

On a Theorem of Shalom

Master Thesis

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ON A THEOREM OF SHALOM

STEPHAN TORNIER

ABSTRACT. We give a concise and transparent proof of a version of Shalom's Theorem 0.3 on Property (T) for quotients of irreducible, cocompact lattices in certain product groups, see Appendix A. In doing so, we rigorously develop continuous and reduced continuous cohomology for locally compact groups in the spirit of relative homological algebra.

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INTRODUCTION

Simple groups are studied in various categories as they constitute building blocks from which other groups may be assembled. For instance, *finite* simple groups have been completely classified and any finite group can be built from simple groups in the sense of the Jordan-Hölder-Schreier theory of composition series.

Also, (topologically) simple *Lie groups* have been classified and by the Levi decomposition theorem any connected, simply-connected Lie group is a semidirect product of a product of simple Lie groups and a solvable Lie group.

In between these two settings, there is a vast amount of infinite but discrete, say *finitely generated*, groups. And not nearly as much is known about simple groups in this setting as is in the two above. A reasonable strategy to find infinite, finitely generated simple groups would be to consider discrete approximations of simple Lie groups, e.g. *lattices*, i.e. discrete, cofinite subgroups. In fact there is the following remarkable theorem due to Margulis, see e.g. [Zim84].

Theorem 0.1 (Margulis). Let G be a connected semisimple Lie group of rank at least two with finite center. Further, let $\Gamma \leq G$ be an irreducible lattice in G . If $N \trianglelefteq \Gamma$ is a normal subgroup of Γ then either N is contained in the finite center of G or Γ/N is finite.

Thus, in a sense, lattices in higher rank semisimple Lie groups are almost simple as discrete groups.

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A similar *normal subgroup theorem*, which does not rely on the ambient group being differentiable or algebraic but is applicable in the category of locally compact groups, was given by Bader and Shalom in [BS06]. Recall, that a (topological) group is called (*topologically*) *just non-compact* if every quotient by a proper (closed) normal subgroup is compact, and similarly (*topologically*) *just infinite* if every quotient by a proper (closed) normal subgroup is finite.

Also, note that locally compact groups are always assumed to be Hausdorff.

Theorem 0.2 (Bader, Shalom). Let G_1 and G_2 be locally compact, compactly generated, topologically just non-compact and non-discrete groups, not both isomorphic to $(\mathbb{R}, +)$. Then any irreducible, cocompact lattice $\Gamma \leq G_1 \times G_2$ is just infinite.

As with Margulis' theorem, the proof of Theorem 0.2 is based on showing that the quotients of Γ have Property (T) and are amenable. The first half was given by Shalom in [Sha00].

Theorem 0.3 (Property (T)). Let G_1 and G_2 be locally compact, compactly generated groups. Let Γ be an irreducible, cocompact lattice in $G_1 \times G_2$ and let N be a normal subgroup of Γ . Then Γ/N has Property (T) if and only if for $i \in \{1, 2\}$ the quotient $G_i/\overline{\text{pr}_i(N)}$ has Property (T) and every continuous homomorphism from G to $(\mathbb{R}, +)$ which vanishes on N is identically zero.

The second half was given by Bader and Shalom in [BS06].

Theorem 0.4 (Amenability). Let G_1 and G_2 be locally compact, compactly generated groups. Let Γ be an irreducible, cocompact lattice in $G_1 \times G_2$ and let N be a normal subgroup of Γ . Then Γ/N is amenable if and only if for $i \in \{1, 2\}$ the quotient $G_i/\overline{\text{pr}_i(N)}$ is amenable.

In this article, we will give a concise and transparent proof of Theorem 0.3 under the additional assumption that G_1 and G_2 are topologically simple, i.e. there are no proper closed normal subgroups, and that every continuous homomorphism from G to $(\mathbb{R}, +)$ vanishes, written $\text{hom}(G, \mathbb{R}) = 0$. The proof is schematically summarized in Appendix A.

The content of this article may be outlined as follows. In Section 1, we briefly introduce the reader to Property (T) and amenability of locally compact groups. Section 2 provides a comprehensive treatment of continuous cohomology of locally compact groups which is essential for the subsequent Section 3 on *reduced* continuous cohomology. Reduced continuous cohomology only slightly modifies continuous cohomology but in doing so allows for a Künneth-type Theorem 3.5 which shall play an essential role in the proof of our version of Theorem 0.3.

In Section 4, we give cohomological characterizations of Property (FH) and Property (T) and alongside provide a full proof of the Delorme–Guichardet theorem. This will enable us to eventually give a concise and transparent proof of our version of Theorem 0.3 in Section 5, see also Appendix A.

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1. PROPERTY (T) AND AMENABILITY

Property (T) was introduced by Kazhdan in 1967 [Kaz67] in order to show that certain lattices are finitely generated and has since become a valuable notion in many areas of mathematics. It exhibits a rich interplay with amenability which was introduced by von Neumann in 1929 [Neu29] in the context of the Banach-Tarski paradox. In this section, we briefly give the very basics on these two notions and examine their interplay. Recent expositions of the material presented here include Zimmer's text on Margulis' theorems [Zim84], the excellent French introduction [dlHVB89] to Property (T), and the extensive treatment [BdlHV08] of the same, partly by the same authors.

1.1. Property (T). Let G be a locally compact group. A (continuous) orthogonal or unitary representation (π, \mathcal{H}) of G *almost has invariant vectors* if for every compact set $K \subseteq G$ and every $\varepsilon > 0$, there is a unit vector $v \in \mathcal{H}$ such that

$$\sup_{g \in K} \|\pi(g)v - v\| < \varepsilon.$$

In this case, v is (K, ε) -invariant.

Example 1.1. As a key example of a unitary representation that almost has invariant vectors, consider the right-regular representation $\varrho_{\mathbb{R}}$ of \mathbb{R} on $L^2(\mathbb{R}, \mu)$ where μ is the Lebesgue measure,

$$\varrho_{\mathbb{R}} : \mathbb{R} \rightarrow \mathrm{U}(L^2(\mathbb{R}, \mu)), \quad t \mapsto (f \mapsto tf) \quad \text{where} \quad (tf)(x) = f(x+t),$$

and the unit vector $f = (b-a)^{-1/2} \chi_{[a,b]}$ for some $b > a$. Let $K \subset \mathbb{R}$ be compact and $\varepsilon > 0$. Then K is contained in an interval $[-c, c]$ for some $c > 0$ and

$$\|\varrho_{\mathbb{R}}(t)f - f\|^2 \leq \frac{2c}{b-a} \quad \text{for all } t \in [-c, c]$$

as is readily checked in the two cases $t \leq b-a$ and $t > b-a$. Hence if $b-a > 2c\varepsilon^{-2}$, then f is (K, ε) -fixed.

Similarly, the unitary representation $(\varrho_{\mathbb{Z}}, L^2(\mathbb{Z}, \mu))$ where \mathbb{Z} is considered as a discrete group and μ is the counting measure almost has invariant vectors.

However, these do not have non-zero invariant vectors since otherwise the respective group would have finite measure.

Definition 1.2 (Property (T)). A locally compact group G has *Property (T)* if every unitary representation of G which almost has invariant vectors actually has a non-zero invariant vector.

Remark 1.3. If G is *second-countable*, it suffices to take separable Hilbert spaces in Definition 1.2: Let (π, \mathcal{H}) be a unitary representation of G with \mathcal{H} non-separable. Then for each $v \in \mathcal{H}$ the closed G -invariant subspace $\mathcal{H}_v = \overline{\mathrm{span} \pi(G)v} \leq \mathcal{H}$ is separable since G is second-countable: Let $(U_n)_{n \in \mathbb{N}}$ be a countable basis of open sets for the topology on G and pick $g_n \in U_n$ for each $n \in \mathbb{N}$. Then $\{\pi(g_n)v \mid n \in \mathbb{N}\}$ is a countable dense subset of \mathcal{H}_v : If $V \subseteq \mathcal{H}_v$ is open, then so is $\{g \in G \mid \pi(g)v \in V\}$ by continuity of π . Hence $\{g \in G \mid \pi(g)v \in V\}$ contains some U_n whence $\pi(g_n)v \in V$.

Since \mathcal{H}_v is G -invariant for every $v \in \mathcal{H}$, applying Zorn's Lemma to the non-empty set of subspaces of \mathcal{H} which are Hilbert direct sums of subspaces of the form \mathcal{H}_v ($v \in \mathcal{H}$) yields a maximal element which is equal to \mathcal{H} .

Hence if (π, \mathcal{H}) almost has invariant vectors, then one of the corestricted unitary representations $(\pi|_{\mathrm{U}(\mathcal{H}_v)}, \mathcal{H}_v)$ ($v \in \mathcal{H}$) with \mathcal{H}_v separable does.

By Example 1.1, \mathbb{R} and \mathbb{Z} are groups *without* Property (T). Compact groups are our first examples of groups *with* Property (T). Our proof uses the Bochner integral, see Appendix C, in particular Example C.3.

Proposition 1.4. Let G be a compact group. Then G has Property (T).

Proof. Let (π, \mathcal{H}) be a unitary representation of G which almost has invariant vectors. We wish to show that π has a non-zero invariant vector. Let $v_0 \in \mathcal{H}$ be a unit vector such that $\sup_{g \in G} \|\pi(g)v_0 - v_0\| < \frac{1}{2}$ and define

$$v = \int_G \pi(g)v_0 \mu(g) \quad \text{satisfying} \quad \langle v, w \rangle = \int_G \langle \pi(g)v_0, w \rangle \mu(g) \quad \text{for all } w \in \mathcal{H}.$$

Then v is G -invariant since for all $h \in G$ and $w \in \mathcal{H}$ we have

$$\langle \pi(h)v, w \rangle = \langle v, \pi(h^{-1})w \rangle = \int_G \langle \pi(hg)v_0, w \rangle \mu(g) = \int_G \langle \pi(g)v_0, w \rangle \mu(g) = \langle v, w \rangle$$

and it satisfies $\|v - v_0\| < \frac{1}{2}$ whence $v \neq 0$ since v_0 is a unit vector. \square

1.1.1. *Inheritance Properties.* We now investigate how Property (T) behaves with respect to extensions and lattices. For a more flexible definition, we introduce Property (T) for pairs (G, H) where H is a subgroup of G .

Definition 1.5. Let G be a locally compact group and let H be a subgroup of G . The pair (G, H) has *Property (T)* if every unitary representation of G which almost has invariant vectors actually has a non-zero H -invariant vector.

The following statement will have many interesting consequences itself. Note that in a category of topological groups by morphism we mean a continuous homomorphism.

Proposition 1.6. Let G and H be locally compact groups. If G has Property (T) and $\varphi : G \rightarrow H$ is a morphism with dense image, then H has Property (T).

Proof. Let (π, \mathcal{H}) be a unitary representation of H which almost has invariant vectors. Then $(\pi\varphi, \mathcal{H})$ is a unitary representation of G which almost has invariant vectors as well; and since G has Property (T), the latter admits a non-zero invariant vector. Therefore, $\text{im } \varphi$ admits a non-zero invariant vector and so does $\overline{\text{im } \varphi} = H$ by continuity. \square

Proposition 1.7. Let $1 \rightarrow G_1 \xrightarrow{i} G \xrightarrow{\text{pr}} G_2 \rightarrow 1$ be a short exact sequence of locally compact groups. Then G has Property (T) if and only if (G, G_1) and G_2 have Property (T). If $G = G_1 \times G_2$ then G has Property (T) if and only if G_1 and G_2 have Property (T).

Proof. Suppose that G has Property (T). Then G_2 has Property (T) by Proposition 1.6 applied to $\text{pr} : G \rightarrow G_2$. Since G_1 may be considered as a subgroup of G via $i : G_1 \rightarrow G$, the pair (G, G_1) has Property (T), too.

Conversely, suppose that (G, G_1) and G_2 have Property (T). Let (π, \mathcal{H}) be a unitary representation of G which almost has invariant vectors. We show that it contains a non-zero invariant vector. Since the pair (G, G_1) has Property (T), the subspace \mathcal{H}^{G_1} of G_1 -invariant vectors in \mathcal{H} is non-zero. Furthermore, the normality of G_1 in G implies that \mathcal{H}^{G_1} is G -invariant. Therefore (π, \mathcal{H}^{G_1}) is again a unitary representation of G ; and it almost has invariant vectors: Otherwise $\mathcal{H}^{G_1 \perp}$ had almost invariant vectors and then a G_1 -invariant vector because (G, G_1) has Property (T). Now, the representation (π, \mathcal{H}^{G_1}) factors through G_2 and therefore contains a non-zero G_2 -invariant vector which then actually is G -invariant. \square

We now turn to lattices for which there is the following remarkable theorem.

Theorem 1.8. Let G be a locally compact, second-countable group and let Γ be a lattice in G . Then G has Property (T) if and only if Γ has Property (T).

Whereas the conclusion from G to Γ is established rather easily, the converse is less elementary and due to Wang [Wan75, Thm. 3.7]. We shall not prove it here.

Proof. (" \Rightarrow "). Suppose that G has Property (T) and let (π, \mathcal{H}) be a unitary representation of Γ which almost has invariant vectors. Then the induction space $\text{Ind}_\Gamma^G \mathcal{H}$ consisting of classes of measurable functions $f : G \rightarrow \mathcal{H}$, modulo equality almost everywhere, satisfying

$$f(x\gamma) = \pi(\gamma^{-1})f(x) \quad \forall \gamma \in \Gamma \quad \forall x \in G \quad \text{and} \quad \int_{G/\Gamma} \|f(x)\|^2 \mu(x\Gamma) < \infty$$

with G -action λ_G defined by $(\lambda_G(x)f)(y) = f(x^{-1}y)$ for all $x, y \in G$ and $f \in \text{Ind}_\Gamma^G \mathcal{H}$ is a unitary representation $(\lambda_G, \text{Ind}_\Gamma^G \mathcal{H})$ of G which almost has invariant vectors as well (this uses second countability, see e.g. [dlHVB89, 3.a.3]).

Since G has Property (T), there is a non-zero invariant vector $f \in \text{Ind}_\Gamma^G \mathcal{H}$, i.e. $\forall x \in G \quad \forall y \in G: f(x^{-1}y) = f(y)$. Since G is locally compact and second-countable, it is σ -finite and hence we may apply Fubini's Theorem to conclude the existence of a conull set $A \subseteq G$ and a non-zero vector $v \in \mathcal{H}$ such that $f(x) = v$ for all $x \in A$. By the invariance properties of f , we further have for every $\gamma \in \Gamma$:

$$f(x\gamma) = v \quad \forall x \in A\gamma^{-1} \quad \text{and} \quad f(x\gamma) = \pi(\gamma^{-1})f(x) \quad \forall x \in B_\gamma \subseteq G \text{ conull}$$

Now, for every $\gamma \in \Gamma$ we may choose $x \in A\gamma^{-1} \cap B_\gamma \cap A$ (note that $\mu(A\gamma^{-1}) = \mu(A)$ since G is unimodular as it contains a lattice (Proposition B.42) and hence $A\gamma^{-1} \cap B_\gamma \cap A$ is conull, thus non-empty) to conclude $v = f(x\gamma) = \pi(\gamma^{-1})f(x) = \pi(\gamma^{-1})v$. Therefore, Γ has Property (T). \square

1.1.2. *Property (T) for $\text{SL}(n, \mathbb{R})$.* We now sketch a proof of the fact that $\text{SL}(n, \mathbb{R})$ has Property (T) for $n \geq 3$, following [dlHVB89, 2.a]. From Theorem 1.8 we thus conclude for instance that $\text{SL}(n, \mathbb{Z})$ ($n \geq 3$) has Property (T), too. More generally, real simple Lie groups of rank at least two and their lattices have Property (T).

Proposition 1.9. The group $\text{SL}(n, \mathbb{R})$ has Property (T) for $n \geq 3$.

The proof of Proposition 1.9 is based on two lemmata. For their formulation, note that $\text{SL}(n, \mathbb{R})$ has the subgroup

$$\text{SL}(n-1, \mathbb{R}) \times \mathbb{R}^{n-1} \hookrightarrow \text{SL}(n, \mathbb{R}), \quad (A, x) \mapsto \begin{pmatrix} A & x \\ 0 & 1 \end{pmatrix}.$$

Lemma 1.10. Let (π, \mathcal{H}) be a unitary representation of $\text{SL}(n, \mathbb{R})$ ($n \geq 2$). If $v \in \mathcal{H}$ is fixed by $\mathbb{R}^{n-1} \leq \text{SL}(n, \mathbb{R})$, then v is fixed by $\text{SL}(n, \mathbb{R})$.

Proof. This follows from the well-known Mautner phenomenon. \square

Lemma 1.11. The pair $(\text{SL}(n-1, \mathbb{R}) \times \mathbb{R}^{n-1}, \mathbb{R}^{n-1})$ has Property (T) for $n \geq 3$.

Our proof of Lemma 1.11 uses results on positive definite functions. Let G be a topological group. Recall that a continuous function $\varphi : G \rightarrow \mathbb{C}$ is called positive definite if for all $m \geq 0$, $g_1, \dots, g_m \in G$ and $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ we have

$$\sum_{1 \leq i, j \leq m} \lambda_i \overline{\lambda_j} \varphi(g_i^{-1} g_j) \geq 0.$$

It is readily checked that for every unitary representation (π, \mathcal{H}) of G and $v \in \mathcal{H}$ the function $g \mapsto \langle v, \pi(g)v \rangle$ is positive definite. As a matter of fact, every positive definite function on a topological group arises this way as the GNS (Gelfand-Naimark-Segal) construction shows, see e.g. [dlHVB89, 5.a.8, 5.a.9]:

Proposition 1.12. Let G be a topological group and $\varphi : G \rightarrow \mathbb{C}$ a continuous, positive-definite function. Then there is a unitary representation $(\pi_\varphi, \mathcal{H}_\varphi)$ of G and a vector $v_\varphi \in \mathcal{H}_\varphi$ such that $\varphi(g) = \langle v_\varphi, \pi_\varphi(g)v_\varphi \rangle$. The pair $((\pi_\varphi, \mathcal{H}_\varphi), v_\varphi)$ is unique up to isomorphism with this property. Furthermore:

- (i) If φ, ψ and χ are continuous, positive-definite functions on G such that $\varphi = \psi + \chi$, then π_ψ and π_χ are subrepresentations of π_φ .
- (ii) If φ is strictly positive constant, then π_φ is the trivial one-dimensional representation.
- (iii) If H is a closed subgroup of G , then $\pi_{\varphi|_H}$ is a subrepresentation of $\pi_\varphi|_H$.
- (iv) Let (π, \mathcal{H}) be a unitary representation of G . If $v \in \mathcal{H}$ and $\varphi : G \rightarrow \mathbb{C}$ is defined by $g \mapsto \langle v, \pi(g)v \rangle$, then π_φ is a subrepresentation of π .

Proof. (Lemma 1.11) Abbreviate $G = \mathrm{SL}(n-1, \mathbb{R})$ and $H = \mathbb{R}^{n-1}$. Let (π, \mathcal{H}) be a unitary representation of G which almost has invariant vectors. Choose an increasing, exhausting sequence $(K_m)_m$ ($m \in \mathbb{N}$) of compact subsets of G and let $v_m \in \mathcal{H}$ be a $(K_m, 1/m)$ -invariant vector. Then the functions $\varphi_m : G \rightarrow \mathbb{C}$, $g \mapsto \langle v_m, \pi(g)v_m \rangle$ are positive definite. Restricting to H and applying the Fourier transform using Bochner's Theorem, yields for each $m \in \mathbb{N}$ a measure μ_m on the dual $\widehat{H} \cong H$ of H . One can show that there exists $m \in \mathbb{N}$ such that $\mu_m(\{0\}) \neq 0$ and hence $\mu_m = c_m \delta_0 + \mu'_m$ for some $c_m > 0$. Applying the inverse transform yields $\varphi_m = c_m + \varphi'_m$ and hence applying Proposition 1.12 shows that $\pi|_H$ contains the trivial one-dimensional representation and therefore a non-zero invariant vector. \square

Proof. (Proposition 1.9). Let (π, \mathcal{H}) be a unitary representation of $\mathrm{SL}(n, \mathbb{R})$ which almost has invariant vectors. Then so does the restriction of π to $\mathrm{SL}(n-1, \mathbb{R}) \ltimes \mathbb{R}^{n-1}$ and hence there is an \mathbb{R}^{n-1} -fixed vector $v \in \mathcal{H}$ by Lemma 1.11. From Lemma 1.10 we then conclude that v is $\mathrm{SL}(n, \mathbb{R})$ -fixed. Hence $\mathrm{SL}(n, \mathbb{R})$ has Property (T). \square

1.2. Amenability. Another property of locally compact groups we are interested in is amenability. There are several equivalent definitions, two of which are most suited for our purposes.

Definition 1.13 (Amenability). A locally compact group G is *amenable* if the right-regular representation $(\varrho_G, L^2(G, \mu))$, where μ is the Haar measure on G , almost has invariant vectors.

It is a non-trivial fact that Definition 1.13 of amenability is equivalent to the following, see [Zim84, Thm. 7.1.8].

Definition 1.14 (Amenability). A locally compact group G is *amenable* if for every continuous action of G on a compact metric space X there is a G -invariant probability measure on X .

From Definition 1.14 we deduce that locally compact abelian groups are amenable.

Proposition 1.15. Let G be a locally compact abelian group. Then G is amenable.

Proof. In view of Definition 1.14, let G act continuously on a compact metric space X . Then G acts continuously on $C(X)$, equipped with the supremum norm, via the left-regular representation $\lambda_G : G \rightarrow \mathrm{Iso}(C(X))$. Therefore, G also acts on the dual space $C(X)^*$ of $C(X)$ via the adjoint representation λ_G^* of λ_G defined by

$$\langle \lambda_G^*(g)\mu, f \rangle = \langle \mu, \lambda_G(g^{-1})f \rangle.$$

for all $\mu \in C(X)^*$ and $f \in C(X)$. Since the set $P(X)$ of probability measures on X is a weak*-compact, convex and λ_G^* -invariant subset of $C(X)^*$, the well-known Kakutani-Markov Theorem implies that it contains a λ_G^* -fixed point, i.e. a G -invariant probability measure on X . \square

Also, we deduce that compact groups are amenable. As with Property (T), our proof uses vector-valued integration, namely the Pettis integral, see [Rud91, I.3.4].

Proposition 1.16. Let G be a compact group. Then G is amenable.

Proof. Let G act continuously on a compact metric space X and let μ be the Haar measure on G . Fix some $\nu_0 \in P(X) \subseteq C(X)^*$. Then

$$\nu = \int_G g_* \nu_0 \mu(g) \quad \text{satisfying} \quad \langle \nu, f \rangle = \int_G \langle g_* \nu_0, f \rangle \mu(g) \quad \text{for all } f \in C(X)$$

is a G -invariant probability measure. \square

1.2.1. Inheritance Properties. The following theorems on how amenability behaves with respect to extensions and lattices should be compared with those for Property (T). See [Zim84, Sec. 4] for full proofs.

Proposition 1.17. Let G and H be locally compact groups. If G is amenable and $\varphi : G \rightarrow H$ is a morphism with dense image, then H is amenable.

Proposition 1.18. Let $1 \rightarrow G_1 \xrightarrow{i} G \xrightarrow{\text{pr}} G_2 \rightarrow 1$ be a short exact sequence of locally compact groups. Then G is amenable if and only if G_1 and G_2 are amenable.

Proposition 1.19. Let G be a locally compact, amenable group and let H be a closed subgroup of G . Then H is amenable.

Proposition 1.20. Let G be a locally compact, second-countable group and let Γ be a lattice in G . Then G is amenable if and only if Γ is amenable.

1.3. Consequences of Property (T). Property (T) and amenability exhibit a rich interplay which leads to several striking consequences.

With Definition 1.13 of amenability, Example 1.1 of groups without Property (T) now generalizes to the following proposition.

Proposition 1.21. Let G be a locally compact, amenable group. Then G has Property (T) if and only if G is compact.

Proof. If G is compact, it has Property (T) by Proposition 1.4.

Conversely, let G be a locally compact, amenable group. If G has Property (T), then the right-regular representation $(\varrho_G, L^2(G, \mu))$ has a non-zero invariant vector. Therefore, G has finite measure and hence is compact by Proposition B.21. \square

Another immediate consequence is the following.

Proposition 1.22. Let G and H be locally compact groups. If G has Property (T) and H is amenable then every morphism $\varphi : G \rightarrow H$ has relatively compact image.

Proof. The closed subgroup $\overline{\text{im } \varphi}$ of H has Property (T) by Proposition 1.6. Hence, being amenable as a closed subgroup of an amenable group by Proposition 1.19 and having Property (T) it is compact by Proposition 1.21. \square

As a consequence of Proposition 1.22 we obtain the following three properties of groups with Property (T).

Corollary 1.23. Let G be a locally compact group with Property (T). Then:

- (i) Every morphism from G to $\mathbb{R}^n \oplus \mathbb{Z}^m$ ($m, n \geq 0$) is trivial.
- (ii) G is unimodular.
- (iii) $G/\overline{[G, G]}$ is compact. If $G = \Gamma$ is discrete then $\Gamma/[\Gamma, \Gamma]$ is finite.

Proof. As to (i), the groups $\mathbb{R}^n \oplus \mathbb{Z}^m$ ($m, n \in \mathbb{N}_0$) are abelian and hence amenable by Proposition 1.15. Therefore, every morphism from G to $\mathbb{R}^n \oplus \mathbb{Z}^m$ has relatively compact image by Proposition 1.22 which hence reduces to $\{0\}$.

For (ii), note that $(\mathbb{R}_{>0}, \cdot)$ is abelian and hence amenable. Thus, the modular function $\Delta_G : G \rightarrow (\mathbb{R}_{>0}, \cdot)$ has relatively compact image whence $\Delta_G \equiv 1$.

As to (iii), the group $G/[G, G]$ is abelian and hence amenable. We conclude that $\text{pr} : G \twoheadrightarrow G/[G, G]$ has (relatively) compact image, i.e. $G/[G, G]$ is compact, and finite in case $G = \Gamma$ is discrete. \square

Remark 1.24. Corollary 1.23 provides more classes of groups *without* Property (T).

- (i) By part (i), a non-trivial free group does not have Property (T).
- (ii) By part (iii), the fundamental group $\Gamma = \pi_1(\Sigma_g)$ of a genus g surface Σ_g does not have Property (T), since $\Gamma/[\Gamma, \Gamma] = \mathbf{H}_1(\Sigma_g, \mathbb{Z}) = \mathbb{Z}^{2g}$.

The following Proposition captures Kazhdan's original interest in Property (T).

Proposition 1.25. Let G be a locally compact group with Property (T). Then G is compactly generated. If $G = \Gamma$ is discrete, then Γ is finitely generated.

Proof. Let \mathcal{C} be the set of all compactly generated, open subgroups of G . Since G is locally compact, every point has a relatively compact neighbourhood. Thus $G = \bigcup_{H \in \mathcal{C}} H$. Furthermore, any $H \in \mathcal{C}$ is open and hence G/H is discrete.

