Doctoral Thesis

On the Analysis, Simulation and Structural Design of Helical Constructions

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On the Analysis, Simulation and Structural Design of Helical Constructions

A thesis submitted to attain the degree of
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(Dr. sc. ETH Zurich)

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2015
Abstract

The present thesis provides analytical and numerical modeling developments on the characterization and simulation of the mechanical response of helical constructions. Moreover, it presents a modeling framework for the assessment of the torsional response bounds of multilayer helical assemblies. Furthermore, it elaborates a methodology for the selection of the braiding pattern of layered helical constructions. Finally, it explicates a modeling scheme for the inference of the structural composition of tendon fascicles.

On the characterization of the helix structural response, a modeling extension is presented, incorporating radial strain as an additional degree of freedom to the commonly favored axial and torsional ones. To that extent, the governing kinematic, constitutive and equilibrium equations are formulated with the use of a beam based model, providing closed-form stiffness term expressions. The extended structural response is further assessed by means of a dedicated planar finite element that not only allows for the simulation of the above broached loading patterns but also for the numerical modeling of thermal loading.

Moreover, the present work provides a modeling scheme for the demarcation of the mechanical response bounds of multilayer helical constructions subject to structural kinematic constraints. In particular, the constructions’ torsional response bounds are analyzed considering a wide range of braiding patterns. The response of the kinematically constrained structures is related by means of scaling factors to the stiffness properties of the unconstrained one, which can be analytically calculated upon closed-form expressions.

Furthermore, a quantitative framework guiding the mechanical design of layered helical assemblies is reported. More specifically, a methodology allowing for the optimization of the structural braiding patterns is presented. The favored structural arrangements are selected so that they maximize the resistance of the arising construction to axial loads and concurrently minimize its torsional propensity. The methodology is used to retrieve favorable structural patterns for helical assemblies comprised of up to five layers, providing a database that covers most practical applications.

Finally, the present thesis explicates a numerical model for the inference of the structural composition of tendon fascicles. In particular, tendon experimental data are coupled to finite element and Bayesian uncertainty quantification modeling. The finite element models allow for the recreation of the available experimental set-ups in a computationally tractable way, while the Bayesian framework allows for a direct comparison amongst hundreds of different model classes. Thereupon, probabilistic bounds for the model parameters are provided, establishing a fundamental linkage between successive tendon hierarchi-
cal levels. By that means, not only a direct correlation between structural composition, mechanical properties and experimental data is achieved, but further, a database for the engineering of artificial tendons is furnished.
Zusammenfassung


Für die Analyse des Strukturverhaltens helixförmiger Konstruktionen wird hier ein existierendes Modell für längs- und Torsionsbelastung um die radiale Deformation erweitert. Hier werden die kinematischen, konstitutiven und Gleichgewichtsgleichungen für eine geschlossene Lösung der Einträge der Struktursteifigkeitsmatrix formuliert. Das ausgebreitete Strukturverhalten ist zusätzlich charakterisiert mittels eines dedizierten planaren finiten Elementmodells, welches nicht nur die Simulation der oben thematisierten Belastungsmuster ermöglicht, sondern auch die numerische Modellierung von Thermalbelastungen.


Ferner wird ein Berechnungsrahmen als Leitmotiv für das mechanische Design der vielschichtigen helixförmigen Gruppen thematisiert. Ausführlicher wird eine Methode präsentiert zwecks der Optimierung der Strukturmustern. Die bevorzugten strukturellen Muster sind so gewählt, dass sie den Widerstand der Konstruktion maximieren und zugleich ihre Drehneigung minimieren. Die Methode wird so angewandt, dass die vorteilhaften Strukturmustern für vielschichtige Gruppen, die aus bis zu fünf Schichten gebildet sind, ausgewählt und verwendet werden, um ein Datenarchiv für alle praktischen Anwendungen zu liefern.

Zum Schluss erklärt die aktuelle These ein numerisches Modell ausführlich, woraus man auf die strukturelle Zusammensetzung der Sehnenfaszikel schliessen kann. Insbesondere werden die Experimentdaten sowohl mit der finiten Elementsimulierung als auch mit der Bayesianischen Wahrscheinlichkeitssimulierung verbunden. Die finiten Elementmod-
elle erlauben die vom Computer gesteuerte Nachbildung der verfügbaren Experimente, während der Bayesianische Rahmen einen direkten Vergleich zwischen hunderten von verschiedenen Modellkategorien erlaubt. Daraufhin werden Wahrscheinlichkeitsgrenzen für die Modellparameter gewährt, die eine fundamentale Verbindung zwischen aufeinander folgenden hierarchischen Sehnenebenen herstellen. Auf diese Weise werden nicht nur direkte Korrelationen zwischen Strukturzusammensetzung, mechanischen Eigenschaften und Experimentdaten erreicht, sondern auch eine Datenbasis für das Entwickeln künstlicher Sehnen geliefert.
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The work presented in this thesis was carried out at the Institute of Mechanical Systems in the Department of Mechanical and Process Engineering at ETH Zurich.

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Zurich, June 2015
Nicos Karathanasopoulos
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Part I

Introduction
The analytical and numerical studies and methods presented in the thesis are placed in context upon a six part division (Parts I-V), each of them containing an introductory section that allows for the part to be self-contained.

More specifically, in the first part (Parts I), an overall introduction to the topic is endeavored. To that extent, after a brief review of the history of helical constructions and their applications, the geometric attributes of helical configurations are explicated in Chapter 1, providing an insight into the structural element studied in all subsequent Chapters.

In the second part of the thesis (Parts II), analytical and numerical modeling advancements on the helix structural response are elaborated. On the analytical modeling side, Chapter 2 characterizes the mechanical response of helical constructions to radial loads by appending the corresponding deformation mode to the commonly considered axial and torsional ones. Furthermore, it demarcates the geometric applicability limits of the elaborated analytical closed-form expressions, while it quantifies the significance of the different contributing response mechanisms (axial, torsional and bending helix cross section stiffness) over a wide range of helix configurations.

Chapter 3 elaborates a planar finite element model for the simulation of the mechanical response of helical structures. Different helix loading schemes are considered, in particular axial, torsional, radial and thermal loading. The planar element is verified with the use of closed-form expressions and upon dedicated commercial finite element models. In Chapter 4 to follow, different applications of the planar finite element modeling are presented, primarily in the context of engineering cables. The analysis addresses helix cross section configurations out of the typically encountered circular ones. Chapter 4 concludes with a study on the computational demands of the numerical approach, highlighting its merits.

In the third part of the thesis (Part III), Chapter 5 analyzes the mechanical response of multilayer helical assemblies under the influence of kinematic constraints. More specifically, their torsional stiffness bounds are studied for a wide range of structural arrangements. The constraints apply at discrete positions along the length of the helical bodies and in particular on the helix cross section normal rotational degree of freedom. The kinematically constrained mechanical response is related to the free of constraints, analytically computed mechanical response with the use of scaling factors.

The fourth part of the thesis (Part IV) deals with the design of layered helical assemblies. Thereupon, Chapter 6 elaborates a quantitative framework for the selection of favorable helical assembly braiding patterns. The structural patterns are selected so
as to maximize the resistance of the arising construction to axial loads and concurrently minimize its torsional propensity. A statistical classification of a wide range of braiding patterns is provided for helical assemblies comprised of up to five layers. The identified optimal structural arrangements yield a database that covers most practical applications.

In the fifth part of the thesis (Part V), biological constructions arranged in helical configurations are simulated. More specifically, Chapter 7 elaborates a numerical scheme for the modeling of tendon fascicles of various compositions and geometric arrangements. Thereupon, experimental observations are coupled to finite element models and to a Bayesian uncertainty quantification framework. The finite element models allow for the recreation of the available experimental set-ups in a computationally tractable way, while the Bayesian framework for a direct comparison amongst hundreds of different model classes pertaining to the different model constructions. By that means, physiologically relevant compositions of the tendon subunits are obtained, providing probabilistic bounds for each of the model parameters. Overall, the explicated methodology makes feasible the correlation of different hierarchical levels, functioning as a benchmark study in the identification of the structural composition of tendons.

The presented work concludes with a general summary of Parts II to V. Chapter 8 recapitulates the principal contributions of the different chapters, discussing potential extensions of the presented work related both to modeling advancements and application specific further derivations. The remaining subsections of the thesis contain the bibliography along with a list of publications as of the current state.
A short history of helical structures

In the long history of load carrying structures, different members have been used to transfer tensile loads, such as trusses and assemblies of helical bodies. Their engineering was guided by the available materials, by the mechanical understanding of each era and first and foremost by “trial and error”, functioning as a primary optimization process.

It is primarily through this process that engineering practice opted almost exclusively for the use of helically braided patterns to construct ropes, suspension bridges or lately electricity power cables. Helical assemblies are endowed with an inherent structural privilege that is of paramount importance for their functionality and resilience. More specifically, they have the prerogative of a high axial strength at their principal loading direction, combined with a substantial flexibility at all other loading directions. As a result, they can sustain and endure oblique loading patterns, as for example wind loads, support motions or lateral excitations better than what continuous constructions of the same material and geometric specifications would.

The engineering merits of helical assemblies were apprehended already in the Mesolithic period (9000-3000 BC) where ropes were used as hoisting and hauling structures. At the time, they were made of multiple natural fibers such as hemp, sisal, manila and cotton depending on the region, as rope remnants found in archaeological excavations of different civilizations indicate. Ancient Egyptians (ca. 2000 B.C.) used ropes made out of leather or palm fibers, while ancient Chinese manufactured bamboo cables to support their suspension bridges. Accordingly, maritime civilizations, as the ones of ancient Greeks and Romans, extensively used helically braided ropes in their sailing boats, while constructed weapons as the catapult that functioned upon helical rope systems.

Rope technology remained almost unaltered for some thousands of years, up to the first industrial revolution, when production was substantially facilitated by a set of technological developments and construction automations. Moreover, improvements to steel-making techniques made high quality steel available at lower costs and larger quantities, so that wire ropes became common in the flourishing mining industry. In the middle of the 19th century, large submarine telegraph cables were constructed following the same principal manufacturing process with the one followed in rope construction. It is around this period that the concept of parallel lay multilayer strands found extensive use in the construction of bridge and lifting cables. In the decades to follow, the networks of electricity power transfer throughout the developed world seized the cable manufacturing inertia, with metallic cables of different size and material mixtures to be used as electricity power conductors.
In the post second-world war societies, the progress of technology broadened the design spectrum of helical constructions. In particular, the development of new anti-corrosive, high tensile strength materials (aramid, carbon and polymer based) offered substantial alternatives to metal design. What is more, it provided the basis for the appropriate tuning of the geometric and mechanical properties of helical assemblies to meet the demands of new arising applications. A characteristic example of the kind can be found in bioengineering, with helically braided constructions used as scaffolds for the restoration of damaged tissues.
Helix geometry

1.1 Centerline representation

The helix centerline geometry can be described means of its position vector $\mathbf{R}$, defined upon the geometric parameters introduced in Fig. 1.1 below:

![Helix centerline geometry](image)

Figure 1.1: Helix centerline geometry
Thereupon, the coordinates of a point along the helix centerline are given as:

\[
\mathbf{R}(s) = \begin{cases} 
X \\ Y \\ Z 
\end{cases} = \begin{cases} 
 a \cos \varphi \\ a \sin \varphi \\ b \varphi 
\end{cases}, \quad \varphi = \frac{s}{\gamma} + \varphi_0, \quad \gamma = \sqrt{a^2 + b^2} \tag{1.1}
\]

where \(b\) stands for the pitch value, a parameter describing the gain in height along the axis \(Z\) per unit angular evolution \(\varphi\). The pitch value is directly related to the helix centerline position \(a\) through the helix angle \(\theta\) upon which the height of the helix for a single period development \(h\) is retrieved:

\[
b = a \tan \theta, \quad h = 2 \pi b \tag{1.2}
\]

In Fig. 1.1, along with the Cartesian frame, the Serret-Frenet frame has been illustrated. The latter is characterized by the Curvilinear orthogonal base vectors \(\mathbf{n}\), \(\mathbf{b}\), \(\mathbf{t}\) standing for the normal, binormal and tangential local vectors respectively. The Serret-Frenet base vectors are related to the Cartesian ones upon trigonometric functions dependent on the helix angle \(\theta\) and on the helix central angle \(\phi\).

\[
\begin{align*}
\mathbf{n} &= -\cos \varphi \mathbf{e}_x - \sin \varphi \mathbf{e}_y \\
\mathbf{b} &= \sin \theta \left[ \sin \varphi \mathbf{e}_x - \cos \varphi \mathbf{e}_y \right] + \cos \theta \mathbf{e}_z \\
\mathbf{t} &= \cos \theta \left[ -\sin \varphi \mathbf{e}_x + \cos \varphi \mathbf{e}_y \right] + \sin \theta \mathbf{e}_z 
\end{align*} \tag{1.3}
\]

Simplifying the above equations with the use of the geometric definitions introduced in Eq. 1.1, the following concise expressions for the local orthonormal base vectors are retrieved:

\[
\begin{align*}
\mathbf{n} &= \begin{cases} 
- \cos \varphi \\ - \sin \varphi \\ 0
\end{cases}, \quad \mathbf{b} = \frac{1}{\gamma} \begin{cases} 
b \sin \varphi \\
-b \cos \varphi \\
a \cos \varphi
\end{cases}, \quad \mathbf{t} = \frac{1}{\gamma} \begin{cases} 
-a \sin \varphi \\
\mathbf{b}
\end{cases} 
\end{align*} \tag{1.4}
\]

Upon the above orthonormal triad, the curvatures \(k_{n|b|t}\) characteristic of the space curve are computed, as follows:

\[
\kappa_n = 0, \quad \kappa_b = \frac{a}{\gamma}, \quad \kappa_t = \frac{b}{\gamma} \tag{1.5}
\]

Considering the variation of the base vectors (Eq. 1.4) along the curvilinear coordinate \(s\) and making use of the curvature definitions provided in Eq. 1.5, the so called Serret-Frenet formulas arise:

\[
\begin{align*}
\begin{pmatrix} 
n_s \\
b_s \\
t_s
\end{pmatrix} = \begin{pmatrix} 
-\kappa \mathbf{t} + \tau \mathbf{b} \\
-\tau \mathbf{n} \\
\kappa \mathbf{n}
\end{pmatrix}
\end{align*} \tag{1.6}
\]

In the above expressions, the curvature along the binormal axis \(\kappa_b\) and the tortuosity along the tangential axis \(\kappa_t\) have been represented by \(\kappa\) and \(\tau\) respectively. The above expressions are used for the calculation of the Christoffel symbols of the 2\textsuperscript{nd} kind, provided in Section 1.5.1.
1.2 Three dimensional representation

A more elaborate description of the helical body is achieved using the local Serret-Frenet frame:

\[ X(x_n, x_b, s) = R(s) + x_n n + x_b b \] (1.7)

In the above expression, \( R \) stands for the position vector of the helix centerline, while \( x_n \) and \( x_b \) stand for the local coordinates along the plane \( nb \) defined by the Serret-Frenet frame. Its gradient \( \partial X/\partial x' \) yields the following non-orthogonal covariant basis:

\[
\begin{align*}
X_{x_n} &= g_1 = n \\
X_{x_b} &= g_2 = b \\
X_s &= g_3 = x_n \tau_b - x_b \tau_n + (1 - x_n \kappa) t
\end{align*}
\] (1.8)

The geometric attributes of the helix three dimensional representation of Eq. 1.7 along with the transformation rules applying between a Cartesian and a Curvilinear helix description are provided in Section 1.5. The non-orthogonal basis (Eq. 1.8) can be written in a tensorial form as:

\[
\begin{pmatrix}
g_1 \\
g_2 \\
g_3
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-\tau x_b & \tau x_n & 1 - k x_n
\end{pmatrix}
\begin{pmatrix}
n \\
b \\
t
\end{pmatrix}
\] (1.9)

The above form indicates that the base vectors and consequently the metric tensor characterizing the parameter space are solely dependent on the helix coordinates \( x_n, x_b \) and on the geometric attributes of the helical body. In other words, the basis of the parameter space defined in Eq. 1.8 is independent of the helix Curvilinear coordinate \( s \), an attribute of primal importance for the developments of Chapter 3.

1.3 Planar projection

Upon the three dimensional helical body representation of Eq. 1.7, helix planar projections can be retrieved. In particular, a planar projection can be obtained with the use of constraining equations descriptive of the projection plane of interest. Below, the constraining equation for a planar projection of the helical body on the Cartesian plane \( Z=0 \) is provided:

\[ Z = 0 \rightarrow \varphi^* = -\frac{\alpha}{b\gamma} x_b \] (1.10)

Substituting the above constraining equation to Eq. 1.7, the following planar domain is obtained:

\[
\begin{pmatrix}
X \\
Y
\end{pmatrix} = (a - x_n) \begin{pmatrix}
\cos \varphi^* \\
\sin \varphi^*
\end{pmatrix} + x_b \frac{b}{\gamma} \begin{pmatrix}
\sin \varphi^* \\
-\cos \varphi^*
\end{pmatrix}, \quad \varphi^* = -\frac{\alpha}{b\gamma} x_b
\] (1.11)
The margins of the above domain are defined on the condition that the local cross section coordinate boundaries \( x_n, x_b \) are known. For the particular case of a circular helix cross section with a radius \( r \), the following relation applies to the coordinates of the helix cross section circumference:

\[
x_n^2 + x_b^2 = r^2
\]  \hspace{1cm} (1.12)

Making use of the above equation as an additional constraint applied to Eq. 1.11, the curve describing the circumference of the helix planar projection is retrieved. Fig. 1.2 shows planar projection circumferences for a helical body with an index \( a/r=3 \) and for different helix angle values \( \theta \):

![Figure 1.2: Helix circular cross section projected circumference plots](image)

The helix projection onto plane \( Z = 0 \) provides a handy framework for the characterization and retrieval of structurally feasible geometric arrangements for multilayer helical constructions, as Section 1.4 below explicates.

