasticities: uncertainty on preferences, uncertainty on income, and uncertainty on an investment project. I consider a decision maker with Risk-Sensitive preferences (Hansen and Sargent 1995), which, like Epstein-Zin preferences (Epstein and Zin 1989), are a special case of the preference representation developed by Kreps and Porteus (1978) and Selden (1978). Contrary to the standard additive expected utility framework, as e.g. employed in the Stern Review and the DICE model, Risk-Sensitive preferences (and Epstein-Zin preferences) allow for the disentanglement of risk and time preferences and thus render it possible to study the effect of a change in risk aversion alone. I enlarge upon the reasons for employing Risk-Sensitive preferences rather than Epstein-Zin preferences in section 2.2.2.

Related research on uncertainty and risk aversion was conducted in the consumption/sav-

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[1] Recall from my note in chapter 1 that I use the words *uncertainty* and *risk* interchangeably, i.e. I do not distinguish between Knightian uncertainty and Knightian risk. All risks and uncertainties mentioned in this dissertation pertain to Knightian risk.

ings literature long before climate change became a widely studied topic in economics. Kihlstrom and Mirman (1974) and Kimball and Weil (2009) examine the effects of changes in risk aversion in the presence of uncertainty on investment returns and labour income, respectively. Bommier et al. (2012) explore the role of risk aversion in a non-parametric approach and investigate their results in the context of income and return uncertainty.[2] Insights from the consumption/savings literature have more recently been applied to discounting in climate economics. Gollier (2002a) complements the discounting debate by accounting for the effect of uncertainty on economic growth and in addition provides indication on the role of risk aversion. Traeger (2014) studies changes in risk aversion more explicitly, accounting for uncertainty on economic growth and return uncertainty.[3,4] The role of risk aversion given uncertainty on future preferences has neither been studied in the consumption/savings nor in the environmental economics literature. Beltratti et al. (1998) and others, however, analyze the effect of increases in preference uncertainty on the preservation of a non-renewable resource.

Similar to Gollier (2002a) and Traeger (2014), the present chapter adds to the discounting debate through a comprehensive treatment of risk aversion. One difference to their frameworks lies in the representation of preferences and in the assumptions imposed on the size and distribution of the uncertainties. Gollier establishes the connection between the discount rate and risk aversion for a decision maker with general Kreps-Porteus-Selden preferences in a setting with small uncertainties. Traeger assumes normally distributed uncertainty and Epstein-Zin preferences. The present chapter features Risk-Sensitive preferences and imposes no assumptions on the size or distribution of the uncertainties. Another difference to the contributions of Gollier and Traeger lies in the types of uncertainty accounted for. I consider preference uncertainty in addition to uncertainty on income and uncertainty on the investment project.

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[2] The effects of increases in income and investment return uncertainty on optimal savings are analyzed by Leland (1968), Sandmo (1970), Selden (1979), Drèze and Modigliani (1972), Kimball (1990), and Bleichrodt and Eeckhoudt (2005).

[3] In context of uncertainty on econmic growth, Traeger (2011) provides insights on the role of risk aversion in a multiperiod framework.

[4] The role of risk aversion is also analyzed in Knapp and Olson (1996) and Epaulard and Pommeret (2003), who examine the response of optimal resource management to changes in risk aversion if net return, technological progress or stock dynamics are uncertain. Furthermore, Ha-Duong and Treich (2004) study the role of risk aversion in the presence of uncertainty on the damage from a pollution stock.

The chapter's general result regarding the link between changes in risk aversion and the discount rate is developed in section 2.2 after describing the setting, the representation of preferences and the derivation of the consumption discount rate. This general result is then applied to specific types of uncertainty in section 2.3, which also provides interpretations and places the findings in the context of existing literature. Section 2.4 concludes the chapter.

## 2.2   Theory

### 2.2.1   Setting

The theoretical analysis is conducted for a decision maker (DM) who is altruistic towards people living in the present $(t = 1)$ and the future $(t = 2)$. His utility is derived from the felicity of the present and the future generation, which is in turn derived from consumption. The DM's purpose is the evaluation of a marginal investment in a two-period endowment economy. The economy is deterministic in the first period, but uncertainty enters the framework in the second period.

The first generation has an exogenous income $y_1$ and invests an amount $e$ in a project with a rate of return $r$. Certain first period consumption is thus $x_1 = y_1 - e$. The second generation consumes the exogenous income $y_2$ and the payoff $(1 + r) e$ from the investment project. The exogenous future income and the investment payoff may be diminished by the random factors $\tilde{\gamma}_y$ and $\tilde{\gamma}_e$, respectively. Uncertain second period consumption is therefore $\tilde{x}_2 = \tilde{\gamma}_y y_2 + \tilde{\gamma}_e (1 + r) e$. The decision maker evaluates the desirability of increasing project investment $e$ by the marginal amount $\varepsilon$, such that first and second period consumption are $x_1 = y_1 - (e + \varepsilon)$ and $\tilde{x}_2 = \tilde{\gamma}_y y_2 + \tilde{\gamma}_e (1 + r) (e + \varepsilon)$. In addition to the two uncertainties entering the evaluation through $\tilde{\gamma}_y$ and $\tilde{\gamma}_e$, I introduce a third random variable in the next subsection, namely $\tilde{a}$, which accounts for uncertainty on future felicity. Throughout this chapter, it is assumed that $y_2, r, \tilde{\alpha} > 0$, $y_1 > e \geq 0$ and $0 < \tilde{\gamma}_y, \tilde{\gamma}_e \leq 1$.

The variables $\tilde{\gamma}_y$, $\tilde{\gamma}_e$ and $\tilde{\alpha}$ are discrete random variables. Formally, a discrete random variable $\tilde{z}$ is a mapping $\Omega \to \mathbb{R}$, where $\Omega$ is the set of states of the world. For simplicity,

$\Omega$ is assumed to be finite. Each state $\omega = 1, 2...N \in \Omega$ realizes with a given probability $l^\omega$, where $\sum_\omega^N l^\omega = 1$. The value of the random variable $\tilde{z}$ when state $\omega$ is realized is denoted by $z^\omega$. The expectation of $\tilde{z}$ is defined by $E_l[\tilde{z}] = \sum_\omega^N l^\omega z^\omega$, and the covariance between two discrete random variables $\tilde{z}_1$ and $\tilde{z}_2$ derives from $cov_l[\tilde{z}_1, \tilde{z}_2] = E_l[\tilde{z}_1 \tilde{z}_2] - E_l[\tilde{z}_1] E_l[\tilde{z}_2]$.

### 2.2.2 Preferences

The standard framework to evaluate uncertain consumption streams $(x_1, \tilde{x}_2)$ is the intertemporally additive expected utility setting with constant elasticity of substitution felicity. This preference representation, however, presupposes that the intertemporal elasticity of substitution ($IES$) is the inverse of the coefficient of Arrow-Pratt relative risk aversion ($RRA$). An increase in risk aversion thus entails a decrease in the $IES$ and cannot be studied individually. To study the effect of a change in risk aversion alone, the additive expected utility framework must be abandoned, and a more flexible preference representation is required.

A possible means of achieving such flexibility involves the preference representations developed in Kreps and Porteus (1978) and Selden (1978). Selden provides a preference representation over certain $\times$ uncertain consumption pairs that allows for the disentanglement of risk and time preferences in a two-period choice problem. Kreps and Porteus axiomatize Selden's preference representation for a $T$-period setting. Their recursive preference representation includes Selden's specification as the two-period special case. Key to the disentanglement of risk aversion and the intertemporal elasticity of substitution in these frameworks is the differentiation between a cardinal atemporal expected utility function $v(\cdot)$ that represents risk preferences and an ordinal intertemporal utility function $U(\cdot)$ that represents time preferences. In a two-period setting, the general representation of Kreps-Porteus-Selden preferences is

$$U(x_1, \tilde{x}_2) = u(x_1) + \beta \phi^{-1}(E_l[\phi(u(\tilde{x}_2))]) \tag{2.1}$$

where $\beta > 0$ is the utility discount factor. The definition $v(\cdot) = \phi(u(\cdot))$ (with $\phi' \geq 0$, $\phi'' \leq 0, v', u' > 0$ and $v'', u'' < 0$) facilitates the disentanglement of risk and time

preferences: Increasing the concavity of $\phi(\cdot)$ enhances the DM's risk aversion through increasing the concavity of $v(\cdot)$ without affecting the curvature of $u(\cdot)$ and thus without affecting the DM's time preferences.[5] If $\phi(\cdot)$ is linear, i.e. if $v(\cdot) = u(\cdot)$, then equation (2.1) reduces to the standard additive expected utility representation in which time and risk preferences are entangled. A decision maker is called temporally risk averse if $\phi(\cdot)$ is strictly concave and is called temporally risk neutral if $\phi(\cdot)$ is linear.

In the present chapter, the *Risk-Sensitive preferences* specification $\phi(z) = -\exp(-kz)$ (Hansen and Sargent 1995) is employed to parameterize the general Kreps-Porteus-Selden preferences as represented by equation (2.1). A more prevalent parameterization is the isoelastic utility specification used by Epstein and Zin (1989) (*Epstein-Zin preferences*) in their extension of Kreps and Porteus (1978) to the infinite horizon setting. The Epstein-Zin parameterization is, however, not monotonic with respect to first-order stochastic dominance, as already discussed by Chew and Epstein (1990, p. 68). Kimball and Weil (2009) illustrate that this may bring about perverted results when studying the role of risk aversion. In particular, the authors show that increasing risk aversion can induce less precautionary savings if the risk on labour income is not restricted to be small. Bommier et al. (2012) ascribe this finding to the failure of the Epstein-Zin parameterization to be 'well ordered in terms of risk aversion'. The non-compliance of this specification with ordinal dominance may imply agents that prefer first-order stochastically dominated lotteries. Within the set of stationary preferences that allow for a disentanglement of the intertemporal elasticity of substitution and risk aversion, Bommier and Le Grand (2014) identify the Risk-Sensitive preference parameterization, as employed in the present chapter, as the only specification that is well ordered in terms of risk aversion.[6]

The framework described thus far only incorporates uncertainty that enters through the argument $\tilde{x}_2$ of the second generation's felicity $u(\cdot)$. However, uncertainty may also exist on the contribution of the future generation's felicity to the intertemporal

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[5]For a more detailed description of the disentanglement of risk and time preferences through the preferences of Kreps and Porteus (1978) and Selden (1978), see Traeger (2009).

[6]Note that the Epstein-Zin preference parameterisation intersects with Hansen and Sargent's Risk-Sensitive preferences for $IES = 1$ and thus for $u(\cdot) = \ln(\cdot)$. In this special case, Epstein-Zin preferences are well ordered in terms of risk aversion.

utility of the DM. Such uncertainty can be thought of as uncertainty regarding future preferences, i.e. uncertainty about the felicity that the future generation gains from a given consumption level or, as in the setting of Beltratti et al. (1998), from a given level of natural resources. The felicity derived from future consumption is then $\tilde{\alpha} u\left(\tilde{x}_2\right)$, where the random multiplicator $\tilde{\alpha}$ determines whether the second generation values a given level of consumption more or less than the present generation. In particular, for $u\left(\cdot\right) > 0$ and if $\alpha^\omega > 1$ ($\alpha^\omega < 1$), the second generation values a given level of consumption more (less) than the present generation. This relation is inverted if $u\left(\cdot\right) < 0$.

Parameterizing equation (2.1) with the Risk-Sensitive preference specification $\phi\left(z\right) = -\exp\left(-kz\right)$ and accounting for uncertainty on future preferences, as represented by $\tilde{\alpha}$, yields

$$U\left(x_1, \tilde{x}_2\right) = u\left(x_1\right) - \frac{\beta}{k} \ln\left(E_l\left[\exp\left(-k\tilde{\alpha} u\left(\tilde{x}_2\right)\right)\right]\right), \qquad (2.2)$$

which is the preference representation upon which the subsequent analysis builds. Under this Risk-Sensitive preference specification, a decision maker is temporally risk averse if $k > 0$ ($\phi\left(\cdot\right)$ strictly concave) and temporally risk neutral if $k \to 0$ ($\phi\left(\cdot\right)$ linear). Increasing $k$ enhances the degree of risk aversion of the DM without affecting his time preferences.

### 2.2.3  Discounting

The consumption discount rate informs the desirability of conducting the additional marginal investment $\varepsilon$. Formally, a DM with preferences as represented by equation (2.2) considers investing $\varepsilon$ desirable if

$$\frac{\partial U\left(x_1, \tilde{x}_2\right)}{\partial \varepsilon} \geq 0,$$

that is, if investing an additional amount $\varepsilon$ in the project increases his intertemporal utility. Solving this derivative yields

$$r \geq DR_{1,2}$$

where

$$DR_{1,2} = \frac{u'(x_1)}{\beta} \left. \frac{E_l\left[\exp\left(-k\tilde{\alpha}u\left(\tilde{x}_2\right)\right)\right]}{E_l\left[\exp\left(-k\tilde{\alpha}u\left(\tilde{x}_2\right)\right)\left(\tilde{\gamma}_e\tilde{\alpha}u'\left(\tilde{x}_2\right)\right)\right]}\right|_{\varepsilon=0} - 1 \tag{2.3}$$

with $\tilde{x}_2 = \tilde{\gamma}_y y_2 + \tilde{\gamma}_e (1+r) e$. A marginal investment $\varepsilon$ is desirable if its rate of return $r$ exceeds the consumption discount rate $DR_{1,2}$, as defined in equation (2.3).[7] The consumption discount rate reflects the marginal rate of substitution between an additional unit $\varepsilon$ of consumption in the first and second period, evaluated at the point at which the additional investment is not yet conducted. This marginal rate of substitution gives account of the assumptions made on preferences, such as the degree of the DM's risk aversion, and assumptions on the economic setting, such as those regarding the considered uncertainties. The consumption discount rate constitutes a convenient tool to measure changes in the desirability of the investment $\varepsilon$ in response to a change in risk aversion.

The term $\tilde{\gamma}_e\tilde{\alpha}u'(\tilde{x}_2)$, which I refer to as the *effective marginal utility,* is an essential part of equation (2.3). It is the state-dependent marginal utility gained by the decision maker in the second period if investments are increased. The higher the effective marginal utility is expected to be, ceteris paribus, the more valuable an additional investment will be, as reflected in a lower discount rate. The expected size of the effective marginal utility depends not only on the distribution of the states of the world and the state-dependent effective marginal utility itself, but also on the preferences of the Risk-Sensitive decision maker. A temporally risk averse decision maker modifies the (statistical) expectation of the effective marginal utility such that the probabilities of the occurrence of good states of the world are undervalued, whereas the probabilities of bad states of the world are overvalued. The more risk averse the decision maker is, the more the probabilities are modified. How good or bad a state of the world is perceived to be depends on a second essential part of equation (2.3), the *effective utility* $\tilde{\alpha}u(\tilde{x}_2)$. I show in the next subsection that the interrelation between the effective utility and the effective marginal utility is key in determining the effect of a change in risk aversion on the consumption discount rate.

To clarify the role of (temporal) risk aversion in the determination of the discount rate

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[7]A more detailed derivation of the discount rate $DR_{1,2}$ can be found in Gollier (2002a).

$DR_{1,2}$, I rewrite equation (2.3) in terms of a weighted sum:

$$DR_{1,2} \;=\; \frac{1}{\beta} \frac{u'(x_1)}{\sum_\omega \pi^\omega \cdot (\gamma_e^\omega \alpha^\omega u'(x_2^\omega))} - 1 \tag{2.4}$$

$$\text{with } \pi^\omega = l^\omega \cdot \frac{\exp(-k\alpha^\omega u(x_2^\omega))}{\sum_{\omega=1}^{N} l^\omega \exp(-k\alpha^\omega u(x_2^\omega))}.$$

The weights $\pi^\omega$ adjust the statistical probabilities $l^\omega$ to account for temporal risk aversion. These weights are interpretable as probabilities since $0 \leq \pi^\omega \leq 1 \; \forall \; \omega$ and $\sum_\omega \pi^\omega = 1$. I call $\pi^\omega$ a *risk aversion adjusted probability*. The fraction in the equation for $\pi^\omega$ is greater than 1 in the worst state of the world ($\alpha^\omega u(x_2^\omega)$ lowest) and smaller than 1 in the best state of the world ($\alpha^\omega u(x_2^\omega)$ highest), which suggests that the risk aversion adjusted probabilities $\pi^\omega$ overvalue the statistical probabilities of bad states and undervalue those of good states. This adjustment is amplified as $k$ increases, i.e. as the decision maker becomes more risk averse. Note that $\pi^\omega$ approaches the statistical probability $l^\omega$ for a temporally risk neutral decision maker ($k \to 0$).[8] Equation (2.4) is then the discount rate of a DM with additive expected utility preferences. Such a decision maker is averse to uncertainty on the argument of second period felicity $u(\cdot)$, but he is neutral towards uncertainty that affects $u(\cdot)$ multiplicatively, such as $\tilde{\alpha}$.

### 2.2.4 Changes in risk aversion

The effect on the decision maker's evaluation of the intertemporal tradeoff in response to a change in his degree of risk aversion is reflected in the derivative of the discount rate $DR_{1,2}$ (equation 2.3) with respect to $k$:

$$\frac{\partial DR_{1,2}}{\partial k} \;=\; \frac{u'(x_1)}{\beta \cdot (E_l[\exp(-k\tilde{\alpha}u(\tilde{x}_2))(\tilde{\gamma}_e\tilde{\alpha}u'(\tilde{x}_2))])^2} \cdot \tag{2.5}$$

$$\begin{pmatrix} E_l[\exp(-k\tilde{\alpha}u(\tilde{x}_2))] E_l[\exp(-k\tilde{\alpha}u(\tilde{x}_2))(\tilde{\gamma}_e\tilde{\alpha}u'(\tilde{x}_2))(\tilde{\alpha}u(\tilde{x}_2))] \\ -E_l[\exp(-k\tilde{\alpha}u(\tilde{x}_2))(\tilde{\alpha}u(\tilde{x}_2))] E_l[\exp(-k\tilde{\alpha}u(\tilde{x}_2))(\tilde{\gamma}_e\tilde{\alpha}u'(\tilde{x}_2))] \end{pmatrix}.$$

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[8] The notion of *risk aversion adjusted probabilities* calls to mind the concept of *risk neutral probabilities* in asset pricing. However, the two concepts are not equivalent. Risk aversion adjusted probabilities $\pi^\omega$ differ from the statistical probabilities $l^\omega$ only if the agent is not temporally risk neutral ($k \neq 0$). If $k = 0$, however, then $\pi^\omega = l^\omega$. By contrast, in asset pricing, a decision maker with $k = 0$ may employ risk neutral probabilities that are not equivalent to the respective statistical probabilities.

Using the risk aversion adjusted probabilities $\pi^\omega$ to define the *risk aversion adjusted expectation operator* $E_\pi\left[g\left(\tilde{z}\right)\right] = \sum_\omega^N \pi^\omega g\left(z^\omega\right)$ for a random variable $\tilde{z}$ and some function $g\left(\tilde{z}\right)$, equation (2.5) can be written as

$$\frac{\partial DR_{1,2}}{\partial k} \quad = \quad \frac{u'\left(x_1\right)}{\beta \cdot \left(E_\pi\left[\tilde{\gamma}_e \tilde{\alpha} u'\left(\tilde{x}_2\right)\right]\right)^2} \cdot \tag{2.6}$$

$$\left(E_\pi\left[\left(\tilde{\alpha}u\left(\tilde{x}_2\right)\right)\left(\tilde{\gamma}_e\tilde{\alpha}u'\left(\tilde{x}_2\right)\right)\right] - E_\pi\left[\tilde{\alpha}u\left(\tilde{x}_2\right)\right]E_\pi\left[\tilde{\gamma}_e\tilde{\alpha}u'\left(\tilde{x}_2\right)\right]\right).$$

In this chapter, a covariance that is constructed from a risk aversion adjusted expectation operator, $cov_\pi\left[g_1\left(\tilde{z}_1\right), g_2\left(\tilde{z}_2\right)\right] = E_\pi\left[g_1\left(\tilde{z}_1\right)g_2\left(\tilde{z}_2\right)\right] - E_\pi\left[g_1\left(\tilde{z}_1\right)\right]E_\pi\left[g_2\left(\tilde{z}_2\right)\right]$, is called a *risk aversion adjusted covariance*. In accordance with this denotation, the second line of equation (2.6) is the risk aversion adjusted covariance between $\tilde{\alpha}u\left(\tilde{x}_2\right)$ and $\tilde{\gamma}_e\tilde{\alpha}u'\left(\tilde{x}_2\right)$: $cov_\pi\left[\tilde{\alpha}u\left(\tilde{x}_2\right), \tilde{\gamma}_e\tilde{\alpha}u'\left(\tilde{x}_2\right)\right]$. If the two random variables *effective utility* and *effective marginal utility* satisfy a specific interrelation, then the sign of this risk aversion adjusted covariance is determinate. Lemma 2 below identifies the comonotonicity characteristics (definition 1) of $\tilde{\alpha}u\left(\tilde{x}_2\right)$ and $\tilde{\gamma}_e\tilde{\alpha}u'\left(\tilde{x}_2\right)$ as this particular interrelation.[9] The proof of lemma 2, which is a close analogue to theorem 43 in Hardy et al. (1934), is relegated to appendix 2.A.

**Definition 1** *(Strict comonotonicity and strict countercomonotonicity)*
*Consider two random variables $Z_1$ and $Z_2$ that are strictly monotonic transformations of a single random variable $\tilde{z}$:*

$$\left(Z_1, Z_2\right) = \left(g_1\left(\tilde{z}\right), g_2\left(\tilde{z}\right)\right).$$

*If $g_1$ and $g_2$ are strictly increasing in $\tilde{z}$, then $Z_1$ and $Z_2$ are called comonotonic.*
*If $g_1$ is strictly increasing and $g_2$ is strictly decreasing in $\tilde{z}$, or vice versa, then $Z_1$ and $Z_2$ are called countercomonotonic.*

**Lemma 2** *(Risk aversion adjusted covariance inequality)*
*Consider two random variables $Z_1$ and $Z_2$ that are strictly monotonic transformations*

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[9]Comonotonicity and countercomonotonicity are defined in various ways in the literature. The definition provided here is based on McNeil et al. (2005), pp. 199, 200.

*of a single random variable $\tilde{z}$. If $Z_1$ and $Z_2$ are strictly comonotonic, then*

$$cov_\pi [Z_1, Z_2] > 0.$$

*The inequality is reversed if $Z_1$ and $Z_2$ are strictly countercomonotonic.*

Applying the insight of lemma 2 to equation (2.6) links the effect of a change in risk aversion on the consumption discount rate to the comonotonicity characteristics of the effective utility and the effective marginal utility. This relation is summarized in proposition 3.

**Proposition 3** *(Effect of a change in risk aversion on $DR_{1,2}$)*
*If the effective utility, $\tilde{\alpha}u(\tilde{x}_2)$, and the effective marginal utility, $\tilde{\gamma}_e\tilde{\alpha}u'(\tilde{x}_2)$, are comonotonic, then the consumption discount rate increases in response to an increase in risk aversion: $\frac{\partial DR_{1,2}}{\partial k} > 0$.*
*If the effective utility, $\tilde{\alpha}u(\tilde{x}_2)$, and the effective marginal utility, $\tilde{\gamma}_e\tilde{\alpha}u'(\tilde{x}_2)$, are countercomonotonic, then the consumption discount rate decreases in response to an increase in risk aversion: $\frac{\partial DR_{1,2}}{\partial k} < 0$.*

**Proof.** The sign of equation (2.6) is the same as that of $cov_\pi [\tilde{\alpha}u(\tilde{x}_2), \tilde{\gamma}_e\tilde{\alpha}u'(\tilde{x}_2)]$ because $\beta$, $u'(x_1)$, $(E_\pi [\tilde{\gamma}_e\tilde{\alpha}u'(\tilde{x}_2)])^2 > 0$. By application of lemma 2, the sign of $cov_\pi [\tilde{\alpha}u(\tilde{x}_2), \tilde{\gamma}_e\tilde{\alpha}u'(\tilde{x}_2)]$ follows from the comonotonicity characteristics of $\tilde{\alpha}u(\tilde{x}_2)$ and $\tilde{\gamma}_e\tilde{\alpha}u'(\tilde{x}_2)$. ∎

Proposition 3 constitutes a general result that is not restricted to specific forms of uncertainty. Information on the comonotonicity characteristics of the effective utility and the effective marginal utility suffices to determine the direction of the effect of a change in risk aversion on the consumption discount rate. The comonotonicity characteristics, however, are conditional on the type of uncertainty accounted for. In the next section, I explore the comonotonicity characteristics for three different types of uncertainty and provide an example involving multiple uncertainties.

## 2.3   Application to specific types of uncertainty

The examination of the comonotonicity characteristics of the effective utility and the effective marginal utility is initially conducted for individual types of uncertainty, i.e. under the assumption that the remaining random variables are deterministic. Linking the results of this examination to proposition 3 yields insights into the effect of a change in risk aversion on the consumption discount rate in the presence of different types of uncertainty. The effective utility corresponding to the three different types of uncertainty is specified as follows:

case 1 ($\tilde{\alpha}$ uncertain, $\gamma_y$, $\gamma_e$ deterministic)   :   $\tilde{\alpha} u(x_2)$ with $x_2 = \gamma_y y_2 + \gamma_e (1 + r) e$

case 2 ($\tilde{\gamma}_y$ uncertain, $\gamma_e$, $\alpha$ deterministic)   :   $\alpha u(\tilde{x}_2)$ with $\tilde{x}_2 = \tilde{\gamma}_y y_2 + \gamma_e (1 + r) e$

case 3 ($\tilde{\gamma}_e$ uncertain, $\gamma_y$, $\alpha$ deterministic)   :   $\alpha u(\tilde{x}_2)$ with $\tilde{x}_2 = \gamma_y y_2 + \tilde{\gamma}_e (1 + r) e$.

These three cases represent uncertain future preferences (case 1), uncertainty on future income (case 2) and uncertainty on the investment project's payoff (case 3).

To assign the comonotonicity characteristics of the effective utility and the effective marginal utility, one must analyze whether they are increasing or decreasing in the considered random variable. This exercise is conducted in appendix 2.B.1, where it is shown that the effective utility is increasing in the random variable $\tilde{\gamma}_y$ and, if $u(\cdot) > 0$, in $\tilde{\alpha}$ . For $e > 0$, the effective utility is also increasing in $\tilde{\gamma}_e$. The effective marginal utility is increasing in $\tilde{\alpha}$, decreasing in $\tilde{\gamma}_y$ and non-monotonic in $\tilde{\gamma}_e$. Combining these insights on the comonotonicity characteristics of the effective utility and the effective marginal utility in the presence of different uncertainties with the general result of this chapter (proposition 3) yields proposition 4.

**Proposition 4** *(Effect of a change in risk aversion on $DR_{1,2}$ for specific uncertainties)*
*The effect of a change in risk aversion on the discount rate is as follows:*

**Case 1:** *If uncertainty exists only on future preferences and if the decision maker is risk averse with respect to this uncertainty, then the discount rate increases in response to an increase in risk aversion.*[10]

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[10]Risk aversion with respect to preference uncertainty means that the decision maker prefers $E[\tilde{\alpha}]$ to

**Case 2:** *If uncertainty exists only on future income, then the discount rate decreases in response to an increase in risk aversion.*

**Case 3:** *If uncertainty exists only on the investment project's payoff and if $e > 0$, then the response of the discount rate to a change in risk aversion depends on the size of the intertemporal elasticity of substitution relative to a threshold:*

*If $IES > \frac{\tilde{\gamma}_e(1+r)e}{\gamma_y y_2 + \tilde{\gamma}_e(1+r)e} \; \forall \; \omega$, then the discount rate increases with risk aversion.*

*If $IES < \frac{\tilde{\gamma}_e(1+r)e}{\gamma_y y_2 + \tilde{\gamma}_e(1+r)e} \; \forall \; \omega$, then the discount rate decreases with risk aversion.*

The following subsections discuss these results and relate them to the literature.

### 2.3.1 Uncertainty on preferences

At the heart of all evaluations of intertemporal tradeoffs lies the attempt to secure the wellbeing of generations living at different times. To this end, a decision maker may want to redistribute resources between present and future generations. The default assumption that underlies such evaluations is that agents living at different times gain the same wellbeing from a given amount of consumption. Rapid societal changes in the past suggest, however, that such an assumption is overly simplistic, especially if the distant future is considered. Whether future developments yield an increase or decrease in the valuation of produced or natural resources is unclear. Nevertheless, the mere presence of uncertainty on future preferences is without much doubt. Solow (1993) emphasizes the relevance of this issue in the context of sustainability. He argues that the notion of sustainability, which involves the quest for a level of future wellbeing that is not below ours, is problematic and vague by nature, as we do not know how the wellbeing of future generations will be determined.

The utility multiplicator $\tilde{\alpha}$ that is considered in this chapter can be interpreted as uncertainty regarding future felicity from consumption. People living in the future could be more dependent on produced goods and thus attach a higher value to their consumption, or they may prefer to live more simply and thus have a lower valuation. Similarly, the felicity derived from the consumption of natural resources could be higher in the future

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$\tilde{\alpha}$. Given $k > 0$, this is the case whenever $u(\cdot) > 0$. The relation in proposition 4 (case 1) is inverted for a DM who is risk loving with respect to $\tilde{\alpha}$; see appendix 2.B.1, 1.

as new insights on their usability are gained, or could be lower as substitutes are discovered. Would a risk averse decision maker who faces such uncertainty invest more for the future generation as his risk aversion increases, to insure for the possibility of increased valuation? Proposition 4 (case 1) suggests that he would not. Uncertainty on future preferences implies that resources that are allocated to the future are put at risk because their valuation is insecure. Therefore, an increasingly risk averse Risk-Sensitive decision maker allocates more consumption to the present generation because he is certain about their valuation.

A small group of authors has studied the effect of uncertain preferences on the optimal allocation of a non-renewable resource. Beltratti et al. (1998) consider the effect of uncertainty on future preferences as specified in the present chapter. However, the intertemporal tradeoff in their approach is evaluated by a decision maker with additive expected utility preferences. The authors find that symmetric uncertainty on future preferences (i.e. mean preserving uncertainty in the sense of Rothschild and Stiglitz) does not affect the optimal consumption/preservation stream.[11]

It follows from proposition 4 (case 1) that the result obtained by Beltratti et al. (1998) is not robust to changes in risk aversion. A simplified two-period version of their model can be nested in the framework of the present chapter, in which it constitutes the special case of temporal risk neutrality ($k \to 0$). Moving from a temporally risk neutral DM to a temporally risk averse DM (i.e. increasing risk aversion) then induces an increase in the discount rate, which implies a decrease in the preservation of a non-renewable resource in the setting of Beltratti et al. The result of Beltratti et al. is thus extended: If the decision maker is temporally risk neutral, then symmetric uncertainty regarding future preferences does not affect the optimal preservation of a non-renewable resource. By contrast, if the decision maker is temporally risk averse, then uncertainty regarding future preferences negatively affects the preservation of a non-renewable resource.

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[11]More complex forms of preference uncertainty are considered by Ayong Le Kama and Schubert (2004), who analyze the effect of uncertainty on the preference for environmental quality relative to the preference for consumption, and by Cunha-E-Sà and Costa-Duarte (2000) and Ayong Le Kama (2012), who consider endogenous uncertainty regarding future preferences.

### 2.3.2 Uncertainty on income, investment payoff, or both

The dimension of economic growth is inherently uncertain. This uncertainty regarding future income levels is even increased by the possibility of detrimental effects from climate change on productive capacities. A forward-looking decision maker may thus want to take measures to insure against these negative effects on future generations, such as investing in the economy's productive capacity or in research on and development of abatement and adaptation capacities. However, the payoff to such investments may be uncertain itself, for example because of uncertainty on technological advancements that determine the effectiveness of abatement and adaptation or because of uncertainty on damages from climate change that decrease the investment project's payoff multiplicatively.

Proposition 4 (cases 2 and 3) clarifies how an increase in the DM's risk aversion affects his willingness to invest for the future in the presence of uncertainty on future income or the investment payoff. If only uncertainty on future income is accounted for (case 2), then increasing the DM's risk aversion unambiguously induces higher precautionary savings, as reflected in a decreasing discount rate. The more risk averse the decision maker is, the more he overvalues the statistical probability of bad states in which the effective marginal utility in the future is relatively high. The increasingly risk averse decision maker therefore allocates more resources to the uncertain future and thus insures against the possibility of low income levels in the second period. If only uncertainty on the investment payoff is accounted for (case 3) and if $e > 0$, then the effect of an increase in risk aversion is ambiguous. The direction of the effect on the discount rate depends on the value of the DM's intertemporal elasticity of substitution relative to the state-dependent value of a threshold. An increase in risk aversion induces an increase in the discount rate if the $IES$ is higher than the value of the threshold in the best state of the world.[12] This outcome always occurs if $IES \geq 1$. An increase in risk aversion then amplifies the DM's aversion to putting resources at risk by investing more in a project with uncertain payoff. Consequently, the DM transfers less resources to the

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[12]The threshold $\frac{\tilde{\gamma}_e(1+r)e}{\gamma_y y_2 + \tilde{\gamma}_e(1+r)e}$ is increasing in $\tilde{\gamma}_e$. The state-dependent threshold thus reaches its highest (lowest) level in the best (worst) state of the world, i.e. when $\tilde{\gamma}_e$ is highest (lowest). If the IES exceeds (falls below) the threshold in the best (worst) state of the world, then it also exceeds (falls below) the thresholds in the other states of the world.

future generation. If the $IES$ is smaller than the value of the threshold in the worst state of the world, then the discount rate is decreasing in risk aversion. A small $IES$ implies a relatively strong desire to smooth deterministic consumption over time. A DM with a relatively low $IES$ thus aims to maintain the future generation's certainty equivalent consumption. As the certainty equivalent consumption decreases as a result of an increase in risk aversion, the DM allocates more resources to the future generation. If the $IES$ lies between the threshold evaluated in the best state of the world and the threshold evaluated in the worst state of the world, then we cannot make a statement on the sign of $\frac{\partial DR_{1,2}}{\partial k}$. Note that the result of proposition 4 (case 3) crucially depends on assuming $e > 0$. In a setting in which initial project investment is nil ($e = 0$) and uncertainty exists only on the investment project's payoff, the discount rate is unaffected by changes in risk aversion.