Denote by $\lambda_{G/H}$ the left-regular representation of G on $l^2(G/H)$. Then

$$(\pi, \mathcal{H}) := \left(\widehat{\bigoplus_{H \in \mathcal{C}} \lambda_{G/H}}, \widehat{\bigoplus_{H \in \mathcal{C}} l^2(G/H)} \right)$$

is a unitary representation of G which almost has invariant vectors: Let K be a compact subset of G . Since G is locally compact, K has a relatively compact neighbourhood which generates some $H_K \in \mathcal{C}$. Then the unit vector $\delta_{e_{H_K}} \in \mathcal{H}$ is K -invariant. Since G has Property (T), there is a non-zero G -invariant vector $v = (v_H)_{H \in \mathcal{C}} \in \mathcal{H}$. If $H \in \mathcal{C}$ is chosen such that $v_H \neq 0$, then $v_H \in l^2(G/H)$ is G -invariant and hence $l^2(G/H)$ is finite. Therefore, G is compactly generated by the union of a compact generating set for H and finitely many coset representatives for G/H . \square

Remark 1.26. Let G be a real simple Lie group of rank at least two, K a maximal compact subgroup of G and Γ a torsion-free lattice of G . Then G has Property (T) by our remarks at the beginning of Section 1.1.2 and therefore so does Γ by Theorem 1.8. Proposition 1.25 and part (iii) of Corollary 1.23 now respectively imply that the locally symmetric space $M = \Gamma \backslash G/K$ has *finitely generated* fundamental group $\pi_1(M) = \Gamma$ and *finite* first homology $\mathbf{H}_1(M, \mathbb{Z}) = \Gamma/[\Gamma, \Gamma]$.

2. CONTINUOUS COHOMOLOGY

In this section we develop a continuous cohomology theory of topological groups G which will be applied in the proof of Theorem 0.3. A good reference is Monod's thorough treatment [Mon01] of continuous bounded cohomology. Classical, yet less detailed references include [HM62], [Bla79], [Gui80] and [BW00].

2.1. Notations and Definitions. A *topological vector space* is a vector space E over \mathbb{R} or \mathbb{C} with a topology such that addition and scalar multiplication are continuous. A *morphism* of topological vector spaces is a continuous linear map.

A G -*module* is a pair (π, E) where E is a topological vector space and π is a homomorphism from G to the group of automorphisms of E . A G -*morphism* of G -modules is a G -equivariant morphism. A G -module (π, E) is *continuous* if the action map $\pi : G \times E \rightarrow E$, $(g, v) \mapsto \pi(g)v$ is continuous.

A *complex* (E^\bullet, d^\bullet) of G -modules is a \mathbb{Z} -indexed sequence

$$\dots \longrightarrow E^{n-1} \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1} \longrightarrow \dots$$

of G -modules and G -morphisms d^n such that $d^{n+1}d^n = 0$ for all $n \in \mathbb{Z}$. The maps d^n are *differentials*. A *right complex* is a complex (E^\bullet, d^\bullet) such that $E^n = 0$ for all $n < 0$, and will also be considered as an \mathbb{N}_0 -indexed sequence. Given a property of G -modules, a complex E^\bullet is said to have that property if all E^n do.

A *morphism of complexes* $\alpha^\bullet : E^\bullet \rightarrow F^\bullet$ is a sequence α^n ($n \in \mathbb{Z}$) of morphisms $\alpha^n : E^n \rightarrow F^n$ such that the following diagram commutes.

$$\begin{array}{ccccccc} \dots & \longrightarrow & E^{n-1} & \longrightarrow & E^n & \longrightarrow & E^{n+1} & \longrightarrow & \dots \\ & & \downarrow \alpha^{n-1} & & \downarrow \alpha^n & & \downarrow \alpha^{n+1} & & \\ \dots & \longrightarrow & F^{n-1} & \longrightarrow & F^n & \longrightarrow & F^{n+1} & \longrightarrow & \dots \end{array}$$

If α^\bullet and β^\bullet are morphisms from (E^\bullet, d^\bullet) to $(F^\bullet, \partial^\bullet)$, a *homotopy* from α^\bullet to β^\bullet is a sequence h^\bullet of morphisms $h^n : E^n \rightarrow F^{n-1}$ ($n \in \mathbb{Z}$) such that $\partial^{n-1}h^n + h^{n+1}d^n = \beta^n - \alpha^n$. In this case, α^\bullet and β^\bullet are *homotopic*.

$$\begin{array}{ccccccc} \dots & \longrightarrow & E^{n-1} & \longrightarrow & E^n & \longrightarrow & E^{n+1} & \longrightarrow & \dots \\ & & \swarrow h^{n-1} & \parallel & \swarrow h^n & \parallel & \swarrow h^{n+1} & \parallel & \swarrow h^{n+2} \\ \dots & \longrightarrow & F^{n-1} & \longrightarrow & F^n & \longrightarrow & F^{n+1} & \longrightarrow & \dots \end{array}$$

Similarly, one defines a G -*morphism of complexes* and a G -*homotopy*. A morphism of complexes $\alpha^\bullet : E^\bullet \rightarrow F^\bullet$ is *null homotopic* if it is homotopic to the zero map. It is a *homotopy equivalence* if there is a morphism $\beta^\bullet : F^\bullet \rightarrow E^\bullet$ such that $\beta^\bullet\alpha^\bullet$ and $\alpha^\bullet\beta^\bullet$ are homotopic to the identity. A *contracting homotopy* of a complex (E^\bullet, d^\bullet) is a homotopy h^\bullet from $0 : E^\bullet \rightarrow E^\bullet$ to $\text{id} : E^\bullet \rightarrow E^\bullet$, i.e. a sequence of morphisms $h^n : E^n \rightarrow E^{n-1}$ ($n \in \mathbb{Z}$) such that $d^{n-1}h^n + h^{n+1}d^n = \text{id}$.

Given a complex (E^\bullet, d^\bullet) , the elements of E^n are *cochains of degree n* . The topological space $\ker d^n$ is denoted $Z^n(G, E)$ and its elements are *cocycles of degree n* . Similarly, $\text{im } d^{n-1}$ is denoted $B^n(G, E)$ and its elements are *coboundaries of degree n* . The n -*th cohomology* of (E^\bullet, d^\bullet) is the quotient topological vector space

$$H^n(E^\bullet) = Z^n(G, E)/B^n(G, E) = \ker d^n / \text{im } d^{n-1}.$$

Morphisms of complexes induce maps in cohomology and homotopic morphisms induce the same maps. A complex (E^\bullet, d^\bullet) is *exact in degree n* if $H^n(E^\bullet) = 0$. It is *exact* if it is exact in all degrees. Given a G -module E , a *resolution* of E is an exact complex $0 \rightarrow E \xrightarrow{\varepsilon} (E^\bullet, d^\bullet)$ where (E^\bullet, d^\bullet) is a right complex, abbreviated (ε, E^\bullet) .

2.2. Homological Algebraic Characterization. Recall that the "ordinary" cohomology of a group G with coefficients E is computed from a sequence which in degree n has G -equivariant maps from G^{n+1} to E . For continuous cohomology we simply require these maps to be continuous.

Definition 2.1. Let G be a topological group and let (π, E) be a continuous G -module. For $n \in \mathbb{N}$, the set $C(G^n, E)$ is a vector space with pointwise addition and scalar multiplication which is turned into a topological one by the compact-open topology. The G -action

$$(xf)(x_1, \dots, x_n) = \pi(x)f(x^{-1}x_1, \dots, x^{-1}x_n)$$

is by linear maps since π acts by linear maps and thus turns $C(G^n, E)$ ($n \in \mathbb{N}$) into a G -module. Consider the complex

$$0 \longrightarrow C(G, E)^G \xrightarrow{d^0} C(G^2, E)^G \xrightarrow{d^1} C(G^3, E)^G \xrightarrow{d^2} \dots$$

of G -invariant vectors where

$$d^n f(x_0, \dots, x_{n+1}) = \sum_{k=0}^{n+1} (-1)^k f(x_0, \dots, \widehat{x}_k, \dots, x_{n+1}).$$

The n -th continuous cohomology ($n \in \mathbb{N}_0$) of G with coefficients E is defined as the n -th cohomology $H_c^n(G, E)$ of the above complex.

The following definitions are designed to allow the computation of continuous cohomology spaces of a topological group G with coefficients in a continuous G -module E from any of a certain class of resolutions of E in the spirit of relative homological algebra.

Definition 2.2. Let A and B be G -modules. A G -morphism $\varphi : A \rightarrow B$ is *admissible* if there is a continuous linear map $\psi : B \rightarrow A$ such that $\varphi\psi\varphi = \varphi$.

$$A \begin{array}{c} \xrightarrow{\varphi} \\ \dashrightarrow \\ \xrightarrow{\psi} \end{array} B.$$

Remark 2.3. Retain the notation of Definition 2.2.

- (i) If $\varphi : A \rightarrow B$ is injective, then $\psi\varphi = \text{id}$.
- (ii) If $\varphi : A \rightarrow B$ is surjective, then $\varphi\psi = \text{id}$.

Proposition 2.4. If a complex (E^\bullet, d^\bullet) admits a contracting homotopy then the maps d^\bullet are admissible.

Proof. Let h^\bullet be a contracting homotopy of (E^\bullet, d^\bullet) . Then

$$d^n h^{n+1} d^n = (\text{id} - h^{n+2} d^{n+1}) d^n = d^n$$

for every $n \in \mathbb{Z}$. Thus h^{n+1} serves to show that d^n is admissible. \square

Definition 2.5. A G -module E is *relatively injective* if for every admissible injective G -morphism $\iota : A \rightarrow B$ and every G -morphism $\alpha : A \rightarrow E$ there is G -morphism β such that $\alpha = \beta\iota$.

$$\begin{array}{ccc} A & \xrightarrow{\iota} & B \\ & \searrow \alpha & \vdots \\ & & E \end{array}$$

With these definitions, all the standard lemmata of homological algebra will, mutatis mutandis, carry over to our continuous case; the end result being:

Theorem 2.6. Let G be a locally compact group and let E be a continuous G -module. Then the following statements hold.

- (i) There is a continuous, relatively injective resolution $0 \rightarrow E \xrightarrow{\varepsilon} E^\bullet$ of E with admissible differentials.
- (ii) For any resolution $0 \rightarrow E \xrightarrow{\varepsilon} E^\bullet$ as in (i), the n -th cohomology ($n \in \mathbb{N}_0$) of the complex $E^\bullet G$ is canonically isomorphic, as a topological vector space, to $H_c^n(G, E)$ as in Definition 2.1.

Part (i) of Theorem 2.6 is proven in Section 2.3.1. Part (ii) will follow from the subsequent exposition of standard lemmata.

First of all, let us record the following result on how relative injectivity can be used if the admissible map is not injective.

Lemma 2.7. Let A and B be G -modules and let $\varphi : A \rightarrow B$ be an admissible G -morphism. Furthermore, let E be a relatively injective G -module. If $\alpha : A \rightarrow E$ is a G -morphism satisfying $\ker \alpha \supseteq \ker \varphi$ then there is a G -morphism $\beta : B \rightarrow E$ such that $\alpha = \beta\varphi$.

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ & \searrow \alpha & \vdots \beta \\ & & E. \end{array}$$

Proof. We view the space $\bar{A} := A/\ker \varphi$ as a G -module with the quotient structure. Then the canonical map $\pi : A \rightarrow \bar{A}$ is a G -morphism and there is a unique injective G -morphism $\bar{\varphi} : \bar{A} \rightarrow B$ such that $\varphi = \bar{\varphi}\pi$.

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ & \searrow \psi & \nearrow \bar{\varphi} \\ & \searrow \pi & \vdots \beta \\ & & \bar{A} \\ & \searrow \alpha & \searrow \bar{\alpha} \\ & & E. \end{array}$$

Since φ is admissible there is a morphism $\psi : B \rightarrow A$ such that $\varphi\psi\varphi = \varphi$. Then the morphism $\pi\psi : B \rightarrow \bar{A}$ satisfies $\bar{\varphi}\pi\psi\bar{\varphi}\pi = \varphi\psi\varphi = \varphi = \bar{\varphi}\pi$. In view of the surjectivity of π this shows that $\bar{\varphi}$ is admissible.

Next, since $\ker \alpha \supseteq \ker \varphi$, there is a G -morphism $\bar{\alpha} : \bar{A} \rightarrow E$ such that $\alpha = \bar{\alpha}\pi$. The relative injectivity of E now provides a G -morphism $\beta : B \rightarrow E$ such that $\bar{\alpha} = \beta\bar{\varphi}$ whence $\alpha = \bar{\alpha}\pi = \beta\bar{\varphi}\pi = \beta\varphi$. \square

Lemma 2.8. Let E be a continuous G -module and let $0 \rightarrow E \xrightarrow{\varepsilon} (E^\bullet, d^\bullet)$ be a resolution of E with admissible differentials. Let

$$0 \longrightarrow F \xrightarrow{\partial^{-1}} F^0 \xrightarrow{\partial^0} F^1 \xrightarrow{\partial^1} \dots$$

be a complex of G -modules starting in degree -1 and let $\alpha : E \rightarrow F$ be a G -morphism. If F^n is relatively injective for all $n \in \mathbb{N}_0$, then there exists a G -morphism of complexes $\alpha^\bullet : (\varepsilon, E^\bullet) \rightarrow (\partial^{-1}, F^\bullet)$ such that $\alpha^{-1} = \alpha$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & E & \xrightarrow{\varepsilon} & E^0 & \xrightarrow{d^0} & E^1 & \xrightarrow{d^1} & \dots \\ & & \downarrow \alpha^{-1} = \alpha & & \downarrow \alpha^0 & & \downarrow \alpha^1 & & \\ 0 & \longrightarrow & F & \xrightarrow{\partial^{-1}} & F^0 & \xrightarrow{\partial^0} & F^1 & \xrightarrow{\partial^1} & \dots \end{array}$$

In the situation of Lemma 2.8 one says that α^\bullet extends α .

Proof. Let $\alpha^{-1} = \alpha$. We proof the existence of α^n ($n \in \mathbb{N}_0$) by induction on n . For $n = 0$, there is a G -morphism $\alpha^0 : E^0 \rightarrow F^0$ such that $\alpha^0 \varepsilon = \partial^{-1} \alpha^{-1} = \partial^{-1} \alpha$ since ε in an admissible injection and F^0 is relatively injective.

Suppose now, that α^k ($-1 \leq k \leq n$) has been constructed. If d^n is injective we can obtain α^{n+1} analogously to how we obtained α^0 . For the general case, consider

$$\begin{array}{ccc} E^n & \xrightarrow{d^n} & E^{n+1} \\ \alpha^n \downarrow & \searrow & \downarrow \alpha^{n+1} \\ F^n & \xrightarrow{\partial^n} & F^{n+1}. \end{array}$$

In order to apply Lemma 2.7, we show that $\ker \partial^n \alpha^n \supseteq \ker d^n$: If $v \in \ker d^n$, then by exactness there is some $w \in E^{n-1}$ such that $d^{n-1} w = v$. This implies $\partial^n \alpha^n v = \partial^n \alpha^n d^{n-1} w = \partial^n \partial^{n-1} \alpha^{n-1} w = 0$ by the induction hypothesis. Hence Lemma 2.7 applies and provides a G -morphism α^{n+1} such that $\partial^n \alpha^n = \alpha^{n+1} d^n$. \square

Lemma 2.9. Retain the above notation. Any two extensions of α are G -homotopic.

Proof. Considering the difference of any two extensions, it suffices to check that any extension α^\bullet of $\alpha = 0 = \alpha^{-1}$ is null G -homotopic. We therefore wish to construct a sequence $h^n : E^n \rightarrow F^{n-1}$ ($n \in \mathbb{Z}_{\geq -1}$) such that $\partial^{n-1} h^n + h^{n+1} d^n = \alpha^n$. Let $h^{-1} = h^0 = 0$ and suppose by induction that h^k ($-1 \leq k \leq n$) has been constructed,

$$\begin{array}{ccccccc} & & E^{n-1} & \xrightarrow{d^{n-1}} & E^n & \xrightarrow{d^n} & E^{n+1} \\ & & \downarrow \alpha^{n-1} & & \downarrow \alpha^n & & \downarrow \alpha^{n+1} \\ & & F^{n-1} & \xrightarrow{\partial^{n-1}} & F^n & & F^{n+1} \\ & \swarrow h^{n-1} & & \swarrow h^n & & \swarrow h^{n+1} & \\ F^{n-2} & \xrightarrow{\partial^{n-2}} & F^{n-1} & \xrightarrow{\partial^{n-1}} & F^n & & F^{n+1} \end{array}$$

As before, we show that Lemma 2.7 is applicable to the triangle

$$\begin{array}{ccc} E^n & \xrightarrow{d^n} & E^{n+1} \\ \alpha^n - \partial^{n-1} h^n \searrow & & \downarrow h^{n+1} \\ & & F^n. \end{array}$$

Namely, let $v \in \ker d^n$ and $w \in E^{n-1}$ such that $d^{n-1} w = v$. Then

$$\begin{aligned} (\alpha^n - \partial^{n-1} h^n)v &= \alpha^n d^{n-1} w - \partial^{n-1} h^n d^{n-1} w \\ &= (\partial^{n-1} h^n d^{n-1} + h^{n+1} d^n d^{n-1})w - \partial^{n-1} h^n d^{n-1} w = 0. \end{aligned}$$

\square

Corollary 2.10. Let E be a continuous G -module, and let (ε, E^\bullet) and (η, F^\bullet) be continuous, relatively injective resolutions of E with admissible differentials. Then there is a G -homotopy equivalence $(\varepsilon, E^\bullet) \rightarrow (\eta, F^\bullet)$ which induces a canonical isomorphism $H^n(E^{\bullet G}) \cong H^n(F^{\bullet G})$ of topological vector spaces for each $n \in \mathbb{N}_0$.

Proof. By Lemma 2.8, there are G -morphisms $\alpha^\bullet : (\varepsilon, E^\bullet) \rightarrow (\eta, F^\bullet)$ and $\beta^\bullet : (\eta, F^\bullet) \rightarrow (\varepsilon, E^\bullet)$ extending the identity on $E = E^{-1} = F^{-1}$. Then both the identity on (ε, E^\bullet) and $\beta^\bullet \alpha^\bullet$ extend the identity on $E = E^{-1}$. Hence, by Lemma 2.9, they are G -homotopic. Similarly, $\alpha^\bullet \beta^\bullet$ is G -homotopic to the identity on (η, F^\bullet) . In particular, α^\bullet and β^\bullet restrict to homotopy equivalences between $E^{\bullet G}$ and $F^{\bullet G}$ and therefore induce isomorphisms $H^n(E^{\bullet G}) \cong H^n(F^{\bullet G})$ for all $n \in \mathbb{N}_0$. \square

The proof of Theorem 2.6 now reduces to the following.

Proof. (Theorem 2.6 (ii)). Applying Corollary 2.10 to $(\eta, F^\bullet) = (\varepsilon, C(G^{\bullet+1}, E))$ as defined in Section 2.3.1 proves part (ii) of Theorem 2.6. \square

2.3. Resolutions. Let G be a locally compact group and let (π, E) be a continuous G -module. In this section we give several complexes from which the continuous cohomology of G with coefficients E may be computed.

2.3.1. The Continuous Homogeneous Resolution. This subsection proves the existence part of Theorem 2.6. We show that the complex in Definition 2.1 comes from a continuous, relatively injective resolution of E with admissible differentials.

For $n \in \mathbb{N}$, the set $C(G^n, E)$ is a vector space with pointwise addition and scalar multiplication which is turned into a topological one by the compact-open topology. The G -action

$$(xf)(x_1, \dots, x_n) = \pi(x)f(x^{-1}x_1, \dots, x^{-1}x_n)$$

is by linear maps since π acts by linear maps. Using local compactness of G one shows that it is continuous.

To prove the relative injectivity of $C(G^n, E)$ ($n \in \mathbb{N}$), consider an extension problem as in Definition 2.5,

$$\begin{array}{ccc} A & \xrightarrow{\quad \iota \quad} & B \\ & \searrow \alpha & \downarrow \beta \\ & & C(G^n, E), \end{array}$$

where σ is as in Definition 2.2. Define $\beta : B \rightarrow C(G^n, E)$ by

$$\beta(b)(x) := \alpha(x_1\sigma(x_1^{-1}b))(x) \text{ where } x = (x_1, \dots, x_n).$$

Then β ranges in continuous maps since for every $b \in B$, $\beta(b)$ may be viewed as a composition of continuous maps:

$$\beta(b) : G^n \xrightarrow{x_1^{-1}b \times \text{id}} B \times G^n \xrightarrow{x_1\sigma \times \text{id}} A \times G^n \xrightarrow{\alpha \times \text{id}} C(G^n, E) \times G^n \xrightarrow{\text{ev}} E,$$

the evaluation map ev being continuous since G is locally compact. Also, β is linear, continuous and G -equivariant: For $g \in G$, $b \in B$ and $x \in G^n$ we have

$$\begin{aligned} \beta(gb)(x) &= \alpha(x_1\sigma(x_1^{-1}gb))(x) = \alpha(g(g^{-1}x_1)\sigma((g^{-1}x_1)^{-1}b))(x) \\ &= (g(\alpha(g^{-1}x_1\sigma((g^{-1}x_1)^{-1}b))))(x) \\ &= \pi(g)\alpha(g^{-1}x_1\sigma((g^{-1}x_1)^{-1}b))(g^{-1}x) \\ &= \pi(g)\beta(b)(g^{-1}x) = (g(\beta(b)))(x). \end{aligned}$$

Eventually, β makes the diagram commute: Let $a \in A$, $x \in G^n$ and compute

$$\beta(\iota(a))(x) = \alpha(x_1\sigma(x_1^{-1}\iota(a)))(x) = \alpha(x_1\sigma(\iota(x_1^{-1}a)))(x) = \alpha(a)(x)$$

where the last equality follows from the fact that $\sigma\iota = \text{id}$, cf. Remark 2.3.

To verify that the differentials are admissible, one checks in view of Proposition 2.4 that the maps

$$h^n : C(G^{n+1}, E) \rightarrow C(G^n, E), \quad (h^n f)(x_1, \dots, x_n) = f(1, x_1, \dots, x_n)$$

($n \in \mathbb{N}_0$) and $h^{-1} := 0$ form a contracting homotopy of the *continuous homogeneous resolution*

$$0 \longrightarrow E \xrightarrow{\varepsilon} C(G, E) \xrightarrow{d^0} C(G^2, E) \xrightarrow{d^1} C(G^3, E) \xrightarrow{d^2} \dots$$

where ε maps $v \in E$ to the constant function on G with value v .

By Theorem 2.6 (ii), the continuous cohomology of G with coefficients E may therefore be computed from the complex given in Definition 2.1.

2.3.2. *The Continuous Inhomogeneous Complex.* Based on the continuous homogeneous resolution we produce another complex from which the continuous cohomology of G with coefficients E may be computed. Namely, we note that $C(G^{n+1}, E)^G$ ($n \in \mathbb{N}$) is isomorphic as a topological vector space to $C(G^n, E)$ via the continuous linear map $\varphi^n : C(G^{n+1}, E)^G \rightarrow C(G^n, E)$ defined by

$$f \mapsto (\varphi^n f) : (x_1, \dots, x_n) \mapsto f(1, x_1, x_1 x_2, \dots, x_1 x_2 \cdots x_n).$$

Indeed, a continuous linear inverse to φ^n is given by $\psi^n : C(G^n, E) \rightarrow C(G^{n+1}, E)^G$,

$$f \mapsto (\psi^n f) : (x_1, \dots, x_{n+1}) \mapsto \pi(x_1) f(x_1^{-1} x_2, \dots, x_n^{-1} x_{n+1}).$$

We may thus introduce the *continuous inhomogeneous complex*

$$0 \longrightarrow E \cong C(G^0, E) \xrightarrow{\partial^0} C(G, E) \xrightarrow{\partial^1} C(G^2, E) \xrightarrow{\partial^2} \dots$$

by defining $\partial^{n-1} := \varphi^n d^{n-1} \psi^{n-1}$ as in the following diagram:

$$\begin{array}{ccc} C(G^n, E)^G & \xrightarrow{d^{n-1}} & C(G^{n+1}, E)^G \\ \varphi^{n-1} \Big\downarrow \psi^{n-1} & & \varphi^n \Big\downarrow \psi^n \\ C(G^{n-1}, E) & \xrightarrow{\partial^{n-1}} & C(G^n, E) \end{array}$$

It has the differential property by definition: For $n \in \mathbb{N}$, we have

$$\partial^n \partial^{n-1} = \varphi^{n+1} d^n \psi^n \varphi^n d^{n-1} \psi^{n-1} = \varphi^{n+1} d^n d^{n-1} \psi^{n-1} = 0.$$

Computing ∂^n ($n \in \mathbb{N}_0$) explicitly, yields for $f \in C(G^n, E)$:

$$\begin{aligned} (\partial^n f)(x_1, \dots, x_{n+1}) &= \pi(x_1) f(x_2, \dots, x_{n+1}) + \sum_{k=1}^n (-1)^k f(x_1, \dots, x_k x_{k+1}, \dots, x_{n+1}) \\ &\quad + (-1)^{n+1} f(x_1, \dots, x_n). \end{aligned}$$

Remark 2.11. Let us state explicitly that $z \in Z^1(G, E)$ if and only if we have $z(gh) = z(g) + \pi(g)z(h)$ for all $g, h \in G$, and that $z \in B^1(G, E)$ if and only if there is some $v \in E$ such that $z(g) = \pi(g)v - v$ for all $g \in G$.

2.3.3. *The L_{loc}^p Homogeneous Resolution.* In this section we require G to be locally compact and σ -compact, and E to be a Banach space. For $n \in \mathbb{N}$ and $p \in [1, \infty)$, the space $L_{loc}^p(G^n, E)$ is a continuous G -module with G -action as in Section 2.3.1 and a Fréchet space, see Appendix C. Defining

$$d^n f(x_0, \dots, x_{n+1}) = \sum_{k=0}^{n+1} (-1)^k f(x_0, \dots, \widehat{x_k}, \dots, x_{n+1})$$

as for the continuous homogeneous resolution, the spaces $L_{loc}^p(G^n, E)$ ($n \in \mathbb{N}$) fit into the following L_{loc}^p homogeneous resolution:

$$0 \longrightarrow E \xrightarrow{\varepsilon} L_{loc}^p(G, E) \xrightarrow{d^0} L_{loc}^p(G^2, E) \xrightarrow{d^1} L_{loc}^p(G^3, E) \xrightarrow{d^2} \dots$$

In fact, if $\chi : G \rightarrow \mathbb{R}_{\geq 0}$ is a compactly supported function of integral 1, then the formulae for the contracting homotopy from Section 2.3.1 carry over to the L_{loc}^p setting in the form $h^n : L_{loc}^p(G^{n+1}, E) \rightarrow L_{loc}^p(G^n, E)$ ($n \in \mathbb{N}_0$),

$$(h^n f)(x_1, \dots, x_n) = \int_G f(g, x_1, \dots, x_n) \chi(g) \mu(g).$$

The relative injectivity mapping problem is solved by

$$\beta(b)(x) = \int_G \alpha(x_1 \sigma(x_1^{-1} b))(x) \chi(x_1) \mu(x_1) \quad \text{where } x = (x_1, \dots, x_n).$$

2.4. Functoriality. We now give several functorial properties of continuous cohomology: covariance with respect to the coefficient module, contravariance with respect to the group and a long exact sequence.