### 1.4 Geometric bounds demarcation

Retrieving the geometric boundaries of a helical body is non trivial as the right plot of Fig. 1.2 schematically suggests. For their complete characterization for single and multihelical constructions, a dedicated calculation scheme is subsequently provided based on the helix planar projection analyzed in Section 1.3.
In particular, the circumferential margin $C'$ of the helix planar projection on plane $Z=0$ is calculated. To that extent, the local cross section coordinates $(x_n, x_b)$ of the point $C'$ are computed using the tangent point $C$ of the circular helix cross section (Fig. 1.3 a)), as follows:

$$\theta_{cp} = \arcsin \frac{r}{a}, \quad \theta'_{cp} = 90^\circ - \theta_{cp}, \quad |x_n| = r \cos \theta'_{cp}, \quad |x_b| = r \sin \theta'_{cp}$$

The bounds of each helical body are thereupon obtained through the central angular domain $2\psi$ (Fig. 1.3 b)) formed by the margins of the helix planar projection (Eq. 1.11):

$$2\psi = 2 \arctan \left( \frac{|Y_c|}{|X_c|} \right)$$

If layered helical constructions are to be engineered, the above central angular domain can be used for the determination of the geometric parameter space combination $(a/r, \theta, N)$ for which a complete layer is formed, working out a minimization process. More specifically, the decimal part of the real number $\chi$ arising from the division of a complete angle $2\pi$ to the central angular domain $2\psi$ is minimized, with its integer part $\lfloor \chi \rfloor$ corresponding to the feasible number of helical bodies $N$ that fit the considered geometry:

$$\chi = \frac{2\pi}{2\psi}, \quad \chi = \left\lfloor \frac{2\pi}{2\psi} \right\rfloor + \left\{ \frac{2\pi}{2\psi} \right\}, \quad f_{\text{min}}(\chi, \lfloor \chi \rfloor = N)$$

Figure 1.3: Geometric bounds: a) Cross section b) Helical body
1.5 Appendix

1.5.1 System Basis

By differentiating the helix position vector (Eq. 1.7) with respect to its variables \( \partial X/\partial x^i \), the general covariant base vectors are derived (can be found in [1]):

\[
\begin{align*}
X_{;x_n} &= g_1 = n \\
X_{;x_b} &= g_2 = b \\
X_{;s} &= g_3 = x_n \tau b - x_b \tau n + (1 - x_n \kappa) t
\end{align*}
\]

(1.16)

with \( \kappa \) and \( \tau \) standing for the curvature and tortuocity of the helix, given in Eq. 1.5, while \( n, b, t \) for the local *Serret-Frenet* base vectors given in Eq. 1.4. The covariant metric tensor \( g_{ij} \) descriptive of the geometric space that the helix position vector defines, is given as follows [2]:

\[
\begin{bmatrix}
1 & 0 & -\tau x_b \\
0 & 1 & \tau x_n \\
-\tau x_b & \tau x_n & \tau^2(x_n^2 + x_b^2) + (1 - \kappa x_n)^2
\end{bmatrix}
\]

(1.17)

while the corresponding contravariant metric tensor \( g^{ij} \) as:

\[
\begin{bmatrix}
g^{11} & g^{12} & g^{13} \\
g^{21} & g^{22} & g^{23} \\
g^{31} & g^{32} & g^{33}
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & \tau x_n & -\tau x_b \\
-\tau^2 x_n x_b & g + (\tau x_n)^2 & \tau x_b \\
\tau x_b & \tau x_n & 1
\end{bmatrix}
\]

(1.18)

where \( g = (1 - \kappa x_n)^2 \) is the determinant of the metric tensor, posing the constrain \( \kappa x_n < 1 \). The *Christoffel* symbols of the second kind \( \Gamma^k_{ij} \) are defined as a function of the base vectors as [2]:

\[
\Gamma^k_{ij} = g_{ij} \cdot \mathbf{g}^k
\]

(1.19)

with the first factor given as:

\[
\begin{align*}
g_{1:1} &= g_{1:2} = g_{2:1} = g_{2:2} = 0 \\
g_{1:3} &= g_{3:1} = -\kappa t + \tau b \\
g_{2:3} &= g_{3:2} = -\tau n \\
g_{3:3} &= [-x_n \tau^2 + (1 - x_n \kappa) \kappa] n - x_b \tau^2 b + x_b \tau \kappa t
\end{align*}
\]

(1.20)
which upon a back-substitution in Eq. 1.19 provides the following expressions for the Christoffel symbols, summarized below in a tensorial form (can be found in [1]):

\[
\Gamma^1_{ij} = \begin{bmatrix} 0 & 0 & -\frac{\kappa x_b}{1 - \kappa x_n} \\ 0 & 0 & -\tau \frac{\kappa (\tau x_b)^2}{1 - \kappa x_n} + \kappa (1 - \kappa x_n) - \tau^2 x_n \\ -\frac{\kappa x_b}{1 - \kappa x_n} & -\tau & 0 \end{bmatrix}
\]

\[
\Gamma^2_{ij} = \begin{bmatrix} 0 & 0 & \frac{\kappa x_b}{1 - \kappa x_n} + \tau \\ 0 & 0 & 0 \\ \frac{\kappa x_b}{1 - \kappa x_n} + \tau & 0 & -\frac{\kappa x_b x_n}{1 - \kappa x_n} - \tau^2 x_b \end{bmatrix}
\]

\[
\Gamma^3_{ij} = \begin{bmatrix} 0 & 0 & -\frac{\kappa}{1 - \kappa x_n} \\ 0 & 0 & 0 \\ -\frac{\kappa}{1 - \kappa x_n} & 0 & \frac{\kappa x_b}{1 - \kappa x_n} \end{bmatrix}
\]

### 1.5.2 Base transformation

The Cartesian basis is correlated to the non-orthogonal Curvilinear basis upon the following transformation tensor \(\mathbf{F}\) [2]:

\[
g_i = \mathbf{F} \cdot \mathbf{e}_i, \quad \text{where} \quad \mathbf{F} = g_i \otimes \mathbf{e}_i
\]

calculated in an explicit form as:

\[
\mathbf{F} = \frac{1}{\gamma} \begin{bmatrix} -\gamma C & bS & x_n S + \gamma \tau x_b C - a S \\ -\gamma S & -b C & -x_n C + \gamma \tau x_b S + a C \\ 0 & a & b \end{bmatrix}
\]

where \(C\) and \(S\) in Eq. 1.23 stand for the cos \(\varphi\) and sin \(\varphi\) respectively. Setting \(x_n = x_b = 0\) to the above transformation matrix \(\mathbf{F}\), the transformation tensor \(\mathbf{F}^*\) is retrieved, relating the Serret Frenet \textbf{nbt} basis to the Cartesian basis. Considering Cartesian vector components \(\mathbf{p}\), covariant \((\text{cov})\), contravariant \((\text{contra})\) and Serret Frenet \textbf{nbt} vector components are obtained, through the following transformations [2]:

\[
\mathbf{p}_{\text{cov}} = \mathbf{F}^T \mathbf{p}, \quad \mathbf{p}_{\text{contra}} = \mathbf{F}^{-1} \mathbf{p}, \quad \mathbf{p}_{\text{nbt}} = \mathbf{F}^* \mathbf{p}
\]
Part II

Helix structural response modeling
An overview on the structural response of helical constructions

The axial and torsional strain mechanical response of helical structures has been characterized both from an analytical and a numerical modeling perspective.

Analytical models have been present already from the first part of the last century [3]. In the half a variety of thin beam theory based models appeared in the literature, providing closed-form stiffness expressions. The latter were given as a function of the axial, torsional and bending helix cross section stiffness.

A modeling scheme that was solely based on the helix cross section axial stiffness was presented by [4, 5], with modeling advancements accounting for the torsional stiffness of the helix cross section to be later on incorporated by [6]. The torsional stiffness role was further analyzed means of both experimental and theoretical parametric studies by [7]. Closed-form expressions accounting for the contribution of the bending helix cross section stiffness were furnished by [8]. Moreover, the effect of locked-coil geometries on the structural response was analyzed in the context of multilayer helical strands by [9]. Noteworthy is the lack of symmetry of the $2 \times 2$ stiffness matrix [6, 8, 10], with the first symmetric model to be presented by [11] upon the developments of Ramsey [12].

On the numerical modeling side, the presented simulation schemes were primarily based on beam and volume finite elements. The axial loading helix response was computed through a helix slice that was discretized with volume elements by [13]. The structural response impact of different kinematic considerations for the deformation of single layer engineering strands was analyzed by [14] with the use of beam elements, while a homogenization modeling approach was elaborated by [15] for periodic beam-like structures. Single layer engineering strands were simulated through a dedicated three dimensional finite element model by [16], confronting the numerically obtained results to the analytically predicted ones. The response of single and double helical configurations to tensile and torsional loads was analyzed by [17], the analysis based on beam elements. Moreover, the mechanical response of wire ropes with an independent core (IWRC) to axial strain was addressed by [18] with the use of three dimensional finite elements. While a large number of volume or beam element based modeling schemes of helical structures has been presented, planar modeling techniques have only recently attracted researcher’s attention. In particular, Treysséde presented a SAFE method extension to helical waveguides where the propagation of elastic waves was addressed using a helix plane [1]. Thereupon, Frikha et al. simulated the structural response of a single layer helical strand to axial straining,
grounding their formulation on a helix special twisted basis [19].

While the coupled axial and torsional response of helical bodies has been extensively analyzed, their radial deformation mode has been disregarded. Chapter 2 of the present work provides an insight into the radial deformation of thin helical structures over the entire span of helix angles. More specifically, closed-form expressions for the respective stiffness terms are provided based on thin beam theory analytical developments. Furthermore, a geometric demarcation of the applicability limits of the elaborated closed-form expressions is provided, complimented by a note on the significance of the different contributing response mechanisms.

On the numerical modeling side, the computational modeling techniques that have appeared in the literature were primarily based on beam or volume elements. The latter, quickly incur a high number of degrees of freedom (dofs), which accordingly entails a substantial numerical cost. Nonetheless, modeling large helical assemblies - an emerging necessity in the field of biomechanics [20], or further coupling mechanical models to statistical analysis schemes (e.g. Bayesian inference patterns) requires that the numerical techniques employed are as computationally “inexpensive” as possible. To that extent, Chapter 3 of the present work develops a planar formulation for the simulation of the mechanical response of helical constructions. Thereupon, different loading patterns are considered, namely axial, torsional, radial and thermal, while the computational merits of the modeling approach are explicated.

The planar finite element model is subsequently used in Chapter 4 to characterize the mechanical response of single and multilayer helical constructions. In particular the stiffness properties of structures comprised of both circular and non-circular helical constituents are assessed.
Analytical modeling

2.1 Analytical modeling extension to account for radial strains

In Fig. 2.1 below, the basic geometric parameters of a helical body are depicted, along with a schematic representation of the deformation modes subsequently analyzed:
Fig. 2.1 depicts the force and moment resultants developed on the helix cross section, denoted as $F_t$, $M_t$ and $F_b$, $M_b$ along the tangential and binormal local axis. The illustrated geometric parameters are defined as follows:

$$h = \ell \sin \theta, \quad a \varphi = \ell \cos \theta, \quad \theta = \arctan \left( \frac{b}{a} \right) \quad (2.1)$$

In the current work, the extended $3 \times 3$ structural response of the helical body is analyzed. In particular, the radial force is appended to the structural response, forming the following system of equations:

$$\begin{bmatrix} F_z \\ M_z \\ F_r \end{bmatrix} = \begin{bmatrix} \kappa_{\varepsilon_z \varepsilon_z} & \kappa_{\varepsilon_z \omega'} & \kappa_{\varepsilon_z \varepsilon_r} \\ \kappa_{\omega' \varepsilon_z} & \kappa_{\omega' \omega'} & \kappa_{\omega' \varepsilon_r} \\ \kappa_{\varepsilon_r \varepsilon_z} & \kappa_{\varepsilon_r \omega'} & \kappa_{\varepsilon_r \varepsilon_r} \end{bmatrix} \begin{bmatrix} \varepsilon_z \\ \omega' \\ \varepsilon_r \end{bmatrix} \quad (2.2)$$

In Section 2.1.1 to follow, the radial strain is related to both the helix geometric arrangement and thereupon to the arising force and moment resultants.

### 2.1.1 Structural interpretation of the radial force notion

The stiffness terms relating axial strain $\varepsilon_z$ and twist $\omega'$ with the resulting force $F_z$ and moment $M_z$ are independent of the number of helix windings. In contrast, an absolute radial force resisting a radial strain $\varepsilon_r$ must increase proportionally to the number of windings considered. To that extent, a normalized form of the radial force is introduced. Starting from equilibrium considerations on a thin helix, it can be noted that it is a circumferential force component $F_c$ that balances the applied radial strain, as Fig. 2.2 below schematically illustrates:

![Figure 2.2: Average radial stress retrieval](image)

The radial pressure $p$ developed is related to the circumferential line load $F_c$, as follows:

$$2a \pi b p = 2F_c \quad \Rightarrow \quad \sigma_r = p = \frac{F_c}{\pi a b} \quad (2.3)$$
where \(2a\) is the diameter and \(\pi b\) the height of the one-half helix winding that is used for equilibrium formulation, while \(\bar{\sigma}_r\) the mean resisting stress to the applied strain. Averaged stresses for the axial force \(F_z\) and moment \(M_z\) appearing in Eq. 2.2 can be accordingly defined by dividing with respect to the area that the projection of the single helix period circumference on plane \(Z=0\) encloses:

\[
\bar{\sigma}_z = \frac{F_z}{\pi a^2}, \quad \bar{m}_z = \frac{M_z}{\pi a^2}, \quad \bar{\sigma}_r = p = \frac{F_c}{\pi a b}
\]

(2.4)

Multiplying the above averaged stresses with the same factor, namely the imaginary cylinder cross sectional area \(\pi a^2\), one obtains:

\[
F_z = \pi a^2 \bar{\sigma}_z, \quad M_z = \pi a^2 \bar{m}_z, \quad F_r = \pi a^2 \bar{\sigma}_r = \frac{a}{b} F_c
\]

(2.5)

where the rightmost equation correlates the radial force to the circumferential force \(F_c\) developed.

### 2.1.2 Kinematic, constitutive and equilibrium equations

Following the analytic theory developments presented in [11] and adding the radial helix deformation mode, the following displacement variations can be defined, defining accordingly the radial, the axial and the torsional helix strain:

\[
\delta \alpha = \epsilon_r a, \quad \delta h = \epsilon_z h, \quad \frac{\delta \varphi}{h} = \frac{\zeta h}{a} \cot \theta
\]

(2.6)

where \(\zeta\) stands for the normalized rotational displacement of the helix. The preceding deformation patterns are related to the helix geometry starting from the variation of the first two relations defined in Eq. 2.1:

\[
\delta h = \delta \ell \sin \theta + \ell \cos \theta \delta \theta
\]

\[
\delta (a \varphi) = \delta a \varphi + a \delta \varphi = \delta \ell \cos \theta - \ell \sin \theta \delta \theta
\]

(2.7)

Substituting the geometric definitions of Eq. 2.6 in the above equation set (Eq. 2.7) and multiplying by \(\cos \theta\) and \(\sin \theta\) the upper and lower equality respectively, one retrieves:

\[
\epsilon_z h \cos \theta = \delta \ell \sin \theta \cos \theta + \ell \cos^2 \theta \delta \theta
\]

\[
\epsilon_r a \varphi \sin \theta + \zeta h \cos \theta = \delta \ell \sin \theta \cos \theta - \ell \sin^2 \theta \delta \theta
\]

(2.8)

Thereafter, subtracting the second equation out of the first yields the angular variation as a function of the deformation modes considered and of geometric terms:

\[
(\epsilon_z - \zeta) h \cos \theta - \epsilon_r a \varphi \sin \theta = \ell \delta \theta
\]

(2.9)

which can be further simplified with the use of the first two equalities of Eq. 2.1:
\[ \delta \theta = (\epsilon_z - \zeta - \epsilon_r) \sin \theta \cos \theta \]  

(2.10)

The above expression relates the helix angular variation to the imposed strains. It constitutes the basic formula for the formulation of the constitutive and equilibrium equations. The normal strain to which the helical body is subject \( \epsilon_{hb} \) along the local tangential axis \( t \) is computed by dividing the first equality of Eq. 2.8 by \( \ell \) and replacing the angular variation expression retrieved in Eq. 2.10:

\[ \epsilon_{hb} = \frac{\delta \ell}{\ell} = \epsilon_z \sin^2 \theta + (\epsilon_r + \zeta) \cos^2 \theta \]  

(2.11)

The flexural and torsional strains that the angular rotation \( \delta \theta \) (Eq. 2.10) induces around the binormal and tangential local axis are subsequently computed, excluding rigid body motion terms which do not contribute to the mechanical power created by angular rotations [12]:

\[
\begin{align*}
\omega_2 &= -\frac{\sin^3 \theta \cos^2 \theta}{a} \epsilon_z + \frac{\cos^2 \theta (1 + \sin^2 \theta)}{a} (\zeta + \epsilon_r) \\
\omega_3 &= -\frac{\sin \theta \cos^3 \theta}{a} \epsilon_z + \frac{\sin^3 \theta \cos \theta}{a} (\zeta + \epsilon_r)
\end{align*}
\]  

(2.12)

The constitutive equations are thereupon written as:

\[
\begin{align*}
F_t &= EA \epsilon_{hb} \\
M_b &= EI \omega_2 \\
M_t &= GJ \omega_3
\end{align*}
\]  

(2.13)

where no separate expression for the binormal shear force \( F_b \) has been used, following the consideration of a zero moment development around the helical body normal axis [11]. The latter suggests that the binormal force is a sole function of the moments developed and hence that its strain energy contribution is already included in the moment description:

\[ F_b = (M_t \sin \theta - M_b \sin \theta) \frac{\cos \theta}{a} \]  

(2.14)

Forming equilibrium and substituting the circumferential force \( F_c \) means of its resulting moment \( F_c = \frac{M_c}{a} \), the following system of equations is obtained:

\[
\begin{align*}
F_z &= F_t \sin \theta + F_b \cos \theta \\
M_z &= M_t \sin \theta + M_b \cos \theta + a (F_t \cos \theta - F_b \sin \theta) \\
F_r &= \frac{a}{b} F_c = M_z / b
\end{align*}
\]  

(2.15)

### 2.1.3 Extended stiffness matrix

Upon substitution of the constitutive equations (Eqs. 2.13) and of the binormal force expression (Eq. 2.14) in the above equilibrium equations (Eqs. 2.15), the following
expressions are retrieved:

\[
F_z = (EA s^3 + \frac{EI}{a^2} s^3 c^4 + \frac{GJ}{a^2} sc^6) \varepsilon_z + (EAac^2 s^3 - \frac{EI}{a} s^2 c^3 (1 + s^2) + \frac{GJ}{a} s^4 c^3) \\frac{\delta \varphi}{h} \\
+ (EA c^2 s - \frac{EI}{a^2} sc^4 (1 + s^2) + \frac{GJ}{a^2} s^3 c^4) \varepsilon_r
\]

\[
M_z = (EAacs^2 - \frac{EI}{a} s^2 c^3 (1 + s^2) + \frac{GJ}{a} s^4 c^3) \varepsilon_z \\
+ (EAa^2 c^2 s + EIc^2 s(1 + s^2)^2 + GJs^7) \\frac{\delta \varphi}{h} + (EAac^3 + \frac{EI}{a} c^3 (1 + s^2)^2 + \frac{GJ}{a} s^6 c) \varepsilon_r
\]

\[
F_r = (EA c^4 s - \frac{EI}{a^2} s^3 c^2 (1 + s^2) + \frac{GJ}{a^2} s^5 c^2) \varepsilon_z + (EAac^3 + \frac{EI}{a} c^3 (1 + s^2)^2 + \frac{GJ}{a} s^6 c) \\frac{\delta \varphi}{h} \\
+ (EA c^4 s + \frac{EI}{a^2} c^4 (1 + s^2)^2 + \frac{GJ}{a^2} s^5 c^2) \varepsilon_r
\]

(2.16)

upon which the following stiffness term expressions are obtained:

\[
\begin{align*}
\kappa_{\varepsilon_z \varepsilon_z} &= EA s^3 + \frac{EI}{a^2} s^3 c^4 + \frac{GJ}{a^2} sc^6 \\
\kappa_{\omega' \omega'} &= E A a^2 c^2 s + EIc^2 s(1 + s^2)^2 + GJs^7 \\
\kappa_{\varepsilon_r \varepsilon_r} &= EA \frac{c^4}{s} + \frac{EI}{a^2} c^4 (1 + s^2)^2 + \frac{GJ}{a^2} s^5 c^2 \\
\kappa_{\omega' \varepsilon_z} &= \kappa_{\varepsilon_z \omega'} = EAacs^2 - \frac{EI}{a} s^2 c^3 (1 + s^2) + \frac{GJ}{a} s^4 c^3 \\
\kappa_{\varepsilon_r \varepsilon_r} &= \kappa_{\varepsilon_r \varepsilon_r} = EA c^2 s - \frac{EI}{a^2} sc^4 (1 + s^2) + \frac{GJ}{a^2} s^3 c^4 \\
\kappa_{\omega' \varepsilon_r} &= \kappa_{\varepsilon_r \omega'} = EAac^3 + \frac{EI}{a} c^3 (1 + s^2)^2 + \frac{GJ}{a} s^6 c
\end{align*}
\]

(2.17)

### 2.1.4 Stiffness terms verifications

A commercial, general purpose finite element program (Ansys) has been used to simulate the helix structural response and verify the extended stiffness matrix. A single helix period \(2\pi\) with an index \(a/r = 20\) \((r=0.05)\) has been constructed, complying with the thin helix considerations made by analytic formulations. The helix geometry has been parametrically varied from closed-coil helix configurations to steep angles, while the boundary conditions applied to the different straining patterns follow the stiffness definition for linear analysis. More specifically, the \(\kappa_{ij}\) stiffness term is defined as the force retrieved in the degree of freedom (dof) \(i\) due to a deformation applied at the dof \(j\) when all other dofs are held restrained. Fig. 2.3 below presents a model snapshot for a helix angle of 45\(^o\).
It should be pointed out that the typical description employed in Eq. 2.2 is dependent on the helix geometry. Therefore, a normalized form of its entries is used, the normalization carried out as follows:

\[
\begin{align*}
K^\ast_{\epsilon_z\epsilon_z} &= \frac{\rho_{zzzz}}{Ea} \\
K^\ast_{\epsilon_z\epsilon_r} &= \frac{\rho_{zzrr}}{Ea} \\
K^\ast_{\epsilon_r\epsilon_r} &= \frac{\rho_{rrrr}}{Ea} \\
K^\ast_{\omega'_z\epsilon_z} &= \frac{\rho_{zzz\omega'}}{Ea} \\
K^\ast_{\epsilon_r\omega'_r} &= \frac{\rho_{rrr\omega'}}{Ea} \\
K^\ast_{\omega'_r\omega'_r} &= \frac{\rho_{r\omega'\omega'}}{Ea^2}
\end{align*}
\] (2.18)

Figs. 2.4 to 2.6 below provide a graphical depiction of the normalized closed-form stiffness term expressions along with the finite element results for all considered deformation modes.