Thus far, different uncertainties have been considered only on an individual basis, i.e. under the assumption that all other variables are deterministic. If some assumptions regarding the relationship between the considered uncertainties are made, then it is possible to use the general result of proposition 3 to analyze the role of risk aversion in the presence of multiple uncertainties. In the context of climate change, we may assume that the uncertainties regarding diverse variables are linked to a random variable that represents future climatic conditions. The comonotonicity characteristics of the effective utility and the effective marginal utility then depend, among other conditions, on the comonotonicity characteristics of the uncertain variables with respect to the random variable 'climate conditions'. Consider, for example, that the future income and investment payoff are uncertain because of their dependence on random climate conditions $\tilde{z}$. A high realization of the random variable $\tilde{z}$ represents an unchanged climate, and a low $\tilde{z}$ indicates detrimental climate change. Countercomonotonicity of $\gamma_y$ and $\gamma_e$ in $\tilde{z}$ ($\gamma'_y(\tilde{z}) > 0$, $\gamma'_e(\tilde{z}) < 0$) is given if the economic income is highest in a world with an unchanged climate, while the payoff from an investment project is highest under very bad climate conditions. In this case, and if economic income varies more strongly with climate conditions than the project's payoff, then an increase in risk aversion unambiguously amplifies the valuation of future consumption (see appendix 2.B.2, 1). However, if $\gamma_y$ and $\gamma_e$ are comonotonic in $\tilde{z}$ ($\gamma'_y(\tilde{z}) > 0$, $\gamma'_e(\tilde{z}) > 0$), i.e. if

both the economic income and the payoff from the investment project are high in good states of the world, then the sign of the derivative of the discount rate with respect to risk aversion depends on the relative magnitudes of the $IES$, the overall variation in consumption and the variation in the project payoff (see appendix 2.B.2, 2).

A result that is consistent with proposition 4 (case 2) is implicit in Gollier's (2002a) examination of the effect of growth uncertainty on the socially optimal discount rate. His result is derived under the assumption that the uncertainty on future income is small. Similarly, Traeger (2014) shows that an increase in temporal risk aversion decreases the social discount rate for mitigation policies in an isoelastic utility (Epstein-Zin preferences) framework if uncertain growth is accounted for. In addition, and in accordance with my remarks on proposition 4 (case 3) regarding the case $e = 0$, Traeger shows that temporal risk aversion with respect to an uncertain marginal investment project has no independent effect on the discount rate. Related to my example of multiple uncertainties, he shows that correlated uncertainty on the investment project induces ambiguous reactions of the discount rate in response to risk aversion changes, depending on the sign and degree of the correlation with growth uncertainty, the variances of both uncertain variables, and the IES. Uncertainty on future income and uncertainty on the marginal investment project are assumed to be (jointly) normally distributed in Traeger's analysis. As mentioned before, Kimball and Weil (2009) suggest that results derived for normal (or small) uncertainties in an isoelastic utility setting may not be extendable to the case of less restricted uncertainties. In particular, the authors show that results on the role of risk aversion in an isoelastic utility setting may be perverted if large uncertainty on future income is considered. In the present chapter, I do not make use of the isoelastic utility specification but rather employ Risk-Sensitive preferences. Thereby I avert the restrictions imposed by the isoelastic utility setting and extend the results obtained by Gollier and Traeger to more general uncertainties. In the present chapter, no assumptions on the size or distribution of the uncertainty are necessary to derive results on the effect of changes in risk aversion. The results of this chapter are consistent with Bommier et al. (2012), who point out that the effect of risk aversion changes in the presence of income uncertainties is monotonic if preferences are well ordered in terms of risk aversion, and with Bommier and Le Grand (2014), who identify the Risk-Sensitive

preference specification to be well ordered in terms of risk aversion.

## 2.4   Conclusion

The prevalent frameworks to assess optimal climate policy, such as the Stern Review (Stern 2007) and Nordhaus' DICE model (Nordhaus 2008), assume a degree of risk aversion that is too low in view of empirical evidence.[13] Changing the degree of risk aversion in these frameworks, however, simultaneously distorts the intertemporal elasticity of substitution. A number of recent contributions have addressed this issue by developing recursive versions of DICE to study the effects of different types of uncertainty under empirically substantiated assumptions regarding risk and time preferences. Ackerman et al. (2013) consider climate uncertainty, Cai et al. (2013) investigate the effect of uncertainty on the economic impact of climate tipping events, Jensen and Traeger (2013) explore long-term growth uncertainty, and Crost and Traeger (2011) examine damage uncertainty. The comparison of the results from these recursive DICE models to the results of the standard DICE model yields insights into the policy implications of increased risk aversion. However, these recursive DICE models are rather complex and account for only individual uncertainties. It is thus difficult to clearly identify the role of risk aversion and to attribute the direction of effects to specific features of the uncertainty.

The framework of the present chapter is a two-period endowment economy in which different uncertainties can be accounted for, both individually and simultaneously. The simplicity of this approach allows for a general result on the link between changes in the decision maker's risk aversion and the consumption discount rate by which the direction of the effect is ascribed to specific characteristics of the considered uncertainties. In particular, I show that the direction of the effect from a change in risk aversion on the consumption discount rate depends on the comonotonicity characteristics of the effective utility and the effective marginal utility. The comonotonicity characteristics in turn differ between the types of uncertainties. Thus, increases in risk aversion may have diverse effects on the evaluation of an intertemporal consumption tradeoff and thus on optimal

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[13]The default assumption in the DICE model is a degree of relative risk aversion of 2 and an intertemporal elasticity of substitution of 0.5. However, empirical evidence from the asset pricing literature suggests a degree of relative risk aversion of 5-10 and an intertemporal elasticity of substitution of 1-1.5 (Bansal and Yaron 2004, Vissing-Jørgensen and Attanasio 2003).

climate policy.

The application of the general result yields new findings regarding the effect of risk aversion changes in the presence of preference uncertainty, and extends existing findings with respect to the presence of income and investment payoff uncertainties. In contrast to Beltratti et al. (1998), I find that preference uncertainty does affect the discount rate positively if a temporally risk averse DM rather than a temporally risk neutral DM is considered. This effect is amplified as the degree of risk aversion increases. Regarding income and investment payoff uncertainty, the application of the general result confirms previous findings of Gollier (2002a) and Traeger (2014), though under less restrictive assumptions regarding the size or distribution of uncertainty. This extension of the results obtained by Gollier and Traeger is rendered possible by employing the Risk-Sensitive preference representation.

# Appendices for chapter 2

## 2.A    Proof of lemma 2

Consider two random variables $Z_1$ and $Z_2$ that are strictly monotonic transformations of a discrete random variable $\tilde{z}$. Denote these functions as $g_1\left(\tilde{z}\right)$ and $g_2\left(\tilde{z}\right)$, respectively. The risk aversion adjusted covariance between $Z_1$ and $Z_2$ is then

$$cov_\pi\left[Z_1, Z_2\right] = cov_\pi\left[g_1\left(\tilde{z}\right), g_2\left(\tilde{z}\right)\right]$$

$$\text{where } cov_\pi\left[g_1\left(\tilde{z}\right), g_2\left(\tilde{z}\right)\right] = E_\pi\left[g_1\left(\tilde{z}\right)g_2\left(\tilde{z}\right)\right] - E_\pi\left[g_1\left(\tilde{z}\right)\right]E_\pi\left[g_2\left(\tilde{z}\right)\right] \qquad (2.7)$$

$$\text{with } E_\pi\left[g\left(\tilde{z}\right)\right] = \sum_{\omega=1}^{N}\pi^\omega g\left(z^\omega\right).$$

The risk aversion adjusted probabilities $\pi^\omega$ are defined by equation (2.4). After some rearrangements, equation (2.7) can be written as

$$cov_\pi\left[g_1\left(\tilde{z}\right), g_2\left(\tilde{z}\right)\right] = \frac{1}{2}\sum_{\omega_i=1}^{N}\sum_{\omega_j=1}^{N}\pi^{\omega_i}\pi^{\omega_j}\left(g_1\left(z^{\omega_j}\right) - g_1\left(z^{\omega_i}\right)\right)\left(g_2\left(z^{\omega_j}\right) - g_2\left(z^{\omega_i}\right)\right).$$

By definition 1, strict comonotonicity between the random variables $Z_1$ and $Z_2$ implies that $g_1\left(\tilde{z}\right)$ and $g_2\left(\tilde{z}\right)$ are both strictly increasing in $\tilde{z}$, which in turn implies that

$$\left(g_1\left(z^{\omega_j}\right) - g_1\left(z^{\omega_i}\right)\right)\left(g_2\left(z^{\omega_j}\right) - g_2\left(z^{\omega_i}\right)\right) > 0$$

and thus

$$cov_\pi\left[g_1\left(\tilde{z}\right), g_2\left(\tilde{z}\right)\right] = cov_\pi\left[Z_1, Z_2\right] > 0.$$

Equivalently, strict countercomonotonicity between the random variables $Z_1$ and $Z_2$ implies that either of the functions $g_1(\tilde{z})$ and $g_2(\tilde{z})$ is strictly increasing in $\tilde{z}$, while the other is strictly decreasing in $\tilde{z}$. This in turn implies that

$$\left(g_1\left(z^{\omega_j}\right) - g_1\left(z^{\omega_i}\right)\right)\left(g_2\left(z^{\omega_j}\right) - g_2\left(z^{\omega_i}\right)\right) < 0$$

and thus

$$cov_\pi\left[g_1(\tilde{z}), g_2(\tilde{z})\right] = cov_\pi\left[Z_1, Z_2\right] < 0.$$

## 2.B  Derivations

### 2.B.1  Comonotonicity characteristics

To determine whether $\tilde{\alpha}u(\tilde{c}_2)$ and $\tilde{\gamma}_e\tilde{\alpha}u'(\tilde{c}_2)$ are comonotonic or countercomonotonic, I specify the type of uncertainty in the future. I consider the three different types of uncertainty introduced in the main part of the chapter:

case 1 ($\tilde{\alpha}$ uncertain, $\gamma_y$, $\gamma_e$ deterministic)  :  $\tilde{\alpha}u(x_2)$ with $x_2 = \gamma_y y_2 + \gamma_e(1+r)e$

case 2 ($\tilde{\gamma}_y$ uncertain, $\gamma_e$, $\alpha$ deterministic)  :  $\alpha u(\tilde{x}_2)$ with $\tilde{x}_2 = \tilde{\gamma}_y y_2 + \gamma_e(1+r)e$

case 3 ($\tilde{\gamma}_e$ uncertain, $\gamma_y$, $\alpha$ deterministic)  :  $\alpha u(\tilde{x}_2)$ with $\tilde{x}_2 = \gamma_y y_2 + \tilde{\gamma}_e(1+r)e$.

The variables $\tilde{\alpha}u(\tilde{x}_2)$ and $\tilde{\gamma}_e\tilde{\alpha}u'(\tilde{x}_2)$ are comonotonic if both are increasing (or decreasing) in a random variable $\tilde{\alpha}$, $\tilde{\gamma}_y$, or $\tilde{\gamma}_e$. The variables are countercomonotonic if either of them is increasing while the other is decreasing in the random variable. Summarizing the results from the comonotonicity examinations 1.-3. below and combining with proposition 3 yields proposition 4 in section 2.3.

1. $\tilde{\alpha}u(x_2)$ with $x_2 = \gamma_y y_2 + \gamma_e(1+r)e$

   $\frac{\partial \tilde{\alpha}u(x_2)}{\partial \tilde{\alpha}} = u(x_2) > 0$ (for $u(\cdot) > 0$)

   $\frac{\partial \gamma_e \tilde{\alpha}u'(x_2)}{\partial \tilde{\alpha}} = \gamma_e u'(\tilde{x}_2) > 0$

   It follows that the effective utility and the effective marginal utility are comonotonic.

   Note that, if $u(\cdot) < 0$, $\tilde{\alpha}u(\tilde{x}_2)$ and $\tilde{\gamma}_e\tilde{\alpha}u'(\tilde{x}_2)$ are countercomonotonic. In this

case, by proposition 3, we have $\frac{\partial DR_{1,2}}{\partial k} < 0$. Given $k > 0$, we must have $u(\cdot) < 0$ whenever the DM is risk loving with respect to $\tilde{\alpha}$.

2. $\alpha u(\tilde{x}_2)$ with $\tilde{x}_2 = \tilde{\gamma}_y y_2 + \gamma_e (1+r)e$

$$\frac{\partial \alpha u(\tilde{x}_2)}{\partial \tilde{\gamma}_y} = \alpha u'(\tilde{x}_2) y_2 > 0$$
$$\frac{\partial \gamma_e \alpha u'(\tilde{x}_2)}{\partial \tilde{\gamma}_y} = \gamma_e \alpha u''(\tilde{x}_2) y_2 < 0$$

It follows that the effective utility and the effective marginal utility are counter-comonotonic.

3. $\alpha u(\tilde{x}_2)$ with $\tilde{x}_2 = \gamma_y y_2 + \tilde{\gamma}_e (1+r)e$ (assume $e > 0$)

$$\frac{\partial \alpha u(\tilde{x}_2)}{\partial \tilde{\gamma}_e} = \alpha u'(\tilde{x}_2)(1+r)e > 0$$

$$\frac{\partial \tilde{\gamma}_e \alpha u'(\tilde{x}_2)}{\partial \tilde{\gamma}_e} = \alpha u'(\tilde{x}_2) + \tilde{\gamma}_e \alpha u''(\tilde{x}_2)(1+r)e$$

then $\frac{\partial \tilde{\gamma}_e \alpha u'(\tilde{x}_2)}{\partial \tilde{\gamma}_e} \gtreqless 0 \Leftrightarrow \alpha u'(\tilde{x}_2) + \tilde{\gamma}_e \alpha u''(\tilde{x}_2)(1+r)e \gtreqless 0$:

$$u'(\tilde{x}_2) + u''(\tilde{x}_2)\gamma_y y_2 + u''(\tilde{x}_2)\tilde{\gamma}_e(1+r)e \gtreqless u''(\tilde{x}_2)\gamma_y y_2$$

$$u''(\tilde{x}_2)\left(\gamma_y y_2 + \tilde{\gamma}_e(1+r)e\right) \gtreqless u''(\tilde{x}_2)\gamma_y y_2 - u'(\tilde{x}_2)$$

$$1 \lesseqgtr \frac{\gamma_y y_2}{\gamma_y y_2 + \tilde{\gamma}_e(1+r)e} - \frac{u'(\tilde{x}_2)}{u''(\tilde{x}_2)\left(\gamma_y y_2 + \tilde{\gamma}_e(1+r)e\right)} = \frac{\gamma_y y_2}{\gamma_y y_2 + \tilde{\gamma}_e(1+r)e} + IES$$

$$\frac{\tilde{\gamma}_e(1+r)e}{\gamma_y y_2 + \tilde{\gamma}_e(1+r)e} \lesseqgtr IES$$

Thus $\frac{\partial \tilde{\gamma}_e \alpha u'(\tilde{x}_2)}{\partial \tilde{\gamma}_e} \gtreqless 0 \Leftrightarrow \frac{\tilde{\gamma}_e(1+r)e}{\gamma_y y_2 + \tilde{\gamma}_e(1+r)e} \lesseqgtr IES \; \forall \; \omega$.

It follows that effective utility and effective marginal utility are countercomonotonic if $IES < \frac{\tilde{\gamma}_e(1+r)e}{\gamma_y y_2 + \tilde{\gamma}_e(1+r)e} \; \forall \; \omega$, and comonotonic if $IES > \frac{\tilde{\gamma}_e(1+r)e}{\gamma_y y_2 + \tilde{\gamma}_e(1+r)e} \; \forall \; \omega$

## 2.B.2 Uncertainty on several variables

Suppose $\gamma_y$, $\gamma_e$ and $\alpha$ are strictly monotonic functions of a discrete random variable $\tilde{z} > 0$, such that we can write $\gamma_y(\tilde{z})$, $\gamma_e(\tilde{z})$, $\alpha(\tilde{z})$. Future consumption, the derivative of the effective utility with respect to $\tilde{z}$, and the derivative of the effective marginal utility

with respect to $\tilde{z}$ are then

$$\tilde{x}_2 = \gamma_y(\tilde{z})\,y_2 + \gamma_e(\tilde{z})\,(1+r)e$$

$$\frac{\partial \alpha(\tilde{z})u(\tilde{x}_2)}{\partial \tilde{z}} = u(\tilde{x}_2)\frac{\partial \alpha(\tilde{z})}{\partial \tilde{z}} + \alpha(\tilde{z})\,u'(\tilde{x}_2)\,y_2\frac{\partial \gamma_y(\tilde{z})}{\partial \tilde{z}}$$

$$+ \alpha(\tilde{z})\,u'(\tilde{x}_2)\,(1+r)e\frac{\partial \gamma_e(\tilde{z})}{\partial \tilde{z}} \tag{2.8}$$

$$\frac{\partial \gamma_e(\tilde{z})\alpha(\tilde{z})u'(\tilde{x}_2)}{\partial \tilde{z}} = \gamma_e(\tilde{z})\,u'(\tilde{x}_2)\frac{\partial \alpha(\tilde{z})}{\partial \tilde{z}} + \alpha(\tilde{z})\,\gamma_e(\tilde{z})\,u''(\tilde{x}_2)\,y_2\frac{\partial \gamma_y(\tilde{z})}{\partial \tilde{z}}$$

$$+ \left[\alpha(\tilde{z})\,u'(\tilde{x}_2) + \alpha(\tilde{z})\,\gamma_e(\tilde{z})\,u''(\tilde{x}_2)\,(1+r)e\right]\frac{\partial \gamma_e(\tilde{z})}{\partial \tilde{z}}. \tag{2.9}$$

Suppose now that $\alpha'(\tilde{z}) = 0$ and $\gamma_y'(\tilde{z})\,y_2 > |\gamma_e'(\tilde{z})\,(1+r)e|\ \forall\ \omega$.

1. If $\gamma_y'(\tilde{z}) > 0$ and $\gamma_e'(\tilde{z}) < 0$, equations (2.8) and (2.9) yield:

   $$\frac{\partial \alpha u(\tilde{x}_2)}{\partial \tilde{z}} = \alpha u'(\tilde{x}_2)\left(\gamma_y'(\tilde{z})\,y_2 + \gamma_e'(\tilde{z})\,(1+r)e\right) > 0$$

   $$\frac{\partial \gamma_e(\tilde{z})\alpha u'(\tilde{x}_2)}{\partial \tilde{z}} = \gamma_e'(\tilde{z})\,\alpha u'(\tilde{x}_2) + \gamma_e(\tilde{z})\,\alpha u''(\tilde{x}_2)\left(\gamma_y'(\tilde{z})\,y_2 + \gamma_e'(\tilde{z})\,(1+r)e\right) \gtreqless 0$$

   $$\gamma_e(\tilde{z})\,u''(\tilde{x}_2)\left(\gamma_y'(\tilde{z})\,y_2 + \gamma_e'(\tilde{z})\,(1+r)e\right) \gtreqless -\gamma_e'(\tilde{z})\,u'(\tilde{x}_2)$$

   $$\frac{\frac{\gamma_y'(\tilde{z})y_2+\gamma_e'(\tilde{z})(1+r)e}{\gamma_y(\tilde{z})y_2+\gamma_e(\tilde{z})(1+r)e}}{\frac{\gamma_e'(\tilde{z})}{\gamma_e(\tilde{z})}} \gtreqless -\frac{u'(\tilde{x}_2)}{u''(\tilde{x}_2)\left(\gamma_y(\tilde{z})y_2+\gamma_e(\tilde{z})(1+r)e\right)} = IES$$

   thus $\frac{\partial \gamma_e(\tilde{z})\alpha u'(\tilde{x}_2)}{\partial \tilde{z}} \gtreqless 0 \Leftrightarrow \frac{\tilde{x}_2'(\tilde{z})\big/\tilde{x}_2(\tilde{z})}{\gamma_e'(\tilde{z})/\gamma_e(\tilde{z})} \gtreqless IES\ \forall\ \omega$.

   This implies: If $\gamma_y'(\tilde{z}) > 0$, $\gamma_e'(\tilde{z}) < 0$: $\frac{\partial DR_{1,2}}{\partial k} < 0$ since $IES > 0 > \frac{\tilde{x}_2'(\tilde{z})\big/\tilde{x}_2(\tilde{z})}{\gamma_e'(\tilde{z})/\gamma_e(\tilde{z})}\ \forall\ \omega$.

   With $e = 0$: $\frac{\partial \alpha u(\tilde{x}_2)}{\partial \tilde{z}} = \alpha u'(\tilde{x}_2)\,\gamma_y'(\tilde{z})\,y_2 > 0$ and

   $$\frac{\partial \gamma_e(\tilde{z})\alpha u'(\tilde{x}_2)}{\partial \tilde{z}} \gtreqless 0 \Leftrightarrow \frac{\gamma_y'(\tilde{z})\big/\gamma_y(\tilde{z})}{\gamma_e'(\tilde{z})/\gamma_e(\tilde{z})} \gtreqless IES\ \forall\ \omega.$$

2. If $\gamma_y'(\tilde{z}) > 0$ and $\gamma_e'(\tilde{z}) > 0$, equations (2.8) and (2.9) yield:

   $$\frac{\partial \alpha u(\tilde{x}_2)}{\partial \tilde{z}} = \alpha u'(\tilde{x}_2)\left(\gamma_y'(\tilde{z})\,y_2 + \gamma_e'(\tilde{z})\,(1+r)e\right) > 0$$

   $$\frac{\partial \gamma_e(\tilde{z})\alpha u'(\tilde{x}_2)}{\partial \tilde{z}} = \gamma_e'(\tilde{z})\,\alpha u'(\tilde{x}_2) + \gamma_e(\tilde{z})\,\alpha u''(\tilde{x}_2)\left(\gamma_y'(\tilde{z})\,y_2 + \gamma_e'(\tilde{z})\,(1+r)e\right) \gtreqless 0$$

   $$\gamma_e(\tilde{z})\,u''(\tilde{x}_2)\left(\gamma_y'(\tilde{z})\,y_2 + \gamma_e'(\tilde{z})\,(1+r)e\right) \gtreqless -\gamma_e'(\tilde{z})\,u'(\tilde{x}_2)$$

   $$\frac{\frac{\gamma_y'(\tilde{z})y_2+\gamma_e'(\tilde{z})(1+r)e}{\gamma_y(\tilde{z})y_2+\gamma_e(\tilde{z})(1+r)e}}{\frac{\gamma_e'(\tilde{z})}{\gamma_e(\tilde{z})}} \lesseqgtr -\frac{u'(\tilde{x}_2)}{u''(\tilde{x}_2)\left(\gamma_y(\tilde{z})y_2+\gamma_e(\tilde{z})(1+r)e\right)} = IES$$

   thus $\frac{\partial \gamma_e(\tilde{z})\alpha u'(\tilde{x}_2)}{\partial \tilde{z}} \gtreqless 0 \Leftrightarrow \frac{\tilde{x}_2'(\tilde{z})\big/\tilde{x}_2(\tilde{z})}{\gamma_e'(\tilde{z})/\gamma_e(\tilde{z})} \lesseqgtr IES\ \forall\ \omega$.

   This implies: If $\gamma_y'(\tilde{z}) > 0$, $\gamma_e'(\tilde{z}) > 0$: $\frac{\partial DR_{1,2}}{\partial k} < (>)\ 0$ if $IES < (>) \frac{\tilde{x}_2'(\tilde{z})\big/\tilde{x}_2(\tilde{z})}{\gamma_e'(\tilde{z})/\gamma_e(\tilde{z})}\ \forall\ \omega$.

With $e = 0$: $\frac{\partial \alpha u(\tilde{x}_2)}{\partial \tilde{z}} = \alpha u'(\tilde{x}_2) \gamma_y'(\tilde{z}) y_2 > 0$ and

$\frac{\partial \gamma_e(\tilde{z}) \alpha u'(\tilde{x}_2)}{\partial \tilde{z}} \gtreqless 0 \Leftrightarrow \frac{\gamma_y'(\tilde{z})/\gamma_y(\tilde{z})}{\gamma_e'(\tilde{z})/\gamma_e(\tilde{z})} \lesseqgtr IES \; \forall \; \omega.$

# Chapter 3

# Extending the Ramsey Equation further: Discounting under Mutually Utility Independent and Recursive Preferences

## 3.1 Introduction

Standard approaches to discounting under certainty originate from the most popular model of intertemporal choice, namely the discounted utility model as introduced by Samuelson (1937) and axiomatized by Koopmans (1960).[1] This model yields the well known Ramsey Equation which aggregates the determinants of the consumption discount rate, impatience and a wealth effect, in an intuitive manner. The predominance of the Ramsey Equation as an organizing principle for discounting sure benefits was recently confirmed by a panel of leading experts on intergenerational discounting (Arrow et al. 2012).

A crucial assumption which is built into the discounted utility model, and thus into the standard approach to discounting under certainty, is preference independence. Prefer-

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[1]See Frederick et al. (2002) for a comprehensive discussion of the discounted utility model's historical origins.

ences over the consumption of one generation are (mutually) preference independent if they are independent of the consumption levels of generations living in the past and in the future. This assumption largely simplifies the preference representation. In a deterministic setting, Koopmans (1960) showed that preference independence constitutes the key axiom for the existence of an additively separable intertemporal utility function.

Recent contributions in the discounting literature emphasize the role of risk and risk aversion. Gollier (2002a, 2002b) motivates an Extended Ramsey Equation which incorporates discounting for reasons of precaution in the presence of growth risk. The additional effect on a one period (instantaneous) consumption discount rate is marginal, however. This insignificance of the growth risk is partly due to an immanent drawback of the additive expected utility framework from which the Extended Ramsey Equation originates. In this framework it is not possible to disentangle risk aversion from the intertemporal elasticity of substitution (IES). In the additive expected utility model, a meaningful degree of risk aversion goes along with an unrealistically small IES. Gollier (2002a), Traeger (2011, 2014) and my analysis in chapter 2 approach this deficit by employing recursive preferences of the Kreps-Porteus type (Kreps and Porteus 1978) to identify the consumption discount rate and its determinants. As the degree of risk aversion can be varied independently of the IES in the Kreps-Porteus framework, it is possible to account for higher degrees of risk aversion. Higher degrees of risk aversion then imply a pronounced effect of growth risk on the consumption discount rate.

Utility independence–the equivalent of preference independence in a risky world–is either implicitly assumed or dismissed without discussion in the cited literature on discounting under growth risk. In Gollier's (2002a, 2002b) Extended Ramsey Equation, utility independence goes along with the time additive structure of the expected utility function. Kreps-Porteus recursive preferences, in contrast, are not utility independent by default. In particular, the Epstein-Zin (Epstein and Zin 1989) parameterization of Kreps-Porteus recursive preferences does not represent utility independent preferences. Without discussing the abandonment of the utility independence assumption, Traeger (2011, 2014) employs the Epstein-Zin (EZ) parameterization to derive an Extended Ramsey Equation for EZ preferences.

The focus of the analysis at hand is on the discount rate of a decision maker whose

preferences are Kreps-Porteus recursive as well as (mutually) utility independent. The Kreps-Porteus framework is chosen for its flexibility with respect to the disentanglement of risk aversion and the IES. Utility independence is postulated as it is a broadly accepted assumption for intertemporal social welfare considerations, such as those underlying the Ramsey Equation and the Extended Ramsey Equation. In a first instance, I show that utility independence restricts a Kreps-Porteus recursive decision maker's preferences to a very specific parametric form, namely to the constant absolute risk aversion form of Hansen and Sargent's (1995) Risk-Sensitive (RS) preferences. Coming from a decision maker with RS preferences, I analyze the instantaneous consumption discount rate in an infinite horizon setting. This is done under the standard assumptions of independently distributed growth risk and constant elasticity of substitution.

I find that the discount rate of the considered decision maker is subject to an effect that is not present in previous approaches to discounting under risk. This effect, which is denoted as the *horizon effect*, may diminish the discount rate to a significant extent. The horizon effect is a function of the length of the time horizon *after* the period of discount, the degree of temporal risk aversion, the variance of growth risk, and the rate of pure time preference. The dependence on the time horizon after the period of discount discloses that the standard practice of cutting off decision problems after a given number of periods is problematic. In particular, Kreps-Porteus recursive frameworks which consider only two-period decision problems for simplification, exclude the horizon effect by construction. In infinite horizon settings like the one considered in this analysis, the role of temporal risk aversion, the role of risk itself, and the role of the rate of pure time preference on the consumption discount rate are amplified in comparison to their roles in the Ramsey Equation, the Extended Ramsey Equation and the Extended Ramsey Equation for EZ preferences (henceforth: the Ramsey Equation and its previous extensions). This point is illustrated through an analytical solution for the consumption discount rate of a decision maker with RS preferences. I refer to this analytical solution as the Extended Ramsey Equation for Risk-Sensitive preferences.

To avoid confusion, note that Gollier (2002a, 2002b) also refers to an effect on the discount rate that is connected to the time horizon. This effect is different from what I have in mind, however. In Gollier's contributions, 'horizon' refers to the time horizon

between the present period and the period to which the discount rate applies. Here, 'horizon' refers to the time horizon after the period for which one discounts. Closer to my understanding of the horizon effect is Traeger (2011). He also points to the fact that the 'planning horizon' after the period for which one discounts may affect the consumption discount rate. While Traeger is aware of the existence of the (planning) horizon effect in a very general Kreps-Porteus recursive setting, he does not study it in detail. In the contrary, as I discuss below, he eliminates the effect by employing an Epstein-Zin parameterization of Kreps-Porteus preferences with homogeneous felicity.

In section 2 I describe the notion of preference and utility independence. I present the terminology and formal definitions in the static context of multiattribute utility theory to familiarize the reader with these concepts. In section 3, I introduce the preferences of the decision maker under consideration. I develop a definition of (mutual) utility independence for preferences over temporal lotteries which is then imposed on a Kreps-Porteus recursive decision maker. I show that the specified preferences are of the Risk-Sensitive type. In section 4, I examine the consumption discount rate of a decision maker with Risk-Sensitive preferences in two steps. First, I prove the existence and the direction of the horizon effect. Second, I derive an Extended Ramsey Equation for RS preferences, discuss its relation to the Ramsey equation and its previous extensions, and emphasize the special role of the rate of pure time preference. Section 5 concludes and suggests future research.

## 3.2 Background: Utility independence

Assumptions of preference or utility independence are standard in the context of utility functions $U(x_1, x_2, ...x_n)$ that aggregate the felicity from different attributes. The representation of preferences over multiple attributes is largely simplified if preferences over a specific attribute (or over lotteries on an attribute) are independent of common levels of other attributes. If mutual preference or utility independence holds, preferences over deterministic attributes and preferences that satisfy the axioms of expected utility theory can be represented through a utility function that is decomposable into smaller units: $U(x_1, x_2, ...x_n) = f(u_1(x_1), u_2(x_2), ...u_n(x_n))$. In particular, an (expected) util-

ity function over $n$ attributes can be decomposed into an additive or multiplicative form if preferences satisfy mutual preference or utility independence.[2]

In this section, I describe the notion of mutual preference and utility independence in the (mostly) static context of multiattribute utility theory (MAUT). The purpose of this description is to familiarize the reader with the basic idea behind these independence concepts. This familiarity will help the understanding of the next section, in which I adjust the definition of mutual utility independence to the temporal and recursive setting of my analysis.

The main reference for independence concepts in the deterministic or expected utility context of MAUT is Keeney and Raiffa's (1976) volume on Decisions with Multiple Objectives. For comprehensive surveys on various independence assumptions, their implications in MAUT and the relevant literature, see Farquhar (1977) and Yilmaz (1978). The definitions of conditional preferences, preference independence and utility independence below are as in Farquhar.

### 3.2.1 Terminology and conditional preferences

Consider a decision maker with preferences $\succeq$ on a set of possible outcomes $X$, which contains $n$ different attribute sets $X_i$ with $i = 1, 2, ...n$. The set of possible outcomes is the Cartesian product of the attribute sets: $X = X_1 \times X_2 \times ... \times X_n$. An element $x_i \in X_i$ is a specific level of attribute $i$. A specific outcome $x \in X$ is written as the $n$-tuple $x = (x_1, x_2, ...x_n)$. In risky situations, the decision maker's preferences $\succeq$ are defined over the set $P$ of lotteries on $X$. An element $p \in P$ is a lottery that assigns probabilities $l^\omega$, with $\omega = 1, ...N$ and $\sum_{\omega=1}^{N} l^\omega = 1$, $l^\omega > 0 \; \forall \; \omega$, to specific outcomes $x^\omega \in X$, such that $p = \sum_{\omega=1}^{N} l^\omega x^\omega$.