2.4.1. Covariance. Let G be a locally compact group and let E and F be continuous G -modules. Considering the continuous homogeneous resolution introduced in Section 2.3.1, a G -morphism $\alpha : E \rightarrow F$ induces via post-composition a G -morphism of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & E & \xrightarrow{\varepsilon} & C(G, E) & \longrightarrow & C(G^2, E) & \longrightarrow & C(G^3, E) & \longrightarrow & \cdots \\ & & \downarrow \alpha & & \downarrow \alpha_* & & \downarrow \alpha_* & & \downarrow \alpha_* & & \\ 0 & \longrightarrow & F & \xrightarrow{\varepsilon} & C(G, F) & \longrightarrow & C(G^2, F) & \longrightarrow & C(G^3, F) & \longrightarrow & \cdots \end{array}$$

and therefore for each $n \in \mathbb{N}_0$ a continuous linear map in cohomology:

$$H_c^n(G, \alpha) : H_c^n(G, E) \rightarrow H_c^n(G, F).$$

This definition is functorial in the sense that $H_c^n(G, \alpha \circ \beta) = H_c^n(G, \alpha) \circ H_c^n(G, \beta)$ and $H_c^n(G, \text{id}) = \text{id}$. Furthermore, it is compatible with the homological algebraic characterization of continuous cohomology of Theorem 2.6:

Proposition 2.12. Retain the above notation. Let (ε, E^\bullet) and (φ, F^\bullet) be continuous, relatively injective resolutions with admissible differentials of E and F respectively. Any extension $\alpha^\bullet : (\varepsilon, E^\bullet) \rightarrow (\varphi, F^\bullet)$ of $\alpha : E \rightarrow F$ as in Lemma 2.8 induces continuous linear maps

$$\alpha_*^n : H^n(E^{\bullet G}) \rightarrow H^n(F^{\bullet G})$$

($n \in \mathbb{N}_0$) such that the following diagrams commute.

$$\begin{array}{ccc} H^n(E^{\bullet G}) & \xrightarrow{\alpha_*^n} & H^n(F^{\bullet G}) \\ \cong \downarrow & & \uparrow \cong \\ H_c^n(G, E) & \xrightarrow{H_c^n(G, \alpha)} & H_c^n(G, F). \end{array}$$

Proof. By Lemma 2.9, any two extensions from $\alpha : E \rightarrow F$ to G -morphisms from (ε, E^\bullet) to (φ, F^\bullet) corresponding to the two paths in the diagram above are G -homotopic and hence induce the same map in cohomology. \square

We remark that occasionally we will abbreviate maps like $H_c^n(G, \alpha)$ which are covariantly induced from a map α by α_* .

2.4.2. Contravariance. Suppose now, that G and H are locally compact groups and that $\varphi : H \rightarrow G$ is a morphism. To any continuous G -module (π, E) we may associate the H -module $\varphi^*E := (\pi\varphi, E)$. Considering again the continuous homogeneous resolution, we then obtain by pre-composition with φ an H -morphism of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & E & \xrightarrow{\varepsilon} & \varphi^*C(G, E) & \longrightarrow & \varphi^*C(G^2, E) & \longrightarrow & \varphi^*C(G^3, E) & \longrightarrow & \cdots \\ & & \downarrow \varphi & & \downarrow \varphi^* & & \downarrow \varphi^* & & \downarrow \varphi^* & & \\ 0 & \longrightarrow & F & \xrightarrow{\varepsilon} & C(H, \varphi^*E) & \longrightarrow & C(H^2, \varphi^*E) & \longrightarrow & C(H^3, \varphi^*E) & \longrightarrow & \cdots \end{array}$$

Since $C(G^n, E)^G \xrightarrow{\varphi^*} C(G^n, E)^H = C(G^n, E)^{\varphi(H)}$ we may, suppressing φ^* from the notation, compose the following two morphisms of complexes

$$\begin{array}{ccccccc}
0 & \longrightarrow & C(G, E)^G & \longrightarrow & C(G^2, E)^G & \longrightarrow & C(G^3, E)^G \longrightarrow \dots \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C(G, E)^H & \longrightarrow & C(G^2, E)^H & \longrightarrow & C(G^3, E)^H \longrightarrow \dots \\
& & \downarrow \varphi^* & & \downarrow \varphi^* & & \downarrow \varphi^* \\
0 & \longrightarrow & C(H, E)^H & \longrightarrow & C(H^2, E)^H & \longrightarrow & C(H^3, E)^H \longrightarrow \dots
\end{array}$$

and thus obtain for each $n \in \mathbb{N}_0$ a continuous linear map in cohomology:

$$\mathrm{H}_c^n(\varphi, E) : \mathrm{H}_c^n(G, E) \rightarrow \mathrm{H}_c^n(H, E),$$

the latter space actually being $\mathrm{H}_c^n(H, \varphi^*E)$. Again, this definition is functorial in the sense that $\mathrm{H}_c^n(\psi \circ \varphi, E) = \mathrm{H}_c^n(\varphi, E) \circ \mathrm{H}_c^n(\psi, E)$ and $\mathrm{H}_c^n(\mathrm{id}, E) = \mathrm{id}$, and it is compatible with the homological algebraic characterization of continuous cohomology of Theorem 2.6:

Proposition 2.13. Retain the above notation. Let (ε, E^\bullet) be a continuous, G -relatively injective resolution of (π, E) and let (η, F^\bullet) be a continuous, H -relatively injective resolution of $(\pi\varphi, E)$. Regarding the former as a complex of H -modules, any extension $\mathrm{id}^\bullet : (\varepsilon, E^\bullet) \rightarrow (\eta, F^\bullet)$ of $\mathrm{id} : (\pi, E) \rightarrow (\pi\varphi, E)$ to an H -morphism of complexes as in Lemma 2.8 induces continuous linear maps

$$\mathrm{H}^n(E^{\bullet G}) \rightarrow \mathrm{H}^n(F^{\bullet H})$$

($n \in \mathbb{N}_0$) such that the following diagrams commute.

$$\begin{array}{ccc}
\mathrm{H}^n(E^{\bullet G}) & \xrightarrow{\mathrm{id}^n} & \mathrm{H}^n(F^{\bullet H}) \\
\cong \downarrow & & \uparrow \cong \\
\mathrm{H}_c^n(G, E) & \xrightarrow[\mathrm{H}_c^n(\varphi, E)]{} & \mathrm{H}_c^n(H, E).
\end{array}$$

Proof. By Lemma 2.9, any two extension of $\mathrm{id} : (\pi, E) \rightarrow (\pi\varphi, E)$ to H -morphisms from $(\varepsilon, \varphi^*E^\bullet)$ to (ν, F^\bullet) corresponding to the two paths in the diagram above are H -homotopic and hence induce the same map in cohomology. \square

As above, we occasionally abbreviate maps like $\mathrm{H}_c^n(\varphi, E)$ which are contravariantly induced from a map φ by φ^* .

2.4.3. Continuous Cohomology Functors. We may now combine the covariance with respect to the coefficients and the contravariance with respect to the group to define continuous cohomology as a sequence of functors between suitable categories.

Namely, let \mathbf{C} be the category which as objects has pairs (G, E) where G is a locally compact group and E is a continuous G -module, and as morphisms from (H, F) to (G, E) has pairs (φ, α) where $\varphi : H \rightarrow G$ and $\alpha : E \rightarrow F$ are morphisms in the respective categories. Then $\mathrm{H}_c^n(-, -)$ ($n \in \mathbb{N}$) is a functor from \mathbf{C} to the category of topological vector spaces $\mathbf{TopVect}$ as follows.

We have already defined $H_c^n(-, -)$ on objects. To define $H_c^n(-, -)$ on morphisms we note that by our definitions the following diagram commutes.

$$\begin{array}{ccc} H_c^n(G, E) & \xrightarrow{\varphi^*} & H_c^n(H, E) \\ \alpha_* \downarrow & & \downarrow \alpha_* \\ H_c^n(G, F) & \xrightarrow{\varphi^*} & H_c^n(H, F). \end{array}$$

The maps $\alpha_* = H_c^n(G, \alpha)$ and $\varphi^* = H_c^n(\varphi, E)$ may therefore be combined to a map

$$H_c^n(\varphi, \alpha) : H_c^n(G, E) \rightarrow H_c^n(H, F),$$

thus defining $H_c^n(-, -) : \mathbf{C} \rightarrow \mathbf{TopVct}$ on morphisms.

2.4.4. Long Exact Sequence. In this section we prove the existence of a long exact sequence in continuous cohomology.

Theorem 2.14. Let G be a locally compact group and let

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

be an exact sequence of continuous G -modules and G -morphisms. Suppose either

- (i) A, B and C are Fréchet spaces and G is σ -compact, or
- (ii) α and β are admissible.

Then there is a family of continuous maps $\tau^n : H_c^n(G, C) \rightarrow H_c^{n+1}(G, A)$ ($n \in \mathbb{N}_0$) such that the following sequence is exact.

$$\dots \xrightarrow{\tau^{n-1}} H_c^n(G, A) \xrightarrow{\alpha_*} H_c^n(G, B) \xrightarrow{\beta_*} H_c^n(G, C) \xrightarrow{\tau^n} H_c^{n+1}(G, A) \longrightarrow \dots$$

The maps τ^n in Theorem 2.14 are called *transgression maps*. The proof of Theorem 2.14 works along the standard lines of homological algebra except that we need to use some results from general topology and functional analysis to make the transgression maps continuous. Namely, recall that if X is a locally compact, σ -compact topological space and F is a Fréchet space then $C(X, F)$ is a Fréchet space, cf. [Bou98b, X1.6, Cor. 3] and [Bou98b, X.3.1, Cor. to Prop. 1].

We shall also need the following generalization of the Bounded Inverse Theorem which for Banach spaces is due to Bartle and Graves [BG52] and in its below form was given by Michael [Mic56, p. 364] as a consequence of a more general Selection Theorem.

Lemma 2.15. Let E and F be Fréchet spaces and let $\varphi : E \rightarrow F$ be a surjective continuous linear map. Then there is a continuous map $\psi : F \rightarrow E$ such that $\varphi\psi = \text{id}$.

In particular we have the following.

Lemma 2.16. Let G be a locally compact, σ -compact topological space and let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be an exact sequence of Fréchet spaces. Then the induced sequence

$$0 \longrightarrow C(G^n, A) \xrightarrow{\alpha_*} C(G^n, B) \xrightarrow{\beta_*} C(G^n, C) \longrightarrow 0$$

is an exact sequence of Fréchet spaces for all $n \in \mathbb{N}_0$.

Proof. The only issue is the surjectivity of β_* . But if σ is a continuous section for β as guaranteed by Lemma 2.15 then $\sigma_* = \sigma \circ (-)$ is a continuous section for $\beta_* = \beta \circ (-)$. Hence β_* is surjective. \square

Proof. (Theorem 2.14). We use the inhomogeneous complex developed in Section 2.3.2 to compute the various cohomology spaces. Suppose condition (i) of the theorem holds, i.e. A , B and C are Fréchet spaces and G is σ -compact. Then by Lemma 2.16, the rows of the following diagram are exact.

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \downarrow \partial_A^{n-1} & & \downarrow \partial_B^{n-1} & & \downarrow \partial_C^{n-1} & \\
0 & \longrightarrow & C(G^n, A) & \xrightarrow{\alpha_*^n} & C(G^n, B) & \xrightleftharpoons[\sigma_*^n]{\beta_*^n} & C(G^n, C) \longrightarrow 0 \\
& \downarrow \partial_A^n & & \downarrow \partial_B^n & & \downarrow \partial_C^n & \\
0 & \longrightarrow & C(G^{n+1}, A) & \xrightleftharpoons[\varrho_*^{n+1}]{\alpha_*^{n+1}} & C(G^{n+1}, B) & \xrightarrow{\beta_*^{n+1}} & C(G^{n+1}, C) \longrightarrow 0 \\
& \downarrow \partial_A^{n+1} & & \downarrow \partial_B^{n+1} & & \downarrow \partial_C^{n+1} & \\
& \vdots & & \vdots & & \vdots &
\end{array}$$

In order to define τ^n , let $c \in Z^n(G, C) = \ker \partial_C^n \subseteq C(G^n, C)$. Let σ_*^n be a continuous section of β_*^n as guaranteed by Lemma 2.15. Since $0 = \partial_C^n \beta_*^n \sigma_*^n c = \beta_*^{n+1} \partial_B^n \sigma_*^n c$ we have $\partial_B^n \sigma_*^n c \in \ker \beta_*^{n+1} = \text{im } \alpha_*^{n+1}$. Because $\text{im } \alpha_*^{n+1} = \ker \beta_*^{n+1}$ is closed, the Bounded Inverse Theorem (or Lemma 2.15) gives a continuous section ϱ_*^{n+1} of the corestriction $\alpha_*^{n+1}|_{\text{im } \alpha_*^{n+1}}$ and one checks that the map

$$\bar{\tau}^n := \varrho_*^{n+1} \circ (\partial_B^n \circ \sigma_*^n)|_{\ker \partial_C^n} \Big|_{\text{im } \alpha_*^{n+1}} : Z^n(G, C) \rightarrow Z^{n+1}(G, A)$$

induces a well-defined (continuous) transgression map τ^n by factoring through $H_c^n(G, C)$ and post-composing with the canonical projection onto $H_c^{n+1}(G, A)$.

If condition (ii) of the theorem holds, i.e. α and β are admissible, then the existence of continuous sections σ_*^n and ϱ_*^n of β_*^n and α_*^{n+1} respectively is immediate and we may define $\bar{\tau}^n$ and τ^n as above. \square

As a particular case of Theorem 2.14 we record the following for future use.

Proposition 2.17. Let G be a locally compact group and let E and F be continuous G -modules. Then for each $n \in \mathbb{N}_0$ the maps

$$H_c^n(G, E \oplus F) \xrightleftharpoons[i_{1*} + i_{2*}]{(\text{pr}_{1*}, \text{pr}_{2*})} H_c^n(G, E) \oplus H_c^n(G, F)$$

are topological isomorphisms.

Proof. We apply Theorem 2.14 (ii) to the exact sequence

$$0 \longrightarrow E \xrightleftharpoons[\text{pr}_1]{i_1} E \oplus F \xrightleftharpoons[i_2]{\text{pr}_2} F \longrightarrow 0.$$

Since the above sequence is split, all transgression maps vanish: Retaining the notation of the proof of Theorem 2.14, an element $\bar{\tau}^n c$ is identified as a coboundary by $i_{1*} \text{pr}_{2*}^n c$. Hence there is for each $n \in \mathbb{N}_0$ an exact sequence

$$0 \longrightarrow H_c^n(G, E) \xrightleftharpoons[\text{pr}_{1*}]{i_{1*}} H_c^n(G, E \oplus F) \xrightleftharpoons[i_{2*}]{\text{pr}_{2*}} H_c^n(G, F) \longrightarrow 0$$

which implies the assertion. \square

2.5. Induction. In this section we give two versions of the classical Eckmann-Shapiro Lemma for continuous cohomology.

2.5.1. Continuous Induction. First, we elaborate on the chapters [BW00, IX.2] and [Bla79, Sec. 8] in order to give the induction theorem for an induction module consisting of continuous functions.

Let G be a locally compact group, Γ a closed subgroup of G and (π, E) a continuous Γ -module. Then the closed subspace

$$\text{Ind}_{\Gamma}^G E = C(G, E)^{\Gamma} = \{f \in C(G, E) \mid f(g\gamma) = \pi(\gamma^{-1})f(g) \ \forall g \in G \ \forall \gamma \in \Gamma\}$$

of $C(G, E)$ is a continuous G -module via $(xf)(y) = \pi(x)f(x^{-1}y)$ whenever $x, y \in G$ and $f \in \text{Ind}_{\Gamma}^G(G, E)$.

Theorem 2.18 (Continuous Induction). Let G be a locally compact, σ -compact group, Γ a closed subgroup of G and (π, E) a continuous Γ -module. Suppose that the canonical projection $\text{pr} : G \rightarrow G/\Gamma$ admits a continuous local section. Then for each $n \in \mathbb{N}_0$ there is a topological isomorphism

$$H_c^n(\Gamma, E) \cong H_c^n(G, \text{Ind}_{\Gamma}^G E).$$

Remark 2.19. Let G be a Hausdorff topological group and let Γ be a discrete subgroup. Then $\text{pr} : G \rightarrow G/\Gamma$ admits a continuous local section.

Our proof of Theorem 2.18 relies on two lemmata. The first one merely consists in unravelling several actions.

Lemma 2.20. Let G be a locally compact group, Γ a closed subgroup of G and (π, E) a continuous Γ -module. Then for every $n \in \mathbb{N}$ and $X := G^n$ the left column maps in the following diagram are Γ -isomorphisms, the middle column maps are $G \times \Gamma$ -isomorphisms and the right column maps are G -isomorphisms, all with the respective actions given aside.

$$\begin{array}{ccc}
C(X, E) & & (\gamma f)(x') = \pi(\gamma)f(\gamma^{-1}x') \\
\parallel \begin{array}{l} i^{-1}f=f(1) \\ (if)(g')(x')=f(x') \end{array} & & \\
C(G, C(X, E))^G & \xrightarrow{\quad} & C(G, C(X, E)) \quad \begin{array}{l} (gf)(g')(x') = f(g^{-1}g')(x') \\ (\gamma f)(g')(x') = \pi(\gamma)f(g'\gamma)(\gamma^{-1}x') \end{array} \\
\parallel \begin{array}{l} (\eta^{-1}f)(g')(x')=f(g', x') \\ (\eta f)(g', x')=f(g')(x') \end{array} & & \\
C(G \times X, E) & \xrightarrow{\quad} & C(G \times X, E) \quad \begin{array}{l} (gf)(g', x') = f(g^{-1}g', x') \\ (\gamma f)(g', x') = \pi(\gamma)f(g'\gamma, \gamma^{-1}x') \end{array} \\
\parallel \begin{array}{l} (\theta^{-1}f)(g', x')=f(g'x', g') \\ (\theta f)(x', g')=f(g', g'^{-1}x') \end{array} & & \\
C(X \times G, E) & \xrightarrow{\quad} & C(X \times G, E) \quad \begin{array}{l} (gf)(x', g') = f(g^{-1}x', g^{-1}g') \\ (\gamma f)(x', g') = \pi(\gamma)f(x', g'\gamma) \end{array} \\
\parallel \begin{array}{l} (\eta f)(x', g')=f(x')(g') \\ (\eta^{-1}f)(x')(g')=f(x', g') \end{array} & & \\
C(X, C(G, E)) & \xrightarrow{\quad} & C(X, C(G, E))^{\Gamma} \\
\parallel \begin{array}{l} (gf)(x')(g') = f(g^{-1}x')(g^{-1}g') \\ (\gamma f)(x')(g') = \pi(\gamma)f(x')(g'\gamma) \end{array} & & \parallel \text{id} \\
C(X, C(G, E))^{\Gamma} & & C(X, C(G, E))^{\Gamma} \\
\parallel \text{id} & & \\
C(X, C(G, E))^{\Gamma} & & C(X, C(G, E))^{\Gamma} \\
(gf)(x')(g') = f(g^{-1}x')(g^{-1}g') & & (gf)(x')(g') = f(g^{-1}x')(g^{-1}g')
\end{array}$$

Our second lemma is contained as Lemma 3.4 in [HM62, Sec. 3]. For the reader's convenience, its proof is rewritten using our notation.

Lemma 2.21. Let G be a locally compact group and Γ a closed subgroup of G . Suppose that G/Γ is paracompact and $\text{pr} : G \rightarrow G/\Gamma$ admits a continuous local section. Then every relatively injective G -module is a relatively injective Γ -module.

Proof. Since every G -module (π, E) injects into the G -module $C(G, E)$, it suffices to show that the latter are relatively injective as Γ -modules.

By the assumptions, there is a locally finite open cover $(U_i)_{i \in I}$ of G/Γ satisfying the following conditions: For every $i \in I$, there is a right Γ -invariant homeomorphism $\varphi_i : \text{pr}^{-1}(U_i) \rightarrow U_i \times \Gamma$ and there is a partition of unity $(\eta_i)_{i \in I}$ subordinate to the cover $(U_i)_{i \in I}$.

Suppose we are given an extension problem as in Definition 2.5 applied to Γ ,

$$\begin{array}{ccc} A & \xrightarrow{\iota} & B \\ & \searrow \alpha & \downarrow \beta \\ & & C(G, E), \end{array}$$

and consider the Γ -morphism $\alpha_i : A \rightarrow C(\text{pr}^{-1}(U_i), E)$, $a \mapsto \alpha(a)|_{\text{pr}^{-1}(U_i)}$. Since φ_i induces a Γ -module isomorphism $C(\text{pr}^{-1}(U_i), E) \cong C(\Gamma, C(U_i, E))$, we see that $C(\text{pr}^{-1}(U_i), E)$ is Γ -injective. Hence we are guaranteed a Γ -module morphism $\beta_i : B \rightarrow C(\text{pr}^{-1}(U_i), E)$ such that $\alpha_i = \beta_i \circ \iota$.

Then for each $i \in I$ we obtain a Γ -module morphism $(\eta_i \circ \text{pr})\beta_i : B \rightarrow C(G, E)$,

$$(\eta_i \circ \text{pr})\beta_i(b) : g \mapsto \begin{cases} (\eta_i \circ \text{pr})(g)\beta_i(b)(g) & g \in \text{pr}^{-1}(U_i) \\ 0 & g \notin \text{pr}^{-1}(U_i) \end{cases},$$

such that $(\eta_i \circ \text{pr})\alpha = (\eta_i \circ \text{pr})\beta_i \circ \iota$ where $(\eta_i \circ \text{pr})\alpha$ is defined analogously.

Since the cover $(U_i)_{i \in I}$ is locally finite, we may define a Γ -module morphism by

$$\beta = \sum_{i \in I} (\eta_i \circ \text{pr})\beta_i : B \rightarrow C(G, E)$$

such that $\alpha = \beta \circ \iota$. Hence $C(G, E)$ is relatively injective as a Γ -module. \square

Proof. (Theorem 2.18). By Section 2.3.1, the continuous cohomology of G with coefficients $\text{Ind}_\Gamma^G E$ may be computed from the complex

$$0 \longrightarrow C(G, \text{Ind}_\Gamma^G E)^G \xrightarrow{d^0} C(G^2, \text{Ind}_\Gamma^G E)^G \xrightarrow{d^1} C(G^3, \text{Ind}_\Gamma^G E)^G \xrightarrow{d^2} \dots$$

Also, by Section 2.3.1 and Lemma 2.21, the continuous cohomology of Γ with coefficients E may be computed from the complex

$$0 \longrightarrow C(G, E)^\Gamma \xrightarrow{d^0} C(G^2, E)^\Gamma \xrightarrow{d^1} C(G^3, E)^\Gamma \xrightarrow{d^2} \dots$$

Since G is σ -compact, G/Γ is paracompact. Thus by Lemma 2.20 there is for every $n \in \mathbb{N}$ a natural isomorphism

$$\begin{aligned} C(G^n, E)^\Gamma &\cong (C(G, C(G^n, E)))^\Gamma \cong \\ &\cong (C(G^n, C(G, E)))^\Gamma \cong C(G^n, C(G, E)^\Gamma)^G \end{aligned}$$

where actually $C(G^n, C(G, E)^\Gamma)^G = C(G^n, \text{Ind}_\Gamma^G E)^G$. These isomorphisms form an isomorphism $C(G^\bullet, E)^\Gamma \cong C(G^\bullet, \text{Ind}_\Gamma^G E)^G$ of complexes. Hence for every $n \in \mathbb{N}_0$:

$$H_c^n(\Gamma, E) \cong H_c^n(G, \text{Ind}_\Gamma^G E).$$

\square

2.5.2. L_{loc}^p Induction. We now give the induction theorem for an induction module consisting of L_{loc}^p functions.

Fix $p \in \mathbb{N}$. Let G be a locally compact, σ -compact, second-countable group, Γ a closed subgroup of G and (π, E) a continuous Γ -module which is a Banach space. Then the closed subspace

$\text{Ind}_{\Gamma}^G(E) = L_{\text{loc}}^p(G, E)^{\Gamma} = \{f \in L_{\text{loc}}^p(G, E) \mid f(g\gamma) = \pi(\gamma^{-1})f(g) \ \forall \gamma \in \Gamma \ \forall g \in G\}$ of $L_{\text{loc}}^p(G, E)$ is a continuous G -module, which is a Fréchet space, with the G -action $(xf)(y) = f(x^{-1}y)$ for all $x \in G$ for almost all $y \in G$ whenever $f \in \text{Ind}_{\Gamma}^G E$.

Theorem 2.22 (L_{loc}^p Induction). Let G be a locally compact, σ -compact, second-countable group, Γ a closed subgroup of G and (π, E) a continuous Γ -module which is a Banach space. Suppose that the canonical projection $\text{pr} : G \rightarrow G/\Gamma$ admits a continuous local section. Then for each $n \in \mathbb{N}_0$ there is a topological isomorphism

$$\mathrm{H}_c^n(\Gamma, E) \cong \mathrm{H}_c^n(G, \text{Ind}_{\Gamma}^G E).$$

Taking the necessary measure-theoretic care, the isomorphisms of Lemma 2.20 and the proof of Theorem 2.18 carry over to the L_{loc}^p case. See [Bla79, Sec. 8].

For us, the significance of the L_{loc}^p induction module $\text{Ind}_{\Gamma}^G E$ lies in the fact that it is a Hilbert space under certain circumstances.

Proposition 2.23. Let G be a locally compact, σ -compact, second-countable group with Haar measure μ . Further, let $\Gamma \leq G$ be a cocompact lattice, (π, E) a unitary representation of Γ and $\sigma : G/\Gamma \rightarrow G$ a measurable section of $\text{pr} : G \rightarrow G/\Gamma$ such that $\mathcal{F} = \sigma(G/\Gamma)$ is relatively compact (for the existence, see e.g. [Par67]). Then $\text{Ind}_{\Gamma}^G E = L_{\text{loc}}^2(G, E)^{\Gamma}$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_{\mathcal{F}} \langle f(g), g(g) \rangle_E \mu(g)$$

Proof. Let $f, g \in L_{\text{loc}}^2(G, E)$. Since \mathcal{F} has compact closure we have

$$\langle f, g \rangle \leq |f|_{\overline{\mathcal{F}}} |g|_{\overline{\mathcal{F}}} < \infty.$$

Hence $\langle -, - \rangle$ is defined. It is positive and hermitian since $\langle -, - \rangle_E$ is.