Figure 2.4: Axial strain stiffness terms
In Fig. 2.6, analytical and numerical modeling results are in excellent agreement over the entire range of helix angles, any discrepancies lying below 0.5% of difference.

2.1.5 On the structural impact of the radial deformation mode

The graphical depiction of the stiffness matrix terms in Figs. 2.4 to 2.6 dictates that the radial terms significance is dependent on the helix angular configuration and therefore on the application of interest. More specifically, it can be noted that for steep helix angles ($\theta \to 90^\circ$) the coupling terms ($k^*_{\varepsilon z\varepsilon r}$, $k^*_{\varepsilon r\omega'}$) approach zero, while the purely radial stiffness term ($k^*_{\varepsilon r\varepsilon r}$) is rather minimal. In other words, the response is primarily tension and torsion dominated dictating that - for steep angular configurations - the radial
deformation mode effect is minor and can be neglected. Conversely, for moderate or low helix angular configurations, the impact of the radial deformation mode cannot be disregarded. More specifically, for moderate helix angles a maximum axial-radial response coupling is noted \((\kappa_{e_r^* e_r}^*)\), an observation that needs to be taken into account in the design of helical assemblies. Further, for low values of the helix angle, a stronger coupling to torsion arises with the pertinent stiffness term \((\kappa_{\omega \epsilon_r^*})\) approaching unity. The latter points to the mental analogy of a ring that is radially strained for which the theoretically predicted ring force is retrieved.

To conclude, the incorporation of the radial deformation in the analysis and design of helical bodies is of value not only in the field of engineering cable construction but more urgently in areas where the configurations employed require that its effect is taken into account, as for example in the realm of biomechanical applications. In this context, scaffold or stent design are indicative examples of the interplay between engineering and medicine where engineering analysis has a significant impact on the applicability, efficiency and safety of the solutions selected. It should be noted though that the hereby provided closed-form expressions are a-priori restrained to linear analysis, constituting thus solely a first approximation for applications where the incorporation of non-linear phenomena is indispensable.

### 2.2 Analytical modeling applicability limits

#### 2.2.1 Slender helix geometry characterization

The analytical closed-formed solutions derived in Section 2.1 are based on Euler-Bernoulli beam theory considerations that are applicable for slender beam geometries. Therefore, their validity is restrained to the geometric space for which the more elaborate Timoshenko beam theory describing non-slender geometries is equivalent to the Euler-Bernoulli one.

Observing the ordinary differential equations that describe the two theories for the static case, it can be noted that their formulations are identical as long as the following term (appearing as an extra term in the Timoshenko formulation) is negligible:

\[
\frac{EI}{\kappa L^2 AG} \ll 1
\]  
(2.19)

where \(E\) and \(G\) stand for the elastic and shear modulus respectively and \(\kappa\) for the Timoshenko shear coefficient (approximately unity for circular sections), while \(A, I\) and \(L\) stand for the helix cross section area, the moment of inertia and the length of the structural member accordingly.

Neglecting the influence of the material moduli ratio in the above expression (that is bounded below unity for typical engineering materials and Poisson ratios), Eq. 2.19 can be reduced to a purely geometrically dependent condition. In particular, for a circular helix cross section of radius \(r\), Eq. 2.19 simplifies to the condition that the ratio of radius \(r\) over the length \(L\) of the member are significantly lower than unity [21]. Elaborating on the above condition for the curved helix geometry, the following expression is retrieved:
\[
\frac{r}{L} \left( \frac{a}{r}, \vartheta \right) = \frac{r}{2\pi \gamma} = \frac{1}{a} \frac{2\pi}{2\sqrt{1 + \tan^2 \vartheta}} = \left( \frac{a}{r} \right)^{-1} \frac{1}{2\pi \sqrt{1 + \tan^2 \vartheta}}
\] (2.20)

The above equation is a function of two distinct parameters, in particular of the helix index \( a/r \) and of the helix angle \( \vartheta \). Fig. 2.7 below schematically depicts the function value over a significant range of the involved parameters:

![Figure 2.7: Slender helix geometric bounds](image)

As Fig. 2.7 indicates, for helix indexes \( a/r \) greater than 10 and for all helix angles or for helix angles greater than 80° and for all helix indexes, trivial function values are obtained, significantly lower than unity (lower than 0.02 thus almost two orders of magnitude). Conversely, the remaining parameter space yields non-trivial values, suggesting that a primal geometric requirement for the Timoshenko beam theory to be simplified and become equivalent to the Euler-Bernoulli one is violated.

As a result, for helical body geometries adhering to the above parameter space, more elaborate modeling approaches need to be followed, accounting for the effect of phenomena that have been currently disregarded, such as for transverse shear effects or for a shifted helix cross section neutral axis for the calculation of the force and moment resultants.
2.2.2 On the significance of the helix cross section stiffness contributions

The closed-form stiffness term expressions obtained in Section 2.1 are explicit functions of three distinct contributions, namely the axial $EA$, torsional $GJ$ and bending $EI$ helix cross section stiffness.

The contribution of each of the above mechanisms is not equally significant over the different helix angular configurations. Taking the ratio of the response mechanism contributions entering the stiffness terms of Eq. 2.17, in particular the ratio of the bending and torsional terms to the axial ones, a quantification of the their relative contribution can be obtained with respect to the helix geometric configuration:

\[
\left(\frac{\kappa_{EI}}{\kappa_{EA}}\right)_{\epsilon_z\epsilon_z} = \frac{1}{4} \left(\frac{r}{a}\right)^2 c^4 \quad \left(\frac{\kappa_{GJ}}{\kappa_{EA}}\right)_{\epsilon_z\epsilon_z} = \frac{1}{2} \left(\frac{r}{a}\right)^2 \kappa \quad \left(\frac{\kappa_{GJ}}{\kappa_{EA}}\right)_{\omega'\epsilon_z} = \frac{1}{2} G \left(\frac{r}{a}\right)^2 c^2 s^2
\]

The above quotients are solely dependent on the inverse of the helix index $a/r$, on trigonometric expressions ($\cos \theta, \sin \theta$), as well as on the ratio of the shear modulus to the elastic modulus.

In the upper part of Fig. 2.8, the ratio of the bending stiffness relevant contributions over the axial ones ($\kappa_{EI}/\kappa_{EA}$) are depicted for all stiffness matrix entries of Eq. 2.21. In the lower part of Fig. 2.8, the response ratio of the torsional stiffness contribution over the axial one ($\kappa_{GJ}/\kappa_{EA}$) is depicted for the $\epsilon_z \epsilon_z, \epsilon_z \omega', \epsilon_r \epsilon_z$ stiffness terms. For that to be done, the elastic to shear modulus ratio appearing in Eq. 2.21 has been simplified considering a Poisson ratio of 0.3.
As Fig. 2.8 indicates, the relative contributions both of the bending and of the torsional helix cross section stiffness for the first column and row of the stiffness matrix, are almost two orders of magnitude lower than the axial one and that over a wide range of the parameter space. In particular for the helix bending cross section stiffness contribution, the upper part of Fig. 2.8 suggests that it is significantly lower than the respective axial one, for all stiffness term entries.

Conversely, Eq. 2.21 dictates that for the lower right $2 \times 2$ part of the helix stiffness matrix, thus for $\kappa_\omega' \omega'$, $\kappa_\omega' \epsilon_r$, and $\kappa_\epsilon_r \epsilon_r$ terms, the torsional stiffness contribution cannot be disregarded, as it is either of the same, or of a higher order of magnitude with respect to the axial one.

Overall, it can be concluded that the bending stiffness contribution is negligible over a substantial span of the parameter space and more generally posed, for any slender geometry designated in Fig. 2.7. Moreover, for the same parameter space, the torsional stiffness contribution can be additively disregarded for the first row and column of the $3 \times 3$ stiffness matrix. As a result, accounting only for helix cross section axial stiffness contribution $E_A$ suffices for the characterization of the helix structural response.
Finite element planar modeling

3.1 Planar finite element modeling weak form

The kinematical equations of the planar model, Eqs. 3.5 through 3.6, including the metrics and Christoffel symbols, Eqs. 1.16 through 1.21, as well as the constitutive equations in curvilinear coordinates, Eqs. 3.7 through 3.10 can be found in F. Treysséde [1].

3.1.1 Linearized curvilinear strains

The non-linear Cauchy-Green strain tensor $\gamma$ is expressed in a Cartesian basis means of the right Cauchy-Green deformation tensor as follows [22]:

$$
\gamma = \frac{1}{2} (\mathbf{C} - \mathbf{I}) = \frac{1}{2} (u_{ij} + u_{ji} + u_{ki}u_{kj}) = \mathbf{E} + \mathbf{O} \left( |\nabla \mathbf{u}|^2 \right)
$$

(3.1)

where $\mathbf{E}$ stands for the linearized Lagrange strain tensor and $\mathbf{O}$ for the higher order terms disregarded for a small strain analysis. In a general convective basis, the strain tensor is defined as follows [22]:

$$
\gamma = \frac{1}{2} (u_{ij} + u_{ji} + g^{kl} u_{ki} u_{lj})
$$

(3.2)

where the symbol $()$ has been used to denote a convective derivative. Neglecting higher order terms, the linearized strain tensor $\epsilon_{ij}$ is obtained:

$$
\gamma_{ij} \simeq \epsilon_{ij} = \frac{1}{2} (u_{ij} + u_{ji})
$$

(3.3)

The derivative of the displacement vector in the general basis is expanded means of the Christoffel symbols of the 2nd kind can be found in Itskov [2]:

$$
\mathbf{u}_{ij} = (u_i g^i)_{,ij} = u_{i,ij} g^i + u_i g^{i,ij} = u_{i,ij} g^i - u_i \Gamma^{i}_{kj} g^k = (u_{i,ij} - u_k \Gamma^{k}_{ij}) g^i
$$

(3.4)

Making use of the symmetry property of the Christoffel symbols, the following expression for the linearized strain tensor is obtained [2]:

31
\[ \epsilon_{ij} = \frac{1}{2} (u_{ij} + u_{ji}) = \frac{1}{2} (u_{i,j} + u_{j,i}) - \Gamma^k_{ij} u_k \] (3.5)

The strain tensor can be written in a matrix notation, separating the variation with respect to the third local coordinate (can be found in [1]):

\[
\begin{pmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\epsilon_{23} \\
\epsilon_{13} \\
\epsilon_{12}
\end{pmatrix} =
\begin{pmatrix}
(\cdot)_{11} & 0 & 0 \\
0 & (\cdot)_{22} & 0 \\
-\Gamma^3_{13} & -\Gamma^3_{23} & -\Gamma^3_{33} \\
-\Gamma^3_{23} & 0 & \frac{1}{2}(\cdot)_{12} \\
-\Gamma^3_{13} & -\Gamma^3_{23} & \frac{1}{2}(\cdot)_{11} - \Gamma^3_{13} \\
\frac{1}{2}(\cdot)_{12} & \frac{1}{2}(\cdot)_{11} & 0
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix} +
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \frac{1}{2}(\cdot)_{33} & 0 \\
\frac{1}{2}(\cdot)_{33} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix}
\] (3.6)

The above form defines two linear operators \( \mathbf{L}_{12} \) and \( \mathbf{L}_3 \), acting on the respective parts of the displacement field.

### 3.1.2 Constitutive equations

A linear elastic and isotropic constitutive formulation is employed, the Cauchy strain tensor \( \boldsymbol{\epsilon} \) related to the Cauchy stress tensor \( \boldsymbol{\sigma} \) means of the material tensor \( \mathbf{C} \) as follows [2]:

\[ \sigma^{ij} = C^{ijkl} \epsilon_{kl} \] (3.7)

with superscripts and subscripts denoting in the general case, contravariant and covariant basis’ components respectively (Section 1.5). The stress vector entering Eq. 3.7 is defined as \( [\sigma^{11} \sigma^{22} \sigma^{33} \sigma^{23} \sigma^{13} \sigma^{12}]^T \), while the strain vector as \( [\epsilon^{11} \epsilon^{22} \epsilon^{33} 2\epsilon^{23} 2\epsilon^{13} 2\epsilon^{12}]^T \).

The pertinent elasticity tensor \( \mathbf{C} \) expression, for an elastic and isotropic material is defined in a Cartesian basis as a function of the Lamé constants and of the Kronecker \( \delta \), [22]:

\[ C^{ijkl} = \Lambda \delta_{ij} \delta_{kl} + G \left[ \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right] \] (3.8)

where:

\[ \Lambda = \frac{E \nu}{(1 + \nu)(1 - 2\nu)}, \quad G = \frac{E}{2(1 + \nu)} \] (3.9)

where \( E \) and \( \nu \) denote the Young modulus and the Poisson ratio respectively. In convective coordinates, the material tensor is dependent on the local basis [2]:

\[ C^{ijkl} = \Lambda g^{ij} g^{kl} + G (g^{ik} g^{jl} + g^{il} g^{jk}) \] (3.10)

with \( g^{ij} \) denoting the entries of the contravariant metric tensor (Section 1.5). It needs to be underlined that the above formulation is valid only upon the assumption of small displacement gradients \( (u_{ij} << 1) \).
3.1.3 Variational formulation

A weak form formulation describing the helix structural response means of the discretization of a cross section of the helical body is subsequently elaborated. To that extent, a helix cross section that lies on the $n$, $b$ plane of the local Serret-Frenet frame is employed as Fig. 3.1 schematically illustrates.

![Figure 3.1: Helix planar finite element mechanical modeling](image)

Thereupon, the concept of *macro* and *micro* strains are utilized. Different loading patterns are described through *macro* strains defined in reference Cartesian coordinates. The complete solution $u_{xyz}$ for an applied strain pattern is composed of two parts:

$$
u_{xyz} = u^{FEM}_{xyz} + \bar{u}_{xyz} \quad (3.11)$$

namely a finite element part and an analytically described displacement field part that is compatible to the applied strain. The above expression can be accordingly written in its helix local curvilinear domain mapping, as follows:

$$u_{123} = u^{FEM}_{123} + \bar{u}_{123} \quad (3.12)$$

Implementing kinematic relations of Eq. 3.6 on the above displacement field (Eq. 3.12), the following *micro* strain field is retrieved:

$$\varepsilon_{123} = \varepsilon^{FEM}_{123} + \bar{\varepsilon}_{123} \quad (3.13)$$
In addition to and independent of the macro and micro strain notion, it is useful to distinguish between mechanical strains $\epsilon^M$, kinematic strains $\epsilon^T$ (or total strains) and free strains $\epsilon^F$, as follows:

$$\epsilon^M_{123} = \epsilon^T_{123} - \bar{\epsilon}^F_{123}$$  \hspace{1cm} (3.14)

where the total strains are connected to mechanical loading due to axial, torsional or radial strains, while free strains are related to thermal or moisture effects. Expanding the above expression for the total strains (Eq. 3.14) with the help of Eq. 3.13, the following expression is retrieved:

$$\epsilon^M_{123} = L_{12} \phi^T \tilde{u}_{123} + \bar{\epsilon}^T_{123} - \bar{\epsilon}^F_{123} = B_{12} \tilde{u}_{123} + \bar{\epsilon}^T_{123} - \bar{\epsilon}^F_{123}$$  \hspace{1cm} (3.15)

where the finite element solution part has been interpolated through the shape functions $\phi$ and any variance of the strains with respect to the helix local curvilinear coordinate $s$ has been dropped. The total potential energy $\Pi$ of the structure is composed of its deformation energy $U$ and of the externally applied work $W$, as follows:

$$\Pi = U - W = \frac{1}{2} \int_{\Omega} \epsilon^M^T C \epsilon^M d\Omega - \int_{\Gamma_{\sigma}} \bar{u}^T \bar{\sigma} d\Gamma_{\sigma} - \int_{\Gamma_u} \hat{u}^T \sigma d\Gamma_u$$  \hspace{1cm} (3.16)

In the above introduced total potential, the external work arises from the prescribed stresses $\hat{\sigma}$ and displacements $\hat{u}$ on the respective domain boundaries as well as from the prescribed macro strains contributing to the volume integral. Substituting the mechanical strain expression of Eq. 3.15 in the energy form of Eq. 3.16 and letting its variation with respect to the unknown solution parameters $\tilde{u}_{123}$ vanish, the following expression arises:

$$\delta \tilde{u}_{123}^T \left[ \sum_{k=1}^{N} \int_{\Omega_k} B^T C (B \tilde{u}_{123} + (\epsilon^T_{123} - \bar{\epsilon}^F_{123})) d\Omega_k - \sum_{k=1}^{N} \int_{\Gamma_{\sigma_k}} \phi^T \bar{\sigma} d\Gamma_{\sigma_k} \right] = 0 \hspace{1cm} (3.17)$$

which yields the following system:

$$K = \sum_{k=1}^{N} \int_{\Omega_k} B^T C B d\Omega_k, \quad r = - \sum_{k=1}^{N} \int_{\Omega_k} B^T C (\epsilon^T_{123} - \bar{\epsilon}^F_{123}) d\Omega_k + \sum_{k=1}^{N} \int_{\Gamma_{\sigma_k}} \phi^T \bar{\sigma} d\Gamma_{\sigma_k} \hspace{1cm} (3.18)$$

In Section 3.2 to follow, expressions for the different straining patterns are elaborated. It needs to be underlined that none of them depends on the out of plane coordinate $s$.

