Beyond Black and Scholes
Uncertainty aversion, delta-vega hedging, and bubbles and crashes

Author(s):
Herrmann, Sebastian

Publication Date:
2016

Permanent Link:
https://doi.org/10.3929/ethz-a-010609303

Rights / License:
In Copyright - Non-Commercial Use Permitted
BEYOND BLACK AND SCHOLES: UNCERTAINTY AVERTION, DELTA-VEGA HEDGING, AND BUBBLES AND CRASHES

A dissertation submitted to

ETH ZURICH

for the degree of
Doctor of Sciences

presented by

SEBASTIAN HERRMANN

MSc ETH in Mathematics
born January 21, 1986
citizen of Germany

accepted on the recommendation of
Prof. Dr. Martin Schweizer examiner
Prof. Dr. H. Mete Soner co-examiner
Prof. Dr. David Hobson co-examiner
Prof. Dr. Johannes Muhle-Karbe co-examiner

2016
Abstract

This thesis is divided into two parts. In the first part, we study the pricing and hedging of derivative securities with uncertainty about the volatility of the underlying asset. We assume that the hedger of a given option is moderately uncertainty-averse: instead of believing in a single model for the evolution of the asset (as in the classical setting) or taking all models from a prespecified class equally seriously (as in the worst-case approach), she downweights models the more they deviate from a reference Black–Scholes (or local volatility) model. The mathematical formulation of a hedging problem corresponding to such preferences amounts to a family of two-player, zero-sum stochastic differential games parametrised by the agent’s uncertainty aversion. In the limit for small uncertainty aversion, we obtain explicit formulas for prices and hedging strategies in terms of the option’s cash gamma. If a liquidly traded option is available as an additional hedging instrument and the reference Black–Scholes model is dynamically recalibrated to the market price of this option, delta-vega hedging is asymptotically optimal and the corresponding indifference price corrections are determined by the disparity between the vegas, gammas, vannas, and volgas of the non-traded and the liquidly traded options.

The second part of the thesis deals with single jump local martingales, their use in the modelling of financial bubbles, and an optimal investment problem in the presence of a financial bubble or crash. Single jump local martingales are processes that follow a given deterministic function $F$ up to some random time (with distribution function $G$) at which they jump and stay constant afterwards. We characterise their (local) martingale properties in terms of conditions on the functions $F$ and $G$. This classification allows an easy construction of strict local martingales, uniformly integrable martingales that are not in $H^1$, etc. We use single jump local martingales to extend the Black–Scholes model by a time-dependent positive excess return which is compensated by a negative jump at a random time representing the bursting of a financial bubble. This model is flexible enough to include specifications such that the asset price process becomes a strict local martingale under a large class of pricing measures. We then study the problem of maximising expected utility from terminal wealth for a power utility investor in this model. Using the convex duality approach, we determine the optimal strategy and the corresponding certainty equivalent up to the solution of an integral equation (or a first-order ODE). A decomposition of the optimal strategy into myopic and hedging demand allows us to analyse the effects of the stochastic investment opportunities.

Acknowledgements

It is my pleasure to express my sincere gratitude to Martin Schweizer and Mete Soner for supervising this thesis. I am particularly grateful for the advice, support, and the encouragements they have provided me over the past years. I extend my thanks to David Hobson and Johannes Muhle-Karbe for readily accepting to act as co-examiners and for their valuable feedback on the thesis. During my time at ETH, I benefited from numerous discussions with several (current and former) colleagues at Group 3 of the Department of Mathematics at ETH. I am especially grateful for the countless hours with Martin Herdegen and Johannes Muhle-Karbe. I also thank both of them and Frank Seifried for the fruitful collaborations that led to this thesis. Finally, I wish to express my deepest gratitude to my mother. Her unconditional love and immeasurable support during my whole life has been invaluable.

Financial support by the Swiss Finance Institute and by the National Centre of Competence in Research “Financial Valuation and Risk Management” (NCCR FINRISK), Project D1 (Mathematical Methods in Financial Risk Management) is gratefully acknowledged. The NCCR FINRISK is a research instrument of the Swiss National Science Foundation.
### III Model Uncertainty, Recalibration, and the Emergence of Delta-Vega Hedging 67

1 Introduction ................................................. 67
2 Problem formulation ........................................ 75
   2.1 Market models for the underlying and its implied volatility 75
   2.2 Model uncertainty setup .................................. 76
   2.3 Hedging problem .......................................... 78
3 Heuristics ..................................................... 81
4 Main results .................................................. 85
   4.1 Notation and assumptions ................................. 85
   4.2 Main result ................................................. 89
   4.3 On the existence of a candidate asymptotic model family . 91
5 Proofs ........................................................... 93
   5.1 Value expansion and almost optimality of the delta-vega hedge ............................ 93
   5.1.1 Notation and preliminaries ............................ 94
   5.1.2 Estimates for the Hamiltonian .......................... 98
   5.1.3 Approximate solution to the HJBI equation ......... 107
   5.1.4 The asymptotic lower bound for the stochastic differential game .............. 110
   5.1.5 The asymptotic upper bound for the stochastic differential game .............. 111
   5.2 Construction of a modified feedback control ......... 115

Appendix
A Linearly constrained quadratic programming ................. 116

### Part Two Bubbles and Crashes

### IV Single Jump Processes and Strict Local Martingales 123

1 Introduction ................................................. 123
2 Analytic preliminaries ....................................... 125
   2.1 Locally absolutely continuous functions ............... 126
   2.2 The function $A^G F$ ...................................... 127
   2.3 Decomposition of locally absolutely continuous functions .... 129
3 Classification of single jump local martingales ............ 131
   3.1 Local martingale property on $[0,t_G]$ .................. 135
   3.2 Sub- and supermartingale property on $[0,t_G]$ ........ 137
   3.3 $H^1$-martingale property on $[0,t_G]$ ............... 138
   3.4 Summary and examples .................................... 140
4 A counter-example in stochastic integration ................ 142
Appendices
A  Elements of real analysis .............................................. 145
B  Elements of (semi-)martingale theory ............................. 146

V  Optimal Investment in a Black–Scholes Model with a Bubble 149
1  Introduction .............................................................. 149
2  Problem formulation ................................................... 153
3  Preliminaries .............................................................. 156
4  Bubbles as strict local martingales and the JLS model ........ 160
5  Main results ............................................................... 166
6  Numerical illustrations .................................................. 174

Appendices
A  Change of filtration ..................................................... 181
B  Single jump local martingales ....................................... 183
C  Analytic results .......................................................... 189
D  Verification and technical details ................................. 197

Bibliography 205
Chapter I

Introduction

The Black–Scholes (BS) model [18, 121] is the benchmark model for continuous-time financial markets. In its basic form, it models the price processes of two traded securities, a risky asset $S$ (e.g., a stock paying no dividends) and a riskless asset $B$ (e.g., a money market account). The risky asset follows a geometric Brownian motion with constant drift rate $\mu$ and constant volatility $\sigma$:

$$\frac{dS_t}{S_t} = \mu \, dt + \sigma \, dW_t.$$

The riskless asset grows at a continuously compounded constant interest rate. Trading is assumed to be frictionless: there are no transaction costs, trading takes place continuously, and both assets can be bought or sold in arbitrary quantities without trades affecting their prices.

Many variations of the BS model have been developed in the literature. A large body considers more complex dynamics for the risky asset: stochastic volatility models depart from the assumption that the volatility of returns is constant, models with jumps allow for discontinuous price processes, etc. Other strands of research analyse the impact of trading frictions: transaction costs, discrete trading, price impact, and so on. This dissertation discusses two extensions of the Black–Scholes world. In Part One, we consider two hedging problems for agents with uncertainty-averse preferences which reflect the idea that the true market dynamics are not known exactly. In Part Two, we develop a tractable extension of the BS model that serves as a stylised model for a financial bubble or crash and solve a corresponding optimal investment problem.

In the remainder of this introduction, we motivate and outline both parts separately. The two chapters of Part One are each self-contained. In Part Two, the results from Chapter IV used in Chapter V are collected in Appendix V.B. The related literature is discussed in each chapter separately.

Part One: Uncertainty Aversion. An accurate forecast of the statistical behaviour of future asset prices is crucial for many decision problems on financial markets. The starting point of classical mathematical finance is a stochastic model for the market dynamics and a mathematical description of the trading
mechanism. To find optimal policies for decision problems, also a performance criterion needs to be specified, which should reflect the decision maker’s preferences for random future payoffs. The two most widely used approaches are the mean-variance and the expected utility criterion. Both model risk-averse preferences in the sense that sure outcomes are preferred over random outcomes (with known probabilities). In their standard form, both criteria are based on a single stochastic model for the financial market to compute the mean and variance or the expected utility of the relevant payoffs. Translated into preferences, this means that the decision maker is convinced that she has perfect knowledge about the market dynamics. Such a decision maker faces only risk or “measurable uncertainty” [106]. In contrast, agents with imperfect knowledge about the true market dynamics are facing (model) uncertainty. In the sequel, we think of models as probability measures on a suitable measurable space \((\Omega, \mathcal{F})\).

The distinction between risk and uncertainty was established in Frank Knight’s seminal work “Risk, Uncertainty, and Profit” [106]. The celebrated Ellsberg paradox [50] shows that the choices of typical decision makers are incompatible with any subjective expected utility criterion. It suggests that typical decision makers are uncertainty-averse in the sense that they prefer to make decisions under risk rather than under uncertainty. Gilboa and Schmeidler [65] manage to resolve the Ellsberg paradox by introducing and axiomatising the multiple priors preferences, which measure the performance of a payoff \(Y\) by its minimal expected utility over some prespecified class \(\mathcal{P}\) (a subclass of all probability measures on \((\Omega, \mathcal{F})\)) of plausible models:

\[
\inf_{P \in \mathcal{P}} E^P [U(Y)] .
\]

In particular, these preferences exhibit a strict dichotomy between plausible models and implausible ones: all models in \(\mathcal{P}\) are taken equally seriously, whereas all other models are dismissed. The standard expected utility framework corresponds to the degenerate case where \(\mathcal{P}\) is a singleton.

In more recent work, Maccheroni, Marinacci, and Rustichini [118] relax the axioms characterising the multiple priors preferences and obtain a performance criterion of the form

\[
\inf_{P} (E^P [U(Y)] + \alpha(P))
\]

for some penalty functional (or ambiguity index) \(\alpha\) mapping models to \([0, +\infty]\). One interpretation is that an agent with preferences as in (2) acts as if she was playing a zero-sum game against a fictitious adversary (e.g., a “malevolent nature”) who can choose \(P\) after the agent has already decided among all available payoffs. However, the fictitious adversary (who tries to minimise the performance criterion) is penalised according to the penalty functional \(\alpha\). Thus, the penalty functional describes how seriously the agent takes different probabilistic scenarios: models \(P\) with \(\alpha(P) = 0\) are taken most seriously, while those with \(\alpha(P) = +\infty\) are dismissed entirely. For example, the multiple priors preferences (1) are recovered by setting \(\alpha(P)\) equal to zero for all \(P \in \mathcal{P}\) and \(+\infty\) otherwise.
By choosing less extreme penalty functionals, the criterion (2) allows to describe more moderate attitudes towards uncertainty.

The most popular choice of penalty functional is based on the relative entropy of a model $P$ with respect to a given reference model $\bar{P}$:

$$\alpha(P) = \frac{1}{\psi} E^P \left[ \log \frac{dP}{d\bar{P}} \right]$$

for a scaling parameter $\psi > 0$; see, e.g., Hansen and Sargent [68, 69]. Here, the agent downweights a model according to its distance (in terms of relative entropy) from the reference model. The scaling parameter $\psi$ describes the magnitude of uncertainty aversion: a small $\psi$ leads to high penalties even for small deviations from the reference model, so that alternative models $P \neq \bar{P}$ are taken less and less seriously as $\psi$ tends to zero.

As the relative entropy is only finite for probability measures that are absolutely continuous with respect to the reference probability measure, this penalty functional excludes any uncertainty about the volatility of the asset price process. Therefore, it is mainly used in conjunction with portfolio choice problems where the drift of the asset price process is important and the uncertainty about the drift is typically much larger than the uncertainty about the volatility. For hedging problems, however, the drift plays only a minor role, while an accurate description of the future volatility is crucial. In Chapter II, we study the pricing and hedging of derivative securities with uncertainty about the volatility of the underlying asset. In analogy to the relative entropy, our performance criterion penalises models based on their “distance” to a reference local volatility model.

In practice, liquidly traded options (e.g., calls and puts) on the underlying are often used as additional hedging instruments for more exotic derivatives. To implement the hedge ratios implied by a particular model, it is essential that the model prices of the liquidly traded options match their market prices well. If the model describes the true dynamics of the underlying perfectly, the model parameters only have to be calibrated at inception. Typically, however, the market prices of liquidly traded options deviate from their model prices after the initial calibration. This problem is handled in practice by frequent recalibration of the model parameters to the market prices of the liquidly traded options. But recalibration is inconsistent with the assumption that the model parameters do not change over time. In Chapter III, we show how uncertainty aversion can be used to consistently incorporate recalibration into the preferences of an agent.

**Overview of Part One.** In Chapter II, which corresponds to the article [77], we consider an agent who has sold a non-traded option and wants to hedge her exposure by trading appropriately in the underlying and a risk-free bank account. Her reference model is of local volatility type and the performance criterion describing her uncertainty aversion penalises deviations of the underlying asset’s true volatility from its reference value. The mathematical formulation of this hedging problem amounts to a family of two-player, zero-sum stochastic differential games parametrised by the agent’s uncertainty aversion. Combining
an asymptotic analysis of the Hamilton–Jacobi–Bellman–Isaacs equation associated to these games with suitably adapted verification arguments allows us to prove a next-to-leading order asymptotic expansion for the optimal values of these games. In the limit for small uncertainty aversion, we find explicit formulas for indifference prices and hedging strategies that take into account the option's susceptibility to volatility misspecification. The leading order price corrections are determined by the expected cash gamma accumulated over the remaining lifetime of the option. The next-to-leading order optimal strategy is a modification of the standard delta hedge in the local volatility reference model. The adjustment hedges against movements of the stock price into zones of high cash gamma.

In Chapter III, which corresponds to the article [76], we consider a similar hedging problem where a traded option is available as an additional hedging instrument. The agent’s reference model is assumed to be a recalibrated BS model in the sense that the volatility parameter of the BS model is dynamically recalibrated to the market price of the liquidly traded option. We establish a leading-order expansion of the optimal values of the corresponding stochastic differential games for small levels of uncertainty aversion. The leading-order term in this expansion is attained by the so-called delta-vega hedge, which neutralises the portfolio's sensitivity to changes in both the asset price and the volatility parameter of the BS model. The corresponding indifference price corrections are determined by the disparity between the so-called vegas, gammas, vannas, and volgas of the non-traded and the liquidly traded options.

Part Two: Bubbles and Crashes. Financial bubbles are typically associated with a disparity between an asset price and its “fundamental value”. The latter is unobservable and often defined as the expected value of the discounted future dividends paid by the asset, where the expectation is computed under some pricing measure “chosen by the market” [90]. It has been argued in the mathematical finance literature (cf. the survey [139] and the references therein) that this form of mispricing can be captured mathematically by modelling financial bubbles as processes that are strict local martingales (i.e., local martingales that are not martingales) under the chosen pricing measure.

The price process of the risky asset in the BS model is a (true) martingale under the unique pricing measure. Therefore, the BS model excludes financial bubbles in the above sense and it is natural to look for tractable modifications that allow for strict local martingale dynamics. This can be achieved either by modifying the volatility of the asset returns (cf., e.g., [123]) or by adding jumps. Volatility modifications have the disadvantage that the resulting processes typically cease to have explicit and tractable representations. In Chapter IV, we study a class of very tractable local martingales whose sole randomness stems from a single jump at a random time. More precisely, we consider processes $\mathcal{M}_t^G = (\mathcal{M}_t^G F)_{t \in [0, \infty)}$ of the form

$$\mathcal{M}_t^G F = F(t) \mathbf{1}_{\{t < \gamma\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{t \geq \gamma\}},$$

(3)

where the jump time $\gamma$ is a $(0, \infty)$-valued random variable with distribution func-
tion $G$, $F : [0, \infty) \to \mathbb{R}$ is a function that is “locally absolutely continuous” with respect to $G$, and the function $\mathcal{A}^G F$ is chosen such that $\mathcal{M}^G F$ becomes a martingale on the right-open interval $[0, t_G]$, where $t_G := \sup\{t \geq 0 : G(t) < 1\} \in (0, \infty]$ denotes the right endpoint of the distribution function $G$. Each trajectory $\mathcal{M}^G F(\omega)$ follows the deterministic function $F$ until just before the random time $\gamma(\omega)$, jumps at time $\gamma(\omega)$, and stays constant at $\mathcal{A}^G F(\gamma(\omega))$ from time $\gamma(\omega)$ on. These single jump local martingales provide the basis for our extension of the BS model.

In Chapter V, we extend the BS model by adding a single jump local martingale to the returns process of the risky asset:

$$\frac{dS_t}{S_{t-}} = \mu \, dt + \sigma \, dW_t + d\mathcal{M}^G \phi.$$  \hspace{1cm} (4)

We assume that the function $\phi$ is increasing, so that $S$ has a negative jump at the time of the crash $\gamma$. Thus, prior to the crash, $S$ has an extra positive excess return $\phi'(t) \, dt$ which is compensated by a negative jump at time $\gamma$. This models—in an idealised way—a main empirical feature of a financial bubble: a strong upward trend is followed by a sharp decline when the bubble bursts.

In the economic literature, it has been argued that bubbles cannot exist in efficient markets because rational agents would immediately sell overpriced assets short and thereby bring market prices back into line with their fundamental values [51]. Consequently, short-selling constraints have been blamed to be a catalyst for financial bubbles (see, e.g., [124]). The naive strategy of holding a constant short position in the inflated asset is based on the idea of profiting from the drop in price when the bubble ultimately bursts and the asset price converges to its fundamental value. However, this strategy is not without risk. As a nonnegative strict local martingale is unbounded from above, the risky asset could grow arbitrarily in the meantime. So by shorting the asset, the investor might go bankrupt before the bubble finally bursts. Another strategy is to “ride the bubble” by investing in the already inflated asset. In fact, [24] find that during the dot-com bubble, successful hedge funds were heavily invested in technology stocks although their trading activities indicate that they were aware of the bubble. This suggests to analyse in a rigorous manner the optimal investment problem in the presence of an asset price bubble, which is the main objective of Chapter V.

**Overview of Part Two.** In Chapter IV, which corresponds to the article [75], we study processes of the form (3) in great generality. Viewed on the closed interval $[0, t_G]$, or equivalently on $[0, \infty]$, $\mathcal{M}^G F$ can be either of the following: not even a semimartingale; a nonintegrable local martingale; an integrable strict local martingale; a uniformly integrable martingale which does not belong to $H^1$; an $H^1$-martingale. We establish a complete characterisation of these five cases in terms of conditions on the input functions $F$ and $G$ (cf. Figure IV.3.1). For example, if $\mathcal{M}^G F$ is integrable and $G$ has no point mass at $t_G$, then $\mathcal{M}^G F$ is a strict local martingale if and only if the limit $\lim_{t \uparrow t_G} F(t)(1 - G(t))$ is nonzero (cf. Lemma IV.3.7). As a probabilistic application, we provide a construction of
a (uniformly integrable) martingale $M$ and a bounded, deterministic integrand $H$ such that the stochastic integral $H \cdot M$ is a strict local martingale.

In Chapter V, which corresponds to the article [74], we study the problem of maximising expected utility from terminal wealth for an investor with constant relative risk aversion (power utility) in the presence of an asset price bubble of the form (4). Using the convex duality approach, we determine the optimal strategy and the corresponding certainty equivalent up to the solution of an integral equation (or a first-order ODE). A decomposition of the optimal strategy into myopic and hedging demand allows us to analyse the effects of the stochastic investment opportunities. On the one hand, investors with relative risk aversion below 1 speculate on an early bursting of the bubble in the sense that their optimal strategy lies below their myopic demand prior to the bursting; it might even involve short selling. On the other hand, investors with relative risk aversion above 1 hedge against a late bursting of the bubble in the sense that their optimal strategy lies above their myopic demand; it might under extreme circumstances even lie above the Merton proportion.

Numerical examples reveal how the optimal strategy and its myopic and hedging demand parts depend on the model parameters. In particular, we show that the optimal strategy is not fundamentally different when the stock price process is a strict local martingale (as opposed to a true martingale) under a certain class of equivalent local martingale measures, including the minimiser for the dual of the utility maximisation problem. In addition, we illustrate the welfare loss compared to the classic BS model and its dependence on the model parameters. We also clarify via our model the connection between the strict local martingale framework for asset price bubbles and the Johansen–Ledoit–Sornette financial bubble model [150].
Part One

Uncertainty Aversion
Chapter II

Hedging with Small Uncertainty Aversion

1 Introduction

Prices on financial markets are, ultimately, driven by human decisions. Whence, every quantitative model is at best a useful approximation. Moreover, even if a given model is well-suited for a particular application, its parameters typically cannot be estimated with arbitrary precision and may change suddenly due to unpredictable shocks. Therefore, it is crucial to assess the impact of model uncertainty and to derive decision rules that explicitly take it into account. The present study tackles this problem for the valuation and hedging of derivatives with uncertainty about the volatility of the underlying.

The most widely used approach in the extant literature, dating back to the seminal papers of Avellaneda, Levy, and Parás [10] and Lyons [116], is the so-called uncertain volatility model (UVM).

1 Here, the single probabilistic model used in the classical setting is replaced by a whole class of different alternatives. These are evaluated with a worst-case approach, both with respect to the “risk” inherent in a given model and with respect to the “Knightian uncertainty” about which model should be applied in the first place. More specifically, the UVM assumes only that the true volatility evolves within a band \( [\sigma_{\text{min}}, \sigma_{\text{max}}] \), without any assumptions on its dynamics. In this setting, one then tries to determine the cheapest hedge that removes all downside risk under any conceivable model, corresponding to infinite aversion against both risk and uncertainty. This approach is well-suited to determine universal no-arbitrage bounds and has initiated a tremendous amount of research; see [79, 80, 60, 45, 81, 1, 12, 132, 137, 16, 46, 62, 134, 19, 85] and the references therein. However, by its very definition it is also very conservative: unless a given model is ruled out a priori, it is treated in exactly the same way as the most plausible alternatives, say point estimates derived from market prices or statistical procedures. If the volatility band is chosen very wide, then the bid-ask spread induced by the worst-case approach is very large and a market maker quoting these bid-ask prices will not find customers in any competitive

\[ \text{\footnotesize\textsuperscript{1}} \text{A related approach is Mykland’s “conservative delta hedging” [128, 129].} \]
environment. If one artificially tightens the band of volatility scenarios to obtain competitive prices, the risk that the true volatility strays out of the band increases. Finally, the worst-case approach often incorporates options’ specific sensitivities to volatility only in a bang-bang fashion. For instance, the ask price of a call option always equals the Black–Scholes price with volatility equal to the maximal value $\sigma_{\text{max}}$. This holds even if the price of the underlying is far away from the strike, where the option’s sensitivity to changes in volatility is very low. A more nuanced attitude towards model uncertainty should reflect that in the context of hedging, agents are typically more concerned about volatility misspecification when it matters – that is, if the option value’s sensitivity to volatility is high.

To address these issues, one can consider preferences that interpolate between the classical and the worst-case approach by weighing different models according to their plausibility. Maccheroni, Marinacci, and Rustichini [118] provide a decision-theoretic, axiomatic foundation for criteria of this kind. They show that uncertainty-averse decision makers rank payoffs $Y$ according to a numerical representation of their preferences of the following form:

$$\inf_P \left( E^P[U(Y)] + \alpha(P) \right).$$  \hspace{1cm} (1.1)

Here, the utility function $U$ and the penalty functional\footnote{This terminology stems from the literature on robust control [69] and also from the robust representation of convex risk measures (see, e.g., [58, Section 5.2]). Note that the penalty is not imposed on the agent, but on her fictitious adversary who minimises over $P$. Hence, the penalty in (1.1) is added and not subtracted.} $\alpha$ describe the decision makers’ attitudes towards risk and uncertainty, respectively. One interpretation is that the decision maker behaves as if she was facing a malevolent opponent (“nature”) who takes advantage of her uncertainty by choosing a probability scenario $P$ that minimises her penalised expected utility. The standard expected utility framework corresponds to the case where $\alpha$ is zero for a single model $P$ and $+\infty$ otherwise. If $\alpha$ is zero for a whole class of models $\mathfrak{P}$ that are considered (equally) reasonable and $+\infty$ otherwise, we are in the worst-case approach and (1.1) reduces to the multiple priors preferences of Gilboa and Schmeidler [65]. Another well-known special case of (1.1) is given by the multiplier preferences of Hansen and Sargent [68], which penalise models based on their relative entropy with respect to a fixed reference model. With this approach, the user is not required to draw a strict line between “reasonable” and “wrong” models; instead, less plausible models are penalised according to how much they differ from the reference model. This specification excludes any uncertainty about volatility as the relative entropy is only finite for models that are absolutely continuous with respect to the reference model. Accordingly, this line of research has mostly focused on portfolio choice problems with uncertainty about future expected returns. With a different penalty functional, however, the same philosophy can also be applied to price and hedge options with uncertainty about the volatility of the underlying, as we do in the present study.

More specifically, we use the following general approach:
(i) Choose a reference model that you believe describes the true dynamics of the stock price reasonably well. Make sure that this model matches the market prices of liquidly traded derivatives on the stock.

(ii) Devise a nonnegative penalty functional that attaches a penalty to each alternative model and reflects how seriously you take it; the higher a model’s penalty, the less plausible it is. Typically, your penalty functional is based on some “distance” to your reference model.

(iii) For each trading strategy, evaluate its performance using (1.1). That is, compute the expected utility from the corresponding terminal Profit&Loss (henceforth P&L; defined as the terminal wealth from trading minus the payoffs of the options you have sold) in each model, add the model’s penalty, and then take the infimum over all models.

(iv) Find a trading strategy that maximises this performance criterion.

In this chapter, we choose a local volatility reference model [48]. This allows to incorporate the spot prices of vanilla options, but still isolates the impact of model uncertainty from additional complexities arising from incompleteness or additional state variables.³

The penalty functionals we consider penalise deviations of the actual (instantaneous) volatility from its counterpart in the reference model. A typical example is the mean-squared distance between those two volatilities:⁴

\[ \alpha(P) = E^P \left[ \frac{1}{2\psi} \int_0^T U'(Y_t)(\sigma_t - \bar{\sigma}(t,S_t))^2 \, dt \right]. \]  

(1.2)

Here, \( Y_t \) is your P&L at time \( t \), defined as wealth from trading minus the reference value of the option at time \( t \), \( \bar{\sigma}(t,S_t) \) is the local volatility of the reference model, and \( \sigma_t \) is the actual instantaneous volatility of returns of the stock price under the model \( P \).⁵ The parameter \( \psi > 0 \) describes the magnitude of uncertainty aversion. Indeed, small values of \( \psi \) lead to high penalties for models that deviate from the reference model, so that those alternative models are taken less and less seriously as \( \psi \) approaches zero.

To obtain explicit formulas, we pass to the limit where uncertainty aversion \( \psi \) tends to zero.⁶ That is, we consider the problem at hand as a perturbation of its classical counterpart for the reference model, and then correct prices and hedges to take into account model uncertainty in an asymptotically optimal manner.

---

³In view of the well-known deficiencies of local volatility models in capturing the dynamics of option prices, cf., e.g., [47, 63], an extension to more general reference models is an important direction for future research.

⁴A similar penalty has been used by [9] in the context of local volatility calibration with prior beliefs.

⁵The term \( U'(Y_t) \) in (1.2) renders the preferences invariant under affine transformations of the utility function; see also Remark 2.6.

⁶Asymptotic analyses of option pricing and hedging problems with the worst-case approach have been carried out by [116, 3, 4, 59].
Figure 1.1: Payoff of a “smooth put” with strike 100 (solid, thick), Black–Scholes value of a 1 year “smooth put” (solid, thin), and first-order bid and ask prices $p_b(\psi)$ and $p_a(\psi)$ (dashed), all as functions of the current stock price; the parameters are $\bar{\sigma} \equiv 20\%$ and $\psi = 10^{-3}$.

For a single vanilla option, hedged by trading in the underlying stock $S$, we establish a rigorous second-order expansion of the value $v(\psi)$ of this hedging problem for small values of the uncertainty aversion parameter $\psi$. The leading-order term in this expansion is attained by the delta hedge in the reference model; almost optimal strategies $\theta^\psi$ are identified by matching the next-to-leading order term. In analogy to perturbation analyses with transaction costs [156], the leading-order coefficient of the expansion as well as the formula for the almost optimal strategy are determined by the solution $\tilde{w}$ to a linear second-order parabolic partial differential equation (PDE) with a source term. The prices at which you are indifferent between a flat position and a long or short position in the option can in turn be described as follows. Starting from some initial capital $x_0$ and a flat position in options, your indifference bid and ask prices $p_b(\psi)$ and $p_a(\psi)$ for the option have the expansions

$$p_b(\psi) = \bar{V} - \tilde{w}\psi + o(\psi) \quad \text{and} \quad p_a(\psi) = \bar{V} + \tilde{w}\psi + o(\psi),$$

where $\bar{V} = \bar{V}(0, S_0)$ is the option price in the reference model. Whence, $\tilde{w}\psi$ is the leading-order discount or premium that you demand as a compensation for exposing yourself to model uncertainty. We can thus interpret $\tilde{w}$ as a measure for the option’s susceptibility to model misspecification and call it the option’s cash equivalent (of small uncertainty aversion). To better understand this correction term, consider its Feynman–Kac representation (cf. Proposition 3.5):

$$\tilde{w}(t, s) = \frac{1}{2} E \left[ \int_t^T (\bar{\sigma}(u, S_u) \bar{\Gamma}_u^s)^2 \, du \right] \bigg| S_t = s \right). \quad (1.3)$$

We see that the option’s sensitivity to volatility uncertainty is determined by the expected volatility-weighted cash gamma $\bar{\Gamma}_u^s := S_u^2 \bar{V}_{ss}(u, S_u)$ accumulated over

---

7In Section 1, we only display the formulas for the special case of the penalty functional (1.2). Our analysis also applies to more general penalty functionals; cf. Section 2.3.
the remaining lifetime of the option.\footnote{Our results can be formally linked to the UVM with a random, time-dependent volatility band depending on the option’s cash gamma and the agent’s uncertainty aversion; cf. Remark 3.6. Note that the cash gamma also plays a crucial role in the asymptotic analysis of other frictions such as discrete rebalancing \cite{15}, transaction costs \cite{156}, price impact \cite{125}, or jumps \cite{29}.} Also note that formula (1.3) is independent of the investor’s utility function $U$. Thus, in the setting of this chapter, uncertainty aversion dominates risk aversion at the leading order.

Let us illustrate our results for a deliberately simple case of a constant volatility reference model and a “smooth put” option, whose payoff is a smoothed version of a standard put option with strike 100.\footnote{Specifically, the payoff is the Black–Scholes value of a standard put option with strike 100 and maturity 1 day.} Figure 1.1 displays the first-order approximations of the indifference bid and ask prices $p_b(\psi)$ and $p_a(\psi)$ for $\psi = 10^{-3}$ as functions of the stock price. It is instructive to compare the corresponding first-order approximation of the bid-ask spread, $2\tilde{w}_\psi$, with the bid-ask spread implied by the UVM. In the left panel of Figure 1.2, the volatility band is chosen such that the at-the-money UVM-spread equals our spread $2\tilde{w}_\psi$. The only reason for this choice is to improve the visual comparability of both spreads. Indeed, the worst-case approach requires the volatility band to be chosen in accordance with the agent’s beliefs, which would arguably yield a wider volatility band. We see that the spread $2\tilde{w}_\psi$ becomes smaller than the UVM-spread if the stock price moves away from the strike to a region where the cash gamma of the option is lower. The right panel of Figure 1.2 compares the spreads for a wider range of stock prices and an even narrower volatility band. Our bid-ask spread exceeds the UVM-spread if the stock price is close to the strike, where the cash gamma and the option’s sensitivity to changes in volatility is high. Conversely, our bid-ask spread is lower than the UVM-spread if the stock price is far away.
II Hedging with Small Uncertainty Aversion

from the strike, where the option’s sensitivity to volatility is low. This illustrates that our approach adjusts to the structure of the option’s sensitivity to volatility misspecification in a more nuanced manner than the UVM.

Let us emphasise, however, that the bid-ask spread implied by our results is not generically close to that of the UVM. For instance, the UVM-spread for two (smooth) put options is twice the spread for a single one, whereas the spread implied by our results scales quadratically in the option’s cash gamma and thus grows by a factor of four (cf. (1.3)). Put differently, if you are already short one option, you require a larger compensation for selling another option of the same type as this makes your position more vulnerable to volatility misspecification.

We next address the corresponding asymptotically optimal hedging strategy. Up to some technical modifications (cf. Theorem 3.3), it is given by the delta hedge derived from the uncertainty-adjusted option price:

\[ \theta_t^\psi = \Delta_t + \bar{w}_s(t, S_t) \psi, \]

where \( \bar{\Delta}_t = \bar{V}_s(t, S_t) \) is the option’s delta in the reference model.\(^{11}\) The adjustment \( \bar{w}_s(t, S_t) \psi \) can be interpreted as a hedge against movements of the stock price into zones of high susceptibility to model misspecification; see Section 3 for more details. Like the price adjustment, the leading-order hedge correction is also determined by uncertainty aversion alone, independent of risk aversion.\(^{12}\)

For the results described so far, we provide a rigorous verification theorem subject to sufficient regularity conditions, which should be seen as a proof of concept. To illustrate the much wider scope of our approach, we also provide formal\(^{13}\) extensions that allow to cover practically relevant settings including exotic options,\(^{14}\) option portfolios, or semi-static hedging with vanilla options and the underlying stock. We find that for many exotics like Asian, lookback, or barrier options, the above representations for the price and hedge corrections remain valid. The only change is that the option’s price in the reference model and its corresponding cash gamma depend on further state variables in these cases.

For other exotics such as options written on the realised variance, further greeks computed in the reference model come into play; cf. Example 4.3. Concerning option portfolios and semi-static hedging, suppose that your book consists of a variety of non-traded options with different maturities and payoff profiles, which you want to hedge by dynamic trading in the stock, and by setting up a static position in some liquidly traded vanilla options. Suppose further that you can buy or sell at time 0 any quantities \( \lambda_1, \ldots, \lambda_M \) of \( M \) different liquid vanilla options, say calls, at prices to which your reference model is calibrated.\(^{15}\) Then given a

\(^{11}\)Here and in the following, subscripts on functions denote the corresponding partial derivatives.

\(^{12}\)The strategy adjustment may become dependent on risk aversion for other penalty functionals; cf. the discussion following Theorem 3.3.

\(^{13}\)A rigorous verification of these results would proceed along the same lines as for the simpler benchmark case discussed here. In order not to drown the ideas in further technicalities resulting from even more state variables, regularity conditions, etc. we do not pursue this here.

\(^{14}\)Worst-case hedges for some exotics have been derived by [79, 23, 36, 35, 84, 83, 82, 151], for example.

\(^{15}\)This is the analogue of the Lagrangian uncertain volatility model [11]; also compare [130].
II.1 Introduction

choice of $\lambda$, the cash equivalent of the combined portfolio is\(^{16}\)

\[
\bar{w}(t, s; \lambda) = \frac{1}{2} E \left[ \int_t^T \left( \bar{\sigma}(u, S_u) \left( \bar{\Gamma}^{s,0}_u - \sum_{i=1}^M \lambda_i \bar{\Gamma}^{s,i}_u \right) \right)^2 \, du \bigg| S_t = s \right], \tag{1.4}
\]

where $\bar{\Gamma}^{s,0}$ is the net cash gamma of your original book (before buying or selling
calls) and $\bar{\Gamma}^{s,i}$ is the cash gamma of the $i$-th call. The corresponding hedge is

\[
\theta^\psi_t = \bar{\Delta}^0_t - \sum_{i=1}^M \lambda_i \bar{\Delta}^i_t + \bar{w}_s(t, S_t; \lambda) \psi,
\]

where $\bar{\Delta}^0$ is the net delta of the original book and $\bar{\Delta}^i$ is the delta of the $i$-th call. Formula (1.4) provides a simple yet theoretically founded criterion to optimise
the resilience of a derivative book against model misspecification. Indeed, since
the cash equivalent is a measure for the portfolio’s susceptibility to model misspecification, minimising (1.4) over $\lambda$ renders your combined portfolio more robust.
This minimisation corresponds to balancing out the cash gamma of your net position by trading appropriately in the liquid options, in analogy to corresponding
results for transaction costs [101, Section 3.1]. The mapping

\[
\bar{\Gamma}^{s,0} \mapsto \inf_{\lambda} \frac{1}{2} E \left[ \int_0^T \bar{\sigma}(u, S_u) \left( \bar{\Gamma}^{s,0}_u - \sum_{i=1}^M \lambda_i \bar{\Gamma}^{s,i}_u \right)^2 \, du \right]
\]

can be interpreted as a “measure of model uncertainty” in the sense of Cont [33].
That is, it satisfies certain desirable properties that, unlike standard risk measures, take into account model-independent hedging strategies and the availability
of options as hedging instruments; cf. Section 4.6 for more details.

From a mathematical point of view, this chapter can be seen as a case study
of an asymptotic analysis of a two-player, zero-sum stochastic differential game
where both the drift and the diffusion coefficients of the state variables are controlled. Instead of the predominating Elliott–Kalton formulation used in the
seminal work of Fleming and Souganidis [57] and many articles thereafter, we
use a weak formulation similar to Pham and Zhang [136]. This approach turns
out to be both more natural and more convenient for the hedging problem at
hand. The candidate expansion of the value function and candidates for almost
optimal controls are derived from an appropriate ansatz substituted into the
Hamilton–Jacobi–Bellman–Isaacs equation. The proof adapts classical verifica-
tion arguments of stochastic optimal control to the asymptotic setting.

The remainder of the chapter is organised as follows. Our framework for
hedging with volatility uncertainty is introduced in Section 2. Subsequently, we
state and discuss our main results in Section 3. A partially heuristic derivation
and formal extensions can be found in Section 4. Finally, the rigorous proofs of
the results from Section 3 are collected in Section 5.

\(^{16}\)The same formula obtains – mutatis mutandis – for exotics of Asian-, lookback-, or barrier-
type, after adding the appropriate state variables to the reference cash gamma.
2 Problem formulation

2.1 Hedging with a local volatility model

Consider an agent (hereafter called “you”) who has written a vanilla option on a stock \( S \) with payoff \( G(S_T) \) at maturity \( T > 0 \). We assume that you can trade frictionlessly in the stock and a bank account with zero interest rate. If you are risk averse, you will try to hedge your exposure to the option by trading “appropriately” in the underlying.

Let us assume that the true dynamics of the stock price process \( S = (S_t)_{t \in [0,T]} \) are governed by the SDE

\[
dS_t = S_t \sigma_t \, dW_t, \quad S_0 = s_0 > 0, \tag{2.1}
\]

for a Brownian motion \( W \) and a volatility process \( \sigma \). A typical approach to hedge options is to postulate a class of models for the stock price dynamics, calibrate it using market prices of liquidly traded derivatives, and act as if the model was correct. A simple, but popular choice for this procedure is the class of local volatility models \([48]\), which assume that the volatility process in (2.1) is a deterministic function of time and the stock price itself, \( \sigma_t = \bar{\sigma}(t, S_t) \). Let us briefly recall how this model can be used to hedge the (sufficiently integrable) option \( G(S_T) \) through self-financing trading in the underlying (cf., e.g., [63] for more details). Self-financing trading strategies are parametrised by progressively measurable processes \( \theta = (\theta_t)_{t \in [0,T]} \) describing the number of shares held at time \( t \). If your initial capital is \( x_0 \) and you trade according to \( \theta \), your wealth at time \( T \) is

\[
x_0 + \int_0^T \theta_t \, dS_t.
\]

Applying Itô’s formula to the process \( \bar{V}(t, S_t) \) for some \( \bar{V} \in C^{1,2}((0, T) \times \mathbb{R}_+) \cap C((0, T] \times \mathbb{R}_+) \) and using that the true dynamics of \( S \) are given by (2.1), we obtain

\[
\bar{V}(t, S_t) = \bar{V}(0, s_0) + \int_0^t \bar{V}_s(u, S_u) \, dS_u + \int_0^t \left( \bar{V}_t(u, S_u) + \frac{1}{2} \sigma_u^2 S_u^2 \bar{V}_{ss}(u, S_u) \right) \, du. \tag{2.2}
\]

Note that if \( \sigma_t = \bar{\sigma}(t, S_t) \), i.e., if your local volatility model is correct, and if \( \bar{V} \) solves the Black–Scholes PDE

\[
\bar{V}_t(t, s) + \frac{1}{2} \bar{\sigma}(t, s)^2 s^2 \bar{V}_{ss}(t, s) = 0, \quad (t, s) \in (0, T) \times \mathbb{R}_+, \tag{2.3}
\]

\[
\bar{V}(T, s) = G(s), \quad s \in \mathbb{R}_+, \tag{2.3}
\]

then (2.2) for \( t = T \) reduces to

\[
G(S_T) = \bar{V}(T, S_T) = \bar{V}(0, s_0) + \int_0^T \bar{V}_s(u, S_u) \, dS_u. \tag{2.4}
\]
Whence, you can perfectly replicate the option payoff $G(S_T)$ by self-financing trading with initial capital $V(0, s_0)$ and trading strategy $\Delta_t := \bar{V}_s(t, S_t)$, the so-called delta hedge. Given that your local volatility model is correct, $\bar{V}(t, S_t)$ is in turn the unique price for the option that is compatible with the absence of arbitrage.

What happens if you delta-hedge even though your local volatility model is not correct? If the true dynamics are given by the SDE (2.1) for some volatility process $\sigma$, but the delta hedge $\bar{\Delta}$ is determined from a local volatility model via the PDE (2.3), then (2.2) can be rewritten as

$$ \bar{V}(t, S_t) = V(0, s_0) + \int_0^t \bar{\Delta}_u dS_u + \frac{1}{2} \int_0^t S_u^2 \bar{V}_{ss}(u, S_u)(\sigma_u^2 - \bar{\sigma}(u, S_u)^2) du. \quad (2.5) $$

For a trading strategy $\theta$, define the Profit & Loss (P&L) $Y_t$ at time $t$ as the difference between the current wealth in your hedge portfolio and the theoretical value of the option in your model:

$$ Y_t = x_0 + \int_0^t \theta_u dS_u - \bar{V}(t, S_t). \quad (2.6) $$

As $\bar{V}(T, S_T) = G(S_T)$, the terminal value $Y_T$ is also your actual P&L at maturity $T$. Substituting (2.5) into (2.6) gives

$$ Y_t = x_0 - \bar{V}(0, s_0) + \int_0^t (\theta_u - \bar{\Delta}_u) dS_u + \frac{1}{2} \int_0^t S_u^2 \bar{V}_{ss}(u, S_u)(\bar{\sigma}(u, S_u)^2 - \sigma_u^2) du. \quad (2.7) $$

The last term in (2.7) describes the hedging error you incur if you use the delta hedge $\theta_t = \bar{\Delta}_t$, starting from initial capital $x_0 = \bar{V}(0, s_0)$.

Let us now formalise that risk-averse investors indeed hedge their exposure to the option. To this end, assume you are certain that the dynamics of the stock price follow your local volatility model, i.e., $\sigma_t = \bar{\sigma}(t, S_t)$, and that you want to maximise your expected utility from your final P&L:

$$ E[U(Y_T)] = E\left[U\left(x_0 + \int_0^T \theta_t dS_t - G(S_T)\right)\right] \rightarrow \max! \quad (2.8) $$

Here, $\theta$ runs through some subset of trading strategies such that the stochastic integral $\int_0^T (\theta_t - \bar{V}_s(t, S_t)) dS_t$ is a supermartingale (to rule out doubling strategies) and $U$ is a (strictly concave) utility function on $\mathbb{R}$. As the option can be perfectly replicated, this is a trivial example of a utility maximisation problem with a random endowment. By Jensen’s inequality, (2.4), and the supermartingale property of $\int_0^T (\theta_t - \bar{V}_s(t, S_t)) dS_t$, we have

$$ E\left[U\left(x_0 + \int_0^T \theta_t dS_t - G(S_T)\right)\right] \leq U\left(x_0 + E\left[\int_0^T (\theta_t - \bar{V}_s(t, S_t)) dS_t\right] - \bar{V}(0, s_0)\right) \leq U(x_0 - \bar{V}(0, s_0)), $$

with equality for $\theta_t = \bar{V}_s(t, S_t) = \bar{\Delta}_t$. This proves the following lemma:
Lemma 2.1. Suppose that $\sigma_t = \bar{\sigma}(t, S_t)$ and that $\tilde{V} \in C^{1,2}((0, T) \times \mathbb{R}_+) \cap C([0, T] \times \mathbb{R}_+)$ solves the PDE (2.3). Then the delta hedge $\bar{\Delta}_t = \tilde{V}_s(t, S_t)$ maximises (2.8) over any set $A \ni \bar{\Delta}$ of strategies $\theta$ such that $\int_0^T (\theta_t - \tilde{V}_s(t, S_t)) dS_t$ is a supermartingale. Moreover:

$$\sup_{\theta \in A} E \left[ U(x_0 + \int_0^T \theta_t dS_t - G(S_T)) \right] = U(x_0 - \tilde{V}(0, s_0)).$$

Remark 2.2. Let us briefly discuss why we assume that $S$ has zero drift. Suppose $dS_t = S_t(\mu_t dt + \bar{\sigma}(t, S_t) dW_t)$ for a suitable drift rate $\mu = (\mu_t)_{t \in [0, T]}$. If $\theta^*$ is an optimal strategy for the pure investment problem

$$E \left[ U(x_0 - \tilde{V}(0, s_0) + \int_0^T \theta_t dS_t) \right] \rightarrow \max,$$

then it follows (using that the replication (2.4) also works with drift) that $\theta^* + \bar{\Delta}$ is an optimal strategy for the mixed hedging/investment problem (2.8), irrespective of the drift rate $\mu$. Whence, without model uncertainty, the hedging component $\bar{\Delta}$ of the optimal strategy does not depend on $\mu$. Therefore, it is expected that also with (small) uncertainty aversion (about this complete reference model), the drift rate only has a small effect on the hedging component.

Assuming that $S$ has zero drift allows us to focus on the hedging rather than the investment component of the optimal strategy. Indeed, with zero drift, there is no incentive to trade the stock other than as a hedging instrument for the option. This is a reasonable approximation at least if the time horizon is not too long so that stock price movements are dominated by Brownian fluctuations.

In Section 2.3, we introduce a modification of the utility maximisation problem (2.8) that takes into account uncertainty about the true dynamics of the stock price. Its formulation requires a setup that can model the flow of observable information and the possibility of different volatility scenarios simultaneously, which we develop in Sections 2.2 and 2.3.

2.2 Volatility uncertainty setup

Fix a time horizon $T > 0$ and constants $s_0 > 0$ and $y_0 \in \mathbb{R}$. Let

$$\Omega = \{ \omega = (\omega^S_t, \omega^Y_t)_{t \in [0, T]} \in C([0, T]; \mathbb{R}^2) : \omega_0 = (s_0, y_0) \}$$

be the canonical space of continuous paths in $\mathbb{R}^2$ starting in $(s_0, y_0)$, endowed with the topology of uniform convergence. Moreover, let $\mathcal{F} = \mathcal{B}(\Omega)$ be the Borel $\sigma$-algebra on $\Omega$. We denote by $S = (S_t)_{t \in [0, T]}$ and $Y = (Y_t)_{t \in [0, T]}$ the first and second component of the canonical process, respectively, i.e., $S_t(\omega) = \omega^S_t$ and $Y_t(\omega) = \omega^Y_t$, and by $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ the (raw) filtration generated by $(S, Y)$. Unless otherwise stated, all probabilistic notions requiring a filtration, such as progressive measurability etc., pertain to $\mathbb{F}$. 
II.2 Problem formulation

Remark 2.3. The measurable space $(\Omega, F)$ together with the filtration $\mathcal{F}$ models the observable variables, the stock price $S$ and the P&L $Y$, independently of any specific choice of probability measure on $(\Omega, \mathcal{F})$. As we shall see, this allows us to use progressively measurable processes as controls. If instead one formulated the hedging problem as a stochastic differential game on a single filtered probability space, one would have to allow Elliott–Kalton strategies (see, e.g., [56]) as controls; these are controls that may depend on the other player’s control in a non-anticipative way. In the context of financial applications, it also seems more natural to treat the state processes as publicly observable, but not necessarily the controls. In our setting, the stock price is quoted on the market and you can easily determine your P&L by looking at your trading account.

Next, we introduce a large class of probability measures on $(\Omega, \mathcal{F})$ which correspond to different choices of strategies $\theta$ and volatility processes $\sigma$. Let $\mathcal{S}$ denote the set of triplets $(\theta, \sigma, P)$ consisting of progressively measurable processes $\theta = (\theta_t)_{t \in [0,T]}$ and $\sigma = (\sigma_t)_{t \in [0,T]}$ and a probability measure $P$ on $(\Omega, \mathcal{F})$ such that $S$ is a (continuous) $P$-martingale with quadratic variation $d\langle S \rangle_t = S_t^2 \sigma_t^2 dt$ and $Y$ is a (continuous) $P$-semimartingale with canonical decomposition

$$Y = y_0 + \int_0^t (\theta_t - \bar{V}_s(t, S_t)) \, dS_t + \int_0^t \frac{1}{2} S_t^2 \bar{V}_{ss}(t, S_t)(\bar{\sigma}(t, S_t)^2 - \sigma_t^2) \, dt \quad (2.10)$$

under $P$. For later reference, note that this implies that under $P$,

$$d\langle S \rangle_t = S_t^2 \sigma_t^2 dt,$$

$$d\langle Y \rangle_t = (\theta_t - \bar{V}_s(t, S_t))^2 S_t^2 \sigma_t^2 dt,$$

$$d\langle S, Y \rangle_t = (\theta_t - \bar{V}_s(t, S_t)) S_t^2 \sigma_t^2 dt. \quad (2.11)$$

For each triplet $(\theta, \sigma, P)$, the probability measure $P$ corresponds to the distribution of the stock price process $S$ and your P&L process $Y$ through their choice of probability measure. “Nature’s” choice of volatility determines the stock price volatility and the drift of the P&L process. Your choice of strategy only affects the volatility of your P&L.

Remark 2.4. This is a setup for a weak formulation of a stochastic differential game similar to [136]. Instead of controlling a process directly, the players can control the distribution of the observable processes $S$ and $Y$ through their choice of probability measure. “Nature’s” choice of volatility determines the stock price volatility and the drift of the P&L process. Your choice of strategy only affects the volatility of your P&L.

2.3 Hedging under volatility uncertainty

The set $\mathcal{S}$ defined in Section 2.2 describes a very large set of possible distributions for the stock price process $S$ and your P&L process $Y$. As some of these scenarios might be implausible (they could, for instance, include doubling strategies), let us fix some subset $\mathcal{S} \subset \mathcal{S}$; for our specific choice, see (3.1) below. Next, fix sets $\mathcal{A}$ and $\mathcal{V}$ of trading strategies and volatilities that you want to take into account. In principle, we could consider pairs $(\theta, \sigma) \in \mathcal{A} \times \mathcal{V}$ for which there is at least
one probability measure \( P \) such that \((\theta, \sigma, P) \in \mathcal{G}\). But as your choice of trading strategy should not restrict “nature’s” choice of volatilities and vice versa, we require that the whole rectangle \( \mathcal{A} \times \mathcal{V} \) has this property:

\[
\mathcal{A} \times \mathcal{V} \subset \mathcal{Z} := \{ (\theta, \sigma) : \mathcal{P}(\theta, \sigma) \neq \emptyset \},
\]

where \( \mathcal{P}(\theta, \sigma) \) denotes the set of probability measures \( P \) such that \((\theta, \sigma, P) \in \mathcal{S}\).

**Remark 2.5.** Note that \( \mathcal{P}(\theta, \sigma) \) can contain more than one \( P \). Indeed, each weak solution of the SDE

\[
dS_t = S_t \sigma_t((S, Y)) \, dW_t,
\]

\[
dY_t = (\theta_t((S, Y)) - \bar{V}_t(t, S_t)) \, dS_t + \frac{1}{2} S_t^2 \bar{V}_{ss}(t, S_t)(\bar{\sigma}(t, S_t)^2 - \sigma_t((S, Y))^2) \, dt,
\]

with initial distribution \( \delta_{(s_0, y_0)} \) gives rise to a measure in \( \mathcal{P}(\theta, \sigma) \). So \( \mathcal{P}(\theta, \sigma) \) is a singleton if and only if the SDE has a solution which is unique in law.

A popular approach to incorporate volatility uncertainty into utility maximisation problems is to treat all volatilities in \( \mathcal{V} \) equally and to maximise the expected utility for the worst-case volatility. In the setting of this chapter, this would lead to

\[
\inf_{\sigma \in \mathcal{V}} \inf_{P \in \mathcal{P}(\theta, \sigma)} E^P \left[ U(Y_T) \right] \longrightarrow \max_{\theta \in \mathcal{A}}!
\]

More generally, one can introduce a penalty functional \( a : \mathcal{V} \to L^0_+(\Omega, \mathcal{F}) \) and consider

\[
\inf_{\sigma \in \mathcal{V}} \inf_{P \in \mathcal{P}(\theta, \sigma)} E^P \left[ U(Y_T) + a(\sigma) \right] \longrightarrow \max_{\theta \in \mathcal{A}}!
\]

This approach allows to weigh volatility scenarios according to their plausibility – the higher the penalty, the less seriously you take the model.\(^{19}\) We suppose that a local volatility model with volatility function \( \bar{\sigma} \) is your best guess for the true dynamics of \( S \). In the following, we consider the penalty functional

\[
a(\sigma) = \frac{1}{\psi} \int_0^T U'(Y_t) f(t, S_t, Y_t; \sigma_t) \, dt.
\]

Here, \( \psi > 0 \) is a parameter and the penalty function \( f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+ \) is a sufficiently smooth function such that \( \zeta \mapsto f(t, s, y; \zeta) \) is strictly convex and has a minimum of 0 at \( \bar{\sigma}(t, s) \), i.e.,

\[
\frac{\partial f}{\partial \zeta}(t, s, y; \bar{\sigma}(t, s)) = 0.
\]  

\(^{17}\)Since \( \mathcal{P}(\theta, \sigma) \) may contain more than one measure, we also have to allow “nature” to choose a specific measure.

\(^{18}\)The penalty functional in the sense of the criterion (1.1) is \( a(P) = E^P \left[ a(\sigma^P) \right] \), where \( \sigma^P \) is the volatility of returns of \( S \) under \( P \).

\(^{19}\)Recall from Footnote 2 that \( a \) penalises the fictitious adversary (“nature”) and not the agent.
In particular, \( a \) does not penalise the reference local volatility model, i.e.,

\[ a(\bar{\sigma}(t, S_t))_{t \in [0,T]} = 0. \]

A typical example for the penalty function is \( f(t, s, y; \varsigma) = \frac{1}{2} (\varsigma - \bar{\sigma}(t, s))^2 \); then, deviations of the volatility from the reference model are penalised in a mean-square sense.\(^{20}\) The function \( f \) describes the “shape” of the uncertainty aversion, while the parameter \( \psi \) quantifies its “magnitude”. Indeed, if \( \psi \) is small, then volatility processes alternative to the reference volatility are penalised heavily, i.e., uncertainty aversion is small.

For each \( \psi > 0, \theta \in \mathcal{A}, \sigma \in \mathcal{V}, \) and \( P \in \mathfrak{P}(\theta, \sigma) \), we define the objective of our hedging problem by

\[ J^\psi(\sigma, P) := E^P \left[ U(Y_T) + \frac{1}{\psi} \int_0^T U'(Y_t) f(t, S_t, Y_t, \sigma_t) \, dt \right] \quad (2.13) \]

and its value by

\[ v(\psi) = v(\psi; \mathcal{A}, \mathcal{V}) := \sup_{\theta \in \mathcal{A}} \inf_{\sigma \in \mathcal{V}} \inf_{P \in \mathfrak{P}(\theta, \sigma)} J^\psi(\sigma, P). \quad (2.14) \]

**Remark 2.6.** Note that the term \( U'(Y_t) \) in the penalty part of the objective (2.13) does not restrict the generality of our results. Indeed, the factor \( U'(Y_t) \) could effectively be removed if desired by working with a penalty function of the form \( f(t, s, y; \varsigma) = \tilde{f}(t, s, y; \varsigma)/U'(y) \) for a suitable function \( \tilde{f} \). We choose the above formulation for the following reasons.

First, in the standard expected utility framework, preferences are invariant under affine transformations of the utility function. The term \( U'(Y_t) \) ensures that this property is preserved for uncertainty-averse decision makers whose preferences are described by (2.13)–(2.14). Second, \( U'(Y_t) \) (and not, e.g., \( U'(y_0) \))\(^{21}\) is the natural choice for a dynamic formulation of the hedging problem (2.14) in terms of a family of conditional problems parametrised by the initial time \( t \), stock price \( S_t = s \), and P&L \( Y_t = y \). Third, our results show that if \( f \) does not depend on the P&L variable \( y \), then the preferences have approximately “constant uncertainty aversion” in the sense that the cash equivalent \( \tilde{w} \) does not depend on the P&L (cf. Proposition 3.5). This would not be the case if one omitted the term \( U'(Y_t) \) in (2.13).

### 3 Main results

Our main result is a second-order expansion of the value \( v(\psi) \) of the hedging problem (2.14) for small values of the uncertainty aversion parameter \( \psi \). Moreover, we

\(^{20}\)Notably, [9] show that penalty functionals of this form can arise as the continuous-time limit of the relative entropy in a discrete-time approximation.

\(^{21}\)Using \( U'(y_0) \) instead of \( U'(Y_t) \) would yield the same expansion for \( v(\psi) \) as in Theorem 3.3 and, formally and at the leading-order, the same almost optimal strategies and volatilities. This is because in the asymptotic limit for small uncertainty aversion, the P&L process converges to a constant.
provide strategies $\theta^\psi$ and volatilities $\sigma^\psi$ that are almost optimal in the sense that their performance coincides with the optimal value up to the next-to-leading order $O(\psi^2)$. These expansions depend on the solutions to two linear second-order parabolic PDEs with source terms. Section 3.1 introduces these PDEs as well as the assumptions underlying our main result. On a first reading, the reader may wish to skip Section 3.1 and jump directly to the statement of the main result in Section 3.2.

3.1 Ingredients and assumptions

In order to present the main ideas in the verification as clearly as possible, we do not strive for minimal technical conditions in the following. Rather, we focus on a set of sufficient conditions and consider only triplets $(\theta, \sigma, P)$ such that the processes $S$, $Y$, and $\sigma$ are $P$-a.s. uniformly bounded by a constant independent of $(\theta, \sigma, P)$. So fix constants $K > s_0 \lor s_0^{-1}$, $y_l < y_0$, and $y_u > y_0$ and define $\mathcal{S} \subset \overline{\mathcal{S}}$ as the set of triplets $(\theta, \sigma, P)$ such that

$$\begin{align*}
\sigma_t(\omega) &\in [0, K], S_t(\omega) \in [K^{-1}, K], Y_t(\omega) \in (y_l, y_u) \\
\text{for } d \tau \times P\text{-a.e. } (t, \omega) \in [0, T] \times \Omega. 
\end{align*}$$

The coefficients of the expansion of $v(\psi)$ are expressed in terms of solutions to two PDEs. Set $D = (0, T) \times (K^{-1}, K) \times (y_l, y_u)$. For each $y \in (y_l, y_u)$, we consider the linear second-order parabolic PDEs

$$\begin{align*}
\hat{w}_t(t, s, y) + \frac{1}{2} \sigma(t, s)^2 s^2 \hat{w}_{ss}(t, s, y) + \hat{g}(t, s, y) &= 0, && (t, s) \in (0, T) \times (K^{-1}, K), \\
\hat{w}(T, s, y) &= 0, && s \in (K^{-1}, K), \\
\hat{w}(t, s, y) &= 0, && t \in (0, T), s \in \{K^{-1}, K\},
\end{align*}$$

and

$$\begin{align*}
\hat{w}_t(t, s, y) + \frac{1}{2} \sigma(t, s)^2 s^2 \hat{w}_{ss}(t, s, y) + \hat{g}(t, s, y) &= 0, && (t, s) \in (0, T) \times (K^{-1}, K), \\
\hat{w}(T, s, y) &= 0, && s \in (K^{-1}, K), \\
\hat{w}(t, s, y) &= 0, && t \in (0, T), s \in \{K^{-1}, K\},
\end{align*}$$

(3.2) and (3.3)
where the source terms \(\tilde{g}, \hat{g} : D \to \mathbb{R}\) are given by

\[
\tilde{g}(t, s, y) := \left(\frac{\sigma(t, s) s^2 \hat{V}_{ss}(t, s)}{2f''(t, s, y; \sigma(t, s))}\right)^2,
\]

\[
\hat{g}(t, s, y) := \frac{1}{6} \frac{\sigma(t, s)^3}{f(3)}(t, s, y; \sigma(t, s)) - \frac{1}{2} \sigma(t, s, y) s^2 \hat{V}_{ss}(t, s)
+ \sigma(t, s, y) \frac{s^2}{2} \left(\hat{V}_{ss}(t, s) \bar{w}_y(t, s, y) - \tilde{w}_{ss}(t, s, y)\right)
- \sigma(t, s, y)^2 \frac{U''(y)}{U(y)} \left(\frac{1}{2} \sigma(t, s) s^2 \hat{\theta}(t, s, y)^2
- \sigma(t, s, y)^2 f''(t, s, y; \sigma(t, s)) \bar{w}(t, s, y)\right),
\]

with

\[
\hat{\theta}(t, s, y) := \tilde{w}_s(t, s, y) + \frac{U'(y)}{U(y)} \bar{w}_y(t, s, y),
\]

\[
\hat{\sigma}(t, s, y) := \frac{\sigma(t, s) s^2 \hat{V}_{ss}(t, s)}{f''(t, s, y; \sigma(t, s))}.
\]

We prove our main result under the following assumptions.

**Assumption 3.1.**

(i) **PDEs:** There are \(\bar{w}, \hat{w} \in C^{1,2}(D) \cap C(\overline{D})\) such that for each \(y \in (y_l, y_u)\), \(\bar{w}(\cdot, \cdot, y)\) and \(\hat{w}(\cdot, \cdot, y)\) are classical solutions to the PDEs (3.2)–(3.3) and

\[
|w_t|, |w_s|, |w_y|, |w_{ss}|, |w_{sy}|, |w_{yy}| \leq K \text{ on } D, \quad w \in \{\bar{w}, \hat{w}\}.
\]

(ii) **Reference local volatility function:** \(\hat{\sigma} : [0, T] \times [K^{-1}, K] \to [0, K]\) is Borel-measurable, there is \(\varepsilon > 0\) such that

\[
\varepsilon \leq \hat{\sigma}(t, s) \leq K - \varepsilon \quad \text{for } (t, s) \in [0, T] \times (K^{-1}, K), \quad \hat{\sigma}(t, K) = \hat{\sigma}(t, K^{-1}) = 0 \quad \text{for } t \in [0, T],
\]

(iii) **Reference value:** \(\bar{V} : [0, T] \times [K^{-1}, K] \to \mathbb{R}\) is Borel-measurable, \(\bar{V} (t, \cdot) \in C^2((K^{-1}, K)) \cap C([K^{-1}, K])\) for all \(t \in (0, T)\), and

\[
|s^2 \bar{V}_{ss}(t, s)| \leq K \quad \text{for } (t, s) \in (0, T) \times (K^{-1}, K).
\]

(iv) **Penalty function:** \(f\) is \(C^4\) in \(\zeta\) and the partial derivatives \(f^{(k)} := \frac{\partial^k}{\partial \zeta^k} f, k = 2, 3, 4,\) satisfy

\[
\frac{1}{K} \leq f''(t, s, y; \zeta) \leq K, \quad \|f^{(3)}(t, s, y; \zeta)\|, \|f^{(4)}(t, s, y; \zeta)\| \leq K \text{ on } D \times [0, K],
\]

\[\text{22Here, we assume that all relevant partial derivatives exist; precise conditions are given in Assumption 3.1.}\]
(v) *Utility function:* $U: \mathbb{R} \to \mathbb{R}$ is $C^3$ with $U' > 0$ and $U'' < 0$ everywhere.

Formally plugging the assumption $\bar{\sigma}(\cdot, K) = \bar{\sigma}(\cdot, K^{-1}) = 0$ into (3.4)–(3.5) and (3.7) motivates to extend the definitions of $\tilde{\sigma}, \tilde{g},$ and $\hat{g}$ by setting

$$\tilde{\sigma}(t, s, y) = \tilde{g}(t, s, y) = \hat{g}(t, s, y) = 0, \quad t \in (0, T), s \in \{K^{-1}, K\}, y \in (y_l, y_u).$$

(3.12)

**Remark 3.2.** (ii)–(iii) are assumptions on the reference volatility and the reference value of the option, while (iv)–(v) are regularity conditions for the objects describing your risk and uncertainty aversion. In contrast, (i) is an assumption on objects derived from these primitives through PDEs. Therefore, let us indicate here what kind of regularity assumptions are sufficient for (i) to hold. We focus on the PDE (3.2) for $\tilde{w}$; the PDE for $\hat{w}$ can be treated analogously. We first fix $y$ and note that the diffusion coefficient $\bar{\sigma}(t, s)^2 s^2$ is bounded away from zero on $(0, T) \times (K^{-1}, K)$ by (3.9). Now, a classical existence and regularity result (see Friedman [61], Theorem 7 in Section 3.3) guarantees the existence of a classical solution $\tilde{w}(\cdot, \cdot, y) \in C^{1,2}((0, T) \times (K^{-1}, K))$ with bounded and Hölder-continuous (in $t$ and $s$) partial derivatives $\tilde{w}_t, \tilde{w}_s, \tilde{w}_{ss}$ provided that the diffusion coefficient and the source term are regular enough, and that the source term is compatible with the zero boundary condition in the sense that $\tilde{g}(t, s, y) = 0$ for $s \in \{K^{-1}, K\}$. Next, one can show that $\tilde{w}$ has the Feynman–Kac representation (see also Proposition 3.5 below)

$$\tilde{w}(t, s, y) = E^{t,s} \left[ \int_t^T \tilde{g}(u, S_u, y) \, du \right],$$

(3.13)

where the expectation is computed under a measure such that $S$ has the dynamics $dS_u = S_u \bar{\sigma}(u, S_u) \, dW_u$ and starts in $S_t = s$. Now, if $\tilde{g}$ is $C^2$ in $y$ with bounded partial derivatives, one can infer from (3.13) that the partial derivatives $\tilde{w}_y, \tilde{w}_{yy}$ exist and are bounded on $D$. Finally, to obtain the existence and bounds for the cross partial derivative $\tilde{w}_{y}^s$, we can differentiate the PDE (3.2) with respect to $y$ to obtain a PDE for $\tilde{w}_y$. Then imposing even further regularity and compatibility conditions, the classical result cited above yields existence and boundedness of $\tilde{w}_{y}^s$.

The uniform boundedness assumptions (3.1) might appear restrictive. However, as the bounds can be chosen arbitrarily large or small, this restriction is of little practical relevance. A simple model satisfying the above assumptions is a geometric Brownian motion which is stopped as soon as it hits $K^{-1}$ or $K$ (for some large $K$). The corresponding local volatility function $\bar{\sigma}(t, s)$ would be a suitable constant on $[0, T] \times (K^{-1}, K)$ and equal to zero on $[0, T] \times \{K^{-1}, K\}$, in accordance with (3.9). Finally, the option payoff has to be regular enough, so that (iii) holds and the arguments outlined in Remark 3.2 go through.

---

23 Hölder continuity uniformly on $(0, T) \times (K^{-1}, K)$ suffices for this step.
3.2 Main result

We are now in a position to state our main result. A possible choice for the sets of trading strategies $\mathcal{A}$ and volatility scenarios $\mathcal{V}$ is provided in Theorem 3.7 below.

**Theorem 3.3.** Suppose that Assumption 3.1 holds and define the stopping time
$$
\tau := \inf\{t \in [0, T] : |Y_t - y_0| \geq 1\} \wedge T.
$$
For each $\psi > 0$, define the strategy $\theta^\psi = (\theta^\psi_t)_{t \in [0, T]}$ and the volatility $\sigma^\psi = (\sigma^\psi_t)_{t \in [0, T]}$ by\(^{24}\)

$$
\begin{align*}
\theta^\psi_t &= \bar{\Delta}_t + \tilde{\theta}(t, S_t, Y_t)1_{(t < \tau)}\psi, \\
\sigma^\psi_t &= \bar{\sigma}(t, S_t) + \tilde{\sigma}(t, S_t, Y_t)\psi,
\end{align*}
$$

where $\bar{\theta}$ and $\bar{\sigma}$ are given in (3.6)–(3.7). Moreover, let $\mathcal{A}$ and $\mathcal{V}$ be sets of progressively measurable processes such that for all $\psi > 0$ small enough, $(\theta^\psi, \sigma^\psi) \in \mathcal{A} \times \mathcal{V} \subset \mathcal{Z}$. Then, as $\psi \downarrow 0$:

$$
v(\psi) = \sup_{\theta \in \mathcal{A}} \inf_{\sigma \in \mathcal{V}} \inf_{P \in \mathcal{P}(\theta, \sigma)} J^\psi(\sigma, P) \\
= U(y_0) - U'(y_0)\tilde{w}(0, s_0, y_0)\psi + U'(y_0)\tilde{w}(0, s_0, y_0)\psi^2 + o(\psi^2)
$$

where $\tilde{w}$ and $\tilde{w}$ are the solutions to the PDEs (3.2)–(3.3). In particular, $\theta^\psi$ is an optimal strategy at the next-to-leading order $O(\psi^2)$ among all strategies in $\mathcal{A}$, and $\sigma^\psi$ is a “worst-case” volatility at the next-to-leading order $O(\psi^2)$ among all volatilities in $\mathcal{V}$.

**Remark 3.4.** An inspection of the proof of Theorem 3.3 shows that the strategy adjustment $\tilde{\theta}$ only affects the performance of the hedge at the next-to-leading order $O(\psi^2)$. Put differently, the delta hedge $\bar{\Delta}$ is already optimal at the leading order $O(\psi)$. Indeed, with controls of the form $\bar{\Delta}_t + \bar{\theta}_t\psi$ and $\bar{\sigma}(t, S_t) + \bar{\sigma}_t\psi$ both the drift and the diffusion coefficient of your P&L are of order $O(\psi)$ (cf. the dynamics of $Y$ in (2.10)). A formal Taylor expansion therefore suggests that the drift dominates the leading order $O(\psi)$ of the value expansion. As your choice of trading strategy only affects the diffusion coefficient while the drift coefficient is determined by the actual volatility, the strategy adjustment therefore only becomes visible at the order $O(\psi^2)$.

The lengthy proof of Theorem 3.3 is postponed to Section 5.1. Here, we first discuss the asymptotic formulas for the value $v(\psi)$ and the corresponding hedging strategy. The first-order terms in the value expansion and the hedge are both determined by the function $\tilde{w}$. Its Feynman–Kac representation in turn allows to identify the main drivers of the hedging problem with small uncertainty aversion:

\(^{24}\)Strictly speaking, $\theta^\psi$ and $\sigma^\psi$ have to be defined for every $\omega \in \Omega$, even those for which $S$ or $Y$ exceeds the bounds (3.1). Outside these bounds, however, the functions $\bar{\Delta}_t, \bar{\theta}, \bar{\sigma}, \bar{\sigma}$ are not defined. As we only consider measures $P$ such that (3.1) holds, we do not make explicit the corresponding straightforward adjustments, which would only hamper readability.
Proposition 3.5 (Feynman–Kac representation). Suppose that Assumption 3.1 holds. Moreover, assume that for each \( t \in [0, T] \), \( \bar{\sigma}(t, \cdot) \) is uniformly continuous on \((K^{-1}, K)\). Then for each \( t \in [0, T] \), \( s \in [K^{-1}, K] \), and \( y \in (y_l, y_u) \),

\[
\tilde{w}(t, s, y) = E^{t,s}\left[ \int_t^T \frac{(\bar{\sigma}(u, S_u)S_u^2\bar{V}_{ss}(u, S_u))^2}{2f''(u, S_u, y; \bar{\sigma}(u, S_u))} \, du \right]
\]

(3.14)

where the expectation is computed under a measure such that \( S \) has the reference dynamics \( dS_u = S_u\bar{\sigma}(u, S_u) \, dW_u \) and starts in \( S_t = s \).

We postpone the proof of Proposition 3.5 to Section 5.2. From the Feynman–Kac representation (3.14) we see that if \( f'' \) is constant, \( \tilde{w} \) measures the expected volatility-weighted cash gamma accumulated over the remaining lifetime of the option. In particular, \( \tilde{w} \) is high whenever it is likely that the stock price will spend a significant amount of time in zones where the option’s cash gamma is high. The volatility weighting means that the cash gamma accrues in “business time” \( \bar{\sigma}(t, S_t)^2 \, dt \), i.e., its effect is amplified in turbulent markets.

Discussion of the almost optimal strategy. The almost optimal strategy \( \theta^\psi \) from Theorem 3.3 has the form

\[
\theta^\psi_t = \bar{\Delta}_t + \left( \bar{w}_s(t, S_t, Y_t) + \frac{U'(Y_t)}{U''(Y_t)} \bar{w}_{sy}(t, S_t, Y_t) \right) 1_{\{t < \tau\}} \psi.
\]

The first summand is simply the delta hedge in the reference model, whereas the second term is a strategy adjustment accounting for volatility uncertainty. The strategy adjustment is only active as long as the P&L process does not deviate too much from its initial value. This is a technical modification that ensures that your P&L stays within a bounded interval under any volatility scenario.\(^{25}\) To understand the substantial features of the strategy adjustment, let us first focus on the case \( \bar{w}_{sy} = 0 \).\(^{26}\) Then the strategy adjustment does not depend on your current P&L, nor on the shape of your utility function, and is simply \( \bar{w}_s(t, S_t) \psi \). Recall that a high (low) \( \tilde{w} \) is associated with a big (small) loss in value due to volatility uncertainty. Suppose that at time \( t \), \( \bar{w}_s(t, S_t) \) is positive. Then an increase (decrease) in the stock price makes your position more (less) vulnerable to volatility misspecification. But as the almost optimal strategy holds \( \bar{w}_s(t, S_t) \psi \) more shares than the pure delta hedge, the higher vulnerability to volatility misspecification in the case of a rising stock price is compensated by an extra profit. Conversely, in the case of a falling stock price, the strategy adjustment leads to a loss compared to the pure delta hedge, which is compensated by the fact that the stock price has moved towards a zone of less vulnerability to volatility misspecification. To summarise, the almost optimal strategy hedges against movements of the stock price into zones of high vulnerability to volatility misspecification. Recalling that

\(^{25}\)Mutatis mutandis, the threshold 1 in the definition of the stopping time \( \tau \) can be replaced by any other constant. The same modification also appears in the asymptotic analysis of models with transaction costs [100].

\(^{26}\)A sufficient condition for \( \bar{w}_{sy} = 0 \) is that \( f''(t, s, y; \bar{\sigma}) \) does not depend on \( y \).
the magnitude of $\tilde{w}$ is mainly determined by the option’s cash gamma, the almost optimal strategy can thus be seen as a hedge against movements of the stock price into zones of high cash gamma.

Now, let us consider the case $\tilde{w}_{sy} \neq 0$. Suppose that the penalty function is of the form $f(t, s, y; \varsigma) = g(t, s; \varsigma)/h(y)$ for suitable functions $g$ and $h > 0$. The previous paragraph corresponds to a constant $h$. Now, $\tilde{w}$ has the Feynman–Kac representation

$$\tilde{w}(t, s, y) = h(y)E^{t,s} \left[ \int_t^T \frac{(\bar{\sigma}(u, S_u)S_u^2\bar{V}_ss(u, S_u))^2}{2g''(u, S_u; \bar{\sigma}(u, S_u))} \, du \right].$$

We see that $h(y)$ acts as an amplifier of your uncertainty aversion. A decreasing (increasing) $h$ can be interpreted as decreasing (increasing) uncertainty aversion (with respect to your P&L), similar to the well-known notions of increasing or decreasing (absolute) risk aversion. Suppose that $h$ is strictly decreasing. If $\tilde{w}_s(t, S_t, Y_t) > 0$, then $\tilde{w}_{sy}(t, S_t, Y_t) < 0$ and the strategy adjustment

$$\left( \tilde{w}_s(t, S_t, Y_t) + \frac{U'(Y_t)}{U''(Y_t)} \tilde{w}_{sy}(t, S_t, Y_t) \right) 1_{\{t<\tau\}}$$

is larger than in the case of $\tilde{w}_{sy} = 0$. As above, this leads to a profit if the stock rises into zones of high $\tilde{w}$ and to a loss if the stock falls into zones of small $\tilde{w}$. Anticipating a decrease in your P&L and therefore an increase in your uncertainty aversion, you want to avoid zones of high vulnerability to volatility misspecification even more than if $h$ was a constant. Therefore, your strategy adjustment is even larger than in the case of constant $h$.

**Indifference prices.** An *indifference bid price* (indifference ask price) (for an option) is a price at which you are indifferent between keeping your current position and buying (selling) the option for that price. We emphasise that indifference prices typically depend on your wealth, the stock price, and on your current asset allocation. We suppose here that you are currently flat, i.e., your current position in options is zero. Then you might demand a premium for adding risk to your portfolio by selling the option short. Throughout, $x_0$ denotes your initial capital (before buying or selling any options) and $V = V(0, s_0)$ is the initial reference value of the option. To ease notation, we write $\tilde{w} = \tilde{w}(0, s_0, x_0)$ and let $v(y_0; \psi)$ denote the value of our hedging problem corresponding to initial P&L $y_0$.