To prove definiteness, let $f \in L_{\text{loc}}^2(G, E)$ and suppose $0 = \langle f, f \rangle = \int_{\mathcal{F}} \|f(g)\|^2 \mu(g)$. Then $f(g) = 0$ for almost all $g \in \mathcal{F}$. Since $G = \mathcal{F}\Gamma$ and Γ is countable by the assumption that G is second-countable, it follows from the Γ -invariance of f that $f(g) = 0$ for almost all $g \in G$. Hence $\langle -, - \rangle$ is positive definite.

Furthermore, $\langle -, - \rangle$ induces the existing topology on $L_{\text{loc}}^2(G, E)$. \square

3. REDUCED CONTINUOUS COHOMOLOGY

It will be useful for our purposes to have a closer look at the associated Hausdorff spaces of continuous cohomology spaces, mostly because of Theorem 3.5 below.

Definition 3.1. Let G be a locally compact group and let E be a continuous G -module. The n -th reduced continuous cohomology ($n \in \mathbb{N}_0$) of G with coefficients E is the Hausdorff space $\overline{H}_c^n(G, E) = H_c^n(G, E)/\{0\}$ associated to $H_c^n(G, E)$.

Remark 3.2. In view of Definition 3.1, we may compute the reduced continuous cohomology from a resolution as $\overline{H}_c^n(G, E) = Z^n(G, E)/\overline{B^n(G, E)}$. We shall write $\overline{B}^n(G, E)$ instead of $\overline{B}^n(G, E)$ for aesthetic reasons.

Just as continuous cohomology, reduced continuous cohomology may be viewed as a sequence of functors $\overline{H}_c^n(-, -)$ ($n \in \mathbb{N}_0$) from the category \mathbf{C} introduced in section 2.4.3 to the category $\mathbf{TopVect}$ of topological vector spaces. On a morphism $(\varphi, \alpha) : (G, E) \rightarrow (H, F)$, these functors are defined by the following diagrams:

$$\begin{array}{ccc} H_c^n(G, E) & \xrightarrow{H_c^n(\varphi, \alpha)} & H_c(H, F) \longrightarrow \overline{H}_c^n(H, F). \\ \downarrow & \searrow \text{dotted} & \nearrow \text{dotted} \\ \overline{H}_c^n(G, E) & & \overline{H}_c^n(H, F) \end{array}$$

For the following, recall that we write α_* (α^*) for a map which is covariantly (contravariantly) induced from a map α . We state three properties of reduced continuous cohomology which are important for our discussion.

Proposition 3.3. Let G be a locally compact group and let E and F be continuous G -modules. Then for each $n \in \mathbb{N}_0$ there is a topological isomorphism

$$\overline{H}_c^n(G, E \oplus F) \xrightarrow[\substack{(pr_{1*}, pr_{2*}) \\ i_{1*} + i_{2*}}]{(pr_{1*}, pr_{2*})} \overline{H}_c^n(G, E) \oplus \overline{H}_c^n(G, F).$$

Proof. Proposition 2.17 provides a topological isomorphism

$$\overline{H}_c^n(G, E \oplus F) = H_c^n(G, E \oplus F)/\overline{\{0\}} \cong (H_c^n(G, E) \oplus H_c^n(G, F))/\overline{\{0\}}.$$

Since the right hand side of the above equation is topologically isomorphic to

$$H_c^n(G, E)/\overline{\{0\}} \oplus H_c^n(G, F)/\overline{\{0\}} = \overline{H}_c^n(G, E) \oplus \overline{H}_c^n(G, F),$$

the assertion follows. \square

Theorem 3.4 (Induction). Let G be a locally compact, σ -compact, second-countable group, Γ a closed subgroup of G and (π, E) a continuous G -module. Suppose that $pr : G \rightarrow G/\Gamma$ admits a continuous local section. Let $\text{Ind}_\Gamma^G E$ be any of the induction modules, $C(G, E)^\Gamma$ or $L_{\text{loc}}^p(G, E)^\Gamma$, introduced in Sections 2.5.1 and 2.5.2. Then for each $n \in \mathbb{N}_0$ there is a topological isomorphism

$$\overline{H}_c^n(\Gamma, E) \cong \overline{H}_c^n(G, \text{Ind}_\Gamma^G(E)).$$

Proof. This is immediate from the corresponding Theorems 2.18 and 2.22. \square

In degree one, there is a Künneth-type theorem for reduced continuous cohomology. The essential argument is due to Shalom, [Sha98, pp. 838/39].

Theorem 3.5. Let G_1 and G_2 be locally compact, compactly generated groups such that $G = G_1 \times G_2$ is unimodular. Let (π, \mathcal{H}) be a unitary representation of G without invariant vectors. Then there are topological isomorphisms

$$\overline{H}_c^1(G, \mathcal{H}) \xrightarrow[\substack{(pr_1, i_2)_* + (pr_2, i_1)_*}]{(i_1, pr_2)_* + (i_2, pr_1)_*} \overline{H}_c^1(G_1, \mathcal{H}^{G_2}) \oplus \overline{H}_c^1(G_2, \mathcal{H}^{G_1}).$$

Proof. The assumption $\mathcal{H}^G = 0$ implies $\mathcal{H}^{G_1} \cap \mathcal{H}^{G_2} = 0$. Then by Proposition 3.3,

$$\overline{H}_c^1(G, \mathcal{H}) \cong \overline{H}_c^1(G, \mathcal{H}^{G_2}) \oplus \overline{H}_c^1(G, \mathcal{H}^{G_1}) \oplus \overline{H}_c^1(G, \mathcal{H} \widehat{\ominus} (\mathcal{H}^{G_2} \oplus \mathcal{H}^{G_1})).$$

where $\widehat{\ominus}$ denotes taking the orthogonal complement. Let us examine the first two spaces of the right hand side: Consider the continuous inhomogeneous complex from Section 2.3.2 and let $z \in Z^1(G, \mathcal{H}^{G_2})$. Then for all $g_1 \in G_1$ and $g_2 \in G_2$ we have

$$z(g_2) + z(g_1) = z(g_2) + \pi(g_2)z(g_1) = z(g_2g_1) = z(g_1g_2) = z(g_1) + \pi(g_1)z(g_2).$$

This implies $z(g_2) = \pi(g_1)z(g_2)$ for all $g_1 \in G_1$ and $g_2 \in G_2$. But $z(g_2) \in \mathcal{H}^{G_2}$ for all $g_2 \in G_2$ and $\mathcal{H}^{G_2} \cap \mathcal{H}^{G_1} = \mathcal{H}^G = 0$, hence $z|_{G_2} \equiv 0$. We therefore have $Z^1(G, \mathcal{H}^{G_2}) \cong Z^1(G_1, \mathcal{H}^{G_2})$ and similarly $B^1(G, \mathcal{H}^{G_2}) \cong B^1(G_1, \mathcal{H}^{G_2})$. Overall,

$$\overline{H}_c^1(G, \mathcal{H}^{G_2}) \cong \overline{H}_c^1(G_1, \mathcal{H}^{G_2}) \quad \text{and} \quad \overline{H}_c^1(G, \mathcal{H}^{G_1}) \cong \overline{H}_c^1(G_2, \mathcal{H}^{G_1}).$$

We now show that $\overline{H}_c^1(G, \mathcal{H} \widehat{\ominus} (\mathcal{H}^{G_2} \oplus \mathcal{H}^{G_1}))$ vanishes, thus completing the proof. Abbreviate $\mathcal{H}' = \mathcal{H} \widehat{\ominus} (\mathcal{H}^{G_2} \oplus \mathcal{H}^{G_1})$ and let $z \in Z^1(G, \mathcal{H}')$. In two steps, we show that z is actually contained in the closure $\overline{B}^1(G, \mathcal{H}')$ of the space of coboundaries of degree one, thus proving the assertion.

- (i) Let $z \in Z^1(G, \mathcal{H}')$. Then $z|_{G_i} \in \overline{B}^1(G_i, \mathcal{H}')$ for $i \in \{1, 2\}$.
- (ii) Let $z \in Z^1(G, \mathcal{H}')$ with $z|_{G_i} \in \overline{B}^1(G_i, \mathcal{H}')$ for $i \in \{1, 2\}$. Then $z \in \overline{B}^1(G, \mathcal{H}')$.

For step (i), consider once more the cocycle identity

$$z(g_2) + \pi(g_2)z(g_1) = z(g_2g_1) = z(g_1g_2) = z(g_1) + \pi(g_1)z(g_2),$$

for all $g_1 \in G_1$ and $g_2 \in G_2$, and rewrite it as

$$(C) \quad (\text{id} - \pi(g_1))z(g_2) = (\text{id} - \pi(g_2))z(g_1) = \pi(g_2)(-z(g_1)) - (-z(g_1))$$

The idea now is, to somehow make the left hand side operator independent of g_1 and invert it, commuting with $\pi(g_2)$, to identify $z|_{G_2}$ as a coboundary.

Let K be a compact, symmetric generating set of G_1 and let μ be the normalized restriction to K of the Haar measure of G . Consider the operator

$$\pi(\mu) : \mathcal{H}' \rightarrow \mathcal{H}', \quad v \mapsto \int_K \pi(g_1)v \mu(g_1).$$

It is bounded of norm at most 1 because

$$\|\pi(\mu)v\| = \left\| \int_K \pi(g_1)v \mu(g_1) \right\| \leq \int_K \|\pi(g_1)v\| \mu(g_1) \leq \|v\|,$$

using that π is a *unitary* representation, and self-adjoint as for all $v, w \in \mathcal{H}'$:

$$\begin{aligned} \langle \pi(\mu)v, w \rangle &= \left\langle \int_K \pi(g_1)v \mu(g_1), w \right\rangle = \int_K \langle \pi(g_1)v, w \rangle \mu(g_1) = \\ &= \int_K \langle v, \pi(g_1^{-1})w \rangle \mu(g_1) = \left\langle v, \int_K \pi(g_1)w \mu(g_1) \right\rangle = \langle v, \pi(\mu)w \rangle \end{aligned}$$

where the second to last equality follows from the symmetry of K and the unimodularity of G , implying $\mu(E) = \mu(E^{-1})$ for any measurable set $E \subset K$.

Furthermore, 1 is not an eigenvalue of $\pi(\mu)$: Suppose there is some non-zero $v \in \mathcal{H}'$ such that $\pi(\mu)v = v$. Expanding the norm in $0 = \|\pi(\mu)v - v\|^2$ yields

$$\begin{aligned} \|v\|^2 &= \text{Re} \langle \pi(\mu)v, v \rangle = \int_K \text{Re} \langle \pi(g_1)v, v \rangle \mu(g_1) \leq \\ &\leq \int_K |\langle \pi(g_1)v, v \rangle| \mu(g_1) \leq \int_K \|\pi(g_1)v\| \|v\| \mu(g_1) = \|v\|^2. \end{aligned}$$

Since $|\langle \pi(g_1)v, v \rangle| - \operatorname{Re}\langle \pi(g_1)v, v \rangle \geq 0$, this implies $|\langle \pi(g_1)v, v \rangle| = \operatorname{Re}\langle \pi(g_1)v, v \rangle$ and hence $\langle \pi(g_1)v, v \rangle = |\langle \pi(g_1)v, v \rangle| = \|\pi(g_1)v\|\|v\|$ for all $g_1 \in E$ where $E \subseteq K$ has full measure. Now the Cauchy-Schwarz equality case implies for all $g_1 \in E$:

$$\pi(g_1)v = \frac{\langle \pi(g_1)v, v \rangle}{\langle v, v \rangle}v = \frac{\|\pi(g_1)v\|\|v\|}{\|v\|^2}v = v.$$

Since π is continuous, E is closed. Hence $K^\circ = E^\circ \subseteq E$: Otherwise there were an open set $U \subseteq K$ not contained in E such that $E^c \cap U \subseteq K$ is non-empty and hence of positive measure. Since K can be chosen so that K° generates G_1 , we obtain a contradiction to the fact that there are no G_1 -invariant vectors in \mathcal{H}' .

Consider now the operator $\operatorname{id} - \pi(\mu) : \mathcal{H}' \rightarrow \mathcal{H}'$. It is bounded of norm at most 2, self-adjoint and does not have 0 as an eigenvalue. By the spectral theorem, $\operatorname{id} - \pi(\mu)$ is unitarily equivalent to a multiplication operator: There is a finite measure space (X, ν) , a function $\varphi \in L^\infty(X, \nu)$ and a unitary map $U : \mathcal{H}' \rightarrow L^2(X, \nu)$ such that $U(\operatorname{id} - \pi(\mu))U^{-1} = M_\varphi$ where $M_\varphi : L^2(X, \nu) \rightarrow L^2(X, \nu)$, $f \mapsto \varphi f$.

Since 0 is not an eigenvalue of $\operatorname{id} - \pi(\mu)$, and hence neither of M_φ , the set $\{x \in X \mid \varphi(x) = 0\}$ has measure zero. Let $E_n = \{x \in X \mid \varphi(x) \geq 1/n\}$ and define

$$\varphi_n(x) = \begin{cases} 1/\varphi(x) & x \in E_n \\ 0 & x \notin E_n \end{cases}.$$

Then the associated multiplication operators M_{φ_n} satisfy

$$(M_{\varphi_n} M_\varphi f)(x) = \begin{cases} f(x) & x \in E_n \\ 0 & x \notin E_n \end{cases} \text{ for all } f \in L^2(X, \nu)$$

and therefore almost invert M_φ in the sense that for each $f \in L^2(X, \nu)$ we have $\|M_{\varphi_n} M_\varphi f - f\|^2 = \int_{X \setminus E_n} \|f(x)\|^2 \nu(x) \rightarrow 0$ because $\mu(E_n) \rightarrow \mu(X)$ and this convergence is in fact uniform on compact subsets of $L^2(X, \nu)$. For every $g_2 \in G_2$, the operator $\pi(g_2)$ commutes with $\operatorname{id} - \pi(\mu)$ and hence $U\pi(g_2)U^{-1}$ commutes with M_φ . Then the $U\pi(g_2)U^{-1}$ ($g_2 \in G_2$) also commute with the M_{φ_n} ($n \in \mathbb{N}$) by the uniqueness of spectral decomposition.

Integrating equation (C) over K and denoting $v = -\int_K z(g_1) \mu(g_1)$ we obtain $(\operatorname{id} - \pi(\mu))z(g_2) = \pi(g_2)v - v$ for all $g_2 \in G_2$. Applying the operators $T_n = U^{-1}M_{\varphi_n}U$ then implies $\pi(g_2)T_nv - T_nv \rightarrow z(g_2)$ uniformly on compact subsets of G_2 . This completes step (i).

For step (ii), we use the interpretation of $Z^1(G, \mathcal{H})$ and $\overline{B}^1(G, \mathcal{H})$ to be developed in Section 4.1: Namely, we suppose that the restricted actions $\varrho_i = (\pi|_{G_i}^{\cup(\mathcal{H}')} , z|_{G_i})$ ($i \in \{1, 2\}$) almost have fixed points and then aim to show that the original action $\varrho = (\pi^{\cup(\mathcal{H}')} , z)$ almost has fixed points, too.

Let $K = K_1 \times K_2 \subseteq G$ be compact and let $\varepsilon > 0$. If there is a ϱ - (K, ε) -fixed point, it is to be found in the set $C \subseteq \mathcal{H}'$ of ϱ_1 - (K_1, ε) -fixed points. This set is non-empty by assumption, closed, convex and ϱ_2 -invariant: For all $x \in C$, $g_1 \in G_1$ and $g_2 \in G_2$:

$$d(g_1(g_2x), g_2x) = d(g_2g_1x, g_2x) = d(g_1x, x) < \varepsilon.$$

Now, let $P_C : \mathcal{H}' \rightarrow C$ be the projection onto C . If $x \in \mathcal{H}'$ is a ϱ_2 - (K_2, ε) -fixed point, then so is $y = P_C(x) \in C$: The facts that C is ϱ_2 -invariant, that ϱ_2 is an *isometric* action and that P_C is a semi-contraction imply for all $g_2 \in G_2$:

$$d(g_2y, y) = d(g_2P_C(x), P_C(x)) = d(P_C(g_2x), P_C(x)) \leq d(g_2x, x) < \varepsilon.$$

Furthermore, y is ϱ - $(K_1 \times K_2, 2\varepsilon)$ -invariant: For all $g = (g_1, g_2) \in G$ we have

$$d(g_1g_2y, y) \leq d(g_1g_2y, g_1y) + d(g_1y, y) = d(g_2y, y) + d(g_1y, y) < 2\varepsilon.$$

□

4. PROPERTY (FH), PROPERTY (T) AND COHOMOLOGY

In this section we give cohomological characterizations of Property (FH), which we recall presently, and Property (T).

4.1. Continuous Affine Isometric Actions. Let G be a locally compact group and let \mathcal{H} be a real or complex Hilbert space. A (continuous) affine isometric action ϱ of G on \mathcal{H} is a (continuous) homomorphism

$$\varrho : G \rightarrow \mathrm{U}(\mathcal{H}) \ltimes \mathcal{H}$$

where $\mathrm{U}(\mathcal{H})$ denotes the orthogonal or unitary group of \mathcal{H} according to whether the latter is real or complex and $\mathrm{U}(\mathcal{H}) \ltimes \mathcal{H}$ is equipped with the product topology, $\mathrm{U}(\mathcal{H})$ carrying the strong operator topology.

Let ϱ be a continuous affine isometric action of G on a Hilbert space \mathcal{H} . In writing $\varrho = (\pi, z)$ where $\pi = \mathrm{pr}_1 \varrho$ and $z = \mathrm{pr}_2 \varrho$ we see, by examining the group law on $\mathrm{U}(\mathcal{H}) \ltimes \mathcal{H}$, that ϱ gives rise to a unitary representation π of G on \mathcal{H} and a continuous function $z : G \rightarrow \mathcal{H}$, satisfying for all $g, h \in G$:

$$z(gh) = z(g) + \pi(g)z(h).$$

Here lies the connection to continuous cohomology. Let $Z^1(G, \mathcal{H}) = \ker \partial^1$ denote the space of cocycles of degree one of the inhomogeneous complex defined in Section 2.3.2. Then $z \in Z^1(G, \mathcal{H})$ if and only if z satisfies the above functional equation, see Remark 2.11. Hence to every element $z \in Z^1(G, \mathcal{H})$ corresponds a continuous affine isometric action $\varrho := (\pi, z)$ and this correspondence is one-to-one. Furthermore, z is an element of $B^1(G, \mathcal{H}) = \mathrm{im} \partial^0 \subseteq Z^1(G, \mathcal{H})$ if and only if the associated action has a fixed point, as is readily checked. As to $\overline{B}^1(G, \mathcal{H})$, we have the following.

Proposition 4.1. Let G , (π, \mathcal{H}) and $\varrho = (\pi, z)$ be as above. Then $z \in \overline{B}^1(G, \mathcal{H})$ if and only if ϱ almost has fixed points.

Proof. Suppose $z \in \overline{B}^1(G, \mathcal{H})$. Let $K \subseteq G$ be compact and $\varepsilon > 0$. Then there is some $b = (g \mapsto \pi(g)v - v) \in B^1(G, \pi)$ such that for all $g \in K$ we have

$$\varepsilon > \|z(g) - b(g)\| = \|z(g) - (\pi(g)v - v)\| = \|\varrho(g)(-v) - (-v)\|.$$

Hence $-v$ is a (K, ε) -fixed point for ϱ .

Conversely, suppose that $v \in \mathcal{H}$ is a (K, ε) -fixed point for ϱ . Then for all $g \in K$,

$$\varepsilon > \|\varrho(g)v - v\| = \|z(g) - (\pi(g)(-v) - (-v))\|,$$

i.e. $z \in \overline{B}^1(G, \mathcal{H})$. □

The following table summarizes our interpretations of $Z^1(G, \mathcal{H})$, $B^1(G, \mathcal{H})$ and $\overline{B}^1(G, \mathcal{H})$ in terms of isometric actions.

$z \in Z^1(G, \pi)$	continuous affine isometric action $\varrho = (\pi, z)$
$z \in B^1(G, \pi)$	ϱ has a fixed point
$z \in \overline{B}^1(G, \pi)$	ϱ almost has fixed points

4.2. Continuous Cohomology and Property (FH). From the discussion of Section 4.1, we obtain the following characterization of Property (FH).

Definition 4.2 (Property (FH)). A locally compact group G has Property (FH) if every continuous affine isometric action of G on a real or complex Hilbert space has a fixed point.

Proposition 4.3. Let G be a locally compact group. Then G has Property (FH) if and only if $H_c^1(G, \mathcal{H}) = 0$ for every orthogonal and every unitary representation (π, \mathcal{H}) of G .

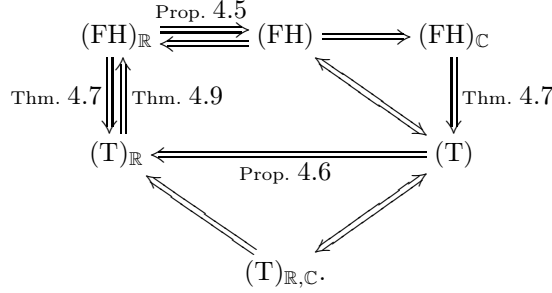
4.3. The Delorme-Guichardet Theorem. Property (FH), introduced in Section 4.2, is of interest to us because of its equivalence to Property (T) for locally compact, σ -compact groups, as in the following theorem.

Theorem 4.4 (Delorme-Guichardet). Let G be a locally compact and σ -compact group. Then G has Property (FH) if and only if G has Property (T).

The two halves of Theorem 4.4 are due to Delorme [Del77] and Guichardet [Gui77]. We shall prove them separately in Theorems 4.7 and 4.9 below. More recent expositions of this material include [dlHVB89] and [BdlHV08]. In proving the Delorme-Guichardet Theorem we will also settle issues arising from whether real and or complex Hilbert spaces are allowed in the definitions of Property (FH) and Property (T). Namely, if we use the notation given by the table

	\mathbb{R}	\mathbb{C}	\mathbb{R}, \mathbb{C}
FH	$(\text{FH})_{\mathbb{R}}$	$(\text{FH})_{\mathbb{C}}$	(FH)
T	$(\text{T})_{\mathbb{R}}$	(T)	$(\text{T})_{\mathbb{R}, \mathbb{C}}$

for the resulting property of locally compact, σ -compact groups, then there are the following relations among these:



Unlabelled implications are either trivial or implied by the labelled ones. *Thus, in fact all these properties are equivalent for locally compact, σ -compact groups.*

Proposition 4.5. Let G be a locally compact group. If G has Property $(\text{FH})_{\mathbb{R}}$ then G has Property (FH).

Proof. Let ϱ be a continuous affine isometric action of G on a complex Hilbert space $(\mathcal{H}, \langle -, - \rangle)$. Then ϱ corestricts to a continuous affine isometric action $\varrho_{\mathbb{R}}$ of G on the realification $(\mathcal{H}_{\mathbb{R}}, \text{Re}\langle -, - \rangle)$ of \mathcal{H} . Since G has Property $(\text{FH})_{\mathbb{R}}$, there is a $\varrho_{\mathbb{R}}$ -fixed point $x \in \mathcal{H}_{\mathbb{R}} = \mathcal{H}$. This x is also ϱ -fixed. Hence G has Property (FH). \square

Proposition 4.6. Let G be a locally compact group. If G has Property (T) then G has Property $(\text{T})_{\mathbb{R}}$.

Proof. Let π be an orthogonal representation of G on a real Hilbert space $(\mathcal{H}, \langle -, - \rangle)$ which almost has invariant vectors. Consider the complexification $(\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}, \langle -, - \rangle_{\mathbb{C}})$ of \mathcal{H} where $\langle v \otimes \alpha, w \otimes \beta \rangle_{\mathbb{C}} = \alpha \overline{\beta} \langle v, w \rangle$ for all $v \otimes \alpha, w \otimes \beta \in \mathcal{H}_{\mathbb{C}}$. Then the complexification $\pi_{\mathbb{C}}$ of π , given by $\pi_{\mathbb{C}}(g)(v \otimes \alpha) = \pi(g)v \otimes \alpha$ ($g \in G$), is a unitary representation of G on $\mathcal{H}_{\mathbb{C}} = \mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}$ which almost has invariant vectors as well. Recall that every vector v of $\mathcal{H}_{\mathbb{C}}$ can be written uniquely in the form $v = v_1 \otimes 1 + v_2 \otimes i$ with $v_1, v_2 \in \mathcal{H}$. Since G has Property (T), there is a non-zero $\pi_{\mathbb{C}}$ -invariant vector $v = v_1 \otimes 1 + v_2 \otimes i$ in $\mathcal{H}_{\mathbb{C}}$ with $v_1, v_2 \in \mathcal{H}$. Then both v_1 and v_2 are π -invariant and at least one of them is non-zero. Hence G has Property $(\text{T})_{\mathbb{R}}$. \square

We now prove the first half of Theorem 4.4, due to Guichardet [Gui77].

Theorem 4.7 (Guichardet). Let G be a locally compact, σ -compact group. If G has Property $(\text{FH})_{\mathbb{R}}$ (Property $(\text{FH})_{\mathbb{C}}$) then G has Property $(\text{T})_{\mathbb{R}}$ (Property (T)).

In view of Section 4.2, the proof of Theorem 4.7 follows easily from the following lemma which is interesting in its own right and thus stated separately.

Lemma 4.8. Let G be a locally compact, σ -compact group and let (π, \mathcal{H}) be an orthogonal (unitary) representation of G which almost has invariant vectors. If $\mathbf{H}_c^1(G, \mathcal{H})$ is Hausdorff, i.e. if $\mathbf{H}_c^1(G, \mathcal{H}) = \overline{\mathbf{H}}_c^1(G, \mathcal{H})$, then π has a non-zero invariant vector.

Proof. (Lemma 4.8). Suppose contrarily that (π, \mathcal{H}) is an orthogonal (unitary) representation of G which almost has invariant vectors but does not have a non-zero invariant vector.

Let $Z^1(G, \mathcal{H})$ denote the space of cocycles of degree one of the inhomogeneous complex defined in Section 2.3.2. Since G is locally compact and σ -compact, $Z^1(G, \mathcal{H})$ is a Fréchet space with the family of semi-norms $\{|\cdot|_K\}_K$ where $K \subseteq G$ is compact and $|\cdot|_K = \sup_{g \in K} \|z(g)\|$ ($z \in Z^1(G, \mathcal{H})$), cf. Section 2.4.4.