For the definitions of preference and utility independence below it will be useful to partition the attribute space and introduce conditional preference relations. The attribute space $i = 1, 2, ...n$ can be partitioned into the nonempty sets $I$ and $\bar{I}$ such that $X = X_1 \times X_2 \times ... \times X_n$ can be expressed as $X = X_I \times X_{\bar{I}}$. The set of lotteries on $X_I$ is then denoted as $P_I$, and $p_I \in P_I$ denotes a specific marginal distribution of $p$ on $X_I$.

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[2]Keeney and Raiffa (1976).

A conditional preference relation is a preference relation that is defined over lotteries in one set, while holding the outcome in a different set fixed. Given a fixed outcome in $X_{\bar{I}}$, an unconditional preference relation $\succeq$ on $P$ can be expressed as a conditional preference relation $\succeq_{x_{\bar{I}}}$ on $P_I$. That is, rather than defining $\succeq$ over lotteries $(p_I, x_{\bar{I}}), (p'_I, x_{\bar{I}}) \in P$ with marginal probabilities $p_I, p'_I \in P_I$ on $X_I$ and probability 1 for the outcome $x_{\bar{I}} \in X_{\bar{I}}$, we can define $\succeq_{x_{\bar{I}}}$ over the marginals $p_I, p'_I \in P_I$. The conditional preference relation $\succeq_{x_{\bar{I}}}$ thus restricts the unconditional preference relation $\succeq$ to those $p \in P$ that assign probability 1 to $x_{\bar{I}}$. Formally

$$p_I \succeq_{x_{\bar{I}}} p'_I \text{ if and only if } (p_I, x_{\bar{I}}) \succeq (p'_I, x_{\bar{I}}) \ \forall \ p_I, p'_I \text{ in } P_I.$$

### 3.2.2 Mutual preference and utility independence

Preferences over outcomes in one attribute set may or may not depend on the specific levels of the remaining attributes. If the preference order over levels in the attribute set $X_I$ is independent of the outcome in a different attribute set $X_{\bar{I}}$, we say that $X_I$ is preference independent of $X_{\bar{I}}$. Formally, preference independence (PI) can be defined as follows:

**Definition 5** *(Preference independence)*
*$X_I$ is preference independent of $X_{\bar{I}}$ if and only if $\succeq_{x_{\bar{I}}} = \succeq_{x'_{\bar{I}}}$ on $X_I \ \forall \ x_{\bar{I}}, x'_{\bar{I}} \in X_{\bar{I}}$.*

Note that preference independence is not a symmetric condition: Given that $X_I$ is preference independent of $X_{\bar{I}}$ we cannot infer that $X_{\bar{I}}$ is preference independent of $X_I$ and vice versa. A symmetric condition, namely mutual preference independence (MPI), is, however, easily constructed:

**Definition 6** *(Mutual preference independence)*
*$X_I$ and $X_{\bar{I}}$ are mutually preference independent if and only if $X_I$ is preference independent of $X_{\bar{I}}$ and $X_{\bar{I}}$ is preference independent of $X_I$.*

Preferences $\succeq$ over $X$ which satisfy MPI on the whole domain (i.e. each subset $X_I \in X$ is PI of its complement $X_{\bar{I}} \in X$) are representable by an additive utility function (Keeney

and Raiffa 1976, theorem 3.6).

Preference independence restricts preferences that are defined over a set of deterministic attributes. The analogue for preferences defined over lotteries is utility independence. If the preference order over lotteries in $P_I$ on $X_I$ is independent of outcomes in $X_{\bar{I}}$, we say that $X_I$ is utility independent of $X_{\bar{I}}$. The definition of utility independence (UI) mirrors that of preference independence. The difference is only in the set over which preferences are defined:

**Definition 7** *(Utility independence)*
$X_I$ *is utility independent of* $X_{\bar{I}}$ *if and only if* $\succeq_{x_{\bar{I}}} = \succeq_{x'_{\bar{I}}}$ *on* $P_I$ $\forall$ $x_{\bar{I}}, x'_{\bar{I}} \in X_{\bar{I}}$.

If utility independence holds, then all conditional preference relations $\succeq_{x_{\bar{I}}}$ on $P_I$ preserve the same order among all $p_I \in P_I$. This includes degenerate lotteries that assign probability 1 to specific levels in the attribute set $X_I$. Thus, whenever $X_I$ is utility independent of $X_{\bar{I}}$, it must also be true that $X_I$ is preference independent of $X_{\bar{I}}$. The converse is not generally true.

Just like preference independence, utility independence is not a symmetric condition: Given that $X_I$ is utility independent of $X_{\bar{I}}$ we cannot infer that $X_{\bar{I}}$ is utility independent of $X_I$ and vice versa. The symmetric condition is called mutual utility independence (MUI):

**Definition 8** *(Mutual utility independence)*
$X_I$ *and* $X_{\bar{I}}$ *are mutually utility independent if and only if* $X_I$ *is utility independent of* $X_{\bar{I}}$ *and* $X_{\bar{I}}$ *is utility independent of* $X_I$.

Preferences $\succeq$ over $P$ which satisfy MUI on the whole domain (i.e. each subset $X_I \in X$ is UI of its complement $X_{\bar{I}} \in X$) and which comply with von Neumann and Morgenstern's expected utility axioms are representable by an additive or multiplicative utility function (Keeney and Raiffa 1976, theorem 6.1). A standard additive expected utility function is mutually utility independent on the whole domain.

### 3.2.3 Temporal context

Consider the aggregation of an infinite number of attributes $x_t$ with $t = 1, 2, 3...$ which differ with respect to the period at which they occur. An intertemporal social welfare function constitutes such an aggregation. A distinctive feature of this aggregation is the temporal order of the attributes. Due to this order, assumptions of utility independence can be given a temporal interpretation. If utility independence is geared towards the past, we speak of history independence, if it is geared towards the future, we call it future independence.

To be more specific, define by $X = X_1 \times X_2 \times X_3...$ the space of possible consumption paths over an infinite horizon. History independence of preferences over lotteries $P_t$ on an attribute set $X_t \in X$ requires that $X_t$ is utility independent of each $X_\tau \in X$ with $\tau < t$. Likewise, future independence of preferences over lotteries $P_t$ on $X_t$ requires that $X_t$ is utility independent of each $X_\tau$ with $\tau > t$. If for some $t, \tau$ with $t < \tau$, it holds that preferences over $P_t$ on $X_t$ are future independent and those over $P_\tau$ on $X_\tau$ are history independent, then it must also be true that $X_t$ and $X_\tau$ are mutually utility independent. If preferences over each $P_t$ on $X_t \in X$ are both future and history independent, then each pair $X_t, X_\tau \in X$ with $t, \tau = 1, 2, ...\infty$ and $t \neq \tau$ is mutually utility independent. We then simply say that preferences over $P$ on $X$ are mutually utility independent on the whole domain.

Preferences which are history and future independent, which satisfy von Neumann and Morgenstern's expected utility axioms, and which are defined over lotteries on intertemporal consumption paths, are representable by an additive or multiplicative intertemporal utility function (Meyer 1976, theorem 9.2). Correspondingly, preferences represented by the standard additive (intertemporal) expected utility function are mutually utility independent on the entire domain.

In a temporal but deterministic context, Koopmans (1960) proved that several axioms, among them a crucial assumption on period independence, warrants the existence of the additive discounted utility model.[3] In the following I extend Koopmans' requirement

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[3]More specifically, Koopmans (1960) showed that stationary, time-consistent, period independent preferences over infinite deterministic consumption paths are representable by $U(x) = \sum_{t=1}^{\infty} \beta^{t-1} u(x_t)$. Under addition of a continuity axiom, he showed that the utility discount factor (rate of pure time

for independence of preferences defined over deterministic consumption paths to a larger domain, namely to the domain of temporal lotteries.

## 3.3 Preferences: Recursive and MUI

Intergenerational decision making involves allocating resources across many different generations. These generations differ with respect to their consumption level as well as with respect to the degree of consumption risk to which they are exposed. A decision maker who optimizes intertemporal welfare evaluates the consumption and risk levels according to his preferences, in particular according to his intertemporal elasticity of substitution (IES) and his degree of risk aversion. These two preference characteristics are entangled in the standard model of intertemporal choice under risk, namely in the additive expected utility model. To model the preferences of an intertemporal decision maker in a more flexible manner, I resort to the recursive utility representation of Kreps and Porteus (1978). A Kreps-Porteus (KP) recursive preference representation enables the disentanglement of a decision makers' degree of risk aversion from the IES.

KP recursive preferences are defined over objects called temporal lotteries. The definitions of mutual preference and utility independence above, however, concern preferences that are defined over deterministic attributes or over lotteries on attribute sets. The results of Keeney and Raiffa (1976), Meyer (1976) and Koopmans (1960) on the decomposition of a utility function when preferences satisfy definitions 5, 6, 7, or 8, are therefore not directly applicable in the context considered here.

To study how mutual utility independence restricts a Kreps-Porteus recursive preference representation, mutual utility independence for preferences over temporal lotteries must be defined. To this end, I denote the set of temporal lotteries by $D$ and write a specific temporal lottery as $(x_1, \tilde{x}_2, \tilde{x}_3, ...) \in D$. A temporal lottery consists of a certain attribute $x_1$ for the initial period and (potentially) uncertain attributes $\tilde{x}_t$ for $t > 1$. Note that the set of degenerate temporal lotteries (deterministic consumption paths) $X^\infty$ is a subset of $D$.[4] Mutual utility independence for preferences over temporal lotteries can be defined

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preference) must be such that $0 < \beta < 1$.

[4]For a more comprehensive discussion of temporal lotteries see Kreps and Porteus (1978), Epstein and Zin (1989), or Bommier and Le Grand (2014).

in the following way:

**Definition 9** *(MUI for preferences over temporal lotteries)*

*Preferences $\succeq$ over the set of temporal lotteries $D$ are mutually utility independent if*

$$(x_1, \tilde{x}_2, ... \tilde{x}_{t-1}, x_t, \tilde{x}_{t+1}...) \quad \succeq \quad \left(x'_1, \tilde{x}'_2, ... \tilde{x}'_{t-1}, x_t, \tilde{x}'_{t+1}...\right)$$

$$\Downarrow$$

$$(x_1, \tilde{x}_2, ... \tilde{x}_{t-1}, x'_t, \tilde{x}_{t+1}...) \quad \succeq \quad \left(x'_1, \tilde{x}'_2, ... \tilde{x}'_{t-1}, x'_t, \tilde{x}'_{t+1}...\right)$$

$\forall \, x_t, x'_t \in X_t.$

Definition 9 is now imposed on Kreps-Porteus recursive preferences. Denote by $\succeq^D$ a preference relation over the set of temporal lotteries $D$. Suppose $\succeq^D$ is KP recursive and let $U^D : D \rightarrow \mathbb{R}$ represent such preferences. Since $U^D$ represents KP recursive preferences, it must satisfy the recursion

$$U^D(x_1, m) = W\left(x_1, E_m\left[U^D\right]\right), \tag{3.1}$$

where $E_m[\cdot]$ is the expectation with respect to the probability measure $m$ on $D$.[5]

Suppose in addition that the considered preference relation $\succeq^D$ satisfies mutual utility independence according to definition 9. Note that the assumption of MUI on $D$ implies MUI on the subdomain $X^\infty \subseteq D$ as well. Given MUI of $\succeq^D$ on $D$, the form of $U^D$ can be narrowed down in two steps.

First, I restrict the form of $U^D$ such that it represents only preferences that are MUI on the subdomain $X^\infty \subseteq D$. To this end, I use Koopmans' (1960) representation result for period independent preferences. His definition of period independence accords to my definition of mutual utility independence.[6] Koopmans shows that a preference relation $\succeq^X$ over $X^\infty$ which satisfies continuity, sensitivity, stationarity and mutual utility inde-

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[5] See e.g. Bommier and Le Grand (2014).

[6] Meyer (1976) shows that the combination of Koopmans' postulates on stationarity and period independence (Koopmans' postulate 3') induce complete pairwise preferential independence. Complete pairwise preferential independence corresponds to the definition of mutual utility independence on the domain $X^\infty$ (see section 3.6.3 in Keeney and Raiffa 1976).

pendence can be represented by an additive discounted utility function $U^X : X^\infty \to \mathbb{R}$:

$$U^X (x_1, x_2 \ldots) = u(x_1) + \beta U^X (x_2, x_3 \ldots). \tag{3.2}$$

Note that $U^D$ and $U^X$ represent the same (mutually utility independent) preferences on $X^\infty$. Since $U^D$ and $U^X$ represent the same ordinal preferences, there exists some increasing $\phi$ such that $U^D = \phi(U^X)$ (Kihlstrom and Mirman 1974). Denoting by $W^D(x, y)$ and $W^X(x, y)$ the aggregators of $U^D$ and $U^X$, and using $U^D = \phi(U^X)$, we can write $W^D(x, y) = \phi(W^X(x, \phi^{-1}(y)))$ and thus[7]

$$U^D(x_1, m) = \phi\left(u(x_1) + \beta\phi^{-1}\left(E_m\left[U^D(x_2, m)\right]\right)\right). \tag{3.3}$$

Equation (3.3) restricts the form of $U^D$ such that it represents only preferences which satisfy MUI on $X^\infty$.

Second, I restrict the form of $U^D$ further such that it represents only preferences that are MUI on the entire domain $D$. To this end, one needs to restrict $\phi$ in such a way that $U^D$ represents preferences with constant absolute risk aversion. I show this in appendix 3.A.1. The implications for a renormalized ordinal KP recursive utility function $U = \phi^{-1}(U^D)$ are summarized in theorem 10.

**Theorem 10** *(Representation of KP recursive preferences that satisfy MUI)*
*Consider a decision maker with preferences that satisfy the Kreps-Porteus recursion (equation 3.1). Suppose that these preferences are mutually utility independent over the set of temporal lotteries $D$ (definition 9). Such preferences can be represented by a utility function $U : D \to \mathbb{R}$ of the following form:*

$$U(x_1, m) = u(x_1) - \frac{\beta}{k} \ln\left(E_m\left[\exp\left(-kU(x_2, m)\right)\right]\right). \tag{3.4}$$

Equation (3.4) is the constant absolute risk aversion form of Hansen and Sargent's (1995) Risk-Sensitive (RS) preferences. The parameter $k$ measures the decision maker's degree

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[7]Let $W^X(x, y) = u(x) + \beta y$ such that $W^D(x, y) = \phi(u(x) + \beta\phi^{-1}(y))$. For $U^D(x_1, m) = W^D(x_1, E_m[U^D])$ this yields equation (3.3).

of temporal risk aversion. We say that the decision maker is temporally risk averse if $k > 0$ and temporally risk loving if $k < 0$. Temporal risk aversion can be understood as aversion towards risk on continuation utility, i.e. on $U(x_2, m)$ in equation (3.4).

For $k = 0$, equation (3.4) nests the additive (intertemporal) expected utility function

$$U(x_1, m) = u(x_1) + \beta E_m [U(x_2, m)]. \tag{3.5}$$

A decision maker with $k = 0$, i.e. an additive expected utility decision maker, is called temporally risk neutral. Such a decision maker is neutral towards risk on continuation utility. Aversion towards risk on consumption $x_t$ is solely governed by the curvature of the felicity function $u(x_t)$, which simultaneously defines the intertemporal elasticity of substitution.

The implications of the mutual utility independence assumption become obvious if one considers the case of independently distributed risk on the attributes $\tilde{x}_t$. If the attributes are statistically independent, the assumption of mutual utility independence on KP recursive preferences implies the additive separability of the respective utility function. In particular, if preferences are represented by equation (3.4) and risk on consumption $\tilde{x}_t$ is independently distributed, the utility function can be written as[8]

$$U(x_1, \hat{x}_2, ...) = u(x_1) + \beta \sum_{t=2}^{\infty} \beta^{t-2} u(\hat{x}_t), \tag{3.6}$$

where $\hat{x}_t$ is certainty equivalent consumption in $t$. For the RS decision maker under consideration, $\hat{x}_t$ is derived from $u(\hat{x}_t) = -\frac{1}{k} \ln(E_{t-1}[\exp(-ku(\tilde{x}_t))])$. If lotteries on $\tilde{x}_t$ are degenerate (if consumption is deterministic), then (3.6) is equivalent to Koopmans' (1960) additive discounted utility function, i.e. equation (3.2).

I am not the first to connect assumptions of mutual utility independence and Kreps-Porteus recursive preferences. Bommier and Le Grand (2014) remark that their Kreps-Porteus recursive preference specification under scrutiny, namely Risk-Sensitive preferences, satisfies mutual utility independence. Above I approached the issue from a different angle, however. I showed more formally that Kreps-Porteus recursive preferences

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[8]See appendix 3.B.1.

which satisfy mutual utility independence must be of the Risk-Sensitive type.[9]

## 3.4 Implications for discounting

I showed above that Kreps-Porteus recursive preferences which satisfy mutual utility independence are restricted to a specific parametric form, namely to that of Risk-Sensitive preferences. In this section I analyze the instantaneous consumption discount rate of a decision maker with such preferences.

An instantaneous consumption discount rate $DR_{1,2}$ compares the effects on intertemporal utility $U(x)$ when consumption in the first and in the second period are marginally changed:

$$DR_{1,2} = -\ln \frac{\partial U(x)/\partial x_2}{\partial U(x)/\partial x_1}. \tag{3.7}$$

I show below that the consumption discount rate of a decision maker with RS preferences is subject to an effect which is not present in the well known Ramsey Equation and its extensions. This effect is denoted as the *horizon effect*.

### 3.4.1 Defining the horizon effect

A horizon effect is present whenever the consumption discount rate is affected by circumstances that realize only after the period for which one discounts. For the instantaneous discount rate $DR_{1,2}$ (equation 3.7), this is the case if the value of period 2 consumption relative to that of consumption in period 1 is subject to circumstance in periods $t \geq 3$.

To formalize the horizon effect, I compare the instantaneous consumption discount rate in two situations, $A$ and $B$. In both situations I consider a decision maker whose preferences are defined over a temporally infinite domain.

In situation $A$, the decision maker faces a world that consists of an infinite number of existing generations. An existing generation $t$ consumes $x_t$ and derives felicity $u(x_t)$

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[9]Related to but quite different from my approach is Traeger (2012). In a finite (rolling) horizon framework with Kreps-Porteus recursive preferences, he derives a constant absolute risk aversion parameterisation of Kreps-Porteus recursive preferences (Risk-Sensitive preferences) from an assumption denoted as 'coinciding last outcome independence'.

from this consumption. The instantaneous consumption discount rate that applies to this situation is denoted as $DR_{1,2}^{\bar{T}=\infty}$, where $\bar{T}$ is the 'last' period in which a generation exists. In this situation, the instantaneous consumption discount rate of a KP recursive decision maker is derived as

$$DR_{1,2}^{\bar{T}=\infty} \quad = \quad -\ln\beta - \ln\frac{E_1\left[\frac{\phi'(U_2)}{\phi'\left(\phi^{-1}(E_1[\phi(U_2)])\right)}u'\left(\tilde{x}_2\right)\right]}{u'\left(x_1\right)} \tag{3.8}$$

$$\text{with } U_t = u\left(\tilde{x}_t\right) + \beta\phi^{-1}\left(E_t\left[\phi\left(U_{t+1}\right)\right]\right) \ \forall \ t = 2, 3, ....$$

Note that equation (3.8) may be subject to circumstances that apply to periods $t \geq 3$ since the continuation utility $U_2$ is a function of these values.

In situation $B$, generations in $t \geq 3$ do not exist.[10] A generation $t$ that does not exist is assigned zero felicity: $u\left(-\right) = 0$.[11] The respective discount rate is denoted as $DR_{1,2}^{\bar{T}=2}$. In this situation, the instantaneous consumption discount rate of a KP recursive decision maker is written as

$$DR_{1,2}^{\bar{T}=2} = -\ln\beta - \ln\frac{E_1\left[\frac{\phi'(u(\tilde{x}_2))}{\phi'\left(\phi^{-1}(E_1[\phi(u(\tilde{x}_2))])\right)}u'\left(\tilde{x}_2\right)\right]}{u'\left(x_1\right)}. \tag{3.9}$$

Note that equation (3.9) is independent of values in periods $t \geq 3$.

In both equations, $\beta$ is the utility discount factor. The term $(-\ln\beta)$ therefore constitutes the rate of pure time preference (the utility discount rate).

Given the description of situations $A$ and $B$, I formally define the horizon effect in the following way:

**Definition 11** *(Horizon effect)*

*An instantaneous consumption discount rate $DR_{1,2}^{\bar{T}=\infty}$ (equation 3.7) is subject to a hori-*

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[10] Alternatively we could assume that generations $t \geq 3$ in situation $B$ do exist and have consumption $\tilde{x}_t$ which is not correlated with consumption in period 2. The implied discounting function would be the same as in the case where we assume that generations in $t \geq 3$ do not exist.

[11] The '$-$' stands for the consumption level of a non-existent generation. Note that this is different from just assuming that an existing generation has zero consumption. For an enlarged discussion of this point see Bommier (2013). He considers preferences that are defined over a finite lifetime, but with an infinite number of possibilites for the length of this lifetime.

*zon effect whenever*

$$DR_{1,2}^{\bar{T}=\infty} \neq DR_{1,2}^{\bar{T}=2}.$$

The comparison of equations (3.8) and (3.9) in light of definition 11 reveals that the discount rate of a KP recursive decision maker is subject to a horizon effect whenever

$$E_1 \left[ \frac{\phi'(U_2)}{\phi'\left(\phi^{-1}\left(E_1\left[\phi(U_2)\right]\right)\right)} u'(\tilde{x}_2) \right] \neq E_1 \left[ \frac{\phi'(u(\tilde{x}_2))}{\phi'\left(\phi^{-1}\left(E_1\left[\phi(u(\tilde{x}_2))\right]\right)\right)} u'(\tilde{x}_2) \right]. \quad (3.10)$$

The fractions on both sides of this inequality adjust the statistical probability of a given state of the world for risk aversion with respect to risk on the continuation utility. The continuation utility is $U_2$ in situation $A$ and $u(\tilde{x}_2)$ in situation $B$. I refer to these fractions as *risk aversion adjustment factors*. The product of the risk aversion adjustment factor and the statistical probability of a given state of the world is called a *risk aversion adjusted probability*.[12]

Equation (3.10) clarifies that a horizon effect may exist whenever the risk aversion adjusted probabilities of a given state of the world are not equivalent for $DR_{1,2}^{\bar{T}=\infty}$ and $DR_{1,2}^{\bar{T}=2}$. Note that the adjustment factors of a decision maker with additive expected utility preferences are 1 in each state of the world since $\phi(\cdot)$ is linear in this case. Hence, the risk aversion adjusted probabilities of such a decision maker are equal to the statistical probabilities and equation (3.10) holds with equality. It follows that the discount rate of an additive expected utility decision maker is never subject to a horizon effect.

### 3.4.2 The discount rate of a Risk-Sensitive decision maker

In theorem 10 I stated that the preferences of a KP recursive and mutually utility independent decision maker are representable by the Risk-Sensitive utility function, as specified in equation (3.4). The discount rate of a RS decision maker is thus restricted to a parametric form with $\phi(z) = -\exp(-kz)$.

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[12]A formal definition of risk aversion adjusted probabilities is provided in appendix 3.A.2.

For the discount rate $DR_{1,2}^{\bar{T}=\infty}$ of situation $A$ above (equation 3.8), this implies the form

$$DR_{1,2}^{\bar{T}=\infty} \quad = \quad -\ln\beta - \ln\frac{E_1\left[\frac{\exp(-kU_2)}{E_1[\exp(-kU_2)]}u'\left(\tilde{x}_2\right)\right]}{u'\left(x_1\right)} \tag{3.11}$$

with $U_t = u\left(\tilde{x}_t\right) - \dfrac{\beta}{k}\ln\left(E_t\left[\exp\left(-kU_{t+1}\right)\right]\right) \ \forall \ t = 2, 3, ....$

The discount rate that corresponds to situation $B$, namely $DR_{1,2}^{\bar{T}=2}$ as specified in (3.9), is written as

$$DR_{1,2}^{\bar{T}=2} = -\ln\beta - \ln\frac{E_1\left[\frac{\exp(-ku(\tilde{x}_2))}{E_1[\exp(-ku(\tilde{x}_2))]}u'\left(\tilde{x}_2\right)\right]}{u'\left(x_1\right)}. \tag{3.12}$$

Equations (3.11) and (3.12) define instantaneous consumption discount rates for a decision maker with KP recursive mutually utility independent (equivalently: Risk-Sensitive) preferences. Under (3.11), the decision maker faces a world that consists of an infinite number of generations. Under (3.12), the decision maker only accounts for the first two generations. Each generation $t \geq 2$ that is taken into account in (3.11) and (3.12) has possibly uncertain consumption $\tilde{x}_t$. Equation (3.11) is the subject under scrutiny in the remaining analysis; equation (3.12) serves as a benchmark to determine the existence, the direction, and the size of the horizon effect.

Before I go to the main analysis, let me point to a number of conditions under which the existence of a horizon effect acting on (3.11) can be excluded. $DR_{1,2}^{\bar{T}=\infty}$ is free from a horizon effect ($DR_{1,2}^{\bar{T}=\infty} = DR_{1,2}^{\bar{T}=2}$) if $\beta = 0$, if $u\left(x_t\right)$ is linear, if there is no risk in period 2, if $k = 0$, or if the risk on $\tilde{x}_t$ is independently distributed. I show and discuss this in appendix 3.B.2. In the next section I assume that these assumptions are not met, hence a horizon effect may exist.

### 3.4.3 Existence and direction of the horizon effect

I examine the instantaneous consumption discount rate $DR_{1,2}^{\bar{T}=\infty}$ of a temporally risk averse Risk-Sensitive decision maker under a set of assumptions that are standard in the discounting literature. In particular, I assume that the decision maker has at least some valuation for generations living in $t \geq 2$ ($\beta > 0$), that the felicity function is concave and characterized by constant elasticity of substitution (CES), and that consumption *growth*

is risky and independently distributed. Note that the riskiness of consumption growth implies that risk on consumption itself cannot be independently distributed.

The discount rate of the considered decision maker is specified in (3.11) with $k > 0$ and $0 < \beta < 1$.[13] The consumption growth rate $g_t = \frac{x_t}{x_{t-1}} - 1 > -1 \ \forall \ t$, is subject to independently distributed risk. Generation $t$ obtains CES felicity from $u(x_t) = \frac{x_t^\rho - 1}{\rho}$ with $\rho < 1$, where $IES = (1 - \rho)^{-1}$ is the intertemporal elasticity of substitution. Given this setting, I examine how the discount rate depends on the horizon in $t \geq 3$. In particular, I prove in appendix 3.A.2 that the horizon effect reduces the discount rate $DR_{1,2}^{\bar{T}=\infty}$ relative to $DR_{1,2}^{\bar{T}=2}$. This finding is formalized in proposition 12:

**Proposition 12** *(Existence and direction of the horizon effect)*

*Consider the discount rate of a RS decision maker (equation 3.11). Assume $k > 0$, $0 < \beta < 1$ and $\tilde{g}_t > -1 \ \forall \ t \geq 2$. The horizon effect exists and reduces the discount rate (i.e. $DR_{1,2}^{\bar{T}=\infty} < DR_{1,2}^{\bar{T}=2}$) for either of the two following specifications:*

*1. $u(x_t) = \frac{x_t^\rho - 1}{\rho}$ $(\rho < 1, IES > 0)$*

   *where $\tilde{g}_2$ is risky and $g_t$ is deterministic $\forall \ t \geq 3$*

*2. $u(x_t) = \ln x_t$ $(\rho = 0, IES = 1)$*

   *where $\tilde{g}_t$ is risky and independently distributed $\forall \ t \geq 2$*

Note that statement 1 in proposition 12 still holds if one employs the more common CES felicity function $u(x_t) = \frac{x_t^\rho}{\rho}$. This is because the discount rate of a RS decision maker is invariant towards the addition of the constant $-\frac{1}{\rho}$ to felicity. Note furthermore that proposition 12 also holds if one substitutes $DR_{1,2}^{\bar{T}=\infty}$ by $DR_{1,2}^{\bar{T}}$ with $\bar{T} = 3, 4, ...\infty$. That is, the findings on the existence and the direction of the horizon effect are not restricted to an infinite horizon setting, but hold for any discount rate that is examined in a setting with $\bar{T} \geq 3$. This is obvious from the proof of proposition 12 in appendix 3.A.2, which does not depend on assuming $\bar{T} = \infty$ but merely presumes that $\bar{T} \geq 3$.

Proposition 12 states that the instantaneous consumption discount rate of a RS decision maker in a standard discounting setting depends on the horizon after the period for which one discounts, i.e. on the horizon after period 2. The standard practise of cutting

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[13]Note that the proof of proposition 12 does not depend on assuming $\beta < 1$. Yet this standard assumption will be convenient for the existence of a limit in the analytical solution of the next section.

off the horizon after the period of discount, i.e. looking at $DR_{1,2}^{\bar{T}=2}$ rather than $DR_{1,2}^{\bar{T}=\infty}$ as in Gollier (2002a) and chapter 2 of this thesis, is therefore problematic in the context considered here. How problematic it is depends on the size of the horizon effect, which I elaborate on in the next section.

On first sight, the existence of the horizon effect may seem to be at odds with the assumption of mutual utility independence: Imposing mutual utility independence on preferences, and hence imposing history and future independence, leads to a discount rate that explicitly depends on the future through the horizon effect. On closer inspection, this result is not surprising. To see this, recall that the combination of MUI and KP recursivity, i.e. assuming RS preferences, implies the additive separability of the decision maker's utility function if risk on consumption $\tilde{x}_t$ is independently distributed. A discount rate that is derived from such an additively separable utility function is not subject to a horizon effect. The independence of preferences over risk in period 2 from the consumption levels in $t \neq 2$, *together* with the statistical independence of consumption risk, imply the absence of a horizon effect. The horizon effect enters the stage only as we give up the statistical independence of consumption risk and instead assume independently distributed risk on growth. Risk on consumption growth in period 2 (whether independently distributed or not) goes along with correlated risk on consumption at each $t \geq 2$. Hence, risk in period 2 does not only affect the riskiness of period 2 consumption, but also affects the riskiness of consumption in $t = 3, 4, ...\infty$ and thereby leads to risk on (continuation) utility $U_3$. The more periods are aggregated in $U_3$, i.e. the longer the horizon is, the bigger is this risk on continuation utility in absolute terms. A Risk-Sensitive decision maker is averse towards risk on continuation utility and thus adjusts the discount rate in accordance with the size of this risk. This is eventually reflected in the horizon effect.

The technicalities behind the horizon effect can be sketched by a simple example. Suppose for simplicity that $\bar{T} = 3$ is the last period in which a generation exists. Only second period consumption growth $\tilde{g}_2$ is risky. Consumption growth in the third period, $g_3$, is deterministic and thus independent of the risk in period 2. Period 3 consumption itself is not independent of period 2 consumption, both are functions of $\tilde{g}_2$. The consumption levels in periods 2 and 3 are given by $\tilde{x}_2 = (1 + \tilde{g}_2) x_1$ and $\tilde{x}_3 = (1 + \tilde{g}_2) (1 + g_3) x_1$. The

discount rate of the RS decision maker can then be written as

$$DR_{1,2}^{\bar{T}=3} = -\ln\beta - \ln \frac{E_1\left[\frac{\exp(-ku(\tilde{x}_2))\exp(-k\beta u(\tilde{x}_3))}{E_1[\exp(-ku(\tilde{x}_2))\exp(-k\beta u(\tilde{x}_3))]}u'(\tilde{x}_2)\right]}{u'(x_1)}.$$

The exponential that contains continuation utility $u(\tilde{x}_3)$ cannot be taken out of the expectation operator in the adjustment factor (the fraction in the numerator) since $\tilde{x}_3$, like $\tilde{x}_2$, is conditional on period 2 information. Hence, period 3 values do not cancel out. The risk aversion adjustment factor is thus a function of the horizon after period 2 and $DR_{1,2}^{\bar{T}=3}$ is consequently subject to a horizon effect.

### 3.4.4   Analytical solution

I restrict the setting of the last section further to derive an analytical solution for the instantaneous consumption discount rate. This allows for a comparison with the Ramsey Equation and its extensions. It also provides insights on the magnitude of the horizon effect and its interrelation with the rate of pure time preference.

In line with the standard in the discounting literature, I assume that growth rates are not only independently but also normally distributed at each point in time, i.e. $\tilde{g}_t \sim N(\mu_t, \sigma_t^2)$. Under this additional assumption and with $u(x_t) = \ln x_t$ ($IES = 1$), it is possible to derive an analytical solution for the instantaneous consumption discount rate of a RS decision maker (equation 3.11).