If you are flat and decide to sell the option for a price $p_a(\psi)$, then your initial P&L is $x_0 + p_a(\psi) - \bar{V}$. Therefore, the equation determining the indifference price $p_a(\psi)$ reads as follows:

$$U(x_0) = v(x_0 + p_a(\psi) - \bar{V}; \psi).$$

Using the expansion of $v$ from Theorem 3.3, straightforward computations yield\(^{27}\)

$$p_a(\psi) = \bar{V} + \tilde{w} \psi + o(\psi). \quad (3.15)$$

\(^{27}\)A second-order expansion for the ask price can also be obtained, but does not offer much additional insight.
If you are flat, your ask price hence exceeds the reference value $\tilde{V}$ of the option by a premium $\tilde{w}\psi + o(\psi)$. We therefore call $\tilde{w}$ the \textit{cash equivalent of small uncertainty aversion}; at the leading order, it is the (normalised) premium (or discount if you are the buyer, see the next paragraph) over the reference value that you demand for assuming a position that is vulnerable to volatility misspecification.

If you are flat and decide to buy the option for a price $p_b(\psi)$, your initial P&L is $x_0 - p_b(\psi) + \tilde{V}$. Buying the option is the same as selling the negative of the option. Moreover, the cash equivalents of the option and its negative coincide because the cash gamma enters the source term (3.4) of the PDE (3.2) for $\tilde{w}$ as a square.\(^{28}\) Thus, $p_b(\psi)$ is determined by

$$U(x_0) = v(x_0 - p_b(\psi) + \tilde{V}; \psi),$$

which yields

$$p_b(\psi) = \tilde{V} - \tilde{w}\psi + o(\psi). \quad (3.16)$$

In analogy to the ask price in (3.15), you demand a discount $\tilde{w}\psi + o(\psi)$ on the reference value $\tilde{V}$ to buy the option. Comparing (3.16) with (3.15), we see that starting from a flat position, your bid-ask spread due to uncertainty aversion is $2\tilde{w}\psi + o(\psi)$.

\textbf{Remark 3.6.} The above results can be formally linked to the uncertain volatility model as follows. Consider the Black–Scholes–Barenblatt equation

$$V^\psi_t(t,s) + \sup_{\varsigma \in \lambda(t,s),\Lambda(t,s)} \frac{1}{2} (\tilde{\sigma}(t,s) + \psi\varsigma)^2 s^2 V^\psi_{ss} = 0, \quad (3.17)$$

$$V^\psi(T,s) = G(s),$$

where $\psi > 0$ is a (small) parameter and $\lambda \leq 0 \leq \Lambda$ are suitable functions. This equation corresponds to the problem of finding the smallest initial capital that allows to superreplicate the option $G(S_T)$ for any volatility process evolving in the random interval

$$[\tilde{\sigma}(t,S_t) + \psi\lambda(t,S_t), \tilde{\sigma}(t,S_t) + \psi\Lambda(t,S_t)];$$

see [116].\(^{29}\) As Lyons [116, Section 5] points out, the solution $V^\psi$ to the Black–Scholes–Barenblatt PDE (3.17) has the formal asymptotic expansion

$$V^\psi(t,s) = \tilde{V}(t,s) + \tilde{V}(t,s)\psi + o(\psi) \quad (\psi \downarrow 0), \quad (3.18)$$

\(^{28}\)This symmetry generally breaks down for the second-order term $\hat{w}$; cf. the corresponding source term (3.5). Hence, for a second-order expansion of the indifference bid price, one has to use the $\tilde{w}$ corresponding to the negative of the option.

\(^{29}\)This specification is a special case of the general “random $G$-expectation” [133].
where $\bar{V}$ is the solution to the Black–Scholes PDE (2.3) and $\tilde{V}$ solves the following linear parabolic PDE with source term:

$$
\tilde{V}_t(t,s) + \frac{1}{2} \tilde{\sigma}(t,s)^2 s^2 \tilde{V}_{ss}(t,s) + \sup_{\varsigma \in [\lambda(t,s), \Lambda(t,s)]} \varsigma \tilde{\sigma}(t,s)^2 s^2 \tilde{V}_s(t,s) = 0,
$$

The expansion (3.18) has been proved by Fouque and Ren [59] for the special case where $\bar{\sigma}$ is a constant, $\Lambda \equiv 1$, and $\lambda \equiv 0$.

In view of (3.18), $\tilde{V}$ can be interpreted as the (normalised) leading-order premium over the reference value that an agent with infinite risk aversion but small volatility band demands as a compensation for assuming a position that is vulnerable to volatility misspecification. This is analogous to the interpretation of $\tilde{w}$ as the cash equivalent of small uncertainty aversion. More specifically, if $f''$ does not depend on $y$ and if we choose

$$
\Lambda(t,s) = -\lambda(t,s) = \frac{1}{2} \bar{\sigma}(t,s) \text{sgn}(\bar{V}_{ss}(t,s)) = \frac{\bar{\sigma}(t,s) s^2 |\bar{V}_{ss}(t,s)|}{2f''(t,s; \bar{\sigma}(t,s))},
$$

then the PDE (3.19) for $\tilde{V}$ reduces to the PDE (3.2) for $\tilde{w}$. Moreover, in this case, our almost optimal strategy $\theta_\psi = \tilde{V}_s(t,s) + \tilde{w}_s(t,s) \psi$ coincides with the delta hedge corresponding to the expansion (3.18). Thus, our results are formally equivalent to those obtained for an infinitely risk averse agent who, however, uses a random and time-dependent volatility band that depends on both the option (through its cash gamma) and her uncertainty aversion (through $f''$).

### 3.3 Existence of probability scenarios

Our main result, Theorem 3.3, is formulated for abstract sets of trading strategies $\mathcal{A}$ and volatility scenarios $\mathcal{V}$, which are (i) large enough to contain the almost optimal strategies $\theta_\psi$ and volatilities $\sigma_\psi$ and (ii) small enough to ensure that a corresponding measure $P \in \mathfrak{P}(\theta, \sigma)$ exists and all the technical prerequisites for our verification theorem are satisfied. In this section, we propose concrete choices $\mathcal{A}_0$ and $\mathcal{V}_0$ that meet these requirements.

To this end, let $\mathcal{A}_0$ denote the set of all real-valued processes $\theta = (\theta_t)_{t \in [0,T]}$ of the form

$$
\theta_t = \tilde{V}_s(t,S_t) + \bar{\theta}_t 1_{t<\tau},
$$

where $\bar{\theta}$ is a bounded, progressively measurable process such that $\bar{\theta}_t : \Omega \to \mathbb{R}$ is continuous for each $t \in [0,T]$, and the stopping time $\tau$ is defined as in Theorem 3.3. In words, every strategy in $\mathcal{A}_0$ has to fall back to the reference delta hedge as soon as the corresponding P&L process $Y$ deviates too far from its initial bounds.

\[^{[116]}\text{Imposes bounds on the instantaneous variance of prices instead of the volatility of returns. Hence, the PDE for } \tilde{V} \text{ there looks slightly different. The PDE presented here is a slight generalisation of the one derived in [59].}\]
value. Prior to the switch, however, the strategy can deviate from the reference hedge in any possibly path-dependent way.

Next, denote by $V_0$ the set of all real-valued processes $\sigma = (\sigma_t)_{t \in [0,T]}$ of the form

$$\sigma_t = \hat{\sigma}(t, S_t, Y_t) 1_{\{S_t \in (K^{-1}, K)\}}, \quad (3.21)$$

where $\hat{\sigma} : [0, T] \times \mathbb{R}^2 \to [0, K]$ is a (globally) Lipschitz continuous function. Hence, for any volatility process in $V_0$, once $S$ leaves $(K^{-1}, K)$, its volatility vanishes and the stock price freezes.

The following theorem shows that under suitable regularity assumptions, $A_0$ and $V_0$ satisfy the assumption in Theorem 3.3:

$$(\theta^\psi, \sigma^\psi) \in A_0 \times V_0 \subset Z \quad \text{for} \ \psi > 0 \ \text{small enough.}$$

Let us emphasise, however, that $\theta^\psi$ and $\sigma^\psi$ remain almost optimal also in any larger classes $A \supset A_0$ and $V \supset V_0$ of trading strategies and volatility scenarios satisfying $A \times V \subset Z$.

**Theorem 3.7.** Suppose that Assumption 3.1 holds, that $\bar{V}_s$, $\bar{V}_{ss}$, and $\bar{\sigma}$ are Lipschitz continuous on $(0, T) \times (K^{-1}, K)$, and that $y_l < y_0 - 2 - \frac{1}{2} K^3 T$ and $y_u > y_0 + 2 + \frac{1}{2} K^3 T$. Then $A_0 \times V_0 \subset Z$, i.e., for any $(\theta, \sigma) \in A_0 \times V_0$, there is a probability measure $P$ on $(\Omega, F)$ such that $(\theta, \sigma, P) \in \mathcal{G}$. If in addition, $f''$ is Lipschitz continuous on $[0, T] \times [K^{-1}, K] \times (y_l, y_u) \times [0, K]$, then $(\theta^\psi, \sigma^\psi) \in A_0 \times V_0$ for all $\psi > 0$ small enough.

The proof of Theorem 3.7 is provided in Section 5.3.

### 4 Heuristics and extensions

In practice, you will rarely need to hedge a single option, but rather a whole book of different contracts. For instance, your book may consist of call and put options at various strikes and maturities and miscellaneous exotic options like Asian options, barrier options, lookback options, or options on the realised variance of the stock returns. Moreover, some options like calls and puts at standardised maturities and with strikes close to the current price of the underlying might be liquidly traded and thus be available as additional hedging instruments for the non-traded options in your book.

Such more involved problems can also be tackled with the methodology of this chapter. Since the corresponding rigorous verification would become even more technical, we only develop these extensions on a heuristic level here. The resulting formulas explain how the sensitivities of an option portfolio affect the cash equivalent of small uncertainty aversion for practically relevant cases. We start with an informal derivation of Theorem 3.3, thereby providing some intuition for the rigorous proof in Section 5.1. Subsequently, we adapt the general procedure to more complicated settings including exotic options of Asian-, lookback-, or...
II.4 Heuristics and extensions

barrier-type, as well as options on the realised variance. We also explain how to
deal with option portfolios as well as with static hedging using vanilla options,
and discuss how this allows to interpret our “cash equivalent” as a “measure of
model uncertainty” in the sense of Cont [33].

4.1 General procedure and the case of Theorem 3.3

Our starting point is the dynamic programming approach to two-player, zero-sum
stochastic differential games. The central idea is to find an asymptotic solution
to the Hamilton–Jacobi–Bellman–Isaacs (henceforth HJBI) equation associated
to the hedging problem and corresponding almost optimal controls. For the con-
venience of the reader, we provide a derivation of the HJBI equation which starts
from a general sufficient criterion for optimality (Proposition 4.1) known as principle of optimality [38] or martingale optimality principle [142, V.15]; see also
[149, Proposition 4.1] for a version of the martingale optimality principle in the
context of a zero-sum game. After that, we explain how the HJBI equation to-
gether with an appropriate ansatz can be used to derive candidates for the value
and the almost optimal controls of our hedging problems. Next, the formulas
provided in Section 3 for the specific case of hedging a single vanilla option are
recovered. Finally, we summarise the general procedure that is used in the re-
mainiing subsections to derive the corresponding candidates also in more general
settings.

A sufficient criterion for optimality. Consider an optimisation problem of
the form

\[ v := \sup_{\theta \in \mathcal{A}} \inf_{\sigma \in \mathcal{V}} \mathbb{E}^{\theta,\sigma} [N_T], \]  

where \( \mathcal{A} \) and \( \mathcal{V} \) are sets of admissible controls, \( N \) is a sufficiently integrable
process and for each \((\theta, \sigma) \in \mathcal{A} \times \mathcal{V} \), \( \mathbb{E}^{\theta,\sigma} [\cdot] \) denotes the expectation under a
given measure \( P^{\theta,\sigma} \).

Proposition 4.1 (Martingale optimality principle). If there is a pair \((\theta^*, \sigma^*) \in
\mathcal{A} \times \mathcal{V} \) such that

(i) for each \( \theta \in \mathcal{A} \), \( N \) is a supermartingale under \( P^{\theta,\sigma^*} \),

(ii) for each \( \sigma \in \mathcal{V} \), \( N \) is a submartingale under \( P^{\theta^*,\sigma} \),

then

\[ N_0 = \mathbb{E}^{\theta^*,\sigma^*} [N_T] = \sup_{\theta \in \mathcal{A}} \inf_{\sigma \in \mathcal{V}} \mathbb{E}^{\theta,\sigma} [N_T] = \inf_{\sigma \in \mathcal{V}} \sup_{\theta \in \mathcal{A}} \mathbb{E}^{\theta,\sigma} [N_T]. \]  

In particular, \( \theta^* \) and \( \sigma^* \) are optimal controls for (4.1) with value \( v = N_0 \).
Proof. As $N$ is a martingale under $P_{\theta^*,\sigma^*}$ by (i) and (ii), the first equality in (4.2) is clear. Moreover, (i) and (ii) imply
\[
\sup_{\theta \in \mathcal{A}} E_{\theta,\sigma^*}^\theta [N_T] \leq N_0 \leq \inf_{\sigma \in \mathcal{V}} E_{\theta^*,\sigma}^\theta [N_T].
\]
Thus,
\[
N_0 \leq \inf_{\sigma \in \mathcal{V}} E_{\theta^*,\sigma}^\theta [N_T] \leq \sup_{\theta \in \mathcal{A}} \inf_{\sigma \in \mathcal{V}} E_{\theta,\sigma}^\theta [N_T] \leq \sup_{\theta \in \mathcal{A}} E_{\theta^*,\sigma^*}^\theta [N_T] \leq N_0,
\]
and the remaining equalities in (4.2) follow.

Derivation of the Hamilton–Jacobi–Bellman–Isaacs equation. Suppose that we are given controls $\theta^*$ and $\sigma^*$ satisfying the conditions of Proposition 4.1. In addition, assume that the dynamics of $N$ under $P_{\theta,\sigma}$ are of the form
\[
dN_t = dM_{\theta,\sigma}^\theta + \nu_{\theta,\sigma}^\theta dt
\]
for some $P_{\theta,\sigma}$-martingale $M_{\theta,\sigma}^\theta$ and a drift rate $\nu_{\theta,\sigma}^\theta$. Then the conditions (i) and (ii) of Proposition 4.1 imply that the drift rates $\nu_{\theta,\sigma}^\theta$ satisfy
\[
\sup_{\theta \in \mathcal{A}} \nu_{\theta,\sigma^*}^\theta \leq 0 \leq \inf_{\sigma \in \mathcal{V}} \nu_{\theta^*,\sigma}^\theta.
\]
Using these inequalities, we find
\[
0 \leq \inf_{\sigma} \nu_{t}^{\theta^*,\sigma^*} \leq \nu_{t}^{\theta^*,\sigma^*} \leq \sup_{\theta} \nu_{t}^{\theta,\sigma^*} \leq 0,
\]
as well as
\[
0 \leq \inf_{\sigma} \nu_{t}^{\theta^*,\sigma} \leq \sup_{\theta} \inf_{\sigma} \nu_{t}^{\theta,\sigma} \leq \inf_{\sigma} \sup_{\theta} \nu_{t}^{\theta,\sigma} \leq \sup_{\theta} \nu_{t}^{\theta,\sigma^*} \leq 0.
\]
Hence, we have equality everywhere and, in particular:
\[
\nu_{t}^{\theta^*,\sigma^*} = \sup_{\theta} \inf_{\sigma} \nu_{t}^{\theta,\sigma} = \inf_{\sigma} \sup_{\theta} \nu_{t}^{\theta,\sigma} = 0. \quad (4.3)
\]
To obtain the drift rates more explicitly, let us now assume that there are sufficiently regular functions $g$ and $w$, and a state process $Z$ that is a (multidimensional) Itô diffusion under each $P_{\theta,\sigma}$, such that for any $(\theta, \sigma) \in \mathcal{A} \times \mathcal{V},
\[
N_t = \int_0^t g(u, Z_u; \theta_u, \sigma_u) \, du + w(t, Z_t) \quad P_{\theta,\sigma}^\theta \text{-a.s.} \quad (4.4)
\]
This form is motivated by the structure of our hedging problem, cf. (4.7) below. More concretely, if $Z$ has dynamics
\[
dZ_t = b(t, Z_t; \theta_t, \sigma_t) \, dt + a(t, Z_t; \theta_t, \sigma_t) \, dW_t
\]
under \(P^{\theta,\sigma}\), then applying Itô's formula under each \(P^{\theta,\sigma}\) to the right-hand side of (4.4) yields

\[
\nu_{t}^{\theta,\sigma} = w_{t}(t, Z_{t}) + H(t, Z_{t}, \nabla w(t, Z_{t}), D^{2} w(t, Z_{t}); \theta_{t}, \sigma_{t}) \tag{4.5}
\]

where \(\nabla w\) and \(D^{2} w\) denote the gradient and the Hessian of \(w(t, z)\) with respect to the \(z\) variable, respectively, and

\[
H(t, z, p, A; \vartheta, \varsigma) := g(t, z; \vartheta, \varsigma) + b(t, z; \vartheta, \varsigma) \cdot p + \frac{1}{2} \text{Trace} \left((aa^{\top})(t, z; \vartheta, \varsigma) A\right).
\]

Substituting (4.5) into (4.3) yields

\[
w_{t}(t, Z_{t}) + \sup_{\vartheta} \inf_{\varsigma} H(t, Z_{t}, \nabla w(t, Z_{t}), D^{2} w(t, Z_{t}); \theta_{t}, \sigma_{t}) = 0.
\]

A sufficient condition for this to hold is that \(w\) satisfies the PDE

\[
w_{t} + \sup_{\vartheta} \inf_{\varsigma} H(t, z, \nabla w, D^{2} w; \vartheta, \varsigma) = 0, \tag{4.6}
\]

which is called the Hamilton–Jacobi–Bellman–Isaacs equation. In addition, candidates for the optimal controls are given by \(\vartheta^{*}(t, Z_{t})\) and \(\varsigma^{*}(t, Z_{t})\), where \(\vartheta^{*}\) and \(\varsigma^{*}\) are the saddle points of the HJBI equation (4.6) in the sense that, for each \((t, z)\):

\[
H(t, z, \nabla w(t, z), D^{2} w(t, z); \vartheta^{*}(t, z), \varsigma^{*}(t, z)) = \sup_{\vartheta} \inf_{\varsigma} H(t, z, \nabla w(t, z), D^{2} w(t, z); \vartheta, \varsigma).
\]

Also note that \(N_{0} = w(0, Z_{0})\) from (4.4), so that \(w(0, Z_{0})\) is a candidate for the value of the optimisation problem (4.1) by Proposition 4.2. Therefore, the function \(w\) is also called the value function of the optimisation problem (4.1).

The above derivation of the HJBI equation is of course merely formal. To prove rigorously that the candidate controls are indeed optimal, a rigorous verification theorem (using for instance the sufficient conditions from Proposition 4.1) is needed.

**Ansatz for asymptotic solution.** For each \(\psi > 0\), consider the hedging problem

\[
v(\psi) := \sup_{\vartheta \in A} \inf_{\sigma \in V} E^{\theta,\sigma} \left[ \frac{1}{\psi} \int_{0}^{T} U'(Y_{t}) f(t, S_{t}, Y_{t}; \sigma_{t}) \, dt + U(Y_{T}) \right], \tag{4.7}
\]

\[\text{31} \text{We would obtain the same results if we interchanged the order of the infimum and the supremum. In the language of two-player, zero-sum stochastic differential games, this indicates that the game “has a value”.}
\]

\[\text{32} \text{For the heuristic derivation in this section, we tacitly assume that for each \((\theta, \sigma), P^{\theta,\sigma}\) attains the infimum in (2.14), so that the additional infimum over measures in (2.14) disappears.}\]
where under each $P_{\theta, \sigma}$, $S$ and $Y$ have dynamics of the form
\begin{align*}
    dS_t &= S_t \sigma_t \, dW_t, \\
    dY_t &= (\theta_t - \bar{\Delta}(t, S_t)) \, dS_t + b(t, S_t; \sigma_t) \, dt,
\end{align*}
for some functions $\bar{\Delta}$ and $b$. The drift rate $b$ is further required to satisfy
\begin{align*}
b(t, s; \bar{\sigma}(t, s)) = 0
\end{align*}
as the theoretical P&L should be “locally drift-less” whenever the true volatility $\sigma_t$ coincides with the reference volatility $\bar{\sigma}(t, S_t)$.

To derive the HJBI equation for (4.7), we first recast the expression inside the expectation of (4.7) into the form (4.4). To this end, let $w(t, s, y)$ be a function satisfying the terminal condition $w(T, s, y) = U(y)$. Then the HJBI equation corresponding to (4.7) reads as follows:
\begin{align*}
w_t + \sup_{\varphi} \inf_{\varsigma} \left\{ \frac{1}{\psi} U' f(\varsigma) + b(\varsigma) w_y + \frac{1}{2} \varsigma^2 s^2 (w_{ss} + 2(\bar{\sigma} - \bar{\Delta}) w_{sy} + (\bar{\sigma} - \bar{\Delta})^2 w_{yy}) \right\} = 0.
\end{align*}
(We suppress the arguments $(t, s, y)$ to ease notation.) Explicit solutions to this equation are typically not available. Therefore, our goal is to obtain an asymptotic solution for small values of the uncertainty aversion parameter $\psi$. More precisely, we want to find strategies $\theta^\psi$, volatilities $\sigma^\psi$, and correction terms $\tilde{w}$ and $\tilde{\sigma}$ such that
\begin{align*}
v(\psi) &= U(y_0) - U'(y_0) \tilde{w}(0, s_0, y_0) \psi + U'(y_0) \tilde{w}(0, s_0, y_0) \psi^2 + o(\psi^2) \\
&= E^\theta^\psi, \sigma^\psi \left[ \frac{1}{\psi} \int_0^T U'(Y_t) f(t, S_t, Y_t; \sigma^\psi_t) \, dt + U(Y_T) \right] + o(\psi^2).
\end{align*}
The first equality is a second-order expansion of the value as a function of the uncertainty aversion parameter $\psi$. The second equality says that the strategy $\theta^\psi$ and the volatility $\sigma^\psi$ are optimal controls at the next-to-leading order $O(\psi^2)$.

We next use the HJBI equation and an appropriate ansatz to derive candidates for the asymptotic expansion of the value function and the almost optimal controls. Recall from Section 2.1 that vanilla options can be perfectly replicated in the reference local volatility model. Hence, without model uncertainty, it is optimal to use the delta hedge for the option, and the value of the hedging problem is simply the utility from the initial P&L; cf. Lemma 2.1. This motivates the following ansatz for the asymptotic expansion of the value function and the almost optimal controls of (4.7):
\begin{align*}
w^\psi(t, s, y) &= U(y) - U'(y) \tilde{w}(t, s, y) \psi, \\
\sigma^\psi(t, s, y) &= \bar{\sigma}(t, s) + \tilde{\sigma}(t, s, y) \psi, \\
\theta^\psi(t, s, y) &= \bar{\Delta}(t, s) + \tilde{\theta}(t, s, y) \psi,
\end{align*}
33This covers most of the specific choices that are dealt with in the following subsections, except for additional state variables needed for some exotic options in Section 4.2. To explain the general procedure, we first focus here on the easiest case with just two state variables, the stock price $S$ and the P&L process $Y$. 

for functions $\tilde{w}, \tilde{\sigma}, \tilde{\theta}$ to be determined. With this ansatz, the HJBI equation yields equations for $\tilde{w}, \tilde{\sigma}, \tilde{\theta}$. Indeed, substituting (4.10)–(4.12) into (4.9) (we assume that $\theta^o$ and $\sigma^\psi$ are point-wise saddle points for the $\sup \inf$), using also the Taylor expansions $f(\zeta) \approx \frac{1}{2} f''(\zeta) (\zeta - \sigma)^2$ (because of (2.12)) and $b(\zeta) \approx b'(\zeta) (\zeta - \sigma)$ (because of the assumption that $b(\bar{\sigma}) = 0$; $b'$ denotes the partial derivative with respect to $\zeta$), and ordering by powers of $\psi$, we obtain

$$-U' \times \left( \tilde{w}_t - \frac{1}{2} f''(\sigma) \sigma^2 - b'(\sigma) \bar{\sigma} + \frac{1}{2} \sigma^2 s^2 \tilde{w}_{ss} \right) \psi + o(\psi) = 0. \tag{4.13}$$

Our candidate for $\bar{\sigma}$ is the minimiser of the $O(\psi)$ term. Solving the first-order condition $f''(\bar{\sigma}) + b'(\bar{\sigma}) = 0$ for $\bar{\sigma}$ yields

$$\bar{\sigma} = \frac{-b'(\sigma)}{f''(\sigma)}. \tag{4.14}$$

Plugging this candidate back into the $O(\psi)$ term in (4.13) and setting the result equal to zero yields a PDE for the cash equivalent $\tilde{w}$:

$$\tilde{w}_t + \frac{1}{2} \sigma^2 s^2 \tilde{w}_{ss} + \frac{b'(\sigma)}{2 f''(\sigma)} = 0. \tag{4.15}$$

To find the candidate for the optimal strategy, in view of the HJBI equation (4.9), we only need to maximise

$$2(\vartheta - \Delta) w_{sy}^\psi + (\vartheta - \Delta)^2 w_{yy}^\psi. \tag{4.16}$$

Substituting the ansatz (4.10) and (4.12) into (4.16), we find

$$\left( 2 \bar{\theta} \frac{\partial^2}{\partial s \partial y} (-U' \tilde{w}) + \bar{\theta}^2 U'' \right) \psi^2 + o(\psi^2).$$

The $O(\psi^2)$ term simplifies to $-2\bar{\theta} (U' \tilde{w}_{sy} + U''' \tilde{w}_s) + \bar{\theta}^2 U''$ and the maximiser of this quadratic equation in $\bar{\theta}$ is

$$\bar{\theta} = \tilde{w}_s + \frac{U'}{U''} \tilde{w}_{sy}. \tag{4.17}$$

Finally, a PDE for the second-order correction term $\tilde{w}$ can be obtained in the same way, using a second-order ansatz $w^o(t, s, y) = U(y) - U'(y) \tilde{w}(t, s, y) \psi + U''(y) \tilde{w}(t, s, y) \psi^2$ for the value function instead of the first-order ansatz (4.10) (the ansatz for the controls remains the same) and setting the $O(\psi^2)$ term in the expanded HJBI equation, evaluated at the candidate strategy and candidate volatility, equal to zero. We omit the lengthy computations.

**The case of Theorem 3.3.** If a single vanilla option needs to be hedged, the P&L dynamics have been derived in Section 2.1. Comparing the dynamics of $Y$ in (2.10) with the form (4.8) considered above, we find that

$$b(t, s; \zeta) = \frac{1}{2} s^2 \tilde{V}_{ss}(t, s)(\bar{\sigma}(t, s)^2 - \zeta^2). \tag{4.18}$$

Thus, $b'(t, s; \bar{\sigma}(t, s)) = -s^2 \tilde{V}_{ss}(t, s) \bar{\sigma}(t, s)$. Plugging this formula for $b'$ into the formulas for $\bar{\sigma}$ in (4.14) and into the PDE for $\tilde{w}$ in (4.15), we recover exactly (3.7) and (3.2). The formula for $\bar{\theta}$ in (4.17) also coincides with (3.6).
Summary. The general procedure to find the candidate controls and the cash equivalent can be summarised as follows.

(i) Introduce as many state variables (e.g., the running maximum or minimum for lookback options, the running integral over the stock price for Asian options, etc.) as necessary to express the theoretical value of each option as a function of time and these state variables.

(ii) Write down the theoretical P&L process $Y$ corresponding to your option position and your trading strategy. Use Itô’s formula to determine the drift and diffusion coefficients for $Y$.

(iii) Write down the HJBI equation corresponding to the control problem.

(iv) Plug an appropriate ansatz as in (4.10)–(4.11) for the value and the volatility into the HJBI equation and expand the result in the $\psi$ variable. Minimise the $O(\psi)$ term over $\tilde{\sigma}$ to obtain the candidate volatility.

(v) Substitute the candidate for $\tilde{\sigma}$ back into the $O(\psi)$ term of the HJBI to find a PDE for the cash equivalent $\tilde{w}$.

(vi) Plug the ansatz (4.12) for the strategy into the expanded HJBI equation. Maximise the $O(\psi^2)$ term over $\tilde{\theta}$ to obtain the candidate strategy.

In the following subsections, we illustrate this approach in a number of practically relevant applications.

4.2 Some exotic options

In this section, we consider some exotic options whose payoffs depend on the whole path of the stock price and not only on the stock price at maturity. For instance, their payoffs could depend on the average of the stock price path over some time period (Asian options), its maximum or minimum (lookback options), or on the realised variance of the stock returns (e.g., a variance swap). The reference values of such options can still be represented as solutions to a PDE, provided one introduces suitable additional state variables that keep track of the path-dependent features of the contract.

Step (i) is to represent the theoretical value of the exotic option under consideration in the local volatility model as a solution to a PDE. To this end, let us introduce a generic additional state variable $A$ with dynamics of the form

$$dA_t = \alpha(t, S_t, A_t, M_t) \, dt + \beta(t, S_t, A_t, M_t) \, d\langle S \rangle_t$$

$$+ \gamma(t, S_t, A_t, M_t) \, dS_t + \delta(t, S_t, A_t, M_t) \, dM_t,$$

where $M_t := \max_{s \in [0,t]} S_s$ is the running maximum of the stock price. Note that we do not merge the $d\langle S \rangle_t$-term with the $dt$-term as $\langle S \rangle_t$ depends on the true volatility $\sigma$. If you want to value and hedge an option with maturity $T$ and payoff of the form $G(S_T, A_T, M_T)$ in the local volatility model, you can essentially
employ the same approach as in Section 2.1. Indeed, applying Itô’s formula to a sufficiently regular function $\bar{V}(t, s, a, m)$, and assuming that the true dynamics of $S$ are $dS_t = S_t \sigma_t dW_t$, we obtain, dropping most arguments:

$$d\bar{V}(t, S_t, A_t, M_t) = (\bar{V}_s + \gamma \bar{V}_a) dS_t + (\bar{V}_m + \delta \bar{V}_a) dM_t$$

$$+ \left\{ \bar{V}_t + (\alpha + \beta \sigma^2 S_t^2) \bar{V}_a + \frac{1}{2} \sigma^2 S_t^2 (\bar{V}_{ss} + 2\gamma \bar{V}_{sa} + \gamma^2 \bar{V}_{aa}) \right\} dt.$$  \hspace{1cm} (4.19)

Note that if $\sigma_t = \bar{\sigma}(t, S_t)$, i.e., if your local volatility model is correct, and if $\bar{V}$ solves the PDE

$$\bar{V}_t + (\alpha + \beta \bar{\sigma}^2 S_t^2) \bar{V}_a + \frac{1}{2} \sigma^2 S_t^2 (\bar{V}_{ss} + 2\gamma \bar{V}_{sa} + \gamma^2 \bar{V}_{aa}) = 0,$$

$$\bar{V}_m + \delta \bar{V}_a = 0 \text{ whenever } s = m, \hspace{1cm} (4.20)$$

then (4.19) yields, using that $M_t$ only increases when $S_t = M_t$,

$$G(S_T, A_T, M_T) = \bar{V}(T, S_T, A_T, M_T) = \bar{V}(0, S_0, A_0, M_0) + \int_0^T \bar{\Delta}(t, S_t, A_t, M_t) dS_t,$$

where $\bar{\Delta} := \bar{V}_s + \gamma \bar{V}_a$. Hence, in the local volatility model and under these assumptions, the option $G$ can be perfectly replicated by self-financing trading with initial capital $\bar{V}(0, S_0, A_0, M_0)$ and trading strategy $\bar{\Delta}$.

Step (ii) is to write down the P&L process $Y$ and to determine its dynamics in terms of the trading strategy $\theta$ and the true volatility $\sigma$. In analogy to the case of a vanilla option treated in Section 2.1, your theoretical P&L at time $t$ is

$$Y_t = x_0 + \int_0^t \theta_u dS_u - \bar{V}(t, S_t, A_t, M_t).$$  \hspace{1cm} (4.21)

Using the PDE (4.20) to substitute $\bar{V}_t$ in (4.19) and plugging the result into (4.21) yields

$$dY_t = (\theta_t - (\bar{V}_s + \gamma \bar{V}_a)) dS_t + S_t^2 \left( \beta \bar{V}_a + \frac{1}{2} (\bar{V}_{ss} + 2\gamma \bar{V}_{sa} + \gamma^2 \bar{V}_{aa}) \right) (\bar{\sigma}^2 - \sigma_t^2) dt.$$  \hspace{1cm} (4.22)

Taking into account $\bar{\Delta} = \bar{V}_s + \gamma \bar{V}_a$ and setting

$$b(\varsigma) := s^2 (\beta \bar{V}_a + \frac{1}{2} (\bar{V}_{ss} + 2\gamma \bar{V}_{sa} + \gamma^2 \bar{V}_{aa})) (\bar{\sigma}^2 - \varsigma^2),$$

this simplifies to

$$dY_t = (\theta_t - \bar{\Delta}) dS_t + b(\varsigma_t) dt.$$

\footnote{Note that this hedge $\bar{\Delta}$ reflects the option’s sensitivity to price moves in the underlying both directly through $S$ and indirectly through the additional state variable $A$.}
Step (iii) is to write down the HJBI equation. Recall that this boils down to using Itô’s formula to derive the drift rate of the process

\[ N_t := \frac{1}{\psi} \int_0^t U'(Y_i) f(t, S_t, Y_t; \sigma_t) \, dt + w^\psi(t, S_t, Y_t, A_t, M_t). \]

We obtain

\[
\begin{align*}
    w^\psi_t + \sup_{\vartheta} \inf_{\varsigma} \left\{ & \frac{1}{\psi} U' f(\varsigma) + b(\varsigma) w^\psi_y + (\alpha + \beta s^2 \varsigma^2) w^\psi_a \\
    & + \frac{1}{2} \varsigma^2 s^2 \left( w^\psi_{ss} + 2(\vartheta - \bar{\Delta}) w^\psi_{sy} + (\vartheta - \bar{\Delta})^2 w^\psi_{yy} \\
    & \quad + 2\gamma w^\psi_{sa} + 2(\vartheta - \bar{\Delta}) w^\psi_{ya} + \gamma^2 w^\psi_{aa} \right) \right\} = 0.
\end{align*}
\]

(4.22)

In analogy to the valuation PDE (4.20) for the exotic option, we also need that \( w^\psi_m + \delta w^\psi_a = 0 \) whenever \( s = m \), so that the \( dM_t \)-term of \( N \) vanishes as well.

Step (iv) is to plug the ansatz

\[
w^\psi(t, s, y, a, m) = U(y) - U'(y) \tilde{w}(t, s, y, a, m) \psi,
\]

\[
\sigma^\psi(t, s, y, a, m) = \bar{\sigma}(t, s) + \bar{\sigma}(t, s, y, a, m) \psi,
\]

into the HJBI equation (4.22), expand the result in the \( \psi \) variable and minimise the \( O(\psi) \) term over \( \tilde{\sigma} \) to find the candidate for the volatility. The \( O(\psi) \) term in the expansions reads as

\[
- U' \times \left( \tilde{w}_t - \frac{1}{2} \frac{f''(\tilde{\sigma})\tilde{\sigma}^2}{f''(\sigma)} - b'(\tilde{\sigma})\tilde{\sigma} + (\alpha + \beta s^2 \tilde{\sigma}^2)\tilde{w}_s \\
\quad + \frac{1}{2} \sigma^2 s^2 \left( \tilde{w}_{ss} + 2\gamma \tilde{w}_{sa} + \gamma^2 \tilde{w}_{aa} \right) \right) \psi.
\]

Minimising this over \( \tilde{\sigma} \) and using the definition of \( b \), we find the candidate volatility:

\[
\sigma^\psi = \bar{\sigma} + \frac{\tilde{\sigma}}{f''(\sigma)} s^2 \left( 2\bar{V}_a + (\bar{V}_{ss} + 2\gamma \bar{V}_{sa} + \gamma^2 \bar{V}_{aa}) \right) \psi.
\]

(4.23)

In step (v), we plug the candidate for \( \tilde{\sigma} \) back into the \( O(\psi) \) term of the HJBI. Then the PDE for \( \tilde{w} \) obtained by setting the \( O(\psi) \) term equal to zero has the following (formal) Feynman–Kac representation:

\[
\tilde{w}(t, s, y, a, m) = E^{t, s, y, a, m} \left[ \int_t^T \frac{\tilde{\sigma}^2}{2f''(\sigma)} \left( S_u^2 \left( 2\bar{V}_a + (\bar{V}_{ss} + 2\gamma \bar{V}_{sa} + \gamma^2 \bar{V}_{aa}) \right) \right) \, du \right].
\]

(4.24)

Here, the expectation is computed under a measure such that \( S \) has reference dynamics with initial conditions \( S_t = s, Y_t = y, A_t = a, M_t = m \).
Finally, step (vi) is to find the candidate strategy by substituting the ansatz
\[ \theta^\psi(t,s,a,m) = \tilde{\Delta}(t,s,a,m) + \tilde{\theta}(t,s,y,a,m)\psi \]
into the HJBI equation and maximising the \(O(\psi^2)\) term over \(\tilde{\theta}\). After some computations, we find
\[ \theta^\psi = \tilde{\Delta} + \left( \tilde{w}_s + \gamma \tilde{w}_a + \frac{U'}{U''}(\tilde{w}_{sy} + \gamma \tilde{w}_{ya}) \right) \psi. \quad (4.25) \]

Let us discuss these results for some more specific examples:

**Example 4.2** (Asian and lookback options). Suppose the payoff of the option depends on the arithmetic average of the stock price over the period \([0,T]\). In this case, we introduce the state variable \(dA_t = S_t dW_t\) and write the payoff in the form \(G(S_T, A_T)\). For example, the payoff of a floating strike Asian call is \((S_T - A_T)/T\)^+. In the above setting, we thus have \(\alpha(t,s,a,m) = s\) and \(\beta = \gamma = \delta = 0\), and we immediately see that the general formulas (4.23), (4.24), and (4.25) for the candidate controls and the cash equivalent \(\tilde{w}\) all reduce to those derived for a single vanilla option, except that the reference value \(\tilde{V}\) of the option depends on the additional state variable, and hence so do the candidate controls and \(\tilde{w}\). As a consequence, the cash gamma of the option is still the central determinant of the cash equivalent of small uncertainty aversion.

Likewise, for lookback-type options like the floating strike lookback put with payoff \(M_T - S_T\), the formulas for the candidate controls and the cash equivalent \(\tilde{w}\) also essentially reduce to those of a vanilla option.

**Example 4.3** (Options on realised variance). Here, the state variable \(dA_t = \frac{1}{S_t} d\langle S \rangle_t\) tracks the cumulative realised variance of returns in the sense that if the true dynamics of \(S\) are \(dS_t = S_t \bar{\sigma} dW_t\), then \(A_t = \int_0^t \bar{\sigma}^2 u \, du\). For example, the variance swap with strike volatility \(\sigma_{\text{strike}}\) has the payoff \(\frac{1}{T} A_T - \sigma_{\text{strike}}^2\); it pays the difference between the average realised variance over the period \([0,T]\) and a given strike variance \(\sigma_{\text{strike}}^2\). In the above setting, we have \(\beta(t,s,a,m) = s^{-2}\) and \(\alpha = \gamma = \delta = 0\). We then obtain from (4.24) that
\[ \tilde{w}(t,s,y,a) = E^{t,s,y,a}\left[ \int_t^T \frac{(\tilde{\sigma}(2\tilde{V}_a + \bar{\Gamma}_s))^2}{2 f''(\bar{\sigma})} \, du \right]. \quad (4.26) \]

We see that the option’s sensitivity with respect to changes in the realised variance and its cash gamma play symmetric roles here. Whence – unlike for models with discrete trading [70] or transaction costs [101] – the quadratic variation of the reference hedge is generally not the right sufficient statistic here; cf. the following Remark 4.4.

**Remark 4.4.** Suppose that the reference model has dynamics \(dS_t = S_t \bar{\sigma} dW_t\) for some general, possibly path-dependent, volatility process \(\bar{\sigma}\). Moreover, assume that \(\tilde{\theta}\) is a replicating strategy for an option \(G(S_T, A_T)\) in that reference model.
If \( \bar{\sigma}_t = \bar{\sigma}(t, S_t) \) is actually of local volatility type, then \( \bar{\theta}_t = \bar{V}_s(t, S_t, A_t) \) and by Itô’s formula, assuming \( \gamma = 0 \) so that \( A \) is of finite variation,

\[
S_t^2 d(\bar{\theta})_t = S_t^2 d(\bar{V}_s(\cdot, S_\cdot, A_\cdot))_t = (\bar{\sigma}_t S_t^2 \bar{V}_{ss}(t, S_t, A_t))^2 dt.
\]

Thus, in view of the representation of the cash equivalent in (3.14), which essentially also holds for Asian, lookback, and barrier options (cf. (4.31) below), one could be led to expect that the cash equivalent for possibly path-dependent volatilities generalises to

\[
E \left[ \int_0^T S_t^2 \frac{2}{2f''(\bar{\sigma}_t)} d(\bar{\theta})_t \right]. \tag{4.27}
\]

However, in Example 4.3, the replicating strategy of the variance swap in the reference model is \( \bar{\theta}_t = \bar{V}_s(t, S_t, A_t) \) and \( \gamma = 0 \), but the cash equivalent (4.26) differs from (4.27). This shows that (4.27) is not the correct general form for the cash equivalent.

### 4.3 Barrier options

The payoff of barrier options depends on whether or not the stock price has hit a given barrier over its lifetime. For instance, a *knock-out call* with barrier \( B \) is an option that has the payoff of a vanilla call option provided that the stock price has not hit the barrier \( B \) at any time before maturity. If the barrier has been hit, the payoff becomes zero. Conversely, the payoff of a *knock-in call* becomes active only if the barrier has been hit before maturity, otherwise the payoff is zero. Barrier options are path-dependent in a rather weak sense. Their payoff depends only on two possible states – whether or not the barrier has been hit. This allows to value barrier options in the local volatility model without introducing additional state variables by imposing suitable boundary conditions.

For simplicity, we focus on knock-out options whose payoff \( G(S_T) \) is knocked out if the stock price breaches the barrier \( B > S_0 \) before maturity. It follows as in Sections 2.1 and 4.2 that the fair value of such an option in the local volatility model can be expressed as the solution to the PDE

\[
\bar{V}_t(t, s) + \frac{1}{2} \bar{\sigma}(t, s)^2 s^2 \bar{V}_{ss}(t, s) = 0, \quad (t, s) \in [0, T) \times (0, B),
\]

\[
\bar{V}(t, B) = 0, \quad t \in [0, T),
\]

\[
\bar{V}(T, s) = G(s), \quad s \in (0, B). \tag{4.28}
\]

\( \bar{V}(t, s) \) represents the value of the knock-out option at time \( t \) provided the stock price is \( s \) and the option has not been knocked out yet. The boundary condition \( \bar{V}(t, B) = 0 \) reflects the fact that the knock-out option becomes worthless if the stock price hits the barrier \( B \) before maturity. If you have sold such an option, your theoretical P&L can be written as

\[
Y_t = x_0 + \int_0^t \theta_u \, dS_u - \bar{V}(t, S_t)1_{(t \leq \rho)}, \tag{4.29}
\]
II.4 Heuristics and extensions

where $\rho$ is the first time that $S$ hits the barrier $B$. Now, note that the boundary condition in (4.28) implies $\bar{V}(t, S_t)1_{\{t \leq \rho\}} = \bar{V}(t \wedge \rho, S_{t \wedge \rho})$. Itô’s formula in turn yields

$$
\bar{V}(t \wedge \rho, S_{t \wedge \rho}) = \bar{V}(0, s_0) + \int_0^t \bar{V}_s(u, S_u)1_{\{u < \rho\}} \, dS_u
+ \int_0^t \left( \bar{V}_t(u, S_u) + \frac{1}{2} \sigma_u^2 S_u^2 \bar{V}_{ss}(u, S_u) \right) 1_{\{u < \rho\}} \, du
$$

Using the PDE (4.28) to substitute the $\bar{V}_t$ term and plugging the result into (4.29) then gives the dynamics of $Y$:

$$
dY_t = (\theta_t - \bar{\Delta}_t) \, dS_t + \frac{1}{2} \Gamma^s_t (\bar{\sigma}(t, S_t)^2 - \sigma^2_t) \, dt,
$$

(4.30)

where $\bar{\Delta}_t = \bar{V}_s(t, S_t)1_{\{t < \rho\}}$ and $\bar{\Gamma}^s_t = S_t^2 \bar{V}_{ss}(t, S_t)1_{\{t < \rho\}}$ are the delta and the cash gamma of the knock-out option, respectively. As one would expect, these quantities are zero after the barrier has been hit.

To find the candidate optimal controls, we have to distinguish two cases. After the barrier has been hit, the option is worthless and no longer needs to be hedged. Therefore, the candidate strategy is simply $0$. With this strategy, your P&L stays constant independently of which volatility “nature” chooses (cf. (4.30)). Therefore, there is no incentive for “nature” to deviate from the reference dynamics and the candidate volatility is simply the reference volatility $\bar{\sigma}$. The HJBI equation corresponding to the hedging problem before the barrier has been hit is the same as in the case of a single vanilla option (but with an additional boundary condition). Therefore, we obtain essentially the same candidates for the optimal controls, namely

$$
\sigma^\psi_t = \bar{\sigma}(t, S_t) + \frac{\bar{\sigma}(t, S_t) \bar{\Gamma}^s_t}{f''(t, S_t; \bar{\sigma}(t, S_t))} \psi,
$$

$$
\theta^\psi_t = \bar{\Delta}_t + \left( \bar{w}_s(t, S_t, Y_t) + \frac{U'(Y_t)}{U''(Y_t)} \bar{w}_y(t, S_t, Y_t) \right) 1_{\{t < \rho\}} \psi,
$$

where the cash equivalent $\bar{w}$ solves the PDE (3.2) with the boundary condition $\bar{w}(t, B, y) = 0$. The (formal) Feynman–Kac representation of this PDE reads as

$$
\bar{w}(t, s, y) = E^{t,s} \left[ \int_t^T \left( \frac{\bar{\sigma}(u, S_u) \bar{\Gamma}^s_u}{2f''(u, S_u; \bar{\sigma}(u, S_u))} \right) \, du \right].
$$

(4.31)

As a result, the expected volatility-weighted cash gamma accumulated over the remaining lifetime of the barrier option is again the major driver of the cash equivalent of small uncertainty aversion.

4.4 Option portfolios

Instead of a single option with maturity $T$, we now consider a whole portfolio of vanilla options with possibly different maturities.\textsuperscript{35} Suppose that you have sold

\textsuperscript{35}Portfolios including exotic options can be treated along the same lines; we do not pursue this here to ease notation.
$N$ options with maturities $T_1, \ldots, T_N \in [0, T]$ and payoffs $G_i(S_{T_1}), \ldots, G_N(S_{T_N})$, respectively. Let $\bar{V}^i(t, s)$ denote the reference value of the option $G_i$, which solves

$$
\bar{V}^i(t, s) + \frac{1}{2} \bar{\sigma}(t, s)^2 s^2 \bar{V}_{ss}^i(t, s) = 0, \quad (t, s) \in [0, T_i) \times \mathbb{R}_+,
$$

$$
\bar{V}^i(T_i, s) = G_i(s), \quad s \in \mathbb{R}_+.
$$

(4.32)

Your theoretical P&L at time $t$ can be expressed as

$$
Y_t = x_0 + \int_0^t \theta_u \, dS_u - \sum_{i=1}^N \bar{V}^i(t \land T_i, S_{t \land T_i}).
$$

(4.33)

We emphasise that for $t = T$, using the terminal conditions of the PDEs (4.32),

$$
Y_T = x_0 + \int_0^T \theta_u \, dS_u - \sum_{i=1}^N G_i(S_{T_i})
$$

is your actual final P&L. We next determine the drift and diffusion parts of $Y$ if the true stock volatility is given by some process $\sigma$. As in Section 2.1, using Itô’s formula and the PDEs (4.32), we find that

$$
\bar{V}^i(t \land T_i, S_{t \land T_i}) = \bar{V}^i(0, s_0) + \int_0^t \bar{\Delta}^i(u, S_u) \, dS_u - \frac{1}{2} \int_0^t \bar{\Gamma}^{s,i}(u, S_u)(\bar{\sigma}(u, S_u)^2 - \sigma_u^2) \, du,
$$

(4.34)

where $\bar{\Delta}^i(u, s) := \bar{V}^i_u(u, s)1_{\{u<T_i\}}$ is the delta of the option $G_i$ and $\bar{\Gamma}^{s,i}(u, s) := s^2 \bar{V}_{ss}^i(u, s)1_{\{u<T_i\}}$ is its cash gamma. Substituting (4.34) into (4.33), we obtain

$$
dY_t = (\theta_t - \bar{\Delta}(t, S_t)) \, dS_t + \frac{1}{2} \bar{\Gamma}^{s}(t, S_t)(\bar{\sigma}(t, S_t)^2 - \sigma_t^2) \, dt,
$$

where $\bar{\Delta}(t, s) := \sum_{i=1}^N \bar{\Delta}^i(t, s)$ is the net delta of your option portfolio and $\bar{\Gamma}^s(t, s) := \sum_{i=1}^N \bar{\Gamma}^{s,i}(t, s)$ is its net cash gamma. We see that the dynamics of $Y$ have exactly the same form as for the case of a single vanilla option (cf. (4.8) and (4.18)). The only difference is that the delta and the cash gamma of the single option are replaced by the net delta and net cash gamma of the option portfolio. Therefore, we obtain analogous candidates for the optimal controls (in feedback form) and the cash equivalent:

$$
\sigma^\psi(t, s, y) = \bar{\sigma}(t, s) + \frac{\bar{\sigma}(t, s)\bar{\Gamma}^s(t, s)}{f''(t, s; \bar{\sigma}(t, s))} \psi,
$$

(4.35)

$$
\theta^\psi(t, s, y) = \bar{\Delta}(t, s) + \left( \bar{\psi}_x(t, s, y) + \frac{U'(y)}{U''(y)} \bar{\psi}_y(t, s, y) \right) \psi,
$$

(4.36)

$$
\bar{\psi}(t, s, y) = E^{t,s} \left[ \int_t^T \mathbb{E}_{t, s, y}(\bar{\sigma}(u, S_u)\bar{\Gamma}^s(u, S_u))^2 \, du \right].
$$

(4.37)
4.5 Static hedging with vanilla options

So far, the only hedging instrument available was the stock. In practice, however, if liquidly traded options are available, these may be used as additional hedging instruments for more complex derivatives. Therefore, we now assume that in addition to trading in the stock you can buy or sell, at time 0, any quantity of $M$ vanilla options with maturities $T_1, \ldots, T_M \in [0, T]$ and payoffs $F_1(S_{T_1}), \ldots, F_M(S_{T_M})$. We suppose that these options are available for prices $p_1, \ldots, p_M$. In the context of worst-case superhedging, this setup is known as the Lagrangian uncertain volatility model \[11\]; also compare \[130\]. For each $i = 1, \ldots, M$, let $\bar{V}^i(t, s)$ be the reference value of the option $F_i$, which solves

$$
\frac{1}{2} \sigma(t, s)^2 s^2 \bar{V}^i_{ss}(t, s) = 0, \quad (t, s) \in [0, T_i) \times \mathbb{R},
$$

$$
\bar{V}^i(T_i, s) = F_i(s), \quad s \in \mathbb{R}.
$$

(4.38)

We require that the reference model is consistent with the observed prices at time 0 in the sense that $\bar{V}^i(0, s_0) = p_i$ for $i = 1, \ldots, M$.\(^{36}\) For notational simplicity, we assume that you have to hedge a portfolio of $N$ vanilla options with maturities $T_{M+1}, \ldots, T_{M+N} \in [0, T]$ and payoffs $G_{M+1}(S_{T_{M+1}}), \ldots, G_{M+N}(S_{T_{M+N}}).$\(^{37}\) We assume that for each $i = M + 1, \ldots, M + N$, the reference value $\bar{V}^i$ of the option $G_i(S_{T_i})$ satisfies the PDE (4.32).

Suppose that you buy $\lambda_i$ options with payoff $F_i(S_{T_i})$ for price $p_i$ at time 0 (a negative $\lambda_i$ indicates a short sale) for $i = 1, \ldots, M$ and follow a self-financing trading strategy $\theta$ for the stock. Then your theoretical P&L at time $t$ is

$$
Y_t = x_0 + \int_0^t \theta_u \, dS_u - \sum_{i=M+1}^{M+N} \bar{V}^i(t \wedge T_i, S_{t \wedge T_i}) + \sum_{i=1}^M \lambda_i \left( \bar{V}^i(t \wedge T_i, S_{t \wedge T_i}) - p_i \right).
$$

(4.39)

Note that the consistency condition $\bar{V}^i(0, s_0) = p_i$ implies that our choice of $\lambda_i$ does not affect the theoretical P&L at time 0. Moreover, by (4.38) and (4.32),

$$
Y_T = x_0 + \int_0^T \theta_u \, dS_u - \sum_{i=M+1}^{M+N} G_i(S_{T_i}) + \sum_{i=1}^M \lambda_i \left( F_i(S_{T_i}) - p_i \right)
$$

is your actual final P&L.

Looking at (4.39), we recognise that (up to linear transformations that can be incorporated into the option payoffs) we are exactly in the setting of an option portfolio discussed in Section 4.4 (with $N$ replaced by $N + M$). Hence, we obtain the same candidate controls (4.35)–(4.36) and cash equivalent (4.37) with net

\(^{36}\)That is, the local volatility model is calibrated to the observed market prices of the liquid options at time 0.

\(^{37}\)Portfolios of barrier options as in \[8\] or other exotics can be treated along the same lines, but require a more extensive notation.
Hedging with Small Uncertainty Aversion

delta

\[ \bar{\Delta} := \sum_{i=M+1}^{M+N} \Delta^i - \sum_{i=1}^{M} \lambda_i \Delta^i \]

and net cash gamma

\[ \bar{\Gamma}^g := \sum_{i=M+1}^{M+N} \bar{\Gamma}^{g,i} - \sum_{i=1}^{M} \lambda_i \bar{\Gamma}^{g,i}. \]

In particular, denoting by \( \bar{\Gamma}^g,0 := \sum_{i=M+1}^{M+N} \bar{\Gamma}^{g,i} \) the net cash gamma of your original book (before buying or selling other options), the cash equivalent of the combined portfolio has the following representation:

\[
\tilde{w}(t, s, y) = E^{t,s} \left[ \int_t^T \left( \bar{\sigma}(u, S_u) (\bar{\Gamma}^{g,0} - \sum_{i=1}^{M} \lambda_i \bar{\Gamma}^{g,i}) (u, S_u) \right)^2 \frac{2 f''(u, S_u, y; \bar{\sigma}(u, S_u))}{\bar{\sigma}''(u, S_u, y; \bar{\sigma}(u, S_u))} \, du \right].
\]

This yields a criterion to manage a portfolio’s sensitivity to volatility uncertainty by trading statically in options: find \( \lambda_i \)'s that minimise \( \tilde{w} \), i.e., that minimise the expected volatility-weighted net cash gamma accumulated over the remaining lifetime of the option portfolio. In Section 4.6, we show that this minimised cash equivalent, viewed as a function of the original portfolio’s net cash gamma, satisfies certain axiomatic properties that have been advocated for measures of model uncertainty for derivatives.

4.6 The cash equivalent as a measure of model uncertainty

Consider a mapping \( \mu \) which assigns a nonnegative number to any contingent claim that has a well-defined value in any model of a given family. Cont [33] calls \( \mu \) a measure of model uncertainty if it satisfies four axioms that reflect the possibility of full or partial hedges in the underlying or in liquidly traded options; cf. [33, Section 4.1] for more details. Here, we restrict attention to the linear space \( \mathcal{X} \) of claims of the form

\[
G = c + \sum_{i=M+1}^{M+N} G_i(S_{T_i}) + \int_0^T \theta_t \, dS_t, \tag{4.40}
\]

where \( N \in \mathbb{N} \), the payoffs \( G_i(S_{T_i}) \) and maturities \( T_i \) are as in Section 4.5, \( c \in \mathbb{R} \) is a constant, and \( \theta \) is a sufficiently regular trading strategy so that the stochastic integral can be defined path-wise.\(^{38}\) We have seen that we can associate to each of these claims its cash gamma \( \bar{\Gamma}^{g,G} \) as the sum \( \sum_{i=M+1}^{M+N} s^2 \bar{V}^g_{s,t}(t, s) 1_{\{t<T_i\}} \) (note that \( c \) and the stochastic integral in (4.40) do not contribute).

\(^{38}\)For instance, if \( \theta \) is of finite variation, then the stochastic integral can be defined path-wise via the integration by parts formula.
Recall the setup and notation of Section 4.5. For notational convenience, we combine the cash gammas $\bar{\Gamma}$, $i = 1, \ldots, M$, into a vector $\bar{\Gamma}$ of functions. Now, define (for some fixed $t, s$) the function $\mu : \mathcal{X} \to \mathbb{R}_+$ by

$$
\mu(G) = \inf_{\lambda \in \mathbb{R}^M} E^{t,s} \left[ \int_t^T \frac{\tilde{\sigma}(u, S_u)(\bar{\Gamma}^G - \lambda \cdot \bar{\Gamma}^G)(u, S_u)^2}{2 f''(u, S_u, y; \tilde{\sigma}(u, S_u))} \, du \right],
$$

which maps a claim $G \in \mathcal{X}$ to its cash equivalent of small uncertainty aversion minimised over all static hedges in liquidly traded options. This mapping fulfills the following (suitably modified) axioms of [33]:

(i) There is no model uncertainty for liquidly traded options:

$$
\mu(F_i(S_T)) = 0 \text{ for all } i = 1, \ldots, M.
$$

Moreover, $\mu(c) = 0$ for all constants $c \in \mathbb{R}$.

(ii) $\mu$ accounts for hedging possibilities provided by dynamic trading in the underlying:

$$
\mu \left( G + \int_0^T \theta_t \, dS_t \right) = \mu(G) \text{ for all trading strategies } \theta \text{ and } G \in \mathcal{X}.
$$

(iii) Diversification decreases the model uncertainty of a portfolio:

$$
\mu(\nu G + (1 - \nu)G') \leq \nu \mu(G) + (1 - \nu) \mu(G')
$$

for all $\nu \in [0, 1]$ and $G, G' \in \mathcal{X}$.

(iv) $\mu$ accounts for hedging possibilities provided by static hedges with liquidly traded options:

$$
\mu \left( G + \sum_{i=1}^M \lambda_i F_i(S_{T_i}) \right) = \mu(G) \text{ for all } \lambda \in \mathbb{R}^M \text{ and } G \in \mathcal{X}.
$$

We also note that by construction, $\mu$ becomes smaller as the set of liquidly traded options expands; this is another natural requirement that has been pointed out in [33]. Properties (i), (ii), and (iv) are immediate from the definition of $\mu$. The convexity property (iii) can be verified as follows. Fix $\nu \in [0, 1]$, $G, G' \in \mathcal{X}$, and denote by $\bar{\Gamma}^G$ and $\bar{\Gamma}^{G'}$ the cash gammas of $G$ and $G'$. Fix $\epsilon > 0$. By the definition of $\mu$, we may choose $\lambda, \lambda' \in \mathbb{R}^M$ such that

$$
\mu(G) \leq E^{t,s} \left[ \int_t^T \frac{\tilde{\sigma}(u, S_u)(\bar{\Gamma}^G - \lambda \cdot \bar{\Gamma}^G)(u, S_u)^2}{2 f''(u, S_u, y; \tilde{\sigma}(u, S_u))} \, du \right] + \frac{\epsilon}{2}
$$

(4.41)

\[39\text{Unlike [33], we disregard bid-ask spreads for the liquidly traded options.}\]
and the analogous inequality with $G$ and $\lambda$ replaced by $G'$ and $\lambda'$ holds as well. Define $\lambda'' := \nu\lambda + (1 - \nu)\lambda'$ and $G'' := \nu G + (1 - \nu)G'$. Then
\[ \tilde{\Gamma}^{G''} - \lambda'' \cdot \bar{\Gamma}^G = \nu \tilde{\Gamma}^{G',G} + (1 - \nu)\tilde{\Gamma}^{G,G'} - (\nu\lambda + (1 - \nu)\lambda') \cdot \bar{\Gamma}^G \]
\[ = \nu (\tilde{\Gamma}^{G,G} - \lambda \cdot \bar{\Gamma}^G) + (1 - \nu)(\tilde{\Gamma}^{G,G'} - \lambda' \cdot \bar{\Gamma}^G). \]
Together with the convexity of $x \mapsto x^2$, this yields
\[ (\tilde{\Gamma}^{G,G} - \lambda'' \cdot \bar{\Gamma}^G)^2 \leq \nu (\tilde{\Gamma}^{G,G} - \lambda \cdot \bar{\Gamma}^G)^2 + (1 - \nu)(\tilde{\Gamma}^{G,G'} - \lambda' \cdot \bar{\Gamma}^G)^2. \] (4.42)

Using the definition of $\mu$, (4.42) and (4.41), we find
\[ \mu(G'') \leq E^{t,s} \left[ \int_t^T \frac{\tilde{\sigma}(u,S_u)(\tilde{\Gamma}^{G,G'} - \lambda'' \cdot \bar{\Gamma}^G)(u,S_u)^2}{2f''(u,S_u,y;\tilde{\sigma}(u,S_u))} du \right] \]
\[ \leq \nu \mu(G) + (1 - \nu)\mu(G') + \varepsilon. \]
The assertion now follows by taking the limit $\varepsilon \downarrow 0$.

## 5 Proofs

In this section, we rigorously prove the results from Section 3. Throughout, we assume that Assumption 3.1 is in force. To ease notation, define for $(t, s, y) \in \mathcal{D} = (0, T) \times (K^{-1}, K) \times (y_i, y_u)$ and $\psi > 0$,
\[ \theta^\psi(t, s, y) := \bar{V}_\psi(t, s) + \bar{\theta}(t, s, y) \psi, \quad \sigma^\psi(t, s, y) := \tilde{\sigma}(t, s) + \tilde{\sigma}(t, s, y) \psi. \] (5.1)

Note that we use the symbols $\theta^\psi$ and $\sigma^\psi$ for both the functions defined in (5.1) and the candidate controls defined in Theorem 3.3. This is, of course, motivated by the relationships
\[ \theta^\psi_t = \theta^\psi(t, S_t, Y_t)1_{\{t < \tau\}} + \bar{\Delta}_t1_{\{t \geq \tau\}} = \bar{\Delta}_t + \tilde{\theta}(t, S_t, Y_t)1_{\{t < \tau\}} \psi, \]
\[ \sigma^\psi_t = \sigma^\psi(t, S_t, Y_t). \]

### 5.1 Value expansion and almost optimality of the candidate strategy

In this section, we prove Theorem 3.3. Throughout, we assume that $\mathcal{A}$, $\mathcal{V}$, and $\psi_c > 0$ are chosen such that $(\theta^\psi, \sigma^\psi) \in \mathcal{A} \times \mathcal{V} \subset \mathcal{Z}$ for every $\psi \in (0, \psi_c)$. The concrete construction of such scenarios summarised in Theorem 3.7 is carried out in Section 5.3.

Define the candidate value function $w : \overline{\mathcal{D}} \times \mathbb{R} \to \mathbb{R}$ by
\[ w(t, s, y; \psi) := w^\psi(t, s, y) := U(y) - U'(y)\bar{w}(t, s, y)\psi + U'(y)\tilde{w}(t, s, y)\psi^2. \] (5.2)

We note that by our assumptions on $\bar{w}$, $\tilde{w}$, and $U$ in Assumption 3.1, there is a constant $K' > 0$ (depending only on $K$, $y_i$, $y_u$, and $U$) such that
\[ |w^\psi_t|, |w^\psi_s|, |w^\psi_y|, |w^\psi_{ss}|, |w^\psi_{sy}|, |w^\psi_{yy}| \leq K' \text{ on } \mathcal{D} \times [-1, 1]. \] (5.3)
Moreover, all partial derivatives of \( w \) in (5.3) are evidently \( C^\infty \) in \( \psi \).

In essence, the proof of Theorem 3.3 boils down to showing that our candidate value function \( w^\psi \), candidate strategy \( \theta^\psi \), and candidate volatility \( \sigma^\psi \) are approximate solutions to the Hamilton–Jacobi–Bellman–Isaacs equation associated to the hedging problem (2.14) in the sense that

\[
w^\psi_t(t, s, y) + \sup_{\theta} \inf_{\varsigma} H^\psi(t, s, y; \theta, \varsigma) = o(\psi^2) \quad \text{as } \psi \downarrow 0, \text{ uniformly in } (t, s, y);
\]

cf. Lemma 5.3 below. Here, for \( \psi \neq 0 \), the Hamiltonian \( H^\psi : D \times \mathbb{R} \times [0, K] \to \mathbb{R} \) is given by

\[
H^\psi(t, s, y; \theta, \varsigma) = \frac{1}{\psi} U'(y) f(t, s, y; \varsigma) + \frac{1}{2} s^2 \bar{V}_{ss}(t, s)(\bar{\sigma}(t, s)^2 - \varsigma^2) w^\psi_y(t, s, y) \\
+ \frac{1}{2} s^2 \bar{V}_{ss}(t, s) \left( w^\psi_{yy}(t, s, y) + 2(\bar{\sigma} - \bar{V}_s(t, s)) w^\psi_{sy}(t, s, y) + (\bar{\sigma} - \bar{V}_s(t, s))^2 w^\psi_{yy}(t, s, y) \right);
\]

recall (4.9) and (4.18) from the heuristic derivation of the HJBI equation in Section 4.1. This part of the proof is purely analytic and is carried out in Section 5.1.1, the main ingredients being the implicit function theorem and Taylor expansions. Then, adapting classical verification arguments to the asymptotic setting allows us to prove the two inequalities

\[
\inf_{\sigma \in \mathcal{V}} \inf_{P \in \mathcal{P}(\theta^\psi, \sigma)} J^\psi(\sigma, P) \geq w^\psi_0 + o(\psi^2), \tag{5.5}
\]

\[
\sup_{\theta \in \mathcal{A}} \inf_{P \in \mathcal{P}(\theta, \sigma^\psi)} J^\psi(\sigma^\psi, P) \leq w^\psi_0 + o(\psi^2), \tag{5.6}
\]

where \( w^\psi_0 := w^\psi(0, s_0, y_0) \); cf. Lemmas 5.7 and 5.8. Denoting by \( \lesssim \) “less or equal up to a term of order \( o(\psi^2) \)”, we obtain from (5.5)–(5.6) that

\[
w^\psi_0 \lesssim \inf_{\sigma \in \mathcal{V}} \inf_{P \in \mathcal{P}(\theta^\psi, \sigma)} J^\psi(\sigma, P) \lesssim \sup_{\theta \in \mathcal{A}} \inf_{P \in \mathcal{P}(\theta^\psi, \sigma^\psi)} J^\psi(\sigma^\psi, P) \gtrsim w^\psi \]

and

\[
w^\psi_0 \lesssim \inf_{\sigma \in \mathcal{V}} \inf_{P \in \mathcal{P}(\theta^\psi, \sigma)} J^\psi(\sigma, P) \lesssim \inf_{P \in \mathcal{P}(\theta^\psi, \sigma^\psi)} J^\psi(\sigma^\psi, P) \lesssim \sup_{\theta \in \mathcal{A}} \inf_{P \in \mathcal{P}(\theta, \sigma^\psi)} J^\psi(\sigma^\psi, P) \lesssim w^\psi.
\]

Hence, we have equality up to a term of order \( o(\psi^2) \) everywhere. In particular,

\[
\sup_{\theta \in \mathcal{A}} \inf_{\sigma \in \mathcal{V}} \inf_{P \in \mathcal{P}(\theta^\psi, \sigma)} J^\psi(\sigma, P) = w^\psi_0 + o(\psi^2) = \inf_{P \in \mathcal{P}(\theta^\psi, \sigma^\psi)} J^\psi(\sigma^\psi, P) + o(\psi^2).
\]

This completes the proof of Theorem 3.3 modulo the proofs of (5.5)–(5.6).
5.1.1 Approximate solution to HJBI equation

We first determine the minimiser of the Hamiltonian with respect to the volatility variable \( \varsigma \), and identify it at the leading order as the candidate \( \sigma_\psi \) from (5.1).

**Lemma 5.1.** Fix \( L > 0 \). Then there are constants \( C_1 > 0 \) and \( \psi_1 > 0 \) (depending on \( L \)) such that for every \( (t, s, y) \in \mathcal{D}, \tilde{\vartheta} \in [-L, L] \), and \( \psi \in (0, \psi_1) \), the function

\[
[0, K] \ni \varsigma \mapsto H_\psi(t, s, y; \tilde{V}_s(t, s) + \tilde{\vartheta} \psi, \varsigma)
\]

has a minimiser \( \varsigma_\psi(t, s, y, \tilde{\vartheta}) \) that satisfies the first-order condition

\[
\frac{\partial H_\psi}{\partial \varsigma}(t, s, y; \tilde{V}_s(t, s) + \tilde{\vartheta} \psi, \varsigma_\psi(t, s, y, \tilde{\vartheta})) = 0,
\]

and

\[
\left| \varsigma_\psi(t, s, y, \tilde{\vartheta}) - \sigma_\psi(t, s, y) \right| \leq C_1 \psi^2.
\]

**Proof.** As \( H_\psi \) is continuous in \( \varsigma \) and \( [0, K] \) is compact, there exists a minimiser \( \varsigma_\psi = \varsigma_\psi(t, s, y, \tilde{\vartheta}) \) of (5.7) for every \( (t, s, y) \in \mathcal{D}, \tilde{\vartheta} \in [-L, L] \), and \( \psi > 0 \). Next, the basic idea is to employ the convexity of the penalty function \( f \) to show that for sufficiently small \( \psi \), \( \varsigma_\psi \) has to lie in the interior of \( [0, K] \). As a consequence, it satisfies the first-order condition.

To make this precise, first note from (3.10) and (5.3) that there is a constant \( K'' > 0 \) such that for all \( (t, s, y) \in \mathcal{D}, \tilde{\vartheta} \in [-L, L] \), and \( \psi \in (-1, 1) \),

\[
\left| \frac{1}{2} s^2 \left( w_{ss}(t, s, y) + 2\tilde{\vartheta} \psi w_{sy}(t, s, y) + (\tilde{\vartheta} \psi)^2 w_{yy}(t, s, y) - \tilde{V}_s(t) \psi(t, s, y) \right) \right| \leq K''.
\]

Since \( \tilde{\sigma}(t, s) \in [0, K] \) on \( [0, T] \times [K^{-1}, K] \) by (3.9), we have

\[
H_\psi(t, s, y; \tilde{V}_s(t, s) + \tilde{\vartheta} \psi, \varsigma_\psi(t, s, y, \tilde{\vartheta})) \leq H_\psi(t, s, y; \tilde{V}_s(t, s) + \tilde{\vartheta} \psi, \tilde{\sigma}(t, s)).
\]

On the one hand, using the definition of \( H_\psi \) together with (2.12) and rearranging terms, this inequality implies

\[
\frac{1}{\psi} U'(y) f(t, s, y; \varsigma_\psi) \leq \left( \tilde{\sigma}(t, s)^2 - (\varsigma_\psi)^2 \right) \frac{1}{2} s^2 
\times \left( w_{ss}(t, s, y) + 2\tilde{\vartheta} \psi w_{sy}(t, s, y) + (\tilde{\vartheta} \psi)^2 w_{yy}(t, s, y) - \tilde{V}_s(t) \psi(t, s, y) \right) 
\leq 2K K'' \left| \tilde{\sigma}(t, s) - \varsigma_\psi \right|.
\]

On the other hand, assumption (3.11) together with (2.12) yields

\[
f(t, s, y; \varsigma_\psi) \geq \frac{1}{2K} (\tilde{\sigma}(t, s) - \varsigma_\psi)^2.
\]
Combining (5.11)–(5.12) and rearranging terms gives

\[ |\tilde{\sigma}(t, s) - \varsigma^w| \leq \frac{4K^2K''}{U'(y_u)} \psi; \]  

(5.13)

note that this inequality is trivially true if \( \tilde{\sigma}(t, s) = \varsigma^w \), so the division by \( |\tilde{\sigma}(t, s) - \varsigma^w| \) in the last step is justified. Now, since \( \tilde{\sigma}(t, s) \) is uniformly in the interior of \([0, K]\) by assumption (3.9), it follows from (5.13) that there is \( \psi_1 \in (0, 1) \) such that for every \((t, s, y) \in D, ~\bar{\vartheta} \in [-L, L] \), and \( \psi \in (0, \psi_1) \), we have \( \varsigma^w(t, s, y, \bar{\vartheta}) \in (0, K) \). This implies that \( \varsigma^w \) satisfies the first-order condition (5.8) or equivalently (through multiplication by \( \psi > 0 \)),

\[
U'(y)f'(t, s, y; \varsigma^w) - s^2\tilde{V}_{ss}(t, s)\varsigma^w(t, s, y)\psi \\
+ \varsigma^w s^2 \left( w_{ss}^\psi(t, s, y) + 2\bar{\vartheta}\psi w_{sy}^\psi(t, s, y) + (\bar{\vartheta}\psi)^2 w_{yy}^\psi(t, s, y) \right) \psi = 0. 
\]  

(5.14)

It remains to prove (5.9). For each \( \lambda = (t, s, y, \bar{\vartheta}) \in D \times [-L, L] \), we define the function \( F_\lambda : (-\psi_1, \psi_1) \times [0, K] \to \mathbb{R}, \ (\psi, \varsigma) \mapsto F_\lambda(\psi, \varsigma)\), by the left-hand side of (5.14) with \( \varsigma^w \) replaced by \( \varsigma \). As \( F_\lambda \) is a polynomial in \( \psi \) and \( f \) is \( C^4 \) in \( \varsigma \), \( F_\lambda \) is \( C^3 \). Fix \( \lambda = (t, s, y, \bar{\vartheta}) \). By construction,

\[ F_\lambda(\psi, \varsigma^w) = 0, \ \psi \in (0, \psi_1). \]  

(5.15)

Now, we want to invoke the implicit function theorem to show that \( \psi \mapsto \varsigma^w \) can be extended via (5.15) to a \( C^3 \) function on \((-\psi_1, \psi_1)\) (choosing \( \psi_1 \) smaller if necessary). To this end, it suffices to show that \( \frac{\partial F_\lambda}{\partial \varsigma} \geq \varepsilon \) for some \( \varepsilon > 0 \). Using (3.11) and (5.10), we obtain for all \( \lambda = (t, s, y, \bar{\vartheta}) \in D \times [-L, L] \) and \( \psi \in (-\psi_1, \psi_1) \),

\[
\frac{\partial F_\lambda}{\partial \varsigma}(\psi, \varsigma) = U'(y)f''(t, s, y; \varsigma) - s^2\tilde{V}_{ss}(t, s)w_{yy}^\psi(t, s, y)\psi \\
+ s^2 \left( w_{ss}^\psi(t, s, y) + 2\bar{\vartheta}\psi w_{sy}^\psi(t, s, y) + (\bar{\vartheta}\psi)^2 w_{yy}^\psi(t, s, y) \right) \psi \\
\geq \frac{U'(y_u)}{K} \psi - 2K''|\psi|. 
\]

Hence, choosing \( \psi_1 \) smaller if necessary, there is \( \varepsilon > 0 \) such that

\[ \frac{\partial F_\lambda}{\partial \varsigma}(\psi, \varsigma) \geq \varepsilon, \ \lambda \in D \times [-L, L], \ \psi \in (-\psi_1, \psi_1), \varsigma \in [0, K], \]  

(5.16)

and the implicit function theorem implies that for each fixed \( \lambda \), \( \psi \mapsto \varsigma^w \) can be extended via (5.15) to \((-\psi_1, \psi_1)\) and is \( C^3 \). As \( F_\lambda(0, \varsigma) = U'(y)f'(t, s, y; \varsigma) \), the uniqueness assertion of the implicit function theorem together with (2.12) also yields that \( \varsigma^w = \tilde{\sigma}(t, s) \). To compute \( \frac{\partial \varsigma^w}{\partial \psi}(0) \), we observe using the first-order
condition (5.14) and the fact that $w^\psi_y(t, s, y) = U'(y)$ for $\psi = 0$, that

$$\begin{align*}
f''(t, s, y; \bar{\sigma}(t, s)) \frac{\partial \varsigma^\psi}{\partial \psi}(0) \\
= \lim_{\psi \to 0} \frac{1}{\psi} \left( f'(t, s, y; \varsigma^\psi) - f'(t, s, y; \varsigma^0) \right) = \lim_{\psi \to 0} \frac{1}{\psi} f'(t, s, y; \varsigma^\psi) \\
= \lim_{\psi \to 0} \frac{1}{U'(y)} \left( s^2 \tilde{V}_{ss}(t, s) \varsigma^\psi w^\psi_y(t, s, y) \\
- \varsigma^\psi s^2 \left( w^\psi_{ss}(t, s, y) + 2 \tilde{\vartheta}_\psi w^\psi_{s\gamma}(t, s, y) + (\tilde{\vartheta}_\psi)^2 w^\psi_{yy}(t, s, y) \right) \right) \\
= s^2 \tilde{V}_{ss}(t, s) \bar{\sigma}(t, s).
\end{align*}$$

Solving for $\frac{\partial \varsigma^\psi}{\partial \psi}(0)$ gives $\frac{\partial \varsigma^\psi}{\partial \psi}(0) = \bar{\sigma}(t, s, y)$. Now, a Taylor expansion of $\varsigma^\psi$ around $\psi = 0$ yields

$$\varsigma^\psi = \bar{\sigma}(t, s, y) + \varsigma(t, s, y) \psi + \frac{1}{2} \frac{\partial^2 \varsigma^\psi}{\partial \psi^2}(\psi_L) \psi^2 = \sigma^\psi(t, s, y) + \frac{1}{2} \frac{\partial^2 \varsigma^\psi}{\partial \psi^2}(\psi_L) \psi^2$$

for some $\psi_L = \psi_L(t, s, y, \tilde{\vartheta}; \bar{\psi})$ between 0 and $\psi \in (0, \psi_1)$. Comparing this with (5.9), it remains to show that $\frac{\partial F^\psi}{\partial \psi}$ can be uniformly bounded in $\lambda = (t, s, y, \tilde{\vartheta}) \in \mathcal{D} \times [-L, L]$ and $\psi \in (0, \psi_1)$. We already know from (5.16) that $\frac{\partial F^\lambda}{\partial \varsigma}$ is bounded away from zero, uniformly over $\lambda \in \mathcal{D} \times [-L, L]$, $\psi \in (-\psi_1, \psi_1)$, and $\varsigma \in [0, \bar{K}]$. In addition, it is straightforward to check that our boundedness assumptions imply that all the second order partial derivatives of $F^\lambda$ are uniformly bounded (in the same sense as above). Therefore, $\frac{\partial^2 \varsigma^\psi}{\partial \psi^2}$ is uniformly bounded by Lemma A.1 (with $M_1 = 0$). This completes the proof. \(\square\)

Conversely, we next determine the maximiser of the Hamiltonian with respect to the strategy variable, show that it coincides at the leading order with the candidate $\theta^\psi$ from (5.1) and that it is independent of the volatility variable.