### 3.2 Mechanical loading patterns planar formulation

In sub-sections 3.2.1 to 3.2.4, the following straining pattern expression are explicated:

- **Axial strain** ($\epsilon_z$)
- **Torsional strain** \((\omega')\)
- **Radial strain** \(\epsilon_r\)
- **Thermal strain** \(\epsilon_{th}\)

## 3.2.1 Axial strain

A homogeneous axial strain \(\epsilon_z\) along the *Cartesian* axis \(Z\) is applied to the helical body. The strain field is compatible with the following displacement field:

\[
\check{\mathbf{u}}_{xyz} = \begin{bmatrix} \check{u}_x \\ \check{u}_y \\ \check{u}_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ Z \end{bmatrix} \epsilon_z \tag{3.19}
\]

The contingent local curvilinear displacement components and their respective displacement gradients are given as follows:

\[
\check{u}_{123} = \begin{bmatrix} 0 \\ \kappa ax_b + \kappa bs \\ \tau ax_b + \tau bs \end{bmatrix} \epsilon_z, \quad \nabla \check{u}_{123} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \kappa a & \kappa b \\ 0 & \tau a & \tau b \end{bmatrix} \epsilon_z \tag{3.20}
\]

upon which the evaluation of the kinematic relations of Eq. 3.6 yields the following imposed total strain field expression:

\[
\check{\epsilon}_{123}^T = \begin{bmatrix} 0 \\ \kappa a \\ \tau b \\ (\kappa b + \tau a) \\ 0 \\ 0 \end{bmatrix} \epsilon_z \tag{3.21}
\]

## 3.2.2 Torsional strain

A homogeneous helix twist \(\omega'\) defines the helix rotation per unit helix curvilinear length \(L_z = bs/\gamma\). The absolute rotation \(\omega\) increases linearly with the helix local curvilinear coordinate \(s\) (Eq. 1.7):

\[
\omega(s) = \omega' \frac{b}{\gamma} s \tag{3.22}
\]

the compatible displacement field to the above introduced torsional strain is given as:

\[
\check{\mathbf{u}}_{xyz} = \begin{bmatrix} \check{u}_x \\ \check{u}_y \\ \check{u}_z \end{bmatrix} = \frac{b}{\gamma} s \begin{bmatrix} -Y \\ X \\ 0 \end{bmatrix} \omega' \tag{3.23}
\]
The curvilinear displacement components are thereupon computed as follows:

\[
\bar{u}_{123} = \tau \begin{bmatrix}
    -bx_b \\
    -b(a - x_n) \\
    b\tau x_b^2 + (a - x_n)^2
\end{bmatrix} \omega' \tag{3.24}
\]

with the respective displacement gradients computed as:

\[
\nabla \bar{u}_{123} = \tau \begin{bmatrix}
    0 & -b s & -bx_b \\
    -b s & 0 & -bx_n^a \\
    -2x_n^a s & 2b s \tau x_b & b\tau x_b^2 + (x_n^a)^2
\end{bmatrix} \omega' \tag{3.25}
\]

where the abbreviation \( x_n^a = a - x_n \) has been used. Evaluation of the kinematic equations of Eq. 3.6 yields the following imposed total strain field expression:

\[
\bar{\epsilon}_{123}^T = \begin{bmatrix}
    1 \\
    \tau b \tau x_b^2 + (a - x_n)^2 \\
    -\tau b (a - x_n) \\
    -\tau bx_b \\
    0
\end{bmatrix} \omega' \tag{3.26}
\]

### 3.2.3 Radial strain

An applied radial strain \( \bar{\epsilon}_r \) is compatible with the following displacement field:

\[
\bar{u}_{xyz} = \begin{bmatrix}
    \bar{u}_x \\
    \bar{u}_y \\
    \bar{u}_z
\end{bmatrix} = \begin{bmatrix}
    X \\
    Y \\
    0
\end{bmatrix} \bar{\epsilon}_r \tag{3.27}
\]

Upon which the curvilinear displacement field vector and its gradient are calculated:

\[
\bar{u}_{123} = \begin{bmatrix}
    -(a - x_n) \\
    b\tau x_b \\
    0
\end{bmatrix} \bar{\epsilon}_r \quad \nabla \bar{u}_{123} = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & b\tau & 0 \\
    0 & 0 & 0
\end{bmatrix} \bar{\epsilon}_r \tag{3.28}
\]

Evaluating the kinematic relations of Eq. 3.6 the following applied total strain field is retrieved:

\[
\bar{\epsilon}_{123}^T = \begin{bmatrix}
    \frac{1}{\tau} (x_n - a)^2 + (\tau x_b)^2 \\
    2\tau (x_n - a) \\
    -2\tau x_b \\
    0
\end{bmatrix} \bar{\epsilon}_r \tag{3.29}
\]
3.2.4 Thermal strain

A homogeneous and isotropic material with a thermal expansion coefficient $\alpha_{th}$ is subject to a strain $\bar{\epsilon}_{th} = \alpha_{th} \Delta T$ due to a temperature change $\Delta T$. The compatible displacement field is given as:

$$\bar{u}_{xyz} = \begin{pmatrix} \bar{u}_x \\ \bar{u}_y \\ \bar{u}_z \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \bar{\epsilon}_{th} \quad (3.30)$$

The curvilinear displacement field components and the displacement field gradient are computed as:

$$\bar{u}_{123} = \begin{pmatrix} -(a - x_n) \\ a \tau s + x_b \\ (bs + ax_b) \tau \end{pmatrix} \bar{\epsilon}_{th} \quad \nabla \bar{u}_{123} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a \tau \\ 0 & a \tau & b \tau \end{bmatrix} \bar{\epsilon}_{th} \quad (3.31)$$

Thereafter, evaluating the kinematic relations of Eq. 3.6, the following applied free strain expression is obtained:

$$\bar{\epsilon}^F_{123} = \begin{pmatrix} 1 \\ 1 \\ \frac{1}{\gamma}(a - x_n)^2 + b \tau (1 + \left( \frac{x_n}{\gamma} \right)^2) \\ 2 \tau x_n \\ -2 \tau x_b \\ 0 \end{pmatrix} \bar{\epsilon}_{th} \quad (3.32)$$
3.3 Finite element modeling specifications

3.3.1 Domain discretization

The helix cross section plane is discretized with linear triangular finite elements as Fig. 3.2 below indicates.

\[ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 1 - \xi - \eta \\ \xi \\ \eta \end{pmatrix} \]  \hspace{1cm} (3.33)

An isoparametric formulation is followed for the domain discretization, so that the points within the modeling domain \((x_n, x_b)\) are mapped upon the same approximation used for the description of the deformation within the element, thus:

\[ u_n(\xi, \eta) = \phi^T \tilde{x}_n \quad u_b(\xi, \eta) = \phi^T \tilde{x}_b \]  \hspace{1cm} (3.34)

where \(\tilde{x}_n\) and \(\tilde{x}_b\) refer to the discretized nodal points of the helix cross section. Displacement derivatives with respect to the cross-sectional coordinates \(x_n, x_b\) are subsequently written as:

\[
\begin{align*}
  u_n(\xi, \eta)_{,x_n} & = u_n(\xi, \eta)_x \xi_{,x_n} + u_n(\xi, \eta)_\eta \eta_{,x_n} & u_n(\xi, \eta)_{,x_b} & = u_n(\xi, \eta)_x \xi_{,x_b} + u_n(\xi, \eta)_\eta \eta_{,x_b} \\
  u_b(\xi, \eta)_{,x_n} & = u_b(\xi, \eta)_x \xi_{,x_n} + u_b(\xi, \eta)_\eta \eta_{,x_n} & u_b(\xi, \eta)_{,x_b} & = u_b(\xi, \eta)_x \xi_{,x_b} + u_b(\xi, \eta)_\eta \eta_{,x_b}
\end{align*}
\]  \hspace{1cm} (3.35)

The above derivatives fill the entries of the inverse of the Jacobian matrix, calculated
below as follows:

\[
J^{-1} = \begin{bmatrix}
\xi_{x_n} & \eta_{x_n} \\
\xi_{x_b} & \eta_{x_b}
\end{bmatrix} = \frac{1}{|J|} \begin{bmatrix}
\tilde{x}_b^3 - \tilde{x}_b^1 & \tilde{x}_b^1 - \tilde{x}_b^2 \\
\tilde{x}_n^1 - \tilde{x}_n^3 & \tilde{x}_n^3 - \tilde{x}_n^1
\end{bmatrix}
\] (3.36)

where the nodal displacement of the triangular element have been used in the rightmost matrix of Eq. 3.36. The determinant of the Jacobian, \(|J|\) equals to the double of the area of the triangle:

\[
|J| = (\tilde{x}_n^2 - \tilde{x}_n^1)(\tilde{x}_b^3 - \tilde{x}_b^1) - (\tilde{x}_b^2 - \tilde{x}_b^1)(\tilde{x}_n^3 - \tilde{x}_n^1)
\] (3.37)

Assessing three degrees of freedom per node, two within the plane \(x_n\) \(x_b\) and a third \((s)\), interpolated upon the same shape functions as the in-plane ones, the following expression is retrieved:

\[
\mathbf{u} = \begin{bmatrix} u_{x_n} \\ u_{x_b} \\ u_s \end{bmatrix} = \Phi^T \tilde{\mathbf{u}} = \begin{bmatrix}
\phi_1 & 0 & 0 & \phi_2 & 0 & 0 & \phi_3 & 0 & 0 \\
0 & \phi_1 & 0 & 0 & \phi_2 & 0 & 0 & \phi_3 & 0 \\
0 & 0 & \phi_1 & 0 & 0 & \phi_2 & 0 & 0 & \phi_3
\end{bmatrix} \begin{bmatrix}
\tilde{u}_{x_n}^1 \\
\tilde{u}_{x_b}^1 \\
\tilde{u}_{x_b}^2 \\
\tilde{u}_{x_b}^3 \\
\tilde{u}_{x_n}^1 \\
\tilde{u}_{x_n}^2 \\
\tilde{u}_{x_n}^3 \\
\tilde{u}_s^1 \\
\tilde{u}_s^2 \\
\tilde{u}_s^3
\end{bmatrix}
\] (3.38)

The above interpolation matrix \(\Phi^T\) along with the linear operator \(
\mathbf{L}_{12}\) (3.6) define the deformation matrix \(\mathbf{B}_{12}\) \((6 \times 9)\), relating nodal displacements to element strains as follows:

\[
\mathbf{B}_{12} = \mathbf{L}_{12} \Phi^T = \begin{bmatrix}
\phi_{1,1} & 0 & 0 & \cdots \\
0 & \phi_{1,2} & 0 & \cdots \\
-\Gamma_{33}^1 \phi_1 & -\Gamma_{33}^2 \phi_1 & -\Gamma_{33}^3 \phi_1 & \cdots \\
-\Gamma_{23}^1 \phi_1 & 0 & \frac{1}{2} \phi_{1,2} & \cdots \\
-\Gamma_{13}^1 \phi_1 & -\Gamma_{13}^2 \phi_1 & \frac{1}{2} \phi_{1,1} & -\Gamma_{13}^3 \phi_1 & \cdots \\
\frac{1}{2} \phi_{1,2} & \frac{1}{2} \phi_{1,1} & 0 & \cdots
\end{bmatrix}
\] (3.39)

where the indeces \(1, 2\) have been used instead of the local coordinates \(x_n\) and \(x_b\).

### 3.3.2 Mesh specifications

The planar mesh is constructed so that finite elements of a quasi-uniform size arise. In particular, for the mesh generation algorithm, two different domain partitioning parameters are defined. More specifically, the desired number of divisions of the helix cross...
section central angle can be freely selected, constructing a certain number of *slices*. Furthermore, the radius of the cross section is subdivided to *rings* that can accordingly tune the radial density of the discretized domain. Fig. 3.3 below illustrates the helix cross section geometry approximation upon different mesh densities.

![Cross section meshes of various densities](image)

Figure 3.3: Cross section meshes of various densities

The node numbering and element formation is done in a spiral, counterclockwise outward pattern. The total number of elements ($N_e$) and nodes ($N_n$) formed as a function of the selected number of *rings* ($N_r$) and *slice* ($N_s$) divisions is given as follows:

$$N_e = \sum_{i=1}^{N_r} (2i - 1) N_s \quad N_n = \sum_{i=1}^{N_r} (i N_s) + 1$$

(3.40)

It needs to be pointed out that the geometric approximation of the helix cross section is achieved means of a minor number of elements. In particular, a minimum of 20 elements yields a 5% geometric error, while for roughly 100 elements the section area is approximated up to numerical accuracy.
3.3.3 Non-circular cross section profiles

While it is helical bodies with circular cross sections that have been primarily analyzed, non-circular cross section geometries appear in both natural formations and engineering applications. An indicative example of the former kind constitute hollow helical arrangements encountered in biological constructions [23]. Accordingly, piping systems structured in hollow helical patterns [24] and locked-coil assemblies of rectangular, trapezoidal or Z-shaped cross sections [9] are characteristic examples of helical constructions engineering applications with cross section shapes other than circular.

The planar finite element model elaborated in Sections 3.1 and 3.2 allows for the simulation of the mechanical response of the above non-circular helical constructions provided that the corresponding mesh generator for the helix cross section is constructed. Fig. 3.4 depicts representative discretized cross sectional domains for circular, trapezoidal and Z-shaped helix cross sections.

![Helix cross section types: a) hollow circular b) trapezoidal c) Z-shaped](image)

It needs to be emphasized that the associated numerical analysis cost remains substantially low despite of the rather complex helix geometry as will be explicated in Section 3.5.

3.4 Planar modeling scheme verification

3.4.1 Stiffness terms comparison

The planar finite element model elaborated in Section 3.1 allows for the simulation not only of the helix axial, torsional and radial straining structural response but also of the mechanical response to thermal straining $\epsilon_{th}$. The latter is described with the use of the same force and moment resultants, thus as $[F_z M_z F_r] = [\kappa_{\epsilon_{z_{\epsilon_{th}}}} \kappa_{\omega_{\epsilon_{th}}} \kappa_{\epsilon_{r_{\epsilon_{th}}}}]^T \epsilon_{th}$. The related stiffness terms can be expressed in a normalized non-dimensional form, the normalization carried out as follows:

$$\kappa^*_{\epsilon_{z_{\epsilon_{th}}}} = \frac{F_z}{\epsilon_{th} E A} \quad \kappa^*_{\omega_{\epsilon_{th}}} = \frac{M_z}{\epsilon_{th} a E A} \quad \kappa^*_{\epsilon_{r_{\epsilon_{th}}}} = \frac{F_r}{\epsilon_{th} E A} \quad (3.41)$$
A commercial general purpose finite element program (*Ansys*) has been used to simulate the helix structural response with the same material and geometry specifications as the ones used in Section 2.1.4. Figs. 3.5 to 3.8 below illustrate the normalized stiffness terms for all strain patterns described in Section 3.2. The planar model predictions are compared with the analytical [11] and numerical modeling ones (Section 2.1.4) over a wide range of helix configurations:

Figure 3.5: Normalized axial strain stiffness terms comparison

Figure 3.6: Normalized torsional strain stiffness terms comparison
The above graphical comparisons, not only verify the modeling approach, but further provide an insight into the thermal structural response. The leftmost plot of Fig. 3.8 indicates that thermal loading triggers primarily the helix axial deformation for steep helix angles approaching 90°. Furthermore, it can be noted that the coupling of the thermal deformation mode to the axial one is substantial over a large range of helix angles, as the leftmost plot of Fig. 3.8 suggests, where approximately a 70% of the corresponding rod stiffness is retained for a helix angle of 45°. Moreover, as the middle plot of Fig. 3.8 suggests, the coupling of thermal loading with torsion fortifies upon decreasing helix angle. In the limiting case of a substantially coiled helix geometry approximating the one of a ring (θ → 0), the corresponding normalized stiffness term asymptotically approaches unity (Fig. 3.8 middle), thus the theoretical value of a ring that is radially strained.
Finally, for a coiled helix construction, thermal straining induces a substantial radial loading that is exponentially increasing upon decreasing helix angle $\theta$, as the rightmost plot of Fig. 3.8 indicates.

3.5 Modeling approach computational merits

The computational complexity of finite element (FE) models is a critical analysis parameter, especially when large structural systems are to be simulated. Assessing the complexity of a model determines whether the latter can be used for a particular task considering the available computational power while it further provides an estimate on the time demanded. The asymptotic complexity - denoted as $O(f(N,W))$ - of a linear FE algorithm neglecting lower order contributions converges to [25]:

$$O_{FEM} = O(NW^2) \quad (3.42)$$

where $N$ stands for the number of nodes and $W$ for the bandwidth of the considered system when a direct solution method is followed. Based on the above simplified relation (Eq. 3.42), a lower bound estimate of the scaling factor relating the computational complexity of volume and planar modeling can be deduced. To that extent, Fig. 3.9 pictures a single period ($2\pi$) of a helix, its geometry approximated with 25 volumetric elements:

![Figure 3.9: A single period of a helical body, approximated with 25 prismatic elements](image)

The helix geometry is approximated with $n$ prismatic volume elements. It is important to mark that the mesh density (thus the element number $n$) needs to be refined
enough for the geometry to be reliably represented. If the prismatic elements are connected with identical cross sections at their interfaces and \( N_h \) helical bodies are modeled, the following approximate relation between the complexities of the volumetric \((v)\) and the planar model \((pl)\) is retrieved:

\[
O_v = 4n N_h O_{pl}
\]  
(3.43)

where the factor 4 reflects the fact that connecting the highest node of one cross section to the lowest node of its adjacent cross doubles the bandwidth. Setting as unity the complexity of the planar modeling approach \((O_{pl} = 1)\), a graphical representation of the scaling factor between the two can be reconstructed, as Fig. 3.10 below depicts:

![Graphical representation](image)

**Figure 3.10: Computational complexity comparison**

The more detailed the geometric approximation \((n)\) or the larger the helical assembly \((N_h)\), the higher the computational resources that volumetric modeling requires, already for low values of the parameters involved, as Fig. 3.10 schematically illustrates. The above analysis testifies to the merits of the planar approach, in particular when iterative numerical processes \((Optimization, Bayesian)\) are to be applied.
Applications in engineering helical strands

4.1 Single layer helical strands

The simplest helical assembly geometry construction encountered in engineering practice consists of a single helical layer. Its structural response has been extensively analyzed with the use of both closed-form solutions and numerical models (Section II). In the left plot of Fig. 4.1, the projection of the planar finite element mesh on the Cartesian plane $Z=0$ is depicted, while the right side of the same graph illustrates a reduced modeling approach for which only the core and a single helical body are simulated:

![Simple strand finite element mesh](image)

Figure 4.1: Simple strand finite element mesh

The above modeling reduction is applicable considering that all helical bodies within the helical layer provide the same mechanical response upon axisymmetric loading, while no intra-layer interaction is taken into consideration.
4.1.1 Single layer strand, case study (1)

Below, a metallic strand with an outer layer comprised of six helical bodies is analyzed. Its material and geometric specifications are summarized in Table 4.1:

Table 4.1: Simple strand material and geometric specifications

<table>
<thead>
<tr>
<th>Layer</th>
<th>$E$ (KN/mm$^2$)</th>
<th>$r$ (mm)</th>
<th>$\nu$</th>
<th>$\theta$ (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core</td>
<td>197.9</td>
<td>1.970</td>
<td>0.3</td>
<td>90.0</td>
</tr>
<tr>
<td>1st</td>
<td>197.9</td>
<td>1.865</td>
<td>0.3</td>
<td>80.8, 77.8, 73.0</td>
</tr>
</tbody>
</table>

Fig. 4.2 below compares the strand stiffness terms as provided by analytical [11] and numerical models [16, 19] in a normalized form. The normalization is carried out with respect to $EA_t$, $6EA_ha$ and $6EA_ha^2 + GJ_c$ for the axial, coupling and torsional stiffness terms respectively. The subscripts $t$, $h$ and $c$ indicate accordingly the total, the helical and the core cross section area.