The map $\beta : \mathcal{H} \rightarrow Z^1(G, \mathcal{H})$, $v \mapsto (g \mapsto \pi(g)v - v)$ is linear, and injective with image $\mathbf{B}^1(G, \mathcal{H})$ by the assumption that π does not have a non-zero invariant vector. Furthermore, β is continuous since for each semi-norm $|\cdot|_K$ on $Z^1(G, \mathcal{H})$ we have

$$|\beta(v)|_K = \sup_{g \in K} \|\pi(g)v - v\| \leq 2\|v\|.$$

Thus, if $\mathbf{B}^1(G, \mathcal{H})$ was closed in $Z^1(G, \mathcal{H})$, the Bounded Inverse Theorem would imply that $\beta^{-1} : \mathbf{B}^1(G, \mathcal{H}) \rightarrow \mathcal{H}$ was continuous as well. However, it is not:

Since π almost has invariant vectors and G is σ -compact, there exists a sequence $(v_n)_n$ of unit vectors in \mathcal{H} such that for every compact subset $K \subseteq G$ we have $|g \mapsto \pi(g)v_n - v_n|_K \rightarrow 0$. Hence β^{-1} is not continuous. \square

Proof. (Theorem 4.7). If G has Property $(\text{FH})_{\mathbb{R}}$ (Property $(\text{FH})_{\mathbb{C}}$) then we have $\mathbf{H}_c^1(G, \mathcal{H}) = 0$ for every orthogonal (unitary) representation (π, \mathcal{H}) of G by Section 4.2. In particular $\mathbf{H}_c^1(G, \mathcal{H})$ is Hausdorff. Thus if (π, \mathcal{H}) is an orthogonal (unitary) representation of G which almost has invariant vectors, it also has a non-zero invariant vector by Lemma 4.8. Hence G has Property $(\text{T})_{\mathbb{R}}$ (Property $(\text{T})_{\mathbb{C}}$). \square

The second half of Theorem 4.4 is due to Delorme [Del77].

Theorem 4.9 (Delorme). Let G be a locally compact group. If G has Property $(\text{T})_{\mathbb{R}}$ then G has Property $(\text{FH})_{\mathbb{R}}$.

We follow the proof presented at [dlHVB89, 4.b]. Let ϱ be a real continuous affine isometric action of G . We will associate an orthogonal representation π of G to ϱ which has the following properties: π almost has invariant vectors and ϱ has a fixed point if and only if π has a non-zero invariant vector. Theorem 4.9 is then immediate from the definitions.

Proposition 4.10. Let G be a locally compact group and let ϱ be a continuous affine isometric action of G on a real Hilbert space \mathcal{H} . For every $t \in \mathbb{R}_{>0}$, there is an orthogonal representation $(\pi_t, (\mathcal{H}_t, \langle \cdot, \cdot \rangle_t))$ of G and a continuous map

$$\gamma_t : \mathcal{H} \rightarrow \mathcal{H}_t, \quad v \mapsto v_t$$

such that $\text{im}(\gamma_t)$ generates \mathcal{H}_t topologically, and for all $v, w \in \mathcal{H}$ we have

$$\langle v_t, w_t \rangle_t = e^{-t\|v-w\|^2} \quad \text{and} \quad \pi_t(g)v_t = (\varrho(g)v)_t.$$

Our proof of Proposition 4.10 utilizes the *exponential* of a Hilbert space.

Proof. Consider the exponential of \mathcal{H} , i.e. the Hilbert space

$$\exp \mathcal{H} := \bigoplus_{n=0}^{\infty} \bigotimes_n \mathcal{H}$$

and the continuous map $\exp : \mathcal{H} \rightarrow \exp \mathcal{H}$ defined for every $v \in \mathcal{H}$ by

$$v \mapsto \left((n!)^{-1/2} \overset{n}{\otimes} v \right)_n = 1 + v + \frac{1}{\sqrt{2!}} v \otimes v + \frac{1}{\sqrt{3!}} v \otimes v \otimes v + \dots$$

Note that $\langle \exp v, \exp w \rangle = e^{\langle v, w \rangle}$. We define $\gamma_t : v \mapsto e^{-t\|v\|^2} \exp(\sqrt{2t}v)$ and let \mathcal{H}_t be the closure of the span of the image of γ_t . Then all assertions follow. \square

In order to prove that ϱ has a fixed point if and only if π_t has a non-zero invariant vector, we record the following lemma.

Lemma 4.11. Let \mathcal{H} be a real Hilbert space and let $(v_n)_n$ be a sequence in \mathcal{H} which converges to infinity, i.e. $\lim_n \|v_n - w\| = \infty \forall w \in \mathcal{H}$. Then for every $t \in \mathbb{R}_{>0}$, the sequence $(v_{n,t})_n$ weakly converges to $0 \in \mathcal{H}_t$, i.e. $\lim_n \langle v_{n,t}, w \rangle = 0 \forall w \in \mathcal{H}_t$.

Proof. If $w_t = \gamma_t(w) \in \text{im } \gamma_t \leq \mathcal{H}_t$, then the assertion $\lim_n \langle v_{n,t}, w_t \rangle = 0$ is immediate from the relation $\langle v_{n,t}, w_t \rangle = e^{-t\|v-w\|^2}$. Otherwise, since the span of $\text{im } \gamma_t$ generates \mathcal{H}_t topologically, let $\varepsilon > 0$ and pick $w_1, \dots, w_m \in \mathcal{H}$ as well as $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ for some m such that $\|w - \sum_{i=1}^m \lambda_i w_{i,t}\| < \varepsilon/2$. Since γ_t ranges in the unit vectors, applying the Cauchy-Schwarz inequality yields

$$|\langle v_{n,t}, w \rangle| \leq \left| \left\langle v_{n,t}, w - \sum_{i=1}^m \lambda_i w_{i,t} \right\rangle \right| + \sum_{i=1}^m |\lambda_i| |\langle v_{n,t}, w_{i,t} \rangle| \leq \frac{\varepsilon}{2} + \sum_{i=1}^m |\lambda_i| e^{-t\|v_n - w_i\|}$$

and hence the assertion since $(v_n)_n$ converges to infinity. \square

Proposition 4.12. Let G be a locally compact group and let ϱ be a continuous affine isometric action of G on a real Hilbert space \mathcal{H} . Fix $t \in \mathbb{R}_{>0}$. Then ϱ has a fixed point if and only if π_t has a non-zero invariant vector.

Proof. If $v \in \mathcal{H}$ is a ϱ -fixed point, then $v_t \in \mathcal{H}_t$ is a π_t -invariant unit vector.

Conversely, let $v \in \mathcal{H}_t$ be a non-zero invariant vector and suppose that ϱ does not have fixed points. Then for every non-zero vector $w \in \mathcal{H}$, the orbit $\varrho(G)w$ is unbounded (the unique center of a bounded orbit would be a fixed point) and hence there is a sequence $(g_n)_n$ in G such that the sequence $(\varrho(g_n)w)_n$ converges to infinity. Hence $0 = \lim_n \langle (\varrho(g_n)w)_t, v \rangle = \lim_n \langle \pi_t(g_n)w_t, v \rangle = \langle w_t \pi(g_n^{-1})v \rangle = \langle w_t, v \rangle$ by Lemma 4.11 for every $w \in \mathcal{H}$, which contradicts $\mathcal{H}_t = \text{span im } \gamma_t$. \square

With the following Lemma, we are now in a position to define the orthogonal representation π associated to ϱ with the claimed properties.

Lemma 4.13. Retain the above notation. Let $A \subseteq \mathcal{H}$ be bounded. Then so is $A_t = \gamma_t(A) \subseteq \mathcal{H}_t$ for every $t \in \mathbb{R}_{>0}$. Furthermore, $\lim_{t \rightarrow 0} \text{diam } A_t = 0$.

Proof. For all $v, w \in A$ we have $\|v_t - w_t\|^2 = 2(1 - e^{-t\|v-w\|^2}) \leq 2(1 - e^{-t(\text{diam } A)^2})$ and hence $(\text{diam } A_t)^2 \leq 2(1 - e^{-t(\text{diam } A)^2})$. \square

Proposition 4.14. Let G be a locally compact group and let ϱ be a continuous affine isometric action of G on a real Hilbert space \mathcal{H} . Then the orthogonal representation

$$(\pi, \mathcal{H}') = \left(\bigoplus_{n=1}^{\infty} \pi_{1/n}, \bigoplus_{n=1}^{\infty} \mathcal{H}_{1/n} \right)$$

of G almost has invariant vectors.

Proof. Let $K \subseteq G$ be compact and $\varepsilon > 0$. Let $v \in \mathcal{H}$. By Lemma 4.13, we may pick $m \in \mathbb{N}$ such that $\text{diam } \gamma_{1/m}(K)v_{1/m} < \varepsilon$. Then the unit vector $v_{1/m} \in \mathcal{H}_{1/m} \leq \mathcal{H}'$ is (K, ε) -invariant since for all $g \in K$:

$$\|\pi_{1/m}(g)v_{1/m} - v_{1/m}\| = \|(\varrho(g)v)_{1/m} - v_{1/m}\| \leq \text{diam}(\varrho(K)v)_{1/m} < \varepsilon.$$

\square

4.4. Reduced Continuous Cohomology and Property (T). We now give a characterization of Property (T) in terms of reduced continuous cohomology.

Theorem 4.15. Let G be a locally compact, second-countable and compactly generated group. Then G has Property (T) if and only if $\overline{H}_c^1(G, \mathcal{H}) = 0$ for every unitary representation (π, \mathcal{H}) of G .

In this generality, Theorem 4.15 was first proven by Shalom [Sha00]. In the case where $G = \Gamma$ is finitely presented, it is due to Mok [Mok95]. We give a proof in the case where $G = \Gamma$ is finitely generated based on an argument due to Kleiner [Kle10]. It uses the notion of an ultralimit which we introduce below.

4.4.1. (Ultra)filters and Ultralimits. First, let us briefly recall the theory of filters, a generalized theory of convergence, from general topology; see e.g. [Bou98a, I.6].

Let X be a set. A *filter on X* is a set of subsets $\mathcal{F} \subseteq \mathcal{P}(X)$ of X which satisfies the following properties:

- (F1) $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$.
- (F2) $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$.
- (F3) $F' \subseteq X, F \in \mathcal{F}, F' \supseteq F \Rightarrow F' \in \mathcal{F}$.

If \mathcal{F} is a filter on X , a subset $\mathcal{B} \subseteq \mathcal{F}$ is a *basis* of \mathcal{F} if every element of \mathcal{F} contains an element of \mathcal{B} . A subset $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis for a filter on X , namely for the filter $\mathcal{F} = \{F \subseteq X \mid \exists B \in \mathcal{B} : F \supseteq B\}$, if and only if for all $B_1, B_2 \in \mathcal{B}$ there is some $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$. A filter is *non-principal* if it does not contain any finite set. A filter \mathcal{F} on X is an *ultrafilter* if it is maximal among all filters on X with respect to inclusion. Equivalently, \mathcal{F} is an ultrafilter if for all $A \subseteq X$, either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$.

Let X and Y be sets, \mathcal{F} and \mathcal{G} filters on X and Y respectively, and $f : X \rightarrow Y$ a map. The set $f(\mathcal{F}) \subseteq \mathcal{P}(Y)$ is a basis for the *image filter* $f_*\mathcal{F}$ of \mathcal{F} . The set $f^{-1}(\mathcal{G}) \subseteq \mathcal{P}(X)$ is a basis for the *trace filter* $f^*\mathcal{G}$ of \mathcal{G} if and only if $f^{-1}(G) \neq \emptyset$ for all $G \in \mathcal{G}$. If \mathcal{F} is an ultrafilter, then so is $f_*\mathcal{F}$. If $i : A \hookrightarrow Y$ is a subset of Y and \mathcal{G} is an ultrafilter containing A , then $i^*\mathcal{G} = \{G \cap A \mid G \in \mathcal{G}\} =: \mathcal{G} \cap A$ is a basis for the trace filter, and this filter is an ultrafilter.

Now, let X be a topological space. A filter \mathcal{F} on X *converges to* $x \in X$ if it contains the neighbourhood system $\mathcal{U}(x)$ of x . In this case, x is a *limit* of \mathcal{F} , written $x \in \lim \mathcal{F}$, the set of limits of \mathcal{F} .

The theory of filters is designed for the following two statements, among others, to hold: A topological space X is Hausdorff if and only if every convergent filter on X has a unique limit, and it is compact if and only if every ultrafilter converges.

We now turn to the notion of an *ultralimit*, cf. [BH09, I.5.6], [FM05, Sec. 3.2] and [Gro93, Sec. 2]. Fix a non-principal ultrafilter ω on \mathbb{N} using Zorn's Lemma.

If X is a topological space and $f : \mathbb{N} \rightarrow X$, $n \mapsto x_n$ is a sequence in X , then $\lim f_*\omega$ is also denoted $\omega\text{-}\lim_n x_n$. Unravelling the definitions, we get using (F3), that $x \in \omega\text{-}\lim_n x_n$ if and only if for all $U \in \mathcal{U}(x)$, the set $\{n \in \mathbb{N} \mid x_n \in U\}$ is contained in ω .

Proposition 4.16. Let $f : \mathbb{N} \rightarrow \mathbb{C}$, $n \mapsto x_n$ be a bounded sequence. Then $\omega\text{-}\lim_n x_n$ exists and is unique. If $\lim_n x_n$ exists in the usual sense, then $\lim_n x_n = \omega\text{-}\lim_n x_n$. Otherwise, $\omega\text{-}\lim_n x_n$ is an accumulation point of $(x_n)_n$.

Proof. Let $i : K \hookrightarrow \mathbb{C}$ be a compact set containing $f(\mathbb{N})$. Then $\omega\text{-}\lim_n x_n$ equals $\lim i^*f_*\omega$. Since ω is an ultrafilter and K contains $f(\mathbb{N})$, so is $i^*f_*\omega$. As K is compact and Hausdorff, $i^*f_*\omega$ converges to a unique limit.

If $\lim_n x_n = x$ exists in the usual sense and $U \in \mathcal{U}(x)$, then $\mathbb{N} - \{n \in \mathbb{N} \mid x_n \in U\}$ is finite and hence not contained in ω by non-principality. Thus $\{n \in \mathbb{N} \mid x_n \in U\}$ is contained in ω since ω is an ultrafilter and hence $x = \omega\text{-}\lim_n x_n$.

If $x = \omega\text{-}\lim_n x_n$, then for all $U \in \mathcal{U}(x)$, the set $\{n \in \mathbb{N} \mid x_n \in U\}$ is contained in ω and hence infinite. Thus x is an accumulation point of $(x_n)_n$. \square

We record the following two elementary calculus-type statements for our setting.

Proposition 4.17. Let $(x_n)_n$ and $(y_n)_n$ be bounded complex sequences. Then

- (i) $\omega\text{-}\lim_n \overline{x_n} = \overline{\omega\text{-}\lim_n x_n}$.
- (ii) $\omega\text{-}\lim_n (x_n + y_n) = \omega\text{-}\lim_n x_n + \omega\text{-}\lim_n y_n$.
- (iii) $\omega\text{-}\lim_n x_n y_n = (\omega\text{-}\lim_n x_n)(\omega\text{-}\lim_n y_n)$.

Proof. Denote the ω -limits of $(x_n)_n$ and $(y_n)_n$ by x and y respectively.

For (i), note that $\mathcal{U}(\overline{x}) = \overline{\mathcal{U}(x)}$. Thus, if $U \in \mathcal{U}(\overline{x})$, then $\{n \in \mathbb{N} \mid \overline{x_n} \in U\}$ equals $\{n \in \mathbb{N} \mid x_n \in \overline{U}\}$ which is in ω by assumption.

For (ii), let $B_\varepsilon(x+y) \subseteq U \in \mathcal{U}(x+y)$. Then $\{n \in \mathbb{N} \mid x_n + y_n \in U\}$ contains $\{n \in \mathbb{N} \mid x_n \in B_{\varepsilon/2}(x)\} \cap \{n \in \mathbb{N} \mid y_n \in B_{\varepsilon/2}(y)\}$ and hence is contained in ω by (F2) and (F3).

For (iii), let $B_\varepsilon(xy) \subseteq U \in \mathcal{U}(xy)$. Since $|x_n y_n - xy| \leq |x_n(y_n - y)| + |y(x_n - x)|$ we may argue as for (ii) with the sets $\{n \in \mathbb{N} \mid x_n y_n \in U\}$, $\{n \in \mathbb{N} \mid x_n \in B_\delta(x)\}$ and $\{n \in \mathbb{N} \mid y_n \in B_{\delta'}(y)\}$ where $\delta < \min\{\varepsilon/(2|y|+2), 1\}$ and $\delta' < \varepsilon/(2|x|+2)$. \square

Proposition 4.18. Let $(x_n)_n$ and $(y_n)_n$ be bounded real sequences such that $x_n \leq y_n$ for almost all n , then $\omega\text{-}\lim_n x_n \leq \omega\text{-}\lim_n y_n$.

Proof. Let $x = \omega\text{-}\lim_n x_n$ and $y = \omega\text{-}\lim_n y_n$. Suppose $y \not\leq x$ and $|x - y| > \varepsilon$. Since $A = \{n \in \mathbb{N} \mid x_n \in B_{\varepsilon/2}(x)\}$ and $B = \{n \in \mathbb{N} \mid y_n \in B_{\varepsilon/2}(y)\}$ are contained in ω , so is $A \cap B$, by (F2). But $A \cap B$ is contained in $\{n \in \mathbb{N} \mid y_n \not\leq x_n\}$ which is finite by assumption; a contradiction to the non-principality of ω . \square

Remark 4.19. If in Proposition 4.18 one assumes that $x_n \leq y_n$ for all $n \in \mathbb{N}$, the non-principality of ω is not needed since $A \cap B$ is empty in this case which is ruled out by (F1).

We now define the notion of an *ultralimit*. Let (X_n, x_n, d_n) ($n \in \mathbb{N}$) be a sequence of pointed metric spaces. On the set

$$(P) \quad X := \left\{ y = (y_n)_n \in \prod_{n=1}^{\infty} X_n \mid \sup_{n \in \mathbb{N}} d_n(y_n, x_n) < \infty \right\}$$

we may introduce the pseudo-metric $d(y, z) := \omega\text{-}\lim_n d_n(y_n, z_n)$; all the axioms, namely positive semi-definiteness, symmetry and the triangle inequality are readily checked using Propositions 4.17 and 4.18.

Therefore, introducing the equivalence relation $y \sim z$ if and only if $d(y, z) = 0$ on X produces a metric space $(X_\omega = X/\sim, d_\omega)$. If $y = (y_n)_n \in X$, we write y_ω for $[(y_n)_n] \in X_\omega$. Then (X_ω, d_ω) has $x_\omega = [(x_n)_n]$ as a natural base point.

Definition 4.20. The *ultralimit of a sequence* (X_n, x_n, d_n) ($n \in \mathbb{N}$) of pointed metric spaces is the pointed metric space $(X_\omega, x_\omega, d_\omega)$, denoted $\omega\text{-}\lim_n (X_n, x_n, d_n)$.

We are going to apply ultralimits in the case where the metric spaces are Hilbert spaces. Proposition 4.22 below shows that the ultralimit of a sequence of pointed Hilbert spaces is a pointed Hilbert space as well.

Proposition 4.21. The ultralimit $(X_\omega, x_\omega, d_\omega)$ of a sequence of pointed metric spaces (X_n, x_n, d_n) ($n \in \mathbb{N}$) is complete.

Proof. Let $(x_\omega^j)_j = ((x_n^j)_n)_j$ be a Cauchy sequence in X_ω . If $(x_\omega^j)_j$ is to converge to $y_\omega \in X_\omega$, we need to show that for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $j > N$ we have $d_\omega(x_\omega^j, y_\omega) < \varepsilon$. Since $d_\omega(x_\omega^j, y_\omega) = \omega\text{-}\lim_n (x_n^j, y_n)$ by definition, it suffices to show that $\{n \in \mathbb{N} \mid d_n(x_n^j, y_n) < \varepsilon\} \in \omega$ for all $j > N$: If then

ω - $\lim_n d_n(x_n^j, y_n)$ was equal to $z \succeq \varepsilon$, the set $\{n \in \mathbb{N} \mid d_n(x_n^j, y_n) \in U\}$, where U is a small neighbourhood of z , would be contained in $\mathbb{N} - \{n \in \mathbb{N} \mid d_n(x_n^j, y_n) < \varepsilon\}$ which is not in ω since ω is an ultrafilter, hence neither is $\{n \in \mathbb{N} \mid d_n(x_n^j, y_n) \in U\}$ by (F3).

We define y_ω as follows. Note that by definition we have

$$\forall \delta > 0 \forall k, l \in \mathbb{N} : A_{k,l,\delta} := \{n \in \mathbb{N} \mid |d_\omega(x_\omega^k, x_\omega^l) - d_n(x_n^k, x_n^l)| < \delta\} \in \omega.$$

By (F2), the sets $A_m := \bigcap_{k,l \geq 1}^m A_{k,l,1/2^m}$ ($m \in \mathbb{N}$), explicitly

$$A_m = \{n \in \mathbb{N} \mid \forall 1 \leq k, l \leq m : |d_\omega(x_\omega^k, x_\omega^l) - d_n(x_n^k, x_n^l)| < 1/2^m\},$$

are contained in ω as well. Inductively defining $B_1 := A_1 = \mathbb{N}$, $B_m := B_{m-1} \cap A_m$, turns the A_m ($m \in \mathbb{N}$) into a decreasing sequence of sets in ω . We set $y_n = x_n^m$ where m is the maximal index such that $n \in B_m$.

Then $(x_\omega^j)_j$ converges to $y_\omega = [(y_n)_n]$: Let $\varepsilon > 0$ and choose N' such that $\varepsilon - 1/2^{N'} > 0$. Furthermore, choose ε' such that $0 < \varepsilon' < \varepsilon - 1/2^{N'}$. Since $(x_\omega^j)_j$ is a Cauchy sequence, there exists an N'' such that for all $k, l > N''$: $d_\omega(x_\omega^k, x_\omega^l) < \varepsilon'$. Set $N = \max\{N', N''\}$. Then for all $j > N$ we have $d_n(x_n^j, y_n) < \varepsilon$ for all $n \in B_j$, hence the assertion: If $n \in B_j$, then $y_n = x_n^k$ for some $k > j$. Therefore,

$$d_n(x_n^j, y_n) = d_n(x_n^j, x_n^k) < \varepsilon' + \frac{1}{2^k} < \varepsilon.$$

□

Proposition 4.22. Let $(\mathcal{H}_n, x_n = 0, \langle -, - \rangle_n)$ ($n \in \mathbb{N}$) be a sequence of pointed real (complex) Hilbert spaces. Then ω - $\lim_n (\mathcal{H}_n, 0, \langle -, - \rangle_n)$ is a pointed real (complex) Hilbert space \mathcal{H}_ω with base point $x_\omega = 0_\omega$ and inner product $\langle v_\omega, w_\omega \rangle_\omega = \omega$ - $\lim_n \langle v_n, w_n \rangle_n$.

Proof. The space \mathcal{H} (see (P)) is a vector space with the pointwise operations and the subset $U = \{v \in \mathcal{H} \mid d(v, 0) = 0\} \subseteq \mathcal{H}$ is a subspace such that $\mathcal{H}/U = \mathcal{H}/\sim = \mathcal{H}_\omega$. Hence the latter is a vector space. The map $\langle -, - \rangle_\omega : \mathcal{H}_\omega \times \mathcal{H}_\omega \rightarrow \mathbb{C}$ is well-defined, and Hermitian or symmetric bilinear by Proposition 4.17 according to whether the \mathcal{H}_n ($n \in \mathbb{N}$) are real or complex. Furthermore, it is readily checked, that $\langle -, - \rangle_\omega$ induces the metric d_ω with respect to which \mathcal{H}_ω is complete by Proposition 4.21. Also, it follows that $\langle -, - \rangle$ is positive definite. □

4.4.2. *Kleiner's Argument.* We now prove, that Theorem 4.15 holds for finitely generated, discrete groups. Let Γ be such a group and let S be a finite, symmetric generating set for Γ . For any action ϱ of Γ on a metric space (X, d) we define the *energy function*

$$E_\varrho : X \rightarrow \mathbb{R}_{\geq 0}, \quad x \mapsto \sum_{s \in S} d(sx, x)$$

We then have the following Proposition, which is based on [Kle10, Thm. A.1].

Proposition 4.23. Let Γ be a finitely generated, discrete group with a finite, symmetric generating set S . Then the following are equivalent.

- (i) Γ has Property (T).
- (ii) There is a constant $c \in (0, \infty)$ such that for every continuous affine isometric action ϱ of Γ on a complex Hilbert space \mathcal{H} and every $x \in \mathcal{H}$, the action ϱ fixes a point in $B_{cE(x)}(x)$.
- (iii) There are constants $c \in (0, \infty)$ and $\lambda \in (0, 1)$ such that for every continuous affine isometric action ϱ of Γ on a complex Hilbert space \mathcal{H} and every $x \in \mathcal{H}$, there is $x' \in B_{cE(x)}(x)$ with $E_\varrho(x') \leq \lambda E_\varrho(x)$.
- (iv) For every continuous affine isometric action ϱ of Γ on a complex Hilbert space, the energy function satisfies $\inf_{x \in \mathcal{H}} E_\varrho(x) = 0$.

Proof. We prove (i) \Leftarrow (ii) \Leftarrow (iii) \Leftarrow (iv) \Leftarrow (i). Suppose, that (ii) holds. Then Γ has Property (FH) $_{\mathbb{C}}$ which implies Property (T) by Theorem 4.7.

If (i) holds, then Γ has Property (FH) $_{\mathbb{C}}$ by Theorem 4.4. This implies (iv): Let ϱ be as stated and let x be a fixed point of ϱ . Then $E_{\varrho}(x) = 0$, hence $\inf_{x \in \mathcal{H}} E_{\varrho}(x) = 0$.

Suppose now, that (iii) holds. Let ϱ be a continuous affine isometric action of Γ on a complex Hilbert space \mathcal{H} and pick $x_0 \in \mathcal{H}$. By (iii), we may inductively define a sequence $(x_n)_n$ in \mathcal{H} such that $x_{n+1} \in B_{cE_{\varrho}(x_n)}(x_n)$ and $E_{\varrho}(x_{n+1}) \leq \lambda E_{\varrho}(x_n)$. Then $E_{\varrho}(x_n) \leq \lambda^n E_{\varrho}(x_0)$ and $d(x_{n+1}, x_n) \leq cE_{\varrho}(x_n) \leq c\lambda^n E_{\varrho}(x_0)$. Therefore, $(x_n)_n$ is a Cauchy sequence. Its limit x satisfies

$$d(x, x_0) \leq \sum_{n=0}^{\infty} d(x_{n+1}, x_n) \leq cE_{\varrho}(x_0) \sum_{k=0}^{\infty} \lambda^k = \frac{cE_{\varrho}(x_0)}{1-\lambda}.$$

By continuity, we have $E_{\varrho}(x) = \lim_n E_{\varrho}(x_n) = 0$ and hence x is fixed by Γ . Therefore (ii) holds with $c' = c/(1-\lambda)$.