**Lemma 5.2.** There are constants $C_1 > 0$ and $\psi_1 > 0$ such that for every $(t, s, y) \in \mathcal{D}$, $\varsigma \in [0, \bar{K}]$, and $\psi \in (0, \psi_1)$, the function

$$\mathbb{R} \ni \vartheta \mapsto H^\psi(t, s, y; \vartheta, \varsigma)$$

has a maximiser $\vartheta^\psi_*(t, s, y)$ independent of $\varsigma$ that satisfies the first-order condition

$$\frac{\partial H^\psi}{\partial \vartheta}(t, s, y; \vartheta^\psi_*(t, s, y), \varsigma) = 0,$$

and

$$\left| \vartheta^\psi_*(t, s, y) - \theta^\psi(t, s, y) \right| \leq C_1 \psi^2.$$

**Proof.** By the definition of $H^\psi$ in (5.4), finding the maximiser of (5.17) is equivalent to finding the maximiser of

$$\mathbb{R} \ni \vartheta \mapsto 2(\vartheta - \tilde{V}_s(t, s)) w^\psi_{s\gamma}(t, s, y) + (\vartheta - \tilde{V}_s(t, s))^2 w^\psi_{yy}(t, s, y).$$

(5.20)
This is simply a quadratic equation in \( \vartheta \) and independent of \( \zeta \). First, we show that the coefficient of the quadratic term is uniformly negative for small \( \psi \). Note from the definition of \( w^\psi \) in (5.2) that
\[
w^\psi_{yy}(t, s, y) = U''(y) + \frac{\partial^2}{\partial y^2} \left( -U'(y)w(t, s, y) + U'(y)\hat{w}(t, s, y)\psi \right).
\]
(5.21)

Since \( U \) is \( C^3 \) and \( U'' < 0 \) there is \( \varepsilon > 0 \) such that \( U'' \leq -2\varepsilon \) on \([y, y]\). By (3.8), the partial derivative on the right-hand side of (5.21) can be bounded uniformly in \((t, s, y) \in \mathcal{D} \) and \( \psi \in (-1, 1) \). Hence, there is \( \psi_1 \in (0, 1) \) such that for all \((t, s, y) \in \mathcal{D} \) and \( \psi \in (-\psi_1, \psi_1) \),
\[
w^\psi_{yy}(t, s, y) \leq -\varepsilon.
\]
(5.22)

Therefore for each \( \psi \in (-\psi_1, \psi_1) \), (5.20) has a maximiser \( \vartheta^\psi(t, s, y) \) (which is also a maximiser of (5.17) if \( \psi \neq 0 \)) that satisfies the first-order condition
\[
w^\psi_{sy}(t, s, y) + (\vartheta^\psi_s(t, s, y) - \hat{V}_s(t, s))w^\psi_{yy}(t, s, y) = 0,
\]
(5.23)

which is equivalent to (5.18).

To prove (5.19), we argue similarly as in the proof of Lemma 5.1 using the implicit function theorem. Define for each \( \lambda = (t, s, y) \in \mathcal{D} \), the function \( F_\lambda : (-\psi_1, \psi_1) \times \mathbb{R} \to \mathbb{R} \) by
\[
F_\lambda(\psi, \delta) = w^\psi_{sy}(t, s, y) + \delta w^\psi_{yy}(t, s, y).
\]

By construction, \( F_\lambda \) is a polynomial in \((\psi, \delta)\) and hence \( C^\infty \). Fix \( \lambda = (t, s, y) \in \mathcal{D} \). By the first-order condition (5.23), we have
\[
F_\lambda(\psi, \vartheta^\psi - \hat{V}_s(t, s)) = 0, \quad \psi \in (-\psi_1, \psi_1),
\]
(5.24)

where \( \vartheta^\psi_s = \vartheta^\psi(t, s, y) \). Since \( \frac{\partial F_\lambda}{\partial \delta}(\psi, \delta) = w^\psi_{yy}(t, s, y) \leq -\varepsilon < 0 \) for every \( \psi \in (-\psi_1, \psi_1) \) by (5.22), the implicit function theorem yields that \( \vartheta^\psi_s \) is \( C^\infty \) in \( \psi \).

As \( F_\lambda(0, \delta) = \delta U''(y) \), the uniqueness assertion of the implicit function theorem also gives \( \vartheta^0_s = \hat{V}_s(t, s) \). To compute \( \frac{\partial \vartheta^\psi_s}{\partial \psi}(0) \), we divide by \( \psi > 0 \) on both sides of (5.24) and let \( \psi \downarrow 0 \). Using also the definition of \( w^\psi \) in (5.2), we obtain
\[
0 = \lim_{\psi \downarrow 0} \frac{1}{\psi} \left( w^\psi_{sy}(t, s, y) + (\vartheta^\psi_s - \hat{V}_s(t, s))w^\psi_{yy} \right)
\]
\[
= -\frac{\partial^2}{\partial s \partial y} (U''(y)\hat{w}(t, s, y)) + \lim_{\psi \downarrow 0} \frac{\partial \vartheta^\psi_s}{\psi} - \vartheta^0_s U''(y)
\]
\[
= -U''(y)\hat{w}_{sy}(t, s, y) - U''(y)\hat{w}_s(t, s, y) + \frac{\partial \vartheta^\psi_s}{\partial \psi}(0)U''(y).
\]

Solving for \( \frac{\partial \vartheta^\psi_s}{\partial \psi}(0) \) gives \( \frac{\partial \vartheta^\psi_s}{\partial \psi}(0) = \tilde{\theta}(t, s, y) \). Now, expanding \( \vartheta^\psi_s \) around \( \psi = 0 \) yields
\[
\vartheta^\psi_s = \hat{V}_s(t, s) + \tilde{\theta}(t, s, y)\psi + \frac{1}{2} \frac{\partial^2 \vartheta^\psi_s}{\partial \psi^2}(\psi_L)\psi^2 = \tilde{\theta}(t, s, y) + \frac{1}{2} \frac{\partial^2 \vartheta^\psi_s}{\partial \psi^2}(\psi_L)\psi^2
\]
for some \( \psi_L = \psi_L(t,s,y;\psi) \) between 0 and \( \psi \in (0,\psi_1) \). Comparing this with (5.19), it remains to show that \( \frac{\partial^2 \vartheta^\psi}{\partial \psi^2} \) can be uniformly bounded in \((t,s,y) \in D\) and \( \psi \in (0,\psi_1) \). We already know that \( \frac{\partial F_\psi}{\partial \psi} = w^\psi_y(t,s,y) \) is bounded away from zero, uniformly over \( \lambda \in D, \psi \in (-\psi_1,\psi_1) \), and \( \delta \in \mathbb{R} \). In addition, using our boundedness assumptions, it is straightforward to check that there is \( M > 0 \) such that for all \( \lambda \in D, \psi \in (-\psi_1,\psi_1) \), and \( \delta \in \mathbb{R} \),

\[
\left| \frac{\partial^2 F_\psi}{\partial^2 \psi^2}(\psi,\delta) \right|, \left| \frac{\partial^2 F_\lambda}{\partial^2 \psi^2}(\psi,\delta) \right| \leq M(1 + \delta), \quad \left| \frac{\partial^2 F_\lambda}{\partial^2 \psi \partial \delta}(\psi,\delta) \right| \leq M,
\]

and that \( \frac{\partial F_\psi}{\partial \psi} \equiv 0 \). Thus, by Lemma A.1 (with \( y_\lambda(\psi) = \vartheta^\psi(t,s,y) - \bar{V}_s(t,s) \)), there is \( M' > 0 \) such that for all \( \lambda \in D \) and \( \psi \in (-\psi_1,\psi_1) \),

\[
\left| \frac{\partial^2 \vartheta^\psi}{\partial \psi^2}(\psi) \right| = \left| \frac{\partial^2 (\vartheta^\psi - \bar{V}_s(t,s))}{\partial \psi^2}(\psi) \right| \leq M'(1 + |\vartheta^\psi - \bar{V}_s(t,s)|).
\]

But \( |\vartheta^\psi - \bar{V}_s(t,s)| = \left| \frac{w^\psi_y(t,s,y)}{w^\psi_y(t,s,y)} \right| \leq \frac{K'}{\varepsilon} \) by (5.23), (5.22), and (5.3). This completes the proof. \( \square \)

We now provide an asymptotic expansion of the HJBI equation at both the delta hedge and the candidate strategy \( \theta^\psi \), both with respect to the candidate volatility \( \sigma^\psi \).

**Lemma 5.3.** As \( \psi \downarrow 0 \), uniformly in \((t,s,y) \in D\),

\[
\begin{align*}
&w^\psi(t,s,y) + H^\psi(t,s,y;\bar{V}_s(t,s),\sigma^\psi(t,s,y)) = O(\psi^2), \quad (5.25) \\
&w^\psi(t,s,y) + H^\psi(t,s,y;\theta^\psi(t,s,y),\sigma^\psi(t,s,y)) = O(\psi^3). \quad (5.26)
\end{align*}
\]

**Proof.** As \( f \) is \( C^4 \) in \( \varsigma \), Taylor’s theorem together with (2.12) yields

\[
\begin{align*}
f(t,s,y;\varsigma) &= \frac{1}{2} f''(t,s,y;\bar{\sigma}(t,s))(\varsigma - \bar{\sigma}(t,s))^2 \\
&\quad + \frac{1}{6} f^{(3)}(t,s,y;\bar{\sigma}(t,s))(\varsigma - \bar{\sigma}(t,s))^3 + \frac{1}{24} f^{(4)}(t,s,y;\varsigma_L)(\varsigma - \bar{\sigma}(t,s))^4
\end{align*}
\]

for some \( \varsigma_L = \varsigma_L(t,s,y;\varsigma) \) between \( \bar{\sigma}(t,s) \) and \( \varsigma \in [0,K] \). Recall that \( f^{(4)} \) is uniformly bounded by (3.11). Hence, using (5.27) for the candidate \( \sigma^\psi(t,s,y) \), we obtain as \( \psi \downarrow 0 \), uniformly in \((t,s,y) \in D\),

\[
\begin{align*}
\frac{1}{\psi} f(t,s,y;\sigma^\psi(t,s,y)) &= \frac{1}{2} f''(t,s,y;\bar{\sigma}(t,s))\bar{\sigma}(t,s)^2 \psi + \frac{1}{6} f^{(3)}(t,s,y;\bar{\sigma}(t,s))\bar{\sigma}(t,s)^3 \psi^2 + O(\psi^3).
\end{align*}
\]

Using this, it is easily seen from the definitions of \( w^\psi \) and \( H^\psi \) that the left-hand sides of (5.25)–(5.26) reduce to polynomials in \( \psi \) up to terms of order \( O(\psi^3) \). Using our boundedness assumptions, it is also straightforward to check that all the
coefficients of these polynomials are uniformly bounded in \((t, s, y) \in D\). Therefore, it suffices to check that the coefficients of the \(O(1)\), \(O(\psi)\) and, in the case of \((5.26)\), \(O(\psi^2)\) terms vanish. One readily verifies that the \(O(1)\) term always vanishes and that the \(O(\psi)\) term reduces in both cases to the PDE (3.2) for \(\hat{w}\). Finally, in the case of \((5.26)\), a lengthy calculation shows that the \(O(\psi^2)\) term reduces to the PDE (3.3) for \(\hat{w}\).

We next analyse the relevant minimised and maximised Hamiltonians if we plug in the leading-order candidate strategies and candidate volatilities from \((5.1)\), respectively.

**Lemma 5.4.** There are constants \(C > 0\) and \(\psi_0 > 0\) such that, for every \((t, s, y) \in D\) and \(\psi \in (0, \psi_0)\):

\[
\begin{align*}
\psi(t, s, y) + \inf_{\varsigma \in [0, K]} H^\psi(t, s, y; \hat{V}_s(t, s), \varsigma) &\geq -C\psi^2, \quad (5.28) \\
\psi(t, s, y) + \inf_{\varsigma \in [0, K]} H^\psi(t, s, y; \theta^\psi(t, s), \varsigma) &\geq -C\psi^3, \quad (5.29) \\
\psi(t, s, y) + \sup_{\varsigma \in \mathbb{R}} H^\psi(t, s, y; \vartheta, \sigma^\psi(t, s, y)) &\leq C\psi^3. \quad (5.30)
\end{align*}
\]

**Proof.** We first derive uniform bounds on the second-order partial derivatives of \(H^\psi\) with respect to \(\vartheta\) and \(\varsigma\). As in the proof of Lemma 5.2 (cf. (5.22)), there is \(\psi_0 \in (0, 1)\) such that for all \((t, s, y) \in D\) and \(\psi \in (0, \psi_0)\), we have \(w^\psi_{yy}(t, s, y) \leq -\varepsilon\). Together with (3.10)–(3.11) and (5.3), this implies that there is \(K_1 > 0\) such that for all \((t, s, y) \in D\), \(\psi \in (0, \psi_0)\), \(\vartheta \in \mathbb{R}\), and \(\varsigma \in [0, K]\),

\[
\frac{\partial^2 H^\psi}{\partial \varsigma^2}(t, s, y; \hat{V}_s(t, s) + \tilde{\vartheta} \psi, \varsigma) = \frac{1}{\psi} U'(y) f''(t, s, y; \varsigma) - s^2 \hat{V}_{ss}(t, s) w^\psi_y(t, s, y) \psi
+ s^2 \left( w^\psi_{ss}(t, s, y) + 2\tilde{\vartheta} \psi w^\psi_{yy} + \tilde{\vartheta}^2 \psi^2 w^\psi_{yy}(t, s, y) \right) \leq \frac{K_1}{\psi}.
\]

In view of (5.3), choosing \(K_1\) larger if necessary, we also have for all \((t, s, y) \in D\), \(\psi \in (0, \psi_0)\), \(\vartheta \in \mathbb{R}\), and \(\varsigma \in [0, K]\),

\[
\frac{\partial^2 H^\psi}{\partial \vartheta \partial \varsigma}(t, s, y; \vartheta, \varsigma) = \varsigma^2 s^2 w^\psi_{yy}(t, s, y) \geq -K_1.
\]

Also note that by definition of \(\tilde{\vartheta}\) in (3.6) and (3.8), there is \(L > 0\) such that \(|\tilde{\vartheta}| \leq L\) on \(D\). Choosing \(\psi_0\) smaller if necessary, we may also assume that the estimates (5.9) and (5.19) of Lemmas 5.1–5.2 (with that value of \(L\)) hold for \(\psi \in (0, \psi_0)\) and that, using Lemma 5.3, there is \(C_2 > 0\) such that for all \((t, s, y) \in D\) and \(\psi \in (0, \psi_0)\),

\[
\begin{align*}
\left| w^\psi(t, s, y) + H^\psi(t, s, y; \hat{V}_s(t, s), \sigma^\psi(t, s, y)) \right| &\leq C_2 \psi^2, \quad (5.33) \\
\left| w^\psi(t, s, y) + H^\psi(t, s, y; \theta^\psi(t, s, y), \sigma^\psi(t, s, y)) \right| &\leq C_2 \psi^3. \quad (5.34)
\end{align*}
\]
We start with the proof of (5.28)–(5.29). Fix \((t, s, y) \in \mathbf{D}, \psi \in (0, \psi_0)\), and \(\tilde{\vartheta} \in [-L, L]\) and define \(h : [0, K] \to \mathbb{R}\) by \(h(\varsigma) = H^\psi(t, s, y; \tilde{V}_s(t, s) + \tilde{\vartheta} \psi, \varsigma)\). Let \(\varsigma^\psi = \varsigma^\psi(t, s, y, \tilde{\vartheta})\) be the minimiser of \(h\) from Lemma 5.1. Expanding \(h(\sigma^\psi)\) around \(\varsigma^\psi\) and using the first-order condition (5.8) gives

\[
h(\sigma^\psi) = h(\varsigma^\psi) + \frac{1}{2} \frac{\partial^2 H^\psi}{\partial \varsigma^2}(\varsigma^\psi - \varsigma^\psi)^2
\]  

(5.35)

for some \(\varsigma_L = \varsigma_L(t, s, y, \tilde{\vartheta}; \psi)\) between \(\sigma^\psi = \sigma^\psi(t, s, y)\) and \(\varsigma^\psi\). Rearranging (5.35) and using (5.31) as well as (5.9) yields \(h(\varsigma^\psi) \geq h(\sigma^\psi) - \frac{1}{2}K_1 C_1^2 \psi^3\). We conclude that for all \((t, s, y) \in \mathbf{D}, \tilde{\vartheta} \in [-L, L]\), and \(\psi \in (0, \psi_0)\),

\[
H^\psi(t, s, y; \tilde{V}_s(t, s) + \tilde{\vartheta} \psi, \varsigma^\psi(t, s, y)) \geq H^\psi(t, s, y; \tilde{V}_s(t, s) + \tilde{\vartheta} \psi, \sigma^\psi(t, s, y)) - \frac{1}{2}K_1 C_1^2 \psi^3.
\]

(5.36)

Combining (5.36) for the choice \(\tilde{\vartheta} = \tilde{\vartheta}(t, s, y)\) with (5.34) yields

\[
w^\psi_i(t, s, y) + \inf_{\varsigma \in [0, K]} H^\psi(t, s, y; \theta^\psi(t, s, y), \varsigma) \\
= w^\psi_i(t, s, y) + H^\psi(t, s, y; \theta^\psi(t, s, y), \varsigma^\psi(t, s, y, \tilde{\vartheta}(t, s, y))) \\
\geq w^\psi_i(t, s, y) + H^\psi(t, s, y; \tilde{V}_s(t, s) + \tilde{\vartheta} \psi, \sigma^\psi(t, s, y)) - \frac{1}{2}K_1 C_1^2 \psi^3 \\
\geq -\left(C_2 + \frac{1}{2}K_1 C_1^2\right) \psi^3.
\]

This proves (5.29). (5.28) follows analogously from (5.36) with \(\tilde{\vartheta} = 0\) and (5.33).

The proof of (5.30) is an almost verbatim repetition of the previous paragraph with the roles of \(\theta\) and \(\sigma\) exchanged. We provide it for completeness. Fix \((t, s, y) \in \mathbf{D}\) and \(\psi \in (0, \psi_0)\) and define the function \(h : \mathbb{R} \to \mathbb{R}\) by \(h(\vartheta) = H^\psi(t, s, y; \vartheta, \sigma^\psi(t, s, y))\). Let \(\vartheta^\psi = \vartheta^\psi(t, s, y)\) be the maximiser of \(h\) from Lemma 5.2. Expanding \(h(\theta^\psi)\) around \(\vartheta^\psi\) and using the first-order condition (5.18) gives

\[
h(\theta^\psi) = h(\vartheta^\psi) + \frac{1}{2} \frac{\partial^2 H^\psi}{\partial \vartheta^2}(\vartheta^\psi - \vartheta^\psi)^2
\]

(5.37)

for some \(\vartheta_L = \vartheta_L(t, s, y; \psi)\) between \(\theta^\psi = \theta^\psi(t, s, y)\) and \(\vartheta^\psi\). Rearranging (5.37) and using (5.32) as well as (5.19) yields \(h(\vartheta^\psi) \leq h(\theta^\psi) + \frac{1}{2}K_1 C_1^2 \psi^4\). We conclude that for all \((t, s, y) \in \mathbf{D}\) and \(\psi \in (0, \psi_0)\),

\[
H^\psi(t, s, y; \vartheta^\psi(t, s, y), \sigma^\psi(t, s, y)) \leq H^\psi(t, s, y; \theta^\psi(t, s, y), \sigma^\psi(t, s, y)) + \frac{1}{2}K_1 C_1^2 \psi^4.
\]

(5.38)
Combining (5.38) with (5.34) proves (5.30) via
\[
\begin{align*}
  w^\psi(t, s, y) + \sup_{\varphi \in \mathbb{R}} H^\psi(t, s, y; \varphi, \sigma^\psi(t, s, y)) \\
  = w^\psi(t, s, y) + H^\psi(t, s, y; \vartheta^\psi(t, s, y), \sigma^\psi(t, s, y)) \\
  \leq w^\psi(t, s, y) + H^\psi(t, s, y; \theta^\psi(t, s, y), \sigma^\psi(t, s, y)) + \frac{1}{2} K_1 C_i^2 \psi^4 \\
  \leq (C_2 + \frac{1}{2} K_1 C_i^2 \psi_0) \psi^3.
\end{align*}
\]

5.1.2 The lower bound (5.5)

Before we can prove the lower bound (5.5), we need two more preliminary results.

**Proposition 5.5.** Fix \((\theta, \sigma, P) \in \mathfrak{S}\).

(i) Let \(\rho := \inf\{t \in [0, T] : S_t \not\in (K^{-1}, K)\} \wedge T\) be the first time that \(S\) leaves \((K^{-1}, K)\). Then \(\rho\) is a stopping time and, for each \(\psi > 0\):

\[
U(Y_\rho) = w^\psi(\rho, S_\rho, Y_\rho) \quad P\text{-a.s.} \tag{5.39}
\]

(ii) The local martingale part of the canonical decomposition of \(Y\) under \(P\), \(\int_0^\rho (\theta_s - \bar{\Delta}_s) \, dS_s\), is a bounded \(P\)-martingale.

**Proof.** (i): It is an easy exercise to show that \(\rho\) is a stopping time for the (non-augmented, non-right-continuous) filtration \(\mathbb{F}\). This uses the fact that all paths of \(S\) are continuous and \((K^{-1}, K)\) is open; cf. [103, Problem 2.7]. To prove (5.39), fix \((\theta, \sigma, P) \in \mathfrak{S}\) and \(\psi > 0\). On \(\{\rho = T\}\), the terminal conditions of \(\hat{w}\) and \(\tilde{w}\) in (3.2)–(3.3) yield

\[
U(Y_T) = w^\psi(T, S_T, Y_T) = w^\psi(\rho, S_\rho, Y_\rho) \quad P\text{-a.s.}
\]

On \(\{\rho < T\}\), we have \(S_\rho \in \{K^{-1}, K\}\) and the boundary conditions of \(\hat{w}\) and \(\tilde{w}\) yield \(w^\psi(\rho, S_\rho, Y_\rho) = U(Y_\rho) P\text{-a.s.}\). It remains to show that \(Y_\rho = Y_T P\text{-a.s.}\).

Recall from (2.10) the dynamics of \(Y\) under \(P\):

\[
Y = y_0 + \int_0^\rho (\theta_s - \bar{V}_s(t, S_s)) \, dS_s + \int_0^\rho \frac{1}{2} S_s^2 \bar{V}_{ss}(t, S_s)(\bar{\sigma}(t, S_t)^2 - \sigma^2_t) \, dt.
\]

By definition of \(\mathfrak{S}\), \(S\) is a local martingale under \(P\). By (3.1), \(S\) is even a \([K^{-1}, K]\)-valued martingale under \(P\). Hence, \(S\) has to stay constant after hitting \(K^{-1}\) or \(K\). In particular, the local martingale term in the dynamics of \(Y\) is constant after time \(\rho\). Moreover, being constant, \(S\) has constant quadratic variation after time \(\rho\), so that by (2.11), \(\sigma_t S_t = 0 \, dt \times P\text{-a.e.} \) on \(\{(t, \omega) : t \geq \rho(\omega)\}\). Together with the assumption (3.9) that \(\bar{\sigma}(. K) \equiv \sigma(\cdot, K^{-1}) \equiv 0\), this yields that the drift term of the dynamics of \(Y\) stays constant after time \(\rho\) as well. Hence, \(Y_\rho = Y_T P\text{-a.s.}\).

(ii): By (3.1), \(Y\) and \(\sigma\) are bounded. Moreover, by (3.10), \(S^2 \bar{V}_{ss}(t, S_t)\) is bounded \(dt \times P\text{-a.e.} \) on \(\{(t, \omega) : t < \rho(\omega)\}\). Since we also know from the proof of part (i) that the drift part of \(Y\) stays constant after time \(\rho\), we conclude that the drift part is bounded. It follows that the local martingale part is also bounded. 

Lemma 5.6. There is a constant $C > 0$ such that, for every $\psi \in (0, \psi_c)$, $\sigma \in \mathcal{V}$, and $P \in \mathfrak{P}(\theta^\psi, \sigma)$ satisfying
\[
J^\psi(\sigma, P) \leq \inf_{\sigma' \in \mathcal{V}} \inf_{P' \in \mathfrak{P}(\theta^\psi, \sigma')} J^\psi(\sigma', P') + 1,
\] (5.40)
we have
\[
P[\tau < T] \leq C \psi
\]
(5.41)
for the stopping time $\tau$ defined in Theorem 3.3.

Proof. As an auxiliary result, we first prove that there is a constant $C_1 > 0$ such that, for every $\psi \in (0, \psi_c)$, $\sigma \in \mathcal{V}$, and $P \in \mathfrak{P}(\theta^\psi, \sigma)$ satisfying (5.40), we have
\[
E^P \left[ \int_0^\rho (\sigma_t - \bar{\sigma}(t, S_t))^2 \, dt \right] \leq C_1 \psi,
\]
(5.42)
where $\rho$ is the first time that $S$ leaves $(K^{-1}, K)$.

To this end, first fix $(t, s, y) \in D$. Expanding the function $\varsigma \mapsto f(t, s, y; \varsigma)$, $\varsigma \in [0, K]$, around $\bar{\sigma}(t, s)$ and using the minimum conditions (2.12), we obtain
\[
f(t, s, y; \varsigma) = \frac{1}{2} f''(t, s, y; \varsigma^L(t, s, y; \varsigma))(\varsigma - \bar{\sigma}(t, s))^2
\]
for some $\varsigma^L(t, s, y; \varsigma) \in [0, K]$. Then the assumption that $\frac{1}{K} \leq f'' \leq K$ on $D \times [0, K]$ from (3.11) gives
\[
\frac{2}{K} f(t, s, y; \varsigma) \leq (\varsigma - \bar{\sigma}(t, s))^2 \leq 2 K f(t, s, y; \varsigma), \quad (t, s, y) \in D, \varsigma \in [0, K].
\]
Using also that $U'(y)$ is positive, bounded, and bounded away from zero over $y \in (y_l, y_u)$, it follows that there is a constant $K'' > 0$ such that
\[
\frac{1}{K''} (\varsigma - \bar{\sigma}(t, s))^2 \leq U'(y) f(t, s, y; \varsigma) \leq K'' (\varsigma - \bar{\sigma}(t, s))^2, \quad (t, s, y) \in D, \varsigma \in [0, K].
\]
(5.43)

Next, set $C_1 := T(K'')^2 K^6 \psi_c + K''(U(y_u) + 1 - U(y_l))$ and fix $\psi \in (0, \psi_c)$, $\sigma \in \mathcal{V}$, and $P \in \mathfrak{P}(\theta^\psi, \sigma)$. Moreover, choose any $P^\psi \in \mathfrak{P}(\theta^\psi, \sigma^\psi)$. On the one hand, using the left-hand inequality in (5.43), the definition of $J^\psi$ in (2.13), and (5.40) gives
\[
E^P \left[ \int_0^\rho (\sigma_t - \bar{\sigma}(t, S_t))^2 \, dt \right] \leq K'' \psi E^P \left[ \frac{1}{\psi} \int_0^\rho U'(Y_t) f(t, S_t, Y_t; \sigma_t) \, dt \right]
\]
\[
\leq K'' \psi (J^\psi(\sigma, P) - E^P [U(Y_T)])
\]
\[
\leq K'' \psi \left( \inf_{\sigma' \in \mathcal{V}} \inf_{P' \in \mathfrak{P}(\theta^\psi, \sigma')} J^\psi(\sigma', P') + 1 - U(y_l) \right).
\]
(5.44)
On the other hand, using first that \( \sigma^\psi \in \mathcal{V} \), and then the right-hand inequality in (5.43), we obtain

\[
\inf_{\sigma' \in \mathcal{V}} \inf_{P \in \mathfrak{P}(\theta^\psi, \sigma')} J^\psi(\sigma', P) \leq J(\sigma^\psi, P^\psi)
= E^P \left[ \frac{1}{\psi} \int_0^T U'(Y_t) f(t, S_t, Y_t; \sigma^\psi_t) \, dt + U(Y_T) \right]
\leq \frac{K''}{\psi} E^P \left[ \int_0^T \left( \sigma^\psi_t - \bar{\sigma}(t, S_t) \right)^2 \, dt \right] + U(y_u).
\]

In view of (5.1), (3.7), and (3.9)–(3.12), we find that

\[
\left| \sigma^\psi_t - \bar{\sigma}(t, S_t) \right| = |\bar{\sigma}(t, S_t, Y_t)| \psi = \left| \frac{\bar{\sigma}(t, S_t) S^2_{t,S} V_{ss}(t, S_t)}{f''(t, S_t, Y_t; \bar{\sigma}(t, S_t))} \right| \psi \leq K^3 \psi \, dt \times P^\psi\text{-a.e.}
\]

Combining this with (5.44)–(5.45) yields (5.42) via

\[
E^P \left[ \int_0^\rho (\sigma_t - \bar{\sigma}(t, S_t))^2 \, dt \right] \leq \frac{K''}{\psi} \left( \frac{K''}{\psi} TK^6 \psi^2 + U(y_u) + 1 - U(y_t) \right)
\leq (TK''K^6E^c + K''(U(y_u) + 1 - U(y_t))) \psi
= C_1 \psi.
\]

To prove (5.41), first note that by definition of \( \bar{\theta} \) in (3.6) and our assumptions on \( \bar{w} \) and \( U \), there is a constant \( L > 0 \) (depending only on \( K, y_t, y_u, \) and \( U \)) such that \( |\bar{\theta}| \leq L \) on \( D \). Moreover, by standard estimates for Itô processes (cf. [142, Lemma V.11.5]), there is a constant \( C' > 0 \) (depending only on \( T \)) such that

\[
E^P \left[ \sup_{\theta \leq \rho} |Y_t - y_0|^2 \right] \leq C' E^P \left[ \int_0^\rho \left( \left( \left( \theta^\psi_t - \bar{V}_s(t, S_t) \right) S_t \sigma_t \right)^2 + \left( \frac{1}{2} S^2_{t,S} V_{ss}(t, S_t) \bar{\sigma}(t, S_t)^2 - \sigma^2_t \right)^2 \right) \, dt \right].
\]

Now, set \( C := C''TL^2K^4E^c + C'K^4C_1 \) and fix \( \psi \in (0, \psi_c) \), \( \sigma \in \mathcal{V} \), and \( P \in \mathfrak{P}(\theta^\psi, \sigma) \) satisfying (5.40). Note that by (3.1) and (3.9)–(3.10), we have \( dt \times P \text{-a.e. on } \{(t, \omega) : t < \rho(\omega)\} \),

\[
\left| \left( \theta^\psi_t - \bar{V}_s(t, S_t) \right) \sigma_t S_t \right| = \left| \bar{\theta}(t, S_t, Y_t) \mathbf{1}_{\{t < \tau\}} \psi \sigma_t S_t \right| \leq L \psi K^2,
\]

\[
\left| \frac{1}{2} S^2_{t,S} V_{ss}(t, S_t) \bar{\sigma}(t, S_t)^2 - \sigma^2_t \right| \leq \frac{1}{2} K \left| \bar{\sigma}(t, S_t) + \sigma_t \right| \left| \bar{\sigma}(t, S_t) - \sigma_t \right|
\leq K^2 \left| \bar{\sigma}(t, S_t) - \sigma_t \right|.
\]

Recall that \( Y \) remains constant after time \( \rho \). Therefore, Markov’s inequality, the
above estimates, and the auxiliary estimate (5.42) yield the assertion:

\[
P[\tau < T] \leq P[\tau \leq \rho] \leq P \left[ \sup_{0 \leq t \leq \rho} |Y_t - y_0|^2 \geq 1 \right] \leq E^P \left[ \sup_{0 \leq t \leq \rho} |Y_t - y_0|^2 \right]
\]
\[
\leq C'T (L\psi K^2)^2 + C'K^4 E^P \left[ \int_0^\rho (\bar{a}(t, S_t) - \sigma_t)^2 \, dt \right]
\]
\[
\leq (C'TL^2 K^4 \psi + C'K^4 C_1) \psi
\]
\[
= C \psi.
\]

Lemma 5.7. As \( \psi \downarrow 0 \), inequality (5.5) holds true:

\[
\inf_{\sigma \in \mathcal{V}} \inf_{P \in \mathcal{P}(\psi, \sigma)} J^\psi(\sigma, P) \geq w_0^\psi + o(\psi^2).
\]

Proof. Fix \( \varepsilon > 0 \). Choose \( C > 0 \) large enough and \( \psi_0 \in (0, \psi_\varepsilon) \) small enough, so that we may use the assertions of Lemmas 5.4 and 5.6. Moreover, choosing \( \psi_0 \) even smaller if necessary, we may assume that \( TC(\psi_0 + C\psi_0) \leq \frac{1}{2} \varepsilon \) and \( \frac{1}{2} \varepsilon \psi_0^2 \leq 1 \). Fix \( \psi \in (0, \psi_0) \). We need to show that

\[
\inf_{\sigma \in \mathcal{V}} \inf_{P \in \mathcal{P}(\psi, \sigma)} J^\psi(\sigma, P) - w_0^\psi \geq -\varepsilon \psi^2.
\]

Choose \( \sigma \in \mathcal{V} \) and \( P \in \mathcal{P}(\psi, \sigma) \) such that

\[
J^\psi(\sigma, P) \leq \inf_{\sigma' \in \mathcal{V}} \inf_{P' \in \mathcal{P}(\psi, \sigma')} J^\psi(\sigma', P') + \frac{1}{2} \varepsilon \psi^2
\]

(in particular, condition (5.40) of Lemma 5.6 holds, so we may use (5.41) later). Then

\[
\inf_{\sigma' \in \mathcal{V}} \inf_{P' \in \mathcal{P}(\psi, \sigma')} J^\psi(\sigma', P') - w_0^\psi \geq J^\psi(\sigma, P) - w_0^\psi - \frac{1}{2} \varepsilon \psi^2.
\]  

Let \( \rho \) be the first time that \( S \) leaves \( (K^{-1}, K) \). By Proposition 5.5 (i), \( U(Y_T) = w^\psi(\rho, S_\rho, Y_\rho) \) \( P \)-a.s. Using this together with the definition of \( J^\psi \) in (2.13), Itô’s formula for the process \( w^\psi(t, S_t, Y_t) \) (up to time \( \rho \)), and the formulas (2.10) and (2.11) (with \( \theta \) replaced by \( \theta^\psi \)) describing the dynamics of \( S \) and \( Y \) under \( P \), we obtain

\[
J^\psi(\sigma, P) - w_0^\psi \geq E^P \left[ \int_0^\rho w^\psi_\xi(t, S_t, Y_t) \, dS_t + \int_0^\rho w^\psi_\eta(t, S_t, Y_t)(\theta_t^\psi - \bar{\Delta}_t) \, dS_t \right]
\]
\[
+ E^P \left[ \int_0^\rho \left( w^\psi_\xi(t, S_t, Y_t) + H^\psi(t, S_t, Y_t; \theta_t^\psi, \sigma_t) \right) \, dt \right],
\]

where \( H^\psi \) is the Hamiltonian defined in (5.4). We claim that both stochastic integrals inside the first expectation on the right-hand side of (5.47) are true martingales under \( P \). First, recall that \( S \) and \( \int_0^\rho (\theta_t^\psi - \bar{\Delta}_t) \, dS_t \) are bounded martingales under \( P \); cf. Proposition 5.5 (ii). Since \( w^\psi_\xi \) and \( w^\psi_\eta \) are bounded by (5.3),
the claim follows. So the first expectation in (5.47) vanishes and it remains to estimate the second term.

Splitting the dt-integral into two parts separated by $\tau \wedge \rho$, using that $\theta^\psi = \theta^\psi(t, S_t, Y_t)$ on $\{ t < \tau \}$ and $\theta^\psi = V^\psi(t, S_t)$ on $\{ t \geq \tau \}$, and applying (5.29), (5.28) and, in the penultimate inequality, (5.41), we obtain

$$J^\psi(\sigma, P) - w_0^\psi \geq -E_P \left[ \int_0^{\rho \wedge \tau} C^3 dt + \int_0^\rho C^2 \rho^2 dt \right] \geq -TC^2(\psi + P[\tau < T])$$

$$\geq -TC^2(\psi + C) \geq -\frac{1}{2} \varepsilon^2. \quad (5.48)$$

Combining (5.48) with (5.46) completes the proof. \(\square\)

5.1.3 The upper bound (5.6)

Lemma 5.8. As $\psi \downarrow 0$, inequality (5.6) holds true:

$$\sup_{\theta \in A} \inf_{P \in \mathcal{P}(\theta, \sigma^\psi)} J^\psi(\sigma^\psi, P) \leq w_0^\psi + o(\psi^2).$$

The proof is analogous to the proof of Lemma 5.7, but easier.

Proof. Fix $\varepsilon > 0$. Choose $C > 0$ large enough and $\psi_0 \in (0, \psi_c)$ small enough, so that we may use the assertions of Lemma 5.4. Moreover, choosing $\psi_0$ even smaller if necessary, we may assume that $TC\psi_0 \leq \frac{1}{2} \varepsilon$. Fix $\psi \in (0, \psi_0)$. We need to show that

$$\sup_{\theta \in A} \inf_{P \in \mathcal{P}(\theta, \sigma^\psi)} J^\psi(\sigma^\psi, P) - w_0^\psi \leq \varepsilon^2. \quad (5.49)$$

Choose $\theta \in A$ such that

$$\inf_{P \in \mathcal{P}(\theta, \sigma^\psi)} J^\psi(\sigma^\psi, P) + \frac{1}{2} \varepsilon^2 \geq \sup_{\theta \in A} \inf_{P \in \mathcal{P}(\theta', \sigma^\psi)} J^\psi(\sigma^\psi, P)$$

and fix any $P \in \mathcal{P}(\theta, \sigma^\psi)$. Then

$$\sup_{\theta' \in A} \inf_{P \in \mathcal{P}(\theta', \sigma^\psi)} J^\psi(\sigma^\psi, P) - w_0^\psi \leq J^\psi(\sigma^\psi, P) - w_0^\psi + \frac{1}{2} \varepsilon^2. \quad (5.49)$$

Let $\rho$ be the first time that $S$ leaves $(K^{-1}, K)$. Recall from the proof of Proposition 5.5 (i) that $S$ has to stay constant (at $K^{-1}$ or $K$) after time $\rho$ (as it is a $[K^{-1}, K]$-valued $P$-martingale). Hence, $\sigma^\psi = \tilde{\sigma}(t, S_t) = 0$ on $\{ t \geq \rho \}$ by (5.1) and (3.12). Then $f(t, S_t, Y_t; \sigma^\psi) = 0$ on $\{ t \geq \rho \}$ by (2.12) and thus

$$J^\psi(\sigma^\psi, P) = E_P \left[ \frac{1}{\psi} \int_0^\rho U'(Y_t) f(t, S_t, Y_t; \sigma^\psi) dt + U(Y_T) \right].$$

Using this and proceeding as in the proof of Lemma 5.7, we obtain

$$J^\psi(\sigma^\psi, P) - w_0^\psi = E_P \left[ \int_0^\rho \psi^\psi(t, S_t, Y_t) dS_t + \int_0^\rho w^\psi(t, S_t, Y_t) \tilde{\rho}_t - \Delta_t dt \right]$$

$$+ E_P \left[ \int_0^\rho \left( w^\psi(t, S_t, Y_t) + H^\psi(t, S_t, Y_t; \tilde{\rho}_t, \sigma^\psi) \right) dt \right]. \quad (5.50)$$
By the same argument as in the proof of Lemma 5.7, the first expectation in (5.50) vanishes and it remains to estimate the second term. Using (5.30) from Lemma 5.4 yields

\[ J(\psi, \sigma, P) - \psi_0 \leq E^P \left[ \int_0^\rho C \psi^3 \, dt \right] \leq TC \psi^3 \leq \frac{1}{2} \varepsilon \psi^2. \quad (5.51) \]

Combining (5.51) with (5.49) completes the proof.

5.2 Feynman–Kac representation

Proof of Proposition 3.5. Fix \((t, s) \in [0, T] \times [K^{-1}, K]\) and let \(\tilde{\sigma}(u, \cdot) : \mathbb{R} \to \mathbb{R}\) be a continuous extension of \(\bar{\sigma}(u, \cdot)|_{(K^{-1}, K)}\) to \(\mathbb{R}\) for each \(u \in [0, T]\). By (3.9), \(\tilde{\sigma}\) can also be chosen bounded. Therefore, a standard result (see, e.g., [98, Theorems 21.9 and 21.7]) yields the existence of a weak solution to the SDE

\[ d \tilde{S}_u = \tilde{S}_u \tilde{\sigma}(u, \tilde{S}_u) \, d \tilde{W}_u, \quad \tilde{S}_t = s. \quad (5.52) \]

That is, there is a filtered probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})\) supporting a Brownian motion \(\tilde{W}\) and a process \(\tilde{S} = (\tilde{S}_u)_{u \in [t, T]}\) satisfying (5.52). Now, define the process \(S = (S_u)_{u \in [t, T]}\) by \(S_u = \tilde{S}_{\rho \wedge u}\) where \(\rho := \inf\{u \in [t, T] : S_u \not\in (K^{-1}, K)\}\) is the first time that \(S\) leaves \((K^{-1}, K)\). Hence, \(S\) evolves like \(\tilde{S}\), but is stopped as soon as it hits \(K^{-1}\) or \(K\). Using that \(\bar{\sigma}(u, \cdot) = 0\) on \(\{K^{-1}, K\}\) by (3.9), it is easy to show that \(S\) has the reference dynamics, i.e.,

\[ dS_u = S_u \bar{\sigma}(u, S_u) \, dW_u, \quad S_t = s. \quad (5.52) \]

Next, fix \(y \in (y_l, y_u)\). Applying Itô’s formula to \(\tilde{w}(u, S_u, y)\) (up to time \(\rho\)) and using the terminal and boundary conditions of \(\tilde{w}\) in (3.2) gives

\[ 0 = \tilde{w}(\rho, S_\rho, y) = \tilde{w}(t, s, y) + \int_t^\rho \tilde{w}_s(u, S_u, y) \, dS_u \]

\[ + \int_t^\rho \left\{ \tilde{w}_t(u, S_u, y) + \frac{1}{2} \bar{\sigma}(u, S_u)^2 S_u^{-1} \tilde{w}_{ss}(u, S_u, y) \right\} \, du. \]

Using the PDE (3.2) to substitute the drift term and rearranging terms, we obtain

\[ \tilde{w}(t, s, y) = -\int_t^\rho \tilde{w}_s(u, S_u, y) \, dS_u + \int_t^\rho \bar{g}(u, S_u, y) \, du. \quad (5.53) \]

Using our definition that \(\bar{g}(t, s, y) = 0\) for \(s \in \{K^{-1}, K\}\) (cf. (3.12)) and the fact that \(S\) stays constant (at \(K^{-1}\) or \(K\)) after time \(\rho\), we may replace the upper limit of the last integral in (5.53) by \(T\). Moreover, by the boundedness of \(\bar{\sigma}\) and \(\tilde{w}\) (cf. (3.8)), the stochastic integral is a martingale under \(\tilde{P}\). Therefore, taking expectations in (5.53) yields the Feynman–Kac representation (3.14) (where the integrand is understood as zero if \(S_u \in \{K^{-1}, K\}\)).
5.3 Existence of probability scenarios

*Proof of Theorem 3.7.* To prove the first assertion, fix \((\theta, \sigma) \in A_0 \times \mathcal{V}_0\). The goal is to construct continuous, adapted processes \(S'\) and \(Y'\) on some probability space \((\Omega', \mathcal{F}', \mathbb{P}', \mathcal{P}')\) satisfying (3.1) as well as

\[
S'_t = s_0 + \int_0^t S'_u \sigma_u((S', Y')) \, dW'_u, \\
Y'_t = y_0 + \int_0^t (\theta_u((S', Y')) - \tilde{V}_s(u, S'_u)) \, ds'_u
+ \int_0^t \frac{1}{2} (S''_u)^2 \tilde{V}_{ss}(u, S'_u) \left( \tilde{\sigma}(u, S'_u)^2 - \sigma_u((S', Y'))^2 \right) \, du,
\]

for a Brownian motion \(W'\). Comparing this with (2.10)–(2.11), we see that the image measure \(P := P' \circ (S', Y')^{-1}\) on \((\Omega, \mathcal{F})\) then satisfies \((\theta, \sigma, P) \in \mathcal{G}\).

We proceed as follows. Note that by (3.20), the diffusion coefficient of \(Y'\) changes to 0 at time \(\tau\). Hence, we first apply a general existence result to obtain a weak solution up to time \(\tau\), ignoring for the moment that \(S'\) or \(Y'\) might exceed the bounds (3.1). Then we employ Theorem B.1 to extend the process after time \(\tau\). Finally, we can stop the extended process at the first time that \(S\) leaves \((K^{-1}, K)\) to obtain a candidate for \((S', Y')\).

First, introduce the cut-off function \(h : \mathbb{R} \to [K^{-1}, K], h(s) := (s \wedge K) \vee K^{-1}\), and recall that \(\theta\) and \(\sigma\) are of the form (3.20) and (3.21), respectively. Consider the SDE

\[
S'^{(1)}_t = s_0 + \int_0^t h(S'^{(1)}_u) \tilde{\sigma}(u, S'^{(1)}_u, Y'^{(1)}_u) \, dW'_u, \\
Y'^{(1)}_t = y_0 + \int_0^t \tilde{\theta}_u((S^{(1)}, Y^{(1)})) h(S'^{(1)}_u) \tilde{\sigma}(u, S'^{(1)}_u, Y'^{(1)}_u) \, dW'_u
+ \int_0^t \frac{1}{2} h(S'^{(1)}_u)^2 \tilde{V}_{ss}(u, h(S'^{(1)}_u)) \left( \tilde{\sigma}(u, h(S'^{(1)}_u))^2 - \tilde{\sigma}(u, S'^{(1)}_u, Y'^{(1)}_u)^2 \right) \, du,
\]

where \(\tilde{V}_{ss}\) and \(\tilde{\sigma}\) are Lipschitz continuous extensions of \(\tilde{V}_{ss}|_{(0, T) \times (K^{-1}, K)}\) and \(\tilde{\sigma}|_{(0, T) \times (K^{-1}, K)}\) to the closure \([0, T] \times [K^{-1}, K]\). By our assumptions, the drift and diffusion coefficients of this SDE are uniformly bounded and continuous on \(\Omega\) for each fixed \(u \in (0, T)\). Therefore, the SDE has a weak solution (see, e.g., [98, Theorems 21.9 and 21.7]). In other words, there exist a filtered probability space \((\Omega', \mathcal{F}', \mathbb{F}', \mathcal{P}')\) carrying a \((\mathbb{P}', \mathbb{F}')\)-Brownian motion \(W'\) and continuous, \(\mathbb{F}'\)-adapted processes \(S'^{(1)}, Y'^{(1)}\) satisfying (5.56)–(5.57). Recall that \(\tau\) is an \(\mathbb{F}\)-stopping time on \((\Omega, \mathcal{F})\). Applying Theorem B.1 to \(X'^{(1)} := (S'^{(1)}, Y'^{(1)})\) and the
\( \mathbb{F} \)-stopping time \( \tau \), there exists an \( \mathbb{F}' \)-adapted process \( X^{(2)} = (S^{(2)}, Y^{(2)}) \) satisfying

\[
S^{(2)}_t = s_0 + \int_0^t h(S^{(2)}_u) \tilde{\sigma}(u, S^{(2)}_u, Y^{(2)}_u) \, dW'_u,
\]

\[
Y^{(2)}_t = y_0 + \int_0^t \tilde{\theta}_u((S^{(2)}_u, Y^{(2)}_u)) h(S^{(2)}_u) \tilde{\sigma}(u, S^{(2)}_u, Y^{(2)}_u) \, dW'_u + \int_0^t \frac{1}{2} h(S^{(2)}_u)^2 \tilde{V}_u(u, h(S^{(2)}_u))(\tilde{\sigma}(u, h(S^{(2)}_u))^2 - \tilde{\sigma}(u, S^{(2)}_u, Y^{(2)}_u)^2) \, du;
\]

note that only the diffusion coefficient of the \( Y \)-component is set to 0 after time \( \tau(X^{(1)}) \) and that the remaining drift and diffusion coefficients are uniformly bounded and Lipschitz continuous in all variables. Next, define \( X' = (S', Y') \) by \( S'_t := S^{(2)}_{t \wedge \rho(X^{(2)})} \) and \( Y'_t := Y^{(2)}_{t \wedge \rho(X^{(2)})} \) where \( \rho := \inf\{t \in [0, T] : S_t \not\in (K^{-1}, K)\} \wedge T \) is the first time that \( S \) leaves \((K^{-1}, K)\); note that \( \rho \) is an \( \mathbb{F} \)-stopping time on \((\Omega, \mathcal{F})\). Then

\[
S'_t = s_0 + \int_0^t h(S'_u) \tilde{\sigma}(u, S'_u, Y'_u) \mathbf{1}_{\{u < \rho(X^{(2)})\}} \, dW'_u,
\]

\[
Y'_t = y_0 + \int_0^t \tilde{\theta}_u((S', Y')) h(S'_u) \tilde{\sigma}(u, S'_u, Y'_u) \mathbf{1}_{\{u < \tau(X^{(1)})\}} \mathbf{1}_{\{u < \rho(X^{(2)})\}} \, dW'_u + \int_0^t \frac{1}{2} h(S'_u)^2 \tilde{V}_u(u, h(S'_u))(\tilde{\sigma}(u, h(S'_u))^2 - \tilde{\sigma}(u, S'_u, Y'_u)^2) \mathbf{1}_{\{u < \rho(X^{(2)})\}} \, du.
\]

Now note that on \( \{u < \rho(X^{(2)})\} \), \( S'_u \in (K^{-1}, K) \), so \( h(S'_u) = S'_u \), \( \tilde{\sigma}(u, S'_u) = \tilde{\sigma}(u, S'_u) \) and \( \tilde{V}_u(u, S'_u) = \tilde{V}_u(u, S'_u) \). Moreover, for each \( \omega' \in \Omega' \), as \( X^{(1)}(\omega') = X^{(2)}(\omega') \) for \( u \leq \tau(X^{(1)}(\omega')) \) by construction of \( X^{(2)} \), Galmarino’s test [43, Theorem IV.100] implies that \( \tau(X^{(1)}(\omega')) = \tau(X^{(2)}(\omega')) \). Similarly, we find that \( \rho(X^{(2)}) = \rho(X') \). Combining these observations with (5.58)–(5.59) yields

\[
S'_t = s_0 + \int_0^t S'_u \tilde{\sigma}(u, S'_u, Y'_u) \mathbf{1}_{\{u < \rho(X')\}} \, dW'_u,
\]

\[
Y'_t = y_0 + \int_0^t \tilde{\theta}_u((S', Y')) S'_u \tilde{\sigma}(u, S'_u, Y'_u) \mathbf{1}_{\{u < (\tau \wedge \rho)(X^{(2)})\}} \, dW'_u + \int_0^t \frac{1}{2} (S'_u)^2 \tilde{V}_u(u, S'_u)(\tilde{\sigma}(u, S'_u)^2 - \tilde{\sigma}(u, S'_u, Y'_u)^2) \mathbf{1}_{\{u < \rho(X')\}} \, du.
\]

First, note that \( \tilde{\sigma}(u, S'_u, Y'_u) \mathbf{1}_{\{u < \rho(X')\}} = \sigma_u((S', Y')) \) by (3.21) and the fact that \( S' \) stays constant after having hit \( K^{-1} \) or \( K \). In particular, \( S' \) satisfies (5.54) as desired. Next, let us analyse the drift term in (5.61). To this end, note that on \( \{u \geq \rho(X')\} \), \( S'_u = S'_{\rho(X')} \in (K^{-1}, K) \), and so \( \tilde{\sigma}(u, S'_u) = 0 \) by (3.9). Thus, the drift term can be rewritten as

\[
\int_0^t \frac{1}{2} (S'_u)^2 \tilde{V}_u(u, S'_u)(\tilde{\sigma}(u, S'_u)^2 - \sigma_u((S', Y'))^2) \, du.
\]
in accordance with (5.55). Finally, let us turn to the diffusion term in (5.61). \( \tau \wedge \rho \) is an \( \mathbb{F} \)-stopping time on \((\Omega, \mathcal{F})\). Moreover, for each \( \omega' \in \Omega' \), as \( X_u^{(2)}(\omega') = X_u'(\omega') \) for \( u \leq \rho(X_u^3(\omega')) \) by construction of \( X' \), Galmarino's test implies that 
\[ (\tau \wedge \rho)(X_u^3(\omega')) = (\tau \wedge \rho)(X'_u(\omega')) \]
Using this, the diffusion term in (5.61) can be rewritten as
\[
\int_0^t \tilde{\theta}_a((S', Y')) 1_{\{u < \tau(X'_u)\}} S_u' \tilde{\sigma}(u, S_u', Y_u') \mathbf{1}_{\{u < \rho(X'_u)\}} \, dW_u' \\
= \int_0^t \left( \tilde{\theta}_a((S', Y')) - \tilde{V}_a(u, S_u') \right) dS_u',
\]
where we use (3.20) and (5.60) in the last step. This shows that \( Y' \) satisfies (5.55) as desired.

It remains to check that \((S', Y')\) does not leave \([K^{-1}, K] \times (y_l, y_u)\). It is clear that \( S' \) evolves in \([K^{-1}, K]\) as the diffusion coefficient in (5.54) is set to zero as soon as \( S' \) hits \( K^{-1} \) or \( K \). Concerning \( Y' \), we see from the definition of \( \tau \) that the diffusion coefficient in (5.55) is set to 0 as soon as \(|Y' - y_0| = 1\). Moreover, using (3.10) and the fact that all volatilities take values in \([0, K]\), the absolute value of the drift term in (5.55) is bounded by \( \frac{1}{2} K^3 T \). It follows that \( Y' \) evolves in \([y_0 - 1 - \frac{1}{2} K^3 T, y_0 + 1 + \frac{1}{2} K^3 T] \subset (y_l, y_u)\). This completes the proof of the first assertion in Theorem 3.7.

For the second assertion, we have to show that \((\theta^\psi, \sigma^\psi) \in \mathcal{A}_0 \times \mathcal{V}_0\) for \( \psi > 0 \) small enough. Recall from Theorem 3.3 that
\[
\theta_t^\psi = \tilde{V}_s(t, S_t) + \tilde{\theta}(t, S_t, Y_t) \mathbf{1}_{\{t < \tau\}} \psi, \\
\sigma_t^\psi = \tilde{\sigma}(t, S_t) + \tilde{\sigma}(t, S_t, Y_t) \psi,
\]
with
\[
\tilde{\theta}(t, s, y) = \tilde{w}_s(t, s, y) + \frac{U'(y)}{U''(y)} \tilde{w}_{sy}(t, s, y), \\
\tilde{\sigma}(t, s, y) = \frac{\tilde{\sigma}(t, s) + 2 \tilde{V}_{ss}(t, s)}{f''(t, s; \tilde{\sigma}(t, s))}.
\]
Under our assumptions, for any \( \psi > 0 \), \( \tilde{\theta}_t := \tilde{\theta}(t, S_t, Y_t) \psi \) is clearly progressively measurable as well as bounded and continuous as a function on \([0, T] \times \Omega\). Thus, \( \theta^\psi \in \mathcal{A}_0 \).

Setting \( \tilde{\sigma}(t, s, y) := \tilde{\sigma}(t, s) + \tilde{\sigma}(t, s, y) \psi \) and using that \( \tilde{\sigma}(t, \cdot) = 0 \) on \( \{K^{-1}, K\} \) by (3.9), we can write \( \sigma_t^\psi = \tilde{\sigma}(t, S_t, Y_t) \mathbf{1}_{\{S_t \in (K^{-1}, K)\}} \). To see that \( \sigma^\psi \in \mathcal{V}_0 \), we thus have to check that \( \tilde{\sigma}(t, s, y) \) is Lipschitz continuous on \((0, T) \times (K^{-1}, K) \times (y_l, y_u)\) (then it can be extended to a Lipschitz continuous function on \([0, T] \times \mathbb{R}^2\) and takes values in \([0, K]\). Lipschitz continuity of \( \tilde{\sigma} \) follows from our assumptions (this uses in particular that \( \frac{1}{K} \leq f'' \leq K \) from (3.11)). Whence, \( \tilde{\sigma} \) is Lipschitz continuous as well. Moreover, \( \tilde{\sigma} \) is bounded. Combining this with (3.9), we conclude that for \( \psi > 0 \) small enough, \( \sigma^\psi \) takes values in \([0, K]\). \(\square\)
A Calculus

The following result is an extension of the implicit function theorem that allows the defining function to depend on a parameter. In particular, it provides parameter-independent bounds for the first and second derivatives of the implicitly defined functions.

**Lemma A.1.** Let $\Lambda \neq \emptyset$ be a set and $U, V$ open subsets of $\mathbb{R}$. For each $\lambda \in \Lambda$, let $F_{\lambda} : U \times V \to \mathbb{R}$ and $y_{\lambda} : U \to V$ be twice continuously differentiable functions with

$$F_{\lambda}(x, y_{\lambda}(x)) = 0, \quad x \in U.$$  \hspace{1cm} (A.1)

If there are constants $M_0 > 1$ and $M_1 \geq 0$ such that for each $\lambda \in \Lambda$ and all $(x, y) \in U \times V$,

$$\left| \frac{\partial F_{\lambda}}{\partial x}(x, y_{\lambda}(x)) \right|, \left| \frac{\partial^2 F_{\lambda}}{\partial x^2}(x, y_{\lambda}(x)) \right| \leq M_0 + M_1|y|, \quad (\forall \lambda \in \Lambda, x \in U),$$

$$\left| \frac{\partial^2 F_{\lambda}}{\partial x \partial y}(x, y_{\lambda}(x)) \right|, \left| \frac{\partial^2 F_{\lambda}}{\partial y^2}(x, y_{\lambda}(x)) \right| \leq M_0, \quad \left| \frac{\partial F_{\lambda}}{\partial y}(x, y_{\lambda}(x)) \right| \geq \frac{1}{M_0},$$  \hspace{1cm} (A.2)

then there is a constant $\tilde{M} > 0$ such that for all $\lambda \in \Lambda$ and $x \in U$,

$$|y'_{\lambda}(x)| \leq \tilde{M}(1 + M_1|y_{\lambda}(x)|) \quad \text{and} \quad |y''_{\lambda}(x)| \leq \tilde{M}(1 + M_1|y_{\lambda}(x)| + M_1|y_{\lambda}(x)|^2).$$

Moreover, if $\frac{\partial^2 F_{\lambda}}{\partial y^2} \equiv 0$, then for all $\lambda \in \Lambda$ and $x \in U$,

$$|y''_{\lambda}(x)| \leq \tilde{M}(1 + M_1|y_{\lambda}(x)|).$$

**Proof.** Taking the derivative of (A.1) with respect to $x$ yields

$$\frac{\partial F_{\lambda}}{\partial x}(x, y_{\lambda}(x)) + \frac{\partial F_{\lambda}}{\partial y}(x, y_{\lambda}(x))y'_{\lambda}(x) = 0, \quad \lambda \in \Lambda, x \in U.$$  \hspace{1cm} (A.3)

Solving this for $y'_{\lambda}(x)$ and using the bounds (A.2) then gives

$$|y'_{\lambda}(x)| \leq M_0(M_0 + M_1|y_{\lambda}(x)|), \quad \lambda \in \Lambda, x \in U.$$  \hspace{1cm} (A.4)

Taking the derivative in (A.3), we obtain for all $x \in U$,

$$\frac{\partial^2 F_{\lambda}}{\partial x^2}(x, y_{\lambda}(x)) + 2\frac{\partial^2 F_{\lambda}}{\partial x \partial y}(x, y_{\lambda}(x))y'_{\lambda}(x)$$

$$+ \frac{\partial^2 F_{\lambda}}{\partial y^2}(x, y_{\lambda}(x))(y'_{\lambda}(x))^2 + \frac{\partial F_{\lambda}}{\partial y}(x, y_{\lambda}(x))y''_{\lambda}(x) = 0.$$  \hspace{1cm} (A.5)

Again, solving for $y''_{\lambda}(x)$ and using the bounds (A.2) as well as (A.4) gives

$$|y''_{\lambda}(x)| \leq \tilde{M}(1 + M_1|y_{\lambda}(x)| + M_1|y_{\lambda}(x)|^2), \quad \lambda \in \Lambda, x \in U,$$  \hspace{1cm} (A.5)

for some sufficiently large constant $\tilde{M}$. Finally, if $\frac{\partial^2 F_{\lambda}}{\partial y^2} \equiv 0$, it is easily seen that the quadratic term in (A.5) vanishes.

\qed
B Stochastic differential equations

Fix an abstract filtered probability space \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)\) carrying an \(r\)-dimensional Brownian motion \(W = (W^1_t, \ldots, W^r_t)_{t \geq 0}\). The goal of this section is to prove the existence of solutions of a class of stochastic differential equations (SDEs) whose coefficients change at a stopping time. More precisely, we consider SDEs of the form

\[
X_t = \xi + \int_0^t \sigma(s, X) \, dW_s + \int_0^t b(s, X) \, ds, \quad t \geq 0, \tag{B.1}
\]

where \(\xi\) is an \(\mathbb{R}^d\)-valued \(\mathcal{F}_0\)-measurable random vector, \(X = (X^1_t, \ldots, X^d_t)_{t \geq 0}\) is a continuous semimartingale in \(\mathbb{R}^d\), and

\[
\begin{align*}
\sigma(s, X) &= \sigma^{(1)}(s, X) \mathbf{1}_{\{s < \tau(X)\}} + \sigma^{(2)}(s, X) \mathbf{1}_{\{s \geq \tau(X)\}}, \\
b(s, X) &= b^{(1)}(s, X) \mathbf{1}_{\{s < \tau(X)\}} + b^{(2)}(s, X) \mathbf{1}_{\{s \geq \tau(X)\}}. \tag{B.2}
\end{align*}
\]

Here, \(\tau\) is a stopping time for the filtration induced by the canonical process on \(C(\mathbb{R}_+; \mathbb{R}^d)\), \(\sigma^{(1)}, b^{(1)}\) are functions on \(\mathbb{R}_+ \times C(\mathbb{R}_+; \mathbb{R}^d)\) that are progressive for the same filtration, and \(\sigma^{(2)}, b^{(2)}\) are measurable functions on \(\mathbb{R}_+ \times \mathbb{R}^d\); all codomains are understood to be of suitable dimension.

First of all, note that we cannot apply general existence results directly to the coefficients \(\sigma\) and \(b\) since the stopping time \(\tau\) is typically not a continuous function on \(C(\mathbb{R}_+; \mathbb{R}^d)\). However, existence for this type of SDE is of course expected provided that solutions exist for both sets of coefficients separately. The obvious idea is to solve the SDE for the first set of coefficients, stop the solution \(\hat{\tau}\), and solve from there the SDE with the second set of coefficients. This can be made precise as follows:

**Theorem B.1.** Suppose that the process \(X^{(1)}\) on \((\Omega, \mathcal{F}, \mathbb{F}; P)\) satisfies

\[
X^{(1)}_t = \xi + \int_0^t \sigma^{(1)}(s, X^{(1)}) \, dW_s + \int_0^t b^{(1)}(s, X^{(1)}) \, ds, \quad t \geq 0. \tag{B.3}
\]

Moreover, assume that there is a constant \(K > 0\) such that for all \(t, t' \geq 0\) and \(x, x' \in \mathbb{R}^d\),

\[
|\sigma^{(2)}(t, x) - \sigma^{(2)}(t', x')| + |b^{(2)}(t, x) - b^{(2)}(t', x')| \leq K|t, x - (t', x')|, \tag{B.4}
\]

\[
|\sigma^{(2)}(t, x)| + |b^{(2)}(t, x)| \leq K(1 + |(t, x)|) \tag{B.5}
\]

where \(|\cdot|\) denotes the Euclidean norm in the suitable dimension. Then there is a continuous, \(\mathbb{F}\)-adapted, \(\mathbb{R}^d\)-valued process \(X\) satisfying (B.1).

**Proof.** The solution prior to time \(\hat{\tau} := \tau(X^{(1)})\) is already given. To construct the part of the solution after time \(\hat{\tau}\), consider the time-shifted filtration \(\hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{t \geq 0}\) defined by \(\hat{\mathcal{F}}_t = \mathcal{F}_{\hat{\tau} + t}\), and the time-shifted \((\hat{\mathbb{F}}, P)\)-Brownian motion \(\hat{W} = (\hat{W}_t)_{t \geq 0}\)
Plugging in (B.3) and (B.6) gives

\[ \hat{W}_t = \left( \frac{\hat{X}}{\hat{A}} \right)_t = \zeta + \int_0^t \begin{pmatrix} \sigma(2)(\hat{A}_s, \hat{X}_s) \\ 0 \\ b(2)(\hat{A}_s, \hat{X}_s) \end{pmatrix} \, d\hat{W}_s + \int_0^t \begin{pmatrix} \sigma(2)(\hat{A}_s, \hat{X}_s) \\ 0 \\ b(2)(\hat{A}_s, \hat{X}_s) \end{pmatrix} \, ds. \]

fulfill the standard Lipschitz and linear growth assumptions that guarantee the existence of a \( \mathcal{P}\)-a.s. unique strong solution for any \( \mathcal{F}_0 \)-measurable random vector \( \zeta \) in \( \mathbb{R}^{d+1} \). In particular, there exists an \( \mathcal{F} \)-progressive process \( \hat{Y} = (\hat{X}, \hat{A}) \) for the initial condition \( \zeta := (X_\hat{t}^{(1)}, \hat{t}) \). Clearly, \( \hat{A}_t = \hat{t} + t \) plays the role of the shifted time variable. A simple time change now yields that \( X_t^{(2)} := \hat{X}_{t-}\mathbf{1}_{\{t \geq \hat{t}\}} \) is \( \mathcal{F} \)-progressive and satisfies

\[ X_t^{(2)} = X_\hat{t}^{(1)} + \int_{\hat{t}}^t \sigma(2)(s, X_s^{(2)}) \, dW_s + \int_{\hat{t}}^t b(2)(s, X_s^{(2)}) \, ds \quad \text{on} \quad \{t \geq \hat{t}\}. \quad \text{(B.6)} \]

Finally, we verify that the process \( X_t := X_t^{(1)} \mathbf{1}_{\{t < \hat{t}\}} + X_t^{(2)} \mathbf{1}_{\{t \geq \hat{t}\}} \) is a solution to the original SDE (B.1). As \( X_t^{(2)} = X_\hat{t}^{(2)} \),

\[ X_t = X_{t \wedge \hat{t}} + X_{t \vee \hat{t}} - X_\hat{t}^{(2)}. \]

Plugging in (B.3) and (B.6) gives

\[
X_t = \xi + \int_0^{t \wedge \hat{t}} \sigma(1)(s, X_s^{(1)}) \, dW_s + \int_0^{t \wedge \hat{t}} b(1)(s, X_s^{(1)}) \, ds \\
+ \int_{\hat{t}}^{t \vee \hat{t}} \sigma(2)(s, X_s^{(2)}) \, dW_s + \int_{\hat{t}}^{t \vee \hat{t}} b(2)(s, X_s^{(2)}) \, ds \\
= \xi + \int_0^t \sigma(1)(s, X_s^{(1)}) \mathbf{1}_{\{s < \hat{t}\}} \, dW_s + \int_0^t b(1)(s, X_s^{(1)}) \mathbf{1}_{\{s < \hat{t}\}} \, ds \\
+ \int_0^t \sigma(2)(s, X_s^{(2)}) \mathbf{1}_{\{s \geq \hat{t}\}} \, dW_s + \int_0^t b(2)(s, X_s^{(2)}) \mathbf{1}_{\{s \geq \hat{t}\}} \, ds. \quad \text{(B.7)}
\]

As \( X_s^{(1)} = X_s \) on \( \{s \leq \hat{t}\} \), Galmarino’s test implies that \( \hat{t} = \tau(X^{(1)}) = \tau(X) \). Moreover, since \( \sigma^{(1)} \) is progressive, \( \sigma^{(1)}(s, X^{(1)}) \) only depends on the path of \( X^{(1)} \) up to time \( s \). Using also the definition of \( \sigma \) in (B.2), we obtain

\[ \sigma^{(1)}(s, X^{(1)}) \mathbf{1}_{\{s < \hat{t}\}} + \sigma^{(2)}(s, X_s^{(2)}) \mathbf{1}_{\{s \geq \hat{t}\}} = \sigma(s, X). \]

Using this and the analogous statement for the drift coefficients, we see from (B.7) that \( X \) is a solution to (B.1). \( \square \)
Chapter III

Model Uncertainty, Recalibration, and the Emergence of Delta-Vega Hedging

1 Introduction

In the context of hedging exotic derivatives, the way mathematical models for financial markets are applied in practice is often inconsistent with the assumptions these models are based on and the way they are analysed in academic research. Classical models prescribe the stochastic behaviour of certain financial variables, e.g., asset prices or interest rates, in terms of deterministic input quantities, the model’s parameters. In practice, however, these deterministic parameters are often not at all treated as deterministic: exotic derivatives traders recalibrate the parameters frequently to the observed market prices of liquidly traded vanilla options and use these options to neutralise the sensitivities of their positions against changes in these parameters (appropriately called out-of-model hedging by Rebonato [140]).

The benchmark Black–Scholes model is typically recalibrated by dynamic updating of the volatility parameter (that the model assumes constant) to the market price of a liquidly traded plain-vanilla option. Vega hedging\footnote{Vega is the sensitivity of the Black–Scholes price with respect to changes in the volatility parameter.} then corresponds to neutralising the sensitivity of the trader’s total position with respect to changes in the volatility parameter. The logical inconsistency of this practice is succinctly summarised by Rebonato [140, Section 1.3.2], for example:\footnote{Davis [39, Section 2. (b)], Musiela and Rutkowski [127, Section 7.1.8], and Wilmott [157, Section 7.10.5] raise the same concern.}

"Needless to say, out-of-model hedging is on conceptually rather shaky ground: if the volatility is deterministic and perfectly known, as many models used to arrive at the price assume it to be, there would be no need to undertake vega hedging. Furthermore, calculating the vega statistics means estimating the dependence on changes in volatility..."
of a price that has been arrived at assuming the self-same volatility to be both deterministic and perfectly known. Despite these logical problems, the adoption of out-of-model hedging in general, and of vega hedging in particular, is universal in the complex-derivatives trading community.”

This chapter provides a consistent justification for the use of Black–Scholes vega by acknowledging from the start that the true dynamics of the underlying are not known with certainty. We suppose that models are deemed more or less plausible depending on their “distance” from a reference Black–Scholes model for the underlying. A new feature is that the volatility parameter of the reference model is dynamically recalibrated to the observed prices of a liquidly traded vanilla option. In the limit for small aversion against this model uncertainty, delta-vega hedging then emerges naturally.

Hedging problem. Consider an agent who has sold a non-traded option on a stock $S$ with payoff $V(S_T)^3$ and has access to three liquidly traded securities to hedge her exposure: the stock $S$, a vanilla option $C$ on the stock (hereafter named “call”), and a bank account with zero interest rate. In practice, the market price of the call is typically quoted in terms of its (Black–Scholes) implied volatility. That is, instead of the market price $C_t$, traders quote the unique $\Sigma_t > 0$ such that

$$C_t = C(t, S_t, \Sigma_t),\tag{1.1}$$

where $C(t, S, \Sigma)$ is the Black–Scholes price of the call corresponding to the volatility parameter $\Sigma$. Whence, instead of modelling the dynamics of the stock and call prices, one can equivalently describe the dynamics of the stock price and the implied volatility of the call, and define the call price via (1.1).

If the stock and the call are traded using a self-financing strategy $\upsilon = (\theta, \phi)$, the corresponding Profit & Loss (P&L) process $Y^\upsilon$ has the following dynamics:

$$dY^\upsilon_t = \theta_t dS_t + \phi_t dC_t - dV(t, S_t, \Sigma_t).$$

Here, $V(t, S_t, \Sigma_t)$ is the Black–Scholes price of the non-traded option $V$ evaluated at the implied volatility $\Sigma_t$ backed out from the price of the call at time $t$. That is, the non-traded option is “marked to model”, whereas the liquidly traded stock and call are “marked to market”. However, at maturity $T$ of the non-traded option, $V(T, S_T, \Sigma_T) = V(S_T)$ is the option payoff so that $Y^\upsilon_T$ coincides with the agent’s actual terminal P&L.

We assume that the agent is uncertain about the dynamics of the stock and the call. To wit, she considers all probability measures $P$ under which the dynamics

\footnote{For simplicity, we restrict ourselves to vanilla options in this introduction. Our main result, Theorem 4.4, is also applicable to a wide range of exotic options like barrier options, lookback options, Asian options, forward-start options, and options on the realised variance of the stock.}
of \((S, \Sigma)\) are governed by\(^4\)

\[
\begin{align*}
\text{d}S_t &= S_t \sigma_t^P \, \text{d}W_t^0, \\
\text{d}\Sigma_t &= \nu_t^P \, \text{d}t + \eta_t^P \, \text{d}W_t^0 + \sqrt{\xi_t^P} \, \text{d}W_t^1,
\end{align*}
\]

for a Brownian motion \((W^0, W^1)\) in \(\mathbb{R}^2\) and a process \(\zeta^P = (\nu^P, \sigma^P, \eta^P, \xi^P)\) satisfying\(^5\)

\[
\nu_t^P \mathcal{C}_\Sigma + \frac{1}{2} S_t^2 \mathcal{C}_{SS}((\sigma_t^P)^2 - \Sigma_t^2) + \sigma_t^P \eta_t^P S_t \mathcal{C}_{SS} + \frac{1}{2} ((\eta_t^P)^2 + \xi_t^P) \mathcal{C}_{\Sigma\Sigma} = 0. \tag{1.3}
\]

The drift condition (1.3) ensures that the call price process \(C\) is a local \(P\)-martingale.\(^6\) Note that the Black–Scholes model corresponds to \(P^0\) with \(\zeta_t^{P_0} = \zeta^0(\Sigma) := (0, \Sigma, 0, 0)\), i.e., the implied volatility is constant and coincides with the spot volatility.

We assume that the agent has moderate risk and uncertainty aversion.\(^7\) Concerning risk aversion, we assume that in any given model, the agent seeks to maximise the expected utility from her terminal P&L. Concerning uncertainty aversion, we suppose that she takes models less seriously the more they deviate from the reference Black–Scholes model. In the spirit of the variational preferences of Maccheroni, Marinacci, and Rustichini [118] and the multiplier preferences of Hansen and Sargent [68],\(^8\) this leads to the following stochastic differential game (SDG):\(^9\)

\[
v(\psi) = \sup_{v=(\theta, \phi)} \inf_P \mathbb{E}^P \left[ U(Y^v_T) + \frac{1}{2\psi} \int_0^T U'(Y^v_t) \left| \zeta_t^P - \zeta_t^{P_0} \right|^2 \, \text{d}t \right]. \tag{1.4}
\]

\(^4\)As is customary in asymptotic analysis, the powers of the processes \(\sigma^P, \nu^P, \eta^P,\) and \(\xi^P\) in the dynamics of \((S, \Sigma)\) are chosen so that all of them have a nontrivial effect on the leading-order term in the asymptotic expansions below. In contrast, using the uncorrelated volatility of implied volatility \(\sqrt{\xi^P}\) instead of the uncorrelated squared volatility \(\xi^P\) would only generate a higher-order effect. This is an artefact of the Black–Scholes model: for any reference model with a nonzero uncorrelated volatility of implied volatility, \(\sqrt{\xi^P}\) would be the natural parametrisation; cf. Remark 3.2 for more details.

\(^5\)Here, the partial derivatives \(\mathcal{C}_\Sigma, \mathcal{C}_{SS}, \mathcal{C}_{\Sigma\Sigma},\) and \(\mathcal{C}_{\Sigma\Sigma}\) of \(\mathcal{C}\) are evaluated in \((t, S_t, \Sigma_t)\).

\(^6\)The local martingale property of the liquidly traded assets is sufficient to exclude arbitrage opportunities. It also ensures that the agent has no incentive to invest in the market but only uses it as a hedging instrument for the non-traded option; cf. Remark 2.1.

\(^7\)In contrast, most of the literature on hedging under model uncertainty studies variants of the uncertain volatility model introduced by Avellaneda, Levy, and París [10] and Lyons [116]. These and many more recent studies (e.g., [60, 45, 132, 137, 16, 134]) look for hedging strategies that dominate the payoff of the non-traded option almost surely for every model of a prespecified class. This worst-case approach corresponds to preferences with infinite risk and uncertainty aversion.