For a general horizon $\bar{T}$, I show in appendix 3.B.3 that the analytical solution is

$$DR_{1,2}^{\bar{T}} = -\ln\beta + \mu_2 - \frac{\sigma_2^2}{2} - \frac{\sigma_2^2}{2}2k - \frac{\sigma_2^2}{2}2k\beta\sum_{\tau=3}^{\bar{T}}\beta^{\tau-3}. \tag{3.13}$$

As $\bar{T} \to \infty$, the geometric series in the last term of equation (3.13) approaches the limit $(1-\beta)^{-1}$. The analytical solution for the discount rate that corresponds to situation $A$ is thus

$$DR_{1,2}^{\bar{T}=\infty} = -\ln\beta + \mu_2 - \frac{\sigma_2^2}{2} - \frac{\sigma_2^2}{2}2k - \frac{\sigma_2^2}{2}2k\frac{\beta}{1-\beta}. \tag{3.14}$$

For $\bar{T} = 2$, i.e. in situation $B$ in which the horizon is cut off after the period to which the discount rate applies, the sum in (3.13) is zero. The analytical solution for the discount

rate that corresponds to situation $B$ is thus

$$DR_{1,2}^{\bar{T}=2} = -\ln\beta + \mu_2 - \frac{\sigma_2^2}{2} - \frac{\sigma_2^2}{2}2k. \tag{3.15}$$

The difference between equations (3.14) and (3.15) constitutes the horizon effect that acts on $DR_{1,2}^{\bar{T}=\infty}$. The horizon effect drives a wedge between the discount rate in an infinite horizon setting and that in a 2-period setting:

$$DR_{1,2}^{\bar{T}=\infty} - DR_{1,2}^{\bar{T}=2} = -\frac{\sigma_2^2}{2}2k\frac{\beta}{1-\beta}. \tag{3.16}$$

Equation (3.16) confirms statement 2 of proposition 12 for a setting with $\tilde{g}_t \sim N\left(\mu_t, \sigma_t^2\right)$. Since $k > 0$ and $0 < \beta < 1$, the horizon effect reduces $DR_{1,2}^{\bar{T}=\infty}$ relative to $DR_{1,2}^{\bar{T}=2}$ as long as second period growth is risky ($\sigma_2^2 \neq 0$).

### 3.4.5 Comparison to the literature

To connect to the Ramsey Equation and its previous extensions, I rewrite equation (3.14) in terms of the utility discount rate (the rate of pure time preference) $\delta = -\ln\beta$. I refer to this analytical solution as the Extended Ramsey Equation for RS preferences.

**Definition 13** *(Extended Ramsey Equation for RS preferences)*
*Given independently and normally distributed risk on $\tilde{g}_t$ and $u\left(x_t\right) = \ln x_t$, the instantaneous consumption discount rate of a RS decision maker (equation 3.11) is written as*

$$DR_{1,2}^{\bar{T}=\infty} = \delta + \mu_2 - \frac{\sigma_2^2}{2} - \frac{\sigma_2^2}{2}2k - \frac{\sigma_2^2}{2}2k\frac{1}{\delta} \tag{3.17}$$

*which is denoted 'Extended Ramsey Equation for RS preferences'.*

The Ramsey Equation (Ramsey 1928) constitutes the most widely accepted organizing principle for deterministic consumption discounting in an intergenerational context (Arrow et al. 2012). The Ramsey equation, which is the consumption discount rate that pertains to the discounted utility model with CES felicity, is written as

$$DR_{1,2}^{RE} = \delta + (1-\rho)\mu_2, \tag{3.18}$$

where $(1 - \rho) = IES^{-1}$ and $\mu_2 = g_2$ since growth is deterministic. The first term in equation (3.18) is the rate of pure time preference. It discounts second period felicity according to the decision maker's degree of impatience or empathic distance. The second term is called the wealth effect. This effect accounts for consumption discounting due to differences in the consumption levels of the first and the second generation. An increase in period 2 consumption is less valuable than an increase in present consumption if the second generation is richer than the present generation ($g_2 > 0$), and if the decision maker is averse towards such consumption inequalities ($IES > 0$).

The first two terms of the Extended Ramsey Equation for RS preferences (equation 3.17) correspond to the $IES = 1$ ($\rho = 0$) specification of the Ramsey Equation (equation 3.18). The last three terms in (3.17) are nil in the deterministic additive discounted utility environment of the Ramsey Equation ($\sigma_2^2 = 0$, $k = 0$).

The Extended Ramsey Equation (Gollier 2002a, 2002b) extends the Ramsey Equation to a world with normally distributed risk on the consumption growth rate $\tilde{g}_2$. It is derived from the intertemporally additive expected utility model with CES felicity, and is written as

$$DR_{1,2}^{ERE} = \delta + (1 - \rho)\, \mu_2 - \frac{\sigma_2^2}{2}\, (1 - \rho)^2 \,. \tag{3.19}$$

The third term in (3.19) reduces the consumption discount rate according to the decision maker's aversion towards second period risk. Note that risk aversion is measured by the inverse of the $IES$ in this setting, i.e. by the factor $(1 - \rho)$ in the last term of (3.19). Since risk aversion cannot be too high in a setting where risk aversion and the $IES$ are entangled, the last term is small for moderate sizes of $\sigma_2^2$.

The first three terms of the Extended Ramsey Equation for RS preferences (3.17) correspond to the $IES = 1$ ($\rho = 0$) specification of equation (3.19). Since the Extended Ramsey Equation yields the discount rate of a decision maker with additive expected utility preferences ($k = 0$), the last two terms of (3.17) are not present in (3.19).

The Extended Ramsey Equation for Epstein-Zin preferences (Traeger 2011, 2014) yields the consumption discount rate of a KP recursive decision maker under an EZ parame-

terization:[14]

$$
\begin{aligned}
DR_{1,2}^{EZ} &= \delta + (1-\rho)\,\mu_2 - \frac{\sigma_2^2}{2}\,(1-\rho)^2 - \frac{\sigma_2^2}{2}\,RIRA\left|1-(1-\rho)^2\right| && \text{if } \rho \neq 0 \quad (3.20) \\
DR_{1,2}^{EZ} &= \delta + \mu_2 - \frac{\sigma_2^2}{2} - \frac{\sigma_2^2}{2}\,(-2\alpha) && \text{if } \rho = 0 \quad (3.21)
\end{aligned}
$$

The respective decision maker is characterized by CES felicity $u(x_t) = \frac{x_t^\rho}{\rho}$ and faces independently and normally distributed growth risk. The decision maker's degree of relative (inter)temporal risk aversion is measured by $RIRA$, which is a function of $\rho$ and of Arrow Pratt risk aversion $(1-\alpha)$. For a given $IES$, a temporally risk averse decision maker ($RIRA > 0$, $\alpha < \rho$) is more risk averse than an additive expected utility decision maker. The discount rate of such a temporally risk averse decision maker is thus smaller than the discount rate that results from Gollier's Extended Ramsey Equation, which is obtained for $RIRA = 0$ ($\alpha = \rho$).

The first four terms of the Extended Ramsey Equation for RS preferences (3.17) resemble the $IES = 1$ ($\rho = 0$) specification of the Extended Ramsey Equation for EZ preferences (equation 3.21). The Extended Ramsey Equation for EZ preferences, however, is not subject to a horizon effect. This is true regardless of the number of periods taken into account in the underlying decision problem. In fact, the planning horizon $\bar{T}$ of the setting in which Traeger (2011) derives equations (3.20) and (3.21) is finite but exceeds the period to which the discount rate applies (here: period 2). My calculations in appendix 3.B.4 confirm the absence of the horizon effect in the discounting function of an EZ decision maker.

These comparisons with the Ramsey Equation and its previous extensions highlight the novelty of the horizon effect. The fifth term in (3.17), which constitutes the horizon effect, is unique to the consumption discount rate $DR_{1,2}^{\bar{T}=\infty}$ of a decision maker with Risk-Sensitive preferences. Since the rate of pure time preference $\delta$ is usually considered to be small, the horizon effect may be quite significant, even for moderate degrees of temporal risk aversion. I enlarge upon this point in the next section.

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[14]See Traeger (2011) for a derivation of this equation in a multiperiod setting or Traeger (2014) for a derivation in a two period setting. Note that Traeger refers to this equation as the 'consumption discount rate in the isoelastic setting with intertemporal risk aversion' rather than as the 'Extended Ramsey Equation for EZ preferences'.

Before I close this section, let me point to an apparent inconsistency that stands out when we compare the Extended Ramsey Equations for RS and EZ preferences with $IES = 1$. To see this apparent inconsistency, note that the Risk-Sensitive and the Epstein-Zin specification of Kreps-Porteus recursive preferences are equivalent if $u(x_t) = \ln x_t$, i.e. if $IES = 1$. One would thus expect to find the same instantaneous consumption discount rate for both specifications in this special case. What I find here, instead, is that the Extended Ramsey Equation for RS preferences is subject to a horizon effect, whereas the Extended Ramsey Equation for EZ preferences is free from a horizon effect.

The cause of this apparent inconsistency is that a homogeneous CES felicity function, $u(x_t) = \frac{x_t^\rho}{\rho}$, is employed for the derivation of (3.20) and (3.21). The homogeneity of this function eliminates the horizon effect in the EZ case, as is evident from the calculations in appendix 3.B.4. A logarithmic felicity function in the contrary, which is often treated as the limit of $u(x_t) = \frac{x_t^\rho}{\rho}$ when $IES = 1$, is not homogeneous. In fact, $u(x_t) = \ln x_t$ is not the limit of the homogeneous CES function $u(x_t) = \frac{x_t^\rho}{\rho}$, but rather the limit of the non-homogeneous CES function $u(x_t) = \frac{x_t^\rho - 1}{\rho}$. These two specifications are often used interchangeably since the addition of the constant $-\frac{1}{\rho}$ to $\frac{x_t^\rho}{\rho}$ does not change preferences over $x_t$. What the addition of this constant does, however, is to eliminate the homogeneity of $u(x_t)$. Without this homogeneity, there may exist a horizon effect, even for a decision maker with EZ preferences.

### 3.4.6 The role of the rate of pure time preference

Much of the disagreement on the adequate size of the consumption discount rate stems from different views on the proper value of the utility discount rate or rate of pure time preference, $\delta = -\ln \beta$. This has recently been illustrated by the debate which surrounds the Stern Review on the Economics of Climate Change (Stern 2007) and Nordhaus' integrated assessment model DICE (Nordhaus 2008). Stern, who concludes that *strong and early action* is appropriate in view of the expected damages from climate change, draws on the prescriptive approach to social discounting when setting the rate of pure time preference to a very low level of $\delta = 0.001$ per annum. The prescriptive approach advocates the stance that there is no ethical justification to value future generations less than current generations, except a small probability for the extinction of the human

race. Nordhaus, in the contrary, suggests that a more conservative *climate policy ramp* is the optimal way to go. His choice of a rather high rate of pure time preference of $\delta = 0.015$ per annum is guided by the descriptive school of thought, which holds that the consumption discount rate should reflect market interest rates.

In the absence of a horizon effect, the connection between the rate of pure time preference and the consumption discount rate is a one to one relationship. Increasing $\delta$ augments the consumption discount rate by the same amount. In this case, the task of the rate of pure time preference is solely to discount the felicity of the generation to which the consumption discount rate applies. In the Ramsey Equation and its previous extensions, all of which are not subject to a horizon effect, $\delta$ takes on this single role.

If the consumption discount rate is subject to a horizon effect as in the Extended Ramsey Equation for RS preferences, the role of the rate of pure time preference is twofold. As in the Ramsey Equation and its previous extensions, $\delta$ accounts for discounting second period felicity. This role is assumed by the first term on the right hand side of equation (3.17). Yet $\delta$ also appears in the term which defines the horizon effect, namely in the last term on the right hand side of equation (3.17). The impact of $\delta$ in this role is such that the absolute magnitude of the horizon effect decreases as $\delta$ increases–in a sense, the horizon effect gets discounted more strongly with higher $\delta$. A smaller absolute magnitude of the horizon effect then implies a bigger discount rate, since the horizon effect impacts $DR_{1,2}^{\bar{T}=\infty}$ negatively. Corollary 14 summarizes this twofold role of the rate of pure time preference in the Extended Ramsey Equation for RS preferences.

**Corollary 14** *(A twofold role of the rate of pure time preference)*
*Consider the Extended Ramsey Equation for Risk-Sensitive preferences (equation 3.17). The rate of pure time preference affects $DR_{1,2}^{\bar{T}=\infty}$ positively through two distinct terms:*

*term 1 of (3.17): the bigger $\delta$ is, the more is period 2 felicity discounted*
*term 5 of (3.17): the bigger $\delta$ is, the smaller is the absolute value of the horizon effect*

The significance of the rate of pure time preference in determining the size of the horizon effect can easily be demonstrated. Compare the size of the horizon effect for Stern's (2007) parameter value to that of Nordhaus (2008). For Stern's $\delta = 0.001$, the fifth term

in equation (3.17) is $-\frac{\sigma_2^2}{2} 2k \cdot 1000$. For Nordhaus' $\delta = 0.015$, the horizon effect takes on the value $-\frac{\sigma_2^2}{2} 2k \cdot 67$. The horizon effect is thus 15 times bigger (in absolute terms) under Stern's low value for the rate of pure time preference.

It is evident from this example that the rate of pure time preference plays an important role in determining the size of the horizon effect. Whether the horizon effect itself plays an important role in determining the size of the instantaneous consumption discount rate however, also depends on the values of $\sigma_2^2$ and $k$.

Kocherlakota (1996) estimates the standard deviation of the consumption growth rate from US time series data to be $\sigma = 3.6\%$. Gollier (2002a) and Traeger (2011, 2014) use this estimate (or a rounded 4%) in the Extended Ramsey Equation and the Extended Ramsey Equation for EZ preferences. With $\sigma_2 = 0.036$, the horizon effect in equation (3.17) takes on the value $-1.3k$ for $\delta = 0.001$ and $-0.09k$ for $\delta = 0.015$.

Choosing an adequate value for $k$ is problematic. The value of $k$ is not only highly relevant in determining the size of the horizon effect, but also largely unexplored. Plugging in 'best guesses' for the value of $k$ may illustrate the significance of the horizon effect in determining the consumption discount rate. Approximating a reasonable range for the value of $k$, however, requires a thorough discussion which extends the scope of the present analysis. I defer this discussion to future research.

## 3.5   Conclusion

I examined the instantaneous consumption discount rate of a decision maker whose preferences are Kreps-Porteus recursive and mutually utility independent. In a first instance, I showed that such preferences are restricted to the Risk-Sensitive preference specification of Hansen and Sargent (1995). I then went on to analyze the discount rate of a decision maker with Risk-Sensitive preferences. The analysis was conducted in a setting with constant elasticity of substitution and independently distributed risk on consumption growth. I showed that this discount rate may be subject to a horizon effect whenever the horizon of the decision maker's intertemporal utility function extends the period to which the discount rate applies. To compare to the Ramsey Equation,

the Extended Ramsey Equation, and the Extended Ramsey Equation for Epstein-Zin preferences, I derived an analytical solution for the discount rate under consideration. To this end, I restricted the setting further such that $IES = 1$ and such that consumption growth risk is normally distributed. The resulting discounting function was denoted as the Extended Ramsey Equation for Risk-Sensitive preferences. On the basis of this analytical solution, I highlighted the twofold role which the rate of pure time preference takes on in a discounting equation that is subject to a horizon effect.

The technicalities that lead to the horizon effect are straightforward. The horizon effect is a direct implication of a number of assumptions which, taken individually, are either standard or are considered to be suitable for intergenerational discounting in my analysis. These assumptions are risk on consumption growth, an infinite horizon, Kreps-Porteus recursivity, and mutual utility independence. Assuming (independently distributed) risk on growth is common in the discounting literature and more in line with reality than an assumption of independently distributed risk on consumption itself. Postulating an infinite horizon is less arbitrary and more general than cutting off the horizon at some period. Employing the Kreps-Porteus recursive framework rather than the additive expected utility model allows for the disentanglement of the degree of risk aversion from the $IES$, and thus for a more flexible parameterization of the decision maker's preferences.

Determining the appropriateness of mutual utility independence in the context of intergenerational discounting requires closer examination. Albeit no such examination exists in the literature, MUI prevails as an assumption on preferences in the most popular discounting equations, namely in the Ramsey Equation and the Extended Ramsey Equation. These equations are derived from intertemporally additive utility functions (the discounted utility model and the additive expected utility model) which build on an implicit or explicit MUI assumption. Critics of the discounted utility model sometimes argue that MUI is too restrictive and fails to comply with preference reversals or habit formation as empirically observed in the preferences of individuals.[15] However, this criticism is geared towards the preferences of an individual over his lifetime consumption, rather than towards the preferences of a decision maker over the consumption of several

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[15]See e.g. Fredrick et al. (2002) and Kleindorfer et al. (1993) who enlarge upon this criticism.

generations. Existing criticism regarding the MUI assumption does therefore not apply in the context of the present analysis.

If the prevalence of MUI in the most popular discounting equations says anything about its validity in intergenerational discounting, one can conclude that it is an appropriate assumption. Furthermore, a first intuition suggests that MUI is an attractive assumption from a normative point of view, especially in the context of intergenerational decision making. Mutual utility independence prevents that preferences that concern one generation are conditioned on the wellbeing of other generations. MUI may therefore be considered to be more egalitarian than assuming some form of dependence.

I conclude with two suggestions for future research. First, the validity of the MUI assumption in the context of intergenerational decision making deserves further study from normative economics and moral philosophy. I showed that imposing MUI on KP recursive preferences has severe implications for the determinants of the discount rate. Second, a thorough discussion on the size of temporal risk aversion in the Risk-Sensitive framework is needed. Given a range of reasonable values of temporal risk aversion, more illuminating conclusions on the size of the horizon effect could be drawn. Thus, more precise statements on the significance of the horizon effect in determining the consumption discount rate could be made.

# Appendices for chapter 3

## 3.A    Proofs

### 3.A.1    Proof of theorem 10

In this proof I show that Kreps-Porteus recursive preferences which satisfy mutual utility independence are representable by (3.4). First, I denote by $_3x = (x_3, x_4, x_5, ...)$, $_3x' = (x'_3, x'_4, x'_5, ...)$ two specific deterministic consumption paths (outcomes) in $_3X = X_3 \times X_4 \times X_5 \times ....$ Using this notation, I consider the specific temporal lotteries $(x_1, p_2, \, _3x)$, $(x_1, p'_2, \, _3x)$, $(x'_1, p_2, \, _3x')$, $(x'_1, p'_2, \, _3x') \in D$ where $x_1$ and $x'_1$ are two specific levels in $X_1$ and $p_2, p'_2 \in P_2$ are two specific lotteries over $X_2$.

Second, I consider a decision maker with KP recursive preferences that are defined on $D$ and satisfy mutual utility independence. Denote these preferences as $\succeq^D$. By the definition of mutual utility independence for temporal lotteries (definition 9) it must be true that

$$(x_1, p_2, \, _3x) \succeq^D (x_1, p'_2, \, _3x) \iff (x'_1, p_2, \, _3x') \succeq^D (x'_1, p'_2, \, _3x').$$

Employing the notion of conditional preferences we can equivalently write

$$\succeq^D_{x_1, _3x} \;\; = \;\; \succeq^D_{x'_1, _3x'} \text{ on } p_2, p'_2 \in P_2, \text{ for (all) } (x_1, \, _3x), (x'_1, \, _3x') \in X_1 \times \, _3X. \quad (3.22)$$

Now let $U^D$ represent $\succeq^D$ and consider $U^D (x_1, m) = W^D (x_1, E_m[U^D])$. Given the temporal lottery $(x_1, p_2, \, _3x)$, we write $U^D (x_1, p_2, \, _3x) = W^D (x_1, E_m[U^D (x_2, \, _3x)])$ where $U^D (x_2, \, _3x) = \phi (U^X (x_2, \, _3x))$ and $U^X (x_2, \, _3x) = u (x_2) + \beta U^X (_3x)$. Now let

$v\left(p_2\right) = E_m\phi\left(u\left(x_2\right) + \beta U^X\left({}_3x\right)\right)$ represent $\succeq^D_{x_1,{}_3x}$. By (3.22) (i.e. by MUI), $v\left(p_2\right)$ represents $\succeq^D_{x_1,{}_3x} \forall\, {}_3x \in {}_3X$ (and $\forall\, x_1 \in X_1$). Put differently, a certainty equivalent $\hat{x}_2$ (as could be derived from $v\left(p_2\right)$), which makes a decision maker with $\succeq^D_{x_1,{}_3x}$ indifferent to receiving the lottery $p_2$, is independent of the specific level of ${}_3x$. This just means that a decision maker with preferences $\succeq^D_{x_1,{}_3x}$ is constantly absolute risk averse, which in turn implies $\phi\left(z\right) = -\exp\left(-kz\right)$. Using $\phi\left(z\right) = -\exp\left(-kz\right)$ in (3.3) and renormalizing by $U = \phi^{-1}\left(U^D\right)$ yields (3.4).

### 3.A.2   Proof of proposition 12

In this section I proof that the horizon effect diminishes the discount rate under the conditions stated in proposition 12. The proof employs the notions of comonotonicity and countercomonotonicity which are defined as follows.

**Definition 15** *(Strict comonotonicity and strict countercomonotonicity)*
*Consider two random variables $Z_1$ and $Z_2$ that are strictly monotonic transformations of a single random variable $\tilde{z}$:*

$$\left(Z_1, Z_2\right) = \left(g_1\left(\tilde{z}\right), g_2\left(\tilde{z}\right)\right).$$

*If $g_1$ and $g_2$ are strictly increasing in $\tilde{z}$, then $Z_1$ and $Z_2$ are called comonotonic.*
*If $g_1$ is strictly increasing and $g_2$ is strictly decreasing in $\tilde{z}$, or vice versa, then $Z_1$ and $Z_2$ are called countercomonotonic.*

Furthermore, the proof uses a lemma that I refer to as the risk aversion adjusted co-variance inequality. Before stating the lemma, I define formally what I mean by a risk aversion adjusted probability, a risk aversion adjusted expectation operator and a risk aversion adjusted covariance. Note that these concepts were introduces and discussed in chapter 2.

A risk aversion adjusted probability twists the statistical probability of a given state of the world $\omega = 1,...N$ to account for temporal risk aversion with respect to continuation utility. First, define $\pi^\omega$ as the product of a statistical probability $l^\omega$ and the risk aversion adjustment factor (see section 3.4) in the respective state $\omega$:

$$\pi^\omega = l^\omega \frac{\phi'\left(U_2^\omega\right)}{\phi'\left(\phi^{-1}\left(E_1\left[\phi\left(U_2\right)\right]\right)\right)}.$$

Second, note that $\pi^\omega$ can be interpreted as a probability whenever $0 \leq \pi^\omega \leq 1 \; \forall \; \omega$ and $\sum_{\omega=1}^N \pi^\omega = 1$. Lastly, note already that if $\phi\left(z\right) = -\exp\left(-kz\right)$ (which is the case for Risk-Sensitive preferences), then $\pi^\omega = l^\omega \frac{\exp\left(-kU_2^\omega\right)}{E_1\left[\exp\left(-kU_2\right)\right]}$, which satisfies the conditions for the interpretation of $\pi^\omega$ as a probability.

A risk aversion adjusted expectation operator for a random variable $\tilde{z}$ and some function $g\left(\tilde{z}\right)$ is then defined as

$$E_\pi\left[g\left(\tilde{z}\right)\right] = \sum_{\omega=1}^N \pi^\omega g\left(z^\omega\right).$$

This expectation operator employs risk aversion adjusted probabilities in the place of statistical probabilities.

Finally, a risk aversion adjusted covariance between two random variables or functions $g_1\left(\tilde{z}_1\right)$ and $g_2\left(\tilde{z}_2\right)$ is a covariance which is constructed from risk aversion adjusted expectation operators:

$$cov_\pi\left[g_1\left(\tilde{z}_1\right), g_2\left(\tilde{z}_2\right)\right] = E_\pi\left[g_1\left(\tilde{z}_1\right) g_2\left(\tilde{z}_2\right)\right] - E_\pi\left[g_1\left(\tilde{z}_1\right)\right] E_\pi\left[g_2\left(\tilde{z}_2\right)\right].$$

We are now ready to state a lemma on the risk aversion adjusted covariance inequality. A proof of this lemma, which is a close analogue to theorem 43 in Hardy et al. (1934), is contained in appendix 2.A of the previous chapter.

**Lemma 16** *(Risk aversion adjusted covariance inequality).*
*Consider two random variables $Z_1$ and $Z_2$ that are strictly monotonic transformations of a single random variable $\tilde{z}$. If $Z_1$ and $Z_2$ are strictly comonotonic, then*

$$cov_\pi\left[Z_1, Z_2\right] > 0.$$

*The inequality is reversed if $Z_1$ and $Z_2$ are strictly countercomonotonic.*

Let us now turn to the actual proof of proposition 12. The decision maker under consideration has mutually utility independent KP recursive preferences. His instantaneous

discount rate for a setting with horizon $\bar{T} > 2$ is thus given by (3.11). Assume that $k > 0$, $0 < \beta < 1$ and $g_t > -1 \; \forall \; t \geq 2$.

To prove statement 1 of proposition 12, assume furthermore that felicity is given by $u(x_t) = \frac{x_t^\rho - 1}{\rho}$ with $\rho < 1$ and that only second period consumption growth $\tilde{g}_2$ is risky. The consumption growth rate in $t \geq 3$ is deterministic.

To prove statement 2 of proposition 12, assume that felicity is given by $u(x_t) = \ln(x_t)$ and that consumption growth $\tilde{g}_t$ in $t \geq 2$ is risky and independently distributed.

<u>proof of statement 1</u>

Suppose $u(x_t) = \frac{x_t^\rho - 1}{\rho}$ with $\rho < 1$ and only period 2 growth is uncertain. Continuation utility $U_2$ can be rewritten in a simple manner since all risk resolves in period 2:

$$U_2 = u(\tilde{x}_2) + \sum_{t=3}^{\bar{T}} \beta^{t-2} u(\tilde{x}_t). \tag{3.23}$$

With $\tilde{x}_2 = (1 + \tilde{g}_2) x_1$, felicity in $t \geq 3$ can be written as

$$u(\tilde{x}_t) = u\left(\tilde{x}_2 \prod_{\tau=3}^{t} (1 + g_\tau)\right) = \frac{\tilde{x}_2^\rho}{\rho} \left[\prod_{\tau=3}^{t} (1 + g_\tau)\right]^\rho - \frac{1}{\rho}. \tag{3.24}$$

Plugging the felicity function (3.24) into the continuation utility $U_2$ (3.23), and (3.23) into the discounting function (3.11), yields $DR_{1,2}^{\bar{T}}$ as

$$DR_{1,2}^{\bar{T}} = -\ln \beta - \ln E_1 \left[ \frac{\exp\left(-ku(\tilde{x}_2) - \frac{k}{\rho}\tilde{x}_2^\rho h(\bar{T})\right)}{E_1\left[\exp\left(-ku(\tilde{x}_2) - \frac{k}{\rho}\tilde{x}_2^\rho h(\bar{T})\right)\right]} \frac{u'(\tilde{x}_2)}{u'(x_1)} \right] \tag{3.25}$$

$$\text{where } h(\bar{T}) = \sum_{t=3}^{\bar{T}} \left(\beta^{t-2} \left[\prod_{\tau=3}^{t} (1 + g_\tau)\right]^\rho\right).$$

In the next step I study how $DR_{1,2}^{\bar{T}}$ changes as the horizon $\bar{T}$ changes. To this end I would need to examine the derivative of $h(\bar{T})$ with respect to $\bar{T}$. However, as the domain of $h(\bar{T})$ is discrete, $h'(\bar{T})$ does not exist. Thus I define a function $\hat{h}(\bar{T}) : \mathbb{R}^+ \to \mathbb{R}$ with $\hat{h}(\bar{T}) = h(\bar{T}) \; \forall \; \bar{T} \in \mathbb{N}$. The function $\hat{h}(\bar{T})$ is assumed to constitute a smooth interpolation between the discrete points defined by $h(\bar{T})$ at all $\bar{T} \in \mathbb{N}$. I then examine

the derivative of $\hat{h}\left(\bar{T}\right)$ rather than that of $h\left(\bar{T}\right)$. Since $\hat{h}\left(\bar{T}\right)$ is strictly increasing, its derivative $\hat{h}'\left(\bar{T}\right)$ is positive.

Substituting all $h\left(\bar{T}\right)$ by $\hat{h}\left(\bar{T}\right)$ and taking the derivative of (3.25) with respect to $\bar{T}$ yields

$$
\frac{\partial DR_{1,2}^{\bar{T}}}{\partial \bar{T}} = \frac{E_1\left[f\left(\tilde{g}_2\right)u'\left(\tilde{x}_2\right)\left(\frac{k}{\rho}\tilde{x}_2^\rho\hat{h}'\left(\bar{T}\right)\right)\right]}{E_1\left[f\left(\tilde{g}_2\right)u'\left(\tilde{x}_2\right)\right]}
$$

$$
-\frac{E_1\left[f\left(\tilde{g}_2\right)\left(\frac{k}{\rho}\tilde{x}_2^\rho\hat{h}'\left(\bar{T}\right)\right)\right]}{E_1\left[f\left(\tilde{g}_2\right)\right]}
$$

$$
\text{where } f\left(\tilde{g}_2\right) = \exp\left(-ku\left(\tilde{x}_2\right) - \frac{k}{\rho}\tilde{x}_2^\rho\hat{h}\left(\bar{T}\right)\right).
$$

Thus

$$
\frac{\partial DR_{1,2}^{\bar{T}}}{\partial \bar{T}} \gtreqless 0
$$

whenever

$$
\frac{E_1\left[f\left(\tilde{g}_2\right)u'\left(\tilde{x}_2\right)\left(\frac{k}{\rho}\tilde{x}_2^\rho\hat{h}'\left(\bar{T}\right)\right)\right]}{E_1\left[f\left(\tilde{g}_2\right)u'\left(\tilde{x}_2\right)\right]} - \frac{E_1\left[f\left(\tilde{g}_2\right)\left(\frac{k}{\rho}\tilde{x}_2^\rho\hat{h}'\left(\bar{T}\right)\right)\right]}{E_1\left[f\left(\tilde{g}_2\right)\right]} \gtreqless 0.
$$

After multiplying the last equation with $E_1\left[f\left(\tilde{g}_2\right)u'\left(\tilde{x}_2\right)\right]/E_1\left[f\left(\tilde{g}_2\right)\right]$ we can write

$$
\left[\begin{array}{c} E_1\left[\frac{f(\tilde{g}_2)}{E_1[f(\tilde{g}_2)]}u'\left(\tilde{x}_2\right)\left(\frac{k}{\rho}\tilde{x}_2^\rho\hat{h}'\left(\bar{T}\right)\right)\right] - \\ E_1\left[\frac{f(\tilde{g}_2)}{E_1[f(\tilde{g}_2)]}u'\left(\tilde{x}_2\right)\right]E_1\left[\frac{f(\tilde{g}_2)}{E_1[f(\tilde{g}_2)]}\left(\frac{k}{\rho}\tilde{x}_2^\rho\hat{h}'\left(\bar{T}\right)\right)\right] \end{array}\right] \gtreqless 0. \tag{3.26}
$$

Now note that $\frac{f(\tilde{g}_2)}{E_1[f(\tilde{g}_2)]} = \frac{\exp\left(-ku(\tilde{x}_2)-\frac{k}{\rho}\tilde{x}_2^\rho\hat{h}(\bar{T})\right)}{E_1\left[\exp\left(-ku(\tilde{x}_2)-\frac{k}{\rho}\tilde{x}_2^\rho\hat{h}(\bar{T})\right)\right]} = \frac{\exp(-kU_2)}{E_1[\exp(-kU_2)]}$ is the risk aversion adjustment factor as mentioned earlier. We can thus rewrite (3.26) in terms of a risk aversion adjusted expectation operator $E_{1,\pi}$:

$$
E_{1,\pi}\left[u'\left(\tilde{x}_2\right)\left(\frac{k}{\rho}\tilde{x}_2^\rho\hat{h}'\left(\bar{T}\right)\right)\right] - E_{1,\pi}\left[u'\left(\tilde{x}_2\right)\right]E_{1,\pi}\left[\frac{k}{\rho}\tilde{x}_2^\rho\hat{h}'\left(\bar{T}\right)\right] \gtreqless 0. \tag{3.27}
$$

Equation (3.27) is the risk aversion adjusted covariance between $u'\left(\tilde{x}_2\right)$ and $\left(\frac{k}{\rho}\tilde{x}_2^\rho h'\left(\bar{T}\right)\right)$, both of which are functions of the single random variable $\tilde{g}_2$. By lemma 16, the sign of the risk aversion adjusted covariance can be determined from the comonotonicity

characteristics of $u'(\tilde{x}_2)$ and $\left(\frac{k}{\rho}\tilde{x}_2^\rho h'(\bar{T})\right)$. The comonotonicity characteristics in turn are determined by the derivatives of $u'(\tilde{x}_2)$ and $\left(\frac{k}{\rho}\tilde{x}_2^\rho h'(\bar{T})\right)$ with respect to the random variable $\tilde{g}_2$. Here we have

$$\frac{\partial u'(\tilde{x}_2)}{\partial \tilde{g}_2} = u''(\tilde{x}_2)x_1 < 0$$

$$\frac{\partial \left(\frac{k}{\rho}\tilde{x}_2^\rho \hat{h}'(\bar{T})\right)}{\partial \tilde{g}_2} = k\tilde{x}_2^{-\rho}\hat{h}'(\bar{T})x_1 > 0,$$

Hence, $u'(\tilde{x}_2)$ and $\left(\frac{k}{\rho}\tilde{x}_2^\rho \hat{h}'(\bar{T})\right)$ are countercomonotonic by definition 15. By lemma 16, countercomonotonicity implies a negative risk aversion adjusted covariance (equation 3.27), which in turn implies $\frac{\partial DR_{1,2}^{\bar{T}}}{\partial \bar{T}} < 0$.

proof of statement 2

Suppose $u(x_t) = \ln x_t$ and consumption growth $\tilde{g}_t$ in $t \geq 2$ is uncertain and independently distributed. Starting with the continuation utility in $\bar{T}$, I plug $U_{\bar{T}}$ into $U_{\bar{T}-1}$, $U_{\bar{T}-1}$ into $U_{\bar{T}-2}$ and so on until I arrive in period $t = 2$:

$$U_2 = \left(\sum_{\tau=2}^{\bar{T}} \beta^{\tau-2}\right)\ln(\tilde{x}_2) - \frac{1}{k}q(\tilde{g}_\tau) \tag{3.28}$$

$$\text{where } q(\tilde{g}_\tau) = \sum_{\tau=3}^{\bar{T}}\left(\beta^{\tau-2}\ln E_{\tau-1}\left[(1+\tilde{g}_\tau)^{-k\sum_\tau^{\bar{T}}\beta^{\bar{T}-\tau}}\right]\right).$$

Equation (3.28) exposes the additive separability of $U_2$ into a first term which collects $\tilde{g}_2$ (note that $\tilde{x}_2 = (1+\tilde{g}_2)x_1$) and a second term, namely $\left(-k^{-1}q(\tilde{g}_\tau)\right)$, which collects $\tilde{g}_t$ for $t \geq 3$. The latter term is independent of risk that reveals in period 2. Hence, upon plugging the continuation utility (3.28) into discounting equation (3.11), all terms containing $q(\tilde{g}_\tau)$ can be taken out of the expectation operator $E_1$ and subsequently cancel out. The instantaneous discount rate for a horizon $\bar{T}$ is thus

$$DR_{1,2}^{\bar{T}} = -\ln\beta - \ln E_1\left[\frac{\exp\left(-kh(\bar{T})\ln(\tilde{x}_2)\right)}{E_1\left[\exp\left(-kh(\bar{T})\ln(\tilde{x}_2)\right)\right]}\frac{u'(\tilde{x}_2)}{u'(x_1)}\right] \tag{3.29}$$

$$\text{where } h(\bar{T}) = \sum_{\tau=2}^{\bar{T}}\beta^{\tau-2}.$$

Equation (3.29) depends on the length of the horizon $\bar{T}$ through $h\left(\bar{T}\right)$.