To prove (iii) \Leftarrow (iv), we assume that (iii) fails. Then (iii) fails in particular for the pair $(c, \lambda) = (n, 1 - 1/n)$, i.e. there exists a continuous affine isometric action $\varrho_n = (\pi_n, z_n)$ of Γ on a complex Hilbert space \mathcal{H}_n and a point $x_n \in \mathcal{H}_n$ such that $E_{\varrho_n}(y) > (1 - 1/n)E_{\varrho_n}(x_n)$ for all $y \in B_{nE_{\varrho_n}(x_n)}(x_n)$. We may assume that $x_n = 0$: Conjugating ϱ_n with $\varrho'_n = (\text{id}, x_n)$ yields $\varrho''_n = \varrho_n \varrho'_n = (\pi_n, \pi_n x_n + z_n - x_n)$ satisfying for all $y \in \mathcal{H}_n$:

$$\begin{aligned} E_{\varrho''_n}(y) &= \sum_{s \in S} d_n(\pi_n(s)y + \pi_n(s)x_n + z(s) - x_n, y) \\ &= \sum_{s \in S} d_n(\pi_n(s)(y + x_n) + z(s), y + x_n) = E_{\varrho_n}(y + x_n). \end{aligned}$$

Assume now, that $x_n = 0$. Let $(\mathcal{H}_n, x_n, (1/E_{\varrho_n}(x_n))^2 \langle -, - \rangle_n)$ be the result of rescaling the metric on \mathcal{H}_n by $1/E_{\varrho_n}(x_n)$. Then $E_{\varrho_n}(x_n) = 1$ and $E_{\varrho_n}(y) > (1 - 1/n)$ for all $y \in B_n(x_n)$.

Now, the ultralimit $(\mathcal{H}_{\omega}, x_{\omega}, \langle -, - \rangle_{\omega}) = \omega\text{-}\lim_n (\mathcal{H}_n, x_n, \langle -, - \rangle_n)$ is a pointed Hilbert space with a continuous affine isometric action ϱ_{ω} : Indeed, \mathcal{H}_{ω} is a pointed Hilbert space by Proposition 4.22. Combining the ϱ_n ($n \in \mathbb{N}$), yields a pointwise action on

$$\mathcal{H} := \left\{ y = (y_n)_n \in \prod_{n=1}^{\infty} \mathcal{H}_n \mid \sup_n d_n(y_n, x_n) < \infty \right\}$$

since for every $y = (y_n)_n \in \mathcal{H}$ and $\gamma \in \Gamma$ we have

$$\begin{aligned} \sup_{n \in \mathbb{N}} d_n(\varrho_n(\gamma)x_n, y_n) &\leq \sup_{n \in \mathbb{N}} d_n(\varrho_n(\gamma)y_n, \varrho_n(\gamma)x_n) + d_n(\varrho_n(\gamma)x_n, x_n) \\ &\leq \sup_{n \in \mathbb{N}} d_n(y_n, x_n) + |\gamma|E_{\varrho_n}(x_n) = \sup_{n \in \mathbb{N}} d_n(y_n, x_n) + |\gamma| < \infty \end{aligned}$$

where $|\gamma|$ denotes the length of γ with respect to S (see also the proof of Proposition 4.24). This pointwise action descends to a continuous affine isometric action ϱ_{ω} on \mathcal{H}_{ω} which satisfies $E_{\varrho_{\omega}}(x_{\omega}) = 1$ since

$$\begin{aligned} E_{\varrho_{\omega}}(x_{\omega}) &= \sum_{s \in S} d_{\omega}(\varrho_{\omega}(s)x_{\omega}, x_{\omega}) = \sum_{s \in S} \omega\text{-}\lim_n d_n(\varrho_n(s)x_n, x_n) = \\ &= \omega\text{-}\lim_n \sum_{s \in S} d_n(\varrho_n(s)x_n, x_n) = \omega\text{-}\lim_n E_{\varrho_n}(x_n) = 1 \end{aligned}$$

and $E_{\varrho_{\omega}}(y_{\omega}) \geq 1$ for all $y_{\omega} \in \mathcal{H}_{\omega}$: If $y_{\omega} = [(y_n)_n]$ then

$$\sup_{n \in \mathbb{N}} d_n(y_n, x_n) < \infty \quad \text{and} \quad E_{\varrho_n}(y) > 1 - \frac{1}{n} \quad \text{for all } y \in B_n(x_n)$$

imply that for every $k \in \mathbb{N}$, almost all y_n satisfy $E_{\varrho_n}(y) > 1 - 1/k$. Hence by Proposition 4.18 we have for every $k \in \mathbb{N}$:

$$E_{\varrho_\omega}(y_\omega) = \omega\text{-}\lim_n E_{\varrho_n}(y_n) \geq 1 - \frac{1}{k}.$$

Therefore $E_{\varrho_\omega}(y_\omega) \geq 1$ for all $y_\omega \in \mathcal{H}_\omega$ which contradicts (iv). \square

Combining Proposition 4.23 with the following result will yield a proof of Theorem 4.15 in the case where $G = \Gamma$ is finitely generated.

Proposition 4.24. Let Γ be a finitely generated, discrete group and let ϱ be a continuous affine isometric action of Γ on a real or complex Hilbert space \mathcal{H} . Then $\inf_{x \in \mathcal{H}} E_\varrho(x) = 0$ if and only if ϱ almost has fixed points.

Proof. Suppose that ϱ almost has fixed points and let $\varepsilon > 0$. Since S is finite and hence compact, there is in particular an (S, ε) -fixed point $v \in \mathcal{H}$. Then

$$E_\varrho(x) = \sum_{s \in S} d(sx, x) \leq |S|\varepsilon.$$

Hence $\inf_{x \in \mathcal{H}} E_\varrho(x) = 0$.

Suppose now, that $\inf_{x \in \mathcal{H}} E_\varrho(x) = 0$. Let $K \subseteq \Gamma$ be compact and let $\varepsilon > 0$. Since Γ is discrete, K is finite and hence there exists $N \in \mathbb{N}$ such that every element $\gamma \in K$ can be written as a product of at most N generators from S , say $\gamma = s_1^\gamma s_2^\gamma \cdots s_{n_\gamma}^\gamma$ where $n_\gamma \leq N$ for all $\gamma \in K$. Since Γ acts isometrically, this implies for all $x \in \mathcal{H}$:

$$\begin{aligned} \sup_{\gamma \in K} d(\gamma x, x) &= \sup_{\gamma \in K} d(s_1^\gamma s_2^\gamma \cdots s_{n_\gamma}^\gamma x, x) \\ &\leq \sup_{\gamma \in K} d(s_1^\gamma s_2^\gamma \cdots s_{n_\gamma}^\gamma x, s_1^\gamma s_2^\gamma \cdots s_{n_\gamma-1}^\gamma x) + d(s_1^\gamma s_2^\gamma \cdots s_{n_\gamma-1}^\gamma x, x) \\ &= \sup_{\gamma \in K} d(s_{n_\gamma}^\gamma x, x) + d(s_1^\gamma s_2^\gamma \cdots s_{n_\gamma-1}^\gamma x, x) \leq \dots \leq N E_\varrho(x). \end{aligned}$$

Because $\inf_{x \in \mathcal{H}} E_\varrho(x) = 0$, we thus obtain a (K, ε) -fixed point. \square

Proof. (Theorem 4.15, $G = \Gamma$ finitely generated). By Proposition 4.23, Γ has Property (T) if and only if for every continuous affine isometric action ϱ of Γ on a complex Hilbert space \mathcal{H} we have $\inf_{x \in \mathcal{H}} E_\varrho(x) = 0$. This is, by Proposition 4.24, equivalent to every such ϱ almost having fixed points which by Proposition 4.1 in turn is equivalent to the vanishing of all reduced continuous cohomology spaces $\overline{H}_c^1(\Gamma, \mathcal{H})$ where (π, \mathcal{H}) is a unitary representation of Γ . \square

Remark 4.25. The Delorme-Guichardet Theorem 4.4 enables us to at least prove the “only if”-direction of Theorem 4.15 in the general case of a locally compact, second-countable and compactly generated group G : If G has Property (T), then G has Property (FH) by Theorem 4.4. Hence $H_c^1(G, \mathcal{H}) = 0$ for every unitary representation (π, \mathcal{H}) of G by Proposition 4.3 and therefore $\overline{H}_c^1(G, \mathcal{H}) = 0$ as well.

5. THE PROOF OF THE THEOREM

We are now in a position to give a proof of Theorem 0.3 under the additional assumptions that G_1 and G_2 are topologically simple, and that every continuous homomorphism from G to $(\mathbb{R}, +)$ vanishes, written $\text{hom}(G, \mathbb{R}) = 0$. The reader is encouraged to simultaneously look at Appendix A which schematically summarizes the proof.

Proof. (Theorem 0.3, G_i topologically simple, $i \in \{1, 2\}$, $\text{hom}(G, \mathbb{R}) = 0$).

By Theorem 4.15 it suffices to show that $\overline{H}_c^1(\Gamma/N, \mathcal{H})$ vanishes for every unitary representation (π, \mathcal{H}) of Γ/N . If $\text{pr} : \Gamma \rightarrow \Gamma/N$ is the canonical projection, then $\text{pr}^* \mathcal{H} = (\pi \text{pr}, \mathcal{H})$ is a unitary representation of Γ such that pr induces an injection

$$\overline{H}_c^1(\Gamma/N, \mathcal{H}) \xrightarrow{\text{pr}^*} \overline{H}_c^1(\Gamma, \text{pr}^* \mathcal{H})$$

We now invoke the L_{loc}^2 Induction Theorem 3.4 for reduced continuous cohomology and thus obtain a unitary representation $(\varrho, \text{Ind}_\Gamma^G \text{pr}^* \mathcal{H})$ of G such that

$$\overline{H}_c^1(\Gamma, \text{pr}^* \mathcal{H}) \cong \overline{H}_c^1(G, \text{Ind}_\Gamma^G \text{pr}^* \mathcal{H})$$

Abbreviate $\mathcal{H}_1 = \text{Ind}_\Gamma^G \text{pr}^* \mathcal{H}$. By Proposition 3.3 we have

$$\overline{H}_c^1(G, \mathcal{H}_1) \cong \overline{H}_c^1(G, \mathcal{H}_1^G) \oplus \overline{H}_c^1(G, \mathcal{H}_1 \hat{\otimes} \mathcal{H}_1^G)$$

If $\mathcal{H}_1^G = 0$, then $\overline{H}_c^1(G, \mathcal{H}_1^G)$ vanishes. If $\mathcal{H}_1^G \neq 0$, then $\overline{H}_c^1(G, \mathcal{H}_1^G)$ vanishes by the assumption that $\text{hom}(G, \mathbb{R}) = 0$: Any cocycle $z : G \rightarrow \mathcal{H}_1^G$ is a continuous homomorphism. Thus, if $z \neq 0$, there is $v \in \mathcal{H}_1$ such that $\text{Re}\langle -, v \rangle \circ z$ is a non-zero continuous homomorphism from G to \mathbb{R} .

Abbreviate $\mathcal{H}_2 = \mathcal{H}_1 \hat{\otimes} \mathcal{H}_1^G$. Since \mathcal{H}_2 does not contain G -invariant vectors, we may apply the Künneth-type Theorem 3.5 to obtain

$$\overline{H}_c^1(G, \mathcal{H}_2) \cong \overline{H}_c^1(G_2, \mathcal{H}_2^{G_1}) \oplus \overline{H}_c^1(G_1, \mathcal{H}_2^{G_2}).$$

We note that $\overline{\text{pr}_2 N}$ and $\overline{\text{pr}_1 N}$ act trivially on $\mathcal{H}_2^{G_1}$ and $\mathcal{H}_2^{G_2}$ respectively: Consider the case of $\overline{\text{pr}_2 N}$ acting on $\mathcal{H}_2^{G_1}$, the other one being handled analogously. We have

$$\mathcal{H}_2 \subseteq L_{\text{loc}}^2(G, \text{pr}^* \mathcal{H})^\Gamma = \{f \in L_{\text{loc}}^2(G, \mathcal{H}) \mid f(g\gamma) = \pi(\text{pr}(\gamma))f(g) \forall \gamma \in \Gamma \forall g \in G\}.$$

Then $f \in \mathcal{H}_2^{G_1}$ satisfies in particular

$$\forall n \in N \forall g \in G : f(gn) = f(g) \quad \text{and} \quad \forall g_1 \in G_1 \forall g \in G : f(g_1g) = f(g),$$

and we aim to show: $\forall n_2 \in \overline{\text{pr}_2 N} \forall g \in G : f(n_2g) = f(g)$. We show that each $n_2 \in \overline{\text{pr}_2 N}$ acts trivially on the dense subspace

$$(C(G, \text{pr}^* \mathcal{H})^\Gamma)^{G_1} \subseteq \mathcal{H}_2^{G_1} = (L_{\text{loc}}^2(G, \text{pr}^* \mathcal{H})^\Gamma)^{G_1}$$

and hence by continuity acts trivially on the whole of $\mathcal{H}_2^{G_1}$.

By the irreducibility of Γ we have $\overline{\text{pr}_i N} \trianglelefteq \overline{\text{pr}_i \Gamma} = G_i$ ($i \in \{1, 2\}$). Let $n_2 \in \overline{\text{pr}_2 N}$. For every $g \in G$, let $n_2^g = g^{-1}n_2g \in \overline{\text{pr}_2 N}$. Pick n_1^g such that $n_1^g n_2^g = n \in \overline{N}$ and let $\tilde{n}_1^g = gn_1^g g^{-1} \in \overline{\text{pr}_1(N)}$. Then for $f \in (C(G, \text{pr}^* \mathcal{H})^\Gamma)^{G_1}$ we have

$$f(n_2g) = f(gn_2^g) = f(\tilde{n}_1^g gn_2^g) = f(gn_1^g n_2^g) = f(gn) = f(g)$$

by the invariance properties and the continuity of f , i.e. n_2 acts trivially on f .

Now, since G_1 and G_2 are topologically simple, we have $\overline{\text{pr}_i N} = G_i$ ($i \in \{1, 2\}$) and hence the spaces $\overline{H}_c^1(G_2, \mathcal{H}_2^{G_1})$ and $\overline{H}_c^1(G_1, \mathcal{H}_2^{G_2})$ vanish by the assumption that $\text{hom}(G, \mathbb{R}) = 0$ as above. Overall, we obtain

$$\overline{H}_c^1(\Gamma/N, \mathcal{H}) \xrightarrow{\text{pr}^*} \overline{H}_c^1(G_2, \mathcal{H}_2^{G_1}) \oplus \overline{H}_c^1(G_1, \mathcal{H}_2^{G_2}) = 0.$$

Therefore, Γ/N has Property (T). \square

APPENDIX A. A "ONE-LINE-PROOF" OF THE THEOREM

$$\begin{array}{c}
 \text{Strategy:} \\
 \text{Theorem 4.15} \\
 \nearrow \\
 \overline{H}_c^1(\Gamma/N, \mathcal{H}) \xrightarrow{\text{pr}^*} \overline{H}_c^1(\Gamma, \text{pr}^* \mathcal{H}) \xrightarrow{\cong} \overline{H}_c^1(G, \underbrace{\text{Ind}_\Gamma^G \text{pr}^* \mathcal{H}}_{=: \mathcal{H}_1}) \xrightarrow{\cong} \overbrace{\overline{H}_c^1(G, \mathcal{H}_1^G)}^{=0} \oplus \overline{H}_c^1(G, \underbrace{\mathcal{H}_1 \hat{\otimes} \mathcal{H}_1^G}_{=: \mathcal{H}_2}) \xrightarrow{\cong} \overline{H}_c^1(G_2, \mathcal{H}_2^{G_1}) \oplus \overline{H}_c^1(G_1, \mathcal{H}_2^{G_2}) = 0 \\
 \searrow \\
 \text{pr} : \Gamma \rightarrow \Gamma/N \\
 \text{induces injection}
 \end{array}$$

Induction: Theorem 3.4
 hom(G, \mathbb{R}) = 0
 Irreducibility, $\text{pr}_i N = G_i$ acts trivially (topological simplicity), $\text{hom}(G, \mathbb{R}) = 0$

Direct sums: Theorem 3.3
 Künneth: Theorem 3.5

APPENDIX B. HAAR MEASURES

This appendix provides a concise introduction to the theory of Haar measures on locally compact Hausdorff groups. We will in particular discuss unimodularity and coset spaces. A good reference is [KL06, Sec. 7]. Further references include [Bou04, Ch. 7] and [Kna02, Ch. VIII].

B.1. Preliminaries. The natural class of groups for which to consider Haar measures is that of locally compact Hausdorff groups, due to Theorem B.14 below.

B.1.1. Locally Compact Hausdorff Groups. After having reviewed the definitions, we show that this class is stable under taking closed subgroups and coset spaces with respect to closed subgroups.

A *topological group* is a group G with a topology such that multiplication and inversion are continuous. As a consequence, left and right multiplication by elements of G as well as inversion are homeomorphisms of G . Therefore, the neighbourhoodsystem of the identity $e \in G$ determines the topology on G . A topological space is *locally compact* if every point has a compact neighbourhood; and it is *Hausdorff* if any two distinct points have disjoint neighbourhoods in which case local compactness is equivalent to every point admitting a relatively compact open neighbourhood, i.e. an open neighbourhood with compact closure.

The class of locally compact Hausdorff groups is stable under taking closed subgroups as follows from the following Proposition. Recall that if X is a topological space and A is a subset of X , we may equip A with the *relative topology*, i.e. $U \subseteq A$ is open if and only if there is an open set $V \subseteq X$, such that $U = A \cap V$.

Proposition B.1. Let X be a locally compact Hausdorff space and let A be a closed subset. Then A is locally compact Hausdorff.

Proof. Recalling that compact subsets of Hausdorff spaces are closed and that closed subsets of compact sets are compact, this is immediate following the definitions. \square

As to coset spaces, we record the following lemma on a property of neighbourhoods that comes with the group structure.

Lemma B.2. Let G be a topological group. Then for every $x \in G$ and every neighbourhood U of $e \in G$, there exists an open neighbourhood V of x with $V^{-1}V \subseteq U$.

Proof. The map $\varphi : G \times G \rightarrow G$, $(g, h) \mapsto g^{-1}h$ is continuous. Hence there are open sets $V_1, V_2 \subseteq G$ such that $V_1^{-1}V_2 = \varphi(V_1 \times V_2) \subseteq U$. Then $V = V_1 \cap V_2$ serves. \square

If G is a topological group and H is a subgroup of G , we equip the set of cosets G/H with the *quotient topology*, i.e. $U \subseteq G/H$ is open if and only if $\pi^{-1}(U) \subseteq G$ is open where $\pi : G \rightarrow G/H$, $g \mapsto gH$. Then π is continuous and open, and left multiplication with $g \in G$ is a homeomorphism of G/H .

Proposition B.3. Let G be a topological group and let H be a closed subgroup of G . Then G/H is Hausdorff.

Proof. Let $xH, yH \in G/H$ be distinct. Then $yHx^{-1} \subseteq G$ is closed and does not contain $e \in G$. Hence, by Lemma B.2, there is an open neighbourhood $V \subseteq G$ of $e \in G$ such that $V^{-1}V \subseteq G - yHx^{-1}$. Then VxH and VyH are disjoint open neighbourhoods of $xH \in G/H$ and $yH \in G/H$ respectively. \square

Proposition B.4. Let G be a locally compact topological group and let H be a subgroup of G . Then G/H is locally compact.

Proof. It suffices to show that $H \in G/H$ has a compact neighbourhood. Since G is locally compact, there is a compact neighbourhood K of $e \in G$. Let V be as in Lemma B.2. Then $\overline{\pi(V)}$ is an open neighbourhood of $H \in G/H$ since π is open. We show that $\overline{\pi(V)}$ is compact. If $gH \in \overline{\pi(V)}$ then $VgH \cap VH \neq \emptyset$ and hence $gH = v_1^{-1}v_2H$ for some $v_1, v_2 \in V$. Thus $\overline{\pi(V)} \subseteq \pi(U)$ which is compact since π is continuous and hence so is $\overline{\pi(V)} \subseteq \pi(U)$. \square

B.1.2. Some Topological Group Theory. We further collect several facts from topological group theory, to be used in the sequel.

First, we state a version of Urysohn's Lemma which guarantees the existence of certain compactly supported functions on locally compact Hausdorff spaces. Recall that if X is a topological space, $f \in C_c(X)$ such that $0 \leq f(x) \leq 1$ for all $x \in X$, $U \subseteq X$ open and $K \subseteq X$ compact, one writes $f \prec U$ if $\text{supp}(f) \subseteq U$ and $K \prec f$ if $f(k) = 1$ for all $k \in K$.

Lemma B.5 (Urysohn). Let X be a locally compact Hausdorff space. If $K \subseteq X$ is compact and $U \subseteq X$ is open such that $K \subseteq U$, then there exists $f \in C_c(G)$ satisfying $K \prec f \prec U$.

Also, we shall need the notion of uniform continuity for functions on topological groups (which comes from giving the group the structure of a uniform space). Let G be a topological group. A function $f : G \rightarrow \mathbb{C}$ is *uniformly continuous on the left (right)* if for all $\varepsilon > 0$ there is an open neighbourhood U of $e \in G$ such that for all $x \in G$ and $g \in U$ we have $|f(gx) - f(x)| < \varepsilon$ ($|f(xg) - f(x)| < \varepsilon$).

Proposition B.6. Let G be a locally compact Hausdorff group. Then any $f \in C_c(G)$ is uniformly continuous on the left and right.

Proof. We prove that f is uniformly continuous on the left, uniform continuity on the right being handled analogously. Let $\varepsilon > 0$. By continuity of f , there is for each $x \in \text{supp } f$ an open neighbourhood U_x of $e \in G$ such that $|f(gx) - f(x)| < \varepsilon/2$ for all $g \in U_x$. For every U_x ($x \in G$), pick a symmetric open neighbourhood V_x of $e \in G$ such that $V_x^2 \subseteq U_x$ using Lemma B.2. Since $\text{supp } f$ is compact, finitely many of the sets $V_x x$ ($x \in \text{supp } f$) cover $\text{supp } f$, say $(V_{x_k} x_k)_{k=1}^n$. Define $V = \bigcap_{k=1}^n V_k$. Then for all $x \in \text{supp } f$ and for all $g \in V$ we have

$$|f(gx) - f(x)| \leq |f(gx) - f(x_k)| + |f(x_k) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

where $k \in \{1, \dots, n\}$ is chosen such that $x \in V_{x_k} x_k$. If $x \notin \text{supp } f$ then for every $g \in V$ either $gx \notin \text{supp } f$ in which case the above inequality is trivial, or $gx \in \text{supp } f$ in which case we set $y = gx$. Then $|f(gx) - f(x)| = |f(g^{-1}y) - f(y)|$ with $y \in \text{supp } f$ and $g^{-1} \in V$; we may then argue as before. \square

Eventually, the following elementary facts will be useful here and there.

Proposition B.7. Let G be a topological group and $A, B \subseteq G$. If A and B are compact, then AB is compact. If either A or B is open, then AB is open.

Proof. If A and B are compact, then so is AB as the image of the compact set (A, B) under the continuous multiplication map from $G \times G$ to G . If either A or B is open, then AB is open as a union of open sets since $\bigcup_{a \in A} aB = AB = \bigcup_{b \in B} Ab$. \square

Proposition B.8. Let G be a locally compact Hausdorff group and let H be a subgroup of G . Further, let $C \subseteq G/H$ be compact. Then there exists a compact set $K \subseteq G$ such that $\pi(K) \supseteq C$.

Proof. We may cover G by relatively compact open sets U_i ($i \in I$). Since π is open and $C \subseteq G/H$ is compact, finitely many of the $\pi(U_i)$ ($i \in I$) cover C , say $(\pi(U_k))_{k=1}^n$. Then $K = \bigcup_{k=1}^n \overline{U_k}$ serves. \square

B.1.3. *Some Measure Theory.* We now review some basic measure theory in order to give the definition of a Haar measure and some first properties.

Let X be a non-empty set. A σ -algebra on X is a set $\mathcal{M} \subseteq \mathcal{P}(X)$ of subsets of X , containing the empty set, which is closed under taking complements and countable unions. A pair (X, \mathcal{M}) where X is a set and \mathcal{M} a σ -algebra on X is a *measurable space*; the sets $E \in \mathcal{M}$ are *measurable*. Given two measurable spaces (X, \mathcal{M}) and (Y, \mathcal{N}) , a map $f : X \rightarrow Y$ is *measurable* if $f^{-1}(F) \in \mathcal{M}$ for all $F \in \mathcal{N}$. As a particular example, let X and Y be topological spaces equipped with their *Borel σ -algebras* $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ respectively, i.e. the σ -algebra generated by the open sets. Then any continuous map from X to Y is measurable. *In the following we shall always equip topological spaces with their Borel σ -algebra.*

A *measure* on a measurable space (X, \mathcal{M}) is a map $\mu : \mathcal{M} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ which is zero on the empty set and countably additive, i.e. whenever $(E_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint measurable sets, then $\mu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$. A triple (X, \mathcal{M}, μ) where (X, \mathcal{M}) is a measurable space and μ is a measure on (X, \mathcal{M}) is a *measure space*. A set of measure zero is a *null set*. The complement of a null set is a *conull set*.

If (X, \mathcal{M}, μ) is a measure space, (Y, \mathcal{N}) a measurable space and $\varphi : X \rightarrow Y$ a measurable map, then $\varphi_* \mu : \mathcal{N} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, $F \mapsto \mu(\varphi^{-1}(F))$ is the *push-forward measure on (Y, \mathcal{N}) under φ* .

The category of measure spaces is designed to allow for the following notion of an integral of certain measurable, complex-valued functions on (X, \mathcal{M}, μ) .

1. If χ_E is the characteristic function of a measurable set $E \in \mathcal{M}$, define

$$\int_X \chi_E(x) \mu(x) = \mu(E).$$

2. If $f = \sum_{i=1}^n \lambda_i \chi_{E_i}$ is a positive real linear combination of characteristic functions of measurable sets, a *simple function*, define

$$\int_X f(x) \mu(x) = \sum_{i=1}^n \lambda_i \int_X \chi_{E_i}(x) \mu(x).$$

3. If $f : X \rightarrow \mathbb{R}$ is measurable and nonnegative, define

$$\int_X f(x) \mu(x) = \sup_{\varphi} \int_X \varphi(x) \mu(x)$$

where φ ranges over all real-valued simple functions on X with $0 \leq \varphi \leq f$.

4. If $f : X \rightarrow \mathbb{R}$ is measurable, decompose

$$f = f_+ - f_- \quad \text{where} \quad f_{\pm}(x) = \max(\pm f(x), 0).$$

If $\int_X |f(x)| \mu(x) < \infty$, define

$$\int_X f(x) \mu(x) = \int_X f_+(x) \mu(x) - \int_X f_-(x) \mu(x).$$

5. If $f : X \rightarrow \mathbb{C}$ is measurable and *integrable*, i.e. $\int_X |f(x)| \mu(x) < \infty$, define

$$\int_X f(x) \mu(x) = \int_X \operatorname{Re}(f(x)) \mu(x) + i \int_X \operatorname{Im}(f(x)) \mu(x).$$

The vector space of classes of measurable, integrable complex-valued functions on X modulo equality on a conull set is denoted by $L^1(X, \mu)$. The integral is a linear map from $L^1(X, \mu)$ to \mathbb{C} . There is the following change of variables formula.