\(^8\)We refer to Section II.1 for more details on these preferences and their relation to the standard expected utility framework as well as the worst-case approach.

\(^9\)Our analysis also applies to somewhat more general penalty terms; cf. (2.13)–(2.15). The term \(U'(Y^v_t)\) renders the preferences invariant under affine transformations of the utility function; cf. Remark 2.6.
Here, \( \psi > 0 \), \( U \) is a utility function, the supremum runs over a suitable class of trading strategies, and the infimum is taken with respect to a suitable class of probability measures satisfying (1.2)–(1.3). One interpretation is that the agent plays a game against a fictitious adversary (a “malevolent nature”) who controls the true dynamics of the liquidly traded assets. However, “extreme” choices of this adversary are penalised by the positive second term in (1.4): the more the chosen model \( P \) deviates from the reference Black–Scholes model \( P^0 \), the higher the penalty for the adversary. The scaling factor \( \psi > 0 \) measures the magnitude of the agent’s uncertainty aversion: small values of \( \psi \) lead to high penalties even for small deviations from the Black–Scholes reference model, which means that alternative models are taken less seriously. Note that as \( \zeta_t^{P^0} = (0, \Sigma_t, 0, 0) \), the reference Black–Scholes model reflects the belief that “the future implied volatility stays at the currently observed level.” Put differently, the reference Black–Scholes model is dynamically recalibrated to the quoted option prices.

A related hedging problem without a liquidly traded call option is studied in Chapter II for a local volatility reference model. There, the fictitious adversary chooses the true spot volatility of the stock, but is penalised according to its distance from the reference local volatility.

**Asymptotics.** To obtain explicit formulas, we pass to the limit where uncertainty aversion \( \psi \) tends to zero.\(^\text{10}\) That is, we consider the hedging problem (1.4) as a small perturbation of the classical hedging problem in the Black–Scholes model and look for hedging strategies and price corrections that take into account the impact of model uncertainty in an asymptotically optimal manner. Our main result, Theorem 4.4, describes a hedging strategy \( \nu^* = (\theta^*, \phi^*) \), a family of models \( (P^\psi)_\psi \), and \( \tilde{w}_0 \geq 0 \) such that, as \( \psi \downarrow 0 \):

\[
\nu(\psi) = U(Y_0) - U'(Y_0)\tilde{w}_0\psi + o(\psi) = E^{P^\psi}\left[ U(Y^\nu_T) + \frac{1}{2\psi} \int_0^T U'(Y^\nu_t) \left| \zeta_T^{P^\psi} - \zeta_t^{P^\psi} \right|^2 dt \right] + o(\psi). \tag{1.5}
\]

The first line in (1.5) is a first-order expansion of the optimal value of the hedging problem for small values of the uncertainty aversion parameter \( \psi \). The second line shows that \( (\nu^*, P^\psi)_\psi \) is an asymptotic saddle point for the family of SDGs (1.4), i.e., the performance of the strategy \( \nu^* \) is optimal at the leading-order \( O(\psi) \) and \( (P^\psi)_\psi \) is a family of leading-order optimal choices for the fictitious adversary. The ask price at which the agent is indifferent between keeping a flat position and selling the option \( V \) has the expansion

\[
p_a(\psi) = V_0 + \tilde{w}_0\psi + o(\psi),
\]

where \( V_0 \) is the Black–Scholes price of the option \( V \) at time 0, evaluated with volatility \( \Sigma_0 \). Thus, \( \tilde{w}_0\psi \) is the leading-order premium that the agent demands as a compensation for exposing herself to model uncertainty. Accordingly, \( \tilde{w}_0 \)

\(^{10}\)Asymptotic analyses of the uncertain volatility model have been carried out by [116, 3, 4, 59].
measures the option’s susceptibility to model misspecification and we call it the cash equivalent (of small uncertainty aversion). We next display and discuss explicit formulas for the hedging strategy $\mathbf{u}^*$, the family of models $(P^\psi)_\psi$, and the cash equivalent $\tilde{w}_0$.

The hedging strategy $\mathbf{u}^* = (\theta^*, \phi^*)$ is the delta-vega hedge for the option $V$:

$$\theta^*_t = V_S(t, S_t, \Sigma_t) - \phi^* C_S(t, S_t, \Sigma_t), \quad \phi^*_t = \frac{V_S}{C_S}(t, S_t, \Sigma_t).$$

To wit, the number of calls $\phi^*$ is chosen so that the net vega of the agent’s position, $\phi^* C_S - V_S$, vanishes. This leaves the agent with a net delta of $-V_S + \phi^* C_S$ which is in turn neutralised by holding $\theta^*$ shares of the underlying, so that the total portfolio is both delta- and vega-neutral.\(^12\) We emphasise that the leading-order optimality of the delta-vega hedge is independent of both the agent’s utility function and her uncertainty aversion parameter $\psi$. While it is important that the agent is risk-averse (otherwise, there would be no need to hedge at all in any given model) and is moderately uncertainty-averse in our sense (vega hedging is redundant without uncertainty aversion), the precise configuration of the agent’s preferences is by and large irrelevant. Moreover, note that the delta-vega hedge is computed with the currently observed implied volatility $\Sigma_t$ of the liquidly traded call, i.e., the Black–Scholes model used to compute the hedge is dynamically recalibrated.

We next address the asymptotically optimal models $(P^\psi)_\psi$. The process $\zeta^{P^\psi}$ describing the model $P^\psi$ satisfies

$$\zeta^{P^\psi}_t = \zeta^0_P + \tilde{\zeta}(t, S_t, \Sigma_t)\psi + o(\psi)$$

for some $\tilde{\zeta} = \tilde{\zeta}(t, S, \Sigma)$ arising from a linearly constrained quadratic programming problem derived from the Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation associated to the SDG (1.4) (the constraints originate from the drift condition (1.3) and the restriction that the uncorrelated squared volatility of implied volatility is nonnegative). The model $P^\psi$ is a perturbation of the Black–Scholes model $P^0$, parametrised by the four processes $\nu^P, \sigma^P, \eta^P$, and $\xi^P$ in (1.2). The explicit formula for $\tilde{\zeta}$ (cf. (4.5)) shows that the asymptotically optimal perturbation exploits the disparity between the vegas, gammas, vannas, and volgas\(^13\) of the non-traded option $V$ and the liquidly traded call while preserving the drift condition (1.3) and the restriction $\xi^P \geq 0$. In fact, if each of these greeks has the same value for both the non-traded option and the liquidly traded call (e.g., if $V$ is a put with the same maturity and strike as the call), then the leading-order optimal perturbation $\zeta$ is zero.

Finally, we discuss the structure of the expansion (1.5) and the cash equivalent $\tilde{w}_0$. As the Black–Scholes model is complete and the traded assets are

---

\(^11\) *Delta* is the sensitivity of a Black–Scholes option value with respect to changes in the price of the underlying.

\(^12\) The vega of the underlying is obviously zero.

\(^13\) *Gamma*, *vanna*, and *volga* are the second-order partial derivatives $\partial^2/\partial S^2$, $\partial^2/(\partial S \partial \Sigma)$, and $\partial^2/\partial \Sigma^2$ of the Black–Scholes value of an option.
local martingales, the zeroth-order term in the expansion (1.5) of \( v(\psi) \) simply is the utility \( U(Y_0) \) generated by the initial P&L. The first-order correction term 
\[-U'(Y_0)\tilde{w}_0\psi\] 
is nonpositive and describes the impact of model uncertainty for small uncertainty aversion. The cash equivalent \( \tilde{w}_0 \) is determined by a linear second-order parabolic partial differential equation (PDE) with a source term. It has the following probabilistic representation:

\[
\tilde{w}_0 = \frac{1}{2} \mathbb{E}^{P_0} \left[ \int_0^T \tilde{g}(t, S_t, \Sigma_0) \, dt \right]
\]

with

\[
0 \leq \tilde{g}(t, S, \Sigma) = -\Sigma(\phi^* S^2 C_{SS} - S^2 V_{SS})\tilde{\sigma}
- \Sigma(\phi^* SC_{SS} - SV_{SS})\tilde{\eta}
- \frac{1}{2}(\phi^* \Sigma \Sigma - \Sigma \Sigma)\tilde{\xi}.
\]

Here, \( \tilde{\zeta}(t, S, \Sigma) = (\tilde{\nu}, \tilde{\sigma}, \tilde{\eta}, \tilde{\xi})(t, S, \Sigma) \) and all functions on the right-hand side of (1.6) are evaluated in \((t, S, \Sigma)\). The cash equivalent \( \tilde{w}_0 \) is thus determined by the expected net cash gamma, net cash vanna, and net volga of the delta-vega hedged position that is accumulated over the lifetime of the option \( V \).\(^{14}\) These three net (cash) greeks are weighted by the leading-order optimal perturbation of the spot volatility, the correlated volatility of implied volatility, and the uncorrelated squared volatility of implied volatility, respectively. The larger \( \tilde{g} \), the larger the cash equivalent \( \tilde{w}_0 \). In particular, a short net gamma position (after vega hedging) is exposed to high spot volatility (positive \( \tilde{\sigma} \)), a short net vanna position is exposed to volatility of implied volatility that is positively correlated with the underlying (positive \( \tilde{\eta} \)), and a short net volga position is exposed to volatility of implied volatility (positive \( \tilde{\xi} \)).\(^{15}\) Conversely, long positions in net gamma or net vanna have the reverse exposures, but a long net volga position is not exposed to volatility of implied volatility because \( \tilde{\xi} \) cannot be negative.

**Techniques.** The HJBI equation associated to the SDG (1.4) involves a pointwise min-max problem for the hedging strategy of the agent and for the control variables of the fictitious adversary. This min-max problem has a nonlinear equality constraint and an inequality constraint that originate from the drift condition (1.3) and the restriction \( \xi^P \geq 0 \), respectively.

\(^{14}\)In contrast, if there is no liquidly traded call available as a hedging instrument, then the option’s cash gamma is the only greek that appears in the probabilistic representation of the cash equivalent; cf. (II.3.14) in Chapter II.

\(^{15}\)According to formula (1.6), a short net volga position is only exposed to the part of the volatility of implied volatility that is uncorrelated with the underlying. However, it can be seen from the proof that the correlated volatility of implied volatility has the same effect, albeit only at the order \( O(\psi^2) \).
Formally passing to the limit as $\psi \downarrow 0$, this problem can be approximated by a linearly constrained quadratic minimisation problem and an unconstrained quadratic maximisation problem. Both of these problems can be solved explicitly and give rise to the delta-vega hedge and candidate controls $(\zeta^\psi, \psi)$ corresponding approximately to the family of models $(P^\psi)\psi$. Plugging these candidates back into the HJBI equation yields a PDE for the first-order term in the expansion of the value function of the SDG.

The rigorous verification of the (asymptotic) optimality of these candidates combines an asymptotic analysis of the HJBI equation with classical verification arguments for SDGs. It is divided into a purely analytic and a probabilistic part. Due to the constraints in the min-max problem, both parts of the proof require substantially different approaches compared to those used in Chapter II. The analytic part uses careful direct estimates and Lagrange duality theory for constrained optimisation problems to show that the candidate value function is asymptotically (in a suitable sense) a solution to the HJBI equation. The probabilistic part of the proof adapts classical verification arguments for SDGs to the asymptotic setting. New difficulties arise now from the fact that the candidate controls of the fictitious adversary do not satisfy the drift condition (1.3) exactly (as the nonlinear constraint is only approximated by a linear one).

**Related literature.** Let us now put our results in context by discussing some of the extant literature on the hedging of exotics using vanilla options. One strand of literature postulates that both the asset price and its spot volatility are stochastic and follow given dynamics driven by two Brownian motions. Stochastic volatility models of this type can typically be completed by using a single plain-vanilla option as an extra hedging instrument in addition to the underlying stock. In Markovian settings, replicating strategies can then be determined in close analogy to the classical Black–Scholes argument. This leads to the so-called “delta-sigma hedge” [86, 148], which neutralises the portfolio’s sensitivity with respect to changes in both the underlying stock price and the spot volatility. This strategy is related to the delta-vega hedge in that it also makes use of the derivative of the option price with respect to “volatility”. Here, however, “volatility” refers to the spot volatility that can (at least in theory) be backed out from the realised variance of the stock. Instead, delta-vega hedging neutralises a portfolio’s sensitivity with respect to changes in the (Black–Scholes) implied volatility that is deduced from the market price of a liquidly traded option. While the spot volatility gives the instantaneous volatility of the stock price, the implied volatility is rather an estimate for the future volatility realised over the whole time interval ranging from today to the maturity of the liquidly traded option. Moreover, in practice, also stochastic volatility models have to be recalibrated once the model and market prices of liquidly traded options diverge.

Another strand of literature studies the robust hedging of exotic derivatives. To wit, these studies look for hedging strategies that work in some large class of models (e.g., any continuous martingale model). The hedging strategies are

---

16See, e.g., [143, 39, 40] for precise conditions.
typically of semi-static form: they allow a static position in a portfolio of calls and puts (often for one maturity and all strikes) and dynamic trading in the underlying.\footnote{Semi-static hedging problems have also been analysed numerically in the context of the \textit{Lagrangian uncertain volatility model} \cite{11, 8}.} For variance swaps, this leads to a robust replicating strategy \cite{131}, whereas robust sub- and superhedging strategies have been determined for various other exotic options (cf., e.g., \cite{79, 23, 36, 35, 28, 84, 83, 82}). In these studies, the goal is to find portfolios that sub- or superreplicate the exotic option in each possible scenario.\footnote{General superhedging duality results in the semi-static context have been obtained, among others, by \cite{1, 12, 46, 62, 19}; see also the references therein.} The underlying preferences therefore correspond to infinite aversion both against risk in a given model and uncertainty about the model itself. In contrast, as in Chapter II, we consider a more moderate attitude towards risk and uncertainty that interpolates smoothly between the worst-case approach and the classical setting with one fixed model. The other major difference is that we allow dynamic trading in a single vanilla option instead of static positions in puts and calls of many strikes.

In practice, even the most liquid at-the-money options have substantially larger bid-ask spreads than the underlying stocks. As a result, a direct implementation of the delta-vega hedge with, e.g., daily rebalancing leads to substantial transaction costs and is found to be inferior to semi-static alternatives in several case studies \cite{35, 135}. As a remedy, the delta-vega hedge needs to be implemented with a suitable “buffer”. That is, rebalancing trades should only take place once the hedge portfolio deviates sufficiently from its frictionless target. The corresponding trading boundaries for Black–Scholes delta-hedging strategies have been determined explicitly in the small-cost limit by \cite{156}; cf. also \cite{101} and the references therein for extensions to more general settings. Extending these tracking results to more general target strategies involving liquid vanilla options is a major challenge for future research. To date, the only result of this kind concerns the dynamic trading of options to reduce transaction costs \cite{66}, which leads to a buffered version of the delta-gamma hedge.

**Organisation of the chapter.** The remainder of the chapter is organised as follows. The mathematical framework for the hedging problem under model uncertainty is introduced in Section 2. Section 3 outlines the heuristic derivation of the asymptotically optimal solution. Our main results are stated and discussed in Section 4. Finally, all proofs are relegated to Section 5.

**Notation.** Vectors $\mathbf{a} \in \mathbb{R}^n$ and vector-valued functions are printed in boldface type. The transpose of a vector $\mathbf{a}$ is denoted by $\mathbf{a}^\top$ and its Euclidean norm by $|\mathbf{a}|$. For the sake of readability, we mostly suppress the arguments of functions in the notation. In calculations and estimates, we typically display the arguments only on the left-most side of (in-)equalities; the omitted arguments should then be clear from the context. Partial derivatives of functions with respect to scalar
variables are denoted by subscripts as in (1.3) and $D_\zeta H$ denotes the gradient of
a function $H(\ldots ; \zeta)$ with respect to the vector variable $\zeta$.

## 2 Problem formulation

To allow for dynamic trading in both the stock and an option on the stock, we consider *market models* for the joint evolution of both assets. Instead of prescribing the dynamics of the option, we follow Schönbucher’s approach [145] and model its Black–Scholes implied volatility. This approach is outlined in Section 2.1 and motivates the precise setup introduced in Section 2.2. The hedging problem is in turn formulated in Section 2.3.

### 2.1 Market models for the underlying and its implied volatility

We consider a financial market with three liquidly traded securities: a stock $S$, an option written on the stock, and a bank account with zero interest rate. The liquidly traded option has a payoff of the form $C(S_T)$ at maturity $T_C$. To avoid confusion with the non-traded option introduced later, this liquidly traded option will be named “call” hereafter. It is market practice to quote option prices in terms of their *(Black–Scholes) implied volatilities*. That is, traders do not quote the market price $p$ of the call $C$, but instead the unique $\Sigma > 0$ such that $p = C(t, S_t, \Sigma)$, where $C(\cdot, \cdot, \Sigma)$ is the solution of the Black–Scholes PDE

$$
C_t(t, S, \Sigma) + \frac{1}{2} \Sigma^2 S^2 C_{SS}(t, S, \Sigma) = 0, \quad (t, S) \in (0, T_C) \times \mathbb{R}_+,
$$

$$
C(T_C, S, \Sigma) = C(S), \quad S \in \mathbb{R}_+,
$$

(2.1)

corresponding to volatility $\Sigma$, maturity $T_C$, and the terminal payoff $C(S_{T_C})$ of the call. Following this practice and Schönbucher’s approach [145], we model the implied volatility rather than the price process of the call. To wit, we assume that the joint dynamics of the stock $S$ and the call’s implied volatility $\Sigma$ are governed by

$$
\text{d}S_t = S_t \sigma_t \text{d}W^0_t,
$$

$$
\text{d}\Sigma_t = \nu_t \text{d}t + \eta_t \text{d}W^0_t + \sqrt{\xi_t} \text{d}W^1_t,
$$

(2.2)

(2.3)

for a bivariate standard Brownian motion $(W^0, W^1)$ and processes $\sigma, \nu, \eta, \xi$. Here, $\sigma$ is the *spot volatility*, and $\nu, \eta,$ and $\xi$ correspond to the *drift of implied volatility*, the *correlated volatility of implied volatility*, and the *uncorrelated squared volatility*.

---

19 Other early articles on risk-neutral dynamics for stochastic implied volatility models include [117, 21, 110]. For more recent developments on arbitrage-free market models for (parts of or the whole) option price surface, we refer the reader to [147, 146, 25, 87, 26, 27, 99] and the references therein.

20 The parametrisation in terms of the *squared volatility* of implied volatility is explained in Remark 3.2.
volatility of implied volatility, respectively. The price process \( C \) of the call in turn is
\[
C_t = C(t, S_t, \Sigma_t).
\]
By Itô’s formula, its dynamics are given by
\[
dC_t = dC(t, S_t, \Sigma_t)
= C_t \text{d}t + C_S \text{d}S_t + C_\Sigma \text{d}\Sigma_t + \frac{1}{2} C_{SS} \text{d}\langle S \rangle_t + C_{SS} \text{d}(S, \Sigma)_t + \frac{1}{2} C_{\Sigma \Sigma} \text{d}\langle \Sigma \rangle_t
= C_S \text{d}S_t + \eta C_\Sigma \text{d}W^0 + \sqrt{\xi} C_\Sigma \text{d}W^1
+ \left\{ C_t + \nu C_\Sigma + \frac{1}{2} \sigma^2 S_t^2 C_{SS} + \sigma \eta S_t C_{SS} + \frac{1}{2} (\eta^2 + \xi) C_{\Sigma \Sigma} \right\} \text{d}t.
\]
We suppose that all liquidly traded assets are local martingales (cf. Footnote 6 and Remark 2.1). Thus, the drift of the liquidly traded call must vanish. Using the PDE (2.1) to substitute \( C_t = C_t(t, S_t, \Sigma_t) \), the following drift condition obtains (cf. [145, Equation (3.6)]):
\[
\nu C_\Sigma + \frac{1}{2} S_t^2 C_{SS}(\sigma^2 - \Sigma^2_t) + \sigma \eta S_t C_{SS} + \frac{1}{2} (\eta^2 + \xi) C_{\Sigma \Sigma} = 0.
\]
In view of (2.5), at most three of the four processes \( \nu, \sigma, \eta, \) and \( \xi \) can be chosen arbitrarily for the resulting model to satisfy the drift condition. Further natural restrictions are \( \sigma > 0, \xi \geq 0, \) and \( \Sigma > 0. \) Note that the standard Black–Scholes model corresponds to the choice \( \nu = \eta = \xi = 0 \) and \( \sigma_t = \Sigma_t = \Sigma_0. \) Then, the drift condition (2.5) is clearly satisfied and spot and implied volatilities are constant and identical.

Remark 2.1. Let us briefly discuss as in Remark II.2.2 why we assume that the traded assets \( S \) and \( C \) have zero drifts. With nonzero drifts, the agent would use the traded assets not only as hedging instruments, but also as investment vehicles. But the real-world drift rates usually have little impact on the hedging component, i.e., the difference between a utility-based hedging strategy and the corresponding utility-based optimal investment strategy. Assuming that the traded assets have zero drifts allows us to focus on hedging rather than optimal investment. Indeed, the agent then has no incentive to trade the stock and the call other than as hedging instruments for the non-traded option.

In the following Section 2.2, we introduce a setup to formulate our hedging problem with uncertainty about the processes \( \nu, \sigma, \eta, \xi. \)

### 2.2 Model uncertainty setup

Fix a time horizon \( T > 0 \) and constants \( S_0 > 0, \Sigma_0 > 0, \) and \( A_0 \in \mathbb{R}. \) Let
\[
\Omega = \{ \omega = (\omega_t^S, \omega_t^\Sigma, \omega_t^A)_{t \in [0, T]} \in C([0, T]; \mathbb{R}^3) : \omega_0 = (S_0, \Sigma_0, A_0) \}\]
be the canonical space of continuous paths in $\mathbb{R}^3$ starting in $(S_0, \Sigma_0, A_0)$, endowed with the topology of uniform convergence. Moreover, let $\mathcal{F}$ be the Borel $\sigma$-algebra on $\Omega$. We denote by $(S_t)_{t \in [0,T]}$, $(\Sigma_t)_{t \in [0,T]}$, and $(A_t)_{t \in [0,T]}$ the first, second, and third component of the canonical process, respectively, i.e., $S_t(\omega) = \omega_t^S$, $\Sigma_t(\omega) = \omega_t^\Sigma$, and $A_t(\omega) = \omega_t^A$. We write $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ for the (raw) filtration generated by $(S, \Sigma, A)$, and denote by $M_t := \sup_{u \in [0,t]} S_u$, $t \in [0,T]$, the running maximum of $S$. Unless otherwise stated, all probabilistic notions requiring a filtration, such as progressive measurability etc., pertain to $\mathbb{F}$. Finally, we write $(X_t)_{t \in [0,T]}$ for the vector-valued process $X_t = (S_t, A_t, M_t, \Sigma_t)$.

**Remark 2.2.** The processes $S$, $M$, and $\Sigma$, model the stock price, its running maximum, and the implied volatility of the traded call, respectively. The process $A$ is an additional state variable that can be used to track exotic features of the non-traded option the agent has to hedge. For instance, an Asian call option $A$ is an additional state variable that can be used to track exotic features of the maximum, and the implied volatility of the traded call, respectively. The process $(\mathcal{V}(t, S_t, A_t))$ of real-valued progressively measurable processes such that:

(a) $S$ and $\Sigma - \int_0^t \nu^P_s \, dt$ are (continuous) local $P$-martingales with quadratic (co-)variations

$$d(S)_t = S_t^2(\sigma_t^P)^2 \, dt,$$

$$d(\Sigma)_t = ((\eta_t^P)^2 + \xi_t^P) \, dt,$$  \hfill (2.6)

(b) $S$ and $\Sigma$ are $P$-a.s. positive;

(c) $\xi^P_t \geq 0$ $P$-a.s.;

(d) the drift condition

$$\nu_t^P \mathcal{C}_\Sigma + \frac{1}{2} S_t^2 \mathcal{C}_{SS}((\sigma_t^P)^2 - \Sigma_t^2) + \sigma_t^P \eta_t^P S_t \mathcal{C}_{SS} + \frac{1}{2} ((\eta_t^P)^2 + \xi_t^P) \mathcal{C}_{\Sigma \Sigma} = 0 \quad (2.7)$$

holds $dt \times P$-a.e. Here, the partial derivatives of $\mathcal{C}$ are evaluated in $(t, S_t, \Sigma_t)$.

A probability measure $P \in \mathfrak{P}^0$ is called a model and the process $\xi^P_t$ is referred to as the control corresponding to the model $P$. Each $P$ represents a market model for the stock price $S$ and the implied volatility $\Sigma$ with dynamics of the form (2.2)–(2.3) (with $\sigma$ replaced by $\sigma^P$ etc.) and (2.7) guarantees that the call price process is a local $P$-martingale (cf. (2.5)).
Definition 2.4. The function $\zeta^0 : \mathbb{R}_+ \to \mathbb{R}^4$ given by $\zeta^0(\Sigma) = (0, \Sigma, 0, 0)^\top$ is called reference feedback control. A probability measure $P \in \mathfrak{P}^{00}$ such that $\zeta^P_t = \zeta^0(\Sigma_t) \, dt \times P$-a.e. is called reference model.

Note that a reference model corresponds to a Black–Scholes model with constant volatility $\sigma_t \equiv \Sigma_t \equiv \Sigma_0$ and trivially satisfies the drift condition (2.7).

Next, we consider a subclass $\mathfrak{P}^0 \subset \mathfrak{P}^{00}$ which (in contrast to $\mathfrak{P}^{00}$) also prescribes the dynamics for the additional state variable $A$ that tracks exotic features of the non-traded option. To this end, we fix Borel functions $\alpha, \beta, \gamma, \delta : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$.

Definition 2.5. $\mathfrak{P}^0 = \mathfrak{P}^0(\alpha, \beta, \gamma, \delta) \subset \mathfrak{P}^{00}$ is the subset of probability measures $P$ such that $A$ is a (continuous) $P$-semimartingale with canonical decomposition

$$dA_t = \left( \alpha + \frac{(\sigma_P^t)^2}{2} \beta \right) \, dt + \gamma \, dS_t + \delta \, dM_t$$

(2.8)

under $P$ (the functions $\alpha, \beta, \gamma, \delta$ are evaluated in $(t, S_t, A_t, M_t)$).

The form (2.8) for the dynamics of $A$ is flexible enough to express Black–Scholes values of, e.g., Asian options, options on the realised variance, or forward-start options by PDE methods. We also note that given sufficiently regular functions $\alpha, \beta, \gamma, \delta$, there is a unique reference model in $\mathfrak{P}^0$.

2.3 Hedging problem

Dynamic model recalibration. Consider an agent who has sold a non-traded option (possibly exotic) on $S$ with sufficiently regular payoffs $V(S_T, A_T, M_T)$ at maturity $T$. She can hedge her exposure by trading dynamically and frictionlessly in the stock, the call, and the bank account.

Among all possible dynamics, the agent considers as most plausible the Black–Scholes model corresponding to the currently observed implied volatility, i.e., $\nu = \eta = \xi = 0$ and $\sigma = \Sigma$ (recall that the drift condition (2.5) holds for this choice). This corresponds to the reference belief that “The future implied volatility stays at the currently observed level.” Note that this differs from the conviction that “The future implied volatility equals the implied volatility observed at time 0.”: the former belief allows for dynamic updating of the observed implied volatility, the latter does not. In particular, at each time $t$, the agent (re-)calibrates her Black–Scholes model to the observed market price of the liquidly traded call option. This is in line with the market practice of frequent recalibration of pricing models to observed option prices. The corresponding Black–Scholes value of the non-traded option can readily be obtained by PDE methods. To this end, let

---

21See Assumption 4.2 for the precise details.

22Recall that $M$ is the running maximum of $S$ and that $A$ is a general state variable with dynamics of the form (2.8) which can track exotic features of the option like the average stock price or the stock price at an intermediate time; cf. Section II.4.2 for examples.
\[ G = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \] be the state space of the process \((S, A, M)\) and for each \(\Sigma > 0\), let \(V(\cdot, \Sigma)\) be a classical solution to the PDE

\[
V_t + (\alpha + \frac{1}{2} \beta \Sigma^2) V_A \\
+ \frac{1}{2} \Sigma^2 S^2 (V_{SS} + 2 \gamma V_{SA} + \gamma^2 V_{AA}) = 0 \quad \text{on} \quad (0, T) \times G, \\
\delta V_A + V_M = 0 \quad \text{on} \quad \{(t, S, A, M) : S \geq M\}, \\
V(T, \cdot, \Sigma) = V \quad \text{on} \quad G. 
\]  

(2.9)

Define the process \(V = (V_t)_{t \in [0, T]}\) by

\[
V_t = V(t, S_t, A_t, M_t, \Sigma_t). 
\]  

(2.10)

Then, as is well known, \(V_t\) is the Black–Scholes value at time \(t\) of the non-traded option \(V\) given the current observation of the stock price \(S_t\), the state variables \(A_t\) and \(M_t\), and the implied volatility \(\Sigma_t\). In other words, the Black–Scholes model used to value the option \(V\) is dynamically recalibrated to the observed call prices.

**Trading strategies and Profit&Loss processes.** A (self-financing) trading strategy is represented by a pair \(\upsilon = (\theta, \phi)\) of real-valued, locally bounded,\(^{23}\) progressively measurable processes \(\theta = (\theta_t)_{t \in [0, T]}\) and \(\phi = (\phi_t)_{t \in [0, T]}\), which describe the number of stocks and calls held by the agent, respectively. Fix a constant \(Y_0 \in \mathbb{R}\) and for each \(P \in \mathfrak{P}_0\) and any trading strategy \(\upsilon\), define the Profit&Loss (P&L) process \(Y^{\upsilon, P}_t = (Y^{\upsilon, P}_t)_{t \in [0, T]}\) by

\[
Y^{\upsilon, P}_t = Y_0 + V_0 + \int_0^t \theta_u dS_u + \int_0^t \phi_u dC_u - V_t. 
\]  

(2.11)

Here, the stochastic integrals are constructed under \(P\). The process \(Y^{\upsilon, P}_t\) describes the value of the agent’s portfolio at time \(t\) under the model \(P\), i.e., her initial capital \(Y_0 + V_0\) (recall the definition of \(V_t\) in (2.10)) plus gains from self-financing trading in the liquidly traded assets (computed under \(P\)) minus the (recalibrated) Black–Scholes value \(V_t\) of the non-traded option at time \(t\). Note that while the position in the liquidly traded assets are “marked to market” and constitute “real values” (because these assets could be liquidated instantly by assumption), the non-traded option has to be “marked to model” and thus only has a “theoretical value”. However, at the maturity \(T\) of the non-traded option, \(V_T\) equals the option’s payoff and the value of the option becomes “real”. In particular, \(Y^{\upsilon, P}_T\) is the agent’s actual terminal wealth.

**Uncertainty aversion.** Fix a model set \(\mathfrak{P} \subset \mathfrak{P}^0\) and a set \(\mathfrak{M}\) of trading strategies. Similarly to Chapter II,\(^{24}\) we assume that the agent ranks trading

\(^{23}\)For locally bounded, progressively measurable integrands, the stochastic integrals in (2.11) are well defined under each measure in \(\mathfrak{P}^0\). The delta-vega hedge considered in our main result, Theorem 4.4, is even continuous.

\(^{24}\)In Chapter II, only the underlying but no liquid call is available for dynamic hedging and the spot volatility is the only control variable of the fictitious adversary.
strategies in $\mathfrak{Y}$ according to a numerical representation of her preferences of the form
\[
\inf_{P \in \mathfrak{P}} E^P \left[ U(Y_T^{\nu,P}) + \frac{1}{\psi} \int_0^T U''(Y_t^{\nu,P}) f(\Sigma_t, \zeta_t^P) \, dt \right],
\]
(2.12)
where $f$ is a suitable function such that for each $\Sigma > 0$, $\mathbb{R}^4 \ni \zeta \mapsto f(\Sigma, \zeta)$ is strictly convex with a unique minimum of 0 at the reference point $\zeta_0^0(\Sigma)$. The utility function $U$ describes the agent’s attitude towards risk in a given model. The infimum over models in $\mathfrak{P}$ together with the penalty term$^{25}$ (the second summand inside the expectation in (2.12)) expresses her attitude towards model uncertainty. The parameter $\psi > 0$ quantifies the magnitude of her uncertainty aversion. Indeed, in the limit $\psi \downarrow 0$, the second summand in (2.12) converges to the indicator $+\infty 1_{\{\zeta \neq \zeta_0(\Sigma)\}}$ and the criterion (2.12) collapses to the standard expected utility under the reference model. In this case, the agent faces no uncertainty aversion at all as she only deems the reference model plausible. Conversely, in the limit $\psi \uparrow \infty$, the penalty term converges to 0 for all $P \in \mathfrak{P}$ and the criterion (2.12) becomes the familiar worst-case expectation $\inf_{P \in \mathfrak{P}} E^P \left[ U(Y_T^{\nu,P}) \right]$. In this case, the agent is very uncertainty-averse in that she regards every model in $\mathfrak{P}$ as equally plausible. The criterion (2.12) interpolates smoothly between these two extreme cases. The reference model is not penalised, while alternative models are underweighted in the agent’s decision making according to their “distance” from the reference model. The interpretation is that the reference model is considered most plausible. Alternative models are taken less seriously, but not ruled out a priori.

For tractability, we focus on the following quadratic specification for the penalty function $f$: $^{26}$
\[
f(\Sigma, \zeta) = \frac{1}{2} (\zeta - \zeta_0(\Sigma))^{\top} \Psi^{-1} (\zeta - \zeta_0(\Sigma))
= \frac{1}{2} (\nu^2/\psi_{\nu} + (\sigma - \Sigma)^2/\psi_{\sigma} + \eta^2/\psi_{\eta} + \xi^2/\psi_{\xi})
\]
(2.13)
where
\[
\Psi = \text{diag}(\psi_{\nu}, \psi_{\sigma}, \psi_{\eta}, \psi_{\xi}) \quad \text{and} \quad \psi_{\nu}, \psi_{\sigma}, \psi_{\eta}, \psi_{\xi} > 0.
\]
(2.14)
The parameters $\psi_{\nu}, \psi_{\sigma}, \psi_{\eta}, \psi_{\xi}$ describe the agent’s relative uncertainty about the true drift of implied volatility, spot volatility, correlated volatility of implied volatility, and uncorrelated squared volatility of implied volatility, respectively. The scaling parameter $\psi$ measures her overall level of aversion against uncertainty.

$^{25}$ Note that the penalty is imposed on the fictitious adversary who chooses the model $P$ after the agent has chosen her trading strategy $\nu$. Alternatively, it can be interpreted as a fictitious bonus for the agent.

$^{26}$ More general functions $f$ are considered in Chapter II, where it becomes apparent that only the locally quadratic structure at the minimum matters for the leading-order asymptotics.
Remark 2.6. Let us argue as in Remark II.2.6 why we include the term $U'(Y_t^{v,P})$ in the penalty term of the numerical representation (2.12). First, in the standard expected utility framework, preferences are invariant under affine transformations of the utility function. The term $U'(Y_t^{v,P})$ ensures that this property is preserved for uncertainty-averse decision makers whose preferences are described by (2.12). Second, $U'(Y_t^{v,P})$ (rather than, e.g., $U'(Y_0)$) is the natural choice for a dynamic formulation of the hedging problem (2.16) in terms of a family of conditional problems parametrised by the initial time $t$, stock price $S_t = s$, and P&L $Y_t^{v,P} = y$. Third, our results show that the preferences described by (2.12) have approximately “constant uncertainty aversion” in the sense that the cash equivalent $\tilde{w}_0$ does not depend on the P&L (cf. Proposition 4.5). This would not be the case if one omitted the term $U'(Y_t^{v,P})$ in (2.12).

Hedging problem. Fix $\psi > 0$. For each trading strategy $v \in \mathcal{Y}$ and each model $P \in \mathcal{P}$, we define the objective of our hedging problem by

$$J^\psi(v, P) := E^P \left[ U(Y_t^{v,P}) + \frac{1}{\psi} \int_0^T U'(Y_t^{v,P}) f(\Sigma_t, \zeta_t^P) \, dt \right]. \quad (2.15)$$

We note that Assumption 4.2 (a) below guarantees that the negative part of the integrand in (2.15) is bounded, so that the expectation is well defined. The value of our hedging problem is

$$v(\psi) = v(\psi; \mathcal{Y}, \mathcal{P}) := \sup_{v \in \mathcal{Y}} \inf_{P \in \mathcal{P}} J^\psi(v, P). \quad (2.16)$$

To wit, the agent wants to find a strategy in $\mathcal{Y}$ that maximises the numerical representation of her preferences (2.12). The goal of this chapter is to find an asymptotic expansion of the value $v(\psi)$ for small levels of uncertainty aversion $\psi$ and to find a trading strategy that achieves the leading-order optimal performance.

3 Heuristics

The asymptotic solution of the family of SDGs (2.16) is related to a linearly constrained quadratic programming problem. In this section, we derive this optimisation problem heuristically from the HJBI equation associated to (2.16). This motivates the definitions of the functions introduced in the subsequent Section 4.

Effective greeks. Let us assume for the moment that the true dynamics of the stock price are given by the Black–Scholes model with some (constant) volatility $\Sigma_0$. Then Itô’s formula and the PDE (2.9) for $\mathcal{V}$ show that the replicating strategy (trading only the stock and the bank account, not the call) of the option with payoff $\mathcal{V}(S_T, A_T, M_T)$ is given by $\theta_t = (\mathcal{V}_S + \gamma \mathcal{V}_A)(t, S_t, A_t, M_t, \Sigma_0)$. In particular,

$^{27}$Formally, this corresponds to directly imposing the penalty in monetary terms, i.e., inside the utility function in (2.12).
the delta $\mathcal{V}_S$ of the option only gives the replicating strategy if $\gamma \equiv 0$, i.e., if the additional state variable $A$ is of finite variation (e.g., for vanilla options like the liquidly traded call, or exotics like barrier, Asian, or lookback options). In general, however, the replicating strategy also has to take into account the indirect sensitivity of the option value with respect to changes in the stock price arising from the additional state variable $A$ (e.g., for a forward-start option as in Example 3.1). Therefore, we call

$$\Delta = \mathcal{V}_S + \gamma \mathcal{V}_A$$

the effective delta of the option $V$. Similarly, we call

$$\Gamma = \mathcal{V}_{SS} + 2\gamma \mathcal{V}_{SA} + \gamma^2 \mathcal{V}_{AA} \quad \text{and} \quad \frac{\partial \Delta}{\partial \Sigma} = \mathcal{V}_{S \Sigma} + \gamma \mathcal{V}_{A \Sigma}$$

the effective gamma and effective vanna of the option $V$, respectively.

Example 3.1 (Forward-start call). A forward-start call with payoff $(S_T - S_{\text{reset}})^+$ is a call option whose strike is set at some future reset date $T_{\text{reset}} \in (0, T)$ (cf., e.g., [127, Section 6.2]). This option payoff can be embedded into our framework by choosing $A_0 = S_0$ and $\gamma(t) = 1_{\{t < T_{\text{reset}}\}}$. Indeed, then $A_t = S_t \wedge T_{\text{reset}}$ and the option payoff can be written as $V(S_T, A_T) = (S_T - A_T)^+$.

Dynamics of the P&L process. In order to write down the HJBI equation associated to the hedging problem, we need the dynamics of the P&L process $Y_{\nu, P}$ for generic strategies $\nu$ and models $P$. Applying Itô’s formula to $Y_{\nu, P}$ (defined in (2.11)) under $P$ (with associated control $\zeta^P$) yields (cf. Lemma 5.2)

$$dY_{\nu, P,t} = (\theta_t - (\Delta(t, X_t) - \phi_t \mathcal{C}_S(t, S_t, \Sigma_t))) \, dS_t + (\phi_t \mathcal{C}_S(t, S_t, \Sigma_t) - \mathcal{V}_S(t, X_t)) \, d\Sigma_{\nu, P,t} - b^t(t, X_t; \zeta^P) \, dt,$$

where $\Sigma_{\nu, P} = \Sigma - \int_0^t \mu_{\nu, P}^P \, du$ is the (continuous) local martingale part of $\Sigma$ under $P$, and (writing $x = (S, A, M, \Sigma) \in \mathbb{R}^4$ and $\zeta = (\nu, \sigma, \eta, \xi) \in \mathbb{R}^4$),

$$b^t(t, x; \zeta) = \nu \mathcal{V}_S + \frac{1}{2}(\beta \mathcal{V}_A + S^2 \Gamma)(\sigma^2 - \Sigma^2) + \sigma \eta S \frac{\partial \Delta}{\partial \Sigma} + \frac{1}{2}(\eta^2 + \xi) \mathcal{V}_{SS}. \quad (3.1)$$

For small uncertainty aversion, models far from the reference model are heavily penalised. Whence, the fictitious adversary needs to choose among small perturbations $\zeta = \zeta^0(\Sigma) + \hat{\zeta} \psi$ of the reference feedback control $\zeta^0(\Sigma) = (0, \Sigma, 0, 0)$. Plugging this perturbation into (3.1), we find

$$b^t(\zeta) = \nu^T \hat{\zeta} \psi + o(\psi), \quad (3.2)$$

where $\nu = (\mathcal{V}_S, \Sigma(\beta \mathcal{V}_A + S^2 \Gamma), \Sigma S \frac{\partial \Delta}{\partial \Sigma}, \frac{1}{2} \mathcal{V}_{SS})$. Note that by expanding the function $b^t(\zeta^0(\Sigma) + \hat{\zeta} \psi)$ around $\psi = 0$, the vector-valued function $\nu$ in (3.2) can also be identified as the gradient $D_\zeta b^t$ evaluated in $\zeta^0(\Sigma)$. 

Remark 3.2. We now explain the use of the uncorrelated squared volatility of implied volatility $\xi$ as a control variable. Equation (3.2) shows that an $O(\psi)$-perturbation of the squared volatility around zero (i.e., a positive fourth component of $\tilde{\xi}$) affects the drift $b^V$ at the order $O(\psi)$ (at least as long as we are in the generic case where $V_{\Sigma\Sigma}$ is nonzero). If we used instead the uncorrelated volatility $\xi^0 := \sqrt{\xi}$ as a basic control variable, then $\xi$ in (3.1) would be replaced by $(\xi')^2$. Following the arguments that lead to (3.2), we would then find for a perturbation of the form $\zeta' = \zeta^0(\Sigma) + \tilde{\zeta}'\psi$ that

$$b^V(\zeta') = (\psi')^\top\tilde{\zeta}'\psi + o(\psi), \quad (3.3)$$

where $\psi'$ is given by $\psi$ with the fourth component replaced by zero. Thus, a perturbation of $\xi'$ around zero of order $O(\psi)$ would then have no impact on the $O(\psi)$ term in the expansion (3.3) of $b^V$. This is an artefact of the Black–Scholes reference model: for any reference model with a nonzero uncorrelated volatility of implied volatility, $\xi^0 \neq 0$, the fourth component of $\psi'$ would generically not vanish, and hence an $O(\psi)$-perturbation of $\xi'$ around $\xi^0$ would affect the drift $b^V$ at the order $O(\psi)$.

HJBI equation. The drift condition (2.7) can be rephrased as $b^C(t, X_t; \zeta^P_t) = 0$ $dt \times P$-a.e., where

$$b^C(t, x; \zeta) = \nu C_S + \frac{1}{2} S^2 C_{SS}(\sigma^2 - \Sigma^2) + \sigma \eta S C_{S\Sigma} + \frac{1}{2} (\eta^2 + \xi) C_{\Sigma\Sigma}.$$ 

In addition, the uncorrelated squared volatility of implied volatility $\xi^P$ (the fourth component of $\zeta^P$) must be nonnegative (cf. Definition 2.3). Hence, the HJBI equation associated to the hedging problem reads as

$$w^V(t, x, y) + \sup_{\psi \in \mathbb{R}^2} \inf_{\zeta \in \mathbb{R}^4: b^C(t, x; \zeta) = 0, \zeta_i \geq 0} H^V(t, x, y; \psi, \zeta) = 0, \quad w^V(T, x, y) = U(y), \quad (3.4)$$

where the function $H^V(t, x, y; \psi, \zeta)$ (spelled out explicitly in (5.15)) depends on first- and second-order partial derivatives of $w^V$ with respect to the space variables $x$ and $y$, and on the drift and diffusion coefficients describing the dynamics of $S, A, M, \Sigma,$ and $Y_{\psi, P}$ under a model $P$ such that $\zeta^P = \zeta$. We refer to Section II.4.1 for a derivation of the HJBI equation from the martingale optimality principle of stochastic optimal control. In essence, the left-hand side of the HJBI equation arises from the drift of the process $w^V(t, X_t, Y_{t; \psi, P}) + \frac{1}{\psi} \int_0^t U'(Y_{t; \psi, P})f(S_u, \zeta^P_u)\, du$ under $P$, which can be computed via Itô’s formula.

Asymptotic ansatz. As the Black–Scholes model is complete and the drift of the liquidly traded assets is zero under each model by assumption, we expect that the zeroth-order term in the expansion of $v(\psi)$ is simply the utility $U(Y_0)$ generated by the initial P&L. Similarly, the optimal control $\zeta$ with zero uncertainty aversion should simply be the reference feedback control $\zeta^0(\Sigma)$. This motivates
the following ansatz for the asymptotic expansion of the value function and the almost optimal feedback control:\footnote{In view of Chapter II, it is expected that $\psi$ (and not, e.g., $\psi^{1/2}$ or $\psi^2$) is the correct power for the expansion of the value function. Alternatively, one could write $\psi^\alpha$ instead of $\psi$ in (3.5) and then find $\alpha = 1$ by matching the powers of the penalty term and the drift term of the P&L process in the expansion of the HJBI equation in such a way that the optimisation over $\tilde{\zeta}$ becomes nontrivial.}

\begin{align}
 w^\psi(t, x, y) &= U(y) - U'(y)\tilde{w}(t, x)\psi, \\
 \zeta^\psi(t, x) &= \zeta^0(\Sigma) + \tilde{\zeta}(t, x)\psi, \tag{3.5}
\end{align}

for functions $\tilde{w}$ and $\tilde{\zeta} = (\tilde{\nu}, \tilde{\sigma}, \tilde{\eta}, \tilde{\xi})$ to be determined. In the reference model, any strategy in the stock and the call that neutralises the net delta qualifies as a replicating strategy. Whence, it is less obvious whether the delta-vega hedge $\nu^\star$ or any other strategy that neutralises the agent’s net delta (e.g., the standard delta hedge without trading in the call) should be the candidate strategy $\nu = (\theta, \phi)$ for the hedging problem. Thus, we leave the choice of $\phi$ open for the moment and just assume that

\begin{equation}
 \theta = \Delta - \phi C_S \tag{3.7}
\end{equation}

neutralises the (effective) net delta. Plugging (3.5)–(3.7) into the HJBI equation (3.4) (using the explicit formula (5.15) for $H^\psi$), dropping the $\sup, \inf$ (we assume that the candidate strategy and control form a saddle point), using the expansion (3.2), and ordering by powers of $\psi$, we obtain

\begin{equation}
 U' \times \left( -\tilde{w}_t - (\alpha + 1/2\beta \Sigma^2)\tilde{w}_S - 1/2\Sigma^2(\tilde{w}_{SS} + 2\gamma\tilde{w}_{SA} + \gamma^2\tilde{w}_{AA}) \\
 + 1/2\tilde{\zeta}^\top \Psi^{-1}\tilde{\zeta} - \nu^\top \tilde{\zeta} - 1/2\xi(\phi C_\Sigma - V_\Sigma)^2 \frac{U''}{U'} \right) \psi + o(\psi) = 0. \tag{3.8}
\end{equation}

Moreover, the constraints in the minimisation part of the HJBI equation transform to

\begin{equation}
 \left( \tilde{\nu} C_\Sigma + \tilde{\sigma} \Sigma S^2 C_{SS} + \tilde{\eta} \Sigma S C_{S\Sigma} + \frac{1}{2} \xi \Sigma C_{\Sigma\Sigma} \right) \psi + o(\psi) = 0 \quad \text{and} \quad \tilde{\zeta} \geq 0. \tag{3.9}
\end{equation}

Our candidates for $\tilde{\zeta}$ and $\phi$ now arise as the saddle point of the min-max problem (minimising over $\tilde{\zeta}$ and maximising over $\phi$) corresponding to the $O(\psi^\alpha)$ term in (3.8) subject to the constraints (3.9). Clearly, the vega hedge $\phi^\star = \frac{\nu^\star}{\psi}$ maximises the $O(\psi)$ term over $\phi \in \mathbb{R}$, irrespective of the choice of $\tilde{\zeta}$. With this choice, the constrained minimisation over $\tilde{\zeta}$ (ignoring the $o(\psi)$ term in the equality constraint in (3.9)) reduces to a linearly constrained quadratic programming problem:

\begin{equation}
 \begin{aligned}
 &\text{minimise} \quad \frac{1}{2} \tilde{\zeta}^\top \Psi^{-1}\tilde{\zeta} - \nu^\top \tilde{\zeta} \\
 &\text{subject to} \quad \tilde{\zeta} \in \mathbb{R}^4, \quad c^\top \tilde{\zeta} = 0, \quad \tilde{\zeta}_4 \geq 0,
\end{aligned} \tag{3.10}
\end{equation}

where $c = (C_\Sigma, \Sigma S^2 C_{SS}, \Sigma S C_{S\Sigma}, \frac{1}{2} C_{\Sigma\Sigma})$. 
Solving the linearly constrained quadratic program. The minimisation problem \((3.10)\) is strictly convex and linearly constrained and thus has a unique minimum. The minimiser \(\tilde{\zeta}^*\) is characterised by the associated Karush–Kuhn–Tucker conditions
\[
\Psi^{-1}\tilde{\zeta}^* - v + \lambda^* c - \mu^* \bar{e}_1 = 0, \quad c^\top \tilde{\zeta}^* = 0, \quad \tilde{\zeta}_4^* \geq 0, \quad \mu^* \geq 0, \quad \mu^* \tilde{\zeta}_4^* = 0,
\]
for some scalars \(\lambda^*\) and \(\mu^*\). It turns out that there is an explicit solution \((\tilde{\zeta}^*, \lambda^*, \mu^*)\) (cf. Lemma A.1 (a)), which motivates our definitions in Section 4.1.

4 Main results

This section contains the mathematically precise statement of our main results. In Section 4.1, we first introduce the required notation and technical assumptions; this notationally heavy part can be skipped at first reading.

4.1 Notation and assumptions

Our main result, Theorem 4.4, provides an asymptotic expansion of the value \(v(\psi)\) from (2.16) for small levels of uncertainty aversion \(\psi\) and an asymptotic saddle point \((\upsilon^*, P^\psi)\psi\), where \(\upsilon^*\) is the delta-vega hedge and \((P^\psi)\psi\) is a suitable family of models. To define the PDE that describes the first-order term of the expansion and to define the quadruple \(\zeta^\psi\) that corresponds approximately (see Definition 4.1 (b) below) to \((P^\psi)\psi\), we need to introduce some notation.

Recall that \(G = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+\) is the state space of the process \((S, A, M)\) and set \(D^0 = (0, T) \times G \times \mathbb{R}_+\). A generic element of \(D^0\) is written as \((t, S, A, M, \Sigma)\) or \((t, x)\) with \(x = (S, A, M, \Sigma)\). The functions \(\Delta, \Gamma, \frac{\partial \Delta}{\partial \Sigma} : D^0 \to \mathbb{R}\) defined by
\[
\Delta(t, x) = \mathcal{V}_S + \gamma \mathcal{V}_A, \quad \Gamma(t, x) = \mathcal{V}_{SS} + 2\gamma \mathcal{V}_{SA} + \gamma^2 \mathcal{V}_{AA}, \quad \frac{\partial \Delta}{\partial \Sigma}(t, x) = \mathcal{V}_{S\Sigma} + \gamma \mathcal{V}_{A\Sigma}
\]
are called the effective delta, effective gamma, and effective vanna of the option \(\mathcal{V}\), respectively; we note that these quantities correspond to the standard greeks if \(\gamma \equiv 0\) like for vanilla, barrier, or lookback options, for example, and refer to Section 3 for a motivation of this terminology in the case \(\gamma \neq 0\). The functions \(c, v : D^0 \to \mathbb{R}^4\) given by
\[
c(t, x) = \left(\mathcal{C}_\Sigma, \Sigma S^2 \mathcal{C}_{SS}, \Sigma SC_{SS}, \frac{1}{2} \Sigma^2 \mathcal{C}_{\Sigma\Sigma}\right)^\top, \quad (4.1)
\]
\[
v(t, x) = \left(\mathcal{V}_\Sigma, \Sigma (\beta \mathcal{V}_A + S^2 \Gamma), \Sigma S \frac{\partial \Delta}{\partial \Sigma}, \frac{1}{2} \Sigma^2 \mathcal{V}_{S\Sigma}\right)^\top, \quad (4.2)
\]
\[29\]Here and in the following, we assume that all relevant partial derivatives of \(\mathcal{C}\) and \(\mathcal{V}\) exist; precise conditions are given in Assumption 4.2 below.
are called the *vega-gamma-vanna-volga vector* of the call and the option \( V \), respectively. With this notation, define the functions \( \lambda, \mu, \bar{\zeta} : \mathbb{D}^0 \to \mathbb{R} \) as follows:

\[
\lambda(t, x) = \begin{cases} 
\frac{c^\top \psi v}{c^\top \psi c} & \text{if } \nu_{\Sigma \Sigma} - \frac{c^\top \psi v}{c^\top \psi c} \Sigma_{\Sigma \Sigma} \geq 0, \\
\frac{c^\top \psi v - \frac{1}{2} \zeta_{\Sigma \Sigma} \Sigma_{\Sigma \Sigma} \psi}{c^\top \psi c - \frac{1}{2} \zeta_{\Sigma \Sigma} \Sigma_{\Sigma \Sigma} \psi} & \text{otherwise},
\end{cases}
\]

(4.3)

\[
\mu(t, x) = \frac{1}{2} (\nu_{\Sigma \Sigma} - \lambda \Sigma_{\Sigma \Sigma})^-, 
\]

(4.4)

\[
\bar{\zeta}(t, x) = \psi (v - \lambda c + \mu \bar{e}_4).
\]

(4.5)

Note that the term \( \mu \bar{e}_4 \) in (4.5) ensures that the fourth component of \( \bar{\zeta} \) is non-negative. Now, fix constants \( 0 < \Sigma < \Sigma_0 < \Sigma \), and define for each \( \psi > 0 \), the candidate feedback control \( \zeta^\psi = (\nu^\psi, \sigma^\psi, \eta^\psi, \xi^\psi) : \mathbb{D}^0 \to \mathbb{R}^4 \) by

\[
\zeta^\psi(t, x) = \zeta^0(\Sigma) + \bar{\zeta} \mathbb{1}_{\{\Sigma < \Sigma < \Sigma\}} \psi.
\]

(4.6)

The indicator \( \mathbb{1}_{\{\Sigma < \Sigma < \Sigma\}} \) is a technical modification that ensures that the implied volatility stays within the interval \( [\Sigma, \Sigma] \) by falling back to the reference feedback control \( \zeta^0(\Sigma) \) (which corresponds to constant implied volatility) as soon as the implied volatility hits the boundary of \( [\Sigma, \Sigma] \). More explicitly, the candidate feedback control can be expressed as

\[
\nu^\psi(t, x) = (\nu_{\Sigma} - \lambda \Sigma_{\Sigma}) \mathbb{1}_{\{\Sigma < \Sigma < \Sigma\}} \psi \nu \psi,
\]

\[
\sigma^\psi(t, x) = \Sigma + \Sigma \left( \beta \Sigma_{\Sigma} + \Sigma^2 \Sigma_{SS} - \lambda \Sigma^2 \Sigma_{SS} \right) \mathbb{1}_{\{\Sigma < \Sigma < \Sigma\}} \psi \sigma \psi,
\]

\[
\eta^\psi(t, x) = \Sigma \left( \Sigma^2 \Sigma_{\Sigma} - \lambda \Sigma_{\Sigma} \Sigma_{SS} \right) \mathbb{1}_{\{\Sigma < \Sigma < \Sigma\}} \psi \eta \psi,
\]

\[
\xi^\psi(t, x) = \frac{1}{2} (\nu_{\Sigma \Sigma} - \lambda \Sigma_{\Sigma \Sigma})^+ \mathbb{1}_{\{\Sigma < \Sigma < \Sigma\}} \psi \xi \psi.
\]

In general, there is no \( P^\psi \in \mathcal{Q}^0 \) such that \( \zeta^\psi \) coincides with the control \( \zeta^{P^\psi} \) corresponding to \( P^\psi \) as the process \( \zeta^{P^\psi}(t, X_t) \) fulfils the drift condition (2.7) only at the order \( O(\psi) \). However, to match the drift condition exactly, one can perturb \( \zeta^\psi \) by a suitable, asymptotically small term. This motivates part (b) of the following definition.

**Definition 4.1.** Let \( \mathfrak{P} \subset \mathcal{Q}^0 \).

(a) For each \( p \geq 1 \), we denote by \( L^p_{\mathfrak{P}} \) the vector space of Borel functions \( K : \mathbb{D}^0 \to \mathbb{R} \) satisfying

\[
\|K\|_{L^p_{\mathfrak{P}}} := \sup_{P \in \mathfrak{P}} \mathbb{E}^P \left[ \int_0^T |K(t, X_t)|^p \, dt \right]^{1/p} < \infty.
\]

(b) A family \( (P^\psi)_{\psi \in (0, \psi_0)} \subset \mathfrak{P} \) for some \( \psi_0 \in (0, 1) \) is called a candidate asymptotic model family (in \( \mathfrak{P} \)) if there is \( K_0 \in L^4_{\mathfrak{P}} \) such that for all \( \psi \in (0, \psi_0) \),

\[
\left| \zeta^P(t, X_t) - \zeta^\psi(t, X_t) \right| \leq K_0(t, X_t) \psi^2 \, dt \times P^\psi\text{-a.e.}
\]
III.4 Main results

The crucial property of a candidate asymptotic model family formalised in Definition 4.1 (b) is that the control $\zeta^P(t, x)$ corresponding to $P^\psi$ is $O(\psi^2)$-close to the candidate control $\zeta^v(t, x)$.

The leading-order coefficient of the asymptotic expansion of $\nu(\psi)$ is given in terms of the solution to a linear second-order parabolic PDE with a source term. Specifically, for each $\Sigma \in [\underline{\Sigma}, \overline{\Sigma}]$, we consider the PDE

$$\tilde{w}_t + (\alpha + \frac{1}{2} \beta \Sigma^2) \tilde{w}_A + \frac{1}{2} \Sigma^2 \sigma^2 (\tilde{w}_{SS} + 2 \gamma \tilde{w}_{SA} + \gamma^2 \tilde{w}_{AA}) + \frac{1}{2} \tilde{g}(\cdot, \Sigma) = 0 \quad \text{on} \; (0, T) \times G,$$

$$\delta \tilde{w}_A + \tilde{w}_M = 0 \quad \text{on} \; \{(t, S, A, M) : S \geq M\},$$

$$\tilde{w}(T, \cdot, \Sigma) = 0 \quad \text{on} \; G,$$

where the source term $\tilde{g} : D^0 \to \mathbb{R}$ is given by

$$\tilde{g}(t, x) = v(t, x) \top \tilde{\zeta}(t, x). \quad (4.8)$$

We prove our main result under the following assumptions.

**Assumption 4.2.** Set $D = (0, T) \times G \times [\underline{\Sigma}, \overline{\Sigma}] \subset D^0$.

(a) **Trading strategy set:** There is a constant $K_\mathcal{P} > 0$ such that for each trading strategy $\nu \in \mathcal{P}$ and each $P \in \mathcal{P}$, $Y^{\nu, P} > -K_\mathcal{P} \; dt \times P$-a.e.

(b) **Model set:** $\mathcal{P} \subset \mathcal{P}^0$ contains a candidate asymptotic model family, a reference model, and there are constants $\nu < 0 < \overline{\nu}$, $0 < \underline{\sigma} < \overline{\sigma}$, $\underline{\eta} < 0 < \overline{\eta}$, and $\overline{\xi} > 0$ such that for each $P \in \mathcal{P}$,

$$\nu^P \in [\nu, \overline{\nu}], \; \sigma^P \in [\underline{\sigma}, \overline{\sigma}], \; \eta^P \in [\underline{\eta}, \overline{\eta}], \; \xi^P \in [0, \overline{\xi}], \; \Sigma \in [\underline{\Sigma}, \overline{\Sigma}] \quad dt \times P$-a.e. \quad (4.9)

(c) **Call PDE:** $T_C \geq T$, i.e., the call expires after the non-traded option, and there is $C \in C^{1,2,2}((0, T_C) \times \mathbb{R}_+ \times \mathbb{R}_+) \cap C([0, T_C] \times \mathbb{R}_+ \times \mathbb{R}_+)$ such that for each $\Sigma \in [\underline{\Sigma}, \overline{\Sigma}]$, $C(\cdot, \Sigma)$ is a classical solution to the PDE (2.1) and

$$C_\Sigma \neq 0 \quad \text{and} \quad |C_\Sigma| \leq K_C \left(|C_\Sigma| + |S^2C_{SS}| + |SC_{SS}|\right) \quad \text{on} \; (0, T) \times \mathbb{R}_+ \times [\underline{\Sigma}, \overline{\Sigma}] \quad (4.10)$$

for some $K_C \in L^2_\mathcal{P}$.

(d) **Non-traded option PDE:** There is $V \in C^{1,2,2,1,2}(D^0) \cap C(\overline{D}^0)$ such that for each $\Sigma \in [\underline{\Sigma}, \overline{\Sigma}]$, $V(\cdot, \Sigma)$ is a classical solution to the PDE (2.9) with

$$|V_\Sigma|, \; |\beta A + S^2(V_{SS} + 2 \gamma V_{SA} + \gamma^2 V_{AA})|, \; |S(V_{SS} + \gamma V_{AS})|, \; |V_{SS}| \leq K_V \quad \text{on} \; D \quad (4.11)$$

for some constant $K_V > 0$.
(e) *Cash equivalent PDE:* There is \( \tilde{w} \in C^{1,2,2,1,2}(D^0) \cap C(D^\infty) \) such that for each \( \Sigma \in [\Sigma, \bar{\Sigma}], \tilde{w}(\cdot, \Sigma) \) is a classical solution to the PDE (4.7),

\[
0 \leq \tilde{w} \leq K_{\tilde{w}} \text{ on } D
\]

for some constant \( K_{\tilde{w}} > 0 \), and

\[
\begin{align*}
\tilde{w}_\Sigma, S(\tilde{w}_S + \gamma \tilde{w}_A), \beta \tilde{w}_A + S^2(\tilde{w}_{SS} + 2\gamma \tilde{w}_{SA} + \gamma^2 \tilde{w}_{AA}), \\
S(\tilde{w}_{SS} + \gamma \tilde{w}_{AS}), \tilde{w}_{\Sigma\Sigma} \in L^4_{\mathfrak{P}}.
\end{align*}
\]

(f) *Utility function:* \( U : \mathbb{R} \to \mathbb{R} \) is \( C^3 \) with \( U' > 0, U'' < 0 \) everywhere and has *decreasing absolute risk aversion,* i.e., \( y \mapsto -\frac{U''(y)}{U'(y)} \) is nonincreasing on \( \mathbb{R} \).

**Remark 4.3.** Let us discuss the various requirements in Assumption 4.2:

(a) This constraint on the agent’s credit line is an admissibility condition for the set of trading strategies. The P&L process \( Y^{v,P} \) is required to be bounded from below, uniformly over all strategies in \( \mathfrak{Y} \) and all models in \( \mathfrak{P} \). We show in Corollary 5.3 that this is satisfied for the P&L process associated to the delta-vega hedge \( v^* \) (cf. (4.12)). Hence, making the constant \( K_{\mathfrak{Y}} \) larger if necessary, the delta-vega hedge can always be added to the set of strategies \( \mathfrak{Y} \).

(b) A construction of a candidate asymptotic model family compatible with (4.9) is outlined in Section 4.3. The existence of uniform bounds on the controls as well as the implied volatility are essential for various steps in the proof of the main result. This is not as big an assumption as it might appear at first glance. Indeed, as the conclusions of our main result do not depend on the choice of these bounds, they can be chosen arbitrarily large.

(c) These regularity assumptions ensure that \( C \) corresponds to the Black–Scholes value of the liquidly traded call. The condition \( C_0 \neq 0 \) guarantees that the delta-vega hedge (cf. (4.12)) is well defined. The second condition in (4.10) ensures that the volga of the call is dominated by the sum of its vega, cash gamma, and cash vanna. For a plain-vanilla call option with payoff \( C(S) = (S - K)^+ \), explicit formulas for these greeks show that this requirement is met if \( \log S \in L^2_{\mathfrak{P}} \). This in turn follows easily from the explicit representation of \( S \) as a stochastic exponential together with the boundedness of the spot volatility from Assumption 4.2 (b).

(d) This is a regularity assumption on the option \( V \) similar to (c). However, we additionally enforce the bounds (4.11) to ensure that the vega-gamma-vanna-volga vector \( v \) is bounded. This assumption is satisfied if the option payoff \( V \) is sufficiently regular.

(e) This assumption posits that a (classical) solution \( \tilde{w} \) to the PDE (4.7) exists and satisfies certain bounds. The validity of this assumption depends on the regularity of the input quantities \( C, V, \alpha, \beta, \gamma, \) and \( \delta \).\[^{30}\]
(f) It is not essential that the utility function is defined on the whole real line. In fact, as we only consider strategies such that the P&L process is bounded from below by $-K_Y$ uniformly over trading strategies and models, we could also work with a (suitably displaced) utility function on $\mathbb{R}_+$. Also note that power and exponential utilities both have decreasing absolute risk aversion.

4.2 Main result

We are now in a position to state our main result, which provides an asymptotic expansion of the value in (2.16) and a corresponding asymptotically optimal policy. The existence of a suitable corresponding model set $\mathfrak{P}$ and a candidate asymptotic model family is considered in Section 4.3 below. Recall from Remark 4.3 (a) that the delta-vega hedge $\nu^*$ can always be included into the set of trading strategies $\mathfrak{Y}$ by making the constant $K_Y$ from Assumption 4.2 (a) larger if necessary.

The number $\tilde{w}_0 := \tilde{w}(0, X_0)$ defined through the solution $\tilde{w}$ to the PDE (4.7) determines the leading-order coefficient in the expansion of the value $v(\psi)$. As it also describes the (normalised) premium that the agent demands as a compensation for exposing herself to model misspecification (cf. the expansion (4.15) of the indifference ask price below), we call it the cash equivalent (of small uncertainty aversion).

**Theorem 4.4.** Let $\mathfrak{Y}$ be a set of trading strategies, $\mathfrak{P} \subset \mathfrak{P}^0$ a model set, and suppose that Assumption 4.2 is satisfied. Define the delta-vega hedging strategy $\nu^* = (\theta^*_t, \phi^*_t)_{t \in [0,T]}$ by

$$
\theta^*_t = \left( \Delta - \frac{\nu_S}{C_S} \right) (t, S_t, A_t, M_t, \Sigma_t),
$$

$$
\phi^*_t = \frac{\nu_S}{C_S} (t, S_t, A_t, M_t, \Sigma_t).
$$

(4.12)

If $\nu^* \in \mathfrak{Y}$ and $(P^\psi)_{\psi \in (0,\psi_0)} \subset \mathfrak{P}$ is a candidate asymptotic model family, then as $\psi \downarrow 0$:

$$
v(\psi) = \sup_{\nu \in \mathfrak{Y}} \inf_{P \in \mathfrak{P}} J^\nu(\nu, P) = \inf_{P \in \mathfrak{P}} \sup_{\nu \in \mathfrak{Y}} J^\nu(\nu, P) + o(\psi)
$$

$$
= U(Y_0) - U'(Y_0) \tilde{w}_0 \psi + o(\psi)
$$

$$
= J^\psi(\nu^*, P^\psi) + o(\psi).
$$

(4.13)

In particular, the delta-vega hedge $\nu^*$ is an optimal strategy at the leading-order $O(\psi)$ among all strategies in $\mathfrak{Y}$, and $P^\psi$ is a leading-order optimal choice of model for the fictitious adversary among all models in $\mathfrak{P}$.

The lengthy proof of Theorem 4.4 is postponed to Section 5.1. The first-order term in the expansion of $v(\psi)$ in (4.13) is determined by the cash equivalent $\tilde{w}_0$. Its probabilistic representation allows to identify the main factors that determine an option’s susceptibility to model misspecification:
Proposition 4.5 (Feynman–Kac representation). Suppose that Assumption 4.2 holds and let $P^0 \in \mathfrak P$ be a reference model. Then

$$
\tilde w_0 = \frac{1}{2} E^{P^0} \left[ \int_0^T \tilde g(t, S_t, A_t, M_t, \Sigma_0) \, dt \right].
$$

Here, the function $\tilde g$ (defined in (4.8)) can be written as

$$
\tilde g(t, S, A, M, \Sigma) = -\Sigma \left( \phi^* S^2 \psi_{SS} - (\beta V_A + S^2 \Gamma) \right) \tilde \sigma
- \Sigma \left( \phi^* S \psi_{S\Sigma} - S \frac{\partial \Delta}{\partial \Sigma} \right) \tilde \eta
- \frac{1}{2} \left( \phi^* \psi_{\Sigma \Sigma} - V_{\Sigma \Sigma} \right) \tilde \xi,
$$

where the functions $(\tilde \nu, \tilde \sigma, \tilde \eta, \tilde \xi) = \tilde \zeta$ are defined in (4.5) and $\phi^* = \frac{\psi_{\Sigma \Sigma}}{\psi_{S \Sigma}}$ is the vega hedge from Theorem 4.4.

Proof. The Feynman–Kac representation is proved in Proposition 5.1 (also note that $\Sigma_t = \Sigma_0 \, dt \times P^0$-a.e. because $P^0$ is a reference model). The representation of $\tilde g$ is the content of Corollary 5.7.

For an interpretation of this representation in the case of $\beta \equiv \gamma \equiv 0$, we refer to the discussion after equation (1.6) in the introduction. If $\gamma \neq 0$ (e.g., for a forward-start call as in Example 3.1), then the effective gamma and effective vanna are in general different from the gamma and vanna of the option. If the option $V$ depends on the realised variance of the stock (e.g., a call on the realised variance), then a term $\beta V_A$ is added to the effective gamma in (4.14).

The next proposition implies that whenever the vega-gamma-vanna-volga vectors of the call and the non-traded option $V$ are collinear, the local impact of uncertainty aversion vanishes at the leading order.

Proposition 4.6. Fix $(t, x) \in D^0$. If the vega-gamma-vanna-volga vectors $c(t, x)$ and $v(t, x)$ are collinear, then $\tilde g(t, x) = 0$.

Proof. Fix $(t, x) \in D^0$ and let $k \in \mathbb R$ such that $v(t, x) = kc(t, x)$. Then by construction (cf. (4.3)–(4.5)), $\lambda(t, x) = k$, $\mu(t, x) = 0$, and $\tilde \zeta(t, x) = 0$. Thus, $\tilde g(t, x) = 0$.

For example, consider the case where the non-traded option is a put with the same strike and maturity as the liquidly traded call. Then the put-call parity implies that the vegas, gammas, vannas, and volgas of both options coincide everywhere. Thus, $\tilde g \equiv 0$ and hence the cash equivalent $\tilde w_0$ vanishes. This is expected as put-call parity also provides a model-free hedge for this situation.

Indifference prices. The indifference ask price (for the non-traded option $V$) is the price at which the agent is indifferent between keeping a flat position and changing her position by selling the non-traded option for that price.
Recall that $V_0$ is the initial reference value of the non-traded option $V$ and that $\tilde{w}_0$ is its cash equivalent. Let $v(y; \psi)$ denote the value of our hedging problem corresponding to initial P&L $y$. If the agent decides to sell the non-traded option for a price $p_a(\psi)$, then her initial P&L for the hedging problem is $Y_0 + p_a(\psi) - V_0$. Therefore, the equation determining the indifference ask price $p_a(\psi)$ reads as follows:

$$U(Y_0) = v(Y_0 + p_a(\psi) - V_0; \psi).$$

Using the expansion of $v$ from Theorem 4.4, straightforward computations yield

$$p_a(\psi) = V_0 + \tilde{w}_0\psi + o(\psi).\quad (4.15)$$

Therefore, $\tilde{w}_0\psi$ is the leading-order premium demanded by the agent as a compensation for exposing herself to model uncertainty.

**Remark 4.7.** Buying an option is the same as selling the negative of that option. However, the cash equivalents corresponding to $V$ and $-V$ are in general different. This asymmetry is caused by the constraint that the uncorrelated squared volatility must be *nonnegative* and the fact that the reference model has zero uncorrelated squared volatility. In other words, the uncorrelated squared volatility can only depart from its reference value in one direction. In contrast, the other control variables can deviate from their reference value in both directions.

### 4.3 On the existence of a candidate asymptotic model family

Our main result, Theorem 4.4, assumes that the set of models $\mathfrak{P}$ contains a candidate asymptotic model family. In this section, we prescribe a set of models $\mathfrak{P}$ and sketch the construction of a candidate asymptotic model family in $\mathfrak{P}$. Fix constants $0 < \Sigma \leq \Sigma_0 < \bar{\Sigma}$, $\nu < 0 < \bar{\nu}$, $0 < \sigma < \bar{\sigma}$, $\eta < 0 < \bar{\eta}$, and $\bar{\xi} > 0$, and let $\mathfrak{P}$ denote the subset of models $P$ in $\mathfrak{P}^0$ such that the bounds (4.9) are satisfied. Under some further regularity assumptions on the greeks of the liquid option, $\mathfrak{P}$ then contains a candidate asymptotic model family.

The construction of the candidate asymptotic model family comprises two steps. The first is to prove that the candidate feedback control $\zeta^\psi$ can be modified by a term of order $O(\psi^2)$ such that the resulting modified feedback control $\tilde{\zeta}^\psi$ satisfies the drift condition (2.7):

**Lemma 4.8.** Let $\mathfrak{P} \subset \mathfrak{P}^0$ be such that (4.9) holds for every $P \in \mathfrak{P}$. Suppose in addition that Assumption 4.2 (c)–(d) holds with (4.10) replaced by the stronger condition that

$$|C_{\Sigma}| \geq 1/K_C \quad \text{and} \quad |S^2 C_{SS}|, |SC_{SS}^\psi|, |C_{\Sigma\Sigma}| \leq K_C \quad \text{on} \quad (0, T) \times \mathbb{R}^+ \times [\Sigma, \bar{\Sigma}], \quad (4.16)$$

for some constant $K_C > 0$. Then there are $\psi_0 > 0$ and functions

$$\tilde{\zeta}^\psi : D^0 \to [\nu, \bar{\nu}] \times [\sigma, \bar{\sigma}] \times [\eta, \bar{\eta}] \times [0, \bar{\xi}], \quad \psi \in (0, \psi_0),$$

such that

$$\tilde{\zeta}^\psi(\psi, \omega) = \zeta^\psi(\psi, \omega) + \tilde{\zeta}^\psi(\psi, \omega),$$

satisfies (2.7).
\( \alpha, \beta, \gamma \) such that for each \( \psi \in (0, \psi_0) \), the restriction \( \tilde{\zeta}^\psi \mid_{(0,T) \times G \times (\Sigma, \Sigma)} \) is continuous and can be extended to a continuous function on \( D^0 = (0, T) \times G \times \mathbb{R}_+ \). Moreover, there is \( K_0 > 0 \) such that for each \( (t, x) = (t, S, A, M, \Sigma) \in D \) and \( \psi \in (0, \psi_0) \),

(a) \( \tilde{\zeta}^\psi (t, x) = \zeta^0 (\Sigma) \) if \( \Sigma \not\in (\Sigma, \Sigma) \), i.e., the modified feedback control falls back to the reference feedback control if the bounds on the implied volatility are reached;

(b) writing \( (\tilde{\nu}^\psi, \tilde{\sigma}^\psi, \tilde{\eta}^\psi, \tilde{\xi}^\psi) = \tilde{\zeta}^\psi (t, x) \), we have

\[
\tilde{\nu}^\psi C_\Sigma + \frac{1}{2} S^2 C_{SS} ((\tilde{\sigma}^\psi)^2 - \Sigma^2) + \tilde{\sigma}^\psi \tilde{\eta}^\psi S C_{S\Sigma} + \frac{1}{2} (\tilde{\eta}^\psi)^2 + \tilde{\xi}^\psi C_{\Sigma \Sigma} = 0,
\]

i.e., the drift condition (2.7) is satisfied for the modified feedback control \( \tilde{\zeta}^\psi \);

(c)

\[
\left| \tilde{\zeta}^\psi (t, x) - \zeta^\psi (t, x) \right| \leq K_0 \psi^2,
\]

(4.17)

\( i.e., the modified feedback control \( \tilde{\zeta}^\psi \) is \( O(\psi^2) \)-close to the candidate \( \zeta^\psi \).

Proof. See Section 5.2. \( \square \)

Let \( \psi_0, K_0 \), and \( \tilde{\zeta}^\psi \) be as in Lemma 4.8. The second step now is to show that the stochastic differential equations (SDEs) corresponding to the modified feedback control \( \tilde{\zeta}^\psi = (\tilde{\nu}^\psi, \tilde{\sigma}^\psi, \tilde{\eta}^\psi, \tilde{\xi}^\psi)^T \) have a weak solution. Fix \( \psi \in (0, \psi_0) \). Writing \( \nu, \sigma, \eta, \xi \) instead of \( \tilde{\nu}^\psi, \tilde{\sigma}^\psi, \tilde{\eta}^\psi, \tilde{\xi}^\psi \) to ease the notation, the relevant SDEs read as

\[
\begin{align*}
\text{d}S_i' &= S_i' \sigma \text{d}W_t^0, \\
\text{d}\Sigma' &= \nu \text{d}t + \eta \text{d}W_t^0 + \sqrt{\xi} \text{d}W_t^1, \\
\text{d}A_i' &= \left( \alpha + \frac{1}{2} \beta \sigma^2 \right) \text{d}t + \gamma S_i' \sigma \text{d}W_t^0 + \delta \text{d}M_t, 
\end{align*}
\]

(4.18)

where \( \alpha, \beta, \gamma, \) and \( \delta \) are evaluated at \( (t, S_i', A_i', M_i') := \sup_{s \in [0,t]} S_s', \nu, \sigma, \eta, \) and \( \xi \) are evaluated at \( (t, S_i', A_i', M_i', \Sigma') \), and \( (W^0, W^1) \) is a bivariate standard Brownian motion.