In Fig. 4.2, the closed form solutions provide an upper stiffness bound for the stiffness terms, a bound that all modeling schemes approach for rather high values of the helix angle. The three dimensional modeling predictions yield respectively a lower bound for the purely axial stiffness term providing an up to 8% lower stiffness with respect to the analytically predicted ones. The finite element results for the coupling and torsional stiffness terms are accordingly lower than the ones retrieved out of the closed-form expressions, with the difference to increase upon decreasing helix angles.
4.1.2 Single layer strand, case study (2)

A single layer strand with a helix angle of 81.8° is subsequently analyzed. In particular, its structural response is retrieved considering different modeling approaches. More specifically, beam homogenization [15] and three dimensional finite element modeling predictions [26] are confronted to analytical [11] and planar finite element modeling results (Section 3.1). Table 4.2 summarizes the material and geometric specifications of the structure:

<table>
<thead>
<tr>
<th>Layer</th>
<th>$E$ (GPa)</th>
<th>$r$ (mm)</th>
<th>$\nu$</th>
<th>$\theta$ (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core</td>
<td>200.0</td>
<td>2.675</td>
<td>0.3</td>
<td>90.0</td>
</tr>
<tr>
<td>1st</td>
<td>200.0</td>
<td>2.590</td>
<td>0.3</td>
<td>81.8</td>
</tr>
</tbody>
</table>

Table 4.2: Simple strand material and geometric specifications, case study (2)

Table 4.3 provides the normalized stiffness term predictions for the different modeling approaches: summarized

<table>
<thead>
<tr>
<th>Models</th>
<th>$\kappa^*<em>{e</em>{\epsilon_{z}}}$ (KN)</th>
<th>$\kappa^*<em>{\omega</em>{\epsilon_{z}}}$ (KN mm)</th>
<th>$\kappa^*<em>{\omega</em>{\omega}}$ (KN mm$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytic</td>
<td>0.974</td>
<td>0.140</td>
<td>0.076</td>
</tr>
<tr>
<td>Nawrocki et al.</td>
<td>0.970</td>
<td>0.120</td>
<td>0.078</td>
</tr>
<tr>
<td>Cartraud et al.</td>
<td>0.961</td>
<td>0.137</td>
<td>0.077</td>
</tr>
<tr>
<td>Planar</td>
<td>0.972</td>
<td>0.132</td>
<td>0.075</td>
</tr>
</tbody>
</table>

In Table 4.3 a maximum relevant difference of approximately ($\approx 1.5\%$) is noted between the stiffness predictions of the different models, indicating that resorting to the rather simplest in analytical formulas provides a reliable prediction of the structural response.

4.2 Multilayer helical strands

The numerical simulation of multilayer helical strands upon the planar model explicated in Section 3.1 relies on the following set of assumptions:

1. The distinct contact pattern of the outer layers to their inner ones is neglected, considering a continuous radial contact for all layers.

2. The response within each helical layer is considered as identical for all comprising helical bodies justifying the modeling reduction to a single helical component for each layer.

3. The analysis is restricted to linear with both geometric or material non-linearities disregarded.
The total axial and torsional stiffness of the helical constructions \((\kappa_{T_{\epsilon_{z}}}, \kappa_{T_{\omega'}})\) is obtained by superposing the contributions of the different layers, as follows:

\[
\begin{bmatrix}
F^T_z \\
M^T_z
\end{bmatrix} = \begin{bmatrix}
\sum_{i=1}^{k} \kappa_{\epsilon_{z}N^i}^i + EA_c \\
\sum_{i=1}^{k} \kappa_{\omega'N^i}^i + GJ_c
\end{bmatrix} \begin{bmatrix}
\epsilon_z \\
\omega'
\end{bmatrix}
\]

where \(N^i\) stands for the number of helical bodies within each helical layer \(i\) and \(\kappa^i\) for the corresponding stiffness contribution of a single helical body within the layer \(i\). Accordingly, the symbols \(A_c\) and \(J_c\) stand for the helix cross section area and for the helix polar moment of inertia.

### 4.2.1 Three layer engineering strand

Below, a helical strand comprised of three outer layers is analyzed. All helical layers have cross sections of equal diameters comprised of 6, 12 and 18 helical bodies respectively, starting from the innermost layer. Fig. 4.3 depicts the projection of the planar mesh on the Cartesian plane \(Z=0\):

![Figure 4.3: Planar reduced mesh of a three layer strand](image)

The material and geometric specifications of the construction are summarized in Table 4.4: where \(Or\) stands for the layer orientation. In Table 4.5, the planar modeling predic-

<table>
<thead>
<tr>
<th>Layers</th>
<th>(E) (GPa)</th>
<th>(\nu)</th>
<th>(\theta(\degree))</th>
<th>(Or(\degree))</th>
<th>(r) (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core</td>
<td>190.0</td>
<td>0.3</td>
<td>/</td>
<td>/</td>
<td>0.545</td>
</tr>
<tr>
<td>1st/2nd/3rd</td>
<td>190.0</td>
<td>0.3</td>
<td>79.2</td>
<td>+</td>
<td>0.500</td>
</tr>
</tbody>
</table>
Table 4.5: Three layer strand stiffness properties

<table>
<thead>
<tr>
<th>Model</th>
<th>$\kappa_{\varepsilon_{\varepsilon}}^T$ (KN)</th>
<th>$\kappa_{\varepsilon_{\varepsilon}}^T$ (KN mm)</th>
<th>$\kappa_{\omega\varepsilon}^T$ (KN mm$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytic</td>
<td>5210</td>
<td>2236</td>
<td>1589</td>
</tr>
<tr>
<td>Planar FEM</td>
<td>5140</td>
<td>2210</td>
<td>1420</td>
</tr>
<tr>
<td>Stanova et al.</td>
<td>5000</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 4.5 suggests that no significant differences arise for the stiffness terms retrieved by the different modeling approaches, any relative difference lying below 5%. Noteworthy is through the comparison among the computational demand of the numerical modeling schemes, where for the planar model 300 nodes suffice for the simulation in contrast to the 342947 nodes required for the volumetric model [27].

### 4.2.2 Locked-coil multilayer strand

The response of a five layer locked-coil helical strand is studied. Its four inner layers are composed of helical bodies with circular cross sections ($c$) encompassed by an outer layer of helical bodies with a Z-shaped cross section ($z$). Fig. 4.4 depicts the projection of the planar finite element mesh on the Cartesian plane $Z=0$:

![Figure 4.4: Planar reduced mesh of a five layer locked-coil strand](image)

The strand is made out of steel with a modulus of $E=200\, KN/mm^2$ and a Poisson ratio value of $\nu = 0.28$. Table 4.6 summarizes the geometric specifications of each helical layer: where the thickness $t$ of the outermost Z-shaped layer is related to the layer’s overall area $A_i$ with the circumference of its centerline $t = A_i/2\pi a$. Table 4.7 enlists the normalized stiffness coefficients provided by the planar modeling along with the ones analytically retrieved [11]. The normalization is carried out with respect to $EA_i$, $EA_1a$ and $EA_1a^2$ for the axial, coupling and torsional stiffness term respectively upon a layerwise devision.
### Table 4.6: Locked-coil helical strand geometric specifications

<table>
<thead>
<tr>
<th>Layer ( j )</th>
<th>( N )</th>
<th>( D/t , (mm) )</th>
<th>( A_i , (mm^2) )</th>
<th>( \theta , (^\circ) )</th>
<th>Shape</th>
<th>( a , (mm) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core</td>
<td>1</td>
<td>3.00</td>
<td>7.06</td>
<td>-</td>
<td>c</td>
<td>-</td>
</tr>
<tr>
<td>1(^{st})</td>
<td>6</td>
<td>2.65</td>
<td>33.10</td>
<td>71.45</td>
<td>c</td>
<td>2.76</td>
</tr>
<tr>
<td>2(^{nd})</td>
<td>12</td>
<td>2.65</td>
<td>66.18</td>
<td>-73.19</td>
<td>c</td>
<td>5.33</td>
</tr>
<tr>
<td>3(^{rd})</td>
<td>18</td>
<td>2.65</td>
<td>99.30</td>
<td>73.90</td>
<td>c</td>
<td>7.93</td>
</tr>
<tr>
<td>4(^{th})</td>
<td>24</td>
<td>2.65</td>
<td>132.37</td>
<td>-72.51</td>
<td>c</td>
<td>10.64</td>
</tr>
<tr>
<td>5(^{th})</td>
<td>28</td>
<td>3.50</td>
<td>301</td>
<td>70.20</td>
<td>z</td>
<td>13.71</td>
</tr>
</tbody>
</table>

### Table 4.7: Normalized stiffness coefficients of locked-coil strand

<table>
<thead>
<tr>
<th>Layer ( j )</th>
<th>( \kappa_{\varepsilon,\varepsilon}^* , FEM )</th>
<th>( \kappa_{\omega,\omega}^* , FEM )</th>
<th>( \kappa_{\omega,\omega}' , FEM )</th>
<th>( \kappa_{\varepsilon,\varepsilon}^* , An. )</th>
<th>( \kappa_{\omega,\omega}^* , An. )</th>
<th>( \kappa_{\omega,\omega}' , An. )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1(^{st})</td>
<td>0.84</td>
<td>0.275</td>
<td>0.094</td>
<td>0.85</td>
<td>0.286</td>
<td>0.096</td>
</tr>
<tr>
<td>2(^{nd})</td>
<td>0.87</td>
<td>-0.260</td>
<td>0.077</td>
<td>0.88</td>
<td>-0.266</td>
<td>0.078</td>
</tr>
<tr>
<td>3(^{rd})</td>
<td>0.88</td>
<td>0.254</td>
<td>0.074</td>
<td>0.89</td>
<td>0.256</td>
<td>0.074</td>
</tr>
<tr>
<td>4(^{th})</td>
<td>0.87</td>
<td>-0.272</td>
<td>0.085</td>
<td>0.87</td>
<td>-0.274</td>
<td>0.086</td>
</tr>
<tr>
<td>5(^{th})</td>
<td>0.82</td>
<td>0.298</td>
<td>0.108</td>
<td>0.83</td>
<td>0.30</td>
<td>0.110</td>
</tr>
</tbody>
</table>

Table 4.8 summarizes the total stiffness terms as numerically and analytically predicted: where no significant relative differences can be observed for all stiffness terms.

### Table 4.8: Locked-coil strand total stiffness

<table>
<thead>
<tr>
<th>Model</th>
<th>( \kappa_{\varepsilon,\varepsilon}^T , (MN) )</th>
<th>( \kappa_{\omega,\varepsilon}^T , (MN , mm) )</th>
<th>( \kappa_{\omega,\omega}'^T , (MN , mm^2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Planar FEM</td>
<td>109.5</td>
<td>196</td>
<td>1603</td>
</tr>
<tr>
<td>Analytic</td>
<td>109.31</td>
<td>197.2</td>
<td>1629</td>
</tr>
</tbody>
</table>
Part III

Structural impact of kinematic constraints
Helix torsional stiffness bounds

5.1 Introduction

Helical structures are encountered in various forms of layered assemblies, such as ropes, cables [28] and electricity power transfer conductors [29]. Their structural response has been characterized with the use of both analytical and numerical models:

On the analytical modeling side, the helix axial and torsional loading response has been assessed by means of closed-form expressions, primarily derived upon thin beam theory considerations. In particular, Hruska provided analytical stiffness expressions that were solely based on the axial stiffness $EA$ of the helix cross section [4]. McConnell et al. emphasized on the role of the helix cross section torsional stiffness $GJ$ [7], while closed-form expressions incorporating contributions of the axial, torsional and bending helix cross section stiffness ($EA$, $GJ$ and $EI$ respectively) were provided by Sathikh et al. [30]. Finally, experimental data on the static axial and torsional loading response were provided for single and three layer helical assemblies [31, 32].

On the numerical modeling side, Jiang et al. [13] elaborated a volume model based on the helical assembly structural symmetry, while a homogenization, beam element based model, applicable for beams with periodic micro-structures was presented by Cartraud et al. [15]. Limitations on the applicability of closed-form expressions were pointed out with the use of three dimensional finite element modeling [16]. Moreover, the effect of different inter-wire motions on the mechanical response of single layer helical assemblies was analyzed, concluding that it is the utter pivoting suppression that can have a substantial influence on the structural response and in particular on its torsional stiffness [14].

The mechanical behavior of multilayer helical assemblies was characterized by closed-form expressions formulated upon the response of the assemblies’s single helical constituents [33]. Furthermore, the mechanical behavior of locked-coil geometries was addressed means of analytically derived simplified routines. The latter were used to quantify the effect of the cross section shape and size on the stiffness coefficients, concluding that any influence is rather minor for all practical applications [9]. Moreover, numerical models for the simulation of two layer [34] and three layer helical assemblies were presented [27] based on three dimensional finite element modeling.
Assessing the stiffness bounds of a helical assembly allows for the computation of the construction’s loading bounds. The latter constitutes the basis for the analysis of their long term behavior. To that extent, Alani et al. studied the effect of the mean axial loading on the endurance of helical strands [35], while the stress state that axial and bending loads induced was correlated to the fatigue life of ropes [36]. Furthermore, fretting damage phenomena were related to the axial and bending loading bounds to which the helical assembly was subject [37]. Finally, experimental studies highlighted the role of torsional loading as a failure mechanism of spiral ropes [38].

The weaving pattern of helical assemblies results in a discrete supporting of all helical layers except for the innermost, as Fig. 5.1 schematically illustrates. Axially loading or externally anchoring the construction entails that the support positions - named as trellis points - are subject to compressive loads. The latter are either self-induced due to the radial forces that axial loading creates, or externally applied at the anchoring positions. As a result, the trellis points are contact boundaries that cannot be generally considered as load free, a conclusion experimentally verified upon post-failure inspection [39].

![Figure 5.1: Multilayer helical assembly support pattern](image)

In the sections to follow, the torsional response of helical bodies positioned in multilayer helical assemblies is analyzed. More specifically, torsional stiffness bounds are derived considering the application of kinematic constraints at discrete positions along the helix development (Section 5.2). In particular, the effect of kinematic constraints applied at the helix cross section normal rotational degree of freedom is assessed for different helical assembly structural arrangements. To that extent, scaling factors are introduced, quantifying the constraint effect with respect to analytical, closed-formed expressions (Section 5.3). A discussion on the obtained values along with concluding remarks follows in Section 5.4.
5.2 Helix torsional structural response

5.2.1 Analytical modeling considerations

Analytical models have provided closed-form expressions for the structural response as a function of different contributing mechanisms, namely of the axial $EA$, torsional $GJ$ and bending $EI$ helix cross section stiffness (Section 5.1). Thereupon, the helix torsional stiffness $\kappa_{\omega'\omega'}$ (see Eq. 2.17) has been analytically defined as follows [30]:

$$\kappa_{\omega'\omega'} = E A a^2 c^2 s + G J s^7 + E I s c^2 \left(1 + s^2\right)^2$$

$$\kappa_{\omega'\omega'}(\theta \to 90^\circ) = G J$$

(5.1)

where the abbreviations $s=\sin \theta$ and $c=\cos \theta$ have been employed. In the above expression $E$ and $G$ stand respectively for the Joung’s modulus and shear modulus, while $A$ for the cross sectional area and $I$ and $J$ for the helix cross section central moments of inertia.

Eq. 5.1 predicts the torsional stiffness of a centrally torqued rod $GJ$ for the geometric margin of a helix with a steep angle approaching $\theta \to 90^\circ$. The result reflects the equilibrium equations and the mechanical considerations upon which the stiffness expression have been derived. In particular that the internal forces and moments developed follow solely the tangential ($F_t$, $M_t$) and binormal local vectors ($F_b$, $M_b$) as Fig. ?? illustrates, while no force or moment arises around the normal vector $n$ of the helix cross section.

5.2.2 Helix kinematic constraints

By applying a kinematic constraint on the normal rotational degree of freedom of the helix cross-section, a local moment $M_n$ develops, as Fig. 5.2 schematically illustrates:

![Helix cross section normal rotational degree of freedom](image)

Figure 5.2: Helix cross section normal rotational degree of freedom

Kinematically constraining a helical body that is positioned at a layer $j$ of a multilayer assembly, at discrete periodic positions along its development, subdivides the helix in segments $AB$, formed between the positions of the constraint application (Fig. 5.3 left). Thereupon, a torsional strain $\omega'$ (Section 5.5) deforms the segment $AB$ to $AB'$ (Fig. 5.3...
middle), with the kinematic constraints represented through moment bearing structural simulacrums, thus through torsional springs or fixed supports (Fig. 5.3 rightmost).

![Figure 5.3: Helical segment torsion](image)

The segment end section moments $M_A^n, M_B^n$ arising from the preceding boundary conditions constitute an additive loading that is disregarded by the analytical, closed-form stiffness expression (Eq. 5.1).

### 5.2.3 Kinematically constrained segment geometry

The locations where kinematic constraints apply are subsequently related to the structural arrangement of the helical assembly. Fig. 5.4 depicts a segment of a helical body positioned at a layer $j$ that is supported to its inner layer $j - 1$ at locations $A$ and $B$ :

![Figure 5.4: Helix trellis support pattern](image)

The length of the created segment $l_c$ depends on the lay angle values $\theta_{(j-1)j}$ of the considered layers, as well as on the normal distance between the supports, as Fig. 5.4 depicts.
Noting that for a complete layer packing, the normal support distance is approximately $2r$, $r$ being the helix cross section radius, the following relation is derived:

$$l_c = \frac{2r}{\cos \theta_c}, \quad \theta_j + \theta_{(j-1)} + \theta_c = 90^\circ, \quad \theta'_{(j-1)j} = 90^\circ - \theta_{(j-1)j}$$  \hspace{1cm} (5.2)

with $\theta'_{(j-1)j}$ standing for the helix angles of the considered layers. Fig. 5.5 depicts the normalized segment length $l_c/r$ bounds for a wide range of helix angle values, typically encountered in multilayer helical assembly constructions (Section 5.1):

![Figure 5.5: Helical segment normalized length](image)

It needs to be noted that the analysis considers only the inward support pattern of the assembly, disregarding the possible creation of segments of shorter lengths due to the support of an outer layer helix $(j+1)$ at an intermediate point $P$ of the analyzed segment, as schematically depicted in Fig. 5.4.
5.3 Structural response assessment

The effect of the rotational constraint elaborated in Section 5.2 is quantified for the limiting case of fixed supporting of the helical segment at its end positions \( A, B \) (Fig. 5.2 right). To that extent, scaling factors \( f_{sc} \) are introduced, relating the torsional stiffness of the kinematically constrained helix \( (c) \) to the closed-form, free of constraints \( (f) \) analytical expression provided in Eq. 5.1:

\[
f_{sc} = \frac{\kappa^c_{\omega'\omega'}}{\kappa^f_{\omega'\omega'}} \quad f_{sc} \geq 1 \tag{5.3}
\]

The scaling factors are thereafter computed for a wide range of helix angles \( \theta_j' \) and index values \( a/r \). Fig. 5.6 depicts the factor values for different combinations of the segment length and of the helix angle values \( \theta_j' \), thus for a wide range of structural arrangements. The results are obtained means of a dedicated volume finite element model elaborated in 5.5:

![Figure 5.6: Torsional stiffness scaling factors](image)
5.4 Discussion and conclusion

Fig. 5.6 demonstrates a primal dependence of the scaling factor on the helix angle of the helical segment. More specifically, kinematically constrained segments with a steep helix angle, yield higher scaling factors for a certain value of the normalized helical length $l_c/r$. An analogous effect is noted by increasing the helix index value $a/r$, so that a constrained segment at an outer layer is stiffer than its corresponding inner segment. For the structural arrangements considered, the torsional stiffness of the constrained structure is higher than the corresponding unconstrained one by a factor bound between 1.6 and 5.6.