The direction of this dependency is studied by taking the derivative of $DR_{1,2}^{\bar{T}}$ with respect to $\bar{T}$. As in the proof of statement 1, I substitute $h\left(\bar{T}\right)$ by its continuous analogue $\hat{h}\left(\bar{T}\right)$. Then,

$$\frac{\partial DR_{1,2}^{\bar{T}=\infty}}{\partial \bar{T}} = \frac{E_1\left[f\left(\tilde{g}_2\right)u'\left(\tilde{x}_2\right)\left(k\hat{h}'\left(\bar{T}\right)\ln\left(\tilde{x}_2\right)\right)\right]}{E_1\left[f\left(\tilde{g}_2\right)u'\left(\tilde{x}_2\right)\right]} - \frac{E_1\left[f\left(\tilde{g}_2\right)\left(k\hat{h}'\left(\bar{T}\right)\ln\left(\tilde{x}_2\right)\right)\right]}{E_1\left[f\left(\tilde{g}_2\right)\right]}$$

$$\text{where } f\left(\tilde{g}_2\right) = \exp\left(-k\hat{h}\left(\bar{T}\right)\ln\left(\tilde{x}_2\right)\right).$$

The direction of the inequality $\frac{\partial DR_{1,2}^{\bar{T}=\infty}}{\partial \bar{T}} \gtreqless 0$ is then equivalent to the direction of the inequality

$$\left[\begin{array}{c} E_1\left[\frac{f(\tilde{g}_2)}{E_1[f(\tilde{g}_2)]}u'\left(\tilde{x}_2\right)\left(k\hat{h}'\left(\bar{T}\right)\ln\left(\tilde{x}_2\right)\right)\right] - \\ E_1\left[\frac{f(\tilde{g}_2)}{E_1[f(\tilde{g}_2)]}u'\left(\tilde{x}_2\right)\right]E_1\left[\frac{f(\tilde{g}_2)}{E_1[f(\tilde{g}_2)]}\left(k\hat{h}'\left(\bar{T}\right)\ln\left(\tilde{x}_2\right)\right)\right] \end{array}\right] \gtreqless 0,$$

which, as in the precedent proof, can be stated as a risk aversion adjusted covariance:

$$E_{1,\pi}\left[u'\left(\tilde{x}_2\right)\left(k\hat{h}'\left(\bar{T}\right)\ln\left(\tilde{x}_2\right)\right)\right] - E_{1,\pi}\left[u'\left(\tilde{x}_2\right)\right]E_{1,\pi}\left[k\hat{h}'\left(\bar{T}\right)\ln\left(\tilde{x}_2\right)\right] \gtreqless 0. \qquad (3.30)$$

Since

$$\frac{\partial u'\left(\tilde{x}_2\right)}{\partial \tilde{g}_2} = u''\left(\tilde{x}_2\right)x_1 < 0$$

$$\frac{\partial\left(kh'\left(\bar{T}\right)\ln\left(\tilde{x}_2\right)\right)}{\partial \tilde{g}_2} = kh'\left(\bar{T}\right)\frac{1}{\left(1+\tilde{g}_2\right)} > 0,$$

$u'\left(\tilde{x}_2\right)$ and $\left(k\hat{h}'\left(\bar{T}\right)\ln\left(\tilde{x}_2\right)\right)$ are countercomonotonic according to definition 15. By lemma 16 it is then implied that equation (3.30) is negative and hence $\frac{\partial DR_{1,2}^{\bar{T}}}{\partial \bar{T}} < 0$.

## 3.B  Derivations

### 3.B.1  Risk-Sensitive preferences for independently distributed $\tilde{x}_t$

Suppose preferences are represented by (3.4) and $\tilde{x}_t$ for $t > 1$ is risky and independently distributed. Plugging continuation utilities $U(x_2, m)$, $U(x_3, m)$,... into the initial utility function $U(x_1, m)$ yields

$$U(x_1, m) = u(x_1) - ...$$
$$...\frac{\beta}{k} \ln \left( E_m \left[ \exp\left(-ku(\tilde{x}_2)\right) \left( E_m \left[ \exp\left(-ku(\tilde{x}_3)\right) \left(E_m \exp\left(-k(...)\right)\right)^\beta \right] \right)^\beta \right] \right).$$

Since $\tilde{x}_t$ is independently distributed, this can be written as

$$U(x_1, m) = u(x_1) - ...$$
$$...\frac{\beta}{k} \ln E_m \exp\left(-ku(\tilde{x}_2)\right) - \frac{\beta^2}{k} \ln \left(E_m \exp\left(-ku(\tilde{x}_3)\right)\right) - \frac{\beta^3}{k} \ln\left(E_m \exp\left(-k(...)\right)\right).$$

Now note that the terms $-\frac{1}{k} \ln\left(E_m \exp\left(-ku(\tilde{x}_t)\right)\right)$ can be substituted for by $u(\hat{x}_t)$ since they determine certainty equivalent consumption $\hat{x}_t$. Thus we can further simplify the last equation and write

$$U(x_1, m) = u(x_1) + \beta u(\hat{x}_2) + \beta^2 u(\hat{x}_3) + ... = u(x_1) + \beta \sum_{t=2}^{\infty} \beta^{t-2} u(\hat{x}_t).$$

### 3.B.2  Absence of the horizon effect

In this section I discuss conditions under which the instantaneous consumption discount rate of a decision maker with Risk-Sensitive preferences (equation 3.11) is not subject to a horizon effect. Although these conditions are fairly obvious, I evolve on them to facilitate the general understanding of the horizon effect.

The requirement for the absence of the horizon effect is that $DR_{1,2}^{\bar{T}=\infty} = DR_{1,2}^{\bar{T}=2}$. This requirement is met under the following specifications.

(i) $\beta = 0$

If $\beta = 0$, the utility function of a RS decision maker (equation 3.4) is $U_1 = u(x_1)$,

independently of the length of the horizon $\bar{T}$ taken into account. The respective decision maker has no valuation for generations living in $t \geq 2$ and therefore applies an infinite discount rate to period 2 consumption values. This is true irrespective of the existence of generations in periods $t \geq 3$. Hence the discount rates $DR_{1,2}^{\bar{T}=\infty}$ and $DR_{1,2}^{\bar{T}=2}$ are equivalent and there is no horizon effect.

### (ii) $u(x_t)$ linear

If $u(x_t)$ is linear, $u'(x_t)$ is a constant and thus independent of $x_t$ (which may or may not be risky). Thus one can write $u'(x_1) = u'(\tilde{x}_2) = c$ which reduces (3.11) to

$$DR_{1,2}^{\bar{T}=\infty} = -\ln\beta - \ln E_1\left[\frac{\exp(-kU_2)}{E_1[\exp(-kU_2)]}\right] = -\ln\beta.$$

This is equivalent to $DR_{1,2}^{\bar{T}=2}$ under linear $u(x_t)$. Hence $DR_{1,2}^{\bar{T}=\infty} = DR_{1,2}^{\bar{T}=2}$.

Intuitively, the absence of the horizon effect is explained by the absence of (risk aversion adjusted) probabilities. Since risk on $u'(\tilde{x}_2)$ plays no role if $u(x_t)$ is linear, there is no role for probabilities or risk aversion adjusted probabilities. The horizon of the decision problem, which enters the discounting equation through the adjustment factor, has therefore no effect on the discount rate.

### (iii) no risk in period 2

If there is no risk in period 2 (but potentially in periods $t > 2$), the risk aversion adjustment factor in equation (3.4) can be written as

$$\frac{\exp(-kU_2)}{E_1[\exp(-kU_2)]} = \frac{\exp(-ku(x_2)) \cdot \exp(\beta\ln(E_2[\exp(-kU_3)]))}{\exp(-ku(x_2)) \cdot \exp(\beta\ln(E_2[\exp(-kU_3)]))} = 1.$$

The expectation operator $E_1$ can be neglected since $x_2$ is certain and since the uncertain continuation utility $U_3$ is transformed into a certainty equivalent by the expectation operator $E_2$. Without the expectation operator $E_1$, the numerator and the denominator cancel each other out. The discount rate is then simply

$$DR_{1,2}^{\bar{T}=\infty} = DR_{1,2}^{\bar{T}=2} = -\ln\beta - \ln\left[\frac{u'(x_2)}{u'(x_1)}\right].$$

The intuition is as in the case where $u(x_t)$ is linear. If there is no risk on $u'(x_2)$, then

there is no role for risk aversion adjustment factors and thus no channel through which the horizon $t \geq 3$ could enter the discounting function.

(iv) $k = 0$

If $k = 0$, the risk aversion adjustment factor $\frac{\exp(-kz)}{E_1[\exp(-kz)]}$ equals one. Thus,

$$DR_{1,2}^{\bar{T}=\infty} = DR_{1,2}^{\bar{T}=2} = -\ln \beta - \ln E_1 \left[ \frac{u'(\tilde{x}_2)}{u'(x_1)} \right].$$

The intuitive explanation is that $k = 0$ restricts $DR_{1,2}^{\bar{T}=\infty}$ to the discount rate of a decision maker who is temporally risk neutral, i.e. a decision maker whose preferences are representable by an additive expected utility function. A discount rate that is derived from an additive utility function only depends on values of the present period and values of the period that is discounted. This means that the instantaneous discount rate $DR_{1,2}^{\bar{T}=\infty}$ is independent of values in periods $t \geq 3$, and thus not subject to a horizon effect.

(v) risk on $\tilde{x}_t$ independently distributed

If risk on $\tilde{x}_t$ is independently distributed, then the discounting equation (3.11) of the Risk-Sensitive decision maker can be written as

$$DR_{1,2}^{\bar{T}=\infty} = -\ln \beta - \ln E_1 \left[ \frac{\exp\left(-ku\left(\tilde{x}_2\right)\right) \cdot \exp\left(\beta \ln\left(E_2\left[\exp\left(-kU_3\right)\right]\right)\right)}{E_1\left[\exp\left(-ku\left(\tilde{x}_2\right)\right)\right] \cdot \exp\left(\beta \ln\left(E_2\left[\exp\left(-kU_3\right)\right]\right)\right)} \frac{u'(\tilde{x}_2)}{u'(x_1)} \right].$$

The exponential function containing $U_3$ was taken out of the expectation operator $E_1$ since the risk contained in $U_3$ is independent of period 2 information. As this exponential appears in the numerator as well as in the denominator, it cancels out and we get $DR_{1,2}^{\bar{T}=\infty} = DR_{1,2}^{\bar{T}=2}$.

The absence of a horizon effect for independently distributed $\tilde{x}_t$ is a direct consequence of the mutual utility independence of the decision maker. I already showed in equation (3.6) of section 3.3 that a RS decision maker who faces independently distributed $\tilde{x}_t$ has an additive utility function. As with $k = 0$, the additivity of the decision maker's utility function implies the absence of a horizon effect.

## 3.B.3 Analytical solution for the discount rate

Consider a RS decision maker with discount rate (3.11). Assume consumption growth $\tilde{g}_t$ is risky and independently distributed in all $t \geq 2$. Furthermore assume that $u(x_t) = \ln x_t$ and $\tilde{g}_t > -1 \ \forall \ t \geq 2$. In the proof of statement 2 of proposition 12, I already showed that the instantaneous discount rate in this setting can be written as

$$DR_{1,2}^{\bar{T}} = -\ln \beta - \ln E_1 \left[ \frac{\exp\left(-kh\left(\bar{T}\right)\ln\left(\tilde{x}_2\right)\right)}{E_1\left[\exp\left(-kh\left(\bar{T}\right)\ln\left(\tilde{x}_2\right)\right)\right]} \frac{u'\left(\tilde{x}_2\right)}{u'\left(x_1\right)} \right]$$

$$\text{where } h\left(\bar{T}\right) = \sum_{\tau=2}^{\bar{T}} \beta^{\tau-2}.$$

With $\tilde{x}_2 = (1 + \tilde{g}_2) x_1$, $u'(x_t) = x_t^{-1}$, after some rearrangements and after using $\ln(1 + \tilde{g}_2) \approx \tilde{g}_2$ (for $\tilde{g}_2$ small), the last equation can be written as

$$DR_{1,2}^{\bar{T}} = -\ln \beta - \ln E_1 \left[ \frac{\exp\left(\left(-kh\left(\bar{T}\right)-1\right)\tilde{g}_2\right)}{E_1\left[\exp\left(-kh\left(\bar{T}\right)\tilde{g}_2\right)\right]} \right].$$

Now assume in addition that $\tilde{g}_t \sim N\left(\mu_t, \sigma_t^2\right) \ \forall \ t \geq 2$. The moment generating function $M_{\tilde{y}}(a) \equiv E\left[\exp(a\tilde{y})\right]$ of a normally distributed random variable $\tilde{y} \sim N\left(\mu, \sigma^2\right)$ is $M_{\tilde{y}}(a) = \exp\left(a\mu + \frac{\sigma^2}{2}a^2\right)$. Using the moment generating function of $\tilde{g}_2$ in the last equation, we can write

$$DR_{1,2}^{\bar{T}} = -\ln \beta - \ln \frac{\exp\left(\left(-kh\left(\bar{T}\right)-1\right)\mu_2 + \frac{\sigma_2^2}{2}\left(-kh\left(\bar{T}\right)-1\right)^2\right)}{\exp\left(-kh\left(\bar{T}\right)\mu_2 + \frac{\sigma_2^2}{2}\left(kh\left(\bar{T}\right)\right)^2\right)}$$

$$= -\ln \beta + \mu_2 - \frac{\sigma_2^2}{2} - \frac{\sigma_2^2}{2} 2kh\left(\bar{T}\right)$$

or equivalently

$$DR_{1,2}^{\bar{T}} = -\ln \beta + \mu_2 - \frac{\sigma_2^2}{2} - \frac{\sigma_2^2}{2} 2k - \frac{\sigma_2^2}{2} 2k\beta \sum_{\tau=3}^{\bar{T}} \beta^{\tau-3}. \tag{3.31}$$

For $\bar{T} = \infty$, we can substitute the sum in (3.31) by the limit of a geometric series, $\lim_{\bar{T}\to\infty} \sum_{t=o}^{\bar{T}} \beta^t = \frac{1}{1-\beta}$, and thus write

$$DR_{1,2}^{\bar{T}=\infty} = -\ln\beta + \mu_2 - \frac{\sigma_2^2}{2} - \frac{\sigma_2^2}{2}2k - \frac{\sigma_2^2}{2}2k\frac{\beta}{1-\beta}.$$

If $\bar{T} = 2$, the sum in (3.31) is zero, and the analytical solution for $DR_{1,2}^{\bar{T}=2}$ is thus

$$DR_{1,2}^{\bar{T}=2} = -\ln\beta + \mu_2 - \frac{\sigma_2^2}{2} - \frac{\sigma_2^2}{2}2k.$$

### 3.B.4   No horizon effect under Epstein-Zin preferences

Epstein-Zin preferences with homogeneous CES felicity $u(x_t) = \frac{x_t^\rho}{\rho}$ are representable by a KP recursive utility function of the form

$$U_t = \frac{x_t^\rho}{\rho} + \beta\left(E_t U_{t+1}^{\frac{\alpha}{\rho}}\right)^{\frac{\rho}{\alpha}}. \tag{3.32}$$

This is the KP recursive EZ preference representation employed by Traeger (2011, 2014) from which (3.20) and (3.21) can be obtained. The instantaneous discount rate of a decision maker with preferences as in (3.32) is given by

$$DR_{1,2}^{\bar{T}>2} = -\ln\beta - \ln\frac{E_1\left[\frac{U_2^{\frac{\alpha}{\rho}-1}}{\left(E_1\left[U_2^{\frac{\alpha}{\rho}}\right]\right)^{1-\frac{\rho}{\alpha}}}u'(\tilde{x}_2)\right]}{u'(x_1)}, \tag{3.33}$$

where uncertain consumption $\tilde{x}_t$ can be written as $\tilde{x}_t = (1+\tilde{g}_2)x_1\frac{\tilde{x}_t}{(1+\tilde{g}_2)x_1}$. Exploiting the homogeneity of $u(\tilde{x}_t)$ we can write $u(\tilde{x}_t) = \rho\frac{((1+\tilde{g}_2)x_1)^\rho}{\rho}u\left(\frac{\tilde{x}_t}{(1+\tilde{g}_2)x_1}\right) = \rho u(\tilde{x}_2)u\left(\frac{\tilde{x}_t}{\tilde{x}_2}\right)$, where the argument of the latter felicity function is statistically independent of $\tilde{g}_2$ since risk on growth is independently distributed. Starting with the $t = \bar{T}$ (terminal period) specification of (3.32), we can solve recursively for continuation utility $U_2$. In each

recursion, the independent distribution of $\tilde{g}_t$ enables us to factor the term $\rho u\left(\tilde{x}_2\right)$ out:

$$
\begin{aligned}
U_{\bar{T}} &= \rho u\left(\tilde{x}_2\right) u(\tilde{x}_{\bar{T}}/\tilde{x}_2) \\
U_{\bar{T}-1} &= \rho u\left(\tilde{x}_2\right)(u(\tilde{x}_{\bar{T}-1}/\tilde{x}_2) + \beta(E_{\bar{T}-1}[u(\tilde{x}_{\bar{T}}/\tilde{x}_2)]^{\frac{\alpha}{\rho}})^{\frac{\rho}{\alpha}}) \\
U_{\bar{T}-2} &= \rho u\left(\tilde{x}_2\right)(u(\tilde{x}_{\bar{T}-2}/\tilde{x}_2) + \beta(E_{\bar{T}-2}[u(\tilde{x}_{\bar{T}-1}/\tilde{x}_2) + \beta(E_{\bar{T}-1}[u(\tilde{x}_{\bar{T}}/\tilde{x}_2)]^{\frac{\alpha}{\rho}})^{\frac{\rho}{\alpha}}]^{\frac{\alpha}{\rho}})^{\frac{\rho}{\alpha}}) \\
U_{\bar{T}-3} &= \rho u\left(\tilde{x}_2\right)(...).
\end{aligned}
$$

Finally we arrive at

$$
U_2 = \rho u\left(\tilde{x}_2\right) h\left(\bar{T}\right), \tag{3.34}
$$

where $h\left(\bar{T}\right)$ is a function that depends on the horizon $\bar{T}$ as well as on the (uncertain) growth rates $\tilde{g}_t$ with $t > 2$. Note that the term $h\left(\bar{T}\right)$ is independent of risk on period 2 growth, $\tilde{g}_2$. Plugging (3.34) into the discounting equation of the EZ decision maker (equation 3.33) yields

$$
DR_{1,2}^{\bar{T}>2} = -\ln\beta - \ln\frac{E_1\left[\dfrac{u(\tilde{x}_2)^{\frac{\alpha}{\rho}-1}h\left(\bar{T}\right)^{\frac{\alpha}{\rho}-1}}{\left(E_1\left[u(\tilde{x}_2)^{\frac{\alpha}{\rho}}h\left(\bar{T}\right)^{\frac{\alpha}{\rho}}\right]\right)^{1-\frac{\rho}{\alpha}}}u'\left(\tilde{x}_2\right)\right]}{u'\left(x_1\right)}.
$$

Since the risk contained in $h\left(\bar{T}\right)$ is independent of the risk in period 2, $h\left(\bar{T}\right)$ can be taken out of the expectation operator and thus cancels out. We are left with

$$
DR_{1,2}^{\bar{T}>2} = -\ln\beta - \ln\frac{E_1\left[\dfrac{u(\tilde{x}_2)^{\frac{\alpha}{\rho}-1}}{\left(E_1\left[u(\tilde{x}_2)^{\frac{\alpha}{\rho}}\right]\right)^{1-\frac{\rho}{\alpha}}}u'\left(\tilde{x}_2\right)\right]}{u'\left(x_1\right)}, \tag{3.35}
$$

which is independent of the horizon after $t = 2$, hence $DR_{1,2}^{\bar{T}>2} = DR_{1,2}^{\bar{T}=2}$. I have thus shown that the instantaneous discount rate $DR_{1,2}^{\bar{T}>2}$ of the KP recursive EZ decision maker with homogeneous felicity $u\left(\tilde{x}_t\right)$ and independently distributed growth risk is not subject to a horizon effect.

# Chapter 4

# A Numerical Assessment
# of the Horizon Effect

## 4.1   Introduction

The descriptive approach to discounting in social cost-benefit analysis holds that the preferences of a social decision maker should be inferred from the revealed preferences of private agents. This is in contrast to the prescriptive approach, which advocates ethical principles as the basis for social preferences. Disputes on the 'correct' approach to discounting in social cost-benefit analysis have a long tradition in the discounting literature, as comprehensively summarized by Arrow et al. (1996) and sketched in chapter 1. Without reciting the pros and cons of the disputants, I take a descriptive approach in the present chapter for practical reasons. Defining social preferences under a prescriptive approach requires a public debate or a deep philosophical analysis. Economists, who are not generally experts on ethical principles, have not much to contribute to this debate other than their private opinion as citizens. What economists can do, however, is to apply their economic toolset in order to reverse-engineer the preferences of private agents from observable data, and employ these parameters in a social welfare function.

The preferences of private agents can be elicited through experiments and microeconomic datasets, or through observing individuals' choices and the resulting prices in financial

markets. The latter type of data is particularly useful to shed light on private agents' intertemporal and risk preferences. Private agents save, trade and invest in the financial markets in order to allocate their lifetime consumption under consideration of the (uncertain) economic circumstances that apply to them. The equilibrium market prices that result from such private interactions reveal information about the parameters of the underlying preferences.

The ultimate purpose of the present chapter is the numerical examination of the instantaneous consumption discount rate of a Risk-Sensitive social agent. Risk-Sensitive preferences, as introduced by Hansen and Sargent (1995), are a special case of Kreps-Porteus recursive preferences (Kreps and Porteus 1978). As such, they enable the disentanglement of an agent's degree of (temporal) risk aversion from his intertemporal elasticity of substitution, and are thus more flexible than standard additive expected utility preferences. The discount rate of a Risk-Sensitive agent has already been examined in chapter 3, albeit only theoretically. It was shown there, that the discount rate of a Risk-Sensitive agent may be subject to a *horizon effect*. In particular, it was shown that the discount rate is decreased by a horizon effect if the Risk-Sensitive agent is strictly temporally risk averse and if his horizon exceeds the period to which the discount rate applies.

In the present chapter, I examine numerically how the horizon effect, and thus a Risk-Sensitive agent's instantaneous consumption discount rate, reacts to changes in the utility discount rate and in the intertemporal elasticity of substitution. This analysis is conducted under a descriptive approach to social discounting. Assuming different parameter combinations of the utility discount rate and the intertemporal elasticity of substitution, I infer a Risk-Sensitive private agent's degree of temporal risk aversion from the risk-free rate of return, as observable in the financial markets. The private agent's preferences are then employed as a descriptor of the preferences of a social agent. However, I postulate an important distinction between the private and the social agent: The choices of the private agent are supposedly driven by a lifetime utility function of horizon $\bar{T}^P$; the choices of the social agent, in the contrary, are driven by a preference model with horizon $\bar{T}^S$, which is assumed to be considerably longer than the (remaining) lifetime of the private agent. Hence, I assume that $\bar{T}^S > \bar{T}^P$. In view of the theoretical insights on the horizon effect from chapter 3, this assumption is expected to imply that the social

agent's instantaneous consumption discount rate is reduced relative to the risk-free rate of return. The numerical results of the present chapter shed further light on the horizon effect, in particular on its magnitude under various preference specifications.

The remainder of this chapter is structured as follows. The theoretical model is outlined in section 4.2. This section should be regarded as a tailored summary of previous insights, and the reader is advised to consult chapter 3 for a more detailed discussion of Risk-Sensitive preferences and the horizon effect. Section 4.3 describes the computational procedure as well as the parameter choices for the main analysis. The numerical results of the main analysis are presented and discussed in section 4.4. Section 4.5 concludes. The appendix contains the presentation and discussion of a number of side analyses, as well as the MATLAB codes that have been developed for the purpose of this chapter.

## 4.2 Theory

### 4.2.1 Setting

The economy considered in this chapter is described by a level of deterministic consumption $x_1$ in the first period and by an uncertain consumption growth rate $\tilde{g}_t = \frac{x_t}{x_{t-1}} - 1$ which is governed by an *iid* (independently and identically distributed) normal distribution, i.e. $\tilde{g}_t \sim N\left(\mu, \sigma^2\right) \ \forall \ t = 1, 2, 3....$ Consumption levels in $t \geq 2$ evolve according to $x_t = x_1 \cdot \Pi_{\tau=2}^{t}\left(1 + \tilde{g}_t\right)$, which implies that the uncertainty on $\tilde{x}_t \ \forall \ t \geq 3$ is not *iid*.

Figure 4.1 depicts the consumption tree of a 4-period economy in which the growth rate can take on either of the values $g^h$, $g^m$, or $g^l$ in each of three uncertain periods $(rndt = 3)$, i.e. in periods 2, 3, and 4. The figure illustrates how growth uncertainty in period $t = 2$ implies non-*iid* uncertainty on the consumption levels in the subsequent nodes, i.e. on the consumption levels in $t \geq 3$. Figure 4.1 will be employed for illustrative purposes in the remainder of this chapter.

$t = 1$     $t = 2$     $t = 3$     $t = 4$

$x_1$

$x_2^h$, $x_2^m$, $x_2^l$    $g^h$, $g^m$, $g^l$

$x_3^{hh}$, $x_3^{hm}$, $x_3^{hl}$, $x_3^{mh}$, $x_3^{mm}$, $x_3^{ml}$, $x_3^{lh}$, $x_3^{lm}$, $x_3^{ll}$

$x_4^{hhh} = x_1(1 + g_h)^3$, $x_4^{hhm}$, $x_4^{hhl}$, $x_4^{hmh}$, $x_4^{hmm}$, $x_4^{hml}$, $x_4^{hlh}$, $x_4^{hlm}$, $x_4^{hll}$, $x_4^{mhh}$, $x_4^{mhm}$, $x_4^{mhl}$, $x_4^{mmh}$, $x_4^{mmm}$, $x_4^{mml}$, $x_4^{mlh}$, $x_4^{mlm}$, $x_4^{mll}$, $x_4^{lhh}$, $x_4^{lhm}$, $x_4^{lhl}$, $x_4^{lmh}$, $x_4^{lmm}$, $x_4^{lml}$, $x_4^{llh}$, $x_4^{llm}$, $x_4^{lll}$
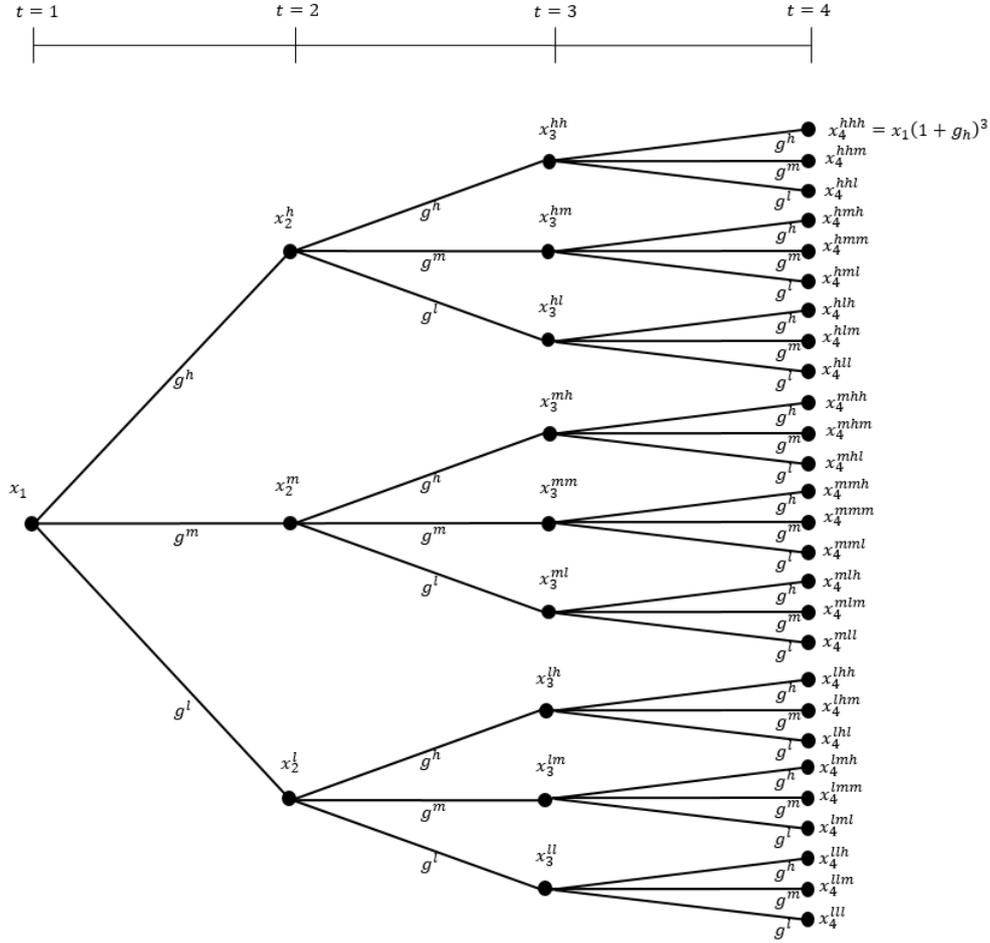
Figure 4.1: Consumption tree with 4 periods and 3 states in each uncertain period.

## 4.2.2 Preferences

I consider two different types of agents $i$ in this chapter, namely a private $(i = P)$ and a social agent $(i = S)$. The two agents are assumed to be exactly equivalent except for the horizon $\bar{T}^i$ over which their preferences are defined. The assumption of equivalence is based on the descriptive approach to social cost-benefit analysis. The descriptive approach holds that the parametrization of a fictive social agent's preference model should be guided by the preferences of the non-fictive members of society (private agents).[1] Following this approach, I employ the (revealed) preferences of a private agent as a descriptor for the preferences of a social agent. The distinction between a private agent's horizon $\bar{T}^P$ and a social agent's horizon $\bar{T}^S$ rests on the idea that a private

---

[1] See e.g. Arrow et al. (1996).

agent accounts for his remaining lifetime utility when making choices, whereas a social agent should also account for the lifetime of the young and the unborn generations when making investment decisions.

The private as well as the social agent are equipped with Risk-Sensitive (RS) preferences in the sense of Hansen and Sargent (1995). RS preferences are a special case of Kreps-Porteus recursive preferences (Kreps and Porteus 1978) which are defined over the set of temporal lotteries $D$. The concepts of temporal lotteries, Risk-Sensitive preferences and Kreps-Porteus recursive preferences have already been introduced and discussed in previous chapters, hence I only repeat the essential components.