Proposition B.9 (Change of variables). Let (X, \mathcal{M}, μ) be a measure space, (Y, \mathcal{N}) a measurable space and $\varphi : X \rightarrow Y$ a measurable map. For every measurable function $f : Y \rightarrow \mathbb{C}$ and every $F \in \mathcal{N}$ we have

$$\int_F f(y) \varphi_* \mu(y) = \int_{\varphi^{-1}(F)} f(\varphi(x)) \mu(x).$$

in case either of the two sides is defined.

Next, we recall Fubini's Theorem which reduces integrating over a product space to integrating over the factors. Let (X, \mathcal{M}, μ) and (Y, \mathcal{N}, ν) be measure spaces. Then so is $(X \times Y, \mathcal{M} \times \mathcal{N}, \mu \times \nu)$ where $(\mu \times \nu)$ is defined by $(\mu \times \nu)(E, F) := \mu(E)\nu(F)$ for all $(E, F) \in \mathcal{M} \times \mathcal{N}$. Also, recall that (X, \mathcal{M}, μ) is σ -finite if X is a countable union of sets of finite measure.

Theorem B.10 (Fubini). Let (X, \mathcal{M}, μ) and (Y, \mathcal{N}, ν) be σ -finite measure spaces. Let $f : X \times Y \rightarrow \mathbb{C}$ be measurable and suppose $\int_X \int_Y |f(x, y)| \nu(y) \mu(x) < \infty$. Then $f \in L^1(X \times Y, \mu \times \nu)$ and

$$\int_X \int_Y f(x, y) \nu(y) \mu(x) = \int_{X \times Y} f(x, y) (\mu \times \nu)(x, y) = \int_Y \int_X f(x, y) \mu(x) \nu(y).$$

Measures on topological spaces which appear in practice often satisfy the following additional regularity properties.

Definition B.11 (Radon measure). A *Radon measure* on a topological space X is a measure on $(X, \mathcal{B}(X))$ which additionally satisfies the following properties:

- (R1) If $K \subseteq X$ is compact, then $\mu(K) < \infty$.
- (R2) If $E \subseteq X$ is measurable, then $\mu(E) = \inf\{\mu(U) \mid U \supseteq E, U \text{ open}\}$.
- (R3) If $U \subseteq X$ is open, then $\mu(U) = \sup\{\mu(K) \mid K \subseteq U, K \text{ compact}\}$.

The importance of Radon measures is also due to the following result of Riesz which often is employed to define a measure on a given space in the first place.

Theorem B.12 (Riesz). Let X be a locally compact Hausdorff space. Further, let $\lambda : C_c(X) \rightarrow \mathbb{C}$ be a positive, i.e. $\lambda(f) \in [0, \infty)$ whenever $f(x) \in [0, \infty)$ for all $x \in X$, linear functional. Then there exists a unique Radon measure μ on X with

$$\lambda(f) = \int_X f(x) \mu(x) \quad \text{for all } f \in C_c(X).$$

Furthermore, μ satisfies

$$\mu(U) = \sup\{\lambda(f) \mid f \prec U\} \quad \text{and} \quad \mu(K) = \inf\{T(f) \mid K \prec f\}$$

for every open set $U \subseteq X$ and every compact set $K \subseteq X$.

B.2. Definition. When dealing with topological groups it is natural to look for measures which are invariant under translation. Such measures always exist for locally compact Hausdorff groups.

Definition B.13 (Haar measure). Let G be a locally compact Hausdorff group. A *left (right) Haar measure on G* is a Radon measure μ on $(G, \mathcal{B}(G))$ which is non-zero on non-empty open sets and invariant under left-translation (right-translation):

- (H1) If $U \subseteq X$ is open, then $\mu(U) \geq 0$.
- (H2) For all $E \in \mathcal{B}(G)$ and $g \in G$: $\mu(gE) = \mu(E)$ ($\mu(Eg) = \mu(E)$).

Theorem B.14 (Haar measure). Let G be a locally compact Hausdorff group. Then there exists a left (right) Haar measure on G which is unique up to strictly positive scalar multiples.

We shall not prove this theorem here. However, we make the following remark.

Remark B.15. Whereas the uniqueness statement of Theorem B.14 is not too hard to establish, the existence proof is more involved and not particularly fruitful. For both, see e.g. [Wei65]. However, there are three classes of locally compact Hausdorff groups for which existence may be established by classical means, see Remark B.20.

Example B.16. Let G be a discrete group. Then the counting measure on G , defined by $\mu : \mathcal{B}(G) = \mathcal{P}(G) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, $E \mapsto |E|$, is a left and right Haar measure.

More examples are to follow in Example B.19. For now, consider the following useful alternative description of Haar measures: Due to Riesz' Theorem B.12, there is a one-to-one correspondence between Haar measures and *Haar functionals*, to be defined below, on a given group which is often used to obtain a Haar measure in the first place. Recall that a topological group G acts on $C_c(G)$ via the left-regular and the right-regular representation $\lambda_G(g)f(x) = f(g^{-1}x)$ and $\varrho_G(g)f(x) = f(xg)$, where $g, x \in G$ and $f \in C_c(G)$.

Definition B.17. Let G be a locally compact Hausdorff group. A *left (right) Haar functional* on G is a non-trivial positive linear functional on $C_c(G)$ which is invariant under λ_G (ϱ_G).

Proposition B.18. Let G be a locally compact Hausdorff group. Then there are the following mutually inverse maps.

$$\Phi : \{\text{Haar measures on } G\} \xrightleftharpoons[\text{Riesz}]{\text{Integration}} \{\text{Haar functionals on } G\} : \Psi$$

Proof. The map Φ is readily checked to range in the positive linear functionals on $C_c(G)$. For λ_G -invariance (ϱ_G -invariance), use the change of variables formula B.9. As to non-triviality, let μ be a left (right) Haar measure on G and let K be a compact neighbourhood of some point in G . Then $\mu(K) \in (0, \infty)$ by (R1) and (H1), and by Urysohn's Lemma B.5 there is $f \in C_c(G)$ such that $K \prec f \prec G$ and therefore $\Phi\mu(f) = \int_G f(g) \mu(g) \geq \mu(K) \gtrsim 0$.

Conversely, if λ is a left (right) Haar functional on G , its non-triviality translates to (H1) for $\mu := \Psi\lambda$ and its invariance under λ_G (ϱ_G) translates to (H2) for μ : Suppose U is a non-empty open set of measure zero with respect to μ . Then any compact set admits a finite cover by left (right) translates of U and hence has measure zero. Thus $\lambda(f) = \int_G f(g) \mu(g) = \int_{\text{supp } f} f(g) \mu(g) = 0$ for all $f \in C_c(G)$, contradicting the non-triviality of λ .

As for invariance, suppose that λ is λ_G -invariant (ϱ_G -invariance being handled analogously) and let $E \in \mathcal{B}(G)$ and $g \in G$. Then by (R2),

$$\mu(gE) = \inf\{\mu(U) \mid U \supseteq gE, U \text{ open}\} = \inf\{\mu(gU) \mid U \supseteq E, U \text{ open}\}.$$

Further, by Theorem B.12 and the λ_G -invariance of λ we have

$$\mu(gU) = \sup\{\lambda(f) \mid f \prec gU\} = \sup\{\lambda(\lambda_G(g)f) \mid f \prec U\} = \mu(U).$$

Hence μ is left invariant. The assertions $\Psi \circ \Phi = \text{id}$ and $\Phi \circ \Psi = \text{id}$ are immediate. \square

Example B.19. Here are further examples of Haar measures.

- (i) On $G = (\mathbb{R}, +)$, a left- and right Haar measure is given by the Lebesgue measure λ which can be defined as the Radon measure associated to the classical Riemann integral $\int_{\mathbb{R}} : C_c(\mathbb{R}) \rightarrow \mathbb{C}$ via Proposition B.18.
- (ii) On $G = (\mathbb{R}^n, +)$, $n \geq 1$, a left- and right Haar measure is given by the n -th power of the Lebesgue measure λ .
- (iii) On $G = (\mathbb{R}^*, \cdot)$, the Lebesgue measure is not left-invariant. However, the map

$$\mu : C_c(G) \rightarrow \mathbb{C}, f \mapsto \int_{\mathbb{R}} f(x) \frac{\lambda(x)}{|x|}$$

can be checked to be a left- and right Haar functional and hence defines a left- and right Haar measure on G by Proposition B.18. Note that the above integral is always finite as the integrand has compact support; use the classical substitution rule to check left- and right-invariance.

- (iv) On $G = \mathrm{GL}(n, \mathbb{R})$, $n \geq 1$, the left- and right Haar functional

$$\mu : C_c(G) \rightarrow \mathbb{C}, f \mapsto \int_G f(X) \frac{\lambda(X)}{|\det X|^n}$$

defines a left- and right Haar functional on G . Here, $\lambda(X) := \prod_{i,j=1}^n \lambda(x_{ij})$ where $X = (x_{ij})_{i,j}$ is the Lebesgue measure on \mathbb{R}^{n^2} of which $\mathrm{GL}(n, \mathbb{R})$ is an open subset; the latter fact is key: The same construction does not work for e.g. $\mathrm{SL}(n, \mathbb{R})$ which is a submanifold of \mathbb{R}^{n^2} of strictly smaller dimension. Again, the integral is finite by compactness of the support of the integrand and invariance is checked by changing variables. Note that the case $G = (\mathbb{R}^*, \cdot)$ is contained via $n = 1$ in this example.

A left- and right Haar measure for $\mathrm{SL}(2, \mathbb{R})$ will be constructed in Example B.35.

Remark B.20. Having established the correspondence between Haar functionals and Haar measures, we now outline existence proofs of Theorem B.14 for compact Hausdorff groups, Lie groups and totally disconnected locally compact separable Hausdorff groups.

- (i) Compact Hausdorff groups. Let G be a compact Hausdorff group. Then G acts continuously on $C(G) = C_c(G)$, equipped with the supremum norm, via the left-regular representation. Therefore, G also acts on the dual space $C(G)^*$ of $C(G)$ via the adjoint representation λ_G^* of λ_G defined by

$$\langle \lambda_G^*(g)\mu, f \rangle = \langle \mu, \lambda_G(g^{-1})f \rangle.$$

for all $\mu \in C(G)^*$ and $f \in C(G)$. Since the set $P(G)$ of probability measures on G is a weak*-compact, convex and λ_G^* -invariant subset of $C(G)^*$, the compact version of the Kakutani-Markov Fixed Point Theorem (e.g. [Zim90, Thm. 2.23]) implies that it contains a λ_G^* -fixed point, i.e. a left-invariant probability measure on G , which turns out to be a left Haar measure on G .

- (ii) Lie groups. Let G be a Lie group with Lie algebra $\mathrm{Lie}(G) \cong \Gamma(\mathrm{T}G)^G$, the space of left-invariant vector fields on G which is isomorphic to $T_e G$ as a vector space. Further, let X_1, \dots, X_n be a basis of $T_e G$ with associated left-invariant vector fields $X_1^G, \dots, X_n^G \in \Gamma(\mathrm{T}G)^G$. Then for each $p \in G$, the tuple $((X_1^G)_p, \dots, (X_n^G)_p)$ is a basis of $T_p G$ and we may for each $i \in \{1, \dots, n\}$ define a 1-form ω_i on G by $(\omega_i)_p((X_j)_p) = \delta_{ij}$; that is, for each $p \in G$, the tuple $((\omega_1)_p, \dots, (\omega_n)_p)$ is the basis of $T_p^* G$ dual to $((X_1^G)_p, \dots, (X_n^G)_p)$. It is readily checked that the left-invariance of X_1^G, \dots, X_n^G implies left-invariance of the ω_i ($i \in \{1, \dots, n\}$) in the sense that $L_g^* \omega_i = \omega_i$ for all $g \in G$ and $i \in \{1, \dots, n\}$. Then so is the n -form $\omega := \omega_1 \wedge \dots \wedge \omega_n$ since \wedge commutes with pullback. Furthermore, one checks that ω is nowhere vanishing. We may then orient G such that ω is positive and hence gives rise to the left Haar functional

$$\lambda_\omega : C_c(G) \rightarrow \mathbb{C}, f \mapsto \int_G f \omega$$

which in turn via Riesz' Theorem B.12 provides a left Haar measure on G , see [Kna02, VIII.2].

- (iii) Totally disconnected locally compact separable Hausdorff groups. Let G be a group of this type. By van Dantzig's theorem, G contains a compact open subgroup K . Assuming G to be non-compact, by separability and openness

of K there are $g_n \in G$ ($n \in \mathbb{N}$) such that $G = \bigsqcup_{n \in \mathbb{N}} g_n K$. Using part (i), let ν be a Haar measure on K and let $\nu_n := g_n \nu$ be the corresponding measure on $g_n K$. For $E \in \mathcal{B}(G)$ define

$$\mu(E) := \sum_{n \in \mathbb{N}} \nu_n(E \cap g_n K) = \sum_{n \in \mathbb{N}} \nu(g_n^{-1} E \cap K)$$

if the sum exists and infinity otherwise. Then μ is a Radon measure on G which is non-zero on non-empty open sets since ν is. Also, μ is left-invariant: Given $g \in G$, there is $\sigma \in S_{\mathbb{N}}$ such that $g g_n K = g_{\sigma(n)} K$. Then

$$\begin{aligned} \mu(g^{-1} E) &= \sum_{n \in \mathbb{N}} \nu(g_n^{-1} g^{-1} E \cap K) = \sum_{n \in \mathbb{N}} \nu(g_{\sigma(n)}^{-1} g g_n g_n^{-1} g^{-1} E \cap K) \\ &= \sum_{n \in \mathbb{N}} \nu(g_{\sigma(n)}^{-1} E \cap K) = \sum_{n \in \mathbb{N}} \nu(g_n E \cap K) = \mu(E). \end{aligned}$$

where the second equality uses K -invariance of ν .

By Remark B.20, compact Hausdorff groups have finite Haar measure. We now show that the converse holds as well.

Proposition B.21. Let G be a locally compact Hausdorff group and let μ be a left (right) Haar measure on G . Then $\mu(G) < \infty$ if and only if G is compact.

Proof. If G is compact, then $\mu(G) < \infty$ by Definition R1. Conversely, suppose that G is not compact and let U be a relatively compact neighbourhood of $e \in G$. Then there is an infinite sequence $(g_n)_{n \in \mathbb{N}}$ of elements of G such that $g_n \notin \bigcup_{k < n} g_k U$; otherwise G would be compact as a finite union of compact sets. Let V be as in Lemma B.2. Then the sets $g_n V$ ($n \in \mathbb{N}$) are pairwise disjoint by the fact that $V V^{-1} \subseteq U$ and the definition of $(g_n)_{n \in \mathbb{N}}$. Therefore, as V has strictly positive measure, G has infinite measure. \square

B.3. Unimodularity. We now address and quantify the question whether left and right Haar measures on a given locally compact Hausdorff group coincide.

Definition B.22. A locally compact Hausdorff group G is *unimodular* if every left Haar measure on G is also a right Haar measure on G and conversely.

Remark B.23. By Theorem B.14, it suffices in Definition B.22 to ask for every left Haar measure on G to also be a right Haar measure.

Proposition B.27 below will provide several classes of unimodular groups. For now, let G be a locally compact Hausdorff group and let μ be a left Haar measure on G . Then for every $g \in G$, the map $\mu_g : \mathcal{B}(G) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, $E \mapsto \mu(Eg)$ is a left Haar measure on G as well. Hence, by uniqueness, there exists a strictly positive real number $\Delta_G(g)$ such that $\mu_g = \Delta_G(g)\mu$, i.e.

$$(M) \quad \mu(Eg) = \mu_g(E) = \Delta_G(g)\mu(E) \quad \text{for all } E \in \mathcal{B}(G).$$

The function $\Delta_G : G \rightarrow \mathbb{R}_{> 0}$ is independent of μ and called *modular function of G* .

Let λ be the left Haar functional associated to μ by Proposition B.18. Then by the change of variable formula B.9 applied to $\varphi = R_{g^{-1}}$, equation (M) immediately translates to

$$(M') \quad \lambda(\varrho_G(g^{-1})f) = \Delta_G(g)\lambda(f) \quad \text{for all } f \in C_c(G).$$

Proposition B.24. Let G be a locally compact Hausdorff group. Then the modular function $\Delta_G : G \rightarrow (\mathbb{R}_{> 0}, \cdot)$ is a continuous homomorphism.

Proof. Let μ be a left Haar measure on G . The homomorphism property is immediate from (M): For all $g, h \in G$ we have

$$\Delta_G(gh)\mu = \mu_{gh} = (\mu_g)_h = \Delta_G(h)\mu_g = \Delta_G(h)\Delta_G(g)\mu = \Delta_G(g)\Delta_G(h)\mu.$$

Evaluating on a set of non-zero finite measure, e.g. a compact neighbourhood of some point, proves the assertion.

As to continuity, let λ be the left Haar functional associated to μ by Proposition B.18. It suffices to check continuity at $e \in G$, since Δ_G is a homomorphism. Let K be compact neighbourhood of $e \in G$. Using Urysohn's Lemma B.5, we choose $\varphi \in C_c(G)$ such that $K \prec \varphi \prec G$ and $\psi \in C_c(G)$ such that $K \text{ supp } \varphi \prec \psi \prec G$ (see Proposition B.7). In particular, φ is uniformly continuous on the right by Proposition B.6: Given $\varepsilon > 0$, let $U \subseteq K$ be a symmetric open neighbourhood of $e \in G$ such that $|\varphi(xg) - \varphi(x)| < \varepsilon$ for all $g \in U$. Then by (M'),

$$|\Delta_G(g) - 1| = \frac{1}{\lambda(\varphi)} |\Delta_G(g)\lambda(\varphi) - \lambda(\varphi)| \leq \frac{1}{\lambda(\varphi)} \lambda(|\varrho_G(g^{-1})\varphi - \varphi|\psi) \leq \varepsilon \frac{\lambda(\psi)}{\lambda(\varphi)}$$

for all $g \in U$. Hence Δ_G is continuous at the identity. \square

Remark B.25. We have noticed that for a locally compact Hausdorff group G with left Haar measure μ and given $g \in G$, the map $\mu_g : \mathcal{B}(G) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, $E \mapsto \mu(Eg)$ is a left Haar measure on G as well. This is an instance of the following more general observation: For every continuous automorphism $\alpha \in \text{Aut}(G)$, the map $\mu_\alpha : \mathcal{B}(G) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, $E \mapsto \mu(\alpha(E))$ is a left Haar measure on G . In this setting, $\mu_g = \mu_{\text{int}(g^{-1})}$ where $\text{int}(g) : G \rightarrow G$, $x \mapsto gxg^{-1}$ denotes conjugation in G by g . One may then introduce the general *modular function* $\text{mod}_G : \text{Aut}(G) \rightarrow (\mathbb{R}_{>0}, \cdot)$ which remains to be a homomorphism and with the appropriate topology on $\text{Aut}(G)$ becomes continuous, see e.g. [Pal01].

We obtain the following useful criterion for unimodularity.

Corollary B.26. Let G be a locally compact Hausdorff group. Then G is unimodular if and only if $\Delta_G \equiv 1$.

Proof. If $\Delta_G \equiv 1$, then G is unimodular by (M) and Remark B.23. Conversely, if G is unimodular, let μ be a Haar measure on G and let E be a compact neighbourhood of some point in G . Then $\mu(E) \in (0, \infty)$ and hence $\Delta_G \equiv 1$ by (M). \square

Corollary B.26 provides us with the following list of classes of unimodular groups. Yet another class will be given in Proposition B.42.

Proposition B.27. Let G be a locally compact Hausdorff group. Then G is unimodular if, in addition, it satisfies one of the following properties: being abelian, compact, topologically simple, discrete, connected semisimple Lie or connected nilpotent Lie.

Proof. Let G be a locally compact Hausdorff abelian group with left Haar measure μ . Since $Eg = gE$ for every subset $E \subseteq G$ and all $g \in G$, the left-invariance of μ implies right-invariance.

If G is compact Hausdorff and μ is a left Haar measure on G , then $\mu(G) \in (0, \infty)$ and hence $\Delta_G \equiv 1$ by (M).

If G is topologically simple, then $\overline{[G, G]}$, which is a closed normal subgroup of G , either equals $\{e\}$ or G . In the former case, G is abelian and hence unimodular; in the latter case, continuity of Δ_G implies:

$$\Delta_G(G) = \Delta_G(\overline{[G, G]}) \subseteq \overline{\Delta_G([G, G])} = \{1\}$$

whence G is unimodular.

For a discrete group, the left Haar measures are the strictly positive scalar multiples of the counting measure which certainly is right-invariant.

Suppose now, that G is a connected semisimple Lie group. Note that in this case the modular function $\Delta_G : G \rightarrow (\mathbb{R} - \{0\}, \cdot)$ is a continuous and hence smooth ([War83, Thm. 3.39]) homomorphism of Lie groups. Thus $d_e \Delta_G : \text{Lie}(G) \rightarrow \mathbb{R}$ is a morphism of Lie algebras. Since $\text{Lie}(G)$ is semisimple and \mathbb{R} is abelian we have

$$d_e \Delta_G(\text{Lie}(G)) = d_e \Delta_G([\text{Lie}(G), \text{Lie}(G)]) = [d_e \Delta_G(\text{Lie}(G)), d_e \Delta_G(\text{Lie}(G))] = \{0\}$$

and hence $\Delta_G \equiv 1$ by the Lie correspondence, passing to the universal cover of G .

For the case of a connected nilpotent Lie group, we appeal to the fact that for any Lie group G we have $\Delta_G(g) = |\det \text{Ad}(g)|$, where $\text{Ad} : G \rightarrow \text{Aut}(\text{Lie}(G))$ is the adjoint representation of G , see e.g. [Kna02, Prop. 8.27] (this follows in the setting of Remark B.20). If, in addition, G is connected and nilpotent, then the exponential map $\exp : \text{Lie}(G) \rightarrow G$ is surjective ([Kna02, Thm. 1.127]) and hence for every $g \in G$ there is some $X \in \text{Lie}(G)$ such that $g = \exp(X)$ and

$$\Delta_G(g) = |\det \text{Ad}(g)| = |\det e^{\text{ad}X}| = e^{\text{tr ad}X} = 1$$

where the last equality follows from $\text{Lie}(G)$ and hence $\text{ad}X$ being nilpotent. \square

The following proposition provides a class of totally disconnected locally compact Hausdorff groups that are unimodular. Recall that if T is a locally finite tree then $\text{Aut}(T)$ is a totally disconnected locally compact separable Hausdorff group with the permutation topology. We adopt Serre's graph theory conventions, see [?].

Proposition B.28. Let $T = (X, Y)$ be a locally finite tree. If $G \leq \text{Aut}(T)$ is closed and locally transitive then G is unimodular.

Proof. Let μ be a left Haar measure on G , see Remark B.20. Since G is locally transitive there is for every triple (x, e_0, e) of a vertex $x \in X$ and edges $e_0, e \in E(x)$ an element $g_e \in G(x)$ such that $g_e e_0 = e$. Then $G(x) = \bigsqcup_{e \in E(x)} g_e G(e_0)$. Since $G(e) = G(\bar{e})$ for all $e \in Y$ we conclude that $\mu(G(e)) = \mu(G(e'))$ for all $e, e' \in Y$. Given $g \in G$ we therefore have

$$\mu(G(e)) = \mu(G(ge)) = \mu(gG(e)g^{-1}) = \mu(G(e)g^{-1}) = \Delta_G(g^{-1})\mu(G(e))$$

and hence G is unimodular. \square

Example B.29. We now provide two related examples of non-unimodular groups.

- (i) Consider the group

$$P := \left\{ \begin{pmatrix} x & y \\ & x^{-1} \end{pmatrix} \middle| x \in \mathbb{R} - \{0\}, y \in \mathbb{R} \right\} \leq \text{SL}(2, \mathbb{R}).$$

Then the functionals $\mu, \nu : C_c(P) \rightarrow \mathbb{C}$, given by

$$\mu : f \mapsto \int_{\mathbb{R}^2} f(X) \frac{\lambda(x)\lambda(y)}{x^2} \quad \text{and} \quad \nu : f \mapsto \int_{\mathbb{R}^2} f(X) \lambda(x)\lambda(y)$$

are left- and right Haar functionals respectively as can be checked by changing variables. However, P is a closed subgroup of $\text{SL}(2, \mathbb{R})$ which is unimodular as a connected simple Lie group by Proposition B.27. We shall shed some light on the origin of this example in Remark B.36.

- (ii) Let T_d be the d -regular tree and let $\omega \in \partial T_d$ be a boundary point of T_d . Set $G := \text{Aut}(T)(\omega)$. Then G is not unimodular: Indeed, let $t \in G$ be a translation of length one towards ω and let $x \in X$ be on the translation axis of t . Then

$$\begin{aligned} \Delta(t) &= \frac{\mu(G(x))}{\mu(G(tx))} = \frac{\mu(G(x))}{\mu(G(x, tx))} \frac{\mu(G(x, tx))}{\mu(G(tx))} \\ &= \frac{[G(x) : G(x, tx)]}{[G(tx) : G(x, tx)]} = \frac{|H(x)tx|}{|H(tx)x|} = \frac{1}{d-1}. \end{aligned}$$

See Remark B.36 for how this relates to part (i).

Utilizing the modular function, we can turn left Haar measures into right Haar measures as in the following Proposition. Let $i : G \rightarrow G$ denote the inversion on G .

Proposition B.30. Let G be a locally compact Hausdorff group with left Haar measure μ . Then $\bar{\mu} = i_*\mu : \mathcal{B}(G) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, $E \mapsto \mu(E^{-1})$ is a right Haar measure on G with associated right Haar functional $\varrho : C_c(G) \rightarrow \mathbb{C}$, $f \mapsto \int_G f(x) \Delta_G(x^{-1}) \mu(x)$. If G is unimodular, then $\bar{\mu} = \mu$.

Proof. The map $\bar{\mu}$ is readily checked to be a right Haar measure on G . The map ϱ is clearly positive and linear. Its non-triviality follows as in the proof of Proposition B.18 using $\Delta_G(g) \geq 0$ for all $g \in G$. As to ϱ_G -invariance, changing variables via Proposition B.9 using $R_{g*}\mu = \mu_{g^{-1}}$ yields

$$\begin{aligned} \varrho(\varrho_G(g)f) &= \int_G f(xg) \Delta_G(x^{-1}) \mu(x) = \int_G f(x) \Delta_G(gx^{-1}) \mu_{g^{-1}}(x) = \\ &= \int_G f(x) \Delta_G(g) \Delta_G(x^{-1}) \Delta_G(g^{-1}) \mu(x) = \int_G f(x) \Delta_G(x^{-1}) \mu(x) = \varrho(f). \end{aligned}$$

for every $f \in C_c(G)$ and $g \in G$. Overall, ϱ is a right Haar functional on G .