The constrained torsional structural response is hyperbolically dependent on the length of the analyzed segment. The hyperbolic profile can be interpreted with the use of the structural simulacrum depicted in Fig. 5.3. In particular, the moments $M^A_n, M^B_n$ developed on the end sections of the segment upon the applied torsional strain $\omega'$ entail that a statically equivalent force arises on the $xy$ plane (Fig. 5.3 left). The force magnitude is reversely analogous to the segment length $l_c$ as the segment moment equilibrium dictates. Therefore, applying the same torsional deformation at segments of higher lengths $l_c$ denotes a hyperbolic decrease of the resulting torsional moment $M_z$, or respectively of the introduced scaling factor $f_{sc}$. For sufficiently large segment length values $l_c$, the torsional response predicted by the analytical closed form expression of Eq. 5.1 is asymptotically retrieved ($f_{sc} \rightarrow 1$).

It needs to be underlined that the provided factors $f_{sc}$ demarcate the upper bounds of the torsional stiffness, as an utter suppression of the local cross section normal rotation $n$ has been considered (Fig. 5.3, right). Combining the upper bound of the torsional stiffness with the analytically computed, free of constraints response (Eq. 5.1), all intermediate mechanical response patterns can be retrieved, provided that the level of kinematic constraining is assessed. The latter is dependent on a series of application specific parameters (axial and radial compressive load intensity, support positions contact surface roughness), an analysis that exceeds the scope of the current work.

To conclude, a framework that allows for the incorporation of kinematic constraints in the assessment of the helix torsional response has been elaborated. In particular, constraints have been considered to apply on the helix cross-section normal rotational degree of freedom, at the helix trellis support positions. Thereupon, the torsional stiffness bounds and therefore the construction’s loading bounds have been demarcated in relation to the structural arrangement of the helical assembly. By that means, a linkage between the impact of the kinematic constraint and the geometric pattern of the structure has been derived. The explicated framework can be of use for the analysis of constructions with braiding patterns and helix cross-section configurations out of the hereby considered range.
5.5 Appendix, Kinematically constrained helix mechanical modeling

A torsional strain $\omega'$ results in an absolute rotation $\omega$ that is linearly dependent on the helix local curvilinear coordinate $s$ (Eq. 1.1), as follows:

$$\omega(s) = \omega' \frac{b}{\gamma} s$$ (5.4)

For the current analysis, a torsional strain $\omega' = 0.01$ has been applied. In particular, the one end section $A$ is held fixed, while the other end section $B$ is subject to the following torsional deformation field (Fig. 5.3):

$$\begin{bmatrix} \bar{B}_x \\ \bar{B}_y \\ \bar{B}_z \end{bmatrix} = \begin{bmatrix} -B_y \\ B_x \\ 0 \end{bmatrix} B_z \omega'$$ (5.5)

The end section position $B$ is uniquely defined through Eq. 1.1 upon the specification of the helix index $a/r$, of the lay angle $\theta$ and of the normalized segment length $l_c/r$.

The mechanical response is computed with the use of a general purpose finite element program (Ansys). The numerical models are parametrically constructed for the different helix index $a/r$ and lay angle values $\theta_{j-1}\ldots\theta_j$. A helix with a circular cross section of radius $r = 0.5\, mm$ has been used with a linear elastic material of modulus $E = 210\, GPa$ and a Poisson ratio $\nu = 0.3$. Solid brick 20-node elements have been employed, with a minimum of 15 divisions along the segment length and a dense discretization of more than 50 elements per segment division. Fig. 5.7 below depicts a snapshot of the helical segment model.

![Figure 5.7: Finite element model](image)

The numerical model is subsequently validated for the case where no kinematic constraints apply. To that extent, boundary conditions allowing for the rotational deformation of the end sections $A, B$, around their local normal axis $n$ are exerted. More specifically, at section $A$, all displacement components of the cross section centerline have been restrained, thus the cross section nodes with $U_y, U_z \approx 0$ (Fig. 5.3 left), while at
section $B$, the imposed rotational deformation is applied only at the nodes that lie on the normal vector $n$ of the section. Fig. 5.8 presents the ratio of the numerically obtained torsional stiffness values to the analytically computed ones (Eq. 5.1), for a wide range of structural arrangements of the analyzed helical segment:

![Figure 5.8: Finite element model verification](image)

Fig. 5.8 indicates that the finite element modeling results are in a very good agreement with the analytically predicted ones (Eq. 5.1), when no kinematic constraints apply. In particular, the maximum relative difference restricts to 6%, noted for low values of the helix angle $\theta$ for which the numerical model provides a more compliant structural response.
Part IV

Structural pattern selection of layered helical assemblies
Optimal structural arrangement of multilayer helical assemblies

6.1 Introduction

Helical assemblies are load carrying structures with applications ranging from ropes and electricity power transfer cables to tissue engineering scaffolds [40, 41]. While a large number of studies have been devoted to the analysis of their mechanical properties, the selection of the structural arrangement itself has largely been disregarded, despite its detrimental impact on both the operational mechanical response and the structure’s long term performance. Whereas an experimental retrieval of optimal structural patterns is infeasible, numerical simulations provide an ideal test-bed for this purpose.

A thorough description of the geometric properties of single, double and triple helical bodies has been provided by [42], an analysis of primal use for the understanding of the structuring of helical assemblies. Helical assemblies are commonly encountered as substructures of larger constructions. Cable-bridge structures are characteristic examples of this kind, for which the cable tension level and placement controls the response of the overall construction [28]. Furthermore, helical assemblies are extensively used in electric power transfer, with their design playing a crucial role in the minimization of power transfer losses [29]. Their extensive use necessitated the characterization of their mechanical response, as the analytical and numerical modeling schemes bibliography indicates, primarily in the context of engineering cables.

Using analytical modeling, Lanteigne provided closed-formed solutions for the quantification of the mechanical response of helically armored cables upon different axial, torsional and bending loads [33]. Accordingly, Raoof et al. developed simplified expressions for the stiffness coefficients of locked-coil strands [9], while Utting and Jones provided a large set of experimental data on single and three layer strands accompanied by closed-form stiffness expressions, [31, 32]. Sathikh et al. elaborated stiffness matrix coefficients for the axial and torsional strain response of helical bodies that hold symmetry considerations of the stiffness matrix taking into account contributions arising from the axial, torsional and bending helix cross section stiffness, [11]. Finally, the mechanical response
of double-helix multi strand constructions to axial and torsional loads was analyzed, under
the assumption that their constituents follow a fiber type response \cite{43, 17}

On the numerical modeling side, Jiang et al. estimated the structural properties
of two layer strands using a reduced computational model that took advantage of the
structural and loading symmetry \cite{34}. Similarly, Stanova et al. worked on the axial
stiffness properties of three layered strands \cite{27}. A study on large spiral cables axial load-
strain curves and failure loads was provided by Judge et al., the analysis based on three
dimensional finite element modeling \cite{44}

More recently, helical assembly applications that go beyond the context of engineering
strands have come to the fore. In particular, in the field of biomechanical engineering,
helically braided scaffolds have been used for the restoration of tendon and ligament tissue
\cite{41}. Moreover, the development of artificial and biological material based applications
such as nanotube helical ropes, has asked for a deeper understanding of their mechanical
response, with bottom-up structural response models appearing in the literature \cite{45}.

A significant number of studies have been devoted to assess the impact of loading
bounds on the endurance and long-term functionality of helical assemblies. Argatov
worked on the effect of interwire contact deformation of single layer rope strands making
use of asymptotic modeling \cite{46}. Alani et al. studied the correlation between the mean
axial loading and the endurance limits of helical assemblies, to point out substantial vari-
ations associated with the helix angle selection of the individual layers \cite{35}. Giglio et al.
\cite{36} derived a linkage between the fatigue life and the stress state of ropes that are subject
to axial and bending loads, suggesting that their bounds are directly related to fretting
damage phenomena \cite{37}. Finally, Chaplin performed a number of experimental studies
that quantified the effect of different loading patterns on the life endurance of spiral ropes,
illustrating the role of torsional loads as a failure mechanism \cite{38}.

In the Sections to follow the engineering of a broad spectrum of helical assembly
constructions is elaborated. In particular layered assemblies comprised of up to five layers
are analyzed in Section 6.2. Amongst all possible structures, torsionally counterbalanced
arrangements of high axial stiffness for two, three, four and five layer constructions are
identified (Section 6.3). A discussion on the retrieved optimal braiding patterns follows
along with concluding remarks in Section 6.4.

6.2 Helical assembly constructions

6.2.1 Parameter search space

The parameter search space of the helical assemblies is subsequently defined. For
each helical layer, the cross section radius of the individual helical bodies \( r_i \) is allowed
to vary by a maximum of 50\% with respect to the radius of the core of the structure
\( r_c \), thus \( 0.5 \leq r_i/r_c \leq 1.5 \). The layer centerline position of each layer \( i \), named as
\( a_i \), is defined as a function of the radius of all helical bodies in the different layers \( j \),
\[ \{r_j\}_{j=1}^i \] and of the core radius \( r_c \), as schematically illustrated in Fig. 6.1. The helix angle
of each layer \( \theta_i \) is accordingly considered to vary within \([70^\circ, 85^\circ]\). The angle selection
allows for the constituents of the assembly to be primarily subject to normal rather than shearing stresses, while it guarantees a high axial strength for the overall construction. Furthermore, different layer orientation combinations $Or_i$ are considered. In particular, each layer can follow either a right or a left handed directionality ($+/−$).

The left part of Fig. 6.1 depicts a two layer helical assembly. The internal structuring of the assembly is illustrated via the section $A-A'$, normal to its evolution axis $Z$ at the right hand side of the graph:

![Diagram of a two layer helical assembly]

Figure 6.1: Multilayer helical assembly geometry

The geometric search space bounds for each of the parameters are summarized in Table 6.1 for helical assemblies comprised of up to five layers. The limiting number of helical bodies $N_i$ positioned at each layer can be readily retrieved upon trigonometric considerations elaborated in Section 2.1.1.

<table>
<thead>
<tr>
<th>Layer $i$</th>
<th>$θ_i$ ($^\circ$)</th>
<th>$a_i/r_c$</th>
<th>$N_i$</th>
<th>$Or_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core</td>
<td>90</td>
<td>/</td>
<td>1</td>
<td>/</td>
</tr>
<tr>
<td>1$^{st}$</td>
<td>[70 85]</td>
<td>[1.5 2.5]</td>
<td>[5 9]</td>
<td>+/−</td>
</tr>
<tr>
<td>2$^{nd}$</td>
<td>[70 85]</td>
<td>[2.5 5.5]</td>
<td>[8 28]</td>
<td>+/−</td>
</tr>
<tr>
<td>3$^{rd}$</td>
<td>[70 85]</td>
<td>[3.5 8.5]</td>
<td>[10 45]</td>
<td>+/−</td>
</tr>
<tr>
<td>4$^{th}$</td>
<td>[70 85]</td>
<td>[4.5 11]</td>
<td>[12 64]</td>
<td>+/−</td>
</tr>
<tr>
<td>5$^{th}$</td>
<td>[70 85]</td>
<td>[5.5 14]</td>
<td>[14 80]</td>
<td>+/−</td>
</tr>
</tbody>
</table>

The orientation combinations studied for a two, three and four layer assembly are enlisted in Table 6.2:
Respectively, the orientation combinations for a five layer structure are summarized in Table 6.3:

<table>
<thead>
<tr>
<th>Orientation Combinations</th>
<th>Two layer</th>
<th>Three layer</th>
<th>Four layer</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>++-++</td>
<td>+-++</td>
<td>+++++</td>
</tr>
<tr>
<td></td>
<td>-++++</td>
<td>+++++</td>
<td>+++++</td>
</tr>
<tr>
<td></td>
<td>+++--</td>
<td>+++--</td>
<td>+++--</td>
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<tr>
<td></td>
<td>+---+</td>
<td>+---+</td>
<td>+---+</td>
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<tr>
<td></td>
<td>--+++</td>
<td>--+++</td>
<td>--+++</td>
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<tr>
<td></td>
<td>-+-++</td>
<td>-+-++</td>
<td>-+-++</td>
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<td></td>
<td>+--+</td>
<td>+--+</td>
<td>+--+</td>
</tr>
<tr>
<td></td>
<td>++--</td>
<td>++--</td>
<td>++--</td>
</tr>
<tr>
<td></td>
<td>+---</td>
<td>+---</td>
<td>+---</td>
</tr>
</tbody>
</table>

It needs to be noted that equi-directional combinations have been excluded as no torsionally balanced constructions can arise. Furthermore, opposite polarity orientation combinations with respect to the enlisted ones have not been taken into account, since the minimization of the total arising moment is of sole interest.

### 6.2.2 Geometrically feasible construction space

The parameter space of the layered arrangements is constructed so as to allow only for compact constructions in their radial and circumferential braiding pattern. Since the retrieval of the feasible constructions is non-trivial, the dedicated calculation scheme presented in Section 1.4 is employed, repeated below for completeness.
In particular, the helix boundaries are demarcated by the intersection of the Cartesian plane $Z=0$ with equation 1.7, which yields:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = (a - x_n) \begin{bmatrix} \cos \varphi^* \\ \sin \varphi^* \end{bmatrix} + x_b \frac{b}{\gamma} \begin{bmatrix} \sin \varphi^* \\ -\cos \varphi^* \end{bmatrix}, \quad \varphi^* = -\frac{\alpha}{b\gamma} x_b \quad (6.1)$$

The circumferential margin $C'$ of the helix intersection with plane $Z=0$ is subsequently calculated. To that extent, the local cross section coordinates $(x_n, x_b)$ of the point $C'$ are computed using the tangent point $C$ of the circular helix cross section (Fig. 6.2 a)), as follows:

$$\theta_{cp} = \arcsin \frac{r}{a}, \quad \theta'_{cp} = 90^\circ - \theta_{cp}, \quad |x_n| = r \cos \theta'_{cp}, \quad |x_b| = r \sin \theta'_{cp} \quad (6.2)$$

The bounds of each helical body are thereafter obtained through the central angular domain $2 \psi$ (Fig. 1.3 b)) formed by the margins of the above defined area (Eq. 6.1):

$$2 \psi = 2 \arctan \left( \frac{|Y_c|}{|X_c|} \right) \quad (6.3)$$

The central angular domain is subsequently used to determine the geometric parameter space combination $(a_i/r_c, \theta_i, N_i)$ for which a complete layer is formed. More specifically, the decimal part of the real number $\chi$ arising from the division of a complete angle $2 \pi$ with the central angular domain $2 \psi$ is minimized, with its integer part ($\lfloor \chi \rfloor$) corresponding to the feasible number of helical bodies $N_i$ that fit the considered geometry:

$$\chi = \frac{2 \pi}{2 \psi}, \quad \chi = \left\lfloor \frac{2 \pi}{2 \psi} \right\rfloor + \left\{ \frac{2 \pi}{2 \psi} \right\}, \quad f_{\min} (\chi, \lfloor \chi \rfloor = N) \quad (6.4)$$

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In Fig. 6.3 the structuring of the feasible helical arrangements is illustrated:

![Tree diagram representation of feasible construction paths](image)

Figure 6.3: Tree diagram representation of feasible construction paths

The analysis has been conducted with the following parameter discretizations. The helix angle $\theta^o$ has been discretized with a step of 0.01° while the helix index $(a_i/r_c)$ using a 0.001 step for the first innermost layer and a 0.01 step for all subsequent layers apart from the outermost, fifth layer, for which a 0.1 step has been used. In the resulting discrete feasible parameter space, the possible states are already in the order of tens of thousands for two layer constructions, following an exponential increase with each additional layer introduced.

### 6.2.3 Assembly mechanical modeling

The helical assembly arrangements have been parametrically constructed, while the analytical structural response expressions of Chapter 2 have been used for the computation of quantities of interest $QoI$, thus for the resulting total force $F^T$ and moment $M^T$ (Eq. 6.5). A linear strain driven analysis ($\epsilon_z$) was made, with all computations run in parallel in the ETH cluster Brutus. For the creation of the parameter space (Section 6.2.1) and for the retrieval of the feasible constructions (Section 6.2.2), the core radius $r_c$ was set to unity.

The total axial force ($F^T$) and moment ($M^T$) developed along the central axis $Z$ of the helical assembly (Fig. 6.1) are calculated by summation over the different layer
contributions $k$, as follows:

$$F^T = F_{\text{core}} + \sum_{i=1}^{k} N_i F_i^z,$$

$$M^T = \sum_{i=1}^{k} N_i F_i^c a_i O_{ri}, \quad i = 1, \cdots k$$

(6.5)

with the involved geometric parameters elaborated in Section 6.2. The circumferential and axial force contributions of each helical body ($F_i^z$, $F_i^c$) are calculated by the force and moment resultants developed on the helix cross section (Fig. 2.1):

$$F_c = F_1 \cos \theta + (M_t \sin \theta - M_b \cos \theta)/a, \quad F_z = F_1 \sin \theta + F_b \cos \theta$$

(6.6)

where no separate expression for the binormal cross sectional local force ($F_b$) has been employed, as the latter is a function of the helix cross section moments [11].

### 6.2.4 Assembly optimality criteria

The identification of favorable structural arrangements hinges upon the following structural objectives. The structures favored:

1. **Maximize** the axial stiffness $\kappa^h_A = \frac{F^T}{\epsilon_z}$, or respectively its normalized counterpart $\kappa^{*h}_A$, with the normalization carried out with respect to the stiffness of a rod of the same total cross sectional area and material ($E A^R$).

2. **Minimize** the resulting moment $M^T$, or respectively its normalized form $M^* = \frac{M^T}{M_{\text{max}}^T}$, where $M_{\text{Max}}^T$ is the maximum arising total moment within the considered structural arrangements, discretely for a two, three, four or five layer structure.

The above objectives ($o$) can be written as follows:

$$o = \begin{cases} 
\kappa^{*h}_A = \frac{\kappa^h_A}{E A^R} \quad (\uparrow) \\
M^* = \frac{M^T}{M_{\text{max}}^T} \quad (\downarrow) 
\end{cases}$$

(6.7)

Assuming equal weights amongst the two structural objectives, an *efficiency factor* $f_e$ can be defined as the quotient of the normalized moment over the normalized axial stiffness of the structural arrangement:

$$f_e = \frac{M^*}{\kappa^{*h}_A} \quad (\downarrow)$$

(6.8)

The above factor allows for a classification of the different structural patterns, where favorable constructions are characterized by efficiency factors that approach zero. To avoid sub-optimal axial stiffness constructions, the normalized axial stiffness has been bound to be above 0.85.
6.3 Optimal structural arrangement patterns

6.3.1 Favorable layer orientation combinations

In order to quantify the effect of the different layer orientation combinations, statistics on the respective efficiency factors $f_e$ are provided. Regarding two layer constructions, the efficiency factor mean value is computed as $\bar{f}_e = 0.39$ with its inter-quantile range being $[f_{e|0.25}, f_{e|0.75}] = [0.22, 0.53]$. As regards three and four layer constructions, mean efficiency factor values $\bar{f}_e$ and inter-quantile ranges are presented in Fig. 6.4:

![Figure 6.4: Efficiency factor statistics for three (left) and four (right) layer constructions](image)

The respective values for five layer assemblies are depicted in Fig. 6.5:

![Figure 6.5: Efficiency factor statistics for five layer constructions](image)
In Sections 6.3.2 to 6.3.6, a subset of optimal structural patterns is provided.