The preferences of a Risk-Sensitive agent $i$ are representable by an intertemporal utility function $U^i : D \to \mathbb{R}$ of the form

$$
\begin{aligned}
U^i_t (x_t, m) &= u(x_t) - \frac{\beta}{k} \ln \left( E_t \left[ \exp \left( -k U^i_{t+1} \right) \right] \right) \ \forall \ t = 1, 2, \dots \left( \bar{T}^i - 1 \right) \quad (4.1) \\
U^i_{\bar{T}^i} (x_{\bar{T}^i}, m) &= u(x_{\bar{T}^i}),
\end{aligned}
$$

where $\beta = \exp(-\delta)$ is a utility discount factor and $\delta$ is the respective utility discount rate (the rate of pure time preference). Furthermore, $m$ is a probability measure over the evolution of future consumption and $k$ measures the agent's degree of temporal risk aversion. An agent is said to be temporally risk averse if $k > 0$, temporally risk neutral if $k = 0$, and temporally risk loving if $k < 0$. Throughout this chapter I assume that the felicity function is given by $u(x_t) = \frac{x_t^\rho - 1}{\rho}$ with $\rho < 1$ (constant elasticity of substitution, CES), where $IES = (1 - \rho)^{-1}$ constitutes the intertemporal elasticity of substitution.[2]

### 4.2.3 The instantaneous consumption discount rate

The object of interest in this chapter is the instantaneous consumption discount rate of a Risk-Sensitive agent. An instantaneous consumption discount rate $(DR_{1,2})$ informs about the value of a consumption increase in the second period (the instantaneous future) relative to the value of a consumption increase in the first period (the present). $DR_{1,2}$

---

[2]Note that the CES felicity function nests the logarithmic felicity function $u(x_t) = \ln x_t$ as the limit case when $\rho \to 0$ $(IES = 1)$.

is formally defined by

$$DR_{1,2} = -\ln \frac{\partial U_1 / \partial x_2}{\partial U_1 / \partial x_1}, \tag{4.2}$$

i.e. it is the rate of change of the marginal rate of substitution. Given a Risk-Sensitive agent $i$ with preference representation (4.1) and horizon $\bar{T}^i$, the instantaneous consumption discount rate is specified by

$$DR_{1,2}^{\bar{T}^i} = -\ln \beta - \ln \frac{E_1 \left[ \frac{\exp\left(-kU_2^i\right)}{E_1\left[\exp\left(-kU_2^i\right)\right]} u'\left(\tilde{x}_2\right) \right]}{u'\left(x_1\right)} \tag{4.3}$$

$$\text{with } U_t^i = u\left(\tilde{x}_t\right) - \frac{\beta}{k} \ln\left(E_t \left[\exp\left(-kU_{t+1}^i\right)\right]\right) \ \forall \ t = 2, 3, ... \left(\bar{T}^i - 1\right)$$
$$\text{and } U_{\bar{T}^i}^i = u\left(\tilde{x}_{\bar{T}^i}\right),$$

where the equations in the second and third line are known as *continuation utilities*.

In the special case of $IES = 1$, equation (4.3) can be solved analytically, which yields the Extended Ramsey Equation for RS preferences:[3]

$$DR_{1,2}^{\bar{T}^i} \approx -\ln \beta + \mu - \frac{\sigma^2}{2} - \frac{\sigma^2}{2}2k - \frac{\sigma^2}{2}2k\beta \sum_{\tau=3}^{\bar{T}^i} \beta^{\tau-3}. \tag{4.4}$$

Note that this analytical solution assumes that the uncertainty on consumption growth $\tilde{g}_t$ is independently and normally distributed in all $t \geq 2$. The moments $\mu$ and $\sigma^2$ which appear in equation (4.4) pertain to the uncertainty on second period growth $\tilde{g}_2$, while the moments that correspond to the uncertainty in periods $t \geq 3$ do not affect the instantaneous consumption discount rate of a RS agent with $IES = 1$. Furthermore note that the approximation '$\approx$' is due to employing the log approximation $\ln\left(1 + \tilde{g}\right) \approx \tilde{g}$ in the derivation of equation (4.4).

The ultimate interest of this chapter lies in the consumption discount rate of a social agent, and in particular in the numerical analysis of this rate's dependence on the horizon $\bar{T}^S$. From the theoretical insights of chapter 3, we know that the discount rate of an agent with intertemporal utility function (4.1) will be decreasing in the horizon $\bar{T}^i$, i.e.

---

[3] See definition 13 and equation 3.14 in chapter 3. Note, however, that the equations in chapter 3 are written for a preference representation with $\bar{T} = \infty$, which is a subcase of the Extended Ramsey Equation for RS preferences with general $\bar{T}$, as derived in appendix 3.B.3 of chapter 3.

it will be subject to a horizon effect. In chapter 3, proposition 12, this effect was shown to be present in a setting with $u(x_t) = \frac{x_t^\rho - 1}{\rho}$ ($\rho < 1$, $IES > 0$) and $\tilde{g}_2$ uncertain ($g_t \ \forall \ t > 2$ deterministic), and in a setting with $u(x_t) = \ln x_t$ ($\rho = 0$, $IES = 1$) and $\tilde{g}_t$ uncertain and independently distributed $\forall \ t$. Given the assumption $\bar{T}^S \neq \bar{T}^P$ of the present chapter, the theory of chapter 3 predicts that the two agents' discount rates will not be equivalent. In particular, for $\bar{T}^S > \bar{T}^P$ the insights of chapter 3 suggest that $DR_{1,2}^{\bar{T}^S} < DR_{1,2}^{\bar{T}^P}$.

### 4.2.4 The risk-free rate of return

Since I employ a descriptive approach to discounting in this chapter, I assume that the social agent's preference parameters $(\delta, IES, k)$ are equivalent to those of the private agent. Calibrating the private agent's preference model (equation 4.1 with $i = P$) to observable data will therefore not only elicit his preference parameters, but also those assumed for the social agent. The preferences of private agents are (in theory) observable from market rates of return. When private agents conduct investments in the financial markets–presumably in order to optimize their lifetime consumption–they reveal information on the preference model that guides their choices. Since the choices of agents are reflected in the equilibrium prices on financial markets, we can employ these prices to reverse-engineer a private agent's preference model.

In the following I assume that the private agent's utility discount rate $\delta$ as well as his intertemporal elasticity of substitution ($IES$) have already been established, whereas his degree of temporal risk aversion $k$ is yet unknown, but is inferable from the risk-free rate of return. Standard asset pricing theory defines the risk-free rate of return $r_f$ by the fundamental pricing equation for an asset with certain payoff and price $p_t = 1$, i.e. by

$$p_t = 1 = E\left[M_{t+1}\left(1 + r_f\right)\right], \tag{4.5}$$

where $M_{t+1}$ is called a pricing kernel. The pricing kernel in turn is defined by

$$E\left[M_{t+1}\right] = \frac{\partial U(x)/\partial x_{t+1}}{\partial U(x)/\partial x_t}, \tag{4.6}$$

which is simply the marginal rate of substitution between consumption in periods $t$ and

$t + 1$.[4] Combining equations (4.5) and (4.6), and setting $t = 1$, yields an equation for the risk-free rate of return, and reveals its equivalence to the instantaneous consumption discount rate as defined in (4.2):

$$r_f = -\ln \frac{\partial U_1(x)/\partial x_2}{\partial U_1(x)/\partial x_1} = DR_{1,2}.$$

Given the risk-free rate of return $r_f$ and the preference parameters $(\delta, IES)$, we can employ the private agent's discounting function (equation 4.3 with $i = P$) to solve for his degree of temporal risk aversion:

$$k^* = \arg\min \left[\left| r_f - DR_{1,2}^{\bar{T}^P} \right|\right]. \tag{4.7}$$

In the numerical analysis below, the private agent's preference tuple $(\delta, IES, k^*)$ is employed to describe the social agent's preferences, and thus to compute the social agent's instantaneous consumption discount rate $DR_{1,2}^{\bar{T}^S}$.

## 4.3 Numerical analysis

### 4.3.1 Computational procedure

The numerical analysis is conducted in three steps which are computed in MATLAB. First, a consumption tree similar to the one illustrated in figure 4.1 is created. Such a consumption tree represents the probabilistic evolution of the economy and thus the circumstances accounted for by an agent when he determines the value of future consumption. Second, under usage of the consumption tree and for exogenously given pairs $(\delta, IES)$, the private agent's preference model with horizon $\bar{T}^P$ is calibrated to the market rate of return. This second step yields $k^*$, such that the tuple $(\delta, IES, k^*)$ can be employed as a preference descriptor for the social agent. Third, the social agent's preference model is parameterized by $(\delta, IES, k^*)$ and the respective instantaneous consumption discount rate of the social agent is derived for a domain of integer valued horizons $\bar{T}^S \in \left[\bar{T}_{\min}^S, ... \bar{T}_{\max}^S\right]$.

---

[4]See e.g. Cochrane's (2005) textbook on asset pricing.

Before I describe these three steps in detail, note that the numerical analysis of this chapter is subject to some computational restrictions. The very existence of the horizon effect, i.e. the dependence of the instantaneous consumption discount rate on circumstances in periods $t > 2$, requires that the probabilistic evolution of the economy up until horizon $\bar{T}^i$ is accounted for in the derivation of any $DR_{1,2}^{\bar{T}i}$. Thus, the size of the created consumption tree must conform with the longest possible horizon, namely with $\bar{T}_{\max}^S$. Furthermore, the presence of *iid* risk on consumption growth in every period $t \geq 2$, and in particular the implied presence of non-*iid* risk on consumption levels, leads to a fanning-out of the consumption tree in every period. The degree to which this fanning out expands the size of the consumption tree depends on the number of states $(n)$ in each uncertain period. To cut a long story short, note that the size of the consumption tree, and thus the dimension of the matrix which contains this tree, increases rapidly in $n$ and $\bar{T}_{\max}^S$.[5]

The numerical analysis is thus restricted in the sense that $n$ and $\bar{T}_{\max}^S$ need to be relatively small. In order to approximate the distribution of the random variable *consumption growth* to a satisfactory degree in spite of the restriction on $n$, I employ the integration method Gaussian quadrature. Contrary to the more standard Monte Carlo procedure, Gaussian quadrature requires a very small number of 'draws' $n$ to approximate a given distribution. I explain this technique in more detail below. In order to cover horizon lengths that are suitable for a social agent, I consider periods of 10 year length. Thus, a social agent with horizon $\bar{T}^S = 100$ years takes into account the circumstances of an economy (a consumption tree) with 10 periods.

**Step 1: Creation of the consumption tree**

A consumption tree as the one illustrated in figure 4.1 reflects the economic circumstances in periods 1 and 2, but also describes the evolution of the economy in periods $t > 2$. The

---

[5] In the main analysis, I conisder $n = 9$ and 10 periods. This assumption results in a consumption tree with $\sum_{t=1}^{10} 9^{t-1} = 435\ 848\ 050$ nodes, which is saved in a matrix of dimension ($387\ 420\ 489 \times 10$). Since the memory on my standard laptop (5.8 GB) is too small to compute and work with matrizes of such size, I resort to ETH's remote desktop system through which I can access a MATLAB application that computes on a machine with 16 cores, 2.6 GHz, and 256 GB memory. However, even on this machine, increasing $n$ or $\bar{T}_{\max}^S$ beyond the assumptions in the main analysis exceeds the computational capacity. I evolve on this issue in appendix 4.A.2.

consumption levels in all periods and all states are inferred from the tuple $(x_1, \mu, \sigma^2)$. The temporal length of a consumption tree is determined by the horizon $\bar{T}^i$ of the agent who evaluates the respective economy. To accommodate the horizon of the private agent as well as a domain of horizons of the social agent, I create a consumption tree that corresponds to the longest horizon of the social agent $(\bar{T}^S_{\max})$ and consider subsets of this tree for shorter horizons.

Between any two consecutive consumption nodes in the tree, consumption growth is uncertain and governed by a normal distribution. The method I employ to approximate this distribution is Gaussian quadrature. As already mentioned, the advantage of this method, in comparison to the standard Monte Carlo procedure, is that a relatively small number of draws is sufficient to approximate a probability distribution to a satisfactory degree.[6] Gaussian quadrature can be understood as a discretization of the state-space of a random variable $\tilde{z}$ with probability density function $w(\tilde{z})$, such that $n$ mass points $z_i$, $i = 1, ...n$, weighted by the quadrature weights $w_i$, represent the distribution.[7] The expectation of a function $f(\tilde{z})$ is then approximated by a weighted sum, such that

$$E\left[f\left(\tilde{z}\right)\right] = \int f\left(z\right) w\left(z\right) dz \approx \sum_{i=1}^{n} w_i f\left(z_i\right).$$

To compute the mass points $z_i$ and the respective weights $w_i$ of the normally distributed random variable $\tilde{g} \sim N\left(\mu, \sigma^2\right)$ at $n$ quadrature nodes, I borrow a routine ($qnwnorm.m$) from Miranda and Fackler's (2002) CompEcon Toolbox.[8] The $n$ mass points $z_i$ ($n$ 'draws' from the distribution of $\tilde{g}$) are employed for the creation of the consumption tree; the weights $w_i$ are used for building the expectations in the next two steps.

The output of step 1 is a matrix $X^{mat}$ of dimension $(n^{rndt} \times \bar{T}^{d,S}_{\max})$ where $rndt$ corresponds to the number of periods in which growth is uncertain and $\bar{T}^{d,S}_{\max}$ is the longest horizon (in terms of periods) considered by the social agent. The matrix $X^{mat}$ con-

---

[6]Under a Monte Carlo procedure, $N$ numbers are drawn from the state-space of a random variable through a random number generator; each number is connected to a probability of occurrence of $\frac{1}{N}$. Since the Monte Carlo procedure builds on the Law of Large Numbers, the number of draws must be large in order to sufficiently approximate a given distribution (see Miranda and Fackler 2002). As explained above, a large number of draws is problematic for the purpose of this paper, since uncertainty exists not only in one period but in several periods. Employing Monte Carlo for the numerical analysis of this chapter would thus imply an enormous consumption tree.

[7]See, e.g., Miranda and Fackler (2002), who explain Gaussian quadrature at length.

[8]The toolbox can be downloaded from http://www4.ncsu.edu/~pfackler/compecon/toolbox.html.

tains deterministic consumption in the initial period $(x_1)$, as well as the uncertain and correlated consumption levels $(\tilde{x}_t)$ in the remaining nodes of the consumption tree.

## Step 2: Calibration to the market rate of return

The purpose of the second step is to find the private agent's degree of temporal risk aversion $k^*$ for a given pair $(\delta, IES)$. To this end, but also for the purpose of the next step, I developed a code which accesses the consumption tree from step 1 to solve recursively for the instantaneous consumption discount rate of a Risk-Sensitive agent with horizon $\bar{T}^i$. In order to compute the instantaneous consumption discount rate of a private agent (equation 4.3 with $i = P$), this code is parameterized by the private agent's horizon $\bar{T}^P$, by his preference pair $(\delta, IES)$, and by some starting value for the degree of temporal risk aversion. The private agent's actual degree of temporal risk aversion $k^*$ is then determined through calibrating the private agent's discount rate to the risk-free rate of return, i.e. through a Newton approximation which solves expression (4.7). The Newton approximation is conducted through MATLAB's routine *fsolve*, which is a standard tool to efficiently solve (systems of) nonlinear equations.

The output of step 2 is the private agent's degree of temporal risk aversion $k^*$ which pertains to a given specification of the preference parameters $(\delta, IES)$, to a specific horizon $\bar{T}^P$, and to an economy described by $\left(r_f, x_1, \mu, \sigma^2\right)$.

## Step 3: Derivation of the social agent's discount rate

Given the consumption tree created in step 1 and the private agent's preference parameters $(\delta, IES, k^*)$, we are ready to conduct the third and final step of the numerical analysis. First, the social agent's discounting function (equation 4.3 with $i = S$) is parameterized 'descriptively' by the private agent's preference tuple $(\delta, IES, k^*)$. Second, the code for the recursive solution of the consumption discount rate, as already developed for step 2, is employed to derive $DR_{1,2}^{\bar{T}^S}$ on a domain of horizons $\bar{T}^S \in \left[\bar{T}_{\min}^S, ... \bar{T}_{\max}^S\right]$. For any $\bar{T}^S$, a subset of the consumption tree which corresponds to the respective horizon length is accessed.

The output of this third and final computational step is the social agent's instantaneous consumption discount rate $DR_{1,2}^{\bar{T}^S}$ given different horizon lengths $\bar{T}^S$.

### Conversion of annual to decadal variables

The choice of parameter values presented in the next subsection is stated in terms of annual variables. However, a period $t$ must not necessarily correspond to a year. In fact, for the computational restrictions outlined above, I consider periods of 10 years (one decade) throughout the main analysis of this chapter.[9] Thus, the annual (''$a$'') parameter values presented in the next subsection must be converted to decadal (''$d$'') values before they can serve as an input to the numerical analysis. The following equations describe this conversion:

$$
\begin{aligned}
x_1^d &= x_1^a \cdot \sum_{j=1}^{10} (1+g^a)^{j-1} \\
r_f^d &= \left(1+r_f^a\right)^{10} - 1 \\
g^d &= (1+g^a)^{10} - 1 \\
\mu^d &= (1+\mu^a)^{10} - 1 \\
\left(\sigma^d\right)^2 &= 10 \cdot (\sigma^a)^2.
\end{aligned}
$$

The variable $x_1^d$ is the certain consumption level in the first decade, $r_f^d$ and $g^d$ are the decadal rate of return and consumption growth, respectively, and $\mu^d$ and $\left(\sigma^d\right)^2$ are the moments of the normally distributed decadal growth rate $\tilde{g}^d$. The respective annual economic variables are $x_1^a$, $r_f^a$, $g^a$, $\mu^a$ and $(\sigma^a)^2$.

Furthermore, the annual utility discount rate $\delta^a$ is transformed into its decadal analogue $\delta^d$ before it is used as an input in the numerical analysis; and the decadal consumption discount rates of the social agent, which constitute the output of the numerical analysis, are converted back to annual values before being presented as results:

$$
\begin{aligned}
\delta^d &= (1+\delta^a)^{10} - 1 \\
DR^a &= \left(1+DR^d\right)^{1/10} - 1.
\end{aligned}
\tag{4.8}
$$

---

[9]Also the side analyses two, three and four (appendizes 4.A.2, 4.A.3, 4.A.4) are conducted for periods of 10 year length. The period length in side analysis one is 1 year (appendix 4.A.1).

### 4.3.2 Parameter choices

The following parameter choices are those employed in the main analysis, the results of which are presented in section 4.4. In addition to the main analysis, I conduct four side analyses. The parameter choices and results of the side analyses are presented and discussed in the appendix. In side analysis one (appendix 4.A.1), I consider a setting with an exogenous degree of temporal risk aversion ($k^{ex}$). The purpose of this side analysis is first, to check whether the numerical analysis of the present chapter complies with the theoretical predictions of chapter 3, and second, to gain some insights on the comparative statics with respect to varying pairs ($\delta^a, IES$). In side analyses two, three and four, I analyze deviations from the main analysis with respect to the number of Gaussian quadrature nodes $n$ (appendix 4.A.2), the horizon of the private agent $\bar{T}^P$ (appendix 4.A.3), and the risk-free rate $r_f^a$ (appendix 4.A.4). Table 4.1 below summarizes the parameter choices in the main as well as in the side analyses.

**Choice of $x_1^a$, $\mu^a$, and $(\sigma^a)^2$:** The initial (first year) annual consumption level is set to $x_1^a = 1$, which is an arbitrary choice from $\mathbb{R}_+$. The particular value of $x_1^a$ does not influence the results as long as it is strictly positive. The moments of the normally, independently, and identically distributed annual consumption growth rate, $\tilde{g}^a \sim N\left(\mu^a, (\sigma^a)^2\right)$, are chosen in line with Kocherlakota (1996). In this contribution on the equity premium puzzle, Kocherlakota uses annual US data from 1889-1978 to estimate the moments of the growth rate of per capita real consumption; he finds $\mu^a = 0.018$ and $(\sigma^a)^2 = 0.00127$. In selecting these variables, I follow the discounting literature, in particular Traeger (2011) and Gollier (2002a), who also cite Kocherlakota (1996) as the source of their assumptions on the moments of $\tilde{g}^a$.

**Choice of $r_f^a$:** One common approach to the parameterization of the risk-free rate $r_f^a$ is to choose it according to the returns to short term (typically 3-months) US Treasury bills. This is also the approach taken by Kocherlakota (1996), who finds that the average real rate of return to bonds is 1%.[10] Another common approach is to consider the returns to 10-year US Treasury notes. Historical (1871-2013) data for these returns is available

---

[10] In estimating this value, Kocherlakota (1996) considers 90-day US Treasury bills from 1931-1978, Treasury certificates from 1920-1931, and 60-day to 90-day Commerical Paper before 1920.

on the homepage of Robert Shiller.[11] From this data, I calculate the average real rate of return to 10-year US Treasury notes to be 2.4%.[12] Taking a pragmatic approach, I choose $r_f^a$ to be 'somewhere in the middle' and set it to $r_f^a = 0.014$ in the main analysis of this chapter.[13]

**Choice of $\bar{T}^P$ and $\bar{T}^S$:** The private agent's horizon is set to $\bar{T}^P = 30$ years (3 decades/periods). The underlying rationale for this choice is the rather intuitive idea that an average agent who invests at the risk-free rate $r_f^a$ takes into account a remaining life-span of 30 years when assessing his investment possibilities.[14] Regarding the social agent's horizon $\bar{T}^S$, I consider a domain of integer values which are separated according to the period length of 10 years. In particular, I consider the domain $\bar{T}^S \in [20, 30, ...100]$ years $(2, 3, ...10$ decades/periods). Note that the main interest of this chapter is in those cases for which $\bar{T}^S > \bar{T}^P$. Nonetheless, the cases $\bar{T}^S \leq \bar{T}^P$ are not excluded from the results.

**Choice of $\delta^a$ and $IES$:** The purpose of the main analysis is to examine the (descriptive) consumption discount rate of a Risk-Sensitive social agent under different assumptions on the pair $(\delta^a, IES)$. To this end, I consider two different values for the utility discount rate, namely a 'low' value of $\delta^a = 0.001$ and a 'high' value of $\delta^a = 0.002$. These utility discount rates are combined with three different values of the intertemporal elasticity of substitution, namely $IES = 0.9$, $IES = 1$, and $IES = 1.1$.[15] This makes for six different parameter specifications over which the comparisons in the main as well as in the side analyses are conducted.[16]

---

[11] http://www.econ.yale.edu/~shiller/data.htm

[12] The data contains the consumer price index as well as the nominal 10-year Treasury rate for the period 1871-2013. I thus derive the inflation rates in the years 1872-2013, the real returns in the respective years, and the average rate of return over the respective period.

[13] In side analysis four (appendix 4.A.4), I consider $r_f^a = 0.02$. Note, however, that side analysis four is not a robustness test, since I change the assumptions on the utility discount rate $\delta^a$ in accordance.

[14] As the specific choice of 30 years is clearly arbitrary, I consider two alternatives, namely $\bar{T}^P = 20$ years and $\bar{T}^P = 40$ years in side analysis three (appendix 4.A.3).

[15] The $r_f^a = 0.02$ case examined in appendix 4.A.4 considers $\delta^a \in [0.0077, 0.0087]$ in combination with $IES \in [0.9, 1, 1.1]$.

[16] These parameter choices are artificial rather than well founded by empirical observations. However, let me highlight that the purpose of the main analysis is not to determine 'the correct' discount rate, but to compare $DR_{1,2}^a \left( \bar{T}^S \right)$ for different pairs $(\delta^a, IES)$. Thus, these paramters are chosen as to fit this practical purpose. In particular, $\delta^a$ and the $IES$ are chosen such that they accomodate $k^* > 0$ under all specifications, and such that the $k^*$ that evolve under different specifications are not too different. The latter criterion requires that the differences between the parameter values in the various specifications are rather small.

|  | main analysis | one | two | three | four |
|---|---|---|---|---|---|
| **economic parameters** |  |  |  |  |  |
| $x_1^a$, initial consumption | 1 | ............. | .......... | .......... | ................... |
| $\mu^a$, growth rate mean | .18 | ............. | .......... | .......... | ................... |
| $(\sigma^a)^2$, growth rate variance | .00127 | ............. | .......... | .......... | ................... |
| $r_f^a$, market rate of return | .014 | - | .......... | .......... | .02 |
| **preference parameters** |  |  |  |  |  |
| $\bar{T}^P$, private horizon (years) | 30 | - | .......... | 20, 40 | ................... |
| $\bar{T}^S$, social horizon (years) | 20,...100 | 2,...1000 | .......... | .......... | ................... |
| $\delta^a$, utility discount rate | .001, .002 | ............. | .......... | .......... | .0077, .0087 |
| $IES$, int. elast. of substit. | .9, 1, 1.1 | ............. | .......... | .......... | ................... |
| $k^{ex}$, exog. risk aversion | - | .02 | - | - | - |
| **other parameters** |  |  |  |  |  |
| period length (years) | 10 | 1 | .......... | .......... | ................... |
| $n$, quadrature nodes | 9 | ............. | 7, 8, 9 | .......... | ................... |

Table 4.1: Parameter choices in the main analysis and in the side analyses.

**Choice of** $n$: The number of Gaussian quadrature nodes is set to $n = 9$. This is the highest number of nodes for which the analyses can be conducted, given the memory restrictions of the employed machine.[17]

Table 4.1 summarizes the (annual) parameter values that are employed in the main and in the side analyses. Regarding the parameter values in the side analyses, only those choices which differ from the ones in the main analysis are presented; hence '.....' should be read as 'as in the main analysis'. Note that the variable $k^{ex}$ (exogenous risk aversion) plays a role only in side analysis one. In all other analyses, $k$ is determined endogenously and denoted as $k^*$.

---

[17]See appendix 4.A.2 for a more detailed discussion of this issue and for a sensitivity analysis with $n \in [7, 8]$.

## 4.4 Results of the main analysis

### 4.4.1 The degree of temporal risk aversion

The calibration of the private agent's discounting function to the risk-risk free rate of return (step 2 of the computational procedure) yields the degree of temporal risk aversion $k^*$. The results of this calibration for each of the six specifications of $(\delta^a, IES)$, denoted as (a, b, c, d, e, f), are presented in table 4.2. The results in table 4.2 can be compared in two directions: First, they can be compared horizontally, i.e. with respect to different values of the utility discount rate $\delta^a$; second, they can be compared vertically, i.e. with respect to different values of the intertemporal elasticity of substitution ($IES$).

The horizontal comparison of the results shows that $k^*$ increases in $\delta^a$. The insights from corollary 14, chapter 3, on the twofold role of the utility discount rate (the rate of pure time preference) suggest that this behavior is likely ascribable to two individual effects.[18] First, a higher utility discount rate implies that second period felicity is discounted more strongly, which leads to an increase in an agent's consumption discount rate. However, when calibrating the private agent's preference model, his consumption discount rate is matched to the risk-free rate, and thus required to remain constant. The positive effect from a higher $\delta^a$ must thus be counterbalanced by a respective change in the free parameter of the calibration procedure, namely $k^*$. Since the discounting function is decreasing in the degree of temporal risk aversion, $k^*$ must increase to counterbalance the effect of a higher $\delta^a$. Second, a higher utility discount rate implies that the horizon effect is 'discounted' more strongly, which induces a decrease in its absolute value.[19] This decrease in the absolute value of the horizon effect must again be counterbalanced by a respective change in $k^*$. Since the degree of temporal risk aversion amplifies the horizon effect, $k^*$ must again increase to counterbalance the effect of an increase in $\delta^a$. In the $IES = 1$ case, these two effects are directly observable in the private agent's Extended Ramsey Equation for RS preferences (equation 4.4 with $i = P$).

The vertical comparison of the results in table 4.2 shows that $k^*$ decreases in the in-

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[18] See also appendix 4.A.1, where I discuss the connection to corollary 14 of chapter 3 in more detail.

[19] Note that the private agent's discounting function (equation 4.3 with $i = P$) is subject to a horizon effect since his horizon of 3 periods (3 decades, 30 years) extends the period to which the discount rate applies, namely the second period.

|  | $\delta^a = 0.002$ | $\delta^a = 0.001$ |
|---|---|---|
| **IES = 0.9** | a:  3.5282 | b:  2.9484 |
| **IES = 1** | c:  2.0750 | d:  1.5700 |
| **IES = 1.1** | e:  1.1558 | f:  0.7006 |

Table 4.2: Endogenous degree of temporal risk aversion under different preference specifications, main analysis.

tertemporal elasticity of substitution. Albeit I have not conducted a sound theoretical analysis with respect to changes in the $IES$, this decrease in $k^*$ can be explained by means of 'informed intuition'. Such intuition suggests that the impact of the $IES$ on the consumption discount rate, and thus on an endogenously determined degree of temporal risk aversion $k^*$, is at least twofold. Consider first the impact of the $IES$ on the consumption discount rate in a setting without risk–note, however, that the described impact will also be present in a setting that involves risk on consumption growth. As described in chapter 1, an increase in the $IES$ implies that wealth differences between the first and the second period affect the discount rate less strongly. Given a positive consumption growth rate, an increase in the $IES$ thus goes along with a decrease in the consumption discount rate. Second, consider a setting with risk on consumption growth. In such a setting, the value of the $IES$ affects the size of the risk on continuation utilities. To see this, note that an increase in the $IES$ leads to a decrease in the curvature of the felicity function $u(x_t)$. A higher $IES$ thus implies that the difference between the (continuation) utility gained on a high-consumption path and that gained on a low-consumption path is relatively big. Thus, given that the high- and the low-consumption paths evolve from two realization of the uncertain growth rate, the risk on continuation utilities increases with the $IES$. A temporally risk averse agent objects to such increases in the risk on continuation utilities, and corresponds by a decrease in the consumption discount rate. Summing up, I explained that the $IES$ affects the consumption discount rate negatively through at least two channels. Hence, to match the consumption discount rate to a given risk-free rate in the calibration procedure, the negative effect of a higher $IES$ must be counterbalanced by the positive effect of a lower $k^*$.

### 4.4.2 The discount rate of the social agent

Figure 4.2 illustrates the effect of a social agent's horizon $\bar{T}^S$ on his instantaneous consumption discount rate. Each of the six graphs corresponds to a different specification of the tuple $(\delta^a, IES, k^*)$, where $k^*$ is determined endogenously as outlined above. The abscissas refer to the domain of horizons of the social agent, namely $\bar{T}^s \in [20, 30, ...100]$ years. Each 10-year segment on the abscissas $(11 - 20, 21 - 30, ...91 - 100)$ pertains to one period/decade. The ordinates map the annual instantaneous consumption discount rate of the social agent, which is a function of the horizon over which his preferences are defined. Note that these annual discount rates are derived from their decadal analogues through equation (4.8). A decadal instantaneous consumption discount rate is the rate at which consumption in the second decade (years $11 - 20$) is discounted relative to consumption in the first decade (years $1 - 10$). The stair-shape of the lines in figure 4.2 is due to this decadal analysis.

The blue lines in the graphs depict the mapping $\bar{T}^S \rightarrow DR_{1,2}^a\left(\bar{T}^S\right)$ under a numerical solution method (described in section 4.3.1, computational procedure); the red lines apply to the analytical solution of $DR_{1,2}^a\left(\bar{T}^S\right)$ which can only be derived in the $IES = 1$ case (see equation 4.4, Extended Ramsey Equation for Risk-Sensitive preferences). The analytical solutions (the red lines in graphs c and e) employ the $k^*$ from the respective numerical solution.[20] The small difference between the red and the blue lines in graphs c and d is due to the log approximation that has been employed for the derivation of the analytical solution.[21]

To ensure the correct interpretation of the figure, consider the graph of specification e. This graph depicts the discount rate of a social agent whose Risk-Sensitive preferences are described by $\delta^a = 0.002$, $IES = 1.1$ and $k^* = 1.1558$. Suppose that the agent is furthermore specified by a horizon of $\bar{T}^S = 60$ years. Given such a horizon, the

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[20] Alternatively, one could derive the degree of temporal risk aversion from the analytical solution itself. Rearranging the Extended Ramsey Equation for RS preferences (equation 4.4 with $i = P$) yields $k^* \approx \left[\mu - r_f - \ln\beta - \frac{\sigma^2}{2}\right] \Big/ \left[\sigma^2\left(1 + \beta\sum_{\tau=3}^{\bar{T}^P}\beta^{\tau-3}\right)\right]$.

[21] Note that the log approximation justifies a small upward bias of the analytical relative to the numerical solution. This bias is visible at $\bar{T}^S = 30$. Potential reasons for the not-exactly-parallel appearance of the red and the blue lines are an amplification of the log approximation error as horizons $\bar{T}^S \neq 30$ (with uncertainty in every period) are considered, and the inexactness of the blue line due to the approximation of the normal distribution (through Gaussian quadrature).
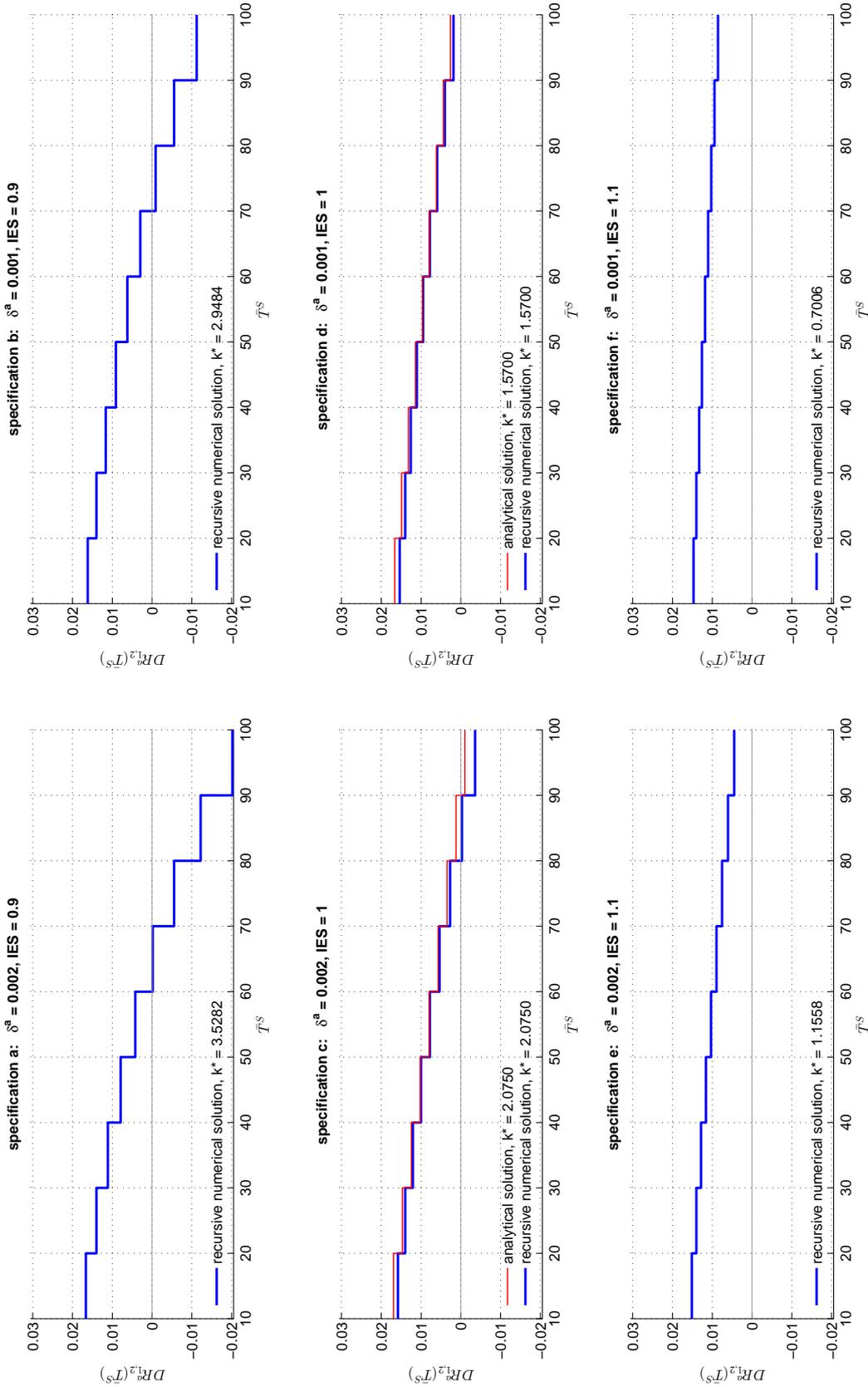
Figure 4.2: Results of the main analysis with $r_f^a = 0.014$, $\bar{T}^P = 30$ years, $n = 9$, and endogenously determined $k^*$.