Suppose there exists a weak solution to (4.18) (starting in \( S_0, \Sigma_0, A_0 \)) with the property that \( \Sigma' \) evolves in \( [\Sigma, \Sigma] \) almost surely and denote by \( P^\psi \) its image measure (under \( (S', \Sigma', A') \)) on the canonical space \( (\Omega, \mathcal{F}) \). Then by construction (cf. Definitions 2.3 and 2.5), \( P^\psi \in \mathcal{P}^0 \) and \( \zeta^\psi_P = \tilde{\zeta}^\psi (t, X_t) \text{d}t \times P^\psi\text{-a.e.} \). Moreover, by Lemma 4.8 and the fact that under \( P^\psi \), \( \Sigma \) evolves in \( [\Sigma, \Sigma] \) almost surely, (4.9) holds for every \( P \in (P^\psi)_{\psi \in (0,\psi_0)} \) and

\[
\left| \zeta^\psi_P (t, X_t) - \zeta^\psi (t, X_t) \right| \leq K_0 \psi^2 \text{d}t \times P^\psi\text{-a.e.}
\]
So \((P_\psi)_{\psi \in (0, \psi_0)}\) is a candidate asymptotic model family in \(\mathfrak{P}\).

It remains to argue the existence of a weak solution to (4.18) with the property that \(\Sigma'\) evolves in \([\Sigma, \Sigma']\). Note that we cannot directly apply standard existence results for weak solutions as the control \(\tilde{\zeta}_{\psi}\) is not continuous in \(\Sigma \in \mathbb{R}_+\). However, one can apply a standard existence result to the SDEs corresponding to the continuous extension of \(\tilde{\zeta}_{\psi}|_{(0,T) \times G \times (\Sigma, \Sigma)}\) to \(D^0\). Then the obvious idea is to stop the resulting weak solution as soon as \(\Sigma'\) hits the boundary of \([\Sigma, \Sigma']\) and restart the SDEs with new dynamics from there. After the restart, we keep \(\Sigma'\) \(\in \{\Sigma, \Sigma'\}\) constant, let \(S'\) evolve like a standard Black–Scholes model with constant volatility \(\Sigma'\), and (assuming suitable Lipschitz and linear growth conditions on the coefficients of the SDE for \(A\); cf. Appendix II.B) find a solution \(A'\) according to the dynamics in (4.18), but with the new dynamics of \(S'\). Then one can check that the constructed process satisfies the SDEs (4.18) with the original feedback control \(\tilde{\zeta}_{\psi}\); see the proof of Theorem II.3.7 for more details in a similar setup.

5 Proofs

This section contains the proofs of our main results. We first establish the value expansion and almost-optimality of the delta-vega hedge asserted in Theorem 4.4. Afterwards, we turn to the construction of the modified feedback control from Lemma 4.8.

5.1 Value expansion and almost optimality of the delta-vega hedge

In this section, we prove Theorem 4.4. Throughout, we assume that Assumption 4.2 is in force, that \(\psi^* \in \mathfrak{Y}\), and that \((P_\psi)_{\psi \in (0, \psi_0)} \subset \mathfrak{P}\) is a candidate asymptotic model family.\(^{31}\) In particular (recall Definition 4.1 (b)), we fix \(1 \leq K_0 \in L^4_\mathfrak{P}\) such that for every \(\psi \in (0, \psi_0)\),

\[
\left| \zeta_{t \psi} - \zeta_{\psi}(t, X_t) \right| \leq K_0(t, X_t)\psi^2 \quad dt \times P_\psi\text{-a.e.} \quad (5.1)
\]

For each \(\psi > 0\), define the candidate value function \(w_\psi : D^\mathfrak{D} \times \mathbb{R} \rightarrow \mathbb{R}\) by

\[
w_\psi(t, x, y) = U(y) - U'(y)\tilde{w}(t, x)\psi \quad (5.2)
\]

and set \(w_0^\psi := w_\psi(0, S_0, A_0, M_0, \Sigma_0, Y_0)\). Suppose for the moment that we have already proved the following two inequalities (cf. Lemmas 5.15 and 5.17):

\[
\inf_{P \in \mathfrak{P}} J_\psi(\psi^*, P) \geq w_0^\psi + o(\psi) \quad \text{as } \psi \downarrow 0, \quad (5.3)
\]

\[
\sup_{P' \in \mathfrak{P}} J_\psi(\psi, P') \leq w_0^\psi + o(\psi) \quad \text{as } \psi \downarrow 0. \quad (5.4)
\]

\(^{31}\)Recall from Remark 4.3 (a) that the delta-vega hedge \(\psi^*\) can always be included into the set of trading strategies \(\mathfrak{Y}\) by making the constant \(K_0\) from Assumption 4.2 (a) larger if necessary. The existence of a candidate asymptotic model family is discussed in Section 4.3.
Denoting by \( \lesssim \) “less or equal up to a term of order \( o(\psi) \)”, we obtain from (5.3)–(5.4) that
\[
\inf_{P \in \mathcal{P}} J^\psi(\mathbf{v}^*, P) \lesssim \sup_{\mathbf{v} \in \mathfrak{V}} \inf_{P \in \mathcal{P}} J^\psi(\mathbf{v}, P) \lesssim \sup_{P \in \mathcal{P}} \inf_{\mathbf{v} \in \mathfrak{V}} J^\psi(\mathbf{v}, P^\psi) \lesssim w^\psi \varepsilon
\]
and
\[
w^\psi \lesssim \inf_{P \in \mathcal{P}} J^\psi(\mathbf{v}^*, P) \lesssim J^\psi(\mathbf{v}^*, P^\psi) \lesssim \sup_{\mathbf{v} \in \mathfrak{V}} J^\psi(\mathbf{v}, P^\psi) \lesssim w^\psi.
\]

Hence, we have equality up to a term of order \( o(\psi) \) everywhere. In particular, assertion (4.13) of Theorem 4.4 holds. This completes the proof of Theorem 4.4 modulo the proof of (5.3)–(5.4). The proof of these two inequalities is based on careful estimates of the HJBI equation associated to the SDG (2.16). Section 5.1.1 introduces the notation used in the rest of the proof as well as some preliminary results. Sections 5.1.2–5.1.3 are purely analytic and provide the required estimates of the HJBI equation. Finally, Sections 5.1.4 and 5.1.5 contain the proofs of the inequalities (5.3) and (5.4).

5.1.1 Notation and preliminaries

Set \( \psi_{\min} = \min(\psi_\nu, \psi_\sigma, \psi_\eta, \psi_\xi) \), \( \psi_{\max} = \max(\psi_\nu, \psi_\sigma, \psi_\eta, \psi_\xi) \) (recall (2.14)), and denote by \( \|Q\|_F \) the Frobenius norm of a matrix \( Q \). Recalling that the squared uncorrelated volatility \( \xi^P \) has to be nonnegative, let \( \mathbf{Z}^0 := \mathbb{R}^3 \times [0, \infty) \) be the natural range for the controls \( \zeta^P \). A generic element of \( \mathbf{Z}^0 \) is always denoted by \( \zeta = (\nu, \sigma, \eta, \xi)^\top \). Next, define the function \( b^C : \mathbf{D}^0 \times \mathbf{Z}^0 \to \mathbb{R} \) by
\[
b^C(t, \mathbf{x}; \zeta) = \nu C_\Sigma + \frac{1}{2} S^2 C_{SS}(\sigma^2 - \Sigma^2) + \sigma \eta S C_{SS} + \frac{1}{2} (\eta^2 + \xi) C_{SS}
\]
\[
= \mathbf{c}(t, \mathbf{x})^\top (\zeta - \zeta^0(\Sigma)) + \frac{1}{2} \begin{pmatrix} \sigma - \Sigma \end{pmatrix}^\top \begin{pmatrix} \mathbf{c}(t, \mathbf{x}) & (S^2 C_{SS} & SC_{SS} & C_{SS}) \end{pmatrix} \begin{pmatrix} \sigma - \Sigma \end{pmatrix};
\]
(5.5)
cf. (4.1) for the definition of the vega-gamma-vanna-volga vector \( \mathbf{c}(t, \mathbf{x}) \) of the call. This definition is motivated by the drift condition (2.7), which states that \( b^C(t, \mathbf{X}_t; \zeta^P) = 0 \) \( dt \times P \)-a.e. for every \( \zeta^P \). For each \( (t, \mathbf{x}) \in \mathbf{D} \), write
\[
\mathbf{Z}^0(t, \mathbf{x}) = \{ \zeta \in \mathbf{Z}^0 : b^C(t, \mathbf{x}; \zeta) = 0 \}
\]
for the set of controls \( \zeta \) that fulfil the drift condition at \( (t, \mathbf{x}) \), and define
\[
\mathbf{Z}^0_{\text{lin}}(t, \mathbf{x}) = \{ \zeta \in \mathbf{Z}^0 : \mathbf{c}(t, \mathbf{x})^\top (\zeta - \zeta^0(\Sigma)) = 0 \},
\]
(5.6)
the set of controls \( \zeta \) that satisfy the “linearised drift condition” at \( (t, \mathbf{x}) \). Next, set
\[
\mathbf{Z} = \lbrack \nu, \nu \rbrack \times \lbrack \sigma, \sigma \rbrack \times \lbrack \eta, \eta \rbrack \times \lbrack 0, \xi \rbrack
\]
for the range of the controls in $\mathfrak{P}$ (cf. Assumption 4.2 (b)) and denote by $Z(t, x) = Z^0(t, x) \cap Z$ and $Z_{lin}(t, x) = Z_{lin}^0(t, x) \cap Z$ the intersections of $Z^0(t, x)$ and $Z_{lin}^0(t, x)$ with $Z$, respectively. Also recall from Definition 2.4 that the reference feedback control is $\zeta^0(\Sigma) = (0, \Sigma, 0, 0)^T$.

We start with the probabilistic representation of the solution to the PDE (4.7) for the cash equivalent $\tilde{w}_0 = \tilde{w}(0, X_0)$.

**Proposition 5.1** (Feynman–Kac representation). Let $P^0 \in \mathfrak{P}$ be a reference model. Then

$$\tilde{w}_0 = \tilde{w}(0, X_0) = \frac{1}{2} E^{P^0} \left[ \int_0^T \bar{g}(t, X_t) \, dt \right]. \quad (5.7)$$

**Proof.** We only sketch the standard proof. Applying Itô’s formula to $\tilde{w}(t, X_t)$ under $P^0$ and using the PDE (4.7) for $\tilde{w}$ shows that

$$\tilde{w}(0, X_0) = \frac{1}{2} \int_0^T \bar{g}(t, X_t) \, dt + \text{(local martingale)}.$$

Using Assumption 4.2 (e), the local martingale term is easily shown to be a martingale. Hence, taking expectations yields the Feynman–Kac representation (5.7). \qed

The next lemma provides the dynamics of the P&L processes:

**Lemma 5.2.** Let $\nu = (\theta, \phi) \in \mathfrak{Y}$ and $P \in \mathfrak{P}$. Then under $P$,

$$dY_{t}^{\nu, P} = \left( \theta_t - (\Delta(t, X_t) - \phi_t C_S(t, S_t, \Sigma_t)) \right) dS_t + \left( \phi_t \bar{C}_S(t, S_t, \Sigma_t) - \bar{V}_S(t, X_t) \right) d\Sigma_t^{c, P} - b^\nu(t, X_t; \zeta^P_t) \, dt. \quad (5.8)$$

Here,

$$\Sigma^{c, P} = \Sigma - \int_0^t \nu_u^P \, du$$

is the (continuous) local martingale part of $\Sigma$ under $P$ and $b^\nu : D^0 \times Z^0 \to \mathbb{R}$ is given by

$$b^\nu(t, X_t; \zeta) = \nu \bar{V}_\Sigma + \frac{1}{2} (\beta \bar{V}_A + S^2 \Gamma)(\sigma^2 - \Sigma^2) + \sigma \eta S \frac{\partial \Delta}{\partial \Sigma} + \frac{1}{2} (\eta^2 + \xi) \bar{V}_\Sigma \Sigma$$

$$= v(t, x)^\top (\zeta - \zeta^0(\Sigma)) + \frac{1}{2} \left( \sigma - \Sigma \right)^\top \left( \begin{array}{cc} \beta \bar{V}_A + S^2 \Gamma & S \frac{\partial \Delta}{\partial \Sigma} \\ S \frac{\partial \Delta}{\partial \Sigma} & \bar{V}_\Sigma \Sigma \end{array} \right) \left( \sigma - \Sigma \right), \quad (5.9)$$

where $v$ is the vega-gamma-vanna-volga vector of the non-traded option (cf. (4.2)).
Proof. Fix \( \psi = (\theta, \phi) \in \mathcal{Y} \) and \( P \in \mathfrak{P} \) and recall from (2.11) and (2.10) that
\[
dY_{t}^{\psi, P} = \theta_{t} dS_{t} + \phi_{t} dC_{t} - dV_{t},
\]
where \( V_{t} = V(t, S_{t}, A_{t}, M_{t}, \Sigma_{t}) \). Thus, it remains to compute the dynamics of \( C \) and \( V \) under \( P \).

First, by (2.4), Itô’s formula (under \( P \)), and the drift condition (2.7), we have
\[
dC_{t} = C_{t} S_{t} dS_{t} + C_{t} \Sigma_{t} d\Sigma_{t}^{c,P}.
\]
(5.11)

Second, applying Itô’s formula to \( V_{t} = V(t, S_{t}, A_{t}, M_{t}, \Sigma_{t}) \) and using the PDE (2.9) to substitute \( V_{t} = V_{t}(t, X_{t}) \) and to eliminate the \( dM_{t} \)-term, we arrive at
\[
dV_{t} = \Delta dS_{t} + V_{t} \Sigma_{t} d\Sigma_{t}^{c,P} + b^{V}(\zeta_{t}^{P}) dt.
\]
(5.12)

Finally, inserting (5.11) and (5.12) into (5.10) yields (5.8). The last equality in the definition (5.9) of \( b^{V} \) is the Taylor expansion of \( b^{V}(\zeta) \) around \( \zeta_{0}(\Sigma) \) and can be verified by computing the gradient and the Hessian of \( b^{V}(\zeta) \) at \( \zeta_{0}(\Sigma) \).

We next analyse the dynamics of the P&L process \( Y_{\psi}^{*}, P \) corresponding to the delta-vega hedge \( \psi^{*} \). To this end we define, for each \( (t, x) \in \mathcal{D}^{0} \):
\[
\psi^{*}(t, x) = \left( \Delta - \frac{V_{t} \Sigma_{t}}{C_{t}^{\Sigma}}, \frac{V_{t} \Sigma_{t}}{C_{t}^{\Sigma}} \right).
\]
(5.13)

Note that, with a slight abuse of notation, we use the symbol \( \psi^{*} \) both for the function defined in (5.13) and the delta-vega hedge defined in Theorem 4.4. This is, of course, motivated by the relationship \( \psi_{t}^{*} = \psi^{*}(t, X_{t}) \).\(^{32}\)

The following corollary to Lemma 5.2 shows that the P&L process \( Y_{\psi^{*}}^{*}, P \) corresponding to the delta-vega hedge \( \psi^{*} \) has no local martingale part and is bounded, uniformly in \( P \in \mathfrak{P} \).

**Corollary 5.3.** There are constants \( \underline{Y}, \overline{Y} \in \mathbb{R} \) such that for each \( P \in \mathfrak{P} \),
\[
Y_{\psi^{*}}^{*}, P \in [\underline{Y}, \overline{Y}] \quad dt \times P\text{-a.e.}
\]
Moreover, under each \( P \in \mathfrak{P} \),
\[
dY_{t}^{\psi^{*}, P} = -b^{V}(t, X_{t}; \zeta_{t}^{P}) dt,
\]
where \( b^{V} \) is defined in (5.9).

Proof. By construction of \( \psi_{t}^{*} = \psi^{*}(t, X_{t}) \), the local martingale part in the dynamics (5.8) of \( Y_{\psi^{*}}^{*}, P \) is zero for each \( P \in \mathfrak{P} \). Thus, it suffices to find a uniform bound (independent of \( P \in \mathfrak{P} \)) for the drift coefficient \( b^{V}(t, X_{t}; \zeta_{t}^{P}) \). But this is immediate from Assumption 4.2 (b) and (d). \( \Box \)

\(^{32}\)With a slight abuse of notation, \( \psi_{t}^{*} \) always denotes the time-\( t \) value of the process \( \psi^{*} \) and not the partial derivative of the function \( \psi^{*} \) with respect to the first variable.
Lemma 5.2 together with the covariations of $S$ and $\Sigma$ in (2.6) and the semi-
martingale decomposition (2.8) of $A$ specifies the joint dynamics of the process
$(S, A, M, \Sigma, Y, \nu, \phi, \xi)$ under $P \in \mathcal{P}$. This allows to write down the Hamilton–Jacobi–
Bellman–Isaacs equation corresponding to the SDG (2.16): for each $\psi > 0$, the HJBI equation reads as

$$w^\psi_t(t, x, y) + \sup_{w \in \mathbb{R}^2} \inf_{\zeta \in \mathbb{Z}^2(\ell, x)} H^\psi(t, x, y; \nu, \psi) = 0,$$

where the Hamiltonian $H^\psi : \mathbb{D}^0 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{Z}^0 \to \mathbb{R}$ is given by

$$H^\psi(t, x, y; \nu, \psi) = \frac{1}{\psi} U'(y)f(\Sigma, \zeta) + \nu w^\psi_\Sigma + (\alpha + \frac{1}{2} \beta \sigma^2) w^\psi_A - b^\psi(\zeta)w^\psi_Y$$

$$+ \frac{1}{2} \sigma^2 S^2[(w^\psi_y + \gamma w^\psi_Y) + \sigma S \psi \psi_\Sigma] + \gamma w^\psi_A + \sigma S \psi \psi_A$$

$$+ \frac{1}{2} S^2(\theta - \Delta \phi C_S)(w^\psi_Y + \gamma w^\psi_Y) + \sigma S \psi \psi_\Sigma + \gamma w^\psi_A + \sigma S \psi \psi_A$$

$$+ \frac{1}{2} \sigma^2 S^2[\theta - \Delta \phi C_S]w^\psi_y + \frac{1}{2}(\eta^2 + \xi)(\phi C_S - \nu \Sigma)w^\psi_Y$$

$$+ \sigma S \psi \psi_\Sigma + \gamma w^\psi_A + \frac{1}{2}(\eta^2 + \xi)(\phi C_S - \nu \Sigma)w^\psi_Y.$$  

(5.15)

We emphasise that our candidate value function $w^\psi$ defined in (5.2) does not solve the HJBI equation (5.14) exactly. However, a key step in the proof of the two inequalities (5.3)–(5.4) is to show that $w^\psi$ is asymptotically (in a suitable sense) a solution to (5.14); cf. Lemmas 5.13 and 5.14 below.

We close this preliminary section by providing an auxiliary lemma that allows
to estimate quantities like $b^\psi(t, x; \zeta)$ or $b^\psi(t, x; \zeta)$ in terms of $|\zeta - \zeta^0(\Sigma)|$.

**Lemma 5.4.** Define the function $q : \mathbb{R}^1 \times \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ by

$$q(\Sigma, \zeta) = \nu a_1 + \frac{1}{2} a_2(\sigma^2 - \Sigma) + \sigma a_3 + \frac{1}{2} (\eta^2 + \xi) a_4,$$

where $a = (a_1, a_2, a_3, a_4)^\top$ and $\zeta = (\nu, \sigma, \eta, \xi)^\top$. Then:

$$|q(\Sigma, \zeta)| \leq \max(1, \Sigma) |a| |\zeta - \zeta^0(\Sigma)| + |a| |\zeta - \zeta^0(\Sigma)|^2.$$

**Proof.** Fix $\Sigma \in \mathbb{R}^1$, $a \in \mathbb{R}^4$, and $\zeta \in \mathbb{R}^4$. As $q$ is quadratic in $\zeta$, we can recast it in matrix form:

$$q(\Sigma, \zeta) = \left(\begin{array}{c} a_1 \\ \Sigma a_2 \\ \Sigma a_3 \\ \frac{1}{2} a_4 \end{array}\right)^\top \left(\begin{array}{c} \sigma - \Sigma \\ \eta \end{array}\right) + \frac{1}{2} \left(\begin{array}{cccc} a_2 & a_3 & \sigma - \Sigma \\ a_3 & a_4 & \eta \end{array}\right).$$

(5.17)
Using the Cauchy–Schwarz inequality, the absolute value of the first summand on the right-hand side of (5.17) is easily estimated by \( \max(1, \Sigma) |\zeta - \zeta^0(\Sigma)|. \) Likewise, using also the compatibility of the Frobenius norm with the Euclidean norm, the absolute value of the second summand is dominated by

\[
\frac{1}{2} \left\| \begin{pmatrix} a_2 & a_3 \\ a_3 & a_4 \end{pmatrix} \right\|_F \left| \begin{pmatrix} \sigma - \Sigma \\ \eta \end{pmatrix} \right|^2 \leq |a| \left| \zeta - \zeta^0(\Sigma) \right|^2.
\]

\[ \square \]

5.1.2 Estimates for the Hamiltonian

In order to prove that the candidate value function is – asymptotically – a solution to the HJBI equation (5.14), we need several estimates for the Hamiltonian \( H^\psi \) defined in (5.15). To this end, we decompose it into four parts:

\[
H^\psi(t, x, y; \nu, \zeta) = U'(y)H^\psi_{t}(t, x; \zeta) + U'(y)H^\psi_{x}(t, x, y; \zeta) - U'(y)H^\psi_{\nu}(t, x; \zeta) = 0.
\]

where

\[
H^\psi_{t}(t, x; \zeta) := \frac{1}{2}\psi(\zeta - \zeta^0(\Sigma)) - \nu(t, x) \psi(t, x) \Sigma - \zeta^0(\Sigma) \Sigma - \zeta^0(\Sigma),
\]

\[
H^\psi_{x}(t, x, y; \zeta) := -\frac{1}{2} \begin{pmatrix} \sigma - \Sigma \\ \eta \end{pmatrix}^\dagger \begin{pmatrix} -\sigma \Sigma + 2\sigma \Sigma^\dagger \Sigma + \Sigma^\dagger \Sigma^\dagger \Sigma \\ \sigma \Sigma + 2\sigma \Sigma^\dagger \Sigma + \Sigma^\dagger \Sigma^\dagger \Sigma \end{pmatrix} \begin{pmatrix} \sigma - \Sigma \\ \eta \end{pmatrix} + \frac{U''(y)}{U'(y)} b^\psi(\zeta) \tilde{w}^\psi,
\]

\[
H^\psi_{\nu}(t, x; \zeta) := \nu \tilde{w}_\Sigma + (\alpha + \frac{1}{2} \beta \sigma^2) \tilde{w}_A + \frac{1}{2} \sigma^2 \tilde{w}_SS + 2\gamma \tilde{w}_{SA} + 2\gamma \tilde{w}_{AA} + \frac{1}{2} (\eta^2 + \xi) \tilde{w}_{\Sigma \Sigma},
\]

\[
H^\psi(t, x, y; \nu, \zeta) := -\frac{w^\psi_{YY}}{2} \begin{pmatrix} \sigma S(\theta - (\Delta - \phi C_S)) \\ \phi C_S - \nu \Sigma \end{pmatrix}^\dagger \begin{pmatrix} 1 \\ \eta^2 + \xi \end{pmatrix} \begin{pmatrix} \sigma S(\theta - (\Delta - \phi C_S)) \\ \phi C_S - \nu \Sigma \end{pmatrix} + \psi U''(y) \begin{pmatrix} \sigma S(\tilde{w}_S + \gamma \tilde{w}_A) \\ \tilde{w}_\Sigma \end{pmatrix}^\dagger \begin{pmatrix} 1 \\ \eta^2 + \xi \end{pmatrix} \begin{pmatrix} \sigma S(\theta - (\Delta - \phi C_S)) \\ \phi C_S - \nu \Sigma \end{pmatrix}.
\]

\( H^\psi_1 \) includes the penalty term (cf. the definition of \( f \) in (2.13)) and the linear \( O(1) \) part of \( b^\psi(\zeta)w^\psi_\nu; H^\psi_2 \) contains the quadratic \( O(1) \) part and the \( O(\psi) \) part of \( b^\psi(\zeta)w^\psi_\nu; H^\psi_3 \) collects all second-order partial derivatives of \( w^\psi \) that involve at least one partial derivative with respect to \( Y; \) and \( H^\psi_4 \) takes care of all remaining partial derivatives of \( w^\psi. \)

For later reference, we note that by the definition of \( H_3 \) and \( \zeta^0, \) the PDE (4.7) for \( \tilde{w} \) can be written as

\[
\tilde{w}_t(t, x) + H_3(t, x; \zeta^0(\Sigma)) + \frac{1}{2} \tilde{g}(t, x) = 0 \quad \text{for } (t, x) \in D.
\]
Moreover, for every \((t, x, y) \in \mathbb{D} \times \mathbb{R}\) and \(\zeta \in \mathbb{Z}\),
\[
H^\psi_1(t, x; \zeta^0(\Sigma)) = 0, \quad H^\psi_2(t, x; \zeta^0(\Sigma)) = 0, \quad H^\psi_4(t, x, y; \psi^*(t, x), \zeta) = 0,
\] (5.21)
by construction of the reference feedback control \(\zeta^0\) (cf. Definition 2.4) and the delta-vega hedge \(\psi^*\) (cf. (5.13)).

**Remark 5.5.** Recall that the HJBI equation (5.14) involves a minimisation over \(\zeta \in \mathbb{Z}(t, x)\) and a maximisation over \(\psi \in \mathbb{R}^2\). The strategy variable \(\psi\) only shows up in the \(H^\psi_4\) term. Moreover, using an ansatz of the form \(\zeta = \zeta^0(\Sigma) + \tilde{\zeta}\psi\), one can check that \(\tilde{\zeta}\) only affects the \(O(\psi)\) term of \(H^\psi\) through \(H^\psi_1\) (provided that \(\psi = \psi^*\) so that the \(H^\psi_4\) term vanishes; cf. (5.21)). The impact of \(\tilde{\zeta}\) through \(H^\psi_1\) and \(H_3\) only appears at higher orders. This distinction is reflected in the proofs of this section as follows.

On the one hand, the estimates for the terms \(H^\psi_2\) and \(H_3\) in Propositions 5.9–5.10 and Corollary 5.11 are rather direct and provide simultaneously asymptotic upper and lower bounds. On the other hand, the proofs of the estimates for \(H^\psi_4\) and, in particular, \(H^\psi_1\) are more difficult as the corresponding bounds arise from optimisation problems over the strategy variables and the controls, respectively. The asymptotic bound for \(H^\psi_1\) is the easier one because \(H^\psi_1\) is quadratic in \(\psi\) and the optimisation is unconstrained. In contrast, the asymptotic bound for \(H^\psi_4\) in Proposition 5.8 arises from the linearly constrained quadratic programming problem (3.10). An additional difficulty stems from the fact that we need this bound to hold for controls \(\zeta \in \mathbb{Z}(t, x)\) that satisfy the nonlinear constraint \(\Psi(t, x; \zeta) = 0\) instead of the linear one (cf. Proposition 5.8 (a)).

We first provide the solution to a linearly constrained quadratic programming problem involving the vega-gamma-vanna-volga vectors \(c(t, x)\) and \(v(t, x)\) that lies at the core of the minimisation part of the HJBI equation. In particular, the candidate feedback control \(\zeta^\psi\) (cf. (4.6)) is a suitably modified version of the minimiser \(\zeta^\psi^*\) (cf. (5.24) below) of this quadratic programming problem; both controls differ only by the indicator \(1_{[\Sigma < \zeta < \bar{\Sigma}]}\) that ensures that \(\zeta^\psi\) falls back to the reference feedback control \(\zeta^0(\Sigma)\) once the implied volatility hits the boundary of \([\Sigma, \bar{\Sigma}]\). Recall the definitions of \(\Psi, c, v, \lambda, \mu, \tilde{\zeta}, \) and \(\zeta^\psi\) in (2.14) and (4.1)–(4.6).

**Lemma 5.6.** For each \((t, x) \in \mathbb{D}\) and \(\psi > 0\), consider the linearly constrained minimisation problem
\[
\text{minimise } H^\psi_1(t, x; \zeta) \quad \text{subject to } \zeta \in \mathbb{Z}^0_{\text{lin}}(t, x).
\] (5.22)
(a) For each \((t, x) \in \mathbb{D}\) and \(\psi > 0\), we have
\[
\min_{\zeta \in \mathbb{Z}^0_{\text{lin}}(t, x)} H^\psi_1(t, x; \zeta) = -\frac{1}{2} \tilde{g}(t, x)\psi
\] (5.23)
and the minimum is attained at
\[
\zeta^\psi^*(t, x) = \zeta^0(\Sigma) + \psi \tilde{\zeta}(t, x).
\] (5.24)
In particular, as \(\zeta^\psi^* \in \mathbb{Z}^0_{\text{lin}}(t, x), c(t, x)^T \tilde{\zeta}(t, x) = 0\) and \(e_1^T \zeta(t, x) \geq 0\).
For each \((t, x) \in D\) and \(\psi > 0\), \((\lambda(t, x), \mu(t, x))\) is a Lagrange multiplier for \((5.22)\) (independent of \(\psi\)), i.e.,

\[
-\frac{1}{2} \bar{g}(t, x) \psi = \inf_{\zeta \in \mathbb{R}^4} L_1^\psi(t, x; \zeta, \lambda(t, x), \mu(t, x)),
\]

where

\[
L_1^\psi(t, x; \zeta, \lambda', \mu') = H_1^\psi(t, x; \zeta) + \lambda' c(t, x) \top (\zeta - \zeta^0(\Sigma)) - \mu' \bar{e}_1^\top (\zeta - \zeta^0(\Sigma))
\]

is the Lagrangian corresponding to the constrained minimisation problem \((5.22)\).

(c) There is \(K_{\bar{g}} > 0\) such that \(0 \leq \bar{g} \leq K_{\bar{g}}\) on \(D\).

(d) There is \(K_\zeta \geq 1\) such that for every \((t, x) \in D\) and \(\psi > 0\),

\[
|\zeta^\psi(t, x) - \zeta^0(\Sigma)| \leq K_\zeta \psi.
\]

(e) For every \((t, x) \in D\) and \(\psi > 0\),

\[
H_1^\psi(t, x; \zeta^\psi(t, x)) = -\frac{1}{2} \bar{g}(t, x) 1_{\{\zeta(\Sigma) = \zeta(\Sigma)\}} \psi.
\]

(f) There is \(K_\lambda \in L^2_{\bar{g}}\) such that for every \((t, x) \in D\),

\[
|\lambda(t, x)| \left\| \begin{pmatrix} S^2 C_{SS} \\ SC_{SE} \\ C_{SE} \end{pmatrix} \right\|_F \leq K_\lambda(t, x). \tag{5.26}
\]

Proof. Recalling the definitions of \(H_1^\psi(t, x; \zeta)\) (cf. \((5.19))\) and \(Z_{lin}(t, x)\) (cf. \((5.6))\) and using the substitution \(z = \zeta - \zeta^0(\Sigma)\), it is easy to see that for each \((t, x) \in D\) and \(\psi > 0\), the minimisation problem \((5.22)\) can be recast as

\[
\begin{align*}
\text{minimise } & \frac{1}{2\psi} z \top \Psi^{-1} z - v(t, x) \top z \\
\text{subject to } & z \in \mathbb{R}^4, z \top c(t, x) = 0, z_i \geq 0.
\end{align*} \tag{5.27}
\]

Note that \((5.27)\) is a linearly constrained minimisation problem of the form \((A.5)\) with \(n = 4, D = \Psi^{-1}/\psi, v = v(t, x),\) and \(c = c(t, x)\). Also note that with this choice of \(D\), we have (denoting by \(d_{\max}\) and \(d_{\min}\) the maximal and minimal element on the diagonal of \(D\), respectively) \(d_{\max} = \psi_{\min}^{-1}/\psi\) and \(d_{\min} = \psi_{\max}^{-1}/\psi\), so that, in particular, \(d_{\max} d_{\min} = \psi_{\max} \psi_{\min}\).

(a) Fix \((t, x) \in D\) and \(\psi > 0\). By Lemma A.1 (a), the minimiser of \((5.27)\) is \(z^* = \bar{z}(t, x)\). After resubstitution, this yields the minimiser \((5.24)\) of the original minimisation problem \((5.22)\). Moreover, by Lemma A.1 (c), the minimum of \((5.27)\) (which clearly coincides with the minimum of \((5.22)\)) is

\[
-\frac{1}{2} v(t, x) \top z^* = -\frac{1}{2} v(t, x) \top \bar{z}(t, x) \psi = -\frac{1}{2} \bar{g}(t, x) \psi;
\]
recall the definition of $\tilde{g}$ in (4.8). For further reference, we also note that the bound on $|z^*|$ from Lemma A.1 (a) translates to

$$\lvert \tilde{\zeta}(t, x) \rvert \leq \psi_{\text{max}} \lvert v(t, x) \rvert. \quad (5.28)$$

(b): This follows immediately from the second assertion of Lemma A.1 (d).

(c): By Assumption 4.2 (d) and the definition of $v(t, x)$, there is a constant $K_\psi > 0$ such that $|v(t, x)| \leq K_\psi$ for all $(t, x) \in D$. Set $K_{\tilde{g}} = \psi_{\text{max}} K_\psi^2$ and fix $(t, x) \in D$ and $\psi > 0$. As $\zeta^0(\Sigma) \in Z^0_{\text{lin}}(t, x)$ and $H_{t, x}^0(\Sigma) = 0$, we have $\tilde{g}(t, x) \geq 0$ by (5.23). On the other hand, using the Cauchy–Schwarz inequality and (5.28), we have for every $(t, x) \in D$:

$$g(t, x) = v(t, x) - \tilde{\zeta}(t, x) \leq |v(t, x)| \lvert \tilde{\zeta}(t, x) \rvert \leq \psi_{\text{max}} |v(t, x)|^2 \leq K_{\tilde{g}}.$$  

(d): Set $K_{\tilde{C}} = \max(\psi_{\text{max}} K_{\tilde{g}}, 1)$ where $K_{\tilde{g}}$ is as in the proof of part (c), and fix $(t, x) \in D$ as well as $\psi > 0$. If $\Sigma \in \{\Sigma, \overline{\Sigma}\}$, then $\zeta^\psi(t, x) = \zeta^0(\Sigma)$ by construction and the assertion is trivial. Otherwise, if $\Sigma \in (\Sigma, \overline{\Sigma})$, then $\zeta^\psi(t, x) = \zeta^{\psi^*}(t, x)$ and (5.28) implies that

$$\lvert \zeta^\psi(t, x) - \zeta^0(\Sigma) \rvert = \lvert \zeta^{\psi^*}(t, x) - \zeta^0(\Sigma) \rvert = \tilde{\zeta}(t, x) \psi \leq \psi_{\text{max}} |v(t, x)| \psi \leq K_{\tilde{C}} \psi.$$  

(e): Fix $(t, x) \in D$ and $\psi > 0$. First, suppose that $\Sigma \in \{\Sigma, \overline{\Sigma}\}$. Then $\zeta^\psi(t, x) = \zeta^0(\Sigma)$ by construction of $\zeta^\psi$ and the assertion follows from the fact that $H_{t, x}^\psi(t, x; \zeta^0(\Sigma)) = 0$ (cf. (5.21)). Second, suppose that $\Sigma \in (\Sigma, \overline{\Sigma})$. Then $\zeta^\psi(t, x) = \zeta^{\psi^*}(t, x)$ and the assertion follows from part (a).

(f): Let $K_{\tilde{\psi}} > 0$ be as in the proof of part (c). Then the bound (A.7) from Lemma A.1 (b) implies that, for every $(t, x) \in D$,

$$\lvert \lambda(t, x) c(t, x) - \mu(t, x) e_4 \rvert \leq (1 + \frac{\psi_{\text{max}}}{\psi_{\text{min}}}) K_{\tilde{\psi}}.$$  

Recall from (4.1) that $c(t, x) = (C_\Sigma, \Sigma S^2 C_{SS}, \Sigma S C_{SD}, \Sigma S C_{SD}^\top C_{SS} C_{SD})^\top$. Clearly, each of the first three components of $\lambda(t, x)c(t, x)$ is bounded in absolute value by the length of the vector $\lambda(t, x)c(t, x) - \mu(t, x)e_4$. Using also that $\Sigma > 0$, we can find a constant $K > 0$ such that for every $(t, x) = (t, S, A, M, \Sigma) \in D$,

$$\lvert \lambda(t, x) C_\Sigma(t, S, \Sigma) \rvert \leq K, \ \lvert \lambda(t, x) S^2 C_{SS}(t, S, \Sigma) \rvert \leq K, \ \lvert \lambda(t, x) S C_{SD}(t, S, \Sigma) \rvert \leq K.$$  

(This argument does not work for the fourth component of $\lambda(t, x)c(t, x)$ due to the presence of the term $\mu(t, x)e_4$ in (5.30).) Set $K_{\lambda}(t, x) = 3K(2 + K_{\tilde{\psi}})$ where $K_{\tilde{\psi}} \in L^2_{\text{q}}$ is as in Assumption 4.2 (c). Clearly, $K_{\lambda} \in L^2_{\text{q}}$.

Now, fix $(t, x) \in D$. Using that the Euclidean norm of a vector is dominated by the sum of the absolute values of each of its entries as well as Assumption 4.2 (c)
to bound $|\mathcal{C}_\Sigma|$.

$$
|\lambda(t, x)| \left\| \begin{pmatrix} S^2\mathcal{C}_{SS} & S\mathcal{C}_{S\Sigma} \\ S\mathcal{C}_{S\Sigma} & \mathcal{C}_{\Sigma\Sigma} \end{pmatrix} \right\|_F = |\lambda(t, x)| \left( |S^2\mathcal{C}_{SS}| + |S\mathcal{C}_{S\Sigma}| + |\mathcal{C}_{\Sigma\Sigma}| \right) \\
\leq |\lambda(t, x)| \left( |S^2\mathcal{C}_{SS}| + 2|S\mathcal{C}_{S\Sigma}| + |\mathcal{C}_{\Sigma\Sigma}| \right) \\
\leq (2 + K_\mathcal{C}(t, x)) |\lambda(t, x)| \left( |\mathcal{C}_\Sigma| + |S^2\mathcal{C}_{SS}| + |S\mathcal{C}_{S\Sigma}| \right).
$$

Combining this with (5.31) and the choice of $K_\lambda$ completes the proof. \qed

**Corollary 5.7.** For each $(t, x) \in D$,

$$
\tilde{g}(t, x) = -\Sigma \left( \phi^* S^2\mathcal{C}_{SS} - (\beta V_A + S^2\Gamma) \right) \tilde{\sigma} \\
- \Sigma \left( \phi^* S\mathcal{C}_{S\Sigma} - S \frac{\partial \Delta}{\partial \Sigma} \right) \tilde{\eta} \\
- \frac{1}{2} \left( \phi^* \mathcal{C}_{\Sigma\Sigma} - V_{\Sigma\Sigma} \right) \tilde{\xi},
$$

where the functions $(\tilde{\nu}, \tilde{\sigma}, \tilde{\eta}, \tilde{\xi}) = \tilde{\zeta}$ are defined in (4.5) and $\phi^* = \frac{V_\Sigma}{\mathcal{C}_\Sigma}$.

**Proof.** We fix $(t, x) \in D$ and drop all arguments in the following to ease the notation. Recall that $\tilde{g} = \mathbf{v}^\top \tilde{\zeta}$ by definition (cf. (4.8)). Moreover, $\mathbf{c}^\top \tilde{\zeta} = 0$ by Lemma 5.6 (a). Hence,

$$
\tilde{g} = (\mathbf{v} - \phi^* \mathbf{c})^\top \tilde{\zeta}.
$$

Note that the first component of $\mathbf{v} - \phi^* \mathbf{c}$ is zero by the choice of $\phi^*$ (the vega hedge neutralises the portfolio vega). Now the assertion follows from the definitions of $\mathbf{c}$ and $\mathbf{v}$ in (4.1) and (4.2). \qed

The remainder of this subsection provides estimates for the four terms $H_{1\psi}^\psi$, $H_{2\psi}^\psi$, $H_{3\psi}^\psi$, and $H_{4\psi}^\psi$. Roughly speaking, part (a) of the first of the following propositions shows that $-\frac{1}{2} \tilde{g}(t, x) \psi$ is not only a lower bound for $H_{1\psi}^\psi(t, x; \zeta)$ over $\zeta \in \mathcal{Z}_{\text{lin}}(t, x)$ (as is shown by Lemma 5.6), but also, up to a term of order $O(\psi^2)$, for $\zeta \in \mathcal{Z}(t, x)$ that are close to $\zeta^0(\Sigma)$. Moreover, part (b) shows that this lower bound is approximately attained by controls $\zeta$ that are close to the candidate feedback control $\zeta^\psi$.

**Proposition 5.8** ($H_{1\psi}^\psi$ estimate).

(a) Let $0 \leq K \in L^4_{\text{lin}}$. There is a nonnegative $K_1 \in L^4_{\text{lin}}$ (depending on $K$) such that for every $(t, x) \in D$, $\zeta \in \mathcal{Z}(t, x)$, and $\psi \in (0, 1)$ satisfying

$$
|\zeta - \zeta^0(\Sigma)| \leq K(t, x) \psi, \quad (5.32)
$$

we have

$$
H_{1\psi}^\psi(t, x; \zeta) \geq -\frac{1}{2} \tilde{g}(t, x) \psi - K_1(t, x) \psi^2.
$$
(b) Let \( 0 \leq \bar{K} \in L^4_{\mathbb{R}} \). There is a nonnegative \( K_1 \in L^2_{\mathbb{R}} \) (depending on \( \bar{K} \)) such that for every \((t, x) \in D, \zeta \in \mathbb{Z}, \psi \in (0, 1)\) satisfying

\[
|\zeta - \zeta^\psi(t, x)| \leq \bar{K}(t, x)\psi^2, \tag{5.33}
\]

we have

\[
H_1^\psi(t, x; \zeta) \leq -\frac{1}{2} \bar{g}(t, x)1_{\{\Sigma \in [\Sigma, \bar{\Sigma}]\}}\psi + K_1(t, x)\psi^2.
\]

Proof. (a): Choose \( 0 \leq K_\lambda \in L^2_{\mathbb{R}} \) as in Lemma 5.6 (f) and set \( K_1(t, x) = \frac{1}{2}K_\lambda(t, x)K(t, x)^2 \). As \( K \in L^4_{\mathbb{R}} \) and \( K_\lambda \in L^2_{\mathbb{R}} \), it follows that \( K_1 \in L^4_{\mathbb{R}} \). Now, fix \((t, x) \in D, \zeta \in \mathbb{Z}(t, x), \psi \in (0, 1)\) satisfying (5.32). As \( \zeta = (\nu, \sigma, \eta, \xi) \in \mathbb{Z}(t, x) \), we have \( b_\psi^\zeta(t, x; \zeta) = 0 \) and \( \xi \geq 0 \). Hence, using also that \( \mu(t, x) \geq 0 \) by definition (cf. (4.4)),

\[
H_1^\psi(t, x; \zeta) \geq H_1^\psi(t, x; \zeta) + \lambda(t, x)b_\psi^\zeta(t, x; \zeta) - \mu(t, x)\xi.
\]

Substituting the expression (5.5) for \( b_\psi^\zeta \) and using the definition (5.25) of the Lagrangian \( L_1^\psi \), we obtain

\[
H_1^\psi(t, x; \zeta) \geq L_1^\psi(t, x; \zeta, \lambda(t, x), \mu(t, x))
\]

\[
+ \frac{\lambda(t, x)}{2} \left( \sigma - \Sigma \right)^\top \begin{pmatrix} S^2C_{SS} & SC_{\Sigma\Sigma} \\ SC_{\Sigma\Sigma} & C_{\Sigma\Sigma} \end{pmatrix} \left( \sigma - \Sigma \right).
\]

The first term on the right-hand side is bounded from below by \(-\frac{1}{2} \bar{g}(t, x)\psi \) by Lemma 5.6 (b). To estimate the second term, we use the Cauchy–Schwarz inequality, the compatibility of the Frobenius norm with the Euclidean norm, and the fact that \( (\sigma - \Sigma, \eta) \) is just the second and third component of \( \zeta - \zeta^\psi(\Sigma) \). As a result, we find that

\[
H_1^\psi(t, x; \zeta) \geq -\frac{1}{2} \bar{g}(t, x)\psi - \frac{1}{2} \lambda(t, x) \left\| \begin{pmatrix} S^2C_{SS} & SC_{\Sigma\Sigma} \\ SC_{\Sigma\Sigma} & C_{\Sigma\Sigma} \end{pmatrix} \right\|_F \left| \zeta - \zeta^\psi(\Sigma) \right|^2.
\]

Finally, condition (5.32) and the bound (5.26) from Lemma 5.6 (f) give

\[
H_1^\psi(t, x; \zeta) \geq -\frac{1}{2} \bar{g}(t, x)\psi - \frac{1}{2} K_\lambda(t, x) K(t, x)^2 \psi^2.
\]

This proves assertion (a) by the choice of \( K_1 \).

(b): Set \( K'(t, x) = K_\zeta + \bar{K}(t, x) \) where \( K_\zeta \) is chosen as in Lemma 5.6 (d). Clearly, \( K' \in L^4_{\mathbb{R}} \). By Assumption 4.2 (d) and the definition of \( v(t, x) \), there is a constant \( K_v > 0 \) such that \( |v(t, x)| \leq K_v \) for all \((t, x) \in D \). Next, set \( K_1(t, x) = (\psi^{-1}\min_{v} K'(t, x) + K_v) \bar{K}(t, x) \). Since \( K', \bar{K} \in L^4_{\mathbb{R}} \), it follows that \( K_1 \in L^2_{\mathbb{R}} \).

Now, fix \((t, x) \in D, \zeta \in \mathbb{Z}, \psi \in (0, 1)\) satisfying (5.33). For brevity, we write \( \zeta^\psi = \zeta^\psi(t, x) \) and \( \zeta^\psi = \zeta^\psi(\Sigma) \). Now, by the multivariate mean-value theorem, there is \( \ell \in [0, 1] \) such that

\[
H_1^\psi(t, x; \zeta) = H_1^\psi(t, x; \zeta^\psi) + D_\zeta H_1^\psi(t, x; \zeta_\ell)^\top (\zeta - \zeta^\psi).
\]

III.5 Proofs 103
III Model Uncertainty, Recalibration, and the Emergence of Delta-Vega Hedging

where \( \zeta_\ell = (1 - \ell) \zeta^0 + \ell \zeta \). By the definition of \( H^\psi_1 \), we have

\[
D_\zeta H^\psi_1(t, x; \zeta_\ell) = \frac{1}{\psi} \Psi^{-1}(\zeta_\ell - \zeta^0) - v(t, x).
\]

By Lemma 5.6 (d) and (5.33),

\[
|\zeta_\ell - \zeta^0| = |\zeta^\psi_\ell - \zeta^0 + \ell(\zeta - \zeta^\psi_\ell)| \leq |\zeta^\psi_\ell - \zeta^0| + \ell |\zeta - \zeta^\psi_\ell|
\]

so that

\[
|D_\zeta H^\psi_1(t, x; \zeta_\ell)| \leq \psi^{-1}K'(t, x) + K_v. \tag{5.35}
\]

Moreover, by Lemma 5.6 (e),

\[
H^\psi_1(t, x; \zeta^\psi_\ell) = -\frac{1}{2} \tilde{g}(t, x) 1_{(\Sigma, \Sigma)}(\zeta^\psi_\ell).
\]

Combining this with (5.35) and (5.33) in (5.34), we obtain

\[
H^\psi_1(t, x; \zeta) \leq -\frac{1}{2} \tilde{g}(t, x) 1_{(\Sigma, \Sigma)}(\zeta) + (\psi^{-1}K'(t, x) + K_v) \tilde{K}(t, x)\psi^2.
\]

This proves assertion (b) by the choice of \( K_1 \). \( \square \)

The next two propositions provide estimates for \( H^\psi_2 \) and \( H_3 \) in terms of the Euclidean distance between the reference feedback control \( \zeta^0(\Sigma) \) and alternatives \( \zeta^\psi_\ell \):

**Proposition 5.9** \( (H^\psi_2 \text{ estimate}). \) There is \( K_2 > 0 \) such that for every \((t, x, y) \in D \times \mathbb{R}, \zeta \in \mathbb{Z}, \) and \( \psi > 0, \)

\[
\left| H^\psi_2(t, x, y; \zeta) \right| 
\leq K_2 |\zeta - \zeta^0(\Sigma)| \left( |\zeta - \zeta^0(\Sigma)| + \frac{U''(y)}{U'(y)} \max \left( 1, |\zeta - \zeta^0(\Sigma)| \right) \psi \right).
\]

**Proof.** Set \( K_2 = \max(K_\varphi, 4\max(1, \Sigma)K_\varphi K_\tilde{w}) \), where \( K_\varphi \) and \( K_\tilde{w} \) are as in Assumption 4.2 (d)–(e). Also fix \((t, x, y) \in D \times \mathbb{R}, \zeta \in \mathbb{Z}, \) and \( \psi > 0. \) Now, first note that \( b^\psi(t, x; \zeta) \) is of the form (5.16) with

\[
a = \left( \nu_{\Sigma}, \beta \nu_{\Lambda} + S^2 \Gamma, \frac{\partial \Delta}{\partial \Sigma}, \nu_{\Sigma \Sigma} \right)^\top.
\]

Hence, by Lemma 5.4, the fact that \( |a| \leq 2K_\varphi \) by Assumption 4.2 (d), and the choice of \( K_2, \)

\[
|b^\psi(t, x; \zeta)| \leq \max(1, \Sigma) |a| |\zeta - \zeta^0(\Sigma)| + |a| |\zeta - \zeta^0(\Sigma)|^2
\]

\[
\leq \frac{K_2}{K_\tilde{w}} |\zeta - \zeta^0(\Sigma)| \max \left( 1, |\zeta - \zeta^0(\Sigma)| \right). \tag{5.36}
\]
Next, using first norm estimates as in the proof of Proposition 5.9 (a) and then Assumption 4.2 (d) to estimate the resulting Frobenius norm by $2K_V$, we find

$$\frac{1}{2} \left| \left( \sigma - \Sigma \right)^{\top} \left( \beta \Sigma_A + S^2 \Gamma \right) \left( \sigma - \Sigma \right) \right| \leq K_V \left| \zeta - \zeta^0(\Sigma) \right|^2. \quad (5.37)$$

Finally, using (5.36)–(5.37) and the fact that $|\tilde{w}| \leq K_{\tilde{w}}$ on $D$ by Assumption 4.2 (e), we obtain

$$\left| H_2^\psi(t, x; \zeta) \right| \leq K_V \left| \zeta - \zeta^0(\Sigma) \right|^2 + K_2 \frac{-U''(y)}{U'(y)} \left| \zeta - \zeta^0(\Sigma) \right| \max \left( 1, \left| \zeta - \zeta^0(\Sigma) \right| \right) \psi \leq K_2 \left| \zeta - \zeta^0(\Sigma) \right| \left( \left| \zeta - \zeta^0(\Sigma) \right| + \frac{-U''(y)}{U'(y)} \max \left( 1, \left| \zeta - \zeta^0(\Sigma) \right| \right) \psi \right). \quad \Box$$

**Proposition 5.10 (H₃ estimate).** There is $K_3 \in L_4^\psi$ such that for every $(t, x) \in D$ and $\zeta \in Z$,

$$\left| H_3(t, x; \zeta) - H_3(t, x; \zeta^0(\Sigma)) \right| \leq K_3(t, x) \left| \zeta - \zeta^0(\Sigma) \right| \max \left( 1, \left| \zeta - \zeta^0(\Sigma) \right| \right).$$

**Proof.** Set $K_3(t, x) = 2 \max(1, \Sigma) |a(t, x)|$ where

$$a(t, x) = \left( \tilde{w}_\Sigma, \beta \tilde{w}_\Sigma + S^2(\tilde{w}_{SS} + 2\gamma \tilde{w}_{SA} + \gamma^2 \tilde{w}_{AA}), S(\tilde{w}_{SS} + \gamma \tilde{w}_{SA}), \tilde{w}_{\Sigma \Sigma} \right)^{\top}.$$

By Assumption 4.2 (e), every component of $a$ is in $L_4^\psi$ and thus also $K_3 \in L_4^\psi$. Now, fix $(t, x) \in D$ and $\zeta \in Z$. It is easy to see that the difference

$$d := H_3(t, x; \zeta) - H_3(t, x; \zeta^0(\Sigma))$$

is of the form (5.16) for $a = a(t, x)$. Hence, by Lemma 5.4 and the choice of $K_3$,

$$|d| \leq \max(1, \Sigma) |a| \left| \zeta - \zeta^0(\Sigma) \right| + |a| \left| \zeta - \zeta^0(\Sigma) \right|^2 \leq K_3(t, x) \left| \zeta - \zeta^0(\Sigma) \right| \max \left( 1, \left| \zeta - \zeta^0(\Sigma) \right| \right). \quad \Box$$

By combining Propositions 5.9 and 5.10, the following corollary guarantees that if $\zeta$ is close to $\zeta^0(\Sigma)$, then $H_2^\psi(t, x; \zeta)$ is of order $O(\psi^2)$ and $H_3(t, x; \zeta)$ can be replaced by $H_3(t, x; \zeta^0(\Sigma))$ and a term of order $O(\psi)$.

**Corollary 5.11.** Let $0 \leq K \in L_4^\psi$. There is $K_{2,3} \in L_4^\psi$ (depending on $K$) such that for every $(t, x, y) \in D \times \mathbb{R}$, $\zeta \in Z$, and $\psi > 0$ satisfying

$$\left| \zeta - \zeta^0(\Sigma) \right| \leq K(t, x) \psi, \quad (5.38)$$

we have

$$\left| H_2^\psi(t, x, y; \zeta) \right| \leq K_{2,3}(t, x) \left( 1 + \frac{U''(y)}{U'(y)} \right) \psi^2,$$

$$\left| H_3(t, x; \zeta) - H_3(t, x; \zeta^0(\Sigma)) \right| \leq K_{2,3}(t, x) \psi.$$
Proof. Choose \( K_2 > 0 \) and \( K_3 \in L^4_\psi \) as in Propositions 5.9 and 5.10. Since \( \mathbf{Z} \) and \( [\Sigma, \overline{\Sigma}] \) are bounded, there is a constant \( K' \geq 1 \) such that for every \( \zeta \in \mathbf{Z} \) and \( \Sigma \in [\Sigma, \overline{\Sigma}] \),

\[
|\zeta - \zeta^0(\Sigma)| \leq K'.
\]

Set \( K_{2,3} = K(t, \mathbf{x}) (K_2 \max(K', K(t, \mathbf{x}))) + K_3(t, \mathbf{x})K' \). It is easy to see that \( K_{2,3} \in L^2_\psi \). Now, fix \((t, \mathbf{x}, \mathbf{y}) \in \mathbf{D} \times \mathbb{R}, \zeta \in \mathbf{Z} \), and \( \psi > 0 \) satisfying (5.38). Then by Proposition 5.9 and (5.38),

\[
|H_{2}^y(t, \mathbf{x}; \zeta)| \leq K_2 K(t, \mathbf{x}) \psi \left( K(t, \mathbf{x}) \psi + \frac{-U''(y)}{U'(y)} K' \psi \right)
\]

\[
\leq K_{2,3}(t, \mathbf{x}) \left( 1 + \frac{-U''(y)}{U'(y)} \right) \psi^2.
\]

Similarly, by Proposition 5.10 and (5.38),

\[
|H_{3}(t, \mathbf{x}; \zeta) - H_{3}(t, \mathbf{x}; \zeta^0(\Sigma))| \leq K_3(t, \mathbf{x}) K(t, \mathbf{x}) K' \psi \leq K_{2,3}(t, \mathbf{x}) \psi.
\]

Finally, Proposition 5.12 below shows that \( H_{4}^\psi \) is bounded from below by 0 up to a term of order \( O(\psi^2) \). Recall from (5.21) that this asymptotic lower bound is attained by the delta-vega hedge, i.e., \( H_{4}^\psi(t, \mathbf{x}; \mathbf{v}^*(t, \mathbf{x}), \zeta) = 0 \).

**Proposition 5.12 (\( H_{4}^\psi \) estimate).** There is a nonnegative \( K_4 \in L^2_\psi \) such that for every \((t, \mathbf{x}, \mathbf{y}) \in \mathbf{D} \times \mathbb{R}, \mathbf{v} \in \mathbb{R}^2, \zeta \in \mathbf{Z} \), and \( \psi > 0 \),

\[
H_{4}^\psi(t, \mathbf{x}; \mathbf{v}, \zeta) \geq U''(y)K_4(t, \mathbf{x}) \psi^2.
\]

**Proof.** We first argue that \( w_{YY}^\psi \leq U'' \) on \( \mathbf{D} \times \mathbb{R} \). As \( U \) has decreasing absolute risk aversion (cf. Assumption 4.2 (f)), we have for each \( y \in \mathbb{R} \),

\[
0 \geq \frac{d}{dy} \left( -\frac{U''(y)}{U'(y)} \right) = -\frac{U'(y)U'''(y) - U''(y)^2}{U'(y)^2}.
\]

In particular, since \( U' > 0 \), we have \( U''' > 0 \). Together with \( \tilde{w} \geq 0 \) (cf. Assumption 4.2 (e)), this yields \( w_{YY}^\psi = U'' - U''' \tilde{w} \psi \leq U'' < 0 \) on \( \mathbf{D} \times \mathbb{R} \).

Now, set \( K_4(t, \mathbf{x}) = \frac{1}{2} K \left( \sigma^2 S^2(\tilde{w}_S + \gamma \tilde{w}_A)^2 + \tilde{w}_\Sigma^2 \right) \) for some constant

\[
K \geq \max_{\zeta \in \mathbf{Z}} \left( 1 + 2\eta^2 + (\eta^2 + \xi^2)^{1/2} \right).
\]

As \( S(\tilde{w}_S + \gamma \tilde{w}_A), \tilde{w}_\Sigma \in L^4_\psi \) by Assumption 4.2 (e), we have \( K_4 \in L^2_\psi \). Next, fix \((t, \mathbf{x}, \mathbf{y}) \in \mathbf{D} \times \mathbb{R}, \zeta \in \mathbf{Z} \), and \( \psi > 0 \). Write

\[
Q \left( \frac{1}{\eta}, \frac{\eta}{\eta^2 + \xi} \right) \quad \text{and} \quad \tilde{w} = \left( \sigma S(\tilde{w}_S + \gamma \tilde{w}_A) \right).
\]

and consider the function \( q: \mathbb{R}^2 \to \mathbb{R} \) given by

\[
q(z) = -\frac{w_{YY}^\psi}{2} z^T Q z + \psi U''(y) \tilde{w}^T Q \tilde{w}.
\]
Clearly, minimising \( q \) over \( \mathbf{z} \in \mathbb{R}^2 \) is equivalent to minimising \( H^\psi_4(t, \mathbf{x}, y; \mathbf{v}, \zeta) \) over \( \mathbf{v} \in \mathbb{R}^2 \) (recall that \( C_\Sigma \neq 0 \) by Assumption 4.2 (c)). Moreover,

\[
\det Q = (\eta^2 + \xi) - \eta^2 = \xi \geq 0 \quad \text{and} \quad \text{Trace } Q = 1 + \eta^2 + \xi > 0,
\]

so that the symmetric matrix \( Q \) is positive semi-definite. It follows that \( q \) is convex and any solution to the first-order condition

\[
Q \left( -w^\psi_{Y,Y} \mathbf{z} + \psi U''(y) \mathbf{\bar{w}} \right) = 0
\]
is a (global) minimiser. As \( \mathbf{z}^* = \frac{U''(y)}{w^\psi_{Y,Y}} \mathbf{\bar{w}} \psi \) solves the first-order condition, we obtain after some algebra that the minimum of \( q \) is

\[
\frac{1}{2} U''(y)^2 \mathbf{\bar{w}}^\top Q \mathbf{\bar{w}} \psi^2.
\]

Using also that \( w^\psi_{Y,Y} \leq U'' < 0 \) on \( D \times \mathbb{R} \), we conclude that for all \( \mathbf{v} \in \mathbb{R}^2 \),

\[
H^\psi_4(t, \mathbf{x}, y; \mathbf{v}, \zeta) \geq \frac{1}{2} U''(y) \mathbf{\bar{w}}^\top Q \mathbf{\bar{w}} \psi^2.
\]

Finally,

\[
\frac{1}{2} |\mathbf{\bar{w}}^\top Q \mathbf{\bar{w}}| \leq \frac{1}{2} |\mathbf{\bar{w}}|^2 \|Q\|_F \leq K_4(t, \mathbf{x})
\]

by the choice of \( K_4 \). Combining the preceding two estimates completes the proof. \( \square \)

### 5.1.3 Approximate solution to the HJBI equation

The following lemma shows that the candidate value function \( w^\psi \) defined in (5.2) is, up to a term of order \( O(\psi^2) \), a "supersolution" to the HJBI equation (5.14). This analytic result is the main ingredient for the proof of the inequality (5.3) in Section 5.1.4.

**Lemma 5.13** (Lower bound). Fix constants \( \underline{Y} \leq \overline{Y} \). There is a nonnegative \( K_{lo} \in L^1_\Psi \) (depending on \( \underline{Y}, \overline{Y} \)) such that for every \( (t, \mathbf{x}, y) \in D \times [\underline{Y}, \overline{Y}] \) and \( \psi \in (0, 1) \),

\[
w^\psi_i(t, \mathbf{x}, y) + \inf_{\zeta \in \mathbb{Z}(t, \mathbf{x})} H^\psi_i(t, \mathbf{x}, y; \mathbf{v}^*(t, \mathbf{x}), \zeta) \geq -K_{lo}(t, \mathbf{x}) \psi^2. \quad (5.39)
\]

**Proof.** As an auxiliary result, we first prove that there is \( K \in L^1_\Psi \) such that for every \( (t, \mathbf{x}, y) \in D \times [\underline{Y}, \overline{Y}], \zeta \in \mathbb{Z}, \) and \( \psi \in (0, 1) \) satisfying

\[
H^\psi(t, \mathbf{x}, y; \mathbf{v}^*(t, \mathbf{x}), \zeta) \leq H^\psi(t, \mathbf{x}, y; \mathbf{v}^*(t, \mathbf{x}), \zeta^0(\Sigma)), \quad (5.40)
\]

we have

\[
|\zeta - \zeta^0(\Sigma)| \leq K(t, \mathbf{x}) \psi. \quad (5.41)
\]
Using Proposition 5.9 and the fact that $Z$ and $[\bar{Y}, \bar{Y}]$ are compact, there is a constant $K'_2 > 0$ such that for every $(t, x, y) \in D \times [\bar{Y}, \bar{Y}], \zeta \in Z$, and $\psi \in (0, 1)$,

$$\left| H^\psi_2(t, x, y; \zeta) \right| \leq K'_2 |\zeta - \zeta^0(\Sigma)|.$$  

Similarly, using Proposition 5.10, there is $K'_3 \in L^4_p$ such that for every $(t, x) \in D$ and $\zeta \in Z$,

$$\left| H^\psi_3(t, x; \zeta) - H^\psi_3(t, x; \zeta^0(\Sigma)) \right| \leq K'_3(t, x) |\zeta - \zeta^0(\Sigma)|.$$  

Now, set $K(t, x) = 2\psi_{\text{max}}(|v(t, x)| + K'_2 + K'_3(t, x))$. Using that $v(t, x)$ is uniformly bounded by Assumption 4.2 (d), it follows that $K \in L^4_p$. Fix $(t, x, y) \in D \times [\bar{Y}, \bar{Y}], \zeta \in Z$, and $\psi \in (0, 1)$ satisfying (5.40). Rearranging (5.40) and using the decomposition (5.18) of $H^\psi$, the fact that the $H^\psi_4$ term vanishes by (5.21), the above estimates for $H^\psi_2$ and $H_3$ as well as a direct estimate for the $H^\psi_1$ term, we find

$$0 \geq \left( H^\psi(t, x, y; v^*(t, x), \zeta) - H^\psi(t, x, y; v^*(t, x), \zeta^0(\Sigma)) \right) / U'(y)$$

$$= H^\psi_1(t, x; \zeta) + H^\psi_2(t, x, y; \zeta) - \left( H^\psi_3(t, x, y; \zeta) - H^\psi_3(t, x, y; \zeta^0(\Sigma)) \right) \psi$$

$$\geq \frac{1}{2\psi\psi_{\text{max}}} |\zeta - \zeta^0(\Sigma)|^2 - v(t, x)^\top (\zeta - \zeta^0(\Sigma)) - (K'_2 + K'_3(t, x)) |\zeta - \zeta^0(\Sigma)| \psi.$$  

By rearranging terms and applying the Cauchy–Schwarz inequality, we obtain

$$|\zeta - \zeta^0(\Sigma)|^2 \leq 2\psi_{\text{max}} \left( |v(t, x)| + K'_2 + K'_3(t, x) \right) |\zeta - \zeta^0(\Sigma)| \psi$$

$$\leq K(t, x) |\zeta - \zeta^0(\Sigma)| \psi$$

and (5.41) follows.

We now turn to the proof of (5.39). Choose $K_{2,3} \in L^2_p$ as in Corollary 5.11 (with $K$ as in the auxiliary result), $K_1 \in L^4_p$ as in Proposition 5.8 (a), and set

$$K_{lo}(t, x) = U'(\bar{Y}) \left( K_1(t, x) + K_{2,3}(t, x) \left( 2 + \frac{-U''(\bar{Y})}{U'(\bar{Y})} \right) \right).$$

Clearly, $K_{lo} \in L^4_p$. Fix $(t, x, y) \in D \times [\bar{Y}, \bar{Y}], \zeta \in Z(t, x)$, and $\psi \in (0, 1)$. First, we note that $\zeta^0(\Sigma) \in Z(t, x)$ and that by (5.20)–(5.21) and Lemma 5.6 (c),

$$w^\psi_1(t, x, y) + H^\psi(t, x, y; v^*(t, x), \zeta^0(\Sigma))$$

$$= -U'(y) (\tilde{w}_1(t, x) + H_3(t, x, y; \zeta^0(\Sigma))) \psi$$

$$= \frac{1}{2} U'(y) \tilde{g}(t, x) \psi \geq 0.$$  

In view of assertion (5.39), we may thus assume that (5.40) is satisfied. In turn, (5.41) holds by the auxiliary result. In particular, we may use the estimates of
Proposition 5.8 (a) (for $H_1^\psi$) and Corollary 5.11 (for $H_3^\psi$ and $H_3$) in the following. These together with the fact that the $H_1^\psi$ term vanishes by (5.21) yield
\[
w_1^\psi(t, x, y) + H^\psi(t, x, y; \psi^*(t, x), \xi) = U'(y) \left( -\bar{w}_1(t, x)\psi + H_1^\psi(t, x; \xi) + H_2^\psi(t, x, y; \xi) - H_3(t, x; \xi)\psi \right) \\
\geq -U'(y) \left( \bar{w}_1(t, x) + H_3(t, x; \psi^0(\Sigma)) + \frac{1}{2}\tilde{g}(t, x) \right) \psi \\
- U'(y) \left( K_1(t, x) + K_{2,3}(t, x) \left( 1 + \frac{-U''(y)}{U'(y)} \right) + K_{2,3}(t, x) \right) \psi^2 \\
\geq -K_{10}(t, x)\psi^2,
\]
where in the last inequality, we also use (5.20) to eliminate the $O(\psi)$ term and the fact that $U$ has decreasing absolute risk aversion (cf. Assumption 4.2 (f)) to estimate the $O(\psi^2)$ term. As $\xi \in Z(t, x)$ was arbitrary, (5.39) follows.

Conversely, the next lemma shows that the candidate value function $w^\psi$ defined in (5.2) is asymptotically a “subsolution” to the HJBI equation (5.14). Here, the asymptotic estimate is of order $O(\psi^2)$ if $\Sigma$ is in the interior of $[\Sigma, \Sigma]$ and of order $O(\psi)$ otherwise. This analytic result is the main ingredient for the proof of the inequality (5.4) in Section 5.1.5.

**Lemma 5.14** (Upper bound). Let $0 \leq \bar{K} \in L^p_\infty$. There is a nonnegative $K_{up} \in L^2_\infty$ (depending on $\bar{K}$) such that for every $(t, x, y) \in D \times \mathbb{R}$, $\xi \in Z$, and $\psi \in (0, 1)$ satisfying
\[
|\xi - \xi^\psi(t, x)| \leq \bar{K}(t, x)\psi^2,
\]
we have
\[
w_1^\psi(t, x, y) + \sup_{\psi \in \mathbb{R}^2} H^\psi(t, x, y; \psi, \xi) \leq K_{up}(t, x)U'(y) \left( 1 + \frac{-U''(y)}{U'(y)} \right) \psi^{1+1(\xi \in \Sigma, \Sigma)}.
\]

**Proof.** Define $K_\xi \geq 1$ as in Lemma 5.6 (d), and set $K(t, x) = \bar{K}(t, x) + K_\xi$. Clearly, $K \in L^2_\infty$. With this choice of $K$, let $K_{2,3} \in L^2_\infty$ be defined as in Corollary 5.11. Moreover, define $K_1 \in L^2_\infty$ as in Proposition 5.8 (b) and $K_4 \in L^2_\infty$ as in Proposition 5.12. In addition, note that there is $K_{\bar{g}} > 0$ such that $0 \leq \bar{g} \leq K_{\bar{g}}$ on $D$ by Lemma 5.6 (c).

Now, set $K_{up}(t, x) = 4 \max \left( K_1(t, x) + 2K_{2,3}(t, x) + K_4(t, x), \frac{1}{2}\bar{K}_{\bar{g}} \right)$. Clearly, $K_{up} \in L^2_\infty$. Fix $(t, x, y) \in D \times \mathbb{R}$, $\psi \in \mathbb{R}^2$, $\xi \in Z$, and $\psi \in (0, 1)$ satisfying (5.42). In particular, condition (5.33) of Proposition 5.8 (b) holds. By Lemma 5.6 (d) and (5.42),
\[
|\xi - \xi^0(\Sigma)| \leq |\xi - \xi^\psi(t, x)| + |\xi^\psi(t, x) - \xi^0(\Sigma)| \\
\leq \bar{K}(t, x)\psi^2 + K_\xi \psi \leq \bar{K}(t, x)\psi,
\]
so that condition (5.38) of Corollary 5.11 is satisfied as well.
Using Propositions 5.8 (b) (for $H_1^\psi$) and 5.12 (for $H_4^\psi$) as well as Corollary 5.11 (for $H_2^\psi$ and $H_3$) to estimate the four summands in the decomposition (5.18) of $H^\psi$, and also (5.20) in the penultimate step, we obtain

\[
\begin{align*}
 w^\psi(t, x, y) + H^\psi(t, x, y; \psi, \zeta) & \leq -U'(y)\tilde{w}_1(t, x)\psi + U'(y)\left(-\frac{1}{2}\tilde{g}(t, x)1_{\{\Sigma \in \Sigma, \Sigma\}}\psi + K_1(t, x)\psi^2\right) \\
 & + U'(y)K_{2,3}(t, x)\left(1 + \frac{-U''(y)}{U'(y)}\right)\psi^2 \\
 & - U'(y)\left(\tilde{w}_1(t, x) - H_3(t, x; \zeta^0(\Sigma)) - K_{2,3}(t, x)\psi\right) - U''(y)K_4(t, x)\psi^2 \\
 & = -U'(y)\left(\tilde{w}_1(t, x) + H_3(t, x; \zeta^0(\Sigma)) + \frac{1}{2}\tilde{g}(t, x)1_{\{\Sigma \in \Sigma, \Sigma\}}\psi\right) \\
 & + U'(y)\left(K_1(t, x) + 2K_{2,3}(t, x) + (K_{2,3}(t, x) + K_4(t, x))\frac{-U''(y)}{U'(y)}\right)\psi^2 \\
 & \leq U'(y)\frac{1}{2}\tilde{g}(t, x)1_{\{\Sigma \in \Sigma, \Sigma\}}\psi + \frac{1}{4}K_{up}(t, x)U'(y)\left(1 + \frac{-U''(y)}{U'(y)}\right)\psi^2 \\
 & \leq \frac{1}{2}K_{up}(t, x)U'(y)\left(1 + \frac{-U''(y)}{U'(y)}\right)\left(1_{\{\Sigma \in \Sigma, \Sigma\}}\psi + \psi^2\right).
\end{align*}
\]

As $\psi \in \mathbb{R}^2$ was arbitrary, the assertion follows easily by distinguishing the cases $\Sigma \in (\underline{\Sigma}, \overline{\Sigma})$ and $\Sigma \in \{\underline{\Sigma}, \overline{\Sigma}\}$ (using that $\psi \in (0, 1)$ in the second case). \hfill \Box

### 5.1.4 The asymptotic lower bound for the stochastic differential game

We are now in a position to establish an asymptotic lower bound for the SDG (2.16), as required for the proof of Theorem 4.4 at the beginning of Section 5.1.

**Lemma 5.15.** As $\psi \downarrow 0$,

\[
\inf_{P \in \mathcal{P}} J^\psi(\psi^*, P) \geq w^\psi_0 + o(\psi).
\]

**Proof.** Choose $Y, \overline{Y}$ as in Corollary 5.3 and, with this choice, let $K_0 \in L^1_\mathbb{P}$ be as in Lemma 5.13. Now, fix $\varepsilon > 0$, $\psi_0^* \in (0, \psi_0)$ such that $\|K_0\|_L^1_\mathbb{P} \psi_0^* \leq \frac{1}{2}\varepsilon$, and let $\psi \in (0, \psi_0^*)$. We need to show that

\[
\inf_{P \in \mathcal{P}} J^\psi(\psi^*, P') - w^\psi_0 \geq -\varepsilon\psi. \tag{5.44}
\]

Choose $P \in \mathcal{P}$ such that $J^\psi(\psi^*, P) - \frac{1}{2}\varepsilon\psi \leq \inf_{P \in \mathcal{P}} J^\psi(\psi^*, P')$. Then

\[
\inf_{P \in \mathcal{P}} J^\psi(\psi^*, P') - w^\psi_0 \geq J^\psi(\psi^*, P) - w^\psi_0 - \frac{1}{2}\varepsilon\psi. \tag{5.45}
\]

Applying Itô’s formula (under $P$) to the process $w^\psi(u, X_u, Y_u\psi^*, P)$ (recall the dynamics of $S, A, M, \Sigma$, and $Y^{\psi^*, P}$ given in (2.6)–(2.8) and Corollary 5.3) and
using the third line in (4.7) (so that the $dM$-integral vanishes) yields for each $u \in [0, T]$,

$$I_u^\psi(v^*, P) := w^\psi(u, X_u, Y_u^{v^*, P}) + \frac{1}{\psi} \int_0^u U'(Y_t^{v^*, P}) f(\Sigma_t, \zeta_t^P) \, dt - w_0^\psi$$

$$= N_u + \int_0^u \left( w^\psi(t, X_t, Y_t^{v^*, P}) + H^\psi(t, X_t, Y_t^{v^*, P}; v^*(t, X_t), \zeta_t^P) \right) \, dt,$$

where

$$N := \int_0^u \left( w^\psi_S(t, X_t, Y_t^{v^*, P}) + \gamma(t, S_t, A_t, M_t) w^\psi_A(t, X_t, Y_t^{v^*, P}) \right) \, dS_t$$

$$+ \int_0^u w^\psi(t, X_t, Y_t^{v^*, P}) \, d\Sigma_t.$$

Note that $\zeta_t^P \in \mathbb{Z}(t, X_t)$ $d\Sigma_t$-a.e. by (2.7). Hence, by Lemma 5.13, for each $u \in [0, T]$,

$$I_u^\psi(v^*, P) \geq N_u - \int_0^u K_{lo}(t, X_t) \, dt \psi^2. \quad (5.47)$$

By construction, $N$ is a local $P$-martingale starting in 0. Suppose for the moment that $N$ is also a submartingale. Then by taking expectations under $P$ on both sides of (5.47) (for $u = T$), we obtain

$$J^\psi(v^*, P) - w_0^\psi \geq -\|K_{lo}\|_{L_p} \psi^2 \geq -\frac{1}{2} \varepsilon \psi.$$

Combining this with (5.45) yields (5.44).

It remains to show that $N$ is a submartingale under $P$. As it is a local martingale, it suffices to show that it is bounded from above by a $P$-integrable random variable. To this end, first note from the definition of $w^\psi$ in (5.2), the fact that $\tilde{w} \geq 0$ on $\mathcal{D}$ by Assumption 4.2 (e), and Assumption 4.2 (f) that $w^\psi \leq U(\overline{Y})$ on $\mathcal{D} \times [\underline{Y}, \overline{Y}]$. Clearly, $U'(\Sigma, \zeta)$ is also uniformly bounded over $y \geq \underline{Y}$, $\Sigma \in [\underline{\Sigma}, \overline{\Sigma}]$, and $\zeta \in \mathbb{Z}$. In view of the definition of $I^\psi(v^*, P)$ in (5.46), the fact that $Y^{v^*, P} \in [\underline{Y}, \overline{Y}]$ $dt \times P$-a.e. by Corollary 5.3, and Assumption 4.2 (b), we conclude that $I^\psi(v^*, P) \leq K_I \, dt \times P$-a.e. for some constant $K_I > 0$. Using this and (5.47), we obtain for each $u \in [0, T]$,

$$N_u \leq K_I + \int_0^T K_{lo}(t, X_t) \, dt.$$

As $K_{lo} \in L_1^p$, $N$ is bounded from above by a $P$-integrable random variable and therefore is a submartingale. This completes the proof.