### 6.3.2 Two layer construction

In Fig. 6.6 favorable two layer constructions with corresponding efficiency factor values $f_e$ in the lowest 2% quantile are depicted:

![Figure 6.6: Two layer assembly optimal structural patterns](image)

---

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6.3.3 Three layer construction

In Fig. 6.7 favorable three layer constructions with efficiency factor values \( f_e \) in the lower 0.5% quantile are presented, for two inner equi-directional layers followed by an outermost layer of opposite directionality:

![Three layer construction diagram](image)

Figure 6.7: Three layer assembly optimal structural patterns, orientation combination ++-

---

6.3.4 Four layer construction

In Fig. 6.8 favorable four layer constructions with efficiency factor values \( f_e \) in the lowest 0.01% quantile are portrayed. The constructions follow a layer orientation combination of three inner equi-directional layers enclosed by an opposite directionality outermost layer:

![Four layer construction diagram](image)
Figure 6.8: Four layer assembly optimal structural patterns, orientation combination ++++-
6.3.5 Absolute optimal structural arrangements for three, four and five layer helical assemblies

In Tables 6.4, 6.5 and 6.6, the most favorable structural arrangements retrieved upon the hereby considered parametric space are tabulated:

Table 6.4: Three layer assembly optimal structural arrangements

| List | $a_{1|2|3}/r_c$ | $\theta_{1|2|3}(\degree)$ | $N_{1|2|3}$ | Or |
|------|----------------|----------------------|-----------|----|
| 1    | 1.517|2.80|4.50 | 84.85|76.56|83.13 | 9|11|15 | + + - |
| 2    | 2.372|4.62|6.93 | 77.40|77.05|82.54 | 5|16|15 | + - + |

Table 6.5: Four layer assembly optimal structural arrangements

| List | $a_{1|2|3|4}/r_c$ | $\theta_{1|2|3|4}(\degree)$ | $N_{1|2|3|4}$ | Or |
|------|------------------|----------------------|-----------|----|
| 1    | 1.966|4.23|6.14|7.75 | 77.72|83.43|78.96|81.15 | 6|10|31|24 | + + + + |
| 2    | 1.730|3.31|4.81|6.36 | 75.12|82.58|71.54|71.51 | 7|12|22|21 | + + + - |
| 3    | 2.393|4.76|6.74|9.05 | 80.18|78.55|73.23|74.46 | 5|15|20|21 | + - + + |
| 4    | 1.580|3.06|4.49|6.42 | 72.32|71.28|78.23|78.23 | 8|10|26|14 | + - + - |

Table 6.6: Five layer assembly optimal structural arrangements

| $a_{1|2|3|4|5}/r_c$ | $\theta_{1|2|3|4|5}(\degree)$ | $N_{1|2|3|4|5}$ | Or |
|-------------------|----------------------|-----------|----|
| 1.721|3.94|6.94|9.14|11.04 | 73.35|83.4|75.8|72.0|76.0 | 7|8|14|39|28 | + + + + + |
| 1.517|2.78|4.72|6.81|8.41 | 84.85|72.3|76.3|75.8|78.8 | 9|11|12|23|37 | + + + + - |
| 1.708|3.09|4.32|5.78|7.48 | 71.06|77.0|83.7|71.0|81.7 | 7|14|24|19|29 | + + + - - |
| 2.321|4.37|6.15|8.40|10.30 | 72.32|73.8|79.2|73.2|84.7 | 5|18|18|21|46 | + - + + |
| 1.916|3.78|5.64|7.45|9.45 | 71.53|75.6|78.3|74.4|74.8 | 6|12|19|25|26 | + + + - |

6.3.6 On the optimality bounds of helical assemblies comprised of layers with helical bodies of quasi-equal cross section radius

In certain applications (e.g. engineering metallic strands [27]), quasi-equal cross section radius values ($r_i$) have been employed, up to a 2% difference with respect to the core radius. Tables 6.7 to 6.10 provide favorable structural arrangements for up to five layer helical assembly constructions in the above curtailed parameter space.
Table 6.7: Two layer assembly

<table>
<thead>
<tr>
<th>List</th>
<th>$a_1/r_c$</th>
<th>$\theta_1(\circ)$</th>
<th>$N_1$</th>
<th>$a_2/r_c$</th>
<th>$\theta_2(\circ)$</th>
<th>$N_2$</th>
<th>Or</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.984</td>
<td>81.60</td>
<td>6</td>
<td>4.00</td>
<td>84.31</td>
<td>12</td>
<td>+-</td>
</tr>
</tbody>
</table>

Table 6.8: Three layer assembly

<table>
<thead>
<tr>
<th>List</th>
<th>$a_{123}/r_c$</th>
<th>$\theta_{123}(\circ)$</th>
<th>$N_{123}$</th>
<th>Or</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.983/3.96/5.96</td>
<td>81.34/74.72/77.38</td>
<td>6/12/18</td>
<td>++-</td>
</tr>
</tbody>
</table>

Table 6.9: Four layer assembly

<table>
<thead>
<tr>
<th>List</th>
<th>$a_{1234}/r_c$</th>
<th>$\theta_{1234}(\circ)$</th>
<th>$N_{1234}$</th>
<th>Or</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.986/4.00/6.03/8.03</td>
<td>82.15/83.72/72.43/83.34</td>
<td>6/12/18/25</td>
<td>+ + -</td>
</tr>
<tr>
<td>2</td>
<td>1.994/4.02/6.04/8.03</td>
<td>84.86/81.79/84.90/83.59</td>
<td>6/12/19/25</td>
<td>+ + + -</td>
</tr>
</tbody>
</table>

Table 6.10: Five layer assembly

<table>
<thead>
<tr>
<th>$a_{12345}/r_c$</th>
<th>$\theta_{12345}(\circ)$</th>
<th>$N_{12345}$</th>
<th>Or</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.991/4.01/6.01/7.98/9.98</td>
<td>83.71/82.5/78.5/73.6/73.3</td>
<td>6/12/19/24/30</td>
<td>+ + + + -</td>
</tr>
<tr>
<td>1.991/4.00/6.00/7.98/9.98</td>
<td>83.71/79.6/82.7/73.6/82.0</td>
<td>6/12/19/24/31</td>
<td>+ - - +</td>
</tr>
</tbody>
</table>

It needs to be noted that even though the above structural patterns are the optimal constructions identified in the above constrained parameter space, their efficiency factor values are more than three orders of magnitude worse than the absolute optimal ones retrieved for the entire parametric space (Table 6.1).

6.4 Discussion and Conclusions

An efficiency quantification of the different layer orientation combinations has been provided in Figs. 6.4 and 6.5, designating the structurally favorable weaving patterns. The figures suggest that structuring the helical assembly upon layers of alternating directionality does not yield on average the most favorable constructions, as the torsional moment counterbalancing is non linearly dependent on the associated geometric parameters.

Furthermore, Figs. 6.6 to 6.8 illustrate that constructions made up of layers of quasi-equal helix cross sections do not yield efficiency factors in the same order of magnitude as the optimal retrieved ones. In other words, extending the design parameter space constitutes a necessity for optimal helical constructions to be retrieved, while resorting to constructions comprised of four or five layers, not only broadens the available optimal design constructions, but also allows for the engineering of constructions of considerably low values of the efficiency factor $f_e$. 

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It should be noted that even though the current study provides a complete enumeration of favorable structural arrangements of up to 5 helical layers, covering most practical applications, extension to structures of a higher layering is also feasible. However, the feasible parameter space increases exponentially with each additional layer, making a brute force calculation of optimal patterns computationally intractable. One would then have to use dynamic optimization techniques to explore the hybrid discrete-continuum parameter search space such as reinforcement learning [47], extensions which are outside the scope of the current work.

In the current analysis an equal weighting of the structural objectives has been followed, as the definition of the efficiency factor dictates. However, an introduction of weights amongst the two objectives is possible, so as to accordingly tailor the selection of the optimal structural arrangement.

The selection of the structural pattern constitutes a primal design step with a considerable impact on the functionality and endurance of the helical assembly. An optimal structural pattern yields a lower internal loading, that allows for the minimization of related fatigue and wear phenomena. Such a design leads to the improvement of the mechanical performance and the prolongation of the structure’s life expectancy, attributes strongly related to cost effective structural solutions.

To conclude, a scheme for the quantification and selection of favorable structural patterns has been presented. Helical assembly constructions of up to five layers were parsed for the favorable parameter space to be identified, furnishing an extended database of optimal structural arrangements that covers a wide range of practical applications. The current work can function not only as a reference but also as a general working framework in the optimization of helical assembly constructions for geometries out of the hereby addressed.
Part V

Bayesian uncertainty quantification of the structural composition of biological helical constructions
Tendon fascicle structural properties inference

7.1 Introduction

Tendons are natural fibrous tissues that function as load transfer members between the muscles and the skeleton. They are ordered in distinct hierarchical levels [48], whose composition and structural arrangement depend on the functional role of the tendon such as positional tendons and energy-storing tendons [49], on its age [50] and also on its health state [51]. The hierarchy of the tendon structural units is illustrated in Figure 7.1:

![Figure 7.1: Tendon hierarchical levels](image)

The material properties of the tendon building blocks have direct implications upon the tendon functionality and efficiency. A thorough understanding of the tissue mechanics
is not feasible without a structural characterization of its subunits. Moreover, a concrete knowledge of the mechanical properties of the different hierarchies is a prerequisite for the selection of materials for any repair and restoration method [52] to provide biocompatible solutions that mimic the physiological functionality of the native tissue [41, 53]. Thus, due to its paramount importance, the characterization of the material properties and functional composition of a tendon has been the subject of several experimental studies.

Experiments characterizing the structural composition of tendons include those of Orgel et al. [54], who established that the fibrous content of the tendon’s subunits is immersed in a matrix, following crimped and helical undulated patterns already at the lowest hierarchical tendon level of fibrils. Yahia et al. and De Campos Vidal et al. [55, 56] used interference and polarized microscopy in bovine and rat tail tendon specimens to infer that the collagen fibril bundles (fibers) and fascicles are also helically structured. The observed helical structuring of these subunits was further corroborated by measurements of their mechanical response. More specifically, tendon fascicle specimens were shown to deform both axially and torsionally upon axial straining, a response characteristic of helical formations [49].

X-Ray imaging studies reported volumetric fibrillar content values between 30\% and 40\%, when a certain level of fiber swelling is taken into account [57]. Contrariwise, electron microscopy studies yielded substantially higher fibrillar contents in the range of 60\% [58]. It needs to be noted that the specimen’s age has been shown to have a direct influence on the mechanical properties [59], while the fiber diameter distribution was uncorrelated with the retrieved mechanical moduli values [60].

Aiming to quantify the tendon mechanical properties, the material moduli values of the different tendon subunits (Fig. 7.1) have been primarily characterized using uniaxial strain experiments. In particular, at the fibrillar scale, the reported moduli values vary over an order of magnitude, from values of 860 ± 450 MPa [61] up to 2.89 ± 0.23 GPa [62]. At the level of fibers, Kato et al. reported mechanical moduli as low as 0.57 ± 0.085 GPa and as high as 2.69 ± 0.42 GPa regarding wet and dry rat tail tendon fiber specimens respectively [63]. Gentleman et al. [64] computed a modulus value of 0.359 ± 0.028 GPa for extruded fibers, and reported a value more than two times the value of Kato et al. 1.17 ± 0.28 GPa for rat tail tendon fibers. At the top hierarchical level of fascicles, experiments report a mechanical moduli of 0.641 ± 0.030 GPa [65] and 0.48 ± 0.07 GPa [66] for rat tail tendon specimens, whereas human fascicle specimens yielded an intermediate estimate of 0.55 ± 0.14 GPa [62].

A further characterization of the mechanical response of the different tendon hierarchical levels has been achieved through the measurement of their volumetric response upon axial straining. The experimental and numerical studies conducted, provided lateral contraction values substantially exceeding the ones predicted by isotropy. More specifically, Lynch et al. [67] measured the Poisson ratio of bovine flexor tendons and reported values up to an order of magnitude greater than the ones predicted assuming an isotropic material. At the scale of fascicles, a 0.8 ± 0.3 Poisson value was measured for rat tail tendon specimens [68], while values within the previous range were also reported for horse tendon fascicles [49]. To complement the experimental investigations, computational studies by Reese et al. using finite element methods [20] suggested that the lateral contraction of
embedded helical fibers is a non-linear, structure dependent function whose values are not bounded by the the isotropic limit of 0.5. The same conclusion is reached and further analyzed using analytical continuous models by Swedberg et al. [69].

In contrast to the substantial scatter observed in the fiber content values, fiber angles $\theta$ (Fig. 7.2) exhibit a rather low variation in their reported values (see Ref.[70] and references therein). It is noteworthy that even though the matrix properties play a detrimental role in the mechanical response and the functionality of the tendon [71], little is known on its mechanical attributes. More specifically, analytical micro-mechanical modeling has provided a matrix modulus estimate of 0.25 MPa [72], while no direct experimental results are available.

Despite the abundance of experimental studies, the exact values of the material properties of the matrix as well as the fascicle to fiber composition ratio remains uncertain. In this work, a specialized finite element model is used for the simulation of the fascicle and fiber under external forces to obtain probabilistic bounds on their material properties and infer their most probable structure. To establish a quantitative link between the finite element model of a fascicle and data, a Bayesian framework for Uncertainty Quantification, model selection and weighted model Predictions is employed (UQ+P) [73, 74]. Bayesian inference stands amongst the prevalent uncertainty quantification techniques that can incorporate both expert knowledge and experimental evidence. It is used for quantifying and calibrating uncertainty models, as well as propagating these uncertainties in engineering simulations to achieve updated robust predictions of system performance, reliability and safety [75].

In particular, a computational model of a fascicle along with experimental data are used to infer its most plausible structural composition using a Bayesian UQ+P framework. The work is structured as follows: In Section 7.2.1, specialized fascicle finite element models are constructed allowing to numerically probe a wide range of fascicle constructions. Thereupon, the structural models are coupled to a Bayesian uncertainty quantification framework (Section 7.2.2) so that a linkage between numerical models and experimental data is derived. By that means, physiologically relevant values for the fascicle structural composition are retrieved (Section 7.3). A discussion on the inferred fascicle synthesis follows, along with concluding remarks (Section 7.4).
7.2 Materials and Methods

7.2.1 Tendon fascicle mechanical modeling

The tendon hierarchical level of fascicles (Fig. 7.1) is simulated as a helical construction consisting of fibers that are immersed in a matrix, as Fig. 7.2 illustrates:

![Figure 7.2: Tendon fascicle geometry](image)

A large number of fascicle mechanical models is built simulating a wide range of structural arrangements and compositions. In particular, fascicles of different fiber contents are constructed, with the content defined as the area covered by the immersed fibers $A_f$ over the total area $A_t$ of a fascicle section $A - A'$ $C_f = \frac{A_f}{A_t}$ (Fig. 7.2 left). Since the fiber content of a fiber varies significantly amongst experiments [57, 59, 58], fascicle models with fiber contents of 30\% up to 60\% with 5\% spacing are constructed. Each fascicle with a different fiber content is assumed to be a different model class.

The immersed fibers for all of the 8 assumed fiber contents are considered to be homogeneously distributed amongst the circular cross section [59, 48]. The specific details regarding the creation of a homogeneous distribution of fibers is described in details in the Section 7.5.1. The fascicle centerline $a$ (Fig. 7.2 left) is selected so that its ratio over the fiber radius $r$ (Fig. 7.2 right) yields a mean index $a/r$ of 100 for all fiber contents $C_f$,
a value in accordance with physiological studies \[76, 48\].

For the entire range of the reported fiber modulus values \( E_f \) to be explored (Section 7.1), a total of 175 fascicle model classes \( M_{C_f}^{E_f} \) is constructed. To that extent, each of the above broached fascicle content values \( C_f \) is subdivided to structural models with fiber modulus values \( E_f \) as low as 350 MPa and as high as 2750 MPa, upon a discretization of 100 MPa, with the different model classes summarized in Table 7.1:

<table>
<thead>
<tr>
<th>Fascicle numerical models ( M_{C_f}^{E_f} )</th>
<th>( C_f(%) )</th>
<th>( E_f ) (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30, 35, ...60</td>
<td>350, 450, ...2750</td>
<td></td>
</tr>
</tbody>
</table>

The above model classes are parametrized with the fascicle angle \( \theta \) and the matrix modulus value \( E_m \) (Section 7.2.2). The angle \( \theta \) is allowed to vary between 50° and 87.5°, while the matrix modulus \( E_m \) amongst [0.1 – 5] MPa \[72, 20\]. Thereupon, prior distributions for each of the above parameters are assigned as explicated in Section 7.2.2. The matrix modulus \( E_m \) is attributed a uniform prior distribution within the above bounds, while the helix angle \( \theta \) a Gaussian distribution with a mean of 70° and a variance of 5.8° \[70\] for the Bayesian analysis to be implemented (Section 7.2.2).

Each one of the fascicle model classes \( M_{C_f}^{E_f} \) are parametrized with the fascicle angle \( \theta \) and the matrix modulus value \( E_m \). Two distinct Quantities of Interest (QoI) are considered to compute the fascicle effective modulus \( E_{eff} \) and its Poisson ratio \( \nu_{eff} \). The effective axial modulus \( E_{eff} \) is computed as the stress \( \sigma_z \) that an axial strain \( \epsilon_z = \delta h/h \) induces (Fig. 7.2), following a linear analysis. Accordingly, the Poisson ratio \( \nu_{eff} \) characterizing the volumetric deformation is calculated by the average transverse strain \( \epsilon_t \) of the fascicle cross section circumference (Fig. 7.2 right) for the applied axial strain \( \epsilon_z \). Both the effective modulus and the Poisson ratio of the fascicle are functions of its structural composition (7.1).

\[
E_{eff} = \frac{\sigma_z}{\epsilon_z} = q_1(E_m, \theta|M_{C_f}^{E_f}), \quad \nu_{eff} = -\frac{\epsilon_t}{\epsilon_z} = q_2(E_m, \theta|M_{C_f}^{E_f}) \quad (7.1)
\]

An in depth analysis of the Poisson ratio dependence on the fascicle structural composition parameter space is provided in Section 7.5.2.

The above QoI are assessed by means of the planar finite element elaborated in Chapter 3, with the analysis ran in ETH Brutus cluster in parallel. To ensure the application of purely axial strain on the tendon fascicle, the boundary conditions applied constrain the radial and torsional deformation of the section nodes lying along the local axis \( b \) and \( n \) respectively (Fig. 7.2 right). A convergence analysis for each of the QoI of Eq. 7.1 revealed that for the fascicle effective modulus \( E_{eff} \), a discretization of one thousand elements is required, while substantially higher mesh densities yielded values that did not differ by more than 1%. The corresponding study for the effective volumetric response designated that a substantially higher mesh density of approximately five thousand elements per section is required for the corresponding convergence level to be reached over the entire range of the parameter space.
7.2.2 Tendon fascicle Bayesian modeling

A parametrized set of 175 fascicle model classes $M = M_{C_f}^{E_f}$ is considered where $\lambda \in \mathbb{R}^2 = (E_m, \theta)$ are the tendon fascicle parameters belonging to each model class, that will be estimated using experimental data. The model parameters $\lambda$ are considered to be uncertain and probability distribution functions (PDF) are introduced to quantify their plausible values. A PDF $\pi(\lambda|M_{C_f}^{E_f})$ is assigned to the model parameters incorporating prior information based on previous knowledge or physical limitations. More specifically, the matrix modulus $E_m$ is attributed a uniform prior distribution within $[0.1 - 5]\text{ MPa}$, while the helix angle $\theta$ a Gaussian distribution with a mean of $70^\circ$ and a variance of $5.8^\circ$.

As explained in [77]: “In Bayesian inference, the probability distribution of the model parameters $\lambda$ is updated based on measurements available for the mechanical properties of the fascicle. Let $D \equiv \hat{y} = \{y_r, r = 1, \ldots, n_y\} \in \mathbb{R}^{n_y}$ be a set of observations (data) available from experiments. The prediction error $e \in \mathbb{R}^{n_y}$ is introduced to characterize the discrepancy between the model predictions $g(\lambda|M_{C_f}^{E_f}) \in \mathbb{R}^{N_g}$ and the corresponding data $\hat{y}$. The observation data and the model predictions satisfy the prediction error equation $\hat{y} = g(\lambda|M_{C_f}^{E_f}) + e$. The prediction error $e = e^d + e^m$ is composed of the experimental error accounting for the measurement error $e^d$ and of the model uncertainties $e^m$ respectively”.