*decadal* discount rate is specified by $DR_{1,2}^d(6)$, where '6' refers to the horizon in terms of decades/periods. From the blue line in graph e, in particular from the fifth step on this line, we can infer that this decadal discount rate corresponds to an annual instantaneous consumption discount rate of $DR_{1,2}^a(60) \approx 1\%$.

In all of the six graphs, we can observe how the assumption on the private agent's horizon is reflected in the social agent's discount rate. For a horizon $\bar{T}^S = \bar{T}^P = 30$ years, the social agent is exactly equivalent to the private agent, and his discount rate therefore corresponds to the risk-free rate of return: $DR_{1,2}^a(30) = r_f^a = 1.4\%$. If the horizon of the social agent is below that of the private agent, we observe that the social agent's discount rate is higher than $r_f^a$. In the more interesting and realistic cases in which $\bar{T}^S > \bar{T}^P = 30$, the social agent's discount rate is lower than $r_f^a$.

This decrease of the instantaneous consumption discount rate relative to the risk-free rate of return is due to the horizon effect. A temporally risk averse Risk-Sensitive agent $i$ takes into account that the risk on period 2 consumption growth translates into correlated risk on the consumption levels in periods $2, 3, ... \bar{T}^i$. Thus, if a bad state of the world (a low consumption growth rate) realizes in period 2, an entirely lower remaining consumption path will be implied. The private Risk-Sensitive agent only fears 30 years of consecutively low consumption levels. The risk-free rate of return reflects this prospect of a relatively short time-span of suffering. The social agent, in contrast, takes into account that his protégés (society) may have to suffer for a considerably longer time-span. As a result of this more 'altruistic' perspective, he discounts future consumption at a rate below $r_f^a$, that is, he values future consumption more than implied by the risk-free rate.

The comparison of the graphs in figure 4.2 suggests that the significance of the horizon effect varies largely across different specifications of the parameters $(\delta^a, IES, k^*)$. In specifications a and b, the horizon effect is most pronounced: For a horizon of $\bar{T}^S = 100$, it leads to consumption discount rates that are as low as $-2\%$ or $-1\%$. In specifications c and d, the discount rate for a horizon of $\bar{T}^S = 100$ years lies around $0\%$. In specifications e and f, the negative impact of the horizon effect is comparatively small for all $\bar{T}^S > 30$, and the social agent's discount rate lies therefore not so much below the risk-free rate of return. A careless glimpse on figure 4.2 may suggest that the absolute value of the horizon effect is decreasing in the $IES$ and increasing in $\delta^a$. This inference is misleading,

however. A low $IES$ and a high $\delta^a$ imply a high degree of temporal risk aversion $k^*$, as explained in the previous discussion of the calibration results. It is this high degree of $k^*$ which brings about a more pronounced horizon effect, and thus a lower consumption discount rate. In fact, lowering the $IES$ or increasing $\delta^a$ for a constant degree of temporal risk aversion has just the opposite effect, i.e. it diminishes the horizon effect. This is suggested by the results of side analysis one, which are presented and discussed in appendix 4.A.1.

From the graphs of specifications a, b and c we can observe that the horizon effect may imply negative consumption discount rates if the horizon $\bar{T}^S$ is long enough, and if the parameters $(\delta^a, IES)$ are such that a relatively high $k^*$ is implied. An agent who is characterized by a negative consumption discount rate values consumption in the future more than consumption in the present. In the standard theory of consumption discounting (based on the standard additive expected utility model, see chapter 1), this is a rather unlikely outcome. There, consumption discount rates can only be negative if the utility discount rate is negative, if consumption grows at a negative rate, or if the risk on future consumption is truly enormous. When the consumption discount rate is based on an agent with Risk-Sensitive preferences, however, we saw in this chapter that the discount rate may be negative although the utility discount rate and the consumption growth rate are positive, and although the variance of consumption growth lies within a standard range.

## 4.5 Conclusion

In this chapter I conducted a numerical examination of a Risk-Sensitive social agent's instantaneous consumption discount rate. In doing so, I assumed a descriptive approach to social discounting. I postulated the existence of a private agent, whose preferences are revealed through the prices in the financial markets, in particular through the risk-free rate of return; and a social agent, whose preferences are aligned along those of the private agent. Crucially, I assumed that the private and the social agent differ with respect to the temporal horizon over which their preference models are defined. In particular, the private agent takes into account his remaining lifetime when conducting

investment decisions, whereas the social agent accounts for a considerably longer horizon when evaluating investments for the benefit of society as a whole.

The numerical analysis involved two main steps. First, the private agent's preference model was calibrated to the risk-free rate of return in order to elicit his degree of temporal risk aversion $(k^*)$. I conducted this calibration for six different combinations of the utility discount rate $(\delta)$ and the intertemporal elasticity of substitution $(IES)$. It was shown that, to match a given risk-free rate of return, the private agent's degree of temporal risk aversion must increase in the utility discount rate, and decrease in the intertemporal elasticity of substitution. Thus, the risk-free rate of return may evolve from the preference model of a Risk-Sensitive private agent with, for example, a high $\delta$, low $IES$, and high $k^*$; or from a preference model with low $\delta$, high $IES$, and low $k^*$.

Second, I employed the six specifications of the Risk-Sensitive private agent's preference tuple $(\delta, IES, k^*)$ to parameterize the Risk-Sensitive social agent's preference model, and thus his instantaneous consumption discounting function. Each of the resulting six specifications of the social agent's discounting function was then examined with respect to its dependence on the social agent's horizon length. It was shown that, for each of the specifications, the social agent's instantaneous consumption discount rate decreases in the horizon. The extend of this decrease, i.e. the magnitude of the horizon effect, however, was shown to vary considerably across the six specifications. For specifications with low $\delta$, high $IES$, and low $k^*$, the horizon effect turned out to have a rather small, yet still non-negligible effect on the social agent's consumption discount rate. In contrast, for specifications with high $\delta$, low $IES$ and high $k^*$, the consumption discount rate was reduced considerably relative to the risk-free rate. Given long horizons, the social agent's instantaneous consumption discount rate may even be decreased to negative levels through the horizon effect.

In summary, the analysis of the present chapter showed that, under a descriptive approach to social discounting and given temporally risk averse Risk-Sensitive private and social agents, there may be reasons to discount future costs and benefits at a consumption discount rate that lies (considerably) below observed market rates of return. The reason for such a decrease in the consumption discount rate relative to the risk-free rate of return is the horizon effect–the notion that a social agent with a long horizon fears

persistence in low consumption levels more than a private agent who takes a relatively short horizon into account. Whether the horizon effect reduces the social agent's discount rate to a large or only to a small extend, depends crucially on the underlying preference assumptions and on the horizons postulated for the private and the social agent.

# Appendices for chapter 4

## 4.A  Side analyses

### 4.A.1  Side analysis one

In this side analysis, the degree of temporal risk aversion is given exogenously and is assumed to be equivalent across the six specifications (a, b, c, d, e, f). This assumption of equivalence enables a ceteris paribus comparison between specifications with different utility discount rates, $\delta^a \in [0.001, 0.002]$; and between specifications with different intertemporal elasticities of substitution, $IES \in [0.9, 1, 1.1]$. Such a ceteris paribus comparison is not possible in the main analysis, since there the degree of temporal risk aversion $k^*$ is determined endogenously, which implies that any $(\delta^a, IES)$ pair is connected to a different $k^*$.

Throughout the present side analysis, I consider a social agent with a domain of horizons $\bar{T}^S \in [2, 3, ...1000]$ who has Risk-Sensitive preferences (equation 4.1 with $i = S$) and is specified by an exogenously given degree of temporal risk aversion $k^{ex} = 0.02$.[22,23] Further assumptions are a period length of 1 year, an initial level of consumption $x_1 = 1$, and a number of Gaussian quadrature nodes of $n = 9$. Note that, since $k^{ex}$ is an exogenous parameter, there is no role for a private agent and a risk-free interest rate.

Figure 4.3 illustrates the workings of the horizon effect within the framework of this side

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[22]Note that it is possible to consider such long horizons in this side analysis since I do not assume that each period is subject to uncertainty on the growth rate. This assumption is made in the main analysis, however, where it restricts the size of the highest horizon $\bar{T}^S_{\max}$ that can be considered. I explained earlier that this restriction is due to computational constraints on the consumption tree.

[23]The choice of $k^{ex} = 0.02$ is rather arbitrary; the only necessary requirement is that it is greater than zero, i.e. that the social agent is strictly tempoally risk averse.
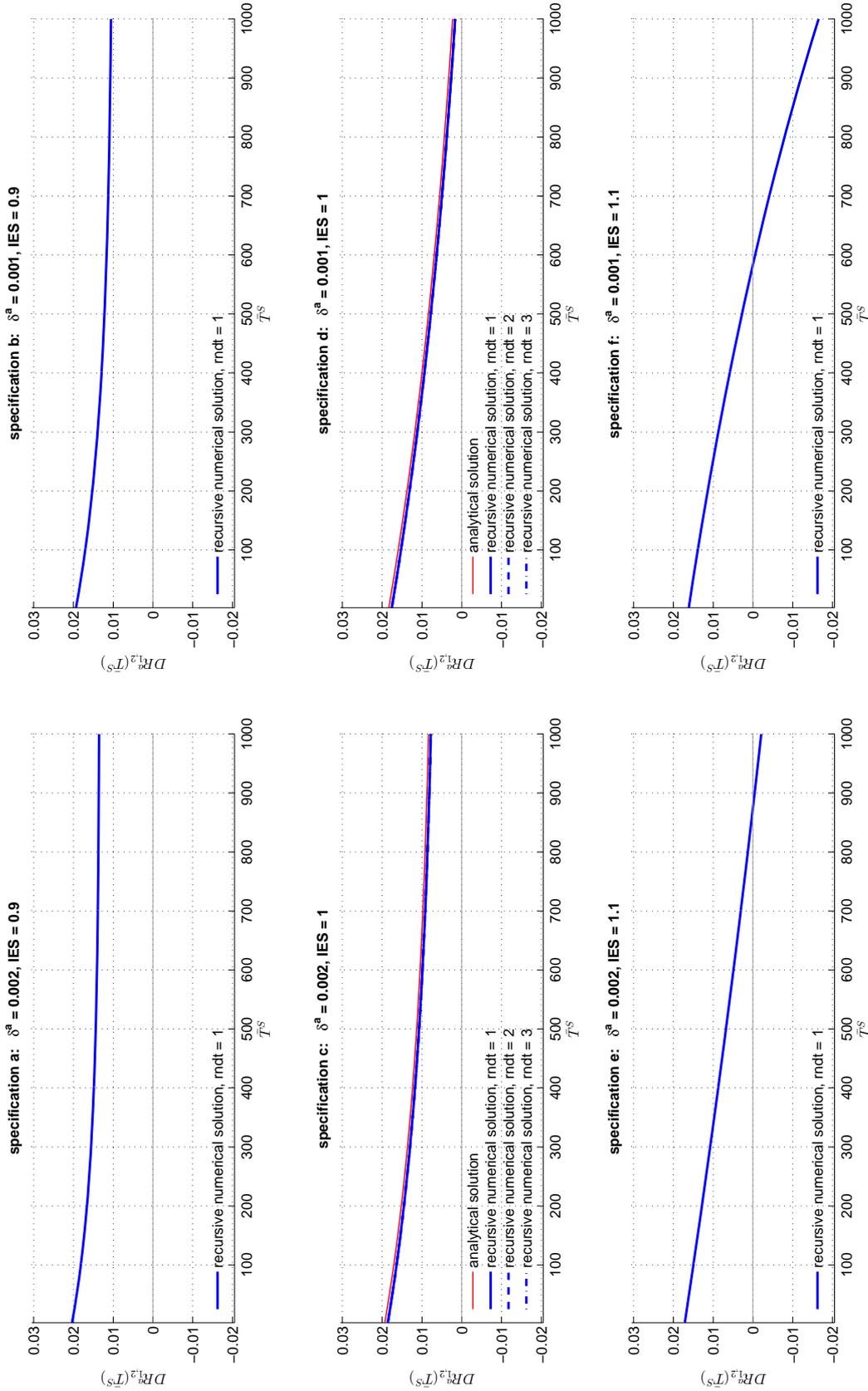
Figure 4.3: Results of side analysis one with $n = 9$ and exogenous $k^{ex} = 0.02$.

analysis. As in the main analysis, each of the six graphs concerns a different $(\delta^a, IES)$ pair, i.e. one of the specifications (a, b, c, d, e, f). The abscissas refer to the domain of horizons $\bar{T}^S$ (in years) of the Risk-Sensitive agent and the ordinates refer to his annual instantaneous consumption discount rate $DR_{1,2}^a (\bar{T}^S)$. Note that the very first value on the abscissas, and thus the very first point at which the lines are drawn, corresponds to $\bar{T}^S = 2$. For $\bar{T}^S = 2$, $DR_{1,2}^a$ is free from a horizon effect by construction, since the horizon is cut off after the period to which the discount rate applies.

**Compliance with the theory of chapter 3**

As an initial step, I check whether the results from this side analysis comply with the theoretical insights of chapter 3, in particular (i) with proposition 12 (existence and direction of the horizon effect), (ii) with definition 13 (the Extended Ramsey Equation for Risk-Sensitive Preferences) and (iii) with corollary 14 (a twofold role of the rate of pure time preference).

(i) compliance with proposition 12, chapter 3

Proposition 12 of chapter 3 concerns a Risk-Sensitive (social) agent who is strictly temporally risk averse. As in the present analysis, the postulated felicity functions $u(\cdot)$ are of the CES type. Proposition 12 states that the instantaneous consumption discount rate of such an agent is decreased by a horizon effect if $\bar{T}^S > 2$ (i.e. if the horizon of the agent extends the period of discount), and if the conditions of either of two specified settings are met.[24] The first setting (statement 1, proposition 12) pertains to an agent with $u(x_t) = \frac{x_t^\rho - 1}{\rho}$ ($\rho < 1$, $IES > 0$) and an economy in which only consumption growth in the second period is uncertain. The second setting (statement 2, proposition 12) involves an agent with $u(x_t) = \ln x_t$ ($\rho = 0$, $IES = 1$) and an economy in which growth is uncertain and independently distributed in every period except the first. To check for the compliance with statement 1, I consider specifications (a, b, c, d, e, f) and assume $rndt = 1$ (number of periods with uncertain growth $= 1$), in particular $\tilde{g}_2 \sim N(0.018, 0.00127)$ and $g_t = 0.018 \ \forall \ t > 2$. To check for the compliance with statement 2, I consider the $IES = 1$ specifications (c, d) and three different economic

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[24] To be exact, the proposition only states that the discount rate for $\bar{T} = \infty$ is smaller than the discount rate for $\bar{T} = 2$. The more general statement, namely that the discount rate declines in $\bar{T}$, is evident from the proof of proposition 12, however. A similar remark is made in section 3.4.3 of chapter 3.

settings which represent special cases of the economy described in statement 2. In particular, I consider $rndt = 1$, $rndt = 2$, and $rndt = 3$, where $\tilde{g}_t \sim N\left(0.018, 0.00127\right)$ in $t = 2$, $t = 2, 3$, and $t = 2, 3, 4$, respectively. As in the first setting, I assume $g_t = 0.018$ in all periods $t$ which are not subject to growth uncertainty. The inspection of figure 4.3 shows that the instantaneous consumption discount rate decreases in $\bar{T}^S$, i.e. $DR_{1,2}^a(2) > DR_{1,2}^a\left(\bar{T}^S\right) \; \forall \; \bar{T}^S > 2$. This observation holds for all specifications, namely for $IES \neq 1$ with $rndt = 1$ as well as for $IES = 1$ with $rndt = 1, 2, 3$. This behavior of the numerical results complies with the theoretical prediction of proposition 12.

(ii) compliance with definition 13, chapter 3

Definition 13 of chapter 3 concerns the analytical solution of the discounting function $DR_{1,2}^a\left(\bar{T}^S\right)$, which can be derived in the $IES = 1$ case under employment of the log-approximation $\ln\left(1 + \tilde{g}\right) \approx \tilde{g}$. This analytical solution was denoted as the Extended Ramsey Equation for RS preferences, as restated in equation (4.4) of the present chapter. The red lines in the graphs of specifications c and d of figure 4.3 pertain to $DR_{1,2}^a\left(\bar{T}^S\right)$ as derived from such an analytical solution. The small difference between the analytical and the numerical solution that can be observed in these graphs is due to the log-approximation employed for the derivation of equation (4.4).[25] The three blue lines in the graphs of specifications c and d pertain to the recursive numerical solutions in the three economic settings described above, i.e. to $DR_{1,2}^a\left(\bar{T}^S\right)$ for an economy with $rndt = 1, 2$, or 3. Only one blue line is visible, however, since the three numerical solutions are exactly equivalent. This equivalence complies with the Extended Ramsey Equation for RS preferences, which is a function of growth uncertainty in period 2 and of the general horizon length $\bar{T}$, but is independent of the presence of growth uncertainty in periods $t > 2$.

(iii) compliance with corollary 14, chapter 3

Corollary 14 of chapter 3 concerns the role of the utility discount rate (rate of pure time preference) in the Extended Ramsey Equation for RS preferences. The standard role of the utility discount rate in a consumption discounting function is to reduce the value of

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[25] The log approximation $\ln\left(1 + \tilde{g}\right) \approx \tilde{g}$ is employed to derive the analytical solution of $DR_{1,2}^a\left(\bar{T}^S\right)$ but plays no role in the numerical solution. Given, e.g., $g = 0.018$, the true solution of $\ln\left(1.018\right)$ is 0.0178, rather than 0.018. Hence the analytical solution, which incorporates this approximation error, overstates the true growth rate in every state of the world. This, in turn, implies a slightly overstated consumption discount rate, as observable in the graphs of specifications c and d of figure 4.3.

second period consumption (increase the instantaneous consumption discount rate) due to a lower valuation of second period felicity relative to first period felicity. This standard effect can be observed in figure 4.3 by comparing the graphs of specifications c and d: The red line of specification c (high $\delta^a$) lies higher than the red line of specification d (low $\delta^a$) at all $\bar{T}^S$, in particular also at $\bar{T}^S = 2$ where no horizon effect applies. Corollary 14 points to a second role of the utility discount rate, which applies only in presence of a horizon effect. The effect of an increase in the utility discount rate in this second role is to decrease the absolute value of the horizon effect, and thereby increase the consumption discount rate. This effect can as well be observed by comparison of the two red lines: The red line of the high-$\delta^a$ specification c is less steep than that of the low-$\delta^a$ specification d. This behavior arises from a less pronounced horizon effect under the high-$\delta^a$ specification c.

In summary, it is indicated that the results from the numerical analyses of the present chapter comply with the theoretical insights of chapter 3. In particular, we have seen that (i) the instantaneous consumption discount rate of a Risk-Sensitive agent decreases in the horizon as predicted by proposition 12 of chapter 3; that (ii) $DR^a_{1,2}\left(\bar{T}^S\right)$ is independent of growth uncertainty in periods $t > 2$ if $IES = 1$, as suggested by the Extended Ramsey Equation for RS preferences (definition 13, chapter 3); and that (iii) the consumption discount rate in the $IES = 1$ case is affected positively by the utility discount rate through two channels, as already stated in corollary 14 of chapter 3.

**Comparative statics with respect to $\delta^a$ and the $IES$**

The equivalence of $k^{ex}$ across specifications (a, b, c, d, e, f) allows for a ceteris paribus comparison with respect to (i) a high and a low $\delta^a$, i.e. $\delta \in [0.001, 0.002]$, and (ii) with respect to an $IES$ that is below, equal to, or above 1, i.e. $IES \in [0.9, 1, 1.1]$. Since this comparison is based on a specific numerical rather than a general theoretical analysis, it should be regarded as suggestive rather than categorical.

Note that the discussion in this subsection is tantamount to the discussion of the calibration results in section 4.4.1. Whenever changes in $\delta^a$ and in the $IES$ affect $DR^a_{1,2}\left(\bar{T}^i\right)$, they must also affect the free parameter $k^*$ in a framework in which the consumption

discount rate is held constant and matched to the risk-free rate. Since the channels through which $(\delta^a, IES)$ affect $k^*$, and thus the channels through which $(\delta^a, IES)$ affect $DR^a_{1,2}\left(\bar{T}^i\right)$, have already been discussed at length in section 4.4.1, the exposition in this section is kept brief.

<u>(i) comparative statics with respect to $\delta^a$</u>

I compare $DR^a_{1,2}\left(\bar{T}^S\right)$ across the different specifications for $\delta^a$, i.e. I compare specification a to b, c to d, and e to f. Each of these comparisons along the graphs in figure 4.3 suggests that, for any $\bar{T}^S$, the instantaneous consumption discount rate is increasing in $\delta^a$. The insights from corollary 14 of chapter 3 suggest that this positive effect is likely attributable to the two roles of the utility discount rate: The standard role of the utility discount rate and its horizon effect specific role, as discussed at length in section 4.4.1.

<u>(ii) comparative statics with respect to the $IES$</u>

I compare $DR^a_{1,2}\left(\bar{T}^S\right)$ across the different specifications for the $IES$, i.e. I compare specification a to c and e, and specification b to d and f. These comparisons indicate that $DR^a_{1,2}\left(\bar{T}^S\right)$ is decreasing in the $IES$. As pointed out in section 4.4.1, 'informed intuition' suggest that this behavior is attributable to at least two channels through which the $IES$ affects the consumption discount rate: First, an increase in the $IES$ implies that wealth differences between the first and the second period affect the value of second period consumption less strongly, hence, given positive consumption growth, the consumption discount rate decreases. This standard effect of the elasticity of substitution is also present in a deterministic setting, as detailed at length in chapter 1. Second, an increase in the $IES$ goes along with an increase in the risk on continuation utilities. Given a temporally risk averse Risk-Sensitive agent, this increase in the risk on continuation utilities brings about a decrease in the consumption discount rate.

In summary, the comparative statics analysis with respect to $\delta^a$ and $IES$ shows that the instantaneous consumption discount rate $DR^a_{1,2}\left(\bar{T}^S\right)$ is (i) increasing in $\delta^a$ and (ii) decreasing in the $IES$. Both interdependencies are partly due to a standard effect which is also present in a deterministic setting, but are in addition attributable to an effect which is specific to a setting that involves uncertainty. Note again that, in contrary to the main analysis, the degree of temporal risk aversion $k^{ex}$ in the present side analysis was exogenously given and constant across all specifications.

|  | **dimension of** $X^{mat}$ | **size of** $X^{mat}$ | **computing time** |
|---|---|---|---|
| **n = 7** | (40 353 607 × 10) | 135 KB | 7 minutes |
| **n = 8** | (134 217 728 × 10) | 470 KB | 24 minutes |
| **n = 9** | (387 420 489 × 10) | 1 439 KB | 67 minutes |

Table 4.3: Computational implications of different $n$, side analysis two.

## 4.A.2 Side analysis two

The number of Gaussian quadrature nodes in the main analysis is set to $n = 9$. This value is chosen as it is the maximum number of nodes for which the computational procedure always runs without problems. For $n = 10$, the procedure is typically interrupted due to memory restrictions. The machine that I use to run the MATLAB codes possesses a memory of 256 GB, but is also employed by other users whose computations may interfere with the memory requirement of my analysis.

In this side analysis, I repeat the computations of the main analysis with $n \in [7, 8]$ in order to assess how sensitive the main analysis is to changes in the number of Gaussian quadrature nodes. All other parameters are as in the main analysis. Table 4.3 outlines the computational implications of the runs with different $n$. The computation time refers to the computation of steps 1, 2, and 3, as described in subsection 4.3.1, and the size of $X^{mat}$ (the matrix which contains the consumption tree) refers to a matrix that is saved as a sparse matrix with double precision.

Table 4.4 contains the output from calibrating the private agent's preference model to $r_f^a$ (step 2 of the computational procedure). The resulting degrees of temporal risk aversion $k^*$ are almost equivalent across the runs for different $n$. For specifications (b, c, d, e, f), the $k^*$ are exactly equivalent up to the fourth decimal point; for specification a, $k^*$ decreases somewhat in the fourth decimal point as $n$ is increased.

Figure 4.4 contains the graphs for the social agent's consumption discount rate under the different preference specifications (step 3 of the computational procedure). The solid blue lines pertain to the results from the main analysis; the dotted and dashed blue lines

|  | $\delta^a = 0.002$ | | | $\delta^a = 0.001$ | | |
|---|---|---|---|---|---|---|
| **IES = 0.9** | a: | $n = 7:$ | 3.5283 | b: | $n = 7:$ | 2.9484 |
|  |  | $n = 8:$ | 3.5282 |  | $n = 8:$ | 2.9484 |
|  |  | $n = 9:$ | 3.5282 |  | $n = 9:$ | 2.9484 |
| **IES = 1** | c: | $n = 7:$ | 2.0750 | d: | $n = 7:$ | 1.5700 |
|  |  | $n = 8:$ | 2.0750 |  | $n = 8:$ | 1.5700 |
|  |  | $n = 9:$ | 2.0750 |  | $n = 9:$ | 1.5700 |
| **IES = 1.1** | e: | $n = 7:$ | 1.1558 | f: | $n = 7:$ | 0.7006 |
|  |  | $n = 8:$ | 1.1558 |  | $n = 8:$ | 0.7006 |
|  |  | $n = 9:$ | 1.1558 |  | $n = 9:$ | 0.7006 |

Table 4.4: Endogeous degree of temporal risk aversion under different preference specifications and different $n$, side analysis two.

pertain to the analyses with $n = 7$ and $n = 8$, respectively.

Regarding the sensitivity of the results of the numerical analysis to changes in $n$, we can make the following observations. For specifications that are connected to a small $k^*$ (d, e, f), the results of the analysis with $n = 7$ are almost equivalent to the results of the main analysis with $n = 9$. This suggests that increasing $n$ even further (e.g. to $n = 10$) would not affect the results of the main analysis under specifications (d, e, f). For specifications that are connected to a higher $k^*$ (a, b, c), the results of the analyses with $n \in [7, 8]$ are subject to a small upward bias of $DR_{1,2}^a\left(\bar{T}^S\right)$ at big $\bar{T}^S$. Yet, close inspection of the graph of specification a in $\bar{T} = 80$ - $\bar{T} = 100$ seems to suggest that the upward bias decreases more than proportionately as $n$ is increased. For the high $k^*$ specifications (a, b, c), it is thus suggested that a further increase in $n$ will yield discount rates $DR_{1,2}^a\left(\bar{T}^S\right)$ that lie slightly below the results of the main analysis at high $\bar{T}^S$.

Overall, side analysis two indicates that a number of quadrature nodes $n = 9$ is sufficient to approximate the horizon effect, and thus the instantaneous consumption discount rate, to a satisfactory degree within the framework of the main analysis. For specifications with low $k^*$, the results are expected to be very robust to increases in $n$; for specifications with high $k^*$, small decreases in the consumption discount rate are expected if the number of quadrature nodes could be increased above $n = 9$.
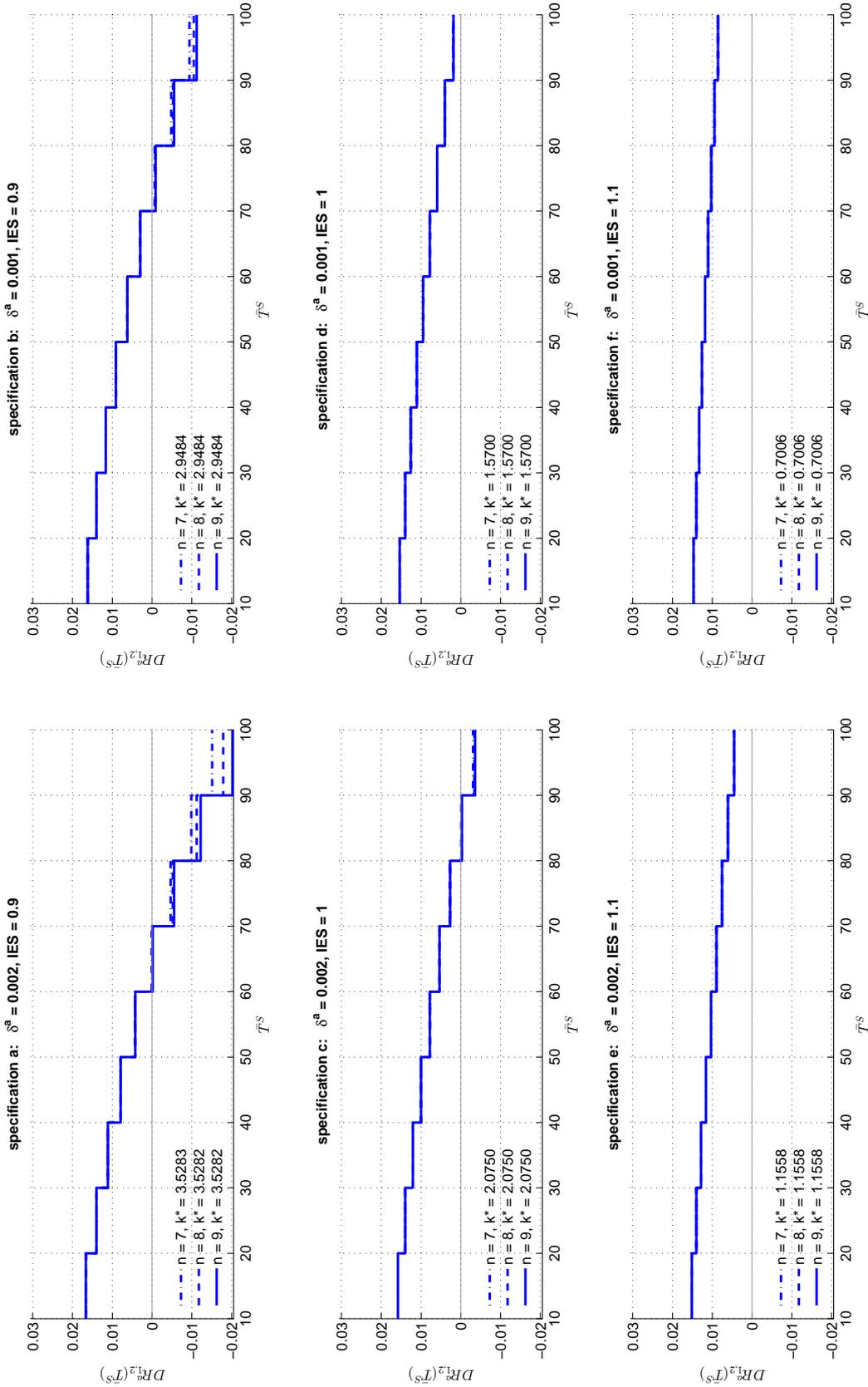
Figure 4.4: Results of side analysis two with $r_f^a = 0.014$, $\bar{T}^P = 30$ years and endogenously determined $k^*$.

| | | $\delta^a = 0.002$ | | $\delta^a = 0.001$ | |
|---|---|---|---|---|---|
| **IES = 0.9** | a: | $\bar{T}^P = 20:$ $\bar{T}^P = 30:$ $\bar{T}^P = 40:$ | 6.9301 3.5282 2.3940 | b: $\bar{T}^P = 20:$ $\bar{T}^P = 30:$ $\bar{T}^P = 40:$ | 5.8198 2.9484 1.9914 |
| **IES = 1** | c: | $\bar{T}^P = 20:$ $\bar{T}^P = 30:$ $\bar{T}^P = 40:$ | 4.1086 2.0750 1.3970 | d: $\bar{T}^P = 20:$ $\bar{T}^P = 30:$ $\bar{T}^P = 40:$ | 3.1246 1.5700 1.0520 |
| **IES = 1.1** | e: | $\bar{T}^P = 20:$ $\bar{T}^P = 30:$ $\bar{T}^P = 40:$ | 2.3070 1.1558 0.7727 | f: $\bar{T}^P = 20:$ $\bar{T}^P = 30:$ $\bar{T}^P = 40:$ | 1.4112 0.7006 0.4659 |

Table 4.5: Endogeous degree of temporal risk aversion under different preference specifications and different $\bar{T}^P$, side analysis three.