5.1.5 The asymptotic upper bound for the stochastic differential game

To establish an asymptotic upper bound for the stochastic differential game (2.16), we first prove that the probability under $P^\psi$ that $\Sigma$ leaves $(\underline{\Sigma}, \overline{\Sigma})$ before time $T$ is of order $O(\psi)$. 

\[ \]
**Proposition 5.16.** Let $\tau := \inf\{t \in [0, T] : \Sigma_t \not\in (\Sigma, \overline{\Sigma})\} \land T$ be the first time that $\Sigma$ leaves $(\Sigma, \overline{\Sigma})$. Then $\tau$ is a stopping time and there is $K_\tau > 0$ such that for every $\psi \in (0, \psi_0)$,

$$P^\psi[\tau < T] \leq K_\tau \psi.$$  

(5.48)

**Proof.** It is an easy exercise to show that $\tau$ is a stopping time for the (non-augmented, non-right-continuous) filtration $\mathcal{F}$. This uses the fact that all paths of $\Sigma$ are continuous and $(\Sigma, \overline{\Sigma})$ is open; cf. [103, Problem 2.7].

Turning to the proof of (5.48), by standard estimates for Itô processes (cf., e.g., [142, Lemma V.11.5]), there is a constant $K > 0$ (depending only on $T$) such that for every $\psi \in (0, \psi_0)$,

$$E^{P^\psi}\left[\sup_{0 \leq t \leq T} |\Sigma_t - \Sigma_0|^2\right] \leq K E^{P^\psi}\left[\int_0^T \left((\nu^P_t)^2 + (\eta^P_t)^2 + \xi^P_t\right) dt\right].$$  

(5.49)

Define $K_\xi \geq 1$ as in Lemma 5.6 (d), and let $K'_0(t, x) = K_0(t, x) + K_\xi \geq 1$ as well as $K_\tau = 2\ell^2 K_0^2 \|K'_0\|_{L^2_{\varphi}}$ with $\ell := \min(\Sigma - \Sigma_0, \Sigma_0 - \Sigma) > 0$. Clearly, $K'_0 \in L^2_{\varphi} \subset L^2_\varphi$, so that $0 \leq K_\tau < \infty$. Fix $\psi \in (0, \psi_0)$. By (5.1) and Lemma 5.6 (d),

$$\left|\zeta^P_t - \zeta^0(\Sigma_t)\right| \leq \left|\zeta^P_t - \zeta^P_t(t, X_t)\right| + \left|\zeta^P(t, X_t) - \zeta^0(\Sigma_t)\right| \leq K'_0(t, X_t) \psi^2 + K_\xi \psi \quad \text{for } \psi \in (0, \psi_0) \text{-a.e.}$$

Recalling that $\zeta^0(\Sigma) = (0, \Sigma, 0, 0)^T$, this estimate yields

$$(\nu^P_t)^2 + (\eta^P_t)^2 + \xi^P_t \leq \left|\zeta^P_t - \zeta^0(\Sigma_t)\right| \leq K'_0(t, X_t)^2 \psi^2 + K'_0(t, X_t) \psi \quad \text{for } \psi \in (0, \psi_0) \text{-a.e.}$$

(5.50)

Moreover, by the definition of $\ell$ and Markov’s inequality,

$$P^\psi[\tau < T] \leq P^\psi\left[\sup_{0 \leq t \leq T} |\Sigma_t - \Sigma_0|^2 \geq \ell^2\right] \leq \ell^2 E^{P^\psi}\left[\sup_{0 \leq t \leq T} |\Sigma_t - \Sigma_0|^2\right].$$  

(5.51)

Combining (5.49)–(5.51) proves (5.48). \[Q.E.D.\]

We are now in a position to establish an asymptotic upper bound for the stochastic differential game (2.16), which completes the proof of Theorem 4.4 at the beginning of Section 5.1.

**Lemma 5.17.** As $\psi \downarrow 0$,

$$\sup_{\psi \in \mathcal{G}} J^\psi(\nu, P^\psi) \leq w^\psi_0 + o(\psi).$$
Proof. Set \( \bar{K} = K_0 \in L^1 \). With this choice of \( \bar{K} \), define \( K_{\bar{K}} \in L^2 \) as in Lemma 5.14. As \( U' \) is decreasing and \( U \) has decreasing absolute risk aversion (cf. Assumption 4.2 (f)), there is \( K_U > 0 \) such that (cf. Assumption 4.2 (a) for the choice of \( K_0 \))

\[
U'(y) \left( 1 - U''(y)/U'(y) \right) \leq K_U, \quad \text{for all } y \geq -K_0. \tag{5.52}
\]

Now, fix \( \varepsilon > 0 \), choose \( \psi_0' \in (0, \psi_0) \) such that

\[
K_U \|K_{\bar{K}}\|_{L^p} \psi_0' + K_U \sqrt{T} \|K_{\bar{K}}\|_{L^2} K_0^{1/2} (\psi_0')^{1/2} \leq \frac{1}{2} \varepsilon
\]

where \( K_0 > 0 \) is as in Proposition 5.16, and let \( \psi \in (0, \psi_0') \). We need to show that

\[
\sup_{\psi' \in \mathcal{Y}} J^\psi(\psi', P^\psi) - w_0^\psi \leq \varepsilon \psi. \tag{5.53}
\]

Choose \( \psi \in \mathcal{Z} \) such that \( J^\psi(\psi, P^\psi) + \frac{1}{2} \varepsilon \psi \geq \sup_{\psi' \in \mathcal{Y}} J^\psi(\psi', P^\psi) \). Then

\[
\sup_{\psi' \in \mathcal{Y}} J^\psi(\psi', P^\psi) - w_0^\psi \leq J^\psi(\psi, P^\psi) - w_0^\psi + \frac{1}{2} \varepsilon \psi. \tag{5.54}
\]

Applying Itô’s formula (under \( P^\psi \)) to the process \( w^\psi(u, X_u, Y_{u,P}^\psi) \) (recall the dynamics of \( S,A,M,\Sigma \) and \( Y_{u,P}^\psi \) given in (2.6)–(2.8) and Lemma 5.2) and using the third line in (4.7) (so that the \( dM \)-integral vanishes) yields for each \( u \in [0, T] \),

\[
I^\psi_u(\psi, P^\psi) := w^\psi(u, X_u, Y_{u,P}^\psi) + \frac{1}{\psi} \int_0^u U'(Y_{t,P}^\psi) f(\Sigma_t, \zeta_t^{P^\psi}) \, dt - w_0^\psi
\]

\[= N_u + \int_0^u \left( w^\psi(t, X_t, Y_{t,P}^\psi) + H^\psi(t, X_t, Y_{t,P}^\psi; \psi, \zeta_t^{P^\psi}) \right) \, dt. \tag{5.55}
\]

Here,

\[
N := \int_0^t \left( w^\psi_S(t, X_t, Y_{t,P}^\psi) + \gamma(t, S_t, A_t, M_t) w^\psi_A(t, X_t, Y_{t,P}^\psi) \right) \, dS_t
\]

\[+ \int_0^t w^\psi_\Sigma(t, X_t, Y_{t,P}^\psi) \, d\Sigma_t^{P^\psi} + \int_0^t w^\psi_Y(t, X_t, Y_{t,P}^\psi) \, dY_t^{c.v,P^\psi}
\]

and \( Y_{c,v,P}^\psi = Y_{v,P}^\psi + \int_0^t b^\psi(t, X_t; \zeta_t^{P^\psi}) \, dt \) is the local martingale part of \( Y_{v,P}^\psi \) under \( P^\psi \).

We want to use Lemma 5.14 to estimate the drift term in the last line of (5.55). Note that condition (5.42) of Lemma 5.14 with \( \zeta \) and \( x \) replaced by \( \zeta_t^{P^\psi} \) and \( X_t \), respectively, is fulfilled \( dt \times P^\psi \)-a.e. by (5.1) and our choice of \( K = K_0 \). Moreover, denoting by \( \tau \) the first time that \( \Sigma \) leaves \( (\Sigma, \bar{\Sigma}) \) (cf. Proposition 5.16),
we have for each $t \in [0, T]$ that $\Sigma_t \in (\Sigma, \Sigma) \ P^\psi$-a.s. on $\{ \tau \geq T \}$. Therefore, by (5.43), (5.52), and Assumption 4.2 (a), for each $u \in [0, T],$

$$I^\psi_u(\mathbf{v}, P^\psi) \leq N_u + \int_0^u K_{\mathrm{up}}(t, \mathbf{X}_t) U'(Y^\psi_{t, P^\psi}) \left( 1 + \frac{-U''(Y^\psi_{t, P^\psi})}{U'(Y^\psi_{t, P^\psi})} \right) dt \left( \psi^2 1_{\{ \tau \geq T \}} + \psi 1_{\{ \tau < T \}} \right) \leq N_u + K_U \int_0^u K_{\mathrm{up}}(t, \mathbf{X}_t) dt \left( \psi^2 1_{\{ \tau \geq T \}} + \psi 1_{\{ \tau < T \}} \right).$$

(5.56)

By construction, $N$ is a local $P^\psi$-martingale starting in 0. Suppose for the moment that $N$ is also a supermartingale. Then by taking expectations under $P^\psi$ on both sides of (5.56) (for $u = T$) and using the Cauchy–Schwarz and Jensen inequalities as well as Proposition 5.16, we obtain

$$J^\psi(\mathbf{v}, P^\psi) - w^\psi_0 \leq K_U \| K_{\mathrm{up}} \|_{L^p} \psi^2 + K_U E^{P^\psi} \left[ \int_0^T K_{\mathrm{up}}(t, \mathbf{X}_t) dt 1_{\{ \tau < T \}} \right] \psi \leq K_U \| K_{\mathrm{up}} \|_{L^p} \psi^2 + K_U E^{P^\psi} \left[ \left( \int_0^T K_{\mathrm{up}}(t, \mathbf{X}_t) dt \right)^2 \right]^{1/2} P^\psi[\tau < T]^{1/2} \psi \leq \left( K_U \| K_{\mathrm{up}} \|_{L^p} \psi + K_U \sqrt{T} \| K_{\mathrm{up}} \|_{L^p} K_{\mathrm{up}}^{1/2} \psi^{1/2} \right) \psi \leq \frac{1}{2} \varepsilon \psi.$$

Combining this with (5.54) yields (5.53).

It remains to show that $N$ is a supermartingale under $P^\psi$. As it is a local martingale, it suffices to show that it is bounded from below by a $P^\psi$-integrable random variable. To this end, first note from the definition of $w^\psi$ in (5.2) and Assumption 4.2 (e) and (f) that for every $(t, \mathbf{x}) \in D$ and $y \geq -K_{\psi}$,

$$w^\psi(t, \mathbf{x}, y) = U(y) - U'(y) \overline{w}(t, \mathbf{x}) \psi \geq U(-K_{\psi}) - U'(-K_{\psi})K_{\overline{w}}.$$

By Assumption 4.2 (a) and the fact that $f \geq 0$, we obtain for each $u \in [0, T],$

$$I^\psi_u(\mathbf{v}, P^\psi) \geq U(-K_{\psi}) - U'(-K_{\psi})K_{\overline{w}} - w^\psi_0 =: K_I.$$

Using this and (5.56) yields for each $u \in [0, T],$

$$N_u \geq K_I - K_U \int_0^T K_{\mathrm{up}}(t, \mathbf{X}_t) dt.$$

As $K_{\mathrm{up}} \in L^2_{\psi}$, $N$ is bounded from below by a $P^\psi$-integrable random variable and therefore is a supermartingale. This completes the proof. \qed
5.2 Construction of a modified feedback control

Proof of Lemma 4.8. For each $\psi > 0$, define the functions $\tilde{\nu}, \tilde{\zeta}^{\psi*}, \tilde{\zeta}^{\psi} : D^0 \to \mathbb{R}$ by

$$\tilde{\nu}(t, x) = \frac{-1}{2C_\Sigma} \left[ \begin{array}{c} \tilde{\sigma} \\ \tilde{\eta} \end{array} \right] \left( \begin{array}{cc} S^2C_{SS} & SC_{S\Sigma} \\ SC_{S\Sigma} & C_{\Sigma\Sigma} \end{array} \right) \left( \begin{array}{c} \tilde{\sigma} \\ \tilde{\eta} \end{array} \right),$$

$$(5.57)$$

$$\tilde{\zeta}^{\psi*}(t, x) = \zeta^0(\Sigma) + \tilde{\zeta} \psi + \tilde{\nu} \tilde{e}_1 \psi^2,$n

$$\tilde{\zeta}^{\psi}(t, x) = \zeta^0(\Sigma) + (\tilde{\zeta} \psi + \tilde{\nu} \tilde{e}_1 \psi^2)1_{[\Sigma < \Sigma \in \mathbb{S}]} = \zeta^\psi + \tilde{\nu} \tilde{e}_1 \psi^21_{[\Sigma < \Sigma \in \mathbb{S}]},$$

$$(5.58)$$

where the functions $\tilde{\sigma}$ and $\tilde{\eta}$ are the second and third component of $\tilde{\zeta}$ defined in (4.5). That is, $\tilde{\zeta}^{\psi}$ arises from $\zeta^{\psi}$ (cf. (4.6)) by a perturbation of the first component (the drift of the implied volatility) by a term of order $O(\psi^2)$.

First, we show the asserted continuity of $\tilde{\zeta}^{\psi}$ and the extension property. It is easy to see from Assumption 4.2 (c) and (d) that the vega-gamma-vanna-volga vectors $c$ and $v$ are continuous on $D^0$. Then also $\lambda$ (it is not hard to show that the two expressions on the right-hand side of (4.3) coincide whenever $V_{\Sigma \Sigma} - \frac{e^T \psi}{e^T c} C_{\Sigma \Sigma} = 0$, $\mu$, and hence $\tilde{\zeta}$ are continuous on $D^0$. Therefore, also $\tilde{\nu}$ and $\tilde{\zeta}^{\psi*}$ are continuous on $D^0$, and it follows that $\tilde{\zeta}^{\psi}$ is continuous on $(0, T] \times G \times \{\Sigma, \bar{\Sigma}\}$. By construction, $\tilde{\zeta}^{\psi*}$ is a continuous extension of $\tilde{\zeta}^{\psi}|_{(0, T) \times G \times \{\Sigma, \bar{\Sigma}\}}$ to $D^0$.

Second, we show that the range of $\tilde{\zeta}^{\psi}$ is contained in $Z = [\mu, \bar{\mu}] \times [\sigma, \bar{\sigma}] \times (|\eta, \bar{\eta}| \times [0, \tilde{\xi}])$ for sufficiently small $\psi \in (0, \psi_0)$. To this end, it suffices to show that $\tilde{\zeta}$ and $\tilde{\nu}$ are bounded on $D = (0, T] \times G \times \{\Sigma, \bar{\Sigma}\}$. First, by (5.28) from the proof of Lemma 5.6 (a), we have

$$\rho(t, x) \leq \psi_{\text{max}} |v(t, x)|, \quad (t, x) \in D.$$

By Assumption 4.2 (d), $|v|$ is bounded on $D$, so that $\tilde{\zeta}$ is bounded on $D$ as well. Second, using the boundedness of $\tilde{\zeta}$ as well as (4.16), it follows from (5.57) that $\tilde{\nu}$ is bounded on $D$ as well. We conclude that there is $\psi_0 \in (0, 1)$ such that $\tilde{\zeta}^{\psi}(t, x) \in Z$ for each $(t, x) \in D$ and $\psi \in (0, \psi_0)$.

Third, we prove part (c). By (5.58), for each $(t, x) \in D$ and $\psi > 0$, we have

$$|\tilde{\zeta}^{\psi}(t, x) - \tilde{\zeta}^{\psi}(t, x)| \leq |\tilde{\nu}(t, x)|\psi^2.$$

As we have argued above that $\tilde{\nu}$ is bounded on $D$, there is $K_0 > 0$ such that (4.17) holds.

Fourth, we show part (b), i.e., that $b^C(t, x; \tilde{\zeta}^{\psi}(t, x)) = 0$. Fix $(t, x) \in D$. If $\Sigma \in \{\Sigma, \bar{\Sigma}\}$, then $\tilde{\zeta}^{\psi}(t, x) = \zeta^0(\Sigma)$ by construction and the assertion is trivial. So suppose that $\Sigma \in (\Sigma, \bar{\Sigma})$. By the representation (5.5) of $b^C$, noting that the second and third components of $\tilde{\zeta}^{\psi}$ coincide with those of $\zeta^{\psi}$, we have

$$b^C(\tilde{\zeta}^{\psi}) = c^T \left( \begin{array}{c} \tilde{\zeta}^{\psi} - \zeta^0(\Sigma) \end{array} \right) + \frac{1}{2} \left( \begin{array}{c} \sigma^{\psi} - \Sigma \\ \eta^{\psi} \end{array} \right) \left( \begin{array}{cc} S^2C_{SS} & SC_{S\Sigma} \\ SC_{S\Sigma} & C_{\Sigma\Sigma} \end{array} \right) \left( \begin{array}{c} \sigma^{\psi} - \Sigma \\ \eta^{\psi} \end{array} \right).$$
As $c^\top (\zeta^\psi - \zeta^0(\Sigma)) = c^\top \tilde{\zeta}^\psi = 0$ by Lemma 5.6 (a), we have $c^\top \left( \tilde{\zeta}^\psi - \zeta^\psi \right)$. Using this together with the fact that $\sigma^\psi = \tilde{\sigma}^\psi$ and $\eta^\psi = \tilde{\eta}^\psi$ (because $\Sigma \in (\Sigma, \Sigma)$) as well as the definition of $\tilde{\nu}$ yields

$$b^c(\tilde{\zeta}^\psi) = c^\top \left( \tilde{\zeta}^\psi - \zeta^\psi \right) - C^\Sigma \tilde{\nu}^2.$$ 

Now by (5.58) and the definition of $c$ (cf. (4.1)), $c^\top (\tilde{\zeta}^\psi - \zeta^\psi) = C^\Sigma \tilde{\nu}^2$, so that $b^c(\tilde{\zeta}^\psi) = 0$.

Finally, part (a) follows immediately from the construction of $\tilde{\zeta}^\psi$.

### A Linearly constrained quadratic programming

**Lagrangian duality.** We recall some basic Lagrange duality results from [14, Section 5.1.5]. Fix $n \in \mathbb{N}$ and functions $f, g, h : \mathbb{R}^n \to \mathbb{R}$. We refer to the problem

$$\begin{align*}
\text{minimise } & f(z) \\
\text{subject to } & z \in \mathbb{R}^n, h(z) = 0, g(z) \leq 0, \\
\end{align*}$$

(A.1)

as the **primal problem** and denote by

$$f^* = \inf \{ f(z) : z \in \mathbb{R}^n, h(z) = 0, g(z) \leq 0 \}$$

its optimal value. The corresponding **Lagrangian** is

$$L(z, \lambda, \mu) = f(z) + \lambda h(z) + \mu g(z), \quad z \in \mathbb{R}^n, \lambda, \mu \in \mathbb{R},$$

and a pair $(\mu^*, \lambda^*)$ is called a **Lagrange multiplier** if

$$f^* = \inf_{z \in \mathbb{R}^n} L(z, \lambda^*, \mu^*) \quad \text{and} \quad \mu^* \geq 0.$$ 

The **dual problem** for (A.1) is

$$\begin{align*}
\text{maximise } & q(\lambda, \mu) \\
\text{subject to } & \lambda \in \mathbb{R}, \mu \geq 0, \\
\end{align*}$$

where the **dual function** $q$ is

$$q(\lambda, \mu) = \inf_{z \in \mathbb{R}^n} L(z, \lambda, \mu), \quad \lambda, \mu \in \mathbb{R}.$$ 

Finally, $q^* = \sup \{ q(\lambda, \mu) : \lambda \in \mathbb{R}, \mu \geq 0 \}$ denotes the optimal value of the dual problem.
A quadratic programming problem with linear equality and inequality constraints. The following lemma provides the solution to a primal problem with a strictly convex quadratic cost function and specific linear equality and inequality constraints.

Lemma A.1. Fix \( n \in \mathbb{N} \), a diagonal matrix \( D = \text{diag}(d_1, \ldots, d_n) \in \mathbb{R}^{n \times n} \) with positive diagonal entries, and vectors \( v = (v_1, \ldots, v_n)\top \) and \( c = (c_1, \ldots, c_n)\top \) in \( \mathbb{R}^n \) such that \( c_i \neq 0 \) for some \( i \in \{1, \ldots, n-1\} \). Moreover, set \( 1_A = 1 \) if \( v_n - \frac{c\top D^{-1}c}{c\top D^{-1}1}c_n < 0 \) and \( 1_A = 0 \) otherwise, and define

\[
\lambda^* = \frac{c\top D^{-1}v - c_n v_n d_n^{-1}1_A}{c\top D^{-1}c - c_n^2 d_n^{-1}1_A},
\]
(\[A.2\])

\[
\mu^* = (v_n - \lambda^* c_n)^-, \tag{A.3}
\]

\[
z^* = D^{-1}(v - \lambda^* c + \mu^* \bar{c}_n). \tag{A.4}
\]

(a) \( z^* \) is the unique optimiser of the primal problem

\[
\text{minimise } \frac{1}{2} z\top Dz - v\top z \tag{A.5}
\]

subject to \( z \in \mathbb{R}^n, c\top z = 0, z_n \geq 0 \),

and satisfies the bound \( |z^*| \leq d_n^{-1} |v| \), where \( d_n = \min(d_1, \ldots, d_n) \).

(b) \((\lambda^*, \mu^*)\) is the unique optimiser of the dual problem for (A.5), which can be written as

\[
\text{maximise } -\frac{1}{2} (v - \lambda c + \mu \bar{c}_n)\top D^{-1}(v - \lambda c + \mu \bar{c}_n) \tag{A.6}
\]

subject to \( \lambda \in \mathbb{R}, \mu \geq 0 \).

The optimiser satisfies the bound

\[
|\lambda^* c - \mu^* \bar{c}_n| \leq \left(1 + \frac{d_{\max}}{d_{\min}}\right) |v|, \tag{A.7}
\]

where \( d_{\max} = \max(d_1, \ldots, d_n) \).

(c) The optimal values of the primal and dual problems coincide (i.e., there is no duality gap) and equal

\[-\frac{1}{2} v\top z^*. \]

(d) The triplet \((z^*, \mu^*, \lambda^*)\) satisfies the optimality conditions

\[
z^* = \arg \min_{z \in \mathbb{R}^n} L(z, \lambda^*, \mu^*), \quad c\top z^* = 0, \quad z^*_n \geq 0, \quad \mu^* \geq 0, \quad \mu^* z^*_n = 0, \tag{A.8}
\]

where \( L \) is the Lagrangian corresponding to the primal problem. Moreover, \((\lambda^*, \mu^*)\) is a Lagrange multiplier for the primal problem.
Proof. First of all, note that the Lagrangian
\[
L(z, \lambda, \mu) = \frac{1}{2} z^\top Dz - v^\top z + \lambda c^\top z - \mu d_n^\top z
\]
corresponding to the primal problem is strictly convex over \(z \in \mathbb{R}^n\). Hence, the dual function \(q(\lambda, \mu) = \inf_{z \in \mathbb{R}^n} L(z, \lambda, \mu)\) can be computed explicitly by substituting the solution \(z'\) to the first-order condition \(0 = D_z L(z', \lambda, \mu) = Dz' - v + \lambda c - \mu d_n\) back into \(L(z', \lambda, \mu)\). This yields
\[
q(\lambda, \mu) = \frac{1}{2}(v - \lambda c + \mu d_n)^\top D^{-1}(v - \lambda c + \mu d_n)
\]
and thus the dual problem takes the form \((A.6)\).

The crucial part of the proof is to show that the triplet \((z^*, \lambda^*, \mu^*)\) satisfies the optimality conditions \((A.8)\). As \(L\) is strictly convex over \(z \in \mathbb{R}^n\), the optimality conditions are equivalent to
\[
Dz^* - v + \lambda^* c - \mu^* d_n = 0, \; c^\top z^* = 0, \; z^*_n \geq 0, \; \mu^* \geq 0, \; \mu^* z^*_n = 0. \tag{A.9}
\]
Recall the definitions of \(\lambda^*, \mu^*, \) and \(z^*\) in \((A.2)-(A.4)\) and note that the assumption that \(c_i \neq 0\) for some \(i \in \{1, \ldots, n - 1\}\) together with the positive definiteness of \(D^{-1}\) ensures that \(\lambda^*\) is well defined. The stationarity condition \(Dz^* - v + \lambda^* c - \mu^* d_n = 0\) holds by definition of \(z^*\). For the other conditions, we distinguish two cases. First, suppose that \(1_A = 0\), i.e., \(v_n - c_n^\top D^{-1} c_n \geq 0\). Then \(\lambda^* = \frac{c_n^\top D^{-1} c_n}{c_n^\top D^{-1} c}, \; \mu^* = 0\), and \(z^*_n = d_n^{-1}(v_n - \lambda^* c_n) \geq 0\). Moreover, \(c^\top z^* = c^\top D^{-1} v - \lambda^* c^\top D^{-1} c = 0\). Second, suppose that \(1_A = 1\), i.e., \(v_n - c_n^\top D^{-1} c_n < 0\), or, equivalently (multiply by \(c_n^\top D^{-1} c\), add and subtract \(c_n^2 v_n d_n^{-1}\), and than divide by \(c_n^\top D^{-1} c - c_n^2 d_n^{-1} > 0\), \(v_n - \lambda^* c_n < 0\). Then \(\mu^* > 0\) and \(z^*_n = 0\) by definition of \(\mu^*\) and \(z^*\). Finally, setting \(c = c^\top D^{-1} c\) and \(v = c^\top D^{-1} v\) for brevity,
\[
c^\top z^* = c^\top D^{-1}(v - \lambda^* c + \mu^* d_n) = v - \lambda^* c + \mu^* c d_n^{-1} = v - \lambda^* c - (v_n - \lambda^* c_n) c d_n^{-1} = v - \lambda^*(1 - c d_n^{-1}) - c_n v_n d_n^{-1} = v - (v_n - c_n v_n d_n^{-1}) - c_n v_n d_n^{-1} = 0.
\]
So, \((A.9)\) holds in both cases. By the characterisation of primal optimal solutions \([14, Proposition 5.1.5]\), this implies that \(z^*\) is an optimiser for the primal problem, that \((\lambda^*, \mu^*)\) is a Lagrange multiplier, and that there is no duality gap. Moreover, \((\lambda^*, \mu^*)\) is an optimiser for the dual problem by a corollary \([14, Proposition 5.1.4 (a)]\) of the weak duality theorem \([14, Proposition 5.1.3]\). As the primal and dual problems are strictly convex and strictly concave, respectively, the optimisers are unique.

Plugging the optimiser \((\lambda^*, \mu^*)\) of the dual problem into the cost function of the dual problem \((A.6)\) and using the definition of \(z^*\), the optimal value \(q^*\) (of both the primal and the dual problem) reads
\[
q^* := -\frac{1}{2}(v - \lambda^* c + \mu^* d_n)^\top D^{-1}(v - \lambda^* c + \mu^* d_n) = -\frac{1}{2}(v - \lambda^* c + \mu^* d_n)^\top z^*.
\]
Now, note that $c^\top z^* = 0$ and $\mu^* e_n^\top z^* = \mu^* z_n^* = 0$ by (A.9). Hence, $q^* = -\frac{1}{2} v^\top z^*$.

Next, using that $z^*$ achieves the optimal value $-\frac{1}{2} v^\top z^*$ for the primal problem and applying the Cauchy–Schwarz inequality, we obtain

$$\frac{1}{2} d_{\min} |z^*|^2 \leq \frac{1}{2} (z^*)^\top D z^* = \frac{1}{2} v^\top z^* \leq \frac{1}{2} |v||z^*|.$$  

This yields the last claim of part (a). Finally, using (A.4), the triangle inequality, and the bound $|z^*| \leq d_{\min}^{-1} |v|$ which we just proved, we obtain

$$|\lambda^* c - \mu^* e_n| = |D z^* - v| \leq |D z^*| + |v| \leq d_{\max} |z^*| + |v| \leq \left(1 + \frac{d_{\max}}{d_{\min}}\right) |v|.$$  

This proves the last claim of part (b) and thereby concludes the proof. \qed
Part Two

Bubbles and Crashes
Chapter IV

Single Jump Processes and Strict Local Martingales

1 Introduction

Strict local martingales, i.e., local martingales which are not martingales, play an important role in mathematical finance, e.g., in the context of modelling financial bubbles [114, 34, 139, 109] or in arbitrage theory [104, 67]. Specific examples of strict local martingales are usually rather complicated, the classical example being the inverse Bessel process [41]. The aim of this chapter is to study a very tractable class of processes and classify their (local) martingale properties. More precisely, we consider single jump local martingales, i.e., processes $M^G_F(t)$ of the form

$$M^G_F(t) = F(t)1_{\{t < \gamma\}} + A^G_F(\gamma)1_{\{t \geq \gamma\}},$$

where the jump time $\gamma$ is a $(0, \infty)$-valued random variable with distribution function $G$, and $F : [0, \infty) \to \mathbb{R}$ is a function that is “locally absolutely continuous” with respect to $G$. In words, each path $M^G_F(\omega)$ follows a deterministic function $F$ up to some random time $\gamma(\omega)$ and stays constant at $A^G_F(\gamma(\omega))$ from time $\gamma(\omega)$ on. The function $A^G_F$ is chosen such that $M^G_F$ becomes a martingale on the right-open interval $[0, t_G)$, where $t_G := \sup\{t \geq 0 : G(t) < 1\} \in (0, \infty]$ denotes the right endpoint of the distribution function $G$. All local martingales studied in this chapter are of the form (1.1).

The two main advantages of single jump local martingales are their flexibility and tractability. They are flexible enough to include examples of processes in well-known martingale spaces. Considered on the closed interval $[0, t_G]$, or equivalently on $[0, \infty]$, $M^G_F$ can be either of the following: not even a semimartingale; a nonintegrable local martingale; an integrable strict local martingale; a uniformly integrable martingale which does not belong to $H^1$; an $H^1$-martingale (and of course an $H^p$-martingale for $p > 1$). Our main result is a complete characterisation of these five cases in terms of conditions on the two input parameters $G$ and $F$ (cf. Figure 3.1). As for tractability, single jump local martingales are particularly suited for explicit calculations. For instance, we give a general, direct
solution to the problem of finding a bounded (deterministic) integrand \( H \) and a martingale \( M \) such that the stochastic integral \( H \cdot M \) is a strict local martingale. Moreover, single jump local martingales are used in [73, Section 5.1] to construct counter-examples to show, using only direct arguments, that neither of the no-arbitrage conditions NA and NUPBR implies the other. Because of their simple structure, these counter-examples also provide more insight into the nature of the underlying result than the more complicated counter-examples already available.

While the distribution function \( G \) of the random time \( \gamma \) is a natural input parameter, the choice of \( F \) as a second input parameter might be less clear. Another natural approach would be to start with a process \( S_t := \delta(\gamma)1_{\{t \geq \gamma\}} \) for a deterministic function \( \delta : [0, \infty) \to \mathbb{R} \). For \( \delta = 1 \), this is done in the literature on credit risk in the definition of the “hazard martingale”, see, e.g., [49, Proposition 2.1]. If \( \delta \) is sufficiently integrable, the compensator (or dual predictable projection; cf. [88]) \( S^p \) exists and \( M := S - S^p \) is a local martingale of the form (1.1). Yet another possibility is to start with a function \( H : (0, \infty) \to \mathbb{R} \) and to express the function \( F = (A^G)^{-1}H \) in terms of \( H \) and \( G \) such that the process

\[
F(t)1_{\{t < \gamma\}} + H(\gamma)1_{\{t \geq \gamma\}}
\]

is a martingale on \([0, t_G]\). This is the parametrisation used in [32] and [42]; cf. the next paragraph. There are at least two reasons why we start our parametrisation with the function \( F \) instead of the jump size \( \delta \) or the function \( H \). First, it turns out that \( F \) and \( G \) are the natural objects to decide whether \( \mathcal{M}^G F \) belongs to a certain (local) martingale space or not. For instance, if \( \mathcal{M}^G F \) is integrable and \( G \) has no point mass at \( t_G \), then \( \mathcal{M}^G F \) being a strict local martingale is equivalent to a nonvanishing limit \( \lim_{t \uparrow t_G} F(t)(1 - G(t)) \) (cf. Lemma 3.7) and \( \mathcal{M}^G F \) being an \( H^1 \)-martingale is equivalent to \( F(\cdot-) \) being \( dG \)-integrable (cf. Lemma 3.9). Second, a natural generalisation is to allow the function \( F \) to be random and to consider the corresponding process in its natural filtration. Then the process can follow different trajectories prior to the random time \( \gamma \), and observing its evolution corresponds to learning the conditional distribution of \( \gamma \) over time. However, if one starts with a process \( S_t = \delta 1_{\{t \geq \gamma\}} \) for a random variable \( \delta \), such a learning effect is much harder to incorporate, because one would have to construct first the desired filtration and then compute the corresponding compensator. If one simply computes the compensator \( S^p \) (if it exists) in the natural filtration of \( S \), then the local martingale \( S - S^p \) only has a single possible trajectory prior to \( \gamma \) and all information is learnt in a single instant at time \( \gamma \).

The study of single jump processes dates back to the classical papers by Dellacherie [42] and Chou and Meyer [32]. Dellacherie [42] (see also Dellacherie and Meyer [43, Chapter IV, No. 104]) starts from the smallest filtration \( \mathbb{F}^\gamma \) with respect to which \( \gamma \) is a stopping time. Among other things, he obtains a single jump local martingale by computing the compensator of the process \( 1_{\{t \geq \gamma\}} \) in this filtration. He also uses single jump processes to give several counter-examples in the general theory of stochastic processes. However, his simplifying assumption that \( t_G = \infty \) immediately excludes the possibility of strict local martingales.
IV.2 Analytic preliminaries

125

(cf. Lemma 3.1). In the same setting, Chou and Meyer [32, Proposition 1] show that any local $\mathbb{F}^\gamma$-martingale null at zero is a (true) martingale on $[0, t_G)$ and of the form (1.2) with

$$F(t) = -\frac{1}{1 - G(t)} \int_{[0,t]} H(v) \, dG(v),$$

and that, conversely, every process of this form is a local $\mathbb{F}^\gamma$-martingale provided that $H$ is “locally” $dG$-integrable (so that (1.3) is well defined) and $\Delta G(t_G) = 0$. Our Theorem 3.5 (a) corresponds to the “converse” statement and shows that the localising sequence can be chosen to consist of stopping times with respect to the natural filtration of $\mathcal{M}^G F$. As this filtration is generally smaller than $\mathbb{F}^\gamma$, we obtain a slightly stronger statement. [32, Proposition 1] also yields that in the case of $t_G < \infty$ and $\Delta G(t_G) > 0$, processes of the form (1.2) are always uniformly integrable martingales provided that $H$ is $dG$-integrable. Our Theorem 3.5 (c) shows that in this case, the process is even an $H^1$-martingale. Single jump martingales also appear in the modelling of credit risk, see, e.g., [17, 49, 91] and Remark 3.2. There the jump time models the default time of a financial asset, and single jump martingales are used to describe the hazard function of the default time. Note that in credit risk modelling only single jump (true) martingales are considered. To the best of our knowledge, our classification of the (local) martingale properties of single jump processes summarised in Figure 3.1 is new.

The remainder of the chapter is structured as follows. Section 2 contains basic definitions and all analytic results necessary for the classification of single jump local martingales given in Section 3. Section 4 presents the counter-example in stochastic integration theory mentioned above.

2 Analytic preliminaries

The proof of the classification of the (local) martingale properties of single jump local martingales is split up into a purely analytic and a stochastic part. In this section, we collect all analytic preliminaries. On a first reading, the reader may wish to go only up to Definition 2.1 and then jump directly to the stochastic part in Section 3.

We always fix a distribution function $G: \mathbb{R} \to [0, 1]$ satisfying $G(0) = 0$ and $G(\infty-) := \lim_{t \to \infty} G(t) = 1$. Recall that its right endpoint is defined by

$$t_G := \sup\{t \geq 0 : G(t) < 1\} \in (0, \infty].$$

For notational convenience, set $G(\infty) := 1$. With this in mind, note that $\Delta G(\infty) = 0$, so that $\Delta G(t_G) > 0$ implies $t_G < \infty$. Also, $dG$ denotes the Lebesgue–Stieltjes measure on $(0, \infty)$ induced by $G$, and $L^1(dG)$ is the space of real-valued $dG$-integrable functions. Note that a Borel-measurable function $\phi: (0, \infty) \to \mathbb{R}$ is $dG$-integrable if and only if it is $dG$-integrable on $(0, t_G)$, since $dG$ is concentrated on $(0, t_G]$ and a possible point mass at $t_G$ does not affect the integrability. We call $\phi$ locally $dG$-integrable, abbreviated by $\phi \in L^1_{\text{loc}}(dG)$, if
\[
\int_{[0,b]} |\phi(v)| \, dG(v) < \infty \text{ for each } b \in (0,t_G). \]
Finally, we set \( \overline{G} := 1 - G \) which is often called the \textit{survival function} of \( G \).

\section{2.1 Locally absolutely continuous functions}

Classically, a Borel-measurable function \( F : [0,\infty) \to \mathbb{R} \) is called absolutely continuous on the interval \((a,b]\) if there exists a Lebesgue-integrable function \( f : (a,b] \to \mathbb{R} \) such that \( F(t) - F(a) = \int_a^t f(v) \, dv \) for all \( t \in (a,b] \); in this case, \( f \) is unique a.e. on \((a,b]\) and is called a density of \( F \). Replacing the Lebesgue measure by \( dG \), we say that \( F \) is \textit{absolutely continuous with respect to} \( G \) on \((a,b]\) if there is a \( dG \)-integrable function \( f : (a,b] \to \mathbb{R} \) such that \( F(t) - F(a) = \int_{(a,t]} f(v) \, dG(v) \) for all \( t \in (a,b] \). Unlike the Lebesgue measure, \( dG \) may have atoms, so that the precise choice of the integration domain in the previous integral is important. Our choice of a left-open and right-closed interval is natural as it forces an absolutely continuous \( F \) to be right-continuous like \( G \). Then \( F \) itself induces a signed Lebesgue–Stieltjes measure \( dF \) on \((a,b]\) which is absolutely continuous with respect to \( dG \) (restricted to \((a,b]\)) in the sense of measures, and \( f \) is a version of the Radon–Nikodym density \( \frac{dF}{dG} \) on \((a,b]\).

The following is a local version of this concept.

\textbf{Definition 2.1.} A Borel-measurable function \( F : [0,\infty) \to \mathbb{R} \) is called \textit{locally absolutely continuous with respect to} \( G \) on \((0,t_G) \), abbreviated as \( F \ll^{\text{loc}} G \), if \( F \) is absolutely continuous with respect to \( G \) on \((0,b]\) for all \( 0 < b < t_G \). In this case, a Borel-measurable function \( f : (0,\infty) \to \mathbb{R} \) is called a \textit{local density of} \( F \) \textit{with respect to} \( G \) if for all \( 0 < b < t_G \), \( f \) is a version of the Radon–Nikodym density \( \frac{dF}{dG} \) on \((0,b]\).

The following result is an easy exercise in measure theory.

\textbf{Lemma 2.2.} Let \( F \ll^{\text{loc}} G \). Then \( F \) is càdlàg and of finite variation on the half-open interval \([0,t_G) \). Moreover, there exists a local density \( f \) of \( F \) with respect to \( G \); it is \( dG \)-a.e. unique on \((0,t_G) \) and locally \( dG \)-integrable with

\[
\int_{(a,b]} f(v) \, dG(v) = F(b) - F(a), \quad 0 \leq a < b < t_G. \tag{2.1}
\]

A local density \( f \) of \( F \ll^{\text{loc}} G \) with respect to \( G \) is only \( dG \)-a.e. unique on \((0,t_G) \) (and not on \((0,t_G]) \) and may not be \( dG \)-integrable on \((0,t_G) \), so it may not be a classical Radon–Nikodym density. Nevertheless, we often write—in slight abuse of notation—\( f = \frac{dF}{dG} \). This is justified on the one hand by the above lemma and on the other hand by the fact that we never consider \( \frac{dF}{dG} \) outside \((0,t_G) \).

\textbf{Remark 2.3.} If \( F \ll^{\text{loc}} G \), then \( F \) need not be càdlàg or of finite variation on the \textit{right-closed} interval \([0,t_G) \). Indeed, define \( G : \mathbb{R} \to [0,1] \) by \( G(t) = t1_{[0,1)}(t) + 1_{[1,\infty)}(t) \), i.e., \( dG \) is a uniform distribution on \((0,1) \) with \( t_G = 1 \), and
IV.2 Analytic preliminaries

Let $F : [0, \infty) \to \mathbb{R}$ be given by $F(t) = 1_{(0,1)}(t) \sin \frac{1}{1-t}$. Then $F \ll G$ with local density

$$\frac{dF}{dG}(v) = 1_{(0,1)}(v) \frac{1}{(1-v)^2} \cos \frac{1}{1-v}.$$ 

However, $F$ is neither càdlàg nor of finite variation on $[0,1]$.

2.2 The function $A^G F$

The first result of this section introduces and analyses the function $A^G F$ appearing in the definition of the process $M^G F$. Its definition is motivated by the idea that $M^G F$ should be a martingale on $[0,t_G]$ provided the function $F$ is nice enough. We refer to the discussion after the proof of Lemma 3.1 for more details.

Lemma 2.4. Let $F \ll G$ and define the function $A^G F : (0, \infty) \to \mathbb{R}$ by

$$A^G F(v) = \begin{cases} 
F(v^-) - \frac{dF}{dG}(v)G(v), & v \in (0,t_G), \\
F(t_G^-)1_{(\Delta G(t_G)>0)}, & v \geq t_G, \text{ if } \lim_{t \uparrow t_G} F(t) \text{ exists in } \mathbb{R}, \\
0, & v \geq t_G, \text{ if } \lim_{t \uparrow t_G} F(t) \text{ does not exist in } \mathbb{R}.
\end{cases} \tag{2.2}
$$

Then $A^G F \in L^1_{\text{loc}}(dG)$ and for all $0 \leq a < b < t_G$,

$$\int_{(a,b]} A^G F(v) dG(v) = \left[-F(v)G(v)\right]_a^b. \tag{2.3}
$$

Thus,

$$A^G F = -\frac{d(FG)}{dG} \text{ dG-a.e. on } (0,t_G).$$

Proof. Note that $A^G F$ is well defined by Lemma 2.2. To prove (2.3), fix $0 \leq a < b < t_G$. The function $F(\cdot-)$ is càglàd and therefore bounded on $(a,b]$, the function $G$ is trivially bounded on $(a,b]$, and the function $\frac{dF}{dG}$ is $dG$-integrable on $(a,b]$ by (2.1). Thus, $A^G F \in L^1_{\text{loc}}(dG)$. Associativity of Lebesgue–Stieltjes integrals together with an integration by parts gives the result via

$$\int_{(a,b]} A^G F(v) dG(v) = \int_{(a,b]} F(v^-) dG(v) - \int_{(a,b]} \frac{dF}{dG}(v)G(v) dG(v)$$

$$= \int_{(a,b]} F(v^-) dG(v) + \left[-F(v)G(v)\right]_a^b - \int_{(a,b]} F(v^-) dG(v). \qed
$$

In general, $A^G F$ is not $dG$-integrable on $(0,t_G]$. The next result lists some equivalent conditions when this is the case and draws an important consequence.
**Lemma 2.5.** Let $F \ll G$. Then the following are equivalent:

(a) $\mathcal{A}^G F \in L^1(dG)$.

(b) $(\mathcal{A}^G F)^- \in L^1(dG)$ and $\limsup_{t \uparrow t_G} F(t) \overline{G}(t) > -\infty$.

(c) $(\mathcal{A}^G F)^+ \in L^1(dG)$ and $\liminf_{t \uparrow t_G} F(t) \overline{G}(t) < \infty$.

Moreover, each of the above implies that the limit $\lim_{t \uparrow t_G} F(t) \overline{G}(t)$ exists in $\mathbb{R}$ and

$$
\int_{(a,t_G)} \mathcal{A}^G F(v) \, dG(v) = F(a) \overline{G}(a) - \lim_{t \uparrow t_G} F(t) \overline{G}(t), \quad a \in [0,t_G).
$$

**(2.4)**

**Proof.** “(a) \(\Rightarrow\) (b), (c)”: If $\mathcal{A}^G F \in L^1(dG)$, then $(\mathcal{A}^G F)^{\pm} \in L^1(dG)$, and dominated convergence and (2.3) give

$$
\int_{(0,t_G)} \mathcal{A}^G F(v) \, dG(v) = \lim_{t \uparrow t_G} \int_{(0,t]} \mathcal{A}^G F(v) \, dG(v) = \lim_{t \uparrow t_G} \left[ -F(v) \overline{G}(v) \right]_0^t = F(0) - \lim_{t \uparrow t_G} F(t) \overline{G}(t).
$$

This shows that $\lim_{t \uparrow t_G} F(t) \overline{G}(t)$ exists, and (2.4) is satisfied first for $a = 0$ and then, by (2.3), for any $a \in (0,t_G)$.

“(b) \(\Rightarrow\) (a)”: Since $(\mathcal{A}^G F)^- \in L^1(dG)$, it suffices to show that

$$
\int_{(0,t_G)} \mathcal{A}^G F(v) \, dG(v) < \infty.
$$

Fatou’s lemma applied to $(\mathcal{A}^G F)^+$, dominated convergence for $(\mathcal{A}^G F)^-$ and (2.3) give

$$
\int_{(0,t_G)} \mathcal{A}^G F(v) \, dG(v) \leq \liminf_{t \uparrow t_G} \int_{(0,t]} \mathcal{A}^G F(v) \, dG(v)
$$

$$
= F(0) - \limsup_{t \uparrow t_G} F(t) \overline{G}(t) < \infty.
$$

“(c) \(\Rightarrow\) (a)”: This is analogous to the proof of “(b) \(\Rightarrow\) (a)”.

The following result provides further characterisations of the $dG$-integrability of $\mathcal{A}^G F$ in the case $\Delta G(t_G) > 0$. In particular, it shows that if $\mathcal{A}^G F$ is $dG$-integrable, then the limit in the second line of the definition of $\mathcal{A}^G F$ in (2.2) exists in $\mathbb{R}$.

**Lemma 2.6.** Let $F \ll G$ and suppose that $\Delta G(t_G) > 0$. The following are equivalent:

(a) $\mathcal{A}^G F \in L^1(dG)$.

(b) $\frac{dF}{dG} \overline{G} \in L^1(dG)$. 
IV.2 Analytic preliminaries

(c) \( \frac{dF}{dG} \in L^1(dG) \).

(d) \( F \) is of finite variation on \([0, t_G]\).

Each of the above implies that the limit \( \lim_{t \uparrow t_G} F(t) \) exists in \( \mathbb{R} \).

Proof. The last statement follows immediately from (d). “(d) \iff (c)” is a standard result in analysis, and “(c) \iff (b)” follows immediately from the fact that the function \( G \) is bounded above by 1 and below by \( G(t_G^-) = \Delta G(t_G) > 0 \) on \((0, t_G)\).

For “(b) \iff (a)”, it suffices to show that the function \( F \) is bounded on the compact interval \([0, t_G]\). (Recall that \( \Delta G(t_G) > 0 \) implies \( t_G < \infty \).) Since \( F \) is càdlàg on \([0, t_G)\), it is enough to show that the limit \( \lim_{t \uparrow t_G} F(t) \) exists in \( \mathbb{R} \).

Assuming (b), this follows from the equivalence “(b) \iff (d)” and the first part of the proof. Assuming (a), this follows via Lemma 2.5 from the fact that the limit \( \lim_{t \uparrow t_G} G(t) \) exists in \( \mathbb{R} \) and that \( \lim_{t \uparrow t_G} G(t) = \Delta G(t_G) > 0 \).

2.3 Decomposition of locally absolutely continuous functions

Let \( F \ll G \). Define the functions \( F^\uparrow, F^\downarrow, |F| : [0, \infty) \to \mathbb{R} \) by

\[
F^\uparrow(t) = \begin{cases} 
\int_{(0,t]} \left( \frac{dF}{dG}(v) \right)^+ \, dG(v) & \text{if } t < t_G, \\
0 & \text{if } t \geq t_G,
\end{cases}
\]

\[
F^\downarrow(t) = \begin{cases} 
\int_{(0,t]} \left| \frac{dF}{dG}(v) \right| \, dG(v) & \text{if } t < t_G, \\
0 & \text{if } t \geq t_G.
\end{cases}
\]

\[
|F|(t) = \begin{cases} 
\int_{[0,t]} \left| \frac{dF}{dG}(v) \right| \, dG(v) & \text{if } t < t_G, \\
0 & \text{if } t \geq t_G.
\end{cases}
\]

\( F^\uparrow, F^\downarrow, |F| \) are well defined by Lemma 2.2, null at 0, nonnegative and increasing on \([0, t_G)\), and satisfy

\[
F1_{[0,t_G]} = F(0)1_{[0,t_G]} + F^\uparrow - F^\downarrow, \\
|F| = F^\uparrow + F^\downarrow.
\] (2.5)

Restricted to \([0, t_G)\), \( |F| \) is simply the total variation of \( F \) and \( F^\uparrow/\downarrow \) is the positive/negative variation of \( F \) shifted to null at 0.

The following result shows that if \( F \ll G \) and \( A^G F \in L^1(dG) \), then the analogous properties hold for \( F^\uparrow, F^\downarrow, |F| \), too.

Lemma 2.7. Let \( F \ll G \) be such that \( A^G F \in L^1(dG) \). Then \( F^\uparrow, F^\downarrow, |F| \ll G \) and \( A^G(F^\uparrow), A^G(F^\downarrow), A^G|F| \) are in \( L^1(dG) \). Moreover,

\[
A^G F = F(0) + A^G(F^\uparrow) - A^G(F^\downarrow) \, dG\text{-a.e.},
\] (2.6)

\[
A^G|F| = A^G(F^\uparrow) + A^G(F^\downarrow) \, dG\text{-a.e.}
\] (2.7)
Lemma 2.5. On the one hand, nonnegativity of $A \leq \infty$ equivalent: $A \leq 0$. Proof. 

Proof. 

\( \text{(a)} \): This is analogous to the proof of \( \text{(b)} \). 

\( \text{(b)} \): If \( A^{G}(F^{\uparrow}) \not\in L^{1}(dG) \), then Lemma 2.5 implies \( \limsup_{t \uparrow t_{G}} F^{\uparrow}(t)G(t) = -\infty \). Choose \( N \in \mathbb{N} \) large enough that \( \int_{(0,t_{G})} |F(v-)|dG(v) < N \), and \( t \in [0,t_{G}) \) such that \( |F(v)|G(v) \geq N \) for all \( v \in [t,t_{G}) \). Using \( G(t_{G}) = 0 \), this gives

\[
\int_{(0,t_{G})} |F(v-)|dG(v) \geq \int_{(t,t_{G})} \frac{|F(v-)|G(v-)}{G(v-)}dG(v) \\
\geq N \int_{(t,t_{G})} \frac{1}{G(t)}dG(v) = N \frac{1 - G(t)}{G(t)} \\
> \int_{(0,t_{G})} |F(v-)|dG(v),
\]

which is a contradiction.

\( \text{(c)} \): This is analogous to the proof of \( \text{(b)} \).
“(a) ⇒ (d)”: This follows immediately from the definition of $A^G F$.

“(d) ⇔ (e)”: Using the definition of $|F|$, Fubini’s theorem and $\Delta G(t_G) = 0$, we obtain

$$
\int_{(0,t_G)} |F|(s-)\,dG(s) = \int_{(0,t_G)} \int_{(0,s)} \left| \frac{dF}{dG}(v) \right| \,dG(v)\,dG(s) \\
= \int_{(0,t_G)} \left| \frac{dF}{dG}(v) \right| \int_{(v,t_G)} dG(s)\,dG(v) \\
= \int_{(0,t_G)} \left| \frac{dF}{dG}(v) \right| (1 - G(v))\,dG(v).
$$

This immediately establishes both directions.

“(e) ⇒ (a)”: On the one hand, (e) implies $F(\cdot-) \in L^1(dG)$ as $|F(v)| \leq |F(0)| + |F(v)|$ for $v \in [0,t_G)$, and on the other hand, (e) implies (d). Now the claim follows from the definition of $A^G F$.

**Remark 2.9.** $F(\cdot-) \in L^1(dG)$ alone does not imply $(A^G F)^\pm \in L^1(dG)$. Indeed, let $G : \mathbb{R} \to [0,1]$ be given by $G(t) = t1_{[0,1)}(t) + 1_{[1,\infty)}(t)$, i.e., dG is a uniform distribution on $(0,1)$ with $t_G = 1$, and define $F : [0,\infty) \to \mathbb{R}$ by $F(t) = 1_{[0,1)}(t)\sin\frac{1}{t^7}$. Then an easy exercise in analysis shows

$$
\int_{(0,1)} |F(v-)|\,dG(v) = \int_0^1 |F(v)|\,dv < \infty,
$$

$$
\int_{(0,1)} (A^G F(v))^\pm\,dG(v) = \int_0^1 \left( \sin \frac{1}{1-v} - \frac{1}{1-v} \cos \frac{1}{1-v} \right)^\pm \,dv = \infty.
$$

## 3 Classification of single jump local martingales

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\gamma$ a fixed $(0,\infty)$-valued random variable with distribution function $G$. The filtration $\mathbb{F}^\gamma = (\mathcal{F}^\gamma_t)_{t\in[0,\infty]}$ given by

$$
\mathcal{F}^\gamma_t = \sigma(\{ \gamma \leq s \} : s \in (0,t])
$$

is the smallest filtration with respect to which $\gamma$ is a stopping time. For any $F \ll G$, define the function $\zeta^F : [0,\infty] \times (0,\infty) \to \mathbb{R}$ by

$$
\zeta^F(t,v) = F(t)1_{\{t<v\}} + A^G F(v)1_{\{v\geq t\}},
$$

where $A^G F : (0,\infty) \to \mathbb{R}$ is defined by

$$
A^G F(v) = \begin{cases} 
F(v-) - \frac{dF}{dG}(v)G(v), & v \in (0,t_G), \\
F(t_G-)1_{\{\Delta G(t_G)>0\}}, & v \geq t_G, \text{ if } \lim_{t\uparrow t_G} F(t) \text{ exists in } \mathbb{R}, \\
0, & v \geq t_G, \text{ if } \lim_{t\uparrow t_G} F(t) \text{ does not exist in } \mathbb{R};
\end{cases}
$$
cf. Lemma 2.4. Note that $\zeta^F$ is Borel-measurable and for each $t \in [0, \infty]$, $\zeta^F(t, \cdot)$ is unique up to $dG$-nullsets (because the local density $\frac{dF}{dG}$ is only $dG$-a.e. unique on $(0, t_G)$). Now define the process $\mathcal{M}^G F = (\mathcal{M}^G F_t)_{t \in [0, \infty]}$ by

$$\mathcal{M}^G F_t = \zeta^F(t, \gamma) = F(t)1_{\{t < \gamma\}} + A^G F(\gamma)1_{\{t \geq \gamma\}}.$$ (3.3)

$\mathcal{M}^G F$ is clearly $\mathbb{F}^\gamma$-adapted and it is easy to see that modifying $A^G F$ on a $dG$-nullset leads to a process that is indistinguishable from the original process $\mathcal{M}^G F$. Every trajectory $\mathcal{M}^G F(\omega)$ is càdlàg and of finite variation on $[0, t_G]$, nonrandom until just before the random time $\gamma(\omega)$, and stays constant at $A^G F(\gamma(\omega))$ from time $\gamma(\omega)$ on. In particular,

$$\mathcal{M}^G_{t_G} F = A^G F(\gamma) \ P\text{-a.s.}$$

The first line in the definition of $A^G F$ is chosen such that $\mathcal{M}^G F$ becomes an $\mathbb{F}^\gamma$-martingale on the right-open interval $[0, t_G)$. This result is well known in the literature (see, e.g., [42]). For the convenience of the reader, we provide a full proof here. In the following Sections 3.1–3.3, we then classify the (local) martingale properties of $\mathcal{M}^G F$ when considered on the closed interval $[0, t_G]$, or, equivalently, on $[0, \infty]$.

**Lemma 3.1.** The process $\mathcal{M}^G F$ is an $\mathbb{F}^\gamma$-martingale on $[0, t_G)$.

**Proof.** For brevity, we set $M := \mathcal{M}^G F$. To check integrability, fix $0 \leq t < t_G$. Then the definition of $M$ and Lemma 2.4 give

$$E[|M_t|] \leq |F(t)|P[t < \gamma] + E[|A^G F(\gamma)|1_{\{t \geq \gamma\}}]$$

$$= |F(t)|(1 - G(t)) + \int_{(0,\gamma]} |A^G F(s)| \, dG(s) < \infty.$$ To check the martingale property for $M$, fix $0 \leq s < t < t_G$. Then $t \geq \gamma$ on \{s $\geq$ $\gamma$\} gives

$$E[M_t | \mathcal{F}_s] = E[M_t 1_{\{s < \gamma\}} + A^G F(\gamma)1_{\{s \geq \gamma\}} \bigg| \mathcal{F}_s]$$

$$= E[M_t 1_{\{s < \gamma\}} \bigg| \mathcal{F}_s] + A^G F(\gamma)1_{\{s \geq \gamma\}} \ P\text{-a.s.}$$

It is not hard to show that \{s $>$ $\gamma$\} is an atom of $\mathcal{F}_s$ (see, e.g., [42], [32] or [43, Chapter IV, No. 104]). Using this and (2.3) gives

$$E[M_t 1_{\{s < \gamma\}} | \mathcal{F}_s] = E[M_t | s < \gamma] 1_{\{s < \gamma\}}$$

$$= E[F(t)1_{\{t < \gamma\}} + A^G F(\gamma)1_{\{s < \gamma\}} \bigg| s < \gamma] 1_{\{s < \gamma\}}$$

$$= F(t)(1 - G(t)) + \int_{(s,\gamma]} A^G F(u) \, dG(u)$$

$$= \frac{F(t)G(t) + \int_{(s,\gamma]} -F(u)G(u) \, d\nu}{G(s)} 1_{\{s < \gamma\}}$$

$$= \frac{F(s)G(s)}{G(s)} 1_{\{s < \gamma\}} = F(s)1_{\{s < \gamma\}} \ P\text{-a.s.}$$
Thus, we may conclude that $E[M_t | \mathcal{F}_s^\gamma] = F(s)1_{\{s<\gamma\}} + \mathcal{A}^GF(\gamma)1_{\{s\geq \gamma\}} = M_s$ $P$-a.s.

We are now in a position to explain the structure of the function $\mathcal{A}^GF$. On the one hand, if $\Delta G(t_G) = 0$, then $\gamma < t_G$ $P$-a.s. and only the first line in the definition of $\mathcal{A}^GF$ is relevant for $\mathcal{M}^GF$. On $(0, t_G)$, $\mathcal{A}^GF$ is chosen such that $\mathcal{M}^GF$ becomes a martingale on the right-open interval $[0, t_G)$. On the other hand, if $\Delta G(t_G) > 0$, then $\gamma = t_G$ with positive probability. Assuming for the moment that $\mathcal{M}^GF$ is a martingale on $[0, t_G]$, the martingale convergence theorem implies that $\mathcal{M}^G_{t_G}F = \lim_{t \uparrow t_G} \mathcal{M}^GF$ $P$-a.s. Evaluating the right-hand side on the event $\{\gamma = t_G\}$ yields $\mathcal{M}^G_{t_G}F = \lim_{t \uparrow t_G} F(t) = F(t_G-)$ on $\{\gamma = t_G\}$. This motivates the second line in the definition of $\mathcal{A}^GF$. The last line is only relevant when $\Delta G(t_G) > 0$ and the left limit $F(t_G-)$ does not exist in $\mathbb{R}$. But then $F$ must be of infinite variation on $[0, t_G)$ and

$$P[\mathcal{M}^G_tF = F(t), t \in [0, t_G)] \geq P[\gamma = t_G] = \Delta G(t_G) > 0,$$

so that $\mathcal{M}^GF$ fails to be a semimartingale on $[0, t_G]$ by Lemma B.6. Note that this is independent of the particular choice $\mathcal{A}^GF(t_G) := 0$.

**Remark 3.2.** Processes of the form $\mathcal{M}^GF$ for particular choices of $F$ play a special role in the modelling of credit risk, see, e.g., [17, 91, 49]. We give two examples. We assume—as is usually done in the literature on credit risk—that $t_G = \infty$. First, for $F := \frac{1}{1-G} = \frac{1}{\beta}$, we have $\mathcal{A}^GF = -\frac{d(F\overline{G})}{dG} = -\frac{d}{dG} = 0$ and

$$\mathcal{M}^G_tF = \frac{1}{1-G(t)}1_{\{t<\gamma\}} + 0 \cdot 1_{\{t\geq \gamma\}} = \frac{1 - 1_{\{t\geq \gamma\}}}{1-G(t)}, \quad t \in [0, \infty).$$

This process is called $\hat{M}$ in [91, Corollary 5.1]. Second, for

$$F(t) := -\int_{[0,t]} \frac{dG(v)}{1-G(v^-)} = -\int_{[0,t]} \overline{G}(v^-)$$

($= \log \overline{G}(t)$ if $G$ is continuous), we have

$$\mathcal{A}^GF(v) = F(v-) - \frac{dF}{dG}(v)\overline{G}(v) = F(v) - \frac{dF}{dG}(v)\overline{G}(v-)$$

$$= F(v) + \frac{1}{G(v-)}\overline{G}(v-) = F(v) + 1 \text{ d}G \text{-a.e.}$$

and

$$\mathcal{M}^G_tF = F(t)1_{\{t<\gamma\}} + (F(\gamma) + 1)1_{\{t\geq \gamma\}}$$

$$= 1_{\{t\geq \gamma\}} - \int_{[0,t\wedge \gamma]} \frac{dG(v)}{1-G(v-)}, \quad t \in [0, \infty).$$

This process is called $M$ in [91, Proposition 5.2].
It is also often assumed that $G$ is absolutely continuous with respect to Lebesgue measure, i.e., $dG(t) = G'(t)\, dt$ for a nonnegative Borel-measurable function $G'$. In this case, the quantity $\kappa^G(t) := \frac{G(t)}{G'(t)}$ is the conditional probability density of the default time, given that default has not occurred up to time $t$, and is often called hazard rate or default intensity. Clearly, any $F^\text{loc} \ll G$ is also locally absolutely continuous with respect to Lebesgue measure, i.e., there is a Borel-measurable function $F'$ such that $dF(t) = F'(t)\, dt$. Now, $\mathcal{M}^G F$ has the following representation in terms of $F$, $F'$ and the hazard rate of $G$:

$$\mathcal{M}^G_t F = F(t)1_{\{t<\gamma\}} + A^G_F(\gamma)1_{\{t\geq \gamma\}}$$

$$= F(t)1_{\{t<\gamma\}} + \left( F(\gamma-) - \frac{dF}{dG}(\gamma)\overline{G}(\gamma) \right) 1_{\{t\geq \gamma\}},$$

$$= F(t)1_{\{t<\gamma\}} + \left( F(\gamma) - \frac{F'(\gamma)}{\kappa^G(\gamma)} \right) 1_{\{t\geq \gamma\}},$$

or alternatively,

$$\mathcal{M}^G_t F = F(t \wedge \gamma) - \frac{F'(\gamma)}{\kappa^G(\gamma)} 1_{\{t\geq \gamma\}}.$$

For the rest of this section (except for Section 3.4), we fix $F^\text{loc} \ll G$ and set $M := \mathcal{M}^G F$ for brevity.

The raw filtration generated by $M$, denoted by $\mathbb{F}^M = (\mathcal{F}^M_t)_{t \in [0, \infty]}$, is the smallest filtration such that $M$ is $\mathbb{F}^M$-adapted. As $M$ is $\mathbb{F}^\gamma$-adapted, $\mathbb{F}^M$ is a subfiltration of $\mathbb{F}^\gamma$.

**Remark 3.3.**

(a) While in the filtration $\mathbb{F}^\gamma$, the value of $\gamma(\omega)$ is known at time $\gamma(\omega)$, this may not be true for the filtration $\mathbb{F}^M$. In $\mathbb{F}^M$, we can only tell the value of $\gamma(\omega)$ at time $\gamma(\omega)$ if we observe a jump of $M(\omega)$ of a certain size at time $\gamma(\omega)$. However, if $\gamma(\omega) < t_G$ and $\frac{dF}{dG}(\gamma(\omega)) = 0$, then $A^G F(\gamma(\omega)) = F(\gamma(\omega)-)$ and $M(\omega)$ has no jump at time $\gamma(\omega)$ (“$\gamma$ occurred, but we did not see it in the path of $M$”). A trivial example is given by $F \equiv 0$. Then $A^G F \equiv 0$, $M \equiv 0$, and $\mathbb{F}^M$ contains no information about $\gamma$ at all.

(b) The filtrations $\mathbb{F}^\gamma$ and $\mathbb{F}^M$ need not be $P$-complete and $\mathbb{F}^M$ need not be right-continuous in general ($\mathbb{F}^\gamma$ is in fact right-continuous, see, e.g., [88, Lemma II.3.24] or [43, Chapter IV, No. 104]). However, most of the results of martingale theory can be proved without these usual conditions. In particular, the martingale convergence theorem and the convergence result for stochastic integrals stated in Lemma B.6 do not rely on them.

By the law of iterated expectations, if $M$ is an $\mathbb{F}^\gamma$-martingale, then it is also an $\mathbb{F}^M$-martingale. However, if $M$ is a local $\mathbb{F}^\gamma$-martingale, then $M$ need not be
a local $\mathbb{F}^M$-martingale. The reason is that the $\mathbb{F}^\gamma$-stopping times in the localising sequence need not be $\mathbb{F}^M$-stopping times. To obtain stronger statements, we distinguish two filtrations in the definition of a local martingale. In particular, $M$ is called an $\mathbb{F}^M$-local $\mathbb{F}^\gamma$-martingale if it is a local $\mathbb{F}^\gamma$-martingale that admits a localising sequence consisting only of $\mathbb{F}^M$-stopping times. We refer to Appendix B for the details and related (partly nonstandard) terminology for (semi-)martingales.

3.1 Local martingale property on $[0, t_G]$

The following preparatory lemma gives conditions for the integrability of $M$ on $[0, t_G]$.

**Lemma 3.4.** The following are equivalent:

(a) The process $M$ is integrable on $[0, t_G]$.

(b) The random variable $M_{t_G}$ is integrable.

(c) $\mathcal{A}^G F \in L^1(dG)$.

**Proof.** “(a) $\Rightarrow$ (b)” is trivial, and “(b) $\Rightarrow$ (a)” holds because $M_t$ is integrable for $t \in [0, t_G)$ by Lemma 3.1. “(b) $\iff$ (c)” follows from $M_{t_G} = \mathcal{A}^G F(\gamma)$ $P$-a.s. and the fact that $\gamma$ has distribution function $G$ under $P$. \qed

**Theorem 3.5.**

(a) If $\Delta G(t_G) = 0$, then $M$ is an $\mathbb{F}^M$-local $\mathbb{F}^\gamma$-martingale on $[0, t_G]$.

(b) If $\Delta G(t_G) > 0$ and $M_{t_G}$ is not integrable, then $M$ fails to be a semimartingale on $[0, t_G]$.

(c) If $\Delta G(t_G) > 0$ and $M_{t_G}$ is integrable, then $M$ is an $H^1$-$\mathbb{F}^\gamma$-martingale on $[0, t_G]$.

**Proof.** (a): We distinguish two cases for $F$. If there exists $t^* \in [0, t_G)$ such that $F(t) = F(t^*)$ for $t \in [t^*, t_G)$, then $\mathcal{A}^G F(v) = F(v-) = F(t^*)$ for $dG$-a.e. $v \in (t^*, t_G)$. Thus, $P$-a.e. path of $M$ is constant on $[t^*, t_G]$. It follows that $M = M^{t^*} -$ $P$-a.s., and so by Lemma 3.1, $M$ is a uniformly integrable $\mathbb{F}^\gamma$-martingale on $[0, t_G]$. If there is no such $t^*$, then there exists a strictly increasing sequence $(t_n)_{n \in \mathbb{N}} \subset [0, t_G)$ such that

$$\lim_{n \to \infty} t_n = t_G \quad \text{and} \quad F(t_n) \neq F(t_{n-1}), \quad n \in \mathbb{N}.$$

For $n \in \mathbb{N}$, define the random time $\tau_n : \Omega \to [0, t_G]$ by

$$\tau_n := t_n 1_{\{M_{t_n} - M_{t_{n-1}} \neq 0\}} + t_G 1_{\{M_{t_n} - M_{t_{n-1}} = 0\}}.$$
Since \( \{M_n - M_{n-1} \neq 0\} \in \mathcal{F}_n^M \), \( \tau_n \) is an \( \mathbb{F}^M \)-stopping time, and
\[
M_n - M_{n-1} = \begin{cases} 
F(t_n) - F(t_{n-1}) & \text{if } \gamma > t_n, \\
A^G F(\gamma) - F(t_{n-1}) & \text{if } t_{n-1} < \gamma \leq t_n, \\
0 & \text{if } \gamma \leq t_{n-1},
\end{cases}
\]
so that
\[
\{\tau_n = t_G\} = \{M_n - M_{n-1} = 0\} \subset \{\gamma \leq t_n\} \subset \{M_{n+1} - M_n = 0\} = \{\tau_{n+1} = t_G\}.
\]
This shows that the sequence \((\tau_n)_{n \in \mathbb{N}}\) is increasing and satisfies
\[
\lim_{n \to \infty} P[\tau_n = t_G] \geq \lim_{n \to \infty} P[\gamma \leq t_{n-1}] = P[\gamma < t_G] = 1;
\]
(3.4)
here, we use the assumption \(\Delta G(t_G) = 0\). Moreover, for \(n \in \mathbb{N}\) and \(s \in [0, t_G]\), it follows from the definition of \(M_n\) that
\[
M_s 1_{\{\gamma \leq t_n\}} = F(s) 1_{\{s < \gamma\}} 1_{\{\gamma \leq t_n\}} + A^G F(\gamma) 1_{\{s \geq \gamma\}} 1_{\{\gamma \leq t_n\}}
\]
\[
= F(t_n \wedge s) 1_{\{t_n \wedge s < \gamma\}} 1_{\{\gamma \leq t_n\}} + A^G F(\gamma) 1_{\{\gamma \leq t_n \wedge s\}} 1_{\{\gamma \leq t_n\}}
\]
\[
= M_{t_n \wedge s} 1_{\{\gamma \leq t_n\}}.
\]
This together with \(\{\tau_n = t_G\} \subset \{\gamma \leq t_n\}\) gives
\[
M_s^\tau_n = M_s 1_{\{\tau_n = t_G\}} + M_{t_n \wedge s} 1_{\{\tau_n = t_G\}} = M_{t_n \wedge s} 1_{\{\gamma \leq t_n\}} 1_{\{\tau_n = t_G\}} + M_{t_n \wedge s} 1_{\{\tau_n = t_G\}} = M_s^\tau_n.
\]
Now the claim follows from (3.4) and the fact that by Lemma 3.1, each \(M_{t_n}^\gamma\) is a uniformly integrable \(\mathbb{F}^\gamma\)-martingale on \([0, t_G]\).

(b): Lemmas 3.4 and 2.6 show that \(F\) is of infinite variation on \([0, t_G]\). Moreover, we have \(P[M_t = F(t), t \in [0, t_G]] \geq P[\gamma = t_G] = \Delta G(t_G) > 0\). Now the claim follows from Lemma B.6.

(c): The assumption that \(M_{t_G}\) is integrable together with Lemma 3.4 gives \(A^G F \in L^1(dG)\). Since \(\Delta G(t_G) > 0\), the limit \(F(t_G -) = \lim_{t \uparrow t_G} F(t)\) exists in \(\mathbb{R}\) by Lemma 2.6 and so there is \(C > 0\) such that \(\sup_{s \in [0, t_G]} |F(s)| \leq C\). Thus,
\[
E \left[ \sup_{0 \leq s \leq t_G} |M_s| \right] = E \left[ \sup_{0 \leq s \leq t_G} \left( |F(s)| 1_{\{s < \gamma\}} + \left| A^G F(\gamma) \right| 1_{\{s \geq \gamma\}} \right) \right]
\]
\[
\leq C + \int_{[0, t_G]} |A^G F(v)| \, dG(v) < \infty.
\]
(3.5)
Moreover, using the definition of \(A^G F\) and \(\Delta G(t_G) > 0\) in the third equality,
\[
\lim_{t \uparrow t_G} M_t = \lim_{t \uparrow t_G} \left( F(t) 1_{\{t < \gamma\}} + A^G F(\gamma) 1_{\{t \geq \gamma\}} \right)
\]
\[
= F(t_G -) 1_{\{\gamma = t_G\}} + A^G F(\gamma) 1_{\{\gamma < t_G\}} = A^G F(\gamma) = M_{t_G} \quad \text{P-a.s.}
\]
Since \(M\) is an \(\mathbb{F}^\gamma\)-martingale on \([0, t_G]\) by Lemma 3.1, combining (3.5) with the martingale convergence theorem shows that \(M\) is an \(H^1-\mathbb{F}^\gamma\)-martingale on the right-closed interval \([0, t_G]\). \(\square\)
3.2 Sub- and supermartingale property on $[0, t_G]$

Definition 3.6. If $M$ is integrable on $[0, t_G]$, the change in mass of $M$ (on $[0, t_G]$) is defined as

$$\Delta \mu := E [M_{t_G}] - E [M_0] = \int_{(0, t_G]} A^G F(v) \, dG(v) - F(0).$$

If $M$ is integrable on $[0, t_G]$, it is a strict local martingale whenever $\Delta \mu \neq 0$. The following result gives a formula that allows to compute $\Delta \mu$ easily.

Lemma 3.7. Suppose that $M$ is integrable on $[0, t_G]$. Then

$$\Delta \mu = - \lim_{t \uparrow t_G} F(t) \overline{G} (t) 1_{\{\Delta G(t_G) = 0\}},$$

and for $0 \leq s < t_G$,

$$E [M_{t_G} | \mathcal{F}_s] - M_s = \frac{\Delta \mu}{G(s)} 1_{\{s < \gamma\}} \text{ P-a.s.} \quad (3.6)$$

Moreover, $M$ is always an integrable $\mathbb{F}^M$-local $\mathbb{F}^\gamma$-martingale on $[0, t_G]$, and more precisely,

(a) $M$ is an $\mathbb{F}^\gamma$-submartingale and a strict local martingale on $[0, t_G]$ if and only if $\Delta \mu > 0$,

(b) $M$ is an $\mathbb{F}^\gamma$-martingale on $[0, t_G]$ if and only if $\Delta \mu = 0$,

(c) $M$ is an $\mathbb{F}^\gamma$-supermartingale and a strict local martingale on $[0, t_G]$ if and only if $\Delta \mu < 0$.

Proof. If $\Delta G(t_G) > 0$, then $M$ is an $H^1$-$\mathbb{F}^\gamma$-martingale on $[0, t_G]$ by Theorem 3.5 (c), and all claims follow. So suppose that $\Delta G(t_G) = 0$. Then $M$ is an $\mathbb{F}^M$-local $\mathbb{F}^\gamma$-martingale on $[0, t_G]$ by Theorem 3.5 (a). Moreover, using $\Delta G(t_G) = 0$ and (2.4) gives

$$\Delta \mu = E [M_{t_G}] - E [M_0] = \int_{(0, t_G]} A^G F(v) \, dG(v) - F(0)$$

$$= \int_{(0, t_G]} A^G F(v) \, dG(v) - F(0) = - \lim_{t \uparrow t_G} F(t) \overline{G} (t).$$

To establish (3.6), fix $0 \leq s < t_G$. Using $M_{t_G} = A^G F(\gamma) \text{ P-a.s.}$, the fact that
\{s < \gamma\} is an atom of \(\mathcal{F}_s^\gamma\), \(\Delta G(t_G) = 0\) and (2.4) gives

\[
E [M_{tg} \mid \mathcal{F}_s] = E [\mathcal{A}^G F(\gamma) \mid s < \gamma] \mathbf{1}_{\{s < \gamma\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{s \geq \gamma\}}
\]

\[
= \int_{(s,t_G)} \mathcal{A}^G F(u) dG(u) \frac{1}{1 - G(s)} \mathbf{1}_{\{s < \gamma\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{s \geq \gamma\}}
\]

\[
= \int_{(s,t_G)} \mathcal{A}^G F(u) dG(u) \frac{1}{1 - G(s)} \mathbf{1}_{\{s < \gamma\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{s \geq \gamma\}}
\]

\[
= \frac{F(s)\overline{G}(s) - \lim_{t\uparrow t_G} F(t)\overline{G}(t)}{\overline{G}(s)} \mathbf{1}_{\{s < \gamma\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{s \geq \gamma\}}
\]

\[
= \frac{F(s)\overline{G}(s) + \Delta \mu}{\overline{G}(s)} \mathbf{1}_{\{s < \gamma\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{s \geq \gamma\}}
\]

\[
= M_s + \frac{\Delta \mu}{\overline{G}(s)} \mathbf{1}_{\{s < \gamma\}} \text{ P-a.s.}
\]

The remaining claims are straightforward. \(\square\)

The next result shows that if \(M\) is integrable on \([0, t_G]\), then it can be naturally decomposed into its initial value \(M_0\) and the difference of two supermartingales starting at 0, i.e., it is a quasimartingale (cf. [44, Theorem VI.40]).