Now, let $\Phi\bar{\mu}$ denote the right Haar functional associated to $\bar{\mu}$ as in Proposition B.18. Then there is a strictly positive real number c such that $\Phi\bar{\mu} = c\varrho$. Applying the change of variables formula B.9, we obtain for all $f \in C_c(G)$:

$$\begin{aligned} \int_G f(x) \bar{\mu}(x) &= c \int_G f(x) \Delta_G(x^{-1}) \mu(x) = c \int_G f(x^{-1}) \Delta_G(x) \bar{\mu}(x) \\ &= c^2 \int_G f(x^{-1}) \Delta_G(x) \Delta_G(x^{-1}) \mu(x) = c^2 \int_G f(x) \bar{\mu}(x). \end{aligned}$$

Let K be a compact symmetric neighbourhood of some point in G and $f \in C_c(G)$ such that $K \prec f \prec G$. Then $\int_G f(x^{-1}) \mu(x) \in (0, \infty)$ and hence $c = 1$. In particular, unimodularity of G implies $\mu = \bar{\mu}$. \square

B.4. Coset spaces. Let G be a locally compact Hausdorff group and let H be a closed subgroup of G . If H is normal in G , there exists a left (right) Haar measure on G/H by Theorem B.14. We now address the question under which circumstances there exists a G -invariant Radon measure on G/H which is non-zero on non-empty open sets if H is not normal in G , and we shall refer to such a measure as a *Haar measure on G/H* by abuse of notation. The following example shows that a Haar measure on G/H may or may not exist.

Example B.31. Let $G = \mathrm{SL}(2, \mathbb{R})$.

- (i) Consider the natural action of G on $X = \mathbb{R}^2 - \{0\}$. Then

$$H := \mathrm{stab}_G((1, 0)^T) = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

and hence $G/H \cong X$ has a Haar measure, namely the restricted two-dimensional Lebesgue measure.

- (ii) On the other hand, G acts on $X = \mathbb{P}^1 \mathbb{R} = \{V \leq \mathbb{R}^2 \mid \dim V = 1\}$. Here,

$$H := \mathrm{stab}_G(\langle e_1 \rangle) = \left\{ \begin{pmatrix} x & y \\ & x^{-1} \end{pmatrix} \mid x \in \mathbb{R} - \{0\}, y \in \mathbb{R} \right\}$$

which is the non-unimodular group of Example B.29. The space $G/H \cong X$ does not admit a Haar measure: For instance, consider the compact subsets $E_1 := \{\langle (1, t)^T \rangle \mid t \in [0, 1]\}$ and $E_2 := \{\langle (t, 1)^T \rangle \mid t \in [0, 1]\}$ of $\mathbb{P}^1 \mathbb{R}$. Then

$$\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix} E_1 = E_1 \cup E_2 \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ 1 & \end{pmatrix} E_1 = E_2.$$

A Haar measure on G/H would assign finite non-zero measure to the compact sets E_1 and E_2 which combined with G -invariance contradicts the above two equalities.

Theorem B.32. Let G be a locally compact Hausdorff group with left Haar measure μ and let H be a closed subgroup of G with left Haar measure ν . Then there exists a Haar measure ξ on G/H if and only if $\Delta_G|_H \equiv \Delta_H$. In this case, ξ is unique up to strictly positive scalar multiples and suitably normalized satisfies for all $f \in C_c(G)$:

$$(W) \quad \int_G f(g) \mu(g) = \int_{G/H} \int_H f(gh) \nu(h) \xi(gH).$$

Proof. (Theorem B.32, “ \Rightarrow ”). If ξ exists as above, then the map

$$\lambda : C_c(G) \rightarrow \mathbb{C}, \quad f \mapsto \int_{G/H} \int_H f(gh) \nu(h) \xi(gH)$$

is a left Haar functional on G and thus defines a left Haar measure μ on G . In particular, $\lambda(\varrho_G(t^{-1})f) = \Delta_G(t)\lambda(f)$ for all $t \in G$ and $f \in C_c(G)$ by (M’). On the other hand, we have for all $t \in H$ and $f \in C_c(G)$:

$$\begin{aligned} \lambda(\varrho_G(t^{-1})f) &= \int_{G/H} \int_H (\varrho_G(t^{-1})f)(gh) \nu(h) \xi(gH) = \\ &= \int_{G/H} \int_H \Delta_H(t)f(gh) \nu(h) \xi(gH) = \Delta_H(t)\lambda(f). \end{aligned}$$

If, by Urysohn’s Lemma B.5, we choose $f \in C_c(G)$ to satisfy $K \prec f \prec G$ where K is a compact neighbourhood of some point in G , then $\int_G f(g) \mu(g) = \lambda(f) \in (0, \infty)$ and hence $\Delta_G|_H \equiv \Delta_H$. \square

The proof of the converse assertion of Theorem B.32 relies on the following description of compactly supported functions on G/H . Once more, Riesz’ Theorem B.12 will be used to produce a measure.

Lemma B.33. Let G be a locally compact Hausdorff group and H a closed subgroup of G with left Haar measure ν . Then the following map is surjective:

$$C_c(G) \rightarrow C_c(G/H), \quad f \mapsto \left(f_H : gH \mapsto \int_H f(gh) \nu(h) \right).$$

Proof. Several things need to be checked. First of all, for all $f \in C_c(G)$ and for all $gH \in G/H$, the integral $\int_H f(gh) \nu(h)$ is independent of the representative of gH and finite. Next, for all $f \in C_c(G)$, the function $f_H f$ is continuous as a parametrized integral as in the proof of the continuity of the modular function. Clearly, $\text{supp } f_H \subseteq p(\text{supp}(f))$ and hence $f_H \in C_c(G/H)$. It remains to prove surjectivity. To this end, let $F \in C_c(G/H)$. Pick $K \subseteq G$ such that $\pi(K) \supseteq \text{supp } F$ (Proposition B.8) and let $\eta \in C_c(G)$ satisfying $K \prec \eta$ (Urysohn’s Lemma). Now define $f \in C_c(G)$ by

$$f : G \rightarrow \mathbb{C}, \quad g \mapsto \begin{cases} \frac{F(gH)\eta(g)}{\eta_H(gH)} & \eta_H(gH) \neq 0 \\ 0 & \eta_H(gH) = 0 \end{cases}$$

Again, we need to show that this function is continuous and has compact support. As for compact support, clearly $\text{supp } f \subseteq \text{supp } \eta$. In fact, if G was compact, we could choose $\eta \equiv 1$. To show that f is continuous, we show that it is continuous at every point of two open sets $U_1 \subseteq G$ and $U_2 \subseteq G$ satisfying $U_1 \cup U_2 = G$. On the set $U_1 := \{g \in G \mid \eta_H(gH) \neq 0\}$ it is continuous as a quotient of continuous functions; and on the set $U_2 := G - KH$ it is continuous as it vanishes there. Further, if $g \notin U_1$, then $0 = \eta_H(gH) = \int_H \eta(gh) \nu(h)$. Since η is a non-negative continuous function,

this implies $\eta(gh) = 0$ for all $h \in H$, hence $g \notin KH$, i.e. $g \in U_2$. Continuity and compact support being established, it remains to show that $f_H \equiv F$. Compute

$$f_H(gH) = \int_H \frac{F(ghH)\eta(gh)}{\eta_H(ghH)} \nu(h) = F(gH) \frac{\int_H \eta(gh) \nu(h)}{\eta_H(gH)} = F(gH).$$

Hence the map $(-)_H : C_c(G) \rightarrow C_c(G/H)$ is surjective. \square

Proof. (Theorem B.32, “ \Leftarrow ”). Let $\sigma : C_c(G/H) \rightarrow C_c(G)$ be a right-inverse for the map $C_c(G) \rightarrow C_c(G/H)$, $f \mapsto f_H$ of Lemma B.33 and consider the map

$$\lambda : C_c(G/H) \rightarrow \mathbb{C}, f \mapsto \int_G (\sigma f)(g) \mu(g).$$

Once λ is independent of σ , it is a positive linear functional. To prove that it is independent of σ , it suffices to show that $\int_G f(g) \mu(g) = 0$ whenever $f_H \equiv 0$. By Lemma B.33 and Urysohn’s Lemma B.5 there exists a function $\eta \in C_c(G)$ such that $(\text{supp } f)H \prec \eta_H \prec G/H$. Then by Proposition B.30 we have

$$\begin{aligned} \int_G f(g) \mu(g) &= \int_G \eta_H(gH) f(g) \mu(g) = \int_G \int_H \eta(gh) f(g) \nu(h) \mu(g) \\ &= \int_G \int_H \eta(gh^{-1}) f(g) \Delta_H(h^{-1}) \nu(h) \mu(g). \end{aligned}$$

We may as well integrate over the compact and hence σ -finite spaces $\text{supp } f \subseteq G$ and $(\text{supp } \eta)^{-1} \text{supp } f \cap H \subseteq H$ (see Proposition B.7). Therefore, Fubini’s Theorem B.10 allows us to continue the above computation by

$$\begin{aligned} &= \int_H \int_G \eta(gh^{-1}) f(g) \Delta_H(h^{-1}) \mu(g) \nu(h) \\ &= \int_H \int_G \eta(g) f(gh) \Delta_H(h^{-1}) \Delta_G(h) \mu(g) \nu(h). \end{aligned}$$

Applying Fubini’s Theorem B.10 again, we deduce using $\Delta_G|_H \equiv \Delta_H$ and $f_H \equiv 0$:

$$= \int_G \eta(g) \int_H f(gh) \nu(h) \mu(g) = \int_G \eta(g) f_H(gH) = 0$$

which completes the proof that λ is a positive linear functional. Hence, by Riesz’ Theorem B.12, there exists a unique Radon measure ξ on G/H such that

$$\begin{aligned} \int_G (\sigma f)(g) \mu(g) &= \lambda(f) = \int_{G/H} f(gH) \xi(gH) = \\ &= \int_{G/H} (\sigma f)_H(gH) \xi(gH) = \int_{G/H} \int_H (\sigma f)(gh) \nu(h) \xi(gH). \end{aligned}$$

for all $f \in C_c(G/H)$. The measure ξ is checked to be non-zero on non-empty open sets and G -invariant, i.e. ξ is a Haar measure on G/H . Since the above equation is independent of σ , we may as well start with a function $f \in C_c(G)$; we have thus proven the existence of a unique Haar measure ξ on G/H satisfying (W). To complete the proof, we need to show that any Haar measure on G/H (not necessarily satisfying (W)) is a strictly positive scalar multiple of ξ : Let ξ_1, ξ_2 be Haar measures on G/H . Then there are left Haar measures μ_1, μ_2 on G satisfying (W) for ξ_1 and ξ_2 respectively (see the converse direction of the proof). By uniqueness, $\mu_2 = c\mu_1$ for some strictly positive real number c . Then ξ_2 and $c\xi_1$ both satisfy (W) for μ_2 . From the uniqueness proven above we conclude $\xi_2 = c\xi_1$. \square

Remark B.34. Retain the notation of Theorem B.32. If G is compact, then the function η in the proof of Lemma B.33 can be chosen to identically equal one. The constructed left Haar functional on G/H is then given by

$$\lambda : C_c(G/H) \rightarrow \mathbb{C}, f \mapsto \int_G \frac{f(gH)}{1_H(gH)} \mu(g) = \frac{1}{\nu(H)} \int_G f(gH) \mu(g).$$

Notice that $\nu(H)$ is finite by Proposition B.21 since H is compact as a closed subset of a compact space. Now, it is a fact (see [KL06, Thm. 7.12]) that the Haar measure ξ on G/H associated to λ can be computed by evaluating λ on characteristic functions. Thus, if $E \subseteq G/H$ is measurable, we have

$$\xi(E) = \frac{\mu(\pi^{-1}(E))}{\nu(H)}, \quad \text{in particular} \quad \xi(G/H) = \frac{\mu(G)}{\nu(H)}.$$

The reader is encouraged to think about how the auxiliary function η mends the issues that arise in the case where G is not compact.

Example B.35. To illustrate the usefulness of Theorem B.32, we now provide a Haar functional for $G := \mathrm{SL}(2, \mathbb{R})$. Recall that G acts transitively on the upper half plane $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ via fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az + b}{cz + d} \quad \text{and} \quad \begin{pmatrix} \sqrt{y} & x\sqrt{y}^{-1} \\ & \sqrt{y} \end{pmatrix} i = x + iy$$

for $x \in \mathbb{R}$ and $y \in \mathbb{R}_{>0}$. Also, one readily verifies that $H := \mathrm{stab}_G(i) = \mathrm{SO}(2, \mathbb{R})$; therefore the maps

$$G/H \rightarrow \mathbb{H}, gH \mapsto gi \quad \text{and} \quad \mathbb{H} \rightarrow G/H, x + iy \mapsto \begin{pmatrix} \sqrt{y} & x\sqrt{y}^{-1} \\ & \sqrt{y} \end{pmatrix}$$

are mutually inverse G -isomorphisms. In fact they are homeomorphisms. Since G is unimodular as a connected semisimple Lie group and H is unimodular as a compact group by Proposition B.27, we by Theorem B.32 conclude the existence of a Haar measure ξ on $G/H \cong \mathbb{H}$. Let ν be the left Haar measure on H . Then the map

$$C_c(G) \rightarrow \mathbb{C}, f \mapsto \int_{G/H} \int_H f(gH) \nu(h) \xi(gH)$$

is a left Haar functional on G . To make this computable, we use the homeomorphisms $H \cong S^1$ and $G/H \cong \mathbb{H}$ to change variables with Proposition B.9, and the fact that the hyperbolic geometry on \mathbb{H} provides an $\mathrm{SL}(2, \mathbb{R})$ -invariant Radon measure on \mathbb{H} . All together, the Haar functional on $G = \mathrm{SL}(2, \mathbb{R})$ then reads

$$f \mapsto \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{2\pi} f \left(\begin{pmatrix} \sqrt{y} & x\sqrt{y}^{-1} \\ & \sqrt{y} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right) d\theta \frac{d\lambda(y) d\lambda(x)}{y^2}.$$

Remark B.36. In the setting of Example B.35 (i), the group P of Example B.29 is the stabilizer in $\mathrm{SL}(2, \mathbb{R})$ of the boundary point of \mathbb{H} associated to the (unit-speed) geodesic $\gamma : [0, \infty) \rightarrow \mathbb{H}$, $t \mapsto i + ie^{it}$. Basically, P translates γ to asymptotic geodesics. More generally, if M is a symmetric space of non-compact type, such as $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$, let $G := \mathrm{Iso}(M)^\circ$, $p \in M$ and $x \in \partial M$ be a boundary point. Then there is the following dichotomy of stabilizers, see e.g. [Ebe96, Sec. 2.17].

$\mathrm{stab}_G(p)$	$\mathrm{stab}_G(x)$
compact	non-compact
connected	not in general connected
not transitive on M	transitive on M
one conjugacy class	in general several conjugacy classes
unimodular	not in general unimodular

B.4.1. Discrete Subgroups. If, in the above discussion, $H = \Gamma$ is a discrete subgroup of G and G is second-countable, then integration over G/Γ can be realized by integrating over a *fundamental domain* for G/Γ in G , to be explained below. We shall always pick the counting measure ν as Haar measure on Γ .

Definition B.37. Let G be a locally compact Hausdorff group and let Γ be a discrete subgroup of G . A *strict fundamental domain* for G/Γ in G is a set $F \in \mathcal{B}(G)$ such that $\pi : F \rightarrow G/\Gamma$ is a bijection. A *fundamental domain* for G/Γ in G is a set $F \in \mathcal{B}(G)$ which differs from a strict fundamental domain by a set of measure zero with respect to any left Haar measure on G .

Proposition B.38. Let G be a locally compact Hausdorff, second-countable group with a discrete subgroup Γ . Then there exists a fundamental domain for G/Γ in G .

Remark B.39. Retain the notation of Proposition B.38. Note that second-countability of G in particular implies that Γ is countable.

Proof. (Proposition B.38). The canonical projection $\pi : G \rightarrow G/\Gamma$ is a local homeomorphism. Combined with second-countability, this implies the existence of an open cover $(U_n)_{n \in \mathbb{N}}$ of G such that $\pi : U_n \rightarrow \pi(U_n)$ is a homeomorphism for every $n \in \mathbb{N}$. Let $F_1 = U_1$ and define inductively $F_n = U_n - U_n \cap \pi^{-1}\pi(\bigcup_{k < n} U_k)$. Then $F := \bigcup_{n \in \mathbb{N}} F_n$ is a fundamental domain for G/Γ in G . \square

Integration over G/Γ now reduces to integration over G as follows.

Proposition B.40. Let G be a locally compact Hausdorff, second-countable group with left Haar measure μ and let Γ be a discrete subgroup of G . Assume that $\Delta_{G|\Gamma} \equiv \Delta_\Gamma$. Further, let F be a fundamental domain for G/Γ in G . Then a Haar measure ξ on G/Γ satisfying (W) exists and is associated to the following functional: $\lambda : C_c(G/\Gamma) \rightarrow \mathbb{C}$, $f \mapsto \int_F f(g\Gamma) \mu(g)$, i.e.

$$\int_{G/\Gamma} f(g\Gamma) \xi(g\Gamma) = \int_F f(g\Gamma) \mu(g) \quad \text{for all } f \in C_c(G/\Gamma).$$

Proof. The functional λ is positive and linear; the associated Radon measure ξ on G/Γ is checked to be non-zero on non-empty open sets and G -invariant. Hence ξ is a Haar measure on G/Γ . To prove that it satisfies (W), note that changing F by a set of measure zero, we may assume that F is a strict fundamental domain. Then G is a countable disjoint union $G = \bigsqcup_{\gamma \in \Gamma} F\gamma$ and hence we have for all $f \in C_c(G)$:

$$\begin{aligned} \int_G f(g) \mu(g) &= \sum_{\gamma \in \Gamma} \int_{F\gamma} f(g) \mu(g) = \sum_{\gamma \in \Gamma} \int_F f(g\gamma) \mu(g) = \int_\Gamma \int_F f(g\gamma) \mu(g) \nu(\gamma) \\ &= \int_F \int_\Gamma f(g\gamma) \nu(\gamma) \mu(g) = \int_F f_\Gamma(g\Gamma) \mu(g) = \int_{G/\Gamma} f_\Gamma(g\Gamma) \xi(g\Gamma) \\ &= \int_{G/\Gamma} \int_\Gamma f(g\gamma) \nu(\gamma) \xi(g\Gamma). \end{aligned}$$

where the second equality follows from the assumption $\Delta_{G|\Gamma} \equiv \Delta_\Gamma \equiv 1$, and the application of Fubini's Theorem B.10 is valid since G is σ -finite as a locally compact, second-countable space and Γ is σ -finite as it is countable. \square

Remark B.41. Retain the notation of Proposition B.40. The assumption $\Delta_{G|\Gamma} \equiv \Delta_\Gamma$ is not automatic. For instance, the subgroup

$$\Gamma := \left\{ \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \mid t \in \mathbb{Z} \right\}$$

of the group P of Example B.29 is isomorphic to \mathbb{Z} and discrete in P . However, for $\gamma = \text{diag}(e^t, e^{-t}) \in \Gamma - \{\text{Id}\}$ we have $\Delta_P(\gamma) = e^{-2t} \neq 1 \equiv \Delta_\Gamma$ by Example B.29.

We end this section with the following result about groups containing lattices: Recall that if G is a locally compact Hausdorff group and Γ is a discrete subgroup of G then Γ is a *lattice in G* if G/Γ supports a finite Haar measure.

Proposition B.42. Let G be a locally compact Hausdorff group. If G contains a lattice, then G is unimodular.

Proof. Let Γ be a lattice in G . Since G/Γ supports a finite Haar measure ξ , Theorem B.32 implies that $\Delta_G|_{\Gamma} \equiv \Delta_{\Gamma} \equiv 1$ and hence $\ker \Delta_G \supseteq \Gamma$. Therefore, Δ_G factors through $G \rightarrow G/\Gamma$ via $\tilde{\Delta}_G : G/\Gamma \rightarrow (\mathbb{R}_{\geq 0}^*, \cdot)$. Then $(\tilde{\Delta}_G)_*\xi$ is a non-zero, finite measure on $\mathbb{R}_{\geq 0}^*$ which is invariant under the image of Δ_G . This forces $\Delta_G \equiv 1$. \square

APPENDIX C. THE BOCHNER INTEGRAL

In this appendix we collect some facts about the Bochner integral and the associated L^p spaces. See e.g. [Yos80, Ch. V].

C.1. Definition. Let (X, \mathcal{M}, μ) be a measure space and let $(E, \|\cdot\|)$ be a Banach space over $k = \mathbb{R}$ or $k = \mathbb{C}$. We wish to single out a class of *integrable* functions $f : X \rightarrow E$ and define an *integral* for functions of this type.

The Bochner integral parallels the measure-theoretic approach for complex-valued functions to a certain extent: A *simple function* $f : X \rightarrow E$ is a measurable function which takes finitely many values. A simple function is *integrable* if $\mu(f^{-1}(x)) < \infty$ for all $x \in E, x \neq 0$ in which case we define its *integral over X* by

$$\int_X f(x) \mu(x) := \sum_{x \in f(X)} \mu(f^{-1}(x))x.$$

Let $S(X, E)$ denote the space of E -valued integrable simple functions on X . Then the above integral is a linear E -valued operator on $S(X, E)$ which is subadditive with respect to $\|\cdot\|$, i.e. for all $f \in S(X, E)$:

$$\left\| \int_X f(x) \mu(x) \right\| \leq \int_X \|f(x)\| \mu(x).$$

For functions other than simple functions we use an approximation process.

Definition C.1. Let (X, \mathcal{M}, μ) be a measure space and E a Banach space. A function $f : X \rightarrow E$ is *strongly measurable* if there exists a sequence $(s_n)_{n \in \mathbb{N}}$ of simple functions $s_n : X \rightarrow E$ such that $f(x) = \lim_n s_n(x)$ for almost all $x \in X$.

For strongly measurable functions we would like to define an integral as a limit of integrals of approximating simple functions. To ensure existence and uniqueness, we impose the following integrability condition.

Definition C.2. Let (X, \mathcal{M}, μ) be a measure space and $(E, \|\cdot\|)$ a Banach space. A strongly measurable function $f : X \rightarrow E$ is *integrable* if there exists a sequence $(s_n)_{n \in \mathbb{N}}$ of integrable simple functions $s_n : X \rightarrow E$ with $\int_X \|f(x) - s_n(x)\| \mu(x) \rightarrow 0$.

Given an integrable function $f : X \rightarrow E$ and a sequence of integrable simple functions $(s_n)_{n \in \mathbb{N}}$ as in Definition C.2, the sequence $(\int_X s_n(x) \mu(x))_{n \in \mathbb{N}}$ is a Cauchy sequence in E and hence converges to a unique limit

$$\int_X f(x) \mu(x) := \lim_n \int_X s_n(x) \mu(x),$$

the *integral of f over X* , which is independent of the chosen sequence. The integrable E -valued functions on X form a vector space on which the above integral is an E -valued linear operator.

We proceed by giving an integrability criterion. The strong measurability condition in Definition C.2 may be verified using the following Theorem of Pettis. Note that if (X, \mathcal{M}, μ) is a measure space and E is a Banach space, a function $f : X \rightarrow E$ is called *weakly measurable* if for all λ in the continuous dual space E^* of E the function $\lambda \circ f : X \rightarrow k$ is measurable. A function $f : X \rightarrow E$ is *almost separably valued* if the image under f of some conull subset of X is separable.

Theorem C.3 (Pettis). Let (X, \mathcal{M}, μ) be a measure space and E a Banach space. A function $f : X \rightarrow E$ is strongly measurable if and only if f is weakly measurable and almost separably valued.

Theorem C.4 (Bochner). Let (X, \mathcal{M}, μ) be a measure space and $(E, \|\cdot\|)$ a Banach space. A strongly measurable function $f : X \rightarrow E$ is integrable if and only if $\|f\| : X \rightarrow k$ is integrable.

C.2. Properties. The Bochner integral is subadditive with respect to the norm and commutes with bounded linear operators.

Proposition C.5. Let (X, \mathcal{M}, μ) be a measure space and let $(E, \|\cdot\|)$ be a Banach space. If $f : X \rightarrow E$ is integrable, then

$$\left\| \int_X f(x) \mu(x) \right\| \leq \int_X \|f(x)\| \mu(x).$$

Proposition C.6. Let (X, \mathcal{M}, μ) be a measure space and let E, F be Banach spaces. Further, let $f : X \rightarrow E$ be integrable and $T : E \rightarrow F$ a bounded linear operator. Then $Tf : X \rightarrow F$ is integrable and

$$T \int_X f(x) \mu(x) = \int_X Tf(x) \mu(x).$$

C.3. Example. Let G be a compact group with Haar measure μ . Further, let (π, E) be a continuous isometric representation of G where E is a Banach space. Given $v \in E$, the function $f : G \rightarrow E$, $g \mapsto \pi(g)v$ is integrable: It is weakly measurable by continuity; also by continuity, its image is a compact metric space, thus separable; hence it is strongly measurable by Pettis' Theorem C.3. Integrability is now immediate from Bochner's Theorem C.4 by the finiteness of μ and the fact that π is an *isometric* representation.

C.4. L^p Spaces. Let (X, \mathcal{M}, μ) be a measure space. Recall that given $p \in [1, \infty)$, the set of equivalence classes of measurable functions $f : X \rightarrow \mathbb{C}$, modulo equality on a conull set, satisfying $\int_X |f(x)|^p \mu(x) < \infty$ is denoted $L^p(X)$.

Now, let $(E, \|\cdot\|)$ be a Banach space. We define $L^p(X, E)$ to be the set of equivalence classes of strongly measurable functions $f : X \rightarrow E$, modulo equality on a conull set, such that $\|f\| \in L^p(X)$. The set $L^p(X, E)$ is shown to be a Banach space with the norm

$$\|f\|_p := \left(\int_X \|f(x)\|^p \mu(x) \right)^{1/p}$$

as in the case of $L^p(X)$.

If $(X, \mathcal{B}(X), \mu)$ is a topological measure space, we may define $L^p_{\text{loc}}(X, E)$ to be the set of equivalence classes of strongly measurable functions $f : X \rightarrow E$, modulo equivalence on a conull set, such that $f|_K \in L^p(K, E)$ for every compact subset $K \subseteq X$. The vector space $L^p_{\text{loc}}(X, E)$ may be equipped with the seminorms

$$\|f\|_{K,p} := \left(\int_K \|f(x)\|^p \mu(x) \right)^{1/p}$$

and is thus turned into a Fréchet space if X is locally compact and σ -compact, see [Bla79, Sec. 1]. In the latter case, $C(X, E)$ is dense in $L^p_{\text{loc}}(X, E)$ for all $p \in [1, \infty)$.

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