### 4.A.3 Side analysis three

The private agent's horizon was set to $\bar{T}^P = 30$ years in the main analysis. Since this is a rather arbitrary assumption, I examine in this side analysis how sensitive the results are to changes in $\bar{T}^P$. In particular, I repeat the computations of the main analysis for $\bar{T}^P \in [20, 40]$, while all other parameters are as in the main analysis.

Table 4.5 contains the results for $k^*$ from step 2 (calibration of the private agent's preference model to the risk-free interest rate) given $\bar{T}^P \in [20, 40]$ and repeats the respective results from the main analyses with $\bar{T}^P = 30$. The table shows that for any of the specifications (a, b, c, d, e, f), the degrees of temporal risk aversion $k^*$ vary considerably for different $\bar{T}^P$. Furthermore, for all of the specifications, we can observe that $k^*$ decreases as $\bar{T}^P$ is increased. The logic behind this behavior is as follows: The absolute magnitude of the horizon effect in the private agent's discounting function (the function for the risk-free rate) increases as the private agent takes into account a longer horizon $\bar{T}^P$. Since the discounting function is calibrated to a common $r_f^a$ for any $\bar{T}^P$, the degree of temporal risk aversion must decrease to counterbalance the effect of a longer horizon $\bar{T}^P$.

Figure 4.5 maps the relationship between the social agent's horizon $\bar{T}^S$ and the discount rate that he applies to second period consumption, $DR_{1,2}^a \left( \bar{T}^S \right)$. The solid blue lines
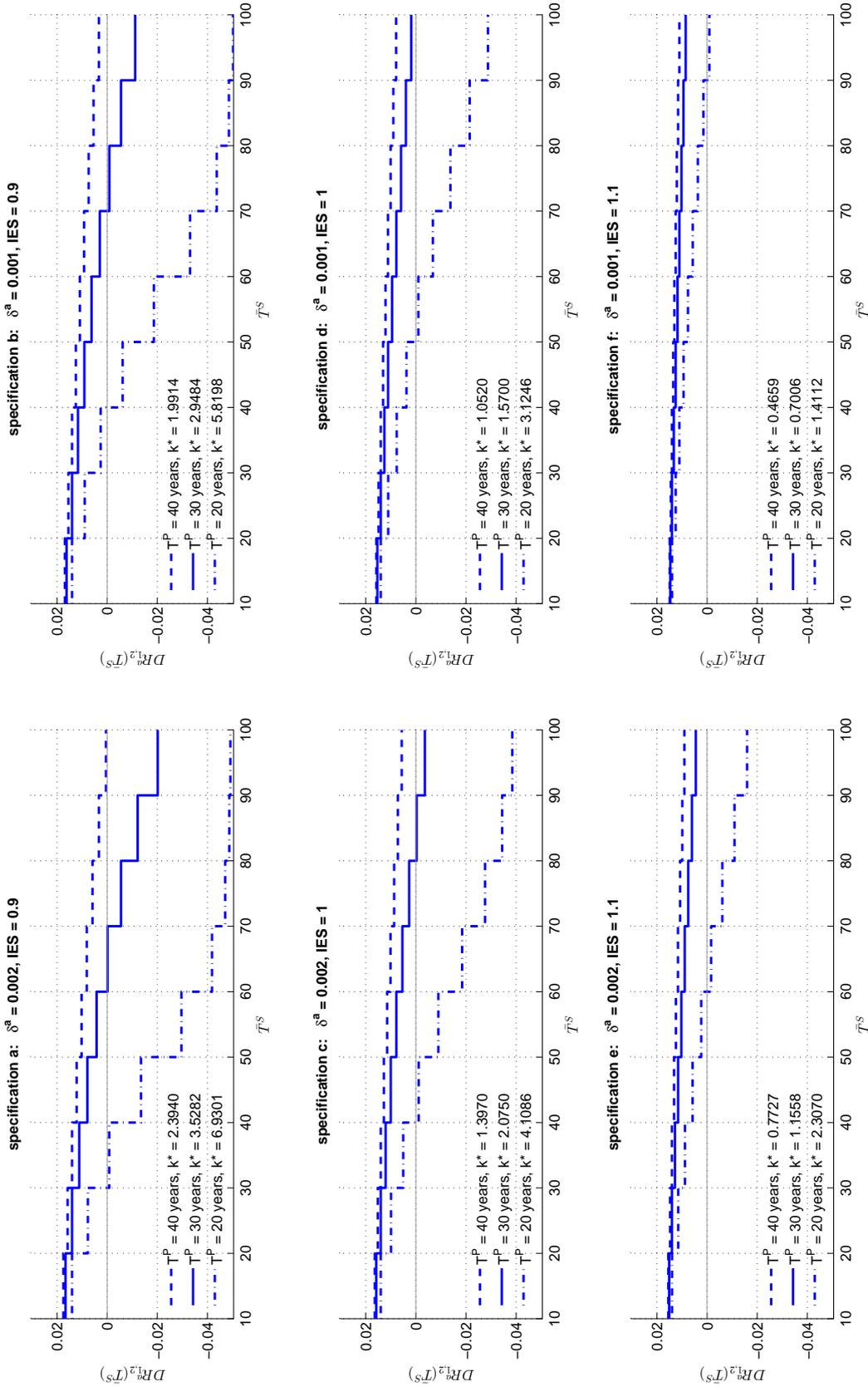
Figure 4.5: Results of side analysis three with $r_f^a = 0.014$, $n = 9$ and endogenously determined $k^*$.

illustrate the results from the main analysis, the dotted and dashed blue lines illustrate the results for $\bar{T}^P = 20$ years and $\bar{T}^P = 40$ years, respectively. In the graphs of all six specifications, the instantaneous consumption discount rate is equivalent to the value of the risk-free rate whenever the social agent's horizon is equivalent to the private agent's horizon. That is, for $\bar{T}^P = 20$ years we get $DR_{1,2}^a\left(\bar{T}^S\right) = 0.014 = r_f^a$ at $\bar{T}^S = 20$, for $\bar{T}^P = 30$ years we get $DR_{1,2}^a(30) = 0.014$, and for $\bar{T}^P = 40$ years we get $DR_{1,2}^a(40) = 0.014$.

The main insight from the results contained in figure 4.5 concerns the relationship between the private agent's horizon $\bar{T}^P$ and the absolute magnitude of the horizon effect that acts on the social agent's consumption discount rate. In all six graphs and at any $\bar{T}^S$, we can observe that the absolute magnitude of the horizon effect decreases, i.e. $DR_{1,2}^a\left(\bar{T}^S\right)$ increases, as higher $\bar{T}^P$ are considered. The reason for this behavior is, of course, that a higher $\bar{T}^P$ is connected to a lower $k^*$, which in turn implies a less pronounced horizon effect. The value of the temporal risk aversion $k^*$ is also the reason for the different spreads between the three lines in the six specifications: The three $k^*$ that correspond to specification a are relatively big, and the difference between them is relatively big as well. The first point implies a generally strong horizon effect, the second point implies a considerable difference between the solutions for different $\bar{T}^P$. In specification f, in the contrary, the size of the three $k^*$ is small, as is the difference between them. Thus, the absolute magnitude of the horizon effect is small for all $\bar{T}^P$, and so is the difference between the three lines.

Summing up, one general conclusion from this side analysis is that the results from the main analysis are very sensitive to changes in $\bar{T}^P$. They are especially sensitive in those specifications that are connected to a high value of $k^*$, i.e. in the specifications that combine a high utility discount rate $\delta^a$ with a low intertemporal elasticity of substitution ($IES$). Furthermore, the insights from the present side analysis highlight, to some extend, the main message of this chapter: Differentiating between a Risk-Sensitive private and a Risk-Sensitive social agent's horizon length drives a wedge between their respective consumption discount rates, and this wedge naturally gets bigger as the difference in their horizons gets bigger.

|  | $\delta^a = \mathbf{0.0087}$ $r_f^a = 0.02$ (side analysis) | $\delta^a = \mathbf{0.002}$ $r_f^a = 0.014$ (main analysis) | $\delta^a = \mathbf{0.0077}$ $r_f^a = 0.02$ (side analysis) | $\delta^a = \mathbf{0.001}$ $r_f^a = 0.014$ (main analysis) |
|---|---|---|---|---|
| **IES = 0.9** | a': 3.6699 | a: 3.5282 | b': 3.0360 | b: 2.9484 |
| **IES = 1** | c': 2.1674 | c: 2.0750 | d': 1.6140 | d: 1.5700 |
| **IES = 1.1** | e': 1.2154 | e: 1.1558 | f': 0.7167 | f: 0.7006 |

Table 4.6: Endogeous degree of temporal risk aversion under different preference specifications and different $r_f^a$, side analysis four.

### 4.A.4 Side analysis four

In the main analysis I employed a risk-free rate of $r_f^a = 0.014$, which was chosen as it lies 'somewhere in the middle' between the returns on short and long term US Treasury securities, i.e. between 1% and 2.4%. As 1.4% may be considered to be a rather low return to a risk-free asset, I repeat the main analysis with a rather high risk-free rate of 2% in this side analysis. Furthermore, I assume that $\delta^a \in [0.0087, 0.0077]$, rather than $\delta^a \in [0.002, 0.001]$ as postulated in the main analysis. This increase in $\delta^a$ is necessary to keep the degree of temporal risk aversion $k^*$ positive, i.e. for the private and the social agent to be temporally risk averse.[26] The exact values of $\delta^a$ are chosen such that the differences $\left( r_f^d - \delta^d \right)$ that result in this side analysis are equivalent to the differences $\left( r_f^d - \delta^d \right)$ that result in the main analysis.

Table 4.6 and figure 4.6 contrast the results of side analysis four to the results of the main analysis. From the results in table 4.6, we can observe that the degree of temporal risk aversion $k^*$ is always higher in the side analysis. In view of the twofold role of the utility discount rate (see the discussions in sections 4.4.1 and 4.A.1), it is suggested that this difference arises from the negative impact of higher $\delta^a$ on the absolute size of the horizon effect: Increases in $\delta^a$ imply a lower absolute value of the horizon effect and must be counterbalanced by an increase in $k^*$.

Figure 4.6 illustrates the development of $DR_{1,2}^a \left( \bar{T}^S \right)$ for the side analysis (dotted blue

---

[26]For $IES = 1$, this can easily be checked by plugging the respective values into the private agent's Extended Ramsey Equation for Risk-Sensitive preferences, namely equation (4.4).
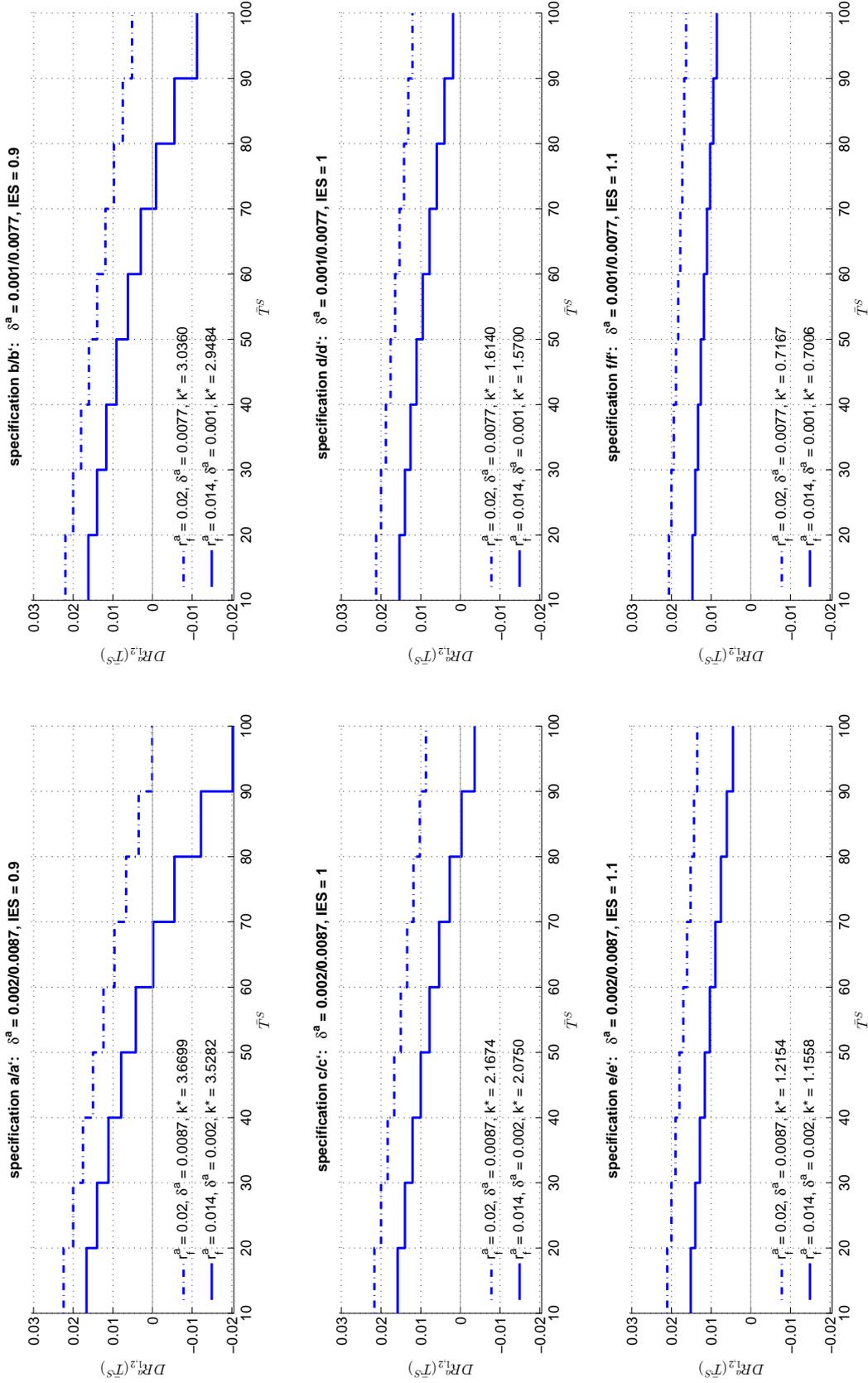
Figure 4.6: Results of side analysis four with $\bar{T}^P = 30$ years, $n = 9$ and endogenously determined $k^*$.

lines) as well as for the main analysis (solid blue lines). For both analyses, the social agent's instantaneous discount rate equals the risk-free rate if his horizon is equivalent to that of the private agent, i.e. $DR_{1,2}^a (30) = r_f^a$. As the solutions for the side and the main analysis, i.e. the dotted and the solid lines, are almost parallel in all six specifications, we can infer that increasing $r_f^a$ (and increasing $\delta^a$ accordingly) does not affect the insights from the main analysis.

The conclusion from side analysis four is thus that, for a higher $r_f^a$, the social agent's discount rate increases at any $\bar{T}^S$, but the observations regarding the generally negative impact of the horizon effect, as well as the observations regarding the comparison across specifications (a, b, c, d, e, f) remain unchanged.

## 4.B   MATLAB codes

This appendix contains the MATLAB codes which I developed for the numerical analysis of this chapter (see the descriptions in section 4.3.1). The main code (appendix 4.B.1) employs the functions *cons_tree5.m* (appendix 4.B.2) which I developed for the creation of the consumption tree, the function *para_sel5.m* (appendix 4.B.3) which specifies the preference parameters, and the function *dr_inst_t5.m* (appendix 4.B.4) which I developed to recursively calculate the instantaneous consumption discount rate of a Risk-Sensitive agent. Note that I have deleted those parts in the main code through which the figures of this chapter are created. Note furthermore that the main code also employs the function *qnwnorm.m* from Miranda and Fackler's (2002) CompEcon Toolbox.

### 4.B.1   Main code

```
%% DESCRIPTION %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% author: Svenja Hector, CER-ETH Zurich, shector@ethz.ch
% date:   23.04.2015

% numerical analysis of the horizon effect for a risk sensitive agent

% uses
```

```matlab
% own m-files: para_sel5, cons_tree5, dr_inst_t5,
% Miranda/Fackler m-files: qnwnorm.m (ckron,gridmake,optget,optset,csize)

close all; clear all; clc;

%% SPECIFICIATION %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% turn processes on (1) or off (0)
save_output  = 1;   % save output/graph/data
make_trees   = 0;   % make consumption trees given maxTsi/maxubsi
find_kstar   = 0;   % find kstar for ub = lb:maxubpi
set_kstar    = 1;   % use endogenously determined kstar in analysis
set_exogenk  = 0;   % set exogenous k, same for all para
find_DR      = 0;   % find DR for maxubsi/maxTsi, given kstarTpi
find_DRanaly = 0;   % use ERE for RS (HE chapter) to determine DR(T)
make_figure  = 1;   % make figure DR(T) (choose set_kstar/set_exogenk also)

% preference parameters
pref     = 1;              % 1 = standard RS, 2 = RS with (1-beta)
para_vec = 1:6;            % specifies ies, delta, k_mid
k_ex     = 0.02;           % risk aversion of si if k exogenous

% economic parameters (annual)
maxTsi  = 10;              % social investor maximum horizon in periods
Tpi     = 3;               % private investor horizon in periods
perl    = 10;              % period length in years
r_an    = 0.014;           % annual market rate of return
n       = 9;               % nodes in Gauss-Hermite quadrature
mu_an   = 0.018;           % mean annual growth rate
var_an  = 0.00127;         % variance of growth rate
sig_an  = sqrt(var_an);    % annual standard deviation of growth rate
x1_an   = 1;               % period 1 consumption
minubsi = maxTsi;          % --> set to maxTsi for all periods to be random,
maxubsi = maxTsi;          %     to 2 if only second period random

% rescale annual values according to period length
mu  = (1+mu_an)^perl - 1;
var = perl*var_an;
sig = sqrt(var);
r   = (1+r_an)^perl - 1;
pot = 0:(perl-1);
x1  = x1_an*sum((1+mu_an).^pot);

% do not change
prec    = 2;               % double or single precision
Tstep   = 1;               % defines horizons at which DR(T) is calculated
Tvec    = 2:Tstep:maxTsi;  % vector of horizons at which DRs are calculated
lb      = 2;               % first uncertain period
maxubpi = Tpi;
minubpi = Tpi;

% [deleted: settings for figures]

%% WARNING & PREPARATIONS %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% warning if consumption trees get very big
if  make_trees == 1 && n^(maxubsi-lb+1)*maxTsi > 2*10^8
    display(['you are about to create a matrix Xsi of dimension (',...
    num2str(n^(maxubsi-lb+1)),',',num2str(maxTsi),')'])
    uiwait(errordlg('Xsi too big? check command window','WARNING'))
end

% rootname for trees, saved in main folder
```

```matlab
root = ['M5_',num2str(n),'_',num2str(mu_an*1000),'_',...
        num2str(var_an*100000),'_',num2str(x1_an),'_',num2str(maxTsi),...
        '_',num2str(perl),'_',num2str(prec),'_',num2str(minubsi),'_',...
        num2str(maxubsi),'_',num2str(k_ex*100)];

% rootname and folder for output (except trees) given specifications above
dataroot = [root,'_',num2str(Tpi),'_',num2str(r_an*100),'_',num2str(pref)];

% start or continue diary, make rootfolder, save specifiations
diaryname = [dataroot,'.doc']; path = pwd;
if save_output == 1;
    % continue diary or start diary in rootfolder/save specifications
    if exist([path '\' dataroot '\' diaryname],'file');
    diary([path '\' dataroot '\' diaryname]);
    fprintf(['\n\n','---------------','\n\n','continue diary','\n\n'])
    else mkdir(dataroot);
    % save specifications
    speciname = ['specifica_',dataroot,'.mat'];
    save([dataroot '/' speciname],'maxTsi','Tpi','perl','r_an','n',...
    'mu_an','sig_an','x1_an','pref','para_vec','prec','Tstep','Tvec',...
    'var_an','lb','minubpi','minubsi','maxubpi','maxubsi','mu','sig',...
    'var','r','x1','k_ex');
    % start diary
    diary([path '\' dataroot '\' diaryname]);
    fprintf(['Diary for specification:','\n','\n'])
    display(['n    = ',num2str(n),     'maxTsi  = ',num2str(maxTsi)])
    display(['mu   = ',num2str(mu_an), 'Tpi     = ',num2str(Tpi),  ])
    display(['var  = ',num2str(var_an),'r       = ',num2str(r_an), ])
    display(['x1   = ',num2str(x1_an), 'para    = '  ])
    display(['pref = ',num2str(pref),  ' ',num2str(para_vec)])
    display([' '])
    display(['maxubpi = ',num2str(maxubpi),' maxubsi = ',num2str(maxubsi)])
    display(['minubpi = ',num2str(minubpi),' minubsi = ',num2str(minubsi)])
    fprintf(['----------------------------------------','\n','\n'])
    end
end

fprintf(['\n','code started:  ',datestr(clock),'\n','\n'])

%% MAKETREES %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% create consumption tree

[x,w] = qnwnorm([n],[mu],[var]);

if make_trees == 1
tic; fprintf([datestr(clock),' - start make_trees','\n'])

for ub = minubsi:maxubsi
    maxT    = maxTsi;
    X       = sparse(cons_tree5(n,mu,maxT,lb,ub,x1,x));
    treename = [root,'_Xmat',num2str(maxTsi),'_',num2str(ub),'.mat'];
    save(treename,'X','-v7.3');
end
clear maxT X treenamesi ub;

fprintf(['time elapsed for make_trees: ',num2str(toc),'s','\n','\n'])
clear tic; clear toc;
end

%% CALIBRATION %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%  find kstar, i.e. the risk aversion of private investor
```

```matlab
if find_kstar == 1
tic; fprintf([datestr(clock),' - start find_kstar','\n'])

kstar_allpara = zeros(1,numel(para_vec));
for pp = 1:numel(para_vec)
para = para_vec(pp);
fprintf([datestr(clock),' - start for parasel = ',num2str(para),'\n'])

% set preference paramters
[delta, beta, ies, rho, B, kmid] = para_sel5(para,pref,perl);

% load consumption tree from file, select private investors horizon
load([root,'_Xmat',num2str(maxTsi),'_',num2str(maxubsi),'.mat']);
Xpi = full(X(1:numel(nonzeros(X(:,Tpi))),1:Tpi));

% find k_star
X       = Xpi;
T       = Tpi;
ub      = maxubpi;
kstar   = fsolve(...
          @(k)dr_inst_t5(X,n,w,x1,T,lb,ub,B,ies,rho,delta,k)-r,kmid);
kstar_allpara(pp) = kstar;

% save relevant data to file
if save_output == 1; save([dataroot,'/kstardata_',dataroot,'_',...
   num2str(para),'.mat'],'kstar'); end
end

if save_output == 1; save([dataroot,'/kstardata_',dataroot,...
   '_all','.mat'],'kstar_allpara'); end

fprintf(['time elapsed for find_kstar: ',num2str(toc),'s','\n','\n'])
clear tic; clear toc
end % end find_kstar

%% FIND DR  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% compute DR(T) given kstar of private investor or exogenous k

if find_DR == 1
tic; fprintf([datestr(clock),' - start find DR','\n'])

naninDRmat = zeros(numel(para_vec),1);
for pp = 1:numel(para_vec)
para = para_vec(pp);
fprintf([datestr(clock),' - start for parasel = ',num2str(para),'\n'])

% set preference paramters
if set_kstar == 1
    load([dataroot,'/kstardata_',dataroot,'_',num2str(para),'.mat']);
    k = kstar;
elseif set_exogenk == 1
    k = k_ex;
end
[delta,beta,ies,rho,B,~] = para_sel5(para,pref,perl);

% compute DR(T)
DR_mat_p = zeros(maxubsi-minubsi+1,length(Tvec)+1);
DR_mat   = zeros(maxubsi-minubsi+1,length(Tvec)+1);
ccc = 0;
for ub = minubsi:maxubsi
fprintf([datestr(clock),' - start for ub = ',num2str(ub),'\n'])
```

```matlab
ccc = ccc + 1;

    % load consumption tree given ub from file
    load([root,'_Xmat',num2str(maxTsi),'_',num2str(ub),'.mat']);
    Xsi = full(X);

    % compute DR(T|k) of social investor
    DR_vec   = zeros(1,length(Tvec)+1);
    DR_vec_p = zeros(1,length(Tvec)+1);
    cc       = 1;
    for T = Tvec
        cc = cc + 1;
        rowsinT       = size(nonzeros(Xsi(:,T)),1);
        X             = Xsi(1:rowsinT,1:T);
        DR_p          = dr_inst_t5(X,n,w,x1,T,lb,ub,B,ies,rho,delta,k);
        DR_vec_p(cc)  = DR_p;
        DR_vec(cc)    = (1+DR_p)^(1/perl)-1;
    end
    DR_mat_p(ccc,:) = DR_vec_p;
    DR_mat(ccc,:)   = DR_vec;
end

% detect NaN in DR_mat
nanmat = isnan(DR_mat);
if ismember(1,nanmat)==1; naninDRmat(pp)= 1; else naninDRmat(pp)=0; end

% save relevant data to file
if save_output == 1; save([dataroot,'/DRdata_',dataroot,'_',...
   num2str(para),'.mat'],'DR_mat','DR_mat_p'); end
end

% save NaN diagnostic to file
if save_output == 1; save([dataroot,'/naninDRmat_',dataroot,'.mat'],...
   'naninDRmat'); end

fprintf([datestr(clock),' - end find_DR','\n'])
fprintf(['time elapsed for find_DR: ',num2str(toc),'s','\n','\n'])
clear tic; clear toc;
end % end find DR

%% ANALYTICAL SOLUTION IES=1 CASE %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

if find_DRanaly == 1
tic; fprintf([datestr(clock),' - start find DRanaly','\n'])

for para = [3 4]

    % set preference paramters
    [delta,beta,ies,rho,B,~] = para_sel5(para,pref,perl);
    if set_kstar == 1
       % use kstar from numerical solution
       load([dataroot,'/kstardata_',dataroot,'_',num2str(para),'.mat']);
       k = kstar;
    elseif set_exogenk == 1
       k = k_ex;
    end

    % calculate DR at all T, given kstar or k_ex
    DR_analy = zeros(1,length(Tvec)+1);
    cc       = 1;
    for T  = Tvec
        cc = cc + 1;
        if T > 2
```

```matlab
            beta_vec = ((1/(1+delta)).*ones(1,T-3+1)).^double(0:T-3);
        elseif T == 2
            beta_vec = 0;
        end
        HE = (var/2)*2*k*(1/(1+delta))*sum(beta_vec);
        DR_analy_p  = delta + mu - (var/2) - (var/2).*2.*k - HE;
        DR_analy(cc)= (1+DR_analy_p).^(1/perl)-1;
    end
end

% save relevant data to file
if save_output == 1
load([dataroot,'/DRdata_',dataroot,'_',num2str(para),'.mat']);
save([dataroot,'/DRdata_',dataroot,'_',num2str(para),'.mat'],...
      'DR_mat','DR_mat_p','DR_analy');
end
end

fprintf([datestr(clock),' - end find_DRanaly','\n'])
fprintf(['time elapsed for find_DRanaly: ',num2str(toc),'s','\n','\n'])
clear tic; clear toc;
end % end find DRanaly

%% MAKE FIGURES %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% [deleted: code to make figure if k endogenous]
% [deleted: code to make figure if k exogenous]

fprintf(['\n','code finished: ', datestr(clock),'\n','\n'])
if save_output == 1; diary off; end
```

## 4.B.2   Function for the creation of the consumption tree

```matlab
function X = cons_tree5(n,mu,maxT,lb,ub,x1,x)

% DESCRIPTION %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% author: Svenja Hector, CER-ETH Zurich, shector@ethz.ch
% date:   02.04.2015
%
% cons_tree5.m creates a matrix of consumption levels at the nodes of a
% probabilistic tree, given horizon maxT and a number of random states in
% each (uncertain) period; growth rates are normally distributed and iid,
% the Gauss quadrature output (n,x) gives the number of states/realizations

% X        : OUTPUT tree of consumption of length maxT
% n,x      : INPUT represents normal distribution: n states with
%            realizations x, x is determined by Gauss quadrature
% mu       : INPUT mean growth rate, also input to Gauss quadrature
% maxT     : INPUT length of tree
% lb, ub   : INPUT defines first (lb) and last (ub) random period
% x1       : INPUT certain period 1 consumption

%% COMPUTATION: CREATION OF CONSUMPTION TREE X_max %%%%%%%%%%%%%%%%%%%%%%%

% matrix of random growth factors
G_max = repmat([1; zeros(n-1,1)],1,maxT);
if n == 1
```

```matlab
G_max(:,2:maxT) = 1+mu;
elseif n > 1
for t = 1:maxT
    if      t == 1;   G_max(1,t) = 1;
    elseif  t <   lb;  G_max(1,t) = 1+mu;
    elseif  t <=  ub;  G_max(:,t) = 1+x;
    elseif  t <= maxT; G_max(1,t) = 1+mu; end
end
end

% matrix of random (correlated) consumption, nodes in tree
X = zeros(n^(ub-lb+1),maxT);
for t = 1:maxT
    if      t == 1;   X(1,t) = x1;
    elseif  t <   lb;  X(1,t) =...
                      kron(nonzeros(X(1,t-1)),nonzeros(G_max(1,t)));
    elseif  t <=  ub;  X(1:(n^(t-lb+1)),t)  =...
                      kron(nonzeros(X(:,t-1)),nonzeros(G_max(:,t)));
    elseif  t <= maxT; X(1:(n^(ub-lb+1)),t) =...
                      kron(nonzeros(X(:,t-1)),nonzeros(G_max(1,t)));
    end
end
end
```

### 4.B.3 Function for the specification of the preference parameters

```matlab
function [delta, beta, ies, rho, B, kmid] = para_sel5(para,pref,perl)

% specification of preference parameters for main5.m, dr_inst_t5.m

if     para==1; ies=0.9; delta_an=0.002; kmid=2;
elseif para==2; ies=0.9; delta_an=0.001; kmid=2;
elseif para==3; ies=1;   delta_an=0.002; kmid=2;
elseif para==4; ies=1;   delta_an=0.001; kmid=2;
elseif para==5; ies=1.1; delta_an=0.002; kmid=2;
elseif para==6; ies=1.1; delta_an=0.001; kmid=2;
end;

delta = (1+delta_an)^perl-1;
rho = 1-(1/ies); beta = 1/(1+delta);
if pref == 1; B = 1; elseif pref == 2; B = (1-beta); end

end
```

### 4.B.4 Function for the recursive calculation of the discount rate

```matlab
function DR = dr_inst_t5(X,n,w,x1,T,lb,ub,B,ies,rho,delta,k)

%% DESCRIPTION %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```matlab
% author: Svenja Hector, CER-ETH Zurich, shector@ethz.ch
% date:    03.04.2015

% the function drinstt derives the instantaneous discount rate of a
% risk sensitive agent with horizon T

% DR         : OUTPUT discount rate of RS agent with horizon T
% X          : INPUT cons. tree accounted for by agent, must be of length T
% n,w        : INPUT n draws from g~N(mu,var), knots are weighted by w
% x1         : INPUT certain period 1 consumption
% T          : INPUT horizon of agent
% lb, ub     : INPUT defines first (lb) and last (ub) random period
% B          : INPUT defines form of utility function
% ies,rho    : INPUT preference parameters for form of felicity function
% delta,beta : INPUT preference parameters for utility discounting

%% COMPUTATION: DERIVATION OF DISCOUNT RATE %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% felicity function and derivative of felicity function
if     ies == 1; u = @log;                            u_prime = @(y) (1/y);
elseif ies ~= 1; u = @(y) ((y.^rho)-1)/(rho);  u_prime = @(y) y.^(rho-1);
end

% utility function
phi             = @(y) -exp(-k.*y);
phi_prime       = @(y) k.*exp(-k.*y);
phi_minone      = @(y) (-1./k).*log(-y);

% preparations
beta_vec    = ((1/(1+delta)).*ones(1,T-ub+1)).^double(0:T-ub);
u_cont      = [[u(X(1,1)); zeros(size(X,1)-1,1)]  u(X(:,2:T))];
u_prime_x2 = arrayfun(u_prime,X(1:n,2));

% continuation utility in last random period: U_last
if     T <  ub; last = T;
       U_last = B.*u_cont(1:(n^(T-1)),last:T);
elseif T >= ub; last = ub;
       U_last = B.*(u_cont(:,last:T)*beta_vec(1,1:(T-ub+1))');
end

% create matrix to collect U in all nodes 1:ub, plug in U_last
U_node          = zeros(n^(last-lb+1),last);
U_node(:,last) = U_last;

% recursively: derive U for all periods 2:(last-1)
for t = (last-1):-1:2;
    % take all phi(Ut+1), group accord. to node t, build E[.]
    E_phi_U=w'*reshape(phi(U_node(1:n^t,t+1)),n,n^(t-1));
    term2 = transpose((1/(1+delta)).*phi_minone(E_phi_U));
    term1 = u_cont(1:(n^(t-1)),t);
    U_node(1:n^(t-1),t) = B.*term1 + term2;
end

% solve discounting function
nom  = phi_prime(U_node(1:n,2));
den  = phi_prime(phi_minone(w'*phi(U_node(1:n,2))));
AF   = nom./den;
MU1  = u_prime(x1);
MU2  = w'*(AF.*u_prime_x2);
DR   = delta - log(MU2/MU1);

end
```

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