Experimental data usually are provided in terms of the mean and the variance of each measured quantity so the maximum entropy principle [78], can be invoked to select a normal distribution for the model error term $e^m$. Similarly, this Gaussian assumption is also well justified for the experimental error term $e^d$ due to the lack of information for assigning an alternative distribution.

As analyzed in [77]: “It is a common assumption that each term is assigned a zero mean, the experimental error is set as $e^d \sim N(0, \Sigma^d)$ and the model uncertainty error as $e^m \sim N(0, \Sigma^m)$, where $\Sigma^d$ and $\Sigma^m$ are taken to be diagonal, due to the fact that measurements and model predictions for different properties, in this case for the effective modulus and Poisson ratio, are independent. The covariance matrix $\Sigma^d = diag(\tilde{\nu}_r^2)$, where $\tilde{\nu}_r^2$ is the variance of the $r$-th observation. It needs to be noted that Bayesian results depend on the choices of the distributions and the correlation structure of the modeling error”. However a detailed study of different assumptions on the distribution of the modeling terms is beyond the scope of the current study. The reader is referred to Ref. [79] for the effect of correlation structure of the covariance matrix on the Bayesian results.

The updated distribution $f(\lambda|D, M_{C_f}^{E_f})$ of the model parameters $\lambda$, given the data $D$, the model class $M_{C_f}^{E_f}$ and prior information about the parameters $\pi(\lambda|M_{C_f}^{E_f})$, is given
from the Bayes theorem, as follows [77]:

\[
f(\lambda|D, M_{CE}) = \frac{f(D|\lambda, M_{CE}^E) \pi(\lambda|M_{CE}^E)}{f(D|M_{CE}^E)}
\]

(7.2)

where as it can be found in [77]: “the \( f(D|\lambda, M_{CE}^E) \) is the likelihood of observing the data \( D \) from a model corresponding to a value \( \lambda \) of the model class \( M_{CE}^E \) and \( f(D|M_{CE}^E) \) is the evidence of the model class \( M_{CE}^E \), selected such that the posterior distribution \( f(\lambda|D, M) \) of the model parameters integrates to one”.

Using the prediction error equation and assuming that the errors terms are independent, the likelihood \( f(D|\lambda, M_{CE}^E) \) of observing the data follows the multi-variable normal distribution:

\[
f(D|\lambda, M_{CE}^E) = \left| \Sigma^{-1/2} \right| \frac{1}{(2\pi)^{N/2}} \exp \left[ -\frac{1}{2} J(\lambda; \hat{y}) \right]
\]

(7.3)

where

\[
J(\lambda; \hat{y}) = \left( \hat{y} - g(\lambda|M_{CE}^E) \right)^T \Sigma^{-1} \left( \hat{y} - g(\lambda|M_{CE}^E) \right)
\]

(7.4)

is the weighted measure of fit between the FEM model predictions and the measured data, \( \Sigma = \Sigma^d + \Sigma^m = diag(\tilde{\nu}^2) \) is the covariance of the prediction error, and \( |.| \) denotes a determinant [77].

Following the Bayesian calibration of each fascicle model, model selection based on the observed data \( D \). Using the 175 competing model classes \( \{M_i\}_{i=1}^{175} = M_{CE}^E \) a ranking based on their probability given the data \( D \) can be performed, using the Bayes factor, as explained in [80, 81]:

\[
Pr(M_i|D) = \frac{f(D|M_i)Pr(M_i)}{f(D|M_1, \ldots, M_p)} = \frac{f(D|M_i)Pr(M_i)}{\sum_{i=1}^p f(D|M_i)Pr(M_i)}
\]

(7.5)

where \( Pr(M_i) \) is the prior probability of each fascicle model class \( \{M_i\}_{i=1}^{175} = M_{CE}^E \). The most probable model class is selected as the one that maximizes \( Pr(M_i|D) \) over \( i \). In the absence of experimental insights, an uninformative uniform prior is assigned over all model classes to be \( Pr(M_i) = 1/175 \). The evidence of each model class is evaluated directly using numerical integration.

Robust posterior predictions of an output QoI \( Q \) are obtained by taking into account the updated (posterior) uncertainties in the model parameters given the measured data \( D \) [82]. Within this work, the robust posterior predictions are conditioned to the data driven Bayesian identification of the uncertainties in the models and their parameters [83]. According to [75]: “Let \( Q(q|\lambda, M) \) be the conditional distribution of \( Q \) given the model parameters \( \lambda \) and the model class \( M \). The posterior robust cumulative distribution
\( Q(q|D, M) \) of the output quantity \( Q \), taking into account the model \( M \) and the data \( D \), is given by:

\[
Q(q|D, M) = \int_{\lambda} Q(q|\lambda, M) f(\lambda|D, M) d\lambda
\]  

(7.6)

The robust estimate \( Q(q|D, M) \) represents an average of the conditional cumulative distribution weighted by the posterior probability distribution \( f(\lambda|D, M) \) of the model parameters [75].

### 7.2.3 Fascicle experimental data

Three data sets \( D \) are compiled using all available experimental data regarding the fascicle structural response. In particular, the fascicle material properties provided by Haraldsson et al. [65] and Derwin et al. [66] are considered either as separate experiments (Data Set 1 and 2 respectively) or as independent experiments (Data Set 3) each of them combined with the fascicle Poisson contraction provided by Cheng et al. [68]. Table 7.2 summarizes the experimental observations considered for each Data Set:

<table>
<thead>
<tr>
<th>Data Set</th>
<th>( E_f \ (MPa) )</th>
<th>( \nu_{fascicle} \ (-) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data Set 1</td>
<td>641 ± 30</td>
<td>0.8 ± 0.3</td>
</tr>
<tr>
<td>Data Set 2</td>
<td>480 ± 70</td>
<td>0.8 ± 0.3</td>
</tr>
<tr>
<td>Data Set 3</td>
<td>641 ± 30, 480 ± 70</td>
<td>0.8 ± 0.3</td>
</tr>
</tbody>
</table>

The distinction between the three data sets is made to assess whether the results change substantially in case one experimental value over the other is favored.
7.3 Results

Three different posterior model probabilities are subsequently calculated, given each individual data sets for all 175 models $Pr \left( \mathcal{E}_{C_f} \mid D_i \right)$. In order to achieve this, the integral of the model marginal likelihood:

$$Pr \left( \mathcal{E}_{C_f} \mid D_i \right) = \int_\lambda f \left( D \mid \lambda, \mathcal{E}_{C_f} \right) \pi \left( \lambda \mid \mathcal{E}_{C_f} \right) d\lambda$$

is evaluated using numerical quadrature over a discrete grid of 800 points in the $(E_m, \theta)$ space. The results are presented in Fig. 7.3.

![Figure 7.3: Fiber modulus probability distribution](image)

The three $7 \times 25$ matrices in Fig. 7.3 are colored by the value of the posterior probability of the models. The most plausible model classes (model classes with posterior probability larger than 0.001, indicate for all data sets a nonlinear inverse correlation of the fiber modulus values with the fiber content. Hence, the most probable model classes occur for fiber modules $E_f$ as low as 1000 MPa for a 60% content and up to 2500 MPa for the lowest volumetric fiber content model class of 30%.

The trend between the two properties as seen in Fig. 7.3 suggests that for fiber modulus values lower than 1000 MPa to be plausible, a fiber content higher than 60% needs to apply so that the experimentally observed data at the fascicle scale can be paired.

For a model and inference results validation, the uncertainty is propagated using weighted model prediction to the quantities of interest in Fig. 7.4:
Figure 7.4: Tendon fascicle effective modulus robust prediction

In the absence of other uncertainties assumed in the model, in the two first cases exactly the input data distribution is retrieved. In the combined data $D_3$ case, the inferred distribution tries to maximize its overlap with both observations on the effective fascicle modulus. However, the numerical model can completely recover the experimental data following the Bayesian calibration.

Fig. 7.5 illustrates for each model class the corresponding matrix modulus values $\hat{E}_m$ that maximize the posterior PDF given each respect model class, thus the following posterior $\hat{E}_m = \arg \max_{E_m} f \left( E_m, \theta | D, M_{C_{ij}}^{E_f} \right)$.
For all data sets, values between 1 and 4 MPa are retrieved, dependent on the content value $C_f$. Extrema values of the material modulus correspond to tendon compositions with vanishing posterior model probability and are therefore highly unlikely. Similarly to Fig. 7.5, the most probable helix angles are plotted in Fig. 7.6:
All data sets indicate that low values of the fiber moduli correspond to low helix angles $\theta$, irrespective of the fiber content $C_f$. Conversely, for statistically plausible model classes with a combination of high fiber modulus values ($E_f \approx 2500$) and low fiber content values ($C_f \approx 0.3$), the most plausible fascicle angles exceed 80° for all data sets. Note that if the experimental observations provided by Jarvinen et al. [70] are used as an additional constrain of the plausible parameter space, content values higher than 45% need to be excluded, along with all corresponding fiber and matrix modulus values.

The above inferred values are in accordance with the prediction of Reese et al. [20]. In their work, they assumed a fixed fiber content of 57% upon which a sensitivity analysis on the parameters was performed, which is the equivalent of performing a prior, non data driven uncertainty propagation. It needs to be emphasized that herein, the inverse route is followed, thus identifying the structural properties through the experimental observations, rather than solving the forward problem.

7.4 Conclusions

In this work, a Bayesian probabilistic framework to perform model selection and infer the relationship between two hierarchical units of tendons was explicated. In particular, the structural composition of tendon fascicles was inferred by using experimental data pertaining to mechanical properties of the tendon and calibrating them to a finite element model. It was observed that the most plausible fascicle model classes have an inverse correlation between the matrix modulus and their fiber content.

The mechanical properties of the fascicle constituents provided in Figs. 7.3 to 7.6 not only characterize the structural composition of a fascicle based on experimentally observed mechanical responses, but further constitute a database for the guidance of tendon reconstruction processes. It needs to be though noted that a more precise, experimentally driven prior knowledge, would allow for the explicated method to provide narrower bounds for the fascicle structural composition. The large range of values inferred for the fascicle synthesis is mostly due to the large experimental uncertainty reported. In order to minimize the measured uncertainty, more experiments with higher accuracy have to be performed, and rather than assuming Gaussianity in the reported experimental sets and be provided with a mean and standard deviation, experimentalists could present the entire data set. This in turn would allow for a more precise parameter selection, while it would make feasible a secondary analysis on the effects arising from other latent design parameters herein neglected, such as the aging process of tendons [50].
7.5 Appendix

7.5.1 Fascicle fiber distribution

All fascicle-fiber FEM designs are constructed assuming a homogeneous distribution of the fibers. In particular, within the circular fascicle cross section of radius \( R \) (Fig. 7.2), \( N \) fibers of radius \( r \) are positioned. Each fiber is parametrized by the \( x_n \), \( y_n \) coordinate of its center within the fascicle cross section (Eq. 1.7). For \( N \) fibers, \( 2N \) coordinates of the fiber centers \((x_i, y_i)_{i=1}^{N}\) need to be positioned.

Acquiring the \( 2N \) unknown coordinates that optimize the homogeneity of the distribution of the circular fibers, belongs to the general category problem of homogeneously packing objects and can be viewed as a classic Thompson problem [84, 85]. Thompson problems regard the distribution of pairwise repulsive ions on the surface of a sphere, and depending on the functional form of the ion-ion repulsion are shown to produce a homogeneous packing of objects contained in the surface. Following the methodology presented in [86], a pair-wise repulsive force \( F_{ij} \) is assigned between the fibers \( i \) and \( j \) varying as a function of the distance from the fibers perimeter \( d - 2r \), \( d \) being the distance between two fiber centers, as follows:

\[
F_{ij} = -\frac{p}{d_{ij}^{-2}} \tag{7.7}
\]

where \( p \) is an arbitrary, non-zero constant number. This repulsive force of Eq. 7.7 applies also between each fiber and the fascicle circumference \( R \), where the fiber-circumference distance \( d_{i-R} \) is defined as the shortest distance of the fiber with the fascicle circumference. The overall potential energy \( V \) associated with all pairwise fiber and fiber-circumference interactions can be then measured as the sum over all non-identical contributions:

\[
V_{total}(d) = \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} pd_{ij}^{-1} + \sum_{i=1}^{N} 2pd_{i-R}^{-1} \tag{7.8}
\]

The homogeneous distribution problem reduces in finding the \( 2N \) coordinates \((x_i, y_i)_{i=1}^{N}\) that minimize the energy functional of Eq. 7.8 with the additional constraint \( x^2 + y^2 < (R - r)^2 \) applying to all pairs \((x_i, y_i)_{i=1}^{N}\). In order to tackle this optimization problem, the Covariance Matrix Adaptation Evolutionary Strategy (CMA-ES) method has been used [87]. The optimization required a minimum of \( 5 \cdot 10^6 \) potential function evaluations for the positioning of a total of forty fibers to be retrieved for the different fascicle content values described in Table 7.1. An illustration of the arising fascicle section profile has been provided on the right part of Fig. 7.2.

7.5.2 Fascicle Poisson ratio

The Poisson ratio of the fascicle hinges on both the material properties and the geometric arrangement of the construction (Eq. 7.1). In the current analysis, the influence of the fascicle constituents \( C_f \), \( E_f \), \( E_m \) and \( \theta \) was considered while any effects arising from the Poisson ratios of the fiber and the matrix were disregarded and set as 0.3 for all analysis models due to the lack of experimental data.
The fascicle structural composition is directly associated to the Poisson values obtained, being a non-linear function of all modeling parameters (Eq. 7.1). Figure 7.7 shows the Poisson values obtained for a fiber content of $C_f = 0.60$ over different fascicle angles and for different ratios of the fiber modulus to the matrix modulus $M_{ratio} = E_f/E_m$, computed for a matrix modulus value of $E_m = 1$:

![Figure 7.7: Fascicle Poisson ratio values](image)

In particular, Fig. 7.4 denotes a strong dependence of the Poisson values on the ratio of the modulus of the fiber to the one of the matrix, along with an additive dependence on the fascicle undulation $\theta$. It is noteworthy that structures with a geometric arrangements close to the one of a rod ($\theta \rightarrow 90^{\circ}$) provide a mechanical response within the isotropic limits (here $\nu_{eff} \rightarrow 0.3$) irrespective of the modulus ratio, suggesting that it is primarily the helical structuring to provide this complex volumetric response, a conclusion pointed out by Reese et al. [20]. Finally, the Poisson value retrieved is associated to constructions fiber content, with the maximum value for the lowest content of 30\% here considered, being approximately 60\% of the maximum noted in Fig. 7.7.
Part VI

Conclusions
Conclusions and outlook

The present thesis has addressed three main thematics. In particular, parts II and III have been devoted to the characterization and simulation of the mechanical response of helical constructions, providing analytical and numerical modeling developments. Part IV has addressed the design of helical assemblies, while part V has set the emphasis on the analysis of helically arranged biological constructions. A recapitalization of the main contributions of each part, along with suggestions for further, closely related research is provided below. More concretely:

Chapter 2 has extended an existing analytical model that incorporates the effect of radial loading to the commonly considered axial and torsional ones. More specifically, closed-form solutions have been provided, relating radial loading to global force and moment resultants. By that means, the structural impact of radial loading has been quantified for a wide range of geometric configurations. However, analytical closed-form expressions describing the structural response to thermal loads have not been furnished. Furthermore, the model has been based on the Euler-Bernoulli beam theory that is a-priori developed to address slender geometries. Even though the geometric space for which the theory applies has been characterized, a model applicable to non-slender helix geometries remains to be derived.

Chapter 3 has elaborated a planar finite element model for the simulation of the mechanical response of helical constructions. Different loading schemes have been considered, addressing the axial, torsional, radial and thermal helix response. The structural element has been verified upon closed-form expressions and dedicated commercial finite element models, while applications to simple and multilayer helical assemblies of various geometries and cross sectional shapes have been presented in Chapter 4. Nevertheless, the numerical scheme has been restrained to linearity, while extensions to non-linear geometric deformations or further to the viscous and plastic domain remain to be developed. Such extensions are of high interest in a wide range of engineering and bio-engineering applications, in particular because of the model’s low numerical cost, which vouches for efficient simulations of the recursive numerical calculations that non-linear formulations require.

Chapter 5 has analyzed the impact of kinematic constraints on the mechanical response
of helical constructions. More specifically, the constraining of the helix cross section normal rotational degree of freedom at discrete positions along the helical body has been elaborated. Thereupon, a wide range of helical assembly braiding patterns have been considered, for which, the torsional response bounds were retrieved. In particular, the torsional stiffness of the kinematically constrained constructions have been related with the use of scaling factors to the stiffness properties of the unconstrained ones that can be analytically calculated, upon closed-form expressions. The analysis has been restricted to the study of the construction’s structural response for the case of utterly restrained local rotations, describing thus only the upper bounds of the mechanical response. As a result, a modeling extension would be of interest, where the kinematic constraining of the helical assembly trellis points is a function of a series of implicitly associated parameters, such as the constructions’ loading and wear state. By that means, a more accurate estimation of the mechanical properties of helical assembly constructions will become feasible.

The 6th Chapter of the thesis has reported a quantitative framework for the design of the braiding pattern of multilayer helical assemblies. More specifically, it has provided a methodology which allows for the selection of structural patterns that maximize the helical construction’s resistance to axial loads while concurrently minimize its torsional propensity. The numerical framework has been applied to helical assemblies comprised of up to five layers, retrieving favorable structural patterns that covers most practical applications. Nonetheless, the objectives that have been hereby considered and therefore the corresponding optimization criteria did not account for constraints related to properties other than the mechanical ones. In other words, the provided framework can be advanced to incorporate application specific criteria. An example of the kind can be found in the realm of electricity power cables design, where constructions with optimal mechanical and electric transfer properties are sought.

The 7th Chapter of the thesis has reported a modeling scheme for the simulation of helical biological constructions. In particular, tendon fascicles of various structural compositions and geometric arrangements have been modeled and coupled to experimental observations upon a Bayesian uncertainty quantification framework. More specifically, the finite element models have allowed for the recreation of the available experimental set-ups in a computationally tractable way, while the Bayesian framework has provided a direct comparison amongst the fascicle models. By that means, probabilistic bounds on the structural composition of the tendon fascicles have been derived, establishing a fundamental linkage between successive tendon hierarchical levels. The methodology can be used for the study of a wide range of parameters affecting the mechanical characterization of biological constructions, such as aging or decease induced structural alterations.
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List of Publications

Journal Publications


Conference Proceedings


N. Karathanasopoulos, “Numerically efficient planar finite element modeling of helical structures, an application to multi-layer engineering strands”, 9th *European Solid Mechanics Conference* (ESMC), Madrid 2015

N. Karathanasopoulos, “Planar numerical modeling of locked-coil helical constructions”, 8th *International congress on Computational Mechanics* (GRACM), Universally of Thessaly Press, Volos 2015

Curriculum Vitae

I was born in August 1988 in Agrinio, part of the Aetolia and Akarnania district, in the west side of Greece. Living in a nearby village, I finished the 3rd Gymnasium of Agrinio in 2006. At the same year, I entered the department of Civil and Environmental Engineering at the University of Patras, where I followed a 5-year program with an emphasis in Structural Engineering. In the summers of 2009 and 2010, I worked part-time in a structural engineering office in Agrinio. After my graduation, in 2011, I moved to California for master studies at UC Berkeley, following the Structural Engineering Mechanics and Materials program (SEMM) of the Civil and Environmental Engineering department. In the summer of 2012, I joined the Mechanical and Process Engineering Department of ETH Zürich following a phd program, which I finished in August 2015.