**Corollary 3.8.** Let \(M = \mathcal{M}^G F\) be integrable on \([0, t_G]\). Set \(M^\uparrow := \mathcal{M}^G (F^\uparrow)\) and \(M^\downarrow := \mathcal{M}^G (F^\downarrow)\). Then \(M^\uparrow\) and \(M^\downarrow\) are \(\mathbb{F}^\gamma\)-supermartingales on \([0, t_G]\), start at 0, and satisfy

\[
M = M_0 + M^\uparrow - M^\downarrow \text{ P-a.s.}
\]

**Proof.** It follows from Lemmas 2.7 and 3.4 that \(M^\uparrow\) and \(M^\downarrow\) are well defined, integrable on \([0, t_G]\) and start at 0. Nonnegativity of \(F^\uparrow, F^\downarrow\) and Lemma 3.7 give the supermartingale property. The decomposition result follows from the definitions of \(M, M^\uparrow\) and \(M^\downarrow\), and from (2.5) and (2.6). \(\square\)

### 3.3 \(H^1\)-martingale property on \([0, t_G]\)

If \(M\) is integrable on \([0, t_G]\) and \(\Delta G(t_G) > 0\), then \(M\) is automatically an \(H^1\)-martingale on \([0, t_G]\) by Theorem 3.5. If \(\Delta G(t_G) = 0\), however, the situation is more delicate.

**Lemma 3.9.** Suppose that \(\Delta G(t_G) = 0\). Then the following are equivalent:

(a) \(M\) is an \(H^1\)-martingale on \([0, t_G]\) (in the sense that there exists a filtration \(\mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty]}\) of \(\mathcal{F}\) such that \(M\) is an \(H^1\)-\(\mathbb{F}\)-martingale on \([0, t_G]\), cf. Definition B.1).

(b) \(M\) is an \(H^1\)-\(\mathbb{F}^\gamma\)-martingale on \([0, t_G]\).

(c) \(F(\cdot) \in L^1(dG)\) and \(\mathcal{A}^G F \in L^1(dG)\).
IV.3 Classification of single jump local martingales 139

Proof. “(b) ⇒ (a)”: This is trivial.

“(a) ⇒ (c)”: Lemma 3.4 yields \( A^G F \in L^1(dG) \). Moreover, using the definitions of \( \zeta^F \) and \( M \) in (3.2) and (3.3) and the fact that \( \Delta G(t_G) = 0 \), we obtain

\[
\int_{(0,t_G)} |F(v-)| \, dG(v) \leq \int_{(0,t_G)} \sup_{t \leq v} |F(t)| \, dG(v) = \int_{(0,t_G)} \sup_{t \leq v} |F^F(t, v)| \, dG(v)
\]

\[
\leq \int_{(0,t_G)} \sup_{t \in [0,t_G]} |\zeta^F(t, v)| \, dG(v) = E \left[ \sup_{t \in [0,t_G]} |M_t| \right] < \infty.
\]

“(c) ⇒ (b)”: As \( M \) is a local \( F^\gamma \)-martingale on \([0, t_G] \) by Theorem 3.5 (a), it suffices to show that \( E \left[ \sup_{t \in [0,t_G]} |M_t| \right] < \infty \). First, note that

\[
|F(t)| \leq |F(0)| + |F|(t) \leq |F(0)| + |F|(v-)
\]

for \( 0 \leq t < v < t_G \). Thus, for \( t \in [0, t_G] \) and \( v \in (0, t_G) \),

\[
|\zeta^F(t, v)| = |F(t)|1_{[t<v]} + |A^G F(v)|1_{[v=t]} \leq |F(0)| + |F|(v-) + |A^G F(v)|.
\]

Using this together with the definition of \( M \) in (3.3), the fact that \( \Delta G(t_G) = 0 \), and the implication “(a) ⇒ (c)” of Lemma 2.8, we get

\[
E \left[ \sup_{t \in [0,t_G]} |M_t| \right] = \int_{(0,t_G)} \sup_{t \in [0,t_G]} |\zeta^F(t, v)| \, dG(v)
\]

\[
\leq |F(0)| + \int_{(0,t_G)} |F|(v-) \, dG(v) + \int_{(0,t_G)} |A^G F(v)| \, dG(v) < \infty.
\]

As a corollary, we obtain a criterion which allows us to construct (uniformly integrable) martingales that are not \( H^1 \)-martingales. A concrete example is given in Example 3.15 below.

Corollary 3.10. Suppose that \( \Delta G(t_G) = 0 \). Assume that \( (A^G F)^- \) or \( (A^G F)^+ \)

belongs to \( L^1(dG) \) and that \( \lim_{t \uparrow t_G} F(t)G(t) = 0 \), but that \( F(\cdot -) \not\in L^1(dG) \). Then \( M \) is an \( F^\gamma \)-martingale but not an \( H^1 \)-martingale on \([0, t_G] \).

Proof. Lemmas 2.5, 3.4 and 3.7 show that \( M \) is an \( F^\gamma \)-martingale on \([0, t_G] \). That \( M \) fails to be an \( H^1 \)-martingale on \([0, t_G] \) follows from Lemma 3.9.

Remark 3.11. Even if \( M \) is an \( H^1 \)-martingale on \([0, t_G] \), one may have \( F \not\in L^1(dG) \). Indeed, define \( G : \mathbb{R} \to [0, 1] \) by \( G(t) = \frac{1}{e-1} \sum_{k=1}^\infty \frac{1}{k!} 1_{[k,\infty)}(t) \) and \( F : [0, \infty) \to \mathbb{R} \) by \( F(t) = \sum_{k=1}^\infty (k-1)! 1_{[k,k+1)}(t) \). Then \( t_G = \infty \) and for \( n \in \mathbb{N} \),

\[
\int_{(0,\infty)} |F(v-)| \, dG(v) = \sum_{n=1}^\infty F(n-1) \Delta G(n) = \frac{1}{e-1} \sum_{n=2}^\infty \frac{1}{n(n-1)} < \infty,
\]

\[
\int_{(0,\infty)} |F(v)| \, dG(v) = \sum_{n=1}^\infty F(n) \Delta G(n) = \frac{1}{e-1} \sum_{n=1}^\infty \frac{1}{n} = \infty.
\]
So $F(\cdot) \in L^1(dG)$ but $F \not\in L^1(dG)$. It remains to show that $M$ is an $H^1$-martingale. In view of Lemma 3.9, the fact that $F(\cdot) \in L^1(dG)$ and the definition of $A^G$, this boils down to proving that

$$
\int_{(0,\infty)} \left| \frac{dF}{dG}(v) \overline{G}(v) \right| \, dG(v) = \sum_{n=1}^{\infty} \left| \frac{\Delta F(n)}{\Delta G(n)} \right| \overline{G}(n) \Delta G(n) = \sum_{n=1}^{\infty} |\Delta F(n)| \overline{G}(n)
$$

is finite. But this is true, because for $n \in \mathbb{N}$,

$$
|\Delta F(n)| = F(n) - F(n-1) \leq F(n) = (n-1)!,
$$

$$
\overline{G}(n) = 1 - G(n) = \frac{1}{e-1} \sum_{k=n+1}^{\infty} \frac{1}{k!} \leq \frac{e}{e-1} \frac{1}{(n+1)!}.
$$

### 3.4 Summary and examples

The flow chart in Figure 3.1 summarises the results of the previous sections. It gives the conditions one has to check in order to determine the (local) martingale properties of $M^G$. In this section, we give examples for four of the five cases one can end up with; examples for the fifth case that $M^G$ is an $H^1$-martingale are easy to find (take, e.g., $F^loc \ll G$ bounded with $A^G$ bounded).

**Example 3.12** (A process which fails to be a semimartingale). Let $G : \mathbb{R} \to [0,1]$ be given by $G(t) = \frac{1}{2} 1_{[0,1)}(t) + 1_{[1,\infty)}(t)$, i.e., the law of the jump time $\gamma$ is a mixture of a uniform distribution on $(0,1)$ and a Dirac measure at 1. In particular, $t_G = 1$ and $\Delta G(t_G) = \frac{1}{2}$. The idea is to choose any $F^loc \ll G$ that is of infinite variation on $[0,t_G]$. Then

$$
P[M_t = F(t), t \in [0,t_G]] \geq P[\gamma = t_G] = \Delta G(t_G) = \frac{1}{2}
$$

by the definition of $M^G$, and Lemma B.6 asserts that $M^G$ fails to be a semimartingale on $[0,t_G]$. (Alternatively, one can use Lemma 2.6 to infer that $A^G \not\in L^1(dG)$ and then apply Lemma 3.4 and Theorem 3.5 (b).) A concrete example is given by $F(t) = 1_{[0,1)}(t) \sin \frac{1}{1-t}$, $t \geq 0$.

For the remaining examples, let $G : \mathbb{R} \to [0,1]$ be given by

$$
G(t) = t 1_{[0,1)}(t) + 1_{[1,\infty)}(t),
$$

i.e., $\gamma$ is uniformly distributed on $(0,1)$. In particular, $t_G = 1$ and $\Delta G(t_G) = 0$. Then for each $F^loc \ll G$, $M^G$ is a local martingale on $[0,1]$ by Theorem 3.5 (a).

**Example 3.13** (A strict local martingale that fails to be integrable). The idea is to find an $F \ll G$ such that $M^G$ is not integrable or, equivalently by Lemma 3.4, $A^G \not\in L^1(dG)$. A concrete example is given by

$$
F(t) = 1_{[0,1)}(t) \sin \frac{1}{1-t}, \quad t \geq 0.
$$
IV.3 Classification of single jump local martingales

Then $F \ll G$, $A^G F(v) = \sin \frac{1}{1-v} - \frac{1}{1-v} \cos \frac{1}{1-v}$ dG-a.e. on $(0, 1)$, and one can show that $A^G F \notin L^1(dG)$. Indeed, it suffices to show that

$$\int_0^1 \left| \frac{1}{1-v} \cos \frac{1}{1-v} \right| dv = \infty,$$

or equivalently, $\int_1^\infty \frac{|\cos x|}{x} dx = \infty$. But for each $k \in \mathbb{N}$, $\int_{(2k-1)\frac{\pi}{2}}^{(2k+1)\frac{\pi}{2}} \left| \frac{\cos x}{x} \right| dx \geq \frac{1}{(2k+1)\frac{\pi}{2}} \int_{(2k-1)\frac{\pi}{2}}^{(2k+1)\frac{\pi}{2}} |\cos x| dx = \frac{2}{(2k+1)^{\frac{\pi}{2}}}$.
and summing over \(k\) leads to an infinite series on the right-hand side.

**Example 3.14** (An integrable strict local martingale). Here the idea is to find \(F \ll G\) with \(F(0) > 0\) and \(A^G F = 0\) \(dG\)-a.e on \((0, 1)\). This means that \(\mathcal{M}^G F\) starts at \(F(0) > 0\) at time 0 and ends up at zero at time 1 \(P\)-a.s. Therefore, \(\mathcal{M}^G F\) cannot be a martingale on \([0, 1]\). A simple example is given by \(F(t) = \frac{1}{1-t} 1_{(0,1)}(t),\) \(t \geq 0\). Then \(F \ll G\),

\[
\Delta \mu = - \lim_{t \uparrow 1} F(t)G(t) = - \lim_{t \uparrow 1} \frac{1-t}{1-t} = -1 \quad \text{and} \quad A^G F = 0 \quad dG\text{-a.e. on } (0, 1).
\]

Thus, by Lemmas 3.4 and 3.7, \(\mathcal{M}^G F\) is an integrable strict local martingale and a supermartingale.

**Example 3.15** (A martingale that fails to be in \(H^1\)). Finding \(F \ll G\) such that \(\mathcal{M}^G F\) is a martingale on \([0, 1]\) that fails to be in \(H^1\) is a bit tricky: if \(F\) grows too slowly, then \(\mathcal{M}^G F\) will be an \(H^1\)-martingale, but if \(F\) grows too quickly, then \(\mathcal{M}^G F\) will be a strict local martingale. The idea is to find an \(F \ll G\) that satisfies the assumptions of Corollary 3.10. Define \(F : [0, \infty) \to \mathbb{R}\) by

\[
F(t) = \frac{1}{(1-t)\log \frac{1}{1-t}} 1_{(0,1)}(t).
\]

Then \(F \ll G\) is nonnegative and increasing on \([0, 1)\), and by monotone convergence,

\[
\int_{(0,1)} |F(v-)| \, dG(v) = \lim_{t \uparrow 1} \int_{0}^{t} F(v) \, dv = \lim_{t \uparrow 1} \left[ \log \left( \frac{\log e}{1-v} \right) \right]_{0}^{t} = \lim_{t \uparrow 1} \log \left( \frac{e}{1-t} \right) = \infty.
\]

Moreover,

\[
A^G F(v) = \frac{1}{(1-v)(\log \frac{e}{1-v})^2} \, dG\text{-a.e. on } (0, 1).
\]

Thus, \(A^G F\) is nonnegative, and therefore \((A^G F)^- \in L^1(dG)\). Finally,

\[
-\Delta \mu = \lim_{t \uparrow 1} F(t)G(t) = \lim_{t \uparrow 1} \frac{1}{\log \frac{1}{1-t}} = 0.
\]

Hence, by Corollary 3.10, \(\mathcal{M}^G F\) is a (uniformly integrable) martingale on \([0, 1]\) but not in \(H^1\).

### 4 A counter-example in stochastic integration

In this section, we consider the following problem from stochastic integration: Does there exist a pair \((M, H)\) where \(M = (M_t)_{t \in [0, 1]}\) is a (true) martingale and \(H = (H_t)_{t \in [0, 1]}\) an integrand with \(0 \leq H \leq 1\) such that the stochastic integral
Using dominated convergence, on a martingale if $H$ is bounded and $M$ is an $H^1$-martingale. Nevertheless, the answer to the above question is positive as is shown in [122, Corollaire VI.21] by an abstract existence proof using the Baire category theorem. It took, however, 30 years until a quite ingenious concrete example was published by Cherny [31]. He constructed the martingale integrator $30$ years until a quite ingenious concrete example was published by Cherny [31].

Starting with an example of this kind, which works for the same as we use in Theorem 4.2 below. The goal of this section is to provide an abstract existence proof using the Baire category theorem. It took, however, 30 years until a quite ingenious concrete example was published by Cherny [31].

Proposition 4.1. Let $F \ll G$ be such that $\mathcal{M}^G F$ is a local martingale on $[0, t_G]$. Moreover, let $J : [0, \infty) \to \mathbb{R}$ be a bounded Borel-measurable function. Define $F^J : [0, \infty) \to \mathbb{R}$ by $F^J(t) = \int_{[0,t]} J(u) \, dF(u)$ for $t \in [0, t_G)$ and $F^J(t) = 0$ for $t \geq t_G$. Then $F^J \ll G$ and $J \cdot \mathcal{M}^G F = \mathcal{M}^G F^J$ $P$-a.s.

Proof. We only establish the result for the case $\Delta G(t_G) = 0$, which corresponds to the setting of Theorem 4.2 below. Since $J$ is bounded and $F \ll G$, $J \in L^1_{\text{loc}}(dF)$, so that $F^J$ is well defined. Clearly, $F^J \ll G$ with local density $\frac{dF^J}{dG} = J \frac{dF}{dG}$. Fix $t \in [0, t_G)$. Using the definition of $\mathcal{M}^G F$, on $\{t < \gamma\}$,

$$J \cdot \mathcal{M}^G F_t = \int_{[0,t]} J(u) \, dF(u) = F^J(t) = \mathcal{M}^G F^J \quad P\text{-a.s.} \quad (4.1)$$

Using dominated convergence, on $\{t \geq \gamma\} \cap \{\gamma < t_G\}$,

$$J \cdot \mathcal{M}^G F_t = \int_{(0,\gamma]} J(u) \, d\mathcal{M}^G F_u + J(\gamma) \Delta \mathcal{M}^G F,$$

$$= \int_{(0,\gamma]} J(u) \, dF(u) + J(\gamma) \left( A^G F(\gamma) - F(\gamma-) \right)$$

$$= F^J(\gamma-) - J(\gamma) \frac{dF}{dG}(\gamma) G(\gamma) = F^J(\gamma-) - \frac{dF^J}{dG}(\gamma) G(\gamma)$$

$$= A^G F^J(\gamma) = \mathcal{M}^G_t F^J \quad P\text{-a.s.} \quad (4.2)$$

$H \cdot M$ is a strict local martingale? By the BDG inequality, $H \cdot M$ is again a martingale if $H$ is bounded and $M$ is an $H^1$-martingale. Nevertheless, the existence proof using the Baire category theorem. It took, however, 30 years until a quite ingenious concrete example was published by Cherny [31]. He constructed the martingale integrator $30$ years until a quite ingenious concrete example was published by Cherny [31].

Moreover, let $J : [0, \infty) \to \mathbb{R}$ be a bounded Borel-measurable function. Define $F^J : [0, \infty) \to \mathbb{R}$ by $F^J(t) = \int_{[0,t]} J(u) \, dF(u)$ for $t \in [0, t_G)$ and $F^J(t) = 0$ for $t \geq t_G$. Then $F^J \ll G$ and $J \cdot \mathcal{M}^G F = \mathcal{M}^G F^J$ $P$-a.s.

Proof. We only establish the result for the case $\Delta G(t_G) = 0$, which corresponds to the setting of Theorem 4.2 below. Since $J$ is bounded and $F \ll G$, $J \in L^1_{\text{loc}}(dF)$, so that $F^J$ is well defined. Clearly, $F^J \ll G$ with local density $\frac{dF^J}{dG} = J \frac{dF}{dG}$. Fix $t \in [0, t_G)$. Using the definition of $\mathcal{M}^G F$, on $\{t < \gamma\}$, $J \cdot \mathcal{M}^G F_t = \int_{[0,t]} J(u) \, dF(u) = F^J(t) = \mathcal{M}^G F^J \quad P\text{-a.s.} \quad (4.1)$

Using dominated convergence, on $\{t \geq \gamma\} \cap \{\gamma < t_G\}$,

$$J \cdot \mathcal{M}^G F_t = \int_{(0,\gamma]} J(u) \, d\mathcal{M}^G F_u + J(\gamma) \Delta \mathcal{M}^G F,$$

$$= \int_{(0,\gamma]} J(u) \, dF(u) + J(\gamma) \left( A^G F(\gamma) - F(\gamma-) \right)$$

$$= F^J(\gamma-) - J(\gamma) \frac{dF}{dG}(\gamma) G(\gamma) = F^J(\gamma-) - \frac{dF^J}{dG}(\gamma) G(\gamma)$$

$$= A^G F^J(\gamma) = \mathcal{M}^G_t F^J \quad P\text{-a.s.} \quad (4.2)$$
As \( \Delta G(t_G) = 0, \gamma < t_G \) P-a.s. and (4.1) together with (4.2) completes the proof.

Throughout the rest of this section, let \( G : \mathbb{R} \to [0, 1] \) be given by
\[
G(t) = t \mathbf{1}_{[0,1]}(t) + \mathbf{1}_{[1,\infty)}(t),
\]
i.e., \( \gamma \) is uniformly distributed on \((0, 1)\). In particular, \( t_G = 1 \) and \( \Delta G(t_G) = 0 \).

Moreover, we always consider the filtration \( \mathbb{F}^\gamma \) introduced in Section 3.

**Theorem 4.2.** Let \( F \ll G \) be such that \( \mathcal{M}^G F \) is a martingale on \([0, 1]\) which is not in \( H^1 \) (cf. Corollary 3.10). Suppose that \( F \) is nondecreasing on \((0, 1)\) and satisfies \( F(0) = 0 \). Define the deterministic integrand \( J = (J_t)_{t \in [0,1]} \) by
\[
J = \sum_{n=0}^{\infty} \mathbf{1}_{[1-2^{-2n}, 1-2^{-(2n+1)}]}.
\]

Then the stochastic integral \( J \bullet \mathcal{M}^G F \) is a strict local martingale on \([0, 1]\).

A concrete example for a function \( F \) satisfying all conditions of Theorem 4.2 is given in Example 3.15. The choice of \( J \) is inspired by Cherny [31].

**Proof.** Proposition 4.1 yields \( F^J \ll G \) and \( J \bullet \mathcal{M}^G F = \mathcal{M}^G F^J \) P-a.s. By Theorem 3.5, \( \mathcal{M}^G F^J \) is a local \( \mathbb{F}^\gamma \)-martingale on \([0, 1]\). It will be a strict local martingale if \( \mathcal{M}^G_F 1^J \) is not integrable, which holds by Lemma 3.4 if and only if
\[
\int_{(0,1)} |A^G F^J(t)| \, dG(t) = \int_0^1 \left| F^J(t) - J(t) \frac{dF}{dG}(t)(1 - G(t)) \right| \, dt = \infty. \tag{4.3}
\]

Since \( \mathcal{M}^G F \) is a martingale on \([0, 1]\), \( A^G F \in L^1(dG) \) by Lemma 3.4 and so
\[
\int_0^1 \left| J(t) F(t) - J(t) \frac{dF}{dG}(t)(1 - G(t)) \right| \, dt = \int_0^1 J(t) |A^G F(t)| \, dt \leq \int_0^1 |A^G F(t)| \, dt = \int_{(0,1)} |A^G F(t)| \, dG(t) < \infty.
\]

In order to establish (4.3), it thus suffices to show that
\[
\int_0^1 \left| F^J(t) - J(t) F(t) \right| \, dt = \int_0^1 \left( (1 - J(t)) F^J(t) + J(t) F^{1-J}(t) \right) \, dt = \infty. \tag{4.4}
\]

For \( n \in \mathbb{N}_0 \), set \( t_n := 1 - 2^{-n} \) and \( t_{-1} := -1 \). We note that
\[
t_m + t_{m+1} = \frac{1}{2}(t_m - t_{m-1}) = \frac{1}{3}(t_m + t_{m+1} - t_{m-1})
\]
for $m \in \mathbb{N}_0$ and that $F^J$ and $F^{1-J}$ are constant on $\{J = 0\}$ and $\{J = 1\}$, respectively. Using this, the nonnegativity of $F^{1-J}$ and $F$, the fact that $F = F^J + F^{1-J}$ is nondecreasing on $(0, 1)$, we obtain

$$
\int_0^1 \left( (1 - J(t))F^J(t) + J(t)F^{1-J}(t) \right) dt
= \sum_{n=0}^{\infty} \left( F^J(t_{2n+1})(t_{2n+2} - t_{2n+1}) + F^{1-J}(t_{2n+1})(t_{2n+1} - t_{2n}) \right)
\geq \frac{1}{2} \sum_{n=0}^{\infty} F(t_{2n+1})(t_{2n+1} - t_{2n}) = \frac{1}{6} \sum_{n=0}^{\infty} F(t_{2n+1})(t_{2n+1} - t_{2n-1})
\geq \frac{1}{6} \int_{(0,1)} F(t) \, dG(t).
$$

(4.5)

Since $M^G F$ is not in $H^1$ by assumption, but $A^G F \in L^1(dG)$, we get $F(\cdot -) \not\in L^1(dG)$ by Lemma 3.9. This implies $F \not\in L^1(dG)$ because $F$ is nondecreasing. Combining this with (4.5) shows that (4.4) holds true. 

\section{Elements of real analysis}

\textbf{Definition A.1.} Let $T \in (0, \infty]$. A function $L : [0, \infty) \to \mathbb{R}$ is called a left-continuous step function on $[0, T)$ if it is of the form

$$
L = \sum_{j=1}^{k} a_j \mathbf{1}_{(t_{j-1}, t_j]},
$$

where $k \in \mathbb{N}$, $0 \leq t_0 < t_1 < \cdots < t_k < T$ and $a_j \in \mathbb{R}$, $j = 1, \ldots, k$. If $F : [0, T] \to \mathbb{R}$ is any other function, we define the elementary integral of $L$ with respect to $F$ on $(0, T]$ by

$$
\int_{(0,T]} L(t) \, dF(t) := \sum_{j=1}^{k} a_j \left( F(t_j) - F(t_{j-1}) \right).
$$

The following result is an easy exercise in analysis.

\textbf{Lemma A.2.} Let $T \in (0, \infty]$ and $F : [0, T] \to \mathbb{R}$ be a function which is of infinite variation on $[0, T]$. Then for each $n \in \mathbb{N}$, there exists a left-continuous step function $L_n : [0, \infty) \to \mathbb{R}$ on $[0, T]$ with $\sup_{t \geq 0} |L_n(t)| \leq 1/n$ and

$$
\int_{(0,T]} L_n(t) \, dF(t) \geq 1.
$$
B Elements of (semi-)martingale theory

Throughout this section, we fix a probability space \((\Omega, \mathcal{A}, P)\) and a time horizon \(T^* \in (0, \infty]\). Moreover, \(I \subset [0, T^*]\) is assumed to be an interval of the form \([0, T]\) or \([0, T^*)\) for some \(T \in (0, T^*)\).

**Definition B.1.** Fix a stochastic process \(X = (X_t)_{t \in [0, T^*]}\) and a filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \in [0, \infty]}\) of \(\mathcal{A}\).

(a) \(X\) is of **finite variation on** \(I\) if for \(P\)-a.e. \(\omega\), the function \(t \mapsto X_t(\omega)\) is of finite variation and càdlàg on \(I\).

(b) \(X\) is called **\(\mathcal{F}\)-adapted on** \(I\) if \(X_t\) is \(\mathcal{F}_t\)-measurable for each \(t \in I\).

(c) \(X\) is called **integrable on** \(I\) if \(E[|X_t|] < \infty\) for each \(t \in I\).

(d) \(X\) is an **\(\mathcal{F}\)-(sub/super)martingale on** \(I\) if for \(P\)-a.e. \(\omega\), the function \(t \mapsto X_t(\omega)\) is càdlàg on \(I\) and \(X\) is \(\mathcal{F}\)-adapted on \(I\), integrable on \(I\), and satisfies the \(\mathcal{F}\)-(sub/super)martingale property on \(I\), i.e.,

\[
E[X_t | \mathcal{F}_s] (\geq, \leq) = X_s \quad \text{for all } s \leq t \text{ in } I.
\]

\(X\) is a (sub/super)martingale on \(I\) if there exists a filtration \(\mathcal{F}' = (\mathcal{F}'_t)_{t \in [0, \infty]}\) of \(\mathcal{A}\) such that \(X\) is an \(\mathcal{F}'\)-(sub/super)martingale on \(I\).

(e) \(X\) is an **\(H^1\)-\(\mathcal{F}\)-martingale on** \(I\) if \(X\) is an \(\mathcal{F}\)-martingale on \(I\) and

\[
E\left[\sup_{t \in I} |X_t|\right] < \infty.
\]

\(X\) is an **\(H^1\)-martingale** if there exists a filtration \(\mathcal{F}' = (\mathcal{F}'_t)_{t \in [0, \infty]}\) of \(\mathcal{A}\) such that \(X\) is an \(\mathcal{H}^1\)-\(\mathcal{F}'\)-martingale on \(I\).

(f) Let \(\mathcal{G} = (\mathcal{G}_t)_{t \in [0, \infty]}\) be a subfiltration of \(\mathcal{F}\). \(X\) is a **\(\mathcal{G}\)-local \(\mathcal{F}\)-martingale on** \(I\) if there exists an increasing sequence of \(\mathcal{G}\)-stopping times \((\tau_n)_{n \in \mathbb{N}}\) with values in \(I \cup \{T\}\) such that for each \(n \in \mathbb{N}\), \(X^\tau_n\) is an \(\mathcal{F}\)-martingale on \(I\), and

(i) in case of \(T \not\in I\), \(\lim_{n \to \infty} \tau_n = T\) \(P\)-a.s.,

(ii) in case of \(T \in I\), \(\lim_{n \to \infty} P[\tau_n = T] = 1\).

In both cases, the sequence \((\tau_n)_{n \in \mathbb{N}}\) is called a **\(\mathcal{G}\)-localising sequence (for \(X\))**. An \(\mathcal{F}\)-local \(\mathcal{F}\)-martingale on \(I\) is simply called a **local \(\mathcal{F}\)-martingale on** \(I\). \(X\) is a **local martingale on** \(I\) if there exists a filtration \(\mathcal{F}' = (\mathcal{F}'_t)_{t \in [0, \infty]}\) of \(\mathcal{A}\) such that \(X\) is a local \(\mathcal{F}'\)-martingale on \(I\). \(X\) is a **strict local martingale on** \(I\) if it is a local martingale on \(I\), but not a martingale on \(I\).
(g) $X$ is an $\mathbb{F}$-semimartingale on $I$ if there are processes $M = (M_t)_{t \in I}$ and $A = (A_t)_{t \in I}$ such that $X = M + A$, where $M$ is local $\mathbb{F}$-martingale on $I$ and $A$ is $\mathbb{F}$-adapted on $I$ and of finite variation on $I$. $X$ is a semimartingale on $I$ if there exists a filtration $\mathbb{F}' = (\mathcal{F}_t')_{t \in [0, \infty]}$ of $\mathcal{A}$ such that $X$ is an $\mathbb{F}'$-semimartingale on $I$.

Whenever we drop the qualifier “on $I$” in the above notations it is understood that $I = [0, T^\ast]$.

The following result is a standard exercise in probability theory.

**Proposition B.2.** Fix a stochastic process $X = (X_t)_{t \in [0, \infty]}$ and a filtration $(\mathcal{F}_t)_{t \in [0, \infty]}$ of $\mathcal{A}$.

(a) Let $\mathbb{F}' = (\mathcal{F}'_t)_{t \in [0, \infty]}$ be a subfiltration of $\mathbb{F}$ such that $X$ is $\mathbb{F}'$-adapted on $I$. If $X$ is an $\mathbb{F}$-(sub/super)martingale on $I$, then $X$ is also an $\mathbb{F}'$-(sub/super) martingale on $I$.

(b) Let $\mathbb{G} = (\mathcal{G}_t)_{t \in [0, \infty]}$, $\mathbb{G}' = (\mathcal{G}'_t)_{t \in [0, \infty]}$, $\mathbb{F}' = (\mathcal{F}'_t)_{t \in [0, \infty]}$ be subfiltrations of $\mathbb{F}$ satisfying $\mathcal{G}_t \subset \mathcal{G}'_t \subset \mathcal{F}'_t \subset \mathcal{F}_t$ for each $t \in I$ and such that $X$ is $\mathbb{F}'$-adapted on $I$. If $X$ is a $\mathbb{G}$-local $\mathbb{F}$-martingale on $I$, then it is also a $\mathbb{G}'$-local $\mathbb{F}'$-martingale on $I$.

**Definition B.3.** Let $T \in (0, \infty]$ and $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty]}$ be a filtration of $\mathcal{A}$. A stochastic process $L = (L_t)_{t \geq 0}$ is called an $\mathbb{F}$-elementary process on $[0, T)$ if there exist $\mathbb{F}$-stopping times $\tau_0, \ldots, \tau_n$ with $0 \leq \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n < T$ and bounded random variables $A_1, \ldots, A_n$ with each $A_j$ being $\mathcal{F}_{\tau_{j-1}}$-measurable such that

$$L = \sum_{j=1}^n A_j 1_{[\tau_{j-1}, \tau_j]}.$$ 

Note that for each $\omega \in \Omega$, the path $L(\omega)$ is a left-continuous step function on $[0, T)$.

**Definition B.4.** Let $T \in (0, \infty)$. Let $X = (X_t)_{t \in [0, \infty]}$ be a stochastic process, $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty]}$ a filtration of $\mathcal{A}$ with respect to which $X$ is adapted on $[0, T]$ and $L = (L_t)_{t \geq 0}$ an $\mathbb{F}$-elementary process on $[0, T)$. Define the $\mathbb{F}$-elementary stochastic integral of $L$ with respect to $X$ on $[0, T]$ by

$$\left( \int_{[0,T]} L_t \, dX_t \right)(\omega) := \int_{[0,T]} L_t(\omega) \, dX_t(\omega).$$

**Lemma B.5.** Let $T \in (0, \infty)$. Let $X = (X_t)_{t \in [0, \infty]}$ be a stochastic process and $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty]}$ a filtration of $\mathcal{A}$. If $X$ is an $\mathbb{F}$-semimartingale on $[0, T]$, then for all $\mathbb{F}$-elementary processes $(L^n_t)_{t \in [0, \infty]}$ on $[0, T)$ satisfying $\lim_{n \to \infty} \sup_{t \geq 0, \omega \in \Omega} |L^n_t(\omega)| = 0$,

$$P-\lim_{n \to \infty} \int_{[0,T]} L^n_t \, dX_t = 0.$$
Proof. This follows immediately from [154, Proposition 7.1.7] which is stronger than our result. Note that [154] work with general filtrations which need not satisfy the usual conditions.

Lemma B.6. Let $X = (X_t)_{t \in [0, \infty]}$ be a right-continuous stochastic process and $T \in (0, \infty)$. Suppose that there exists a deterministic function $F : [0, T] \to \mathbb{R}$ such that $F$ is of infinite variation on $[0, T]$ and $P[X(t) = F(t), t \in [0, T)] =: \epsilon > 0$. Then $X$ is not a semimartingale on $[0, T]$.

Proof. Seeking a contradiction, suppose there exists a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0, \infty]}$ of $\mathcal{A}$ with respect to which $X$ is a semimartingale on $[0, T]$. Since $F$ is of infinite variation on $[0, T]$, by Lemma A.2, for each $n \in \mathbb{N}$ there exists a left-continuous step function $L_n : [0, \infty) \to \mathbb{R}$ on $[0, T)$ with $\sup_{t \geq 0} |L_n(t)| \leq 1/n$ and

$$\int_{[0,T]} L_n(t) \, dF(t) \geq 1.$$ 

For each $n \in \mathbb{N}$, define the $\mathcal{F}$-elementary process $(L^n_t)_{t \in [0, \infty)}$ on $[0, T)$ by $L^n_t(\omega) := L_n(t)$. Then $\lim_{n \to \infty} \sup_{t \geq 0, \omega \in \Omega} |L^n_t(\omega)| = 0$, but

$$P \left[ \int_{[0,T]} L^n_t \, dX_t \geq 1 \right] \geq P \left[ \int_{[0,T]} L_n(t) \, dF(t) \geq 1, X(t) = F(t), t \in [0, T) \right] = P[X(t) = F(t), t \in [0, T)] = \epsilon \quad \text{for all } n \in \mathbb{N}.$$ 

This implies in particular that $(\int_{[0,T]} L_n(t) \, dX_t)_{n \in \mathbb{N}}$ does not converge to 0 in probability. Hence, $X$ fails to be an $\mathcal{F}$-semimartingale on $[0, T]$ by Lemma B.5, and we arrive at a contradiction. 

$\square$
Chapter V

Optimal Investment in a Black–Scholes Model with a Bubble

1 Introduction

This chapter studies the problem of maximising expected utility from terminal wealth for a power utility investor in the presence of an asset price bubble. The two main objectives are to determine the optimal strategy and the certainty equivalent of trading in this market, and to understand how the model parameters and the investor’s relative risk aversion affect the qualitative behaviour of the solution.

Model description and main features. We propose an extension of the classical Black–Scholes model by adding an extra positive excess return rate which is compensated by a negative jump at a random time, representing the bursting of the bubble. In other words, we complement the returns process $\mu t + \sigma W_t$, a scaled Brownian motion with a positive drift, by adding an independent martingale of finite variation with a single negative jump. Up to regularity assumptions, the distribution of the jump time as well as the extra excess return rate can be chosen arbitrarily. The jump size is then determined by the martingale condition. The main features of the model are:

(a) It is flexible enough to include specifications such that the asset price process becomes a strict local martingale (i.e., a local martingale that is not a martingale) under a large class of equivalent local martingale measures. This is interesting as strict local martingale models reflect the disparity between the price of an asset and its “fundamental value” that is typically associated with an asset price bubble (cf. the survey article by Protter [139]).

(b) The Johansen–Ledoit–Sornette (JLS) financial bubble model [150] is contained as a special case (cf. Section 4). In particular, our model provides a solid mathematical footing for the JLS model and allows to analyse to what extent it can be embedded in the strict local martingale framework. We find that although the original JLS model is not compatible with the strict
local martingale framework, there are strict local martingale modifications that preserve the key assumption of the JLS model that the hazard rate of the jump time follows a so-called log-periodic power law (Theorem 4.2).

(c) The extra positive drift and the negative jump describe—in an idealised way—a main empirical feature of a bubble, which is a strong upward trend followed by a sharp decline when the bubble bursts. It may thus be used as an instructive stylised example to study qualitative effects of financial bubbles.

(d) Although the additional random factor renders our model incomplete, it is still tractable enough to allow (semi-)explicit calculations. In particular, it permits a semi-explicit solution to an optimal investment problem whose solution can display sometimes surprising effects (see the following paragraph).

Solution to the optimal investment problem. We provide an explicit formula for the optimal strategy in terms of the solution to an integral equation (or to a first-order ODE); see Theorem 5.1. Moreover, we decompose the optimal strategy into myopic and hedging demands (Theorem 5.3). The myopic demand is the strategy followed by an investor who at each point in time acts as if he had an infinitesimally short investment horizon. In other words, such an investor bases his decisions solely on the local behaviour of the asset price process, irrespective of any possible changes in the investment opportunity set (cf. [120]) at future times. The hedging demand is simply the difference between the optimal strategy and the myopic demand; it describes how investors take into account that the investment opportunity set can change up to their time horizon. We show that in our model, the myopic demand, expressed as fraction of the investor’s wealth invested in the stock, always lies between 0 and the “Merton proportion” (the optimal fraction of wealth held in the stock by an investor with constant relative risk aversion in the Black–Scholes model). More interestingly, we show that the sign of the hedging demand only depends on the investor’s relative risk aversion $p$. If risk aversion is high ($p \geq 1$), then the hedging demand is nonnegative, meaning that the investor must invest more into the bubble than what is prescribed by his myopic demand, and if risk aversion is low ($p \leq 1$), then the hedging demand is nonpositive. We explain this effect, which might seem to be counter-intuitive at first glance, in detail after Theorem 5.3. These results have an important consequence. In the case $p \leq 1$, the optimal strategy is always below the Merton proportion, but may involve short selling. However, in the economically more relevant case $p \geq 1$, the optimal strategy never involves short selling, but may lie above the Merton proportion. In other words, investors with relative risk aversion $p \geq 1$ ride the bubble instead of attacking it. This theoretical insight is in line with the empirical findings of [24] that hedge funds were heavily invested in the stocks of the dot-com bubble despite being aware of
the presence of the bubble.\footnote{\cite{152} draw the same empirical conclusion from data describing the trading activities of a well-informed bank riding the South Sea bubble in 1720.}

In the context of optimal investment with constant relative risk aversion in diffusion models, the myopic demand of the optimal strategy always has the \textit{mean-variance representation}

\[ \frac{\text{instantaneous expected return}}{\text{relative risk aversion} \times \text{instantaneous variance of the returns}}. \]

In the presence of jumps, however, such a representation seems to be missing in the literature. We provide a mean-variance representation of the myopic demand in our setting which rests on the insight that while the instantaneous expected return is still computed under the real-world measure $P$, the instantaneous variance of the returns should be computed under a suitable, preference-dependent equivalent local martingale measure (ELMM); see Theorem 5.4. We emphasise that this is consistent with the representation in diffusion models because there the instantaneous variance is invariant under equivalent changes of measure.

On the basis of numerical illustrations, we discuss some stylised facts of the optimal strategy and its decomposition into myopic and hedging demand as well as their dependence on the model parameters. In particular, we find that the optimal strategy is not fundamentally different when the stock price process is a strict local martingale (as opposed to a true martingale) under a large class of equivalent local martingale measures including the minimiser for the dual of the utility maximisation problem. In addition, we describe the welfare loss incurred by an investor in our model when compared to an investor in the Black–Scholes model with the same instantaneous expected return and the same instantaneous continuous variance.

\textbf{Techniques and extensions.} The convex duality approach was introduced to portfolio optimisation in incomplete markets by [71, 72] and advanced by [102, 108] and others; cf. [141] for a very readable introduction. We use this approach both for the derivation and the verification of the optimal strategy. To this end, it is crucial to construct a parametric (sub-)class of ELMMs that is big enough to contain the optimiser of a dual minimisation problem (Theorem 3.2); in our case, the parameter is a deterministic function. Given this class, one can then use the optimality conditions of convex duality to derive an equation for the optimal parameter and a relation between the parameter and the trading strategy for the investment problem. Once the equation is derived, one has to prove the existence of a solution and then verify that the corresponding trading strategy is indeed optimal.

\textbf{Related literature and alternative interpretations.} Financial bubbles have been studied for a long time both from an empirical and a theoretical perspective. For a first overview on the vast literature about this topic, we refer to the entry on Crashes and Bubbles in the Encyclopedia of Quantitative Finance.
[97] and to [144] for a very informative article on the history of bubbles and their treatment in the economic literature.

In the recent mathematical finance literature, bubbles have often been modelled by stochastic processes which are strict local martingales under some equivalent local martingale measure; cf. the survey article [139] and the references therein. Our model is flexible enough to allow bubbles in this sense; more precisely, if the jump distribution and the extra excess return are chosen in a suitable way, our model becomes a strict local martingale under a whole subclass of ELMMs including the minimiser for the dual of the investment problem. Another strand of literature on financial bubbles originated from the idea of fitting asset prices to a so-called log-periodic power law in order to predict the end of a possible bubble. This led to the development of the JLS financial bubble model [94, 93]. The fact that our framework contains the JLS model as a special case allows us to analyse to what extent the JLS model can be embedded in the strict local martingale framework. We clarify the connection between these two so far disconnected theories in Section 4.

For all the reasons given above, our model is mainly motivated by asset price bubbles. Nonetheless, the optimal investment problem could alternatively be viewed in the context of crash risk or partial default risk. In that case, the random time $\gamma$ of the jump would be interpreted as the time of the crash or of the default of the risky asset. Optimal investment in a Black–Scholes model under the threat of a crash was first studied by [107] and then generalised in several directions; see, e.g., [149, 13] and the references therein. In this strand of the literature, the time of the crash is assumed to be uncertain in the Knightian sense, i.e., the probability distribution of $\gamma$ is unknown$^2$ and the optimisation follows a worst-case approach. For standard utility functions, the worst-case optimal strategy can be obtained explicitly up to the solution of a first-order ODE. Our problem is different in that we assume that the distribution of $\gamma$ is known a priori.$^3$ So we are in a setting of a risky (as opposed to uncertain) crash.

If the inter-arrival times of subsequent crashes are assumed to be exponentially distributed, the corresponding optimal investment problem has been studied in the context of jump-diffusion models by [113, 37], among others. Using the affine structure of the underlying price dynamics, the authors obtain explicit strategies up to the solution of a first-order ODE. For general (as opposed to exponential) distributions of crash times, the optimal investment problem has recently been studied in the context of default risk [111] and counterparty risk [92]; here, $\gamma$ is interpreted as the time of default of the risky asset. In both articles, the optimal strategy is characterised in terms of a solution to a BSDE (with jumps). In fact, our setup can be seen as a special case of [92]. Note, however, that our method of solving the problem (convex duality) is different from theirs (dynamic programming and BSDEs) and our solution is more explicit than theirs (in cases comparable to our setup); see [92, Section 4.3].

---

$^2$The maximal relative size of the crash, however, is known from the beginning and strictly smaller than 1.

$^3$Moreover, we allow general (deterministic) drift rates.
Organisation of the chapter. The rest of the chapter is organised as follows. Section 2 fixes the probabilistic setup and notation and describes the financial market and the investment problem. Section 3 contains preliminary results on convex duality and a description of a (sub-)class of ELMMs for our financial market. Section 4 clarifies the connections of our model to the JLS model and the strict local martingale framework. Section 5 states the main results on the optimal investment problem and presents a non-rigorous derivation of the optimal strategy. Section 6 contains numerical illustrations. Most of the proofs are relegated to Appendices A–D.

2 Problem formulation

Probabilistic setup. Fix a finite time horizon $T > 0$ and let $(\Omega, \mathcal{F}, P)$ be a probability space carrying a Brownian motion $W = (W_t)_{t \in [0,T]}$ and an independent random variable $\gamma$ taking values in $(0,T]$. Define the (raw) filtrations $\mathbb{F}^W = (\mathcal{F}^W_t)_{t \in [0,T]}$, $\mathbb{F}^\gamma = (\mathcal{F}^\gamma_t)_{t \in [0,T]}$ and $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ by $\mathcal{F}^W_t = \sigma(W_u : 0 \leq u \leq t)$, $\mathcal{F}_t^\gamma = \sigma(1_{\{\gamma \leq u\}} : 0 \leq u \leq t)$ and $\mathcal{F}_t = \sigma(\mathcal{F}_t^\gamma, \mathcal{F}_t^W)$. Note that $\mathbb{F}^W$ and $\mathbb{F}^\gamma$ are independent under $P$ and that $\gamma$ is a stopping time with respect to $\mathbb{F}^\gamma$ and $\mathbb{F}$. We denote the distribution function of $\gamma$ under $P$ by $G$ and assume that $G \in C^2[0,T]$ and $G' > 0$ on $[0,T]$; note that the law of $\gamma$ may have a point mass at $T$, in which case $\Delta G(T) > 0$. For convenience, we define the function $\kappa^G : [0,T) \rightarrow (0,\infty)$ by

$$
\kappa^G(t) = (-\log(1 - G(t)))' = \frac{G'(t)}{1 - G(t)}.
$$

In the credit risk literature, $1 - G$ is called the survival function of $\gamma$, and $\kappa^G$ is called the hazard rate of $\gamma$ (see, e.g., [17, Chapter 5]). Unless otherwise stated, all probabilistic notions requiring a probability measure and/or a filtration (e.g., (local) martingale properties of processes) pertain to $P$ and/or $\mathbb{F}$.

Single jump local martingales of finite variation. Our stock price process will be driven by the Brownian motion $W$ and a local martingale of finite variation which has a single jump at time $\gamma$. These single jump local martingales play a major role in this chapter. We introduce them here and collect their key properties in Appendix B; for a more general treatment of this type of processes, we refer to Chapter IV and [73]. For $F \in C^1[0,T]$, define the process $\mathcal{M}^G_F = (\mathcal{M}^G_tF)_{t \in [0,T]}$ by

$$
\mathcal{M}^G_tF = F(t)1_{\{t < \gamma\}} + \mathcal{A}^G_F(\gamma)1_{\{t \geq \gamma\}},
$$

where the function $\mathcal{A}^G_F : [0,T] \rightarrow \mathbb{R}$ is given by

$$
\mathcal{A}^G_F(v) = \begin{cases} 
F(v) - \frac{F(v)}{\kappa^G(v)} , & v \in [0,T), \\
F(v-)1_{\{\Delta G(T) > 0\}}, & v = T, \text{ if } \lim_{t \uparrow T} F(t) \text{ exists in } \mathbb{R}, \\
0, & v = T, \text{ if } \lim_{t \uparrow T} F(t) \text{ does not exist}.
\end{cases}
$$
Note that even though the function $F$ is only defined on the half-open interval $[0, T)$, the process $\mathcal{M}^G F$ is defined on the closed interval $[0, T]$. Each trajectory $\mathcal{M}^G F(\omega)$ follows the deterministic function $F$ on $[0, \gamma(\omega))$, jumps to $\mathcal{A}^G F(\gamma(\omega))$ at the random time $\gamma(\omega)$ and is constant on $[\gamma(\omega), T]$. Under mild assumptions on $F$ and $G$, $\mathcal{M}^G F$ is a local $(P, \mathcal{F}^\gamma)$-martingale (see Proposition B.2 or, more generally, Theorem IV.3.5), and so by independence of $\mathbb{F}^W$ and $\mathbb{F}^\gamma$ also a local $(P, \mathcal{F})$-martingale.

**Remark 2.1.** Single jump local martingales have been studied in the literature before. However, we use a different parametrisation than, e.g., Chou and Meyer [32], who start with the function $\mathcal{A}^G F$ instead of $F$. It turns out that starting with $F$ is the more natural option for the characterisation of the (local) martingale properties of $\mathcal{M}^G F$ (as carried out in Chapter IV). In particular, we obtain an easier characterisation of strict local martingales, i.e., local martingales that are not martingales (cf. Proposition B.2 (b) (ii)). We refer to Section IV.1 for a more detailed explanation.

**Financial market.** We consider a financial market consisting of a positive riskless asset (“bond”) $B = (B_t)_{t \in [0, T]}$, which is taken as numéraire and without loss of generality normalised to 1, and a risky asset (“stock”) $S = (S_t)_{t \in [0, T]}$ whose dynamics (in units of the numéraire) are given by

$$dS_t = S_t(\mu \, dt + \sigma \, dW_t + d\mathcal{M}^G \phi_t), \quad S_0 = 1,$$

where $\mu, \sigma > 0$ and $\phi \in C^2[0, T)$ satisfies

$$0 \leq \phi' \leq \kappa^G \text{ on } [0, T).$$

(2.5)

We may assume without loss of generality that $\phi(0) = 0$. To prevent possible confusions, we stress that $S$ and $\mathcal{M}^G \phi$ live on the closed interval $[0, T]$, even though $\phi$ is only defined on the half-open interval $[0, T)$.

Note that the randomness in $\mathcal{M}^G \phi$ stems from $\gamma$, interpreted as the time when the bubble bursts, which we also refer to as crash. The dynamics of the returns process $R = (R_t)_{t \in [0, T]}$ of $S$, defined by $R_t = \mu t + \sigma W_t + \mathcal{M}^G \phi$, can be summarised as follows: Prior to $\gamma$, it consists of a drift $(\mu + \phi'(t)) \, dt$ and a random fluctuation $\sigma \, dW_t$; at time $\gamma$, there is a nonpositive jump $\Delta \mathcal{M}^G \phi_\gamma = -\delta(\gamma) 1_{\{\gamma < T\}}$, where $\delta : [0, T) \to [0, 1]$ defined by

$$\delta(t) := \phi(t) - \mathcal{A}^G \phi(t) = \frac{\phi'(t)}{\kappa^G(t)}$$

(2.6)

describes the absolute size of the jump of $\mathcal{M}^G \phi$ if it happens at time $t \in [0, T)$; and after $\gamma$, it consists of a drift $\mu \, dt$ and a random fluctuation $\sigma \, dW_t$, i.e., it satisfies the same dynamics as the returns process of a standard Black–Scholes model. Put differently, compared to the returns process of a standard Black–Scholes model with parameters $\mu$ and $\sigma$, $R$ has a nonnegative extra drift $\phi'(t) \, dt$ prior to $\gamma$, and at time $\gamma$, there is a nonpositive jump of size $-\delta(\gamma) 1_{\{\gamma < T\}}$. This
models—in an idealised way—a main empirical feature of a bubble, which is a strong upward trend followed by a sharp decline at bursting. For this reason, we call \( \phi' \) the \emph{instantaneous pre-crash excess return}. Moreover, we call \( \mu \) the \emph{instantaneous expected return} and \( \sigma^2 \) the \emph{instantaneous continuous variance} (of the return). Using \( \delta \), we can reformulate (2.5) as

\[
0 \leq \delta \leq 1 \text{ on } [0, T),
\]

which shows that the left inequality in (2.5) ensures that the instantaneous pre-crash excess return is nonnegative, whereas the right inequality ensures that the stock price is always nonnegative. If the right inequality is strict for all \( t \in [0, T) \), the stock price is even positive.

\textbf{Remark 2.2.} The names for \( \phi' \), \( \mu \) and \( \sigma \) can be also justified as follows: For fixed \( t \in (0, T) \),

\[
\lim_{h \downarrow 0} \frac{1}{h} E \left[ R_t - R_{t-h} \mid \mathcal{F}_{t-h} \right] = \mu \text{ P-a.s.,} \tag{2.7}
\]

\[
\lim_{h \downarrow 0} \frac{1}{h} E \left[ (R_t - R_{t-h} - \mu h) 1_{\{t < \gamma\}} \mid \mathcal{F}_{t-h} \right] = \phi'(t) 1_{\{t \leq \gamma\}} \text{ P-a.s.,} \tag{2.8}
\]

\[
\lim_{h \downarrow 0} \frac{1}{h} E \left[ (R_t - R_{t-h} - \mu h)^2 \mid \mathcal{F}_{t-h} \right] = \sigma^2 + \phi'(t) \delta(t) 1_{\{t \leq \gamma\}} \text{ P-a.s.} \tag{2.9}
\]

Note that the limits in (2.7)--(2.9) have to be taken from the left (and not from the right as in [120, Assumption 10], where the returns processes are continuous), since the instantaneous quantities should be predictable. Moreover, note that the instantaneous variance of \( R \) is the sum of its continuous variance \( \sigma^2 \) and its (time-dependent) jump variance \( \phi'(t) \delta(t) 1_{\{t \leq \gamma\}} \) (cf. Corollary 3.3). For the proof of (2.7)--(2.9), note that \( \sigma W \) and \( \mathcal{M}^G \phi \) are square-integrable \((P, \mathcal{F})\)-martingales by independence of \( \mathcal{F}^W \) and \( \mathcal{F}^\gamma \) and Proposition B.3. Thus, (2.7) follows from the martingale property of \( \sigma W \) and \( \mathcal{M}^G \phi \), (2.8) follows from (B.2) using that by independence of \( \mathcal{F}^W_t \) and \( \mathcal{F}^\gamma_t \), for \( h \in (0, t) \),

\[
E \left[ (R_t - R_{t-h} - \mu h) 1_{\{t < \gamma\}} \mid \mathcal{F}_{t-h} \right]
= E \left[ \sigma(W_t - W_{t-h}) \mid \mathcal{F}_{t-h}^W \right] E \left[ 1_{\{t < \gamma\}} \mid \mathcal{F}_{t-h}^\gamma \right] \\
+ E \left[ (\mathcal{M}^G_t \phi - \mathcal{M}^G_{t-h} \phi) 1_{\{t \leq \gamma\}} \mid \mathcal{F}_{t-h} \right]
= E \left[ (\mathcal{M}^G_t \phi - \mathcal{M}^G_{t-h} \phi) 1_{\{t < \gamma\}} \mid \mathcal{F}_{t-h}^\gamma \right] \text{ P-a.s.,}
\]

and (2.9) follows from (B.3) using that by orthogonality of \( \sigma W \) and \( \mathcal{M}^G \phi \), for \( h \in (0, t) \),

\[
E \left[ (R_t - R_{t-h} - \mu h)^2 \mid \mathcal{F}_{t-h} \right]
= E \left[ \sigma^2(W_t - W_{t-h})^2 \mid \mathcal{F}_{t-h} \right] + E \left[ (\mathcal{M}^G_t \phi - \mathcal{M}^G_{t-h} \phi)^2 \mid \mathcal{F}_{t-h}^\gamma \right] \text{ P-a.s.}
\]
Trading in the market and investment problem. We consider a small investor with initial capital $x > 0$, who can trade in the financial market described above. For any $\mathbb{F}$-predictable, real-valued process $\pi = (\pi_t)_{t \in [0,T]}$ which is integrable with respect to the returns process $R$, let $X^\pi = (X^\pi_t)_{t \in [0,T]}$ be the unique solution to the SDE

$$\frac{dX^\pi_t}{X^\pi_t} = \pi_t \frac{dS_t}{S_t} = \pi_t dR_t, \quad X^\pi_0 = x. \quad (2.10)$$

We call $\pi$ an admissible strategy if $X^\pi$ is positive. In this case, we can interpret $X^\pi$ as the wealth process corresponding to a self-financing strategy for the market $(B,S)$ (with initial capital $x$) and $\pi_t$ as the fraction of wealth invested in the stock at time $t$.

We assume that the investor has a constant relative risk aversion with parameter $p > 0$. The corresponding utility function is given by

$$U(x) = \begin{cases} \frac{1}{1-p} x^{1-p} & \text{if } p \neq 1, \\ \log x & \text{if } p = 1, \end{cases} \quad x > 0.$$ 

The investor’s goal is to maximise the expected utility $E \left[U(X^\pi_T)\right]$ over all admissible strategies $\pi$:

$$E \left[U(X^\pi_T)\right] \to \max \pi \quad (2.11)$$

3 Preliminaries

Sufficient optimality conditions. We use the method of convex duality both for the derivation and the verification of the optimal strategy. Instead of the very deep general result of Kramkov and Schachermayer [108] for general incomplete semimartingale models, we only use the following elementary result giving a sufficient condition for optimality.

Proposition 3.1. Let $\hat{\pi} = (\hat{\pi}_t)_{t \in [0,T]}$ be an admissible strategy, $\hat{Q}$ an equivalent local martingale measure (ELMM) and $\hat{\varepsilon} > 0$. If

$$(OC1) \quad U'(X^\hat{\pi}_T) = \hat{\varepsilon} \frac{d\hat{Q}}{d\hat{P}} \quad \text{and} \quad (OC2) \quad E^\hat{Q} [X^\hat{\pi}_T] = x,$$

then $\hat{\pi}$ maximises the expected utility $E \left[U(X^\pi_T)\right]$ over all admissible strategies $\pi$.

The ELMM $\hat{Q}$ appearing in the above result is also called the dual minimiser corresponding to the problem (2.11). The result remains true for general utility functions $U$ satisfying mild technical conditions.

Proof. Define the convex dual $V : (0, \infty) \to \mathbb{R}$ of $U$ by

$$V(\bar{z}) = \sup_{\bar{x} > 0} \left(U(\bar{x}) - \bar{x}\bar{z}\right).$$

Clearly, \( U(\tilde{x}) \leq V(\tilde{z}) + \tilde{x} \tilde{z} \) for any \( \tilde{x}, \tilde{z} > 0 \) with equality if \( U'(\tilde{x}) = \tilde{z} \). Moreover, for any admissible strategy \( \pi \), under any ELMM \( Q \), the wealth process \( X^\pi \) is a local martingale bounded from below and hence a supermartingale, starting in \( x \). Thus, for any \( z > 0 \),

\[
E[U(X_t^\pi)] \leq E \left[ V \left( z \frac{dQ}{dP} \right) \right] + E^Q [zX_t^\pi] \leq E \left[ V \left( z \frac{dQ}{dP} \right) \right] + zx. \tag{3.1}
\]

If both \( U'(X_t^\pi) = z \frac{dQ}{dP} \) and \( E^Q [X_t^\pi] = x \) hold, then both inequalities in (3.1) turn into equalities, and \( \pi \) is optimal for (2.11). \( \square \)

**Equivalent local martingale measures.** In order to apply Proposition 3.1 in a specific incomplete model, it is important to have a good knowledge about the model’s ELMMs. Therefore, our next goal is to describe a (sub-)class of ELMMs for the financial market (2.4).

**Theorem 3.2.** Let \( y \in C^1([0, T]) \) with \( \inf_{t \in [0, T]} y(t) > -1 \) be such that

\[
\int_0^T (\phi'(u)y(u))^2 \, du < \infty \quad \text{and} \quad \int_0^T 1_{\{\Delta G(t) > 0\}} \kappa^G(u)(1 + y(u)) \, du < \infty. \tag{3.2}
\]

Define the functions \( \zeta : [0, T) \to (0, \infty) \) and \( H : [0, \infty) \to [0, 1] \) and the process \( Z = (Z_t)_{t \in [0, T]} \) by

\[
\zeta(t) = \exp \left( -\int_0^t \kappa^G(u)y(u) \, du \right), \tag{3.3}
\]

\[
H(t) = 1 - \exp \left( -\int_0^t \kappa^G(u)(1 + y(u)) \, du \right) 1_{\{t < T\}}, \tag{3.4}
\]

\[
Z_t = \mathcal{E}_t \left( -\int_0^1 \frac{1}{\sigma} (\mu - \phi'(u)y(u)) 1_{\{u \leq \gamma, a < T\}} \, dW_a \right) \mathcal{M}_t^G \zeta. \tag{3.5}
\]

Then \( Z \) is a positive \( P \)-martingale starting at 1. Define the measure \( Q \approx P \) on \( \mathcal{F}_T \) by \( \frac{dQ}{dP} = Z_T \). Then \( S \) is a local \( Q \)-martingale and satisfies the SDE

\[
dS_t = S_{t-} \left( \sigma dW_t^Q + d\mathcal{M}^H \left( \int_0^t \phi'(u)(1 + y(u)) \, du \right)_t \right), \tag{3.6}
\]

where \( W^Q = W + \int_0^1 \frac{1}{\sigma} (\mu - \phi'(u)y(u)) 1_{\{u \leq \gamma, a < T\}} \, du \) is a \( Q \)-Brownian motion, \( \gamma \) has distribution function \( H \) under \( Q \), and \( \mathcal{M}^H \left( \int_0^t \phi'(u)(1 + y(u)) \, du \right) \) is a square-integrable \( Q \)-martingale.

**Proof.** For convenience, define the function \( j : [0, T]^2 \to \mathbb{R} \) by

\[
j(t, v) = \frac{1}{\sigma} (\mu - \phi'(t)y(t)) 1_{\{t \leq \nu, t < T\}}
\]

and set \( Z^1 := \mathcal{E} \left( -\int_0^T j(u, \gamma) \, dW_u \right) \) and \( Z^2 := \mathcal{M}^G \zeta. \)
First, \( Z = Z^1 Z^2 \) is a \((P, \mathbb{F})\)-martingale by Lemma A.1 (a) (i) with \( Y^1 = Z^1 \) and \( Y^2 = Z^2 \), using that \( Z^2 \) is a positive \((P, \mathbb{F}^\gamma)\)-martingale by Proposition B.4. Clearly, \( Z_0 = Z^1_0 = Z^2_0 = 1 \), and \( Z^2 \) is also a \((P, \mathbb{F})\)-martingale by independence of \( \mathbb{F}^W \) and \( \mathbb{F}^\gamma \) under \( P \).

Second, define \( Q^1 \approx P \) on \( \mathcal{F}_T \) by \( \frac{dQ^1}{dP} = Z^1_T \). Clearly, \( Q^1 \approx Q \) with \( \frac{dQ}{dQ^1} = Z^2_T \). By Girsanov’s theorem (from \( P \) to \( Q^1 \)), \( W - ( - \int_0^T j(u, \gamma) \, du ) = W^Q \) is a \( Q^1 \)-Brownian motion, and again by Girsanov’s theorem (from \( Q^1 \) to \( Q \)) and the fact that \( Z^2 \) is purely discontinuous, \( W^Q - \int_0^T \frac{1}{Z^2_u} \, d[Z^2] \) is also a local \( Q \)-martingale. By Lévy’s characterisation of Brownian motion, it is even a \( Q \)-Brownian motion.

Third, define \( Q^\gamma \approx P \) on \( \mathcal{F}_T \) (and on \( \mathcal{F}_T^\gamma \)) by \( \frac{dQ^\gamma}{dP} = Z^2_T \). Then \( \gamma \) has distribution function \( H \) under \( Q^\gamma \) by Proposition B.4, and also under \( Q \) by Lemma A.1 (b) (i), applying the latter for \( X^{2,Q} = 1_{\{\gamma \leq t\}} \) and \( s = 0 \).

Finally,

\[
\mathcal{M}^Q \phi + \int_0^T 1_{\{u \leq \gamma\}} \phi'(u) y(u) \, du = \mathcal{M}^H \left( \int_0^T \phi'(u) (1 + y(u)) \, du \right)
\]

is a square-integrable \((Q^\gamma, \mathbb{F}^\gamma)\)-martingale by Proposition B.4, and so also a square-integrable \((Q, \mathbb{F})\)-martingale by Lemma A.1 (b) (ii). Now (3.6) follows from the definition of \( W^Q \) and the dynamics of \( S \) in (2.4). \( \square \)

In order to better understand the dynamics of the returns process \( R \) under \( Q \), we compute the instantaneous variances of its continuous and its jump part relative to \( Q \).

**Corollary 3.3.** Let \( y, H, Q \) and \( W^Q \) be as in Theorem 3.2. Then \( R^{c,Q} = \sigma W^Q \) is the continuous martingale part, \( R^{d,Q} = \mathcal{M}^H \left( \int_0^T \phi'(u) (1 + y(u)) \, du \right) \) is the purely discontinuous martingale part of \( R \) under \( Q \); their instantaneous variances are given for \( t \in (0, T) \) by

\[
\lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}^Q \left[ (R^{c,Q}_t - R^{c,Q}_{t-h})^2 \bigg| \mathcal{F}_{t-h} \right] = \sigma^2 \quad Q\text{-a.s.}, \tag{3.7}
\]

\[
\lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}^Q \left[ (R^{d,Q}_t - R^{d,Q}_{t-h})^2 \bigg| \mathcal{F}_{t-h} \right] = \phi'(t)^2 \frac{1 + y(t)}{\kappa^G(t)} 1_{\{t \leq \gamma\}} =: (\tau^Q_t)^2 \quad Q\text{-a.s.} \tag{3.8}
\]

**Proof.** The assertion about \( R^{c,Q} \) and \( R^{d,Q} \) follows immediately from Theorem 3.2, (3.7) is straightforward, and (3.8) follows from Proposition B.3 (using Lemma A.1 (b) (i) for the change of filtration and (B.9) to switch from \( \kappa^H \) to \( \kappa^G \)) via

\[
\lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}^Q \left[ (R^{d,Q}_t - R^{d,Q}_{t-h})^2 \bigg| \mathcal{F}_{t-h} \right] = \frac{\phi'(t)^2 (1 + y(t))^2}{\kappa^H(t)} 1_{\{t \leq \gamma\}} = \phi'(t)^2 \frac{1 + y(t)}{\kappa^G(t)} 1_{\{t \leq \gamma\}} \quad P\text{-a.s.} \quad \square
\]

Next, we provide necessary and sufficient conditions for the stock price to be a \( Q \)-martingale or a strict local \( Q \)-martingale.
Corollary 3.4. Let $y$ and $Q$ be as in Theorem 3.2.

(a) If $\Delta G(T) > 0$, then $S$ is always a $Q$-martingale.

(b) If $\Delta G(T) = 0$, assume in addition that there exist constants $\epsilon \in (0, 1]$ and $C \geq 1$ such that

$$\epsilon \leq 1 + y(t) \leq C + \frac{C}{\phi'(t)} 1_{\{\kappa^G(t) < C \phi'(t)\}}, \quad t \in [0, T). \quad (3.9)$$

Then:

- $S$ is a $Q$-martingale if and only if
  $$\int_0^T (\kappa^G(u) - \phi'(u)) \, du = \infty.$$

- $S$ is a strict local $Q$-martingale if and only if
  $$\int_0^T (\kappa^G(u) - \phi'(u)) \, du < \infty.$$

Proof. First, by (3.6) and the fact that $M^H (\int \phi'(u)(1 + y(u)) \, du)$ is purely discontinuous, $S = \mathcal{E}(\sigma W^Q) \mathcal{E} \left( M^H \left( \int_0^\cdot \phi'(u)(1 + y(u)) \, du \right) \right)$ $P$-a.s. (3.10)

Next, by Corollary 3.6 below, (3.10) is a $Q$-martingale if and only if the second factor is. To this end, note that the second factor is of the form $M^H \eta$ for $\eta = \exp \left( \int \phi'(u)(1 + y(u)) \, du \right)$ by Proposition B.5. Now, if $\Delta G(T) > 0$, then $M^H \eta$ is a $Q$-martingale by Proposition B.2 (b) (i) (and Lemma A.1 (b) (ii) for the change of filtration). So we have (a). Otherwise, if $\Delta G(T) = 0$, Corollary B.6 (and Lemma A.1 (b) (ii) for the change of filtration) show that $M^H \eta$ is a $Q$-martingale if and only if $\int_0^T (\kappa^G(u) - \phi'(u)) \, du = +\infty$. This gives (b). □

We illustrate Corollary 3.4 by giving an example where $S$ is a strict local $Q$-martingale. Another example in the context of the JLS model is given in Example 4.3 below.

Example 3.5. Let $\gamma$ be uniformly distributed on $[0, 1]$, i.e., $T = 1$, $G(t) = t$ on $[0, 1]$, and $\phi \in C^2[0, 1]$ given by $\phi(t) = -\log(1 - t) - t$. Then $\phi'(t) = \frac{1}{1 - t} - 1 = \kappa^G(t) - 1$ fulfils assumption (2.5), and so by Corollary 3.4, $S$ is a strict local martingale under any ELMM $Q$ corresponding to some $y \in C^1[0, T]$ with $\inf_{t \in [0, T]} y(t) > -1$ and satisfying (3.2) and (3.9). Note that $\delta(t) = t$, the relative size of the jump of $S$ if it happens at time $t \in [0, 1)$, increases linearly from 0 to 1: the later the bubble bursts, the larger is the relative jump size at bursting.

Finally, we present a technical corollary to Theorem 3.2 describing a subclass of local $Q$-martingales. Given here for notational convenience, it is used only in the proofs of Corollary 3.4 and Lemma D.3, and can be skipped on first reading.
Corollary 3.6. Let \( y, H, Q \) and \( W^Q \) be as in Theorem 3.2. Let \( k : [0, T]^2 \to \mathbb{R} \) be of the form \( k(t, v) = k_1(t) + k_2(t)1_{\{t \leq v, t < T\}} \), where \( k_1, k_2 : [0, T] \to \mathbb{R} \) are measurable functions satisfying \( \int_0^T k_i^2(u) \, du < \infty \), \( i = 1, 2 \), and let \( \eta \in C^1[0, T) \) be such that \( \int_0^T |A^H\eta(u)|H'(u) \, du < \infty \). Then

\[
\mathcal{E} \left( \int_0^T k(u, \gamma) \, dW^Q_u \right) \mathcal{M}^H\eta
\]

is a local \( Q \)-martingale. It is a \( Q \)-martingale if and only if \( \mathcal{M}^H\eta \) is a \( Q \)-martingale.

Proof. Let \( j, Z, Z^1, Z^2 \) and \( Q^\gamma \) be as in the proof of Theorem 3.2. Set \( \tilde{Z}^1 := \mathcal{E} \left( \int_0^T k(u, \gamma) \, dW^Q_u \right) \), \( \tilde{Z}^2 := \mathcal{M}^H\eta \), \( Y^1 := Z^1 \tilde{Z}^1 \) and \( Y^2 := Z^2 \mathcal{M}^H\eta \). Then \( \tilde{Z}^2 = \mathcal{M}^H\eta \) is a local \((Q^\gamma, \mathbb{F}^\gamma)\)-martingale by Proposition B.2 (a) (using that \( \gamma \) has distribution function \( H \) under \( Q^\gamma \)), and a short calculation gives \( Y^1 = \mathcal{E} \left( \int_0^T (k - j)(u, \gamma) \, dW_u \right) \) \( \mathcal{P} \)-a.s.

We have to show that \( \tilde{Z}^1 \tilde{Z}^2 \) is a local \((Q, \mathbb{F})\)-martingale, and that \( \tilde{Z}^1 \tilde{Z}^2 \) is a \((Q, \mathbb{F})\)-martingale if and only if \( \tilde{Z}^2 \) is a \((Q, \mathbb{F})\)-martingale, or equivalently by Lemma A.1 (b) (ii), a \((Q^\gamma, \mathbb{F}^\gamma)\)-martingale. By Bayes’ theorem, it suffices to show that \( Y^1 Y^2 = Z^1 \tilde{Z}^2 \tilde{Z}^2 \) is a local \((P, \mathbb{F})\)-martingale and that it is a \((P, \mathbb{F})\)-martingale if and only if \( Y^2 = Z^2 \tilde{Z}^2 \) is a \((P, \mathbb{F}^\gamma)\)-martingale. Recalling that \( Z^2_T = \frac{dQ^\gamma}{dP} \), Bayes’ theorem yields that \( Y^2 = Z^2 \tilde{Z}^2 \) is a local \((P, \mathbb{F}^\gamma)\)-martingale, and Lemma A.1 (a) (ii) and (i) with \( k \) replaced by \( k - j \) completes the proof. \( \square \)

4 Bubbles as strict local martingales and the JLS model

Financial bubbles are often associated with a disparity between the price of an asset and its “fundamental value”. It has been argued in the mathematical finance literature that this form of mispricing can be captured very generally by modelling asset prices as processes that are strict local martingales under some ELMM; see Loewenstein and Willard [115], Cox and Hobson [34], Heston, Loewenstein, and Willard [78], Jarrow, Protter, and Shimbo [89, 90], Protter [139], and the references therein. In another strand of the literature (especially econophysics), the Johansen–Ledoit–Sornette (JLS) financial bubble model (see [150] and the references therein) has received a lot of attention. However, the JLS model does not mention strict local martingales at all. In this section, we show that the JLS model is a special case of the model introduced in Section 2, and we discuss to what extent it can be embedded in the strict local martingale framework.

Strict local martingale models in our setting. The class of processes for the stock price \( S \) introduced in Section 2 is rich enough to include specifications of the parameters \( G \) and \( \phi \) such that \( S \) is a strict local martingale under certain ELMMs. Indeed, if \( Q \) is the ELMM constructed in Theorem 3.2 for a suitable
function $y$ and if $y$ satisfies (3.9) (e.g., $y \equiv 0$), then Corollary 3.4 shows that $S$ is a strict local martingale under $Q$ if and only if $\Delta G(T) = 0$ and the function $\kappa^G - \phi'$ is integrable on $(0, T)$. We refer to Example 3.5 for a concrete specification.

The Johansen–Ledoit–Sornette model. The JLS model proposes\footnote{The following specification is taken from [150] (up to changes in notation); the original specification in [93] is slightly different and in particular has no explicit Brownian component.} that the price process of a financial asset can be modelled as the sum of its “fundamental value” (which is not further specified) and a bubble component which has the dynamics (under $P$)

$$
\frac{dS_t}{S_t} = \phi'(t) \, dt + \sigma \, dW_t - \delta \, dJ_t,
$$

where $\phi'$ is a deterministic function, $J_t = 1_{\{t \geq \gamma\}}$ jumps from 0 to 1 at the time $\gamma$ of the crash, and the constant $\delta \in (0, 1)$ is the relative loss of the bubble component at the time of the crash. The time of the crash $\gamma$ is a nonnegative random variable independent\footnote{This assumption is not made explicitly in the JLS model, but is natural as the postulated form of the hazard rate (4.2) does not depend on $W$.} of the Brownian motion $W$ with a distribution function $G$ that is sufficiently regular on $(0, T)$. The corresponding hazard rate

$$
\kappa^G(t) = \frac{G'(t)}{1 - G(t)}
$$

describes the conditional probability of the crash occurring in the next instant, given that the crash has not happened yet. A key assumption is that the hazard rate follows a log-periodic power law (LPPL),

$$
\kappa^G(t) = B'|T - t|^{m-1} + C'|T - t|^{m-1} \cos (\varpi \log(T - t) - \psi') ,
$$

where $B', C', m, T, \varpi$ and $\psi'$ are suitable real parameters; we refer to [150, Section 2.1] for interpretations. The JLS model confines the parameter $m$ to the interval $(0, 1)$ so that the hazard rate diverges to infinity as $t \uparrow T$ but remains integrable on $(0, T)$ (this restriction is discussed below). The parameters have to be chosen such that the hazard rate is always nonnegative (cf. [153]; this constraint has been ignored in many of the early articles). The “critical time” $T > 0$ is interpreted as the end of the bubble regime [150], and the crash can happen at any time before $T$. Finally, it is assumed that $S$ is a (true) martingale. Formal arguments then yield that $\phi'(t) = \delta \kappa^G(t)$, which is in accordance with the relation (2.6).

Using our notation for single jump local martingales, we can combine the drift term and the jump term in (4.1) to arrive at\footnote{To be precise, we assume here that $S$ evolves like a geometric Brownian motion after the crash; the JLS model does not specify what happens after the crash.}

$$
\frac{dS_t}{S_t} = \sigma \, dW_t + d\mathcal{M}^G \phi, \quad (4.3)
$$
where $\mathcal{M}^G\phi$ is a single jump local martingale as introduced in (2.2) and $\phi$ is the primitive of $\phi'$ with $\phi(0) = 0$, i.e.,

$$\phi(t) = \int_0^t \phi'(u) \, du = \delta \int_0^t \kappa^G(u) \, du.$$  

We conclude that the JLS model is a special case of (2.4) with zero instantaneous expected return $\mu = 0$, a hazard rate of the form (4.2), and $\phi'$ chosen such that $\delta(t)$, the absolute size of the jump of $\mathcal{M}^G\phi$ if it happens at time $t \in [0, T)$, is a constant in $(0, 1)$. We also note that the JLS model requires that there is always a positive probability that no crash occurs (this is equivalent to an integrable hazard rate; see the discussion of the restriction $m \in (0, 1)$ below).

We are now in a position to analyse to what extent the JLS model can be embedded in the strict local martingale framework.

**The JLS model is never a strict local martingale.** Although the JLS model assumes from the start that the bubble component is a (true) martingale, one could instead start from the specification (4.3) and ask whether the martingale property follows as a consequence. To this end, we note that by Proposition B.1,

$$\kappa^G \text{ is integrable on } (0, T) \iff \Delta G(T) > 0. \quad (4.4)$$

As the JLS model requires $m \in (0, 1)$, the hazard rate $\kappa^G$ defined in (4.2) is integrable on $(0, T)$, and thus $\Delta G(T) > 0$ by (4.4). Now, by Corollary 3.4 (a) (with $y \equiv 0$, so that $P = Q$; in particular, the distribution function of $\gamma$ under $Q$ is still $G$), $S$ is a (true) martingale (under $P = Q$). However, we show below that strict local martingales can emerge if the assumption $m > 0$ is dropped.

**The log-periodic power law for the logarithm of the conditional expectation.** Using the formulation (4.3), we can also rigorously show that the log-periodic power law (4.2) of the hazard rate carries over to another log-periodic power law for the logarithm of the conditional expectation of the bubble component at some time $t \in (0, T)$, given the event that the crash has not happened yet. Using independence of $\gamma$ and $W$, the conditional expectation of $S_t$, given that $t < \gamma$, is computed as follows:

$$E[S_t \, | \, t < \gamma] = \frac{1}{1 - G(t)} E[S_0 E_t(\sigma W + \mathcal{M}^G\phi)1_{\{t < \gamma\}}]$$

$$= \frac{S_0}{1 - G(t)} E[E_t(\sigma W) \exp(\phi(t))1_{\{t < \gamma\}}]$$

$$= S_0 \exp(\phi(t)).$$

Hence, the logarithm of the expected value of the bubble component given that the crash has not happened yet reads as

$$I(t) := \log E[S_t \, | \, t < \gamma] = \log S_0 + \phi(t) = \log S_0 + \delta \int_0^t \kappa^G(u) \, du.$$  

---

7See, e.g., [153, 22, 150] for a formal derivation.
Substituting the LPPL form (4.2) of the hazard rate and using that $m \in (0, 1)$, an integration gives

$$I(t) = A + B|T - t|^m + C|T - t|^m \cos (\varpi \log(T - t) - \psi),$$

(4.5)

where $B = -\delta B'/m$, $C = -\delta C'/\sqrt{m^2 + \varpi^2}$, and $A$ and $\psi$ are constants depending on $A'$, $B'$, $C'$, $m$, $T$, $\varpi$, $\psi'$ and $S_0$ (cf. equation (6) in [150]).

**Remark 4.1.** Equation (4.5) is at the root of the literature on log-periodic power laws in the context of financial bubbles. In 1996, Bouchaud, Johansen, and Sornette [20] and Feigenbaum and Freund [55] independently suggested that the log price of a financial asset prior to a large crash can be fitted by a log-periodic power law (4.5). The main objective is then to obtain a prediction for the “critical time” $T$ which is interpreted as the “most probable time for the crash” [95] (because the hazard rate explodes at $T$). This approach has been widely used (see [150, 52] for an overview) and intensely debated in the literature (see in particular [54, 96, 53] and also [22]).

**Discussion of the restriction $m \in (0, 1)$.** In [150, Section 2.2], it is argued that $m$ should lie in the interval $(0, 1)$. The authors state that $m < 1$ is necessary to obtain an accelerating hazard rate. While this is certainly true, Brée and Joseph [22] point out that $m < 1$ should not be an *a priori* restriction when fitting the LPPL (4.5) to data. A best fit with $m \geq 1$ should rather be used to reject the model.

Here, we are concerned with the restriction $m > 0$. It is argued in [150] that $m > 0$ is necessary to obtain that the bubble component “remains finite at all time, including $t_c = T$” (p. 4419). However, we claim that if $m \leq 0$, then $\gamma < T$ P-a.s. Indeed, if $m \leq 0$, then the hazard rate (4.2) is nonintegrable on $(0, T)$ and then $G(T-) = 1$ by Proposition B.1, so that $\gamma < T$ P-a.s. In words, the crash happens strictly before the “critical time” $T$ with probability 1. Hence, also in this case the bubble component stays finite at all times and the argument of [150] does not justify eliminating the case $m \leq 0$ *a priori*. The authors of [150] also claim that the property of the JLS model that there is a positive probability that no crash occurs “makes it rational for investors to remain invested, knowing that a bubble is developing and that a crash is looming [because ...] there is a chance for investors to gain from the ramp-up of the bubble and walk away unscathed” (p. 4419). However, even if a crash happens almost surely before time $T$, it can similarly be argued that it is rational for investors to ride the bubble, knowing that the bubble will surely burst before time $T$, as long as they reduce their position before time $T$. With this strategy they simply bet on the event that the bubble only bursts after they have closed their position. In fact, our Theorem 5.3 shows that at least investors with relative risk aversion larger than 1 follow such a strategy as long as the underlying asset has a positive

---

*We note that no distinction between the fundamental value and the bubble component has been made in the early articles. Moreover, sometimes the price is fitted instead of the log price.*
instantaneous expected return.\(^9\) We emphasise that shorting the bubble is not an arbitrage opportunity in the case where the bubble bursts almost surely before time \(T\) (after all, the bubble component is a local martingale). For instance, the naive strategy of holding a (constant) short position in the bubble leads to bankruptcy with positive probability because the bubble can grow arbitrarily if it bursts sufficiently late.

In the remainder of this section, we consider modifications of the JLS model with \(m \leq 0\) that can lead to strict local martingale models. The limiting case of \(m = 0\) has already been suggested by Ausloos, Boveroux, Minguet, and Vandewalle [5, 6, 7]. They propose to replace the LPPL (4.2) by

\[
I(t) = A + B \log(T-t) + C \log(T-t) \cos(\varpi \log(T-t) - \psi).
\]

The corresponding hazard rate,

\[
\kappa^G(t) = B'|T-t|^{-1} + C'|T-t|^{-1} \cos(\varpi \log(T-t) - \psi'),
\]

is nonintegrable on \((0, T)\) and hence \(\gamma < T\) \(P\)-a.s. by Proposition B.1. To the best of our knowledge, the case \(m < 0\) has not been studied in the literature so far.

**Strict local martingale modifications of the JLS model.** We have seen above that the assumption \(m > 0\) in the JLS model immediately excludes the possibility of strict local martingales. We have also argued in the previous paragraph that the justification for imposing this restriction \(a\ priori\) is rather weak. We now drop the restriction \(m \in (0, 1)\) and show that there are modifications of the JLS model that lead to strict local martingale models while preserving the log-periodic power law (4.2) for the hazard rate. Throughout, we fix \(G \in C^2[0, T]\) and \(\phi \in C^1[0, T]\) as in Section 2 and consider a bubble component \(S\) with dynamics

\[
\frac{dS_t}{S_{t^-}} = \sigma dW_t + dM^G_t \phi_t, \quad S_0 = 1.
\]

In particular, the relative jump size \(\delta(t) = \frac{\phi'(t)}{\kappa^G(t)}\) if the crash happens at time \(t \in [0, T)\) is not necessarily a constant as in the JLS model, but can be time-dependent.

Assuming that the hazard rate \(\kappa^G\) has a log-periodic power law (4.2), the following result gives necessary and sufficient conditions for \(S\) to be a strict local martingale. Moreover, it shows that in that case, the relative jump size \(\delta(t)\) must essentially converge to 1 as \(t \uparrow T\). In other words, for each \(\varepsilon > 0\), there is a positive probability that the bubble component loses a fraction \(1 - \varepsilon\) of its value at the time of the crash.

\(^9\)It is well known that risk-averse agents (with a finite credit line) never invest in an asset with zero instantaneous expected return.
Theorem 4.2. Suppose that the hazard rate $\kappa^G$ is of the form

$$\kappa^G(t) = B'|T - t|^{m-1} + C'|T - t|^{m-1} \cos(\varpi \log(T - t) - \psi'), \ t \in [0, T),$$

(4.6)

for real parameters $B'$, $C'$, $m$, $\varpi$ and $\psi'$ with $|C'| < B'$ (so that $\kappa^G > 0$ on $[0, T)$). Then $S$ is a nonnegative strict local martingale with a nonpositive jump if and only if

$$m \leq 0 \quad \text{and} \quad \varphi'(t) = \kappa^G(t) - \vartheta(t)$$

(4.7)

for some Borel function $\vartheta : [0, T) \to \mathbb{R}$ that is integrable on $(0, T)$ and satisfies $0 \leq \vartheta \leq \kappa^G$ on $[0, T)$. Moreover, in this case, $\limsup_{t \uparrow T} \delta(t) = 1$.

Proof. In view of the form (4.6) for $\kappa^G$ and the property $|C'| < B'$, we first note that $m \leq 0$ is equivalent to $\kappa^G$ being nonintegrable on $(0, T)$. This together with Proposition B.1 yields

$$m \leq 0 \Leftrightarrow \Delta G(T) = 0.$$  

(4.8)

Moreover, by the dynamics of $S$, the construction of the process $\mathcal{M}^G \varphi$, and the formula for the stochastic exponential, $S$ being a nonnegative process with a nonpositive jump is equivalent to $0 \leq \delta \leq 1$ on $[0, T)$. Substituting $\delta = \frac{\varphi'}{\kappa^G}$, adding $-1$ everywhere and multiplying with $-\kappa^G < 0$, we find the equivalence

$$0 \leq \delta \leq 1 \text{ on } [0, T) \Leftrightarrow 0 \leq \kappa^G - \varphi' \leq \kappa^G \text{ on } [0, T).$$

(4.9)

Now suppose that $S$ is a nonnegative strict local martingale with a nonpositive jump and set $\vartheta(t) = \kappa^G(t) - \varphi'(t), \ t \in [0, T)$. Then $0 \leq \vartheta \leq \kappa^G$ on $[0, T)$ by (4.9). Moreover, by Corollary 3.4 (b) (with $y \equiv 0$), $\Delta G(T) = 0$ and $\int_0^T \vartheta(u) \, du < \infty$. This together with (4.8) shows (4.7) and the integrability of $\vartheta$.

Conversely, suppose that (4.7) holds for an integrable $\vartheta : [0, T) \to \mathbb{R}$ satisfying $0 \leq \vartheta \leq \kappa^G$ on $[0, T)$. By (4.9) and (4.7), the latter condition is equivalent to $0 \leq \delta \leq 1$ on $[0, T)$, so that $S$ is nonnegative with a nonpositive jump. Moreover, $\kappa^G - \varphi' = \vartheta$ is integrable on $(0, T)$ and $m \leq 0$. This together with (4.8) and Corollary 3.4 (b) (with $y \equiv 0$) shows that $S$ is a strict local martingale.

It remains to show that in the case of a strict local martingale, we have $\limsup_{t \uparrow T} \delta(t) = 1$. So suppose that (4.7) holds for an integrable $\vartheta : [0, T) \to \mathbb{R}$ that satisfies $0 \leq \vartheta \leq \kappa^G$ on $[0, T)$. Then $\delta = 1 - \frac{\varphi'}{\kappa^G}$ on $[0, T)$ and it suffices to show that $\liminf_{t \uparrow T} \frac{\vartheta(t)}{\kappa^G(t)} = 0$. Clearly, $\liminf_{t \uparrow T} \frac{\vartheta(t)}{\kappa^G(t)} \geq 0$. Seeking a contradiction, suppose that there is $\varepsilon > 0$ such that $\liminf_{t \uparrow T} \frac{\vartheta(t)}{\kappa^G(t)} \geq \varepsilon$. Then there is $t_0 \in (0, T)$ such that

$$\vartheta(t) \geq \frac{\varepsilon}{2} \kappa^G(t), \quad t \in (t_0, T).$$

(4.10)

Recall that $\kappa^G$ is nonintegrable on $(0, T)$. As $\kappa^G$ is continuous on $[0, T)$, it is also nonintegrable on $(t_0, T)$. But then by (4.10) and the fact that $\vartheta \geq 0$ on $[0, T)$, also $\vartheta$ is nonintegrable on $(0, T)$. This is a contradiction. \qed
As $\kappa^G$ is nonintegrable in the strict local martingale case, the second condition in (4.7) implies that also $\phi'$ must be nonintegrable. In particular, $\phi(t)$ must explode as $t$ approaches $T$.

**Example 4.3.** The simplest function $\vartheta$ satisfying the conditions in Theorem 4.2 is $\vartheta \equiv 0$. This leads to a rather extreme behaviour: the bubble component $S$ always jumps to $0$ when the crash happens because the relative jump size is $\delta(t) = \frac{\phi'(t)}{\kappa^G(t)} = 1$ for $t \in [0, T)$. Another choice for which the relative jump size is always strictly below $1$ is $\vartheta(t) \equiv 1$ (provided $\kappa^G \geq 1$ on $[0, T)$). Then the relative jump size is $\delta(t) = \frac{\phi'(t)}{\kappa^G(t)} = 1 - \kappa^G(t)^{-1}$ and $S$ remains positive after the crash.

**Conclusion.** We have shown that the JLS model in its original form is not compatible with the idea of modelling financial bubbles as strict local martingales. However, we have then argued that the reasoning for the a priori restriction of the parameter $m$ to the interval $(0, 1)$ is not very convincing. Therefore, we have considered a generalisation of the JLS model (which is a special case of the model introduced in Section 2) that preserves the key log-periodic power law for the hazard rate, but is flexible enough to yield strict local martingales. In this setup, we have shown that strict local martingales appear if the parameter $m$ is nonpositive and the extra drift $\phi'(t)$ grows sufficiently fast.

## 5 Main results

In this section, we characterise the optimal strategy for the investment problem (2.11), provide a decomposition into its myopic and hedging demands, and calculate the corresponding certainty equivalent. In addition, we provide a heuristic derivation of Theorem 5.1.

All our results are explicit up to the solution of an integral equation (or a first-order ODE), which will be given in terms of the auxiliary functions $a, b, m, n : [0, T) \times [-1, \infty) \times (0, \infty) \to \mathbb{R}$ defined by

$$a(t, y, p) = 1 - \frac{1}{p\sigma^2} \kappa^G(t) (\mu - \phi'(t)y) = 1 - \delta(t) \frac{1}{p\sigma^2} (\mu - \phi'(t)y),$$

$$b(t, y, p) = \left(1 + \frac{1}{p} \right) a(t, y, p),$$

$$m(t, y, p) = (1 + y)^{\frac{1}{2}} a(t, y, p),$$

$$n(t, y, p) = -\frac{1 - p}{2p^2\sigma^2} \phi'(t)y^2 + \kappa^G(t) (b(t, y, p) - 1).$$

Here, $t$, $y$ and $p$ stand for the time parameter, the parameter describing the optimal strategy and the risk aversion parameter, respectively. For an economic interpretation of $a$, see Remark 5.2 below. The function $m$ is directly linked to the myopic part of the optimal strategy (see Theorem 5.3 below). The functions $b$ and $n$ are more of technical nature and do not have a direct economic interpretation.
Existence and uniqueness of the optimal strategy. First, we formulate the general existence and uniqueness result for the optimal strategy.

**Theorem 5.1.** Fix $p \in (0, \infty)$. There exists a unique function $\hat{y} \in C^1[0, T)$ with $\hat{y} > -1$ satisfying the integral equation

$$m(t, y(t), p) = \exp \left(- \int_0^T n(u, y(u), p) \, du\right), \quad t \in [0, T). \tag{5.5}$$

The strategy $\hat{\pi} = (\hat{\pi}_t)_{t \in [0, T]}$ defined in terms of $\hat{y}$ by

$$\hat{\pi}_t = \frac{1}{p\sigma^2} \left(\mu - \phi'(t)\hat{y}(t)\mathbf{1}_{\{t \leq \gamma, t < T\}}\right) \tag{5.6}$$

is admissible and maximises the expected utility $E\left[U(X^{\hat{\pi}}_T)\right]$ over all admissible strategies $\pi$. Moreover, $\hat{y}$ satisfies (3.2) and (3.9).

**Remark 5.2.** When we speak about a solution $\hat{y}$ to the integral equation (5.5), we tacitly impose that $\int_0^T |n(u, \hat{y}(u), p)| \, du < \infty$. Then (5.5), the definition of $m$ in (5.3) and the requirement $\hat{y} > -1$ imply that $a(t, \hat{y}(t), p) > 0$, $t \in [0, T)$. Economically, the latter property means that the investor's wealth is positive after the bubble has burst. Indeed, on $\{\gamma = t\}$, since the stock loses a fraction $\delta(t)$ of its value at time $t$, the wealth at time $t$ is given by

$$(1 - \hat{\pi}_t)X^\hat{\pi}_{t-} + \hat{\pi}_t X^\hat{\pi}_{t-} (1 - \delta(t)) = X^\hat{\pi}_{t-} (1 - \delta(t)\hat{\pi}_t) = X^\hat{\pi}_{t-} a(t, \hat{y}(t), p).$$

**Proof of Theorem 5.1.** The idea is to construct a triplet $(\hat{\pi}, \hat{Q}, \hat{z})$ which satisfies the assumptions of Proposition 3.1 and thereby yields an optimiser for the investment problem (2.11). We proceed in three steps: first, we construct a (unique) solution to the integral equation (5.5); second, we construct a triplet $(\hat{\pi}, \hat{Q}, \hat{z})$; third, we verify that this triplet satisfies conditions (OC1) and (OC2) from Proposition 3.1. (The integral equation (5.5) as well as the triplet $(\hat{\pi}, \hat{Q}, \hat{z})$ are formally derived at the end of this section.)

**Step 1.** Theorem C.6 shows in full detail that (5.5) has a unique solution $\hat{y} > -1$ satisfying (3.2) and (3.9). Here, we only outline the main difficulties and ideas. By taking logarithms on both sides, differentiating with respect to $t$ and rearranging terms, the integral equation (5.5) is easily transformed into an ODE of the form

$$y'(t) = f(t, y(t)), \quad t \in [0, T). \tag{5.7}$$

It is important to note that since (5.5) need not be defined for $t = T$, also $f$ need not be defined for $t = T$. However, formally letting $t \uparrow T$ in (5.5), we find the “terminal condition”

$$\lim_{t \uparrow T} m(t, y(t), p) = 1. \tag{5.8}$$

The fact that this “terminal condition” both is implicit and can only be expressed as a limit renders the ODE nonstandard. Proving existence of a solution $\hat{y}$ to the
ODE (5.7) can, however, be reduced to finding a pair \((y^*, y_*)\) of so-called *backward upper* and *backward lower solutions* to (5.7) (cf. Lemma C.2). The construction of suitable \(y^*\) and \(y_*\) so that the solution \(\hat{y}\) also satisfies (5.8) is the main technical difficulty of this first step of the proof.

**Step 2.** Now we construct a triplet \(\left(\hat{\pi}, \hat{Q}, \hat{z}\right)\) as follows. First, by Step 1, \(\hat{y}\) satisfies the assumptions of Theorem 3.2 (note that (3.9) implies \(\inf_{t \in [0, T]} \hat{y}(t) > -1\)), which yields an explicit ELMM \(\hat{Q}\) for \(S\). Second, we have to check that \(\hat{\pi}\) defined in (5.6) is integrable with respect to the returns process \(R\) and that it is admissible. The first assertion is clear from the fact that \(\hat{y}\) satisfies (3.2). For the second assertion, Lemma D.1 identifies the wealth process \(X_{\hat{\pi}}\) in terms of \(\hat{y}\) and shows that it remains positive; the proof is mainly computational. Third, define \(\hat{z} > 0\) via

\[
\hat{z}^{-\frac{1}{p}} = \mathcal{m}(0, \hat{y}(0), p) \exp \left(-\left(1-p\right)\frac{\mu^2}{2p^2\sigma^2}T\right);
\]

(5.9)

note that \(m(0, \hat{y}(0), p) > 0\) since \(\hat{y}\) solves (5.5).

**Step 3.** The verifications of (OC1) and (OC2) are carried out in Lemmas D.2 and D.3, respectively. The major difficulty of this step of the proof is to show that the candidate wealth process \(X^*\) is a \(\hat{Q}\)-martingale (i.e., (OC2)). The proof of (OC1) is mainly computational.

**Myopic and hedging demand of the optimal strategy.** A frequent goal in the context of optimal investment problems is to understand the qualitative behaviour of the optimal strategy. To this end, optimal strategies are often decomposed into the sum of a myopic demand and a hedging demand (see for instance [2, Section III], [105, Equation (19)], [30, Equation (14)], [112, Corollary 3]). In discrete time, the myopic demand is the optimal strategy of an investor who treats each period as if it were the last, irrespective of the conditional distribution of any future returns (cf. Mossin [126]). In a continuous-time setting, the myopic demand at time \(t\) can be defined as the limit (if it exists) of the optimal strategy when the investment horizon \(T-t\) goes to zero. One can show that in our setting, this corresponds to letting \(T \downarrow t\) in the integral equation (5.5) (as one would expect formally). So the solution to the limiting equation (5.10) below can be used to define the *myopic demand* via (5.11). Then the *hedging demand* is defined as the difference between the optimal strategy and the myopic demand (cf. (5.12)). The following theorem states interesting consequences of this decomposition.

**Theorem 5.3.** Fix \(p \in (0, \infty)\). There exists a unique function \(y^m \in C^1[0, T]\) with \(y^m \geq 0\) satisfying the equation

\[
m(t, y^m(t), p) = 1.
\]

(5.10)

Let \(\hat{y}\) be as in Theorem 5.1. The processes \(\pi^m = (\pi^m_t)_{t \in [0, T]}\) and \(\pi^h = (\pi^h_t)_{t \in [0, T]}\)
defined in terms of \( y^m \) and \( \hat{y} \) by

\[
\pi^m_t = \frac{1}{p\sigma^2} \left( \mu - \phi'(t)y^m(t) \right) \mathbf{1}_{\{t \leq \gamma, t < T\}}, \tag{5.11}
\]

\[
\pi^h_t = \hat{\pi}_t - \pi^m_t = \frac{1}{p\sigma^2} \phi'(t)(y^m(t) - \hat{y}(t)) \mathbf{1}_{\{t \leq \gamma, t < T\}} \tag{5.12}
\]

are called the myopic demand and the hedging demand of the optimal strategy \( \hat{\pi} \).

(a) The myopic demand satisfies

\[
0 < \pi^m \leq \frac{\mu}{p\sigma^2}, \tag{5.13}
\]

where on \( \{t \leq \gamma, t < T\} \), the right inequality is an equality if and only if \( \phi'(t) = 0 \).

(b) The hedging demand satisfies

\[
\pi^h \leq 0 \text{ for } p \in (0, 1), \quad \pi^h = 0 \text{ for } p = 1, \quad \text{and } \pi^h \geq 0 \text{ for } p > 1. \tag{5.14}
\]

Moreover, if \( \limsup_{t \uparrow T} G'(t) < \infty \), then \( \lim_{t \uparrow T} \pi^m = 0 \) P-a.s.

\[\text{Proof.}\] The existence and uniqueness of \( y^m \) follow from Lemma C.3, and nonnegativity of \( y^m \) follows from Corollary C.4. The inequalities (5.13) in (a) follow from the mean-variance representation (5.18) below of the myopic demand. On \( \{t \leq \gamma, t < T\} \), we have \( \pi^m_t = \frac{1}{p\sigma^2} \left( \mu - \phi'(t)y^m(t) \right) \) and this is equal to \( \frac{\mu}{p\sigma^2} \) if and only if \( \phi'(t)y^m(t) = 0 \), which by Corollary C.4 is equivalent to \( \phi'(t) = 0 \). The inequalities (5.14) follow from Theorem C.6 (noting that \( y^m = y^* \) for \( p \leq 1 \) and \( y^m = y^* \) for \( p > 1 \)). The second assertion in (b) is trivial if \( \Delta G(T) = 0 \). If \( \Delta G(T) > 0 \), Corollary C.5 shows that \( \lim_{t \uparrow T} \phi'(t)(y^*(t) - y_*(t)) = 0 \), and so a fortiori \( \lim_{t \uparrow T} \phi'(t)(y^m(t) - \hat{y}(t)) = 0 \) since \( y_* \leq \hat{y} \leq y^* \) on \( [0, T) \) by Theorem C.6. This completes the proof.

Let us comment on the above result. First, Theorem 5.1 shows that the optimal strategy \( \hat{\pi} \) is generally given in terms of the solution to an integral equation (or an ODE). By contrast, to find the myopic demand of the optimal strategy, it suffices to solve an equation for each \( t \).

Second, our interpretation of the myopic demand in continuous time suggests that the hedging demand should approach 0 at the time horizon \( T \), and this holds true under a very mild technical assumption on \( G \).

Third, the economic interpretation of the behaviour of the hedging demand is as follows. After the bubble has burst, the model behaves like a Black–Scholes model with instantaneous expected return \( \mu \) and instantaneous continuous variance \( \sigma^2 \). Before the crash, the instantaneous expected return is still \( \mu \), but the total instantaneous variance of returns exceeds \( \sigma^2 \) due to the bubble component \( \mathcal{M}^G \phi \). Hence, any risk-averse investor will favour the Black–Scholes market over our market (indeed, the certainty equivalent of trading in our market in Theorem 5.5 below displays a discount with respect to the Black–Scholes certainty
Optimal Investment in a Black–Scholes Model with a Bubble equivalent). The later the bubble bursts, the less time the investor has to invest in the Black–Scholes market. Consequently, it is favourable for the investor if the bubble bursts early and unfavourable if it bursts late or never.

Now, an investor with a high relative risk aversion \( p > 1 \) has a nonnegative hedging demand \( \pi^h \). Hence, in the favourable event that the bubble bursts early, he will lose more money than if he had chosen the myopic demand, while having hardly profited from the (nonnegative) instantaneous pre-crash excess return \( \phi' \). However, in the unfavourable event that the bubble bursts late or never, he profits significantly from the (nonnegative) instantaneous pre-crash excess return \( \phi' \) by investing more than the myopic demand; this compensates him for the small amount of time he can invest in the bubble-free market. In this sense, investors with high relative risk aversion hedge against a late bursting of the bubble.

On the other hand, an investor with a low relative risk aversion \( p < 1 \) has a nonpositive hedging demand. An early bursting of the bubble is favourable to him in two ways. First, as above, he can invest in the bubble-free market for a longer time period after the crash. Second, he loses at the time of the crash less money (or even gains money in case of a short position coming from a hedging demand that exceeds the myopic demand in absolute value) than if he had followed the myopic demand. However, if the bubble bursts late or never, his optimal strategy performs worse than the myopic demand, because he profits significantly less from the instantaneous pre-crash excess return \( \phi' \). In this sense, investors with low relative risk aversion speculate on an early bursting of the bubble.

In the limiting case of logarithmic utility \( (p = 1) \), the investor neither hedges against nor speculates on the timing of the crash; his optimal strategy equals the myopic demand, reflecting the well-known fact that log-investors behave myopically. Moreover, the equation \( m(t, y(t), 1) = 1 \) reduces to a quadratic equation in \( y(t) \), whose unique solution with \( y > -1 \) is given by \( \hat{y}(t) = 0 \) if \( \phi'(t) = 0 \) and

\[
\hat{y}(t) = \frac{1}{2\phi'(t)} \left( \mu - \phi'(t) - \sigma^2 \kappa(t) \phi'(t) + \sqrt{\left( \mu - \phi'(t) - \sigma^2 \kappa(t) \phi'(t) \right)^2 + 4\mu \phi'(t)} \right)
\]

if \( \phi'(t) > 0 \).

**Mean-variance representation of the myopic demand.** In diffusion models of the form \( \frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t \, dW_t \) (with general drift and volatility processes \( (\mu_t)_{t \in [0,T]} \) and \( (\sigma_t)_{t \in [0,T]} \)), it is well known that the myopic demand of a power-utility investor is given by \( \frac{1}{p} \frac{\mu_t}{\sigma_t^2} \), i.e., by the fraction of the instantaneous expected return \( \mu_t \) and the instantaneous variance \( \sigma_t^2 \), scaled by the inverse of the risk aversion parameter \( p \) (see, e.g., [119, 105, 30]). So the myopic demand is rightly said to have a mean-variance representation. In the presence of jumps, this representation no longer holds true. In the case of logarithmic utility, however, a similar representation can be obtained if one computes the instantaneous variance not under \( P \), but under the dual minimiser \( Q \). Indeed, in our model, Theorem 5.4
below implies that
\[ \hat{\pi}_t = \pi^m_t = \frac{\mu}{\sigma^2 + (\tau_{\tilde{Q}_t})^2}, \]
i.e., the myopic demand (which equals the optimal strategy in this case) is given by the instantaneous expected return \( \mu \) relative to \( P \), divided by the instantaneous variance relative to the dual minimiser \( Q \) (cf. (3.7)–(3.8)). It can be shown that this representation holds true (up to some technical assumptions) for general semimartingale models.

This raises the question whether the myopic demand can also be written as a mean-variance ratio for \( p \neq 1 \). We proceed to show that in our model, this can be achieved if the instantaneous variance is computed under a suitable ELMM (which is a perturbation of the dual minimiser \( \tilde{Q} \)). Note that this is fully consistent with the mean-variance representation in diffusion models because there the instantaneous variance is invariant under equivalent changes of measure.

In preparation for the mean-variance representation of the myopic demand \( \pi^m \), define the function \( \tilde{y}^m : [0, T) \to (-1, \infty) \) by
\[ \tilde{y}^m(t) = y^m(t)h(y^m(t), p), \]
where \( h : (-1, \infty) \times (0, \infty) \to (0, \infty) \) is given by
\[
\begin{align*}
  h(y, p) &= \begin{cases} 
    \frac{1}{p} + \frac{1}{p((1+y)^{\frac{1}{p}} - 1)} - \frac{1}{y} & \text{if } y \neq 0, \\
    1 + p & \text{if } y = 0.
  \end{cases}
\end{align*}
\]
Using a Taylor expansion of the function \( y \mapsto (1+y)^{\frac{1}{p}} \) (of order 3) in 0, it is not difficult to check that \( h \in C^1((-1, \infty) \times (0, \infty)) \). Moreover, one can show (cf. Appendix D) that for \( y \in (-1, \infty) \),
\[
  1 \leq h(y, p) \leq \frac{1}{p} \quad \text{for } p \in (0, 1),
\]
\[
  h(y, 1) = 1,
\]
\[
  \frac{1}{p} \leq h(y, p) \leq 1 \quad \text{for } p \in (1, \infty).
\]

**Theorem 5.4.** Fix \( p \in (0, \infty) \). Then \( \tilde{y}^m \) satisfies (3.2) and (3.9). Let \( \tilde{Q}^m \) be the ELMM from Theorem 3.2 corresponding to \( \tilde{y}^m \). Then
\[ \pi_t^m = \frac{1}{p} \frac{\mu}{\sigma^2 + (\tau_{\tilde{Q}_t})^2}, \]
where \( (\tau_{\tilde{Q}_t})^2 = \phi'(t)^2 \frac{1 + \tilde{y}^m(t)}{\kappa'(t)} \mathbf{1}_{\{t \leq t, t < T\}} \) is the instantaneous jump variance of the stock return relative to \( \tilde{Q}^m \), see (3.8).

The proofs of (5.17) as well as Theorem 5.4 and the following Theorem 5.5 are mainly computational and can be found at the end of Appendix D.
Certainty equivalent. Finally we calculate the certainty equivalent of the optimal strategy $\hat{\pi}$.

**Theorem 5.5.** If $p = 1$, the certainty equivalent of trading in the market is

$$U^{-1}\left( E\left[ U(X_{T}^{\hat{\pi}})\right]\right) = x \exp\left(\frac{\mu^{2}}{2\sigma^{2}}T\right) \times \exp\left(-\int_{0}^{T} \phi'(u)^{2}\hat{y}(u)^{2}\left(1 - G(u)\right) du\right)$$

$$\times \exp\left(-\int_{0}^{T} \left(\log(1 + \hat{y}(u)) - \frac{\hat{y}(u)}{1 + \hat{y}(u)}\right)G'(u) du\right).$$

(5.19)

If $p \neq 1$, the certainty equivalent of trading in the market is

$$U^{-1}\left( E\left[ U(X_{T}^{\hat{\pi}})\right]\right) = x \exp\left(\frac{\mu^{2}}{2p\sigma^{2}}T\right) \times m(0, \hat{y}(0), p)^{-\frac{T}{2p\sigma^{2}}}.$$  

(5.20)

The different factors in (5.19) have a clear economic interpretation. The first is the certainty equivalent of the “Merton strategy” $\frac{\mu}{\sigma}$ in the Black–Scholes model. It is shown in the proof (see the end of Appendix D) that the product of the first and the second factor is the certainty equivalent of the strategy $\hat{\pi}$ in the Black–Scholes model, so that the second factor alone describes the relative certainty equivalent loss due to trading the strategy $\hat{\pi}$ (instead of $\frac{\mu}{\sigma}$) in the Black–Scholes model. Finally, the third factor expresses the certainty equivalent loss due to the presence of the bubble component $M^G\phi$.

In the case of general power utility, the first factor in (5.20) is again the certainty equivalent of the optimal “Merton strategy” $\frac{\mu}{p\sigma}$ in the Black–Scholes model, and the second one describes the combined relative certainty equivalent loss due to trading with the strategy $\hat{\pi}$ in the Black–Scholes model and due to the presence of the bubble component $M^G\phi$.

**Derivation of the integral equation (5.5).** We proceed to provide a heuristic derivation of the integral equation (5.5) and the optimal strategy (5.6). To this end, by virtue of Proposition 3.1, we assume that a triplet $(\pi, Q, z)$ consisting of an admissible strategy $\pi$, an ELMM $Q$ for $S$ belonging to the class considered in Theorem 3.2 and a number $z > 0$ satisfies the first optimality condition

$$(OC1) \quad U'(X_{T}^{\pi}) = z \frac{dQ}{dP}.$$  

We proceed in three steps; for ease of notation, we often drop arguments (in particular time) in the calculations.

**Step 1.** Since $Q$ belongs to the class of ELMMs considered in Theorem 3.2, there exists a nice function $y \in C^{1}[0, T]$ such that the density process $Z$ of $Q$ with respect to $P$ is given by

$$Z = E\left(-\int_{0}^{\gamma} \frac{1}{\sigma} \left(\mu - \phi' y, 1_{\{u \leq \gamma, u < T\}}\right) dW_u \right) M^{G}\zeta,$$

(5.21)

Note that for the derivation of the optimal strategy we do not need to consider the second optimality condition (OC2) in Proposition 3.1. (OC2) is only needed for the verification.
where \( \zeta(t) = \exp \left( - \int_0^t \kappa G y \, du \right) \). By (OC1), \( X_T^\pi = (U')^{-1}(zZ_T) \), and so by (5.21), after some algebra,

\[
X_T^\pi = x \mathcal{E}_T \left( \int_0^T \pi_u \sigma \, dW^Q_u \right) \times \mathcal{E}_T \left( \int_0^T \pi_u \, d\mathcal{M}_u^H \left( \int_0^u \phi'(1 + y) \, dv \right) \right),
\]

where \( J_0 := \frac{1}{\sigma^s} z^{-\frac{1}{2}} \exp \left( \frac{1 - p}{2p \sigma^s} \mu^2 \right) \) and the function \( J : [0, T] \to (0, \infty) \) is defined by

\[
J(v) = \begin{cases} 
\exp \left( \int_v^T \frac{1 - p}{2p \sigma^s} \phi'(y - 2\mu + \frac{1}{p} \kappa G y) \, dv \right) & \text{if } v < T, \\
\exp \left( \int_0^T \frac{1 - p}{2p \sigma^s} \phi'(y - 2\mu + \frac{1}{p} \kappa G y) \, dv \right) & \text{if } v = T.
\end{cases}
\]

**Step 2.** By the SDE (2.10) for the wealth process \( X^\pi \) and the dynamics (3.6) of \( S \) under \( Q \),

\[
X_T^\pi = x \mathcal{E}_T \left( \int_0^T \pi_u \sigma \, dW^Q_u \right) \times \mathcal{E}_T \left( \int_0^T \pi_u \, d\mathcal{M}_u^H \left( \int_0^u \phi'(1 + y) \, dv \right) \right),
\]

where \( H \) denotes the distribution function of \( \gamma \) under \( Q \). Comparing (5.22) and (5.24), we make the educated guess that the first and second respective factors coincide and that the integrands of the “d \( W^Q \)-terms” coincide. This gives that the optimal strategy \( \pi \) satisfies

\[
\pi_t = \frac{1}{\sigma^s} (\mu - \phi'(t)y(t)1_{\{t \leq \gamma, \gamma < T\}}), \quad t \in [0, T].
\]

As \( \pi \) follows a deterministic function up to \( \gamma \), by (a formal application of) Proposition B.5,

\[
\mathcal{E}_T \left( \int_0^T \pi_u \, d\mathcal{M}_u^H \left( \int_0^u \phi'(1 + y) \, dv \right) \right) = \mathcal{M}_T^H \xi,
\]

where \( \xi(t) = \exp \left( \int_0^t \frac{1}{\sigma^s} (\mu - \phi'y) \phi'(1 + y) \, du \right), \ t \in [0, T] \). Next, using (B.9) as well as the definition (5.1) of \( a \) yields, after some algebra,

\[
X_T^\pi = x \mathcal{E}_T \left( \int_0^T \pi_u \sigma \, dW^Q_u \right) \times K(\gamma),
\]

where the function \( K : [0, T] \to (0, \infty) \) is defined by

\[
K(v) = \begin{cases} 
\exp \left( \int_v^T \frac{1}{\sigma^s} (\mu - \phi'y) \phi'(1 + y) \, du \right) a(v, y(v), p) & \text{if } v < T, \\
\exp \left( \int_0^T \frac{1}{\sigma^s} (\mu - \phi'y) \phi'(1 + y) \, du \right) & \text{if } v = T.
\end{cases}
\]

**Step 3.** Equating the second factors on the right-hand sides of (5.22) and (5.25) gives

\[
\frac{K(v)}{J(v)} = J_0, \quad v \in [0, T].
\]
Using (5.27) for \( v < T \) and \( v = T \) and rearranging terms yields
\[
\frac{K(v)/K(T)}{J(v)/J(T)} = 1, \quad v \in [0, T) .
\] (5.28)

The definitions of \( J \), \( K \), \( m \) and \( n \) in (5.23), (5.26), (5.3) and (5.4) give, after some algebra,
\[
\frac{K(v)/K(T)}{J(v)/J(T)} = m(v, y(v), p) \exp \left( \int_{v}^{T} n(u, y(u), p) \, du \right), \quad v \in [0, T). 
\]

Finally, plugging this into (5.28) and rearranging terms yields the integral equation (5.5).

6 Numerical illustrations

In this section, we use numerical illustrations to answer the following five questions:

(1) How does a shift in the model parameters affect the optimal strategy and its myopic and hedging demands?

(2) Can the optimal strategy involve short selling or investing more than the Merton proportion?

(3) What are the consequences, for the myopic demand, of vanishing, bounded and exploding instantaneous jump volatilities?

(4) Does the optimal strategy distinguish fundamentally between whether or not the price process is a strict local martingale under the dual minimiser \( \hat{Q} \) (or under \( \tilde{Q}^{m} \))?

(5) How big is the welfare loss of trading in our model in comparison to optimal investment in a Black–Scholes model? And how does the welfare loss depend on shifts in the model parameters?

Recall that after the bubble has burst, it is optimal to keep a constant fraction of wealth \( \frac{\mu \sigma^2}{\varpi} \) (the Merton proportion) in the stock. So in the following, we are only interested in the optimal strategy before the crash. Thus, all the plots in the following tables show the fraction of wealth invested in the stock (depending on time \( t \)), conditional on the event that the bubble has not burst yet.

(1) **Comparative statics of the myopic and hedging demands.** Theorem 5.3 states that the sign of the hedging demand \( \pi^h \) is determined by the investor’s relative risk aversion \( p \). Thus, we provide illustrations for the cases of high \(( p > 1 )\) and low \(( p < 1 )\) risk aversion. The limiting case of logarithmic utility \(( p = 1 )\) always leads to a vanishing hedging demand and so the optimal strategy equals the myopic demand. It turns out that the qualitative behaviour
Table 6.1: Optimal strategies (top row), myopic demands (middle row) and hedging demands (bottom row) for high relative risk aversion ($p = 4$); the line strength corresponds to the size of the parameter given in the head of each column (dotted lines represent the smallest value, etc.) with default parameters $\mu = 0.1$, $\sigma = 0.2$ and $\alpha = 0.2$. The setup is $T = 1$, $G(t) = 1 - \exp(-t)$ and $\phi'(t) = \alpha$; in particular, the relative jump size is $\delta(t) \equiv \alpha$.

of the optimal strategy in this case closely resembles the behaviour of the myopic demand of the optimal strategy in the case $p \neq 1$. Therefore, we omit illustrations for the case $p = 1$.

Tables 6.1 and 6.2 depict the optimal strategy before the crash as well as its decomposition into myopic and hedging demands for various choices of $\mu$, $\sigma$ and $\alpha$, a parameter describing the relative jump size of the stock price process $S$, for high ($p > 1$) and low ($p < 1$) risk aversion, respectively. As one expects from the mean-variance representation (5.18), the myopic demand is increasing in the instantaneous expected return $\mu$ and decreasing in the instantaneous continuous volatility $\sigma$ as well as in the relative jump size $\alpha$. Note that the myopic part is constant in Tables 6.1 and 6.2. This is because equation (5.10) determining the myopic demand becomes independent of time $t$ for our choice of $G$ and $\phi'$. In general, the myopic demand will not be constant (cf. Table 6.4).

The qualitative behaviour of the hedging demand, however, depends crucially on the relative risk aversion. In the case of high risk aversion ($p > 1$), the hedging demand is always nonnegative and has the same monotonicity properties as the myopic demand. In the case of low risk aversion ($p < 1$), the hedging demand is
(2) Short selling and investing more than the Merton proportion. The optimal strategy includes short selling if and only if the investor’s relative risk aversion \( p \) is smaller than 1 and the (nonpositive) hedging demand exceeds the (nonnegative) myopic demand in absolute value. The interpretation is that the investor is happy to speculate on an early bursting of the bubble; he is willing to accept an outcome worse than that of the myopic strategy in case the bubble bursts late for the possibility of much better gains in case the bubble bursts early. Table 6.2 shows that short selling is amplified by “good” post-crash investment opportunities, i.e., low \( \sigma \) and high \( \mu \). For high relative risk aversion (\( p > 1 \)), the myopic and hedging demands are always nonnegative (by Theorem 5.3); hence the optimal strategy never involves short selling.

When \( p > 1 \), the optimal strategy may lie above the Merton proportion (Table 6.3). At first glance, this might be surprising as the volatility of our model
Table 6.3: Under extreme circumstances, the optimal strategy before the crash (solid, left panel) may lie above the Merton proportion (dashed). The middle and right panels show the corresponding myopic and hedging demands, respectively. The setup is $T = 1$, $G(t) = 1 - \exp(-t)$ and $\phi'(t) = 0.2t$. The parameters are $\mu = 0.3$, $\sigma = 0.05$ and $p = 4$.

is higher than in the corresponding Black–Scholes model due to the presence of the extra jump component. However, on closer inspection, this effect can be explained by a combination of a high myopic demand at time 0 and a hedging demand that is sufficiently increasing close to time 0.

(3) Dependence of the myopic demand on the instantaneous jump variance. The dependence of the myopic demand on the instantaneous jump variance is given in (5.18). If the latter vanishes at the time horizon, the pre-crash myopic demand converges to the Merton proportion. If it converges to a finite, positive number, then the pre-crash myopic demand converges to a finite value between 0 and the Merton proportion, and if it explodes, then the pre-crash myopic demand converges to 0. Thus, the behaviour of the pre-crash myopic demand can be qualitatively very different. Table 6.4 displays the pre-crash myopic demand and the corresponding instantaneous jump volatilities for the three cases.

(4) No fundamental distinction between strict local martingales and true martingales. Financial bubbles are often defined as stochastic processes that are strict local martingales under the chosen pricing measure; see the introduction of Section 4. Therefore, it is natural to ask whether an investor’s optimal strategy depends fundamentally on whether or not the price process is a strict local martingale under the dual minimiser $\hat{Q}$.

The solid lines in Table 6.5 illustrate the optimal strategy and its decomposition into myopic and hedging demand in the case where $S$ is a strict local martingale under the dual minimiser $\hat{Q}$, under $Q^m$, and in fact under any ELMM $Q$ obtained via Theorem 3.2 under the additional condition (3.9). For $\alpha = 1$, the setup of Table 6.5 coincides with the example given after Corollary 3.4. However, for any $\alpha \in [0, 1)$, the stock price process $S$ is a true martingale under $\hat{Q}$ (by Corollary 3.4), and the dotted lines in Table 6.5 depict the optimal strategy and
\[ \phi'(t) = 0.2 \quad \phi'(t) = 0.2 \frac{1}{\sqrt{1-t}} \quad \phi'(t) = 0.2 \frac{1}{1-t} \]

Table 6.4: Myopic demands of the optimal strategy (solid, top row) before the crash for different choices of \( \phi' \), Merton proportion (dashed), and corresponding instantaneous jump volatilities \( \tilde{\tau}^m := \tilde{\tau}^{Q^m} \) of the stock returns process relative to \( \tilde{Q}^m \) (bottom row). The setup is \( T = 1 \), \( G(t) = t \) and the \( \phi' \) are given above; in particular, the relative jump sizes are \( \delta(t) = 0.2(1-t) \), \( \delta(t) = 0.2\sqrt{1-t} \) and \( \delta(t) = 0.2 \), respectively. The parameters are \( \mu = 0.1 \), \( \sigma = 0.2 \) and \( p = 4 \).

Table 6.5: The optimal strategy does not distinguish qualitatively between \( S \) being a strict local martingale or a true martingale under the dual minimiser \( \hat{Q} \) or under \( \tilde{Q}^m \). The setup is \( T = 1 \), \( G(t) = t \) and \( \phi'(t) = \alpha(\frac{1}{1-t} - 1) \); in particular, the relative jump size is \( \delta(t) = \alpha t \). The solid lines correspond to \( \alpha = 1 \), for which \( S \) is a strict local martingale under \( \hat{Q} \) (and under \( \tilde{Q}^m \)), the dotted lines correspond to \( \alpha = 0.7 \), for which \( S \) is a true martingale under \( \hat{Q} \) (and under \( \tilde{Q}^m \)). The dashed lines represent the Merton proportion. The parameters are \( \mu = 0.1 \), \( \sigma = 0.2 \) and \( p = 4 \).
its decomposition into myopic and hedging demand for $\alpha = 0.7$. The pictures show that the qualitative behaviour of the optimal strategies is quite similar. In fact, the optimal strategies converge (numerically) as $\alpha \uparrow 1$.

So at least from our idealising point of view in the context of optimal investment, investors’ actions do not seem to clearly distinguish between the presence of strict local martingales and true martingales.

\begin{table}[h]
\centering
\begin{tabular}{ccc}
$p = 0.25$ & $p = 1$ & $p = 4$
\end{tabular}
\end{table}

Table 6.6: Dependence of the relative equivalent safe rate loss ($r\text{ESRL}$) on $\mu$ and $\sigma$ for $\alpha = 0.1$ (dotted), $\alpha = 0.2$ (dot-dashed), $\alpha = 0.4$ (dashed) and $\alpha = 0.8$ (solid). The setup is $T = 1$, $G(t) = 1 - \exp(-t)$ and $\phi'(t) = \alpha$. The parameters are $\sigma = 0.2$ (top row) and $\mu = 0.1$ (bottom row).

**Comparative statics of the welfare loss relative to the Black–Scholes model.** By Theorem 5.5, the addition of a bubble component to the Black–Scholes model reduces the certainty equivalent of trading in the market. We aim to analyse the influence of the model parameters on this welfare loss. A natural quantity to compare different markets is the equivalent safe rate. If $CE$ denotes the certainty equivalent of trading in some market with initial capital $x$ and time horizon $T$, then the equivalent safe rate is defined as the unique solution $r := ESR$ to the equation $xe^{rT} = CE$. In other words, the investor is indifferent between trading in this market and receiving a safe annualised return $r$ on his initial capital.\(^{11}\)

Let $CE^{BS} = x \exp\left(\frac{\mu^2}{2\sigma^2} T\right)$ denote the certainty equivalent of trading in a

\(^{11}\)In a different setting, [64] define the equivalent safe rate slightly differently: they look at the “long-run” equivalent safe rate, i.e., the limit as $T \uparrow \infty$. 
In a Black–Scholes market. The corresponding \textit{equivalent safe rate} is then given by

\[
ESR_{BS} = \frac{1}{T} \log \left( \frac{CE_{BS}}{x} \right) = \frac{\mu^2}{2p\sigma^2}.
\]

Denoting the certainty equivalent of trading in our market given in (5.19) and (5.20) by \( CE \), the corresponding equivalent safe rate is given by

\[
ESR = \frac{1}{T} \log \left( \frac{CE}{x} \right) = ESR_{BS} - \left\{ \begin{array}{ll}
\frac{p^2}{1-p} \frac{1}{T} \log m(0, \hat{y}(0), p) & \text{if } p \neq 1,
\frac{1}{T} \int_0^T \left( \frac{\varphi'(u)\hat{y}(u)}{2\sigma^2\varphi(u)} + \log(1 + \hat{y}(u)) - \frac{\hat{y}(u)}{1+\hat{y}(u)} \right) G'(u) \, du & \text{if } p = 1.
\end{array} \right.
\]

(6.1)

In order to improve comparability over different sets of parameters, we consider the \textit{relative equivalent safe rate loss} \( r_{ESRL} = 1 - \frac{ESR}{ESR_{BS}} \) below; it is a relative measure for the incurred losses of trading in our market compared to trading in a Black–Scholes market. It follows from (6.1) that

\[
r_{ESRL} = \frac{2\sigma^2}{\mu^2} \times \left\{ \begin{array}{ll}
\frac{p^2}{1-p} \frac{1}{T} \log m(0, \hat{y}(0), p) & \text{if } p \neq 1,
\frac{1}{T} \int_0^T \left( \frac{\varphi'(u)\hat{y}(u)}{2\sigma^2\varphi(u)} + \log(1 + \hat{y}(u)) - \frac{\hat{y}(u)}{1+\hat{y}(u)} \right) G'(u) \, du & \text{if } p = 1.
\end{array} \right.
\]

Table 6.6 illustrates the dependence of the \( r_{ESRL} \) on the model parameters \( \mu \) and \( \sigma \) as well as on the parameter \( \alpha \) describing the relative jump size of the stock price process \( S \). In the case \( p = 4 \), the \( r_{ESRL} \) is increasing in \( \alpha \) and decreasing in \( \sigma \), while being almost constant in \( \mu \). This is because the diffusive part becomes more and more dominant against the jump part with decreasing \( \alpha \) or increasing \( \sigma \), and the \( r_{ESRL} \) becomes smaller.

In the case \( p = 0.25 \), the dependencies are much less clear. On the one hand, if \( \mu \) is sufficiently small and/or \( \sigma \) is sufficiently large, then the \( r_{ESRL} \) is increasing in \( \alpha \). The reason is that in this case, as has been observed above (cf. Table 6.2), the optimal strategy does not involve short selling. Therefore, the higher \( \alpha \), the higher the losses at the bursting of the bubble, and the investor prefers small jump sizes; this means that the \( r_{ESRL} \) is increasing in \( \alpha \).

On the other hand, if \( \mu \) is high enough and/or \( \sigma \) is low enough, so that the optimal strategy involves a significant short position for a significant amount of time, then the investor’s wealth is likely to increase at the bursting of the bubble. Under these circumstances, the investor prefers larger jump sizes; in other words, the \( r_{ESRL} \) is decreasing in \( \alpha \).

An interesting observation is that for small \( \sigma \), i.e., when the jump part dominates the diffusive part, investors with a small relative risk aversion lose only a small fraction of their \( ESR \) compared to an investment into a Black–Scholes market. On the contrary, investors with high relative risk aversion face huge losses in their \( ESR \) for small \( \sigma \). Again, the reason is that investors with low relative risk aversion are happy to take considerable short positions when \( \sigma \) is very small, which leads to huge gains in the event that the bubble bursts early, while the optimal strategy for investors with high relative risk aversion never involves short selling.
A Change of filtration

Recall from Section 2 that the (raw) filtrations $\mathbb{F}^W = (\mathcal{F}^W_t)_{t \in [0,T]}$, $\mathbb{F}^\gamma = (\mathcal{F}^\gamma_t)_{t \in [0,T]}$ and $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ are defined by

$$\mathcal{F}^W_t = \sigma (W_u : 0 \leq u \leq t), \quad \mathcal{F}^\gamma_t = \sigma (1_{\{\gamma \leq u\}} : 0 \leq u \leq t), \quad \mathcal{F}_t = \sigma (\mathcal{F}^W_t, \mathcal{F}^\gamma_t),$$

and that $\mathbb{F}^W$ and $\mathbb{F}^\gamma$ are independent under $P$. The key message of the following technical result is that (local) $\mathbb{F}^\gamma$-martingales are (local) $\mathbb{F}$-martingales not only under $P$ but also under certain equivalent measures $Q \approx P$, under which $\mathbb{F}^W$ and $\mathbb{F}^\gamma$ are no longer independent.

**Lemma A.1.** Let the function $k : [0,T]^2 \to \mathbb{R}$ be of the form

$$k(t,v) = \hat{k}(t) + \check{k}(t) 1_{\{t \leq v, t < T\}},$$

where $\hat{k}, \check{k} \in L^2([0,T])$. Set $Y^1 := \mathcal{E} \left( \int_0^\cdot k(u, \gamma) \, dW_u \right)$ and let $Z^2$ be a positive $\mathbb{F}^\gamma$-martingale with $Z^2_0 = 1$.

(a) Let $Y^2$ be an $\mathbb{F}^\gamma$-adapted càdlàg process.

(i) The following are equivalent:

- $Y^2$ is an $\mathbb{F}^\gamma$-martingale;
- $Y^2$ is an $\mathbb{F}$-martingale;
- $Y^1 Y^2$ is an $\mathbb{F}$-martingale.

(ii) If $Y^2$ is a local $\mathbb{F}^\gamma$-martingale, then $Y^2$ and $Y^1 Y^2$ are local $\mathbb{F}$-martingales.

(b) Define $Q^\gamma, Q \approx P$ on $\mathcal{F}_T$ by $dQ^\gamma = Z^2_0$ and $dQ = Y^1_0 Z^2_0$.\footnote{Note that (a) (i) with $Y^2 := Z^2$ shows that $Y^1 Z^2$ is a positive $\mathbb{F}$-martingale with $Y^1_0 Z^2_0 = 1$.} Let $X^{2,Q}$ be an $\mathcal{F}^\gamma_T$-measurable random variable and $Y^{2,Q}$ an $\mathbb{F}^\gamma$-adapted càdlàg process.

(i) $X^{2,Q}$ is $Q$-integrable if and only if it is $Q^\gamma$-integrable, and in this case,

$$E^Q \left[ X^{2,Q} \mid \mathcal{F}_s \right] = E^{Q^\gamma} \left[ X^{2,Q} \mid \mathcal{F}_s \right] \quad P\text{-a.s.}, \quad s \in [0,T]. \quad (A.1)$$

(ii) $Y^{2,Q}$ is a (square-integrable) $(Q^\gamma, \mathbb{F}^\gamma)$-martingale if and only if it is a (square-integrable) $(Q, \mathbb{F})$-martingale.

(iii) If $Y^{2,Q}$ is a local $(Q^\gamma, \mathbb{F}^\gamma)$-martingale, then it is also a local $(Q, \mathbb{F})$-martingale.

Proof. First, we show that an $\mathcal{F}^\gamma_T$-measurable random variable $X^2$ is integrable if and only if $Y^1 X^2$ is so, and in this case,

$$E \left[ Y^1 X^2 \mid \mathcal{F}_s \right] = Y^1_s E \left[ X^2 \mid \mathcal{F}^\gamma_s \right] \quad P\text{-a.s.}, \quad s \in [0,T]. \quad (A.2)$$
By linearity, we may assume that $X^2$ is nonnegative. Then the first assertion follows from (A.2) for $s = 0$. To establish (A.2), fix $s \in [0, T]$ and set

$$C_s := \{C = C^W \cap \gamma \leq u : C^W \in \mathcal{F}_s^W, u \leq s\}.$$

Then $C_s$ is an intersection-closed generator of $\mathcal{F}_s$, and by $\mathcal{F}_s$-measurability of both sides of (A.2), positivity of $Y_s^1$ and a monotone class argument, it suffices to show that

$$E \left[ \frac{Y_t^1}{Y_s^1} X^2 1_C \right] = E \left[ X^2 \mid \mathcal{F}_s \right] 1_C \quad \text{for all } C \in C_s \cup \{\Omega\}. \quad (A.3)$$

To establish (A.3), for fixed $v \in [0, T]$, set $Y^{1,v} = \mathcal{E} \left( \int_0^v k(u, v) \, dW_u \right)$. By the assumption on $k$ and Novikov's condition, each $Y^{1,v}$ is a positive $\mathbb{F}$-martingale with $Y_0^{1,v} = 1$ and hence satisfies

$$E \left[ \frac{Y_t^1}{Y_s^1} Y^{1,v} 1_A \right] = E \left[ Y^{1,v} 1_A \right], \quad s \in [0, T], A \in \mathcal{F}_s. \quad (A.4)$$

Moreover, by independence of $\mathcal{F}_T^\gamma = \sigma(\gamma)$ and $W$, a monotone class argument and (A.4),

$$E \left[ \frac{Y_t^1}{Y_s^1} \mid \mathcal{F}_T^\gamma \right] = E \left[ \frac{Y_t^{1,v}}{Y_s^{1,v}} \right] = 1 \quad P\text{-a.s.,} \quad s \in [0, T]. \quad (A.5)$$

Now, (A.3) for $C = \Omega$ follows from $\mathcal{F}_T^\gamma$-measurability of $X^2$ and (A.5) via

$$E \left[ \frac{Y_t^1}{Y_s^1} X^2 \right] = E \left[ E \left[ \frac{Y_t^1}{Y_s^1} \mid \mathcal{F}_T^\gamma \right] X^2 \right] = E \left[ X^2 \right] = E \left[ E \left[ X^2 \mid \mathcal{F}_s \right] \right].$$

If $C = C^W \cap \{\gamma \leq u\}$, where $C^W \in \mathcal{F}_s^W$ and $u \leq s$, then $Y_t^1/Y_s^1 = Y_t^{1,0}/Y_s^{1,0}$ on $C$ since $k(t, v) = k(t) = k(t, 0)$ for $t > v$. Moreover, $Y_t^{1,0}/Y_s^{1,0} 1_{C^W}$ is $\mathcal{F}_s^W$-measurable and $X^2 1_{\{\gamma \leq u\}}$ and $E \left[ X^2 \mid \mathcal{F}_s \right] 1_{\{\gamma \leq u\}}$ are $\mathcal{F}_s^\gamma$-measurable. This, independence of $\mathcal{F}_T^W$ and $\mathcal{F}_T^\gamma$ and (A.4) for $A = C^W$ yield

$$E \left[ \frac{Y_t^1}{Y_s^1} X^2 1_C \right] = E \left[ \frac{Y_t^{1,0}}{Y_s^{1,0}} X^2 1_C \right] = E \left[ \frac{Y_t^{1,0}}{Y_s^{1,0}} 1_{C^W} \right] E \left[ X^2 1_{\{\gamma \leq u\}} \right]$$

$$= E \left[ 1_{C^W} E \left[ X^2 \mid \mathcal{F}_s^\gamma \right] 1_{\{\gamma \leq u\}} \right] = E \left[ E \left[ X^2 \mid \mathcal{F}_s \right] 1_{C^W} 1_{\{\gamma \leq u\}} \right]$$

$$= E \left[ E \left[ X^2 \mid \mathcal{F}_s \right] 1_C \right].$$

Second, we establish (a). By the first part of the proof, $X^2 := Y_t^2$ is integrable if and only if $Y_t^1 Y_t^2$ is so, and in this case,

$$E \left[ Y_t^1 Y_t^2 \mid \mathcal{F}_s \right] = Y_s^1 E \left[ Y_t^2 \mid \mathcal{F}_s \right] \quad P\text{-a.s.,} \quad s \in [0, T]. \quad (A.6)$$

Since $Y_s^1 > 0 \ P\text{-a.s.},$ (A.6) shows that $Y^2$ is an $\mathbb{F}^\gamma$-martingale if and only if $Y^1 Y^2$ is an $\mathbb{F}$-martingale, and for $Y^1 \equiv 1$ (i.e., for $k \equiv 0$), this implies that $Y^2$ is an $\mathbb{F}^\gamma$-martingale if and only if it is an $\mathbb{F}$-martingale. So we have (i). To establish
(ii), let \( \tau \) be a \([0, T]\)-valued \( \mathbb{F}\)-stopping time. Then by the \( \mathbb{F}\)-martingale property of \( Y^1 \) (which follows from Novikov’s condition and the assumptions on \( k \)) and (A.2) for \( X^2 = |Y^2_\tau| \) and \( s = 0 \),

\[
E \left[ Y^1_\tau | Y^2_\tau \right] = E \left[ E \left[ Y^1_\tau | \mathcal{F}_\tau \right] | Y^2_\tau \right] = E \left[ E \left[ Y^1_\tau | Y^2_\tau \right] | \mathcal{F}_\tau \right] = E \left[ Y^1_\tau | Y^2_\tau \right] = E \left[ |Y^2_\tau| \right].
\]

This implies that \( Y^1_\tau Y^2_\tau \) is integrable if and only if \( Y^2_\tau \) is so. Now if the stopped process \((Y^2)^\tau\) is an \( \mathbb{F}\)-martingale, by the \( \mathbb{F}\)-martingale property of \( Y^1 \), the \( \mathbb{F}\)-martingale property of \((Y^2)^\tau\) and (A.2) for \( X := (Y^2)^\tau \), for \( s \in [0, T] \),

\[
E \left[ Y^1_\tau Y^2_\tau 1_{\{\tau > s\}} | \mathcal{F}_s \right] = E \left[ E \left[ Y^1_\tau Y^2_\tau 1_{\{\tau > s\}} | \mathcal{F}_{\tau \wedge s} \right] | \mathcal{F}_s \right] = E \left[ E \left[ Y^1_\tau Y^2_\tau 1_{\{\tau > s\}} | \mathcal{F}_{\tau \wedge s} \right] | \mathcal{F}_s \right] = E \left[ Y^1_\tau (Y^2)^\tau_{\tau \wedge s} 1_{\{\tau > s\}} \right] = Y^1_s E \left[ (Y^2)^\tau_{\tau \wedge s} \right] 1_{\{\tau > s\}} \quad \text{P-a.s.}
\]

Thus, \((Y^1)^\tau(Y^2)^\tau\) is an \( \mathbb{F}\)-martingale because

\[
E \left[ (Y^1)^\tau(Y^2)^\tau \mid \mathcal{F}_s \right] = Y^1_s Y^2_\tau 1_{\{\tau \leq s\}} + E \left[ Y^1_\tau Y^2_\tau 1_{\{\tau > s\}} \mid \mathcal{F}_s \right] = (Y^1)^\tau_s (Y^2)^\tau_s \quad \text{P-a.s.}
\]

For \( Y^1 \equiv 1 \) (i.e., for \( k \equiv 0 \)), this also implies that \( (Y^2)^\tau \) is an \( \mathbb{F}\)-martingale. So if \((\tau_n)_{n \in \mathbb{N}}\) is a localising sequence for \( Y^2 \) in \( \mathbb{F}\), it is also a localising sequence for \( Y^1 \) and \( Y^1 Y^2 \) in \( \mathbb{F}\), and we have (ii).

Finally, we establish (b). For (i), set \( X^2 = Z^2_{\tau_1} \). Then by Bayes’ theorem, (A.2) for \( s = 0 \) and again Bayes’ theorem,

\[
E^Q \left[ |X^2| \right] = E^P \left[ Y^1_\tau Z^2_\tau | X^2 \right] = E^P \left[ Y^1_\tau | X^2 \right] = E^P \left[ |X^2| \right] \]

\[
= E^P \left[ Z^2_\tau | X^2 \right] = E^Q \left[ |X^2| \right],
\]

which shows that \( X^2 \) is \( Q\)-integrable if and only if it is \( Q^\gamma\)-integrable. Now the same argument yields (A.1) using (A.2) for general \( s \in [0, T] \).

For (ii) and (iii), set \( Y^2 = Z^2 Y^2 \). Then by Bayes’ theorem, \( Y^2 \) is a local \((\gamma, \mathbb{F})\)-martingale if and only if \( Y^2 \) is a local \((P, \mathbb{F})\)-martingale. Likewise by Bayes’ theorem, \( Y^2 \) is a local \((\mathbb{Q}, \mathbb{F})\)-martingale if and only if \( Y^1 Y^2 \) is a local \((\mathbb{P}, \mathbb{F})\)-martingale. Now (ii) and (iii) follow from (a) (i) and (ii) using also the fact that \( Y^2 \gamma^\tau \) is \( Q\)-integrable if and only if it is \( Q^\gamma\)-integrable; this follows from (A.1) for \( s = 0 \) and \( X^2 = (Y^2)^2 \) using the fact that a martingale on a finite time horizon is square-integrable if and only if it is square-integrable at the final time.

\[ \square \]

## B Single jump local martingales

In this section, we collect some results on single jump local martingales, i.e., processes of the form \( \mathcal{M}_G^\gamma F = F(t)1_{\{t < \gamma\}} + A^G \bar{F}(\gamma) 1_{\{t \geq \gamma\}} \) (cf. Section 2). They are partly simplified versions of results in Chapter IV and [73]. Recall that \( G \) is the distribution function of \( \gamma \), \( \kappa^G(v) = \frac{G(v)}{1-G(v)} \) is the corresponding hazard
rate, and \( A^G F(v) = F(v) - \frac{F'(v)}{\kappa^G(v)} \) for \( v < T \). The underlying filtration is always \( \mathbb{F}^\gamma = (\mathcal{F}^\gamma_t)_{t \in [0,T]} \). To prevent possible confusions, we stress that all processes of the form \( \mathcal{M}^G \) are defined on the closed interval \([0,T]\) even though \( F \) is only defined on the half-open interval \([0,T)\). In particular, all assertions on (local) martingale properties of \( \mathcal{M}^G \) refer to the closed interval \([0,T]\).

We begin with a characterisation of the condition \( \Delta G(T) = 0 \) in terms of integrability of the hazard rate.

**Proposition B.1.** The following are equivalent:

(a) The hazard rate \( \kappa^G \) is nonintegrable on \((0,T)\).

(b) \( G(T-) = 1 \).

(c) \( \Delta G(T) = 0 \).

**Proof.** As \( \gamma \) is \((0,T]\)-valued, \( G(T) = 1 \), so that the equivalence “(b) \iff (c)” is trivial. Next, by the definition of \( \kappa^G \), \( \kappa^G(t) = -\frac{d}{dt} \log(1 - G(t)) \) for \( t \in [0,T) \).

Integrating both sides over \((0,T)\) yields (with \( \log 0 := -\infty \))

\[
\int_0^T \kappa^G(u) \, du = - \log(1 - G(T-))
\]

and proves the equivalence “(a) \iff (b)". \( \square \)

**Martingale properties.** First, we give conditions for \( \mathcal{M}^G \) to be a (local or square-integrable) martingale. If \( \mathcal{M}^G \) is square-integrable, we provide a formula for its predictable quadratic variation \( \langle \mathcal{M}^G \rangle \) and show that it can be obtained as integrated instantaneous variance.

**Proposition B.2.** Let \( F \in C^1[0,T) \).

(a) The process \( \mathcal{M}^G \) is an integrable local martingale if and only if

\[
\int_0^T |A^G F(u)| G'(u) \, du < \infty.
\]

A sufficient condition for \( \int_0^T |A^G F(u)| G'(u) \, du < \infty \) is that \( F \) and \( A^G F \) are bounded from below on \((0,T)\). This is automatically satisfied if \( \mathcal{M}^G \) is nonnegative.

(b) Suppose that \( \mathcal{M}^G \) is an integrable local martingale.

(i) If \( \Delta G(T) > 0 \), then \( \mathcal{M}^G \) is a martingale and the limit \( \lim_{t \uparrow T} F(t) \) exists in \( \mathbb{R} \).

(ii) If \( \Delta G(T) = 0 \), then \( \mathcal{M}^G \) is a martingale if and only if

\[
\lim_{t \uparrow T} F(t)(1 - G(t)) = 0.
\]
Proof. (a): By Lemma IV.3.4, $\mathcal{M}^G F$ is integrable if and only if
\[
\int_0^T |\mathcal{M}^G F(u)|G'(u) \, du < \infty.
\]
Moreover, in this case $\mathcal{M}^G F$ is automatically a local martingale by Lemma IV.3.7. The second assertion follows from the implication “(b) $\Rightarrow$ (a)” of Lemma IV.2.5.

(b): Part (i) follows from (a), Lemma IV.2.6, and Theorem IV.3.5 (c); part (ii) follows from Lemma IV.3.7 (b).

Proposition B.3. Let $F \in C^1[0,T)$ be such that $\int_0^T \left( \frac{F'(u)}{\kappa^G(u)} \right)^2 G'(u) \, du < \infty$.

Then $\mathcal{M}^G F$ is a square-integrable $P$-martingale and its predictable quadratic variation (relative to $P$) is given by
\[
\langle \mathcal{M}^G F \rangle_t = \int_0^t \frac{F'(u)^2}{\kappa^G(u)} 1_{\{u \leq \gamma\}} \, du.
\]
Moreover, for $t \in (0,T)$, we have the short-time asymptotics
\[
\lim_{h \downarrow 0} \frac{1}{h} E \left[ (\mathcal{M}^G_{t+h} F - \mathcal{M}^G_{t-h} F)^2 \mid \mathcal{F}_t \right] = \frac{(F'(t))^2}{\kappa^G(t)} 1_{\{t \leq \gamma\}} \quad P\text{-a.s.}
\]

Proof. First, as $\int_0^T G'(u) \, du = 1 - \Delta G(T) \leq 1$, $\int_0^T \frac{F'(u)}{\kappa^G(u)} G'(u) \, du < \infty$ by the hypothesis and Jensen’s inequality. Thus, $\mathcal{M}^G F$ is an $H^1$-martingale by Lemmas IV.2.6, IV.3.4, and Theorem IV.3.5 (c) if $\Delta G(T) > 0$, and by Lemmas IV.2.8 and IV.3.9 if $\Delta G(T) = 0$. Moreover, since $\mathcal{M}^G F$ is purely discontinuous with a single jump at $\gamma$ on $\{\gamma < T\}$, its optional quadratic variation satisfies
\[
[\mathcal{M}^G F]_t = \sum_{0 < \epsilon \leq t} (\Delta \mathcal{M}^G F_\epsilon)^2 = \left( \frac{F'(\gamma)}{\kappa^G(\gamma)} \right)^2 1_{\{\gamma \leq t, \gamma < T\}} \quad P\text{-a.s.,} \quad t \in [0,T].
\]
In particular,
\[
E \left[ [\mathcal{M}^G F]_T \right] = \int_0^T \left( \frac{F'(u)}{\kappa^G(u)} \right)^2 G'(u) \, du < \infty.
\]
Thus, by the Burkholder–Davis–Gundy inequality, $\mathcal{M}^G F$ is a square-integrable martingale.

Second, to establish (B.1), define the function $B : [0,T) \to \mathbb{R}$ by
\[
B(t) = \int_0^t \frac{F'(u)^2}{\kappa^G(u)} \, du.
\]
Then $B$ is increasing, in $C^1[0,T)$ and $B(0) = 0$. Moreover,
\[
\mathcal{A} G B(v) = B(v) - \frac{F'(v)^2}{\kappa^G(v)^2}, \quad v \in [0,T),
\]
and $\mathcal{A}^G B(T) = B(T-) 1_{\Delta G(T)>0}$. As

$$\int_0^T (\mathcal{A}^G B(u)) - G'(u)\,du \leq \int_0^T \left( \frac{F'(u)}{\kappa^G(u)} \right)^2 G'(u)\,du < \infty$$

and $\limsup_{t\uparrow T} B(t)(1 - G(t)) > -\infty$, $\int_0^T |\mathcal{A}^G B(u)| \,G'(u)\,du < \infty$ by the implication "(b) ⇒ (a)" of Lemma IV.2.5. Thus, $\mathcal{M}^G B$ is an integrable local martingale by Proposition B.2 (a). Writing

$$\mathcal{M}^G_t B = B(t) 1_{\{t<\gamma\}} + \mathcal{A}^G B(\gamma) 1_{\{\gamma\leq t, \gamma<T\}} + \mathcal{A}^G B(T) 1_{\{\gamma=T=t\}}$$

$$= B(t \wedge \gamma) 1_{\{\gamma<\gamma\}} + B(T-) 1_{\{\gamma=T=t\}} - \left( \frac{F'(\gamma)}{\kappa^G(\gamma)} \right)^2 1_{\{\gamma\leq t, \gamma<T\}}$$

$$= \int_0^t \frac{F'(u)^2}{\kappa^G(u)} 1_{\{u\leq t\}}\,du - \left( \frac{F'(\gamma)}{\kappa^G(\gamma)} \right)^2 1_{\{\gamma\leq t, \gamma<T\}}, \quad t \in [0, T],$$

shows that $\int_0^t \frac{F'(u)^2}{\kappa^G(u)} 1_{\{u\leq t\}}\,du$ is the compensator of $[\mathcal{M}^G F] = \left( \frac{F'(\gamma)}{\kappa^G(\gamma)} \right)^2 1_{\{\gamma\leq t, \gamma<T\}}$, and so (B.1) follows from the definition of the predictable quadratic variation.

Third, to establish (B.2), fix $t \in (0, T)$. Since $\mathcal{M}^G_u F = F(t)$ for $u < \gamma$ and $F \in C^1[0, T)$,

$$\lim_{h \downarrow 0} \frac{1}{h} \left( \mathcal{M}^G_t F - \mathcal{M}^G_{t-h} F \right) 1_{\{t<\gamma\}} = \lim_{h \downarrow 0} \frac{1}{h} \left( F(t) - F(t-h) \right) 1_{\{t<\gamma\}}$$

$$= F'(t) 1_{\{t<\gamma\}} \quad \text{P-a.s.}$$

and $\frac{F(t) - F(t-h)}{h}$ is uniformly bounded by a constant for $h \in (0, t]$. Now (B.2) follows from Hunt’s lemma [44, Theorem V.45] and the fact that

$$E \left[ F'(t) 1_{\{t<\gamma\}} \bigg| \bigvee_{u<t} \mathcal{F}^\gamma_u \right] = F'(t) E \left[ 1_{\{t\leq\gamma\}} \bigg| \bigvee_{u<t} \mathcal{F}^\gamma_u \right] = F'(t) 1_{\{t\leq\gamma\}},$$

where we have used that $\{ t = \gamma \}$ is a $\mathcal{P}$-nullset and that $1_{\{t\leq\gamma\}}$ is $\bigvee_{u<t} \mathcal{F}^\gamma_u$-measurable because $\gamma$ is an $\mathcal{F}^\gamma$-stopping time.

Finally, to establish (B.3), by square-integrability of $\mathcal{M}^G F$ and the definition of the predictable quadratic variation, for fixed $h \in (0, t]$, 

$$\frac{1}{h} E \left[ (\mathcal{M}^G_t F - \mathcal{M}^G_{t-h} F)^2 \bigg| \mathcal{F}^\gamma_{t-h} \right] = E \left[ \frac{(\mathcal{M}^G_t F)_t - (\mathcal{M}^G_{t-h} F)_{t-h}}{h} \bigg| \mathcal{F}^\gamma_{t-h} \right].$$

Moreover, by (B.1), and since $F', \kappa^G \in C[0, T]$ and $\kappa^G > 0$,

$$\lim_{h \downarrow 0} \frac{1}{h} \left( \frac{(\mathcal{M}^G_t F)_t - (\mathcal{M}^G_{t-h} F)_{t-h}}{h} \right) = \lim_{h \downarrow 0} \frac{1}{h} \int_{t-h}^t \frac{F'(u)^2}{\kappa^G(u)} 1_{\{u\leq t\}}\,du$$

$$= \frac{F'(t)^2}{\kappa^G(t)} 1_{\{t\leq\gamma\}} + \text{P-a.s.},$$

and $\frac{F'(t)^2}{\kappa^G(t)} 1_{\{t\leq\gamma\}}$ is uniformly bounded by a constant. Now (B.3) follows from Hunt’s lemma as above.

Note that if $F \in C^1[0, T)$ with $0 \leq F' \leq \kappa^G$, then $F$ and $\mathcal{A}^G F$ are bounded from below, and $\mathcal{M}^G F$ is a square-integrable martingale by Proposition B.3.
Change of measure and removal of drift. The next result deals with the change of measure and the removal of drift for single jump local martingales.

**Proposition B.4.** Let \( F, y \in C^1[0, T) \) be such that \( 0 \leq F' \leq \kappa^G \) and

\[
\inf_{t \in [0,T)} y(t) > -1.
\]

Moreover, if \( \Delta G(T) > 0 \), assume that

\[
\int_0^T |F'(u)y(u)| \, du < \infty \quad \text{and} \quad \int_0^T \kappa^G(u)(1 + y(u)) \, du < \infty. \tag{B.4}
\]

Define the functions \( \zeta : [0, T) \to (0, \infty) \) and \( H : [0, \infty) \to [0, 1] \) by

\[
\zeta(t) = \exp\left(-\int_0^t \kappa^G(u)y(u) \, du\right), \tag{B.5}
\]

\[
H(t) = 1 - \exp\left(-\int_0^t \kappa^G(u)(1 + y(u)) \, du\right)1_{\{t < T\}}. \tag{B.6}
\]

Then \( \zeta \) is positive and \( \mathcal{M}^G \zeta \) is a positive \( P \)-martingale starting at 1. Define the measure \( Q^\gamma \approx P \) on \( F_T^1 \) by \( \frac{dQ^\gamma}{dP} = \mathcal{M}^G \zeta \). Then \( \gamma \) has distribution function \( H \) under \( Q^\gamma \), and for \( t \in [0, T) \),

\[
\mathcal{A}^G \zeta(t) = \zeta(t)(1 + y(t)), \tag{B.7}
\]

\[
1 - H(t) = \zeta(t)(1 - G(t)), \tag{B.8}
\]

\[
\kappa^H(t) := \frac{H'(t)}{1 - H(t)} = \kappa^G(t)(1 + y(t)). \tag{B.9}
\]

Moreover,

\[
\mathcal{M}^G F + \int_0^1 1_{\{u < \gamma\}} F'(u)y(u) \, du = \mathcal{M}^H \left( \int_0^1 F'(u)(1 + y(u)) \, du \right) \tag{B.10}
\]

is a square-integrable \( Q^\gamma \)-martingale.

**Proof.** We apply the more general “removal of drift” result [73, Theorem 4.2]. To this end, we define \( A \in C^1[0, T) \) by \( A(t) = \int_0^t F'(u) y(u) \, du \) and declare that \( 0/0 := 0 \). Then \( f := \frac{dF}{dG} = \frac{F'}{G'} \) and \( a := \frac{dA}{dG} = fy \) on \( [0, T) \). Note that if \( \Delta G(T) > 0 \), then by (B.4),

\[
\int_0^T |a(u)|G'(u) \, du = \int_0^T |f(u)y(u)|G'(u) \, du = \int_0^T |F'(u)y(u)| \, du < \infty,
\]

\[
\int_0^T |f(u)|G'(u) \, du = \int_0^T F'(u) \, du \leq \int_0^T \kappa^G(u) \, du = -\log \Delta G(T) < \infty,
\]

and so the assumptions in the first line of [73, Theorem 4.2] are satisfied. In addition, clearly \( \{f = 0\} \cap (0, T) \subset \{a = 0\} \), \( \frac{y}{f} = y > -1 \) on \( (0, T) \) and
\(\int_0^b \left| \frac{a(u)}{f(u)} \right| G'(u) \, du < \infty\) for each \(b \in (0, T)\), i.e., the conditions (4.8)–(4.10) (and trivially also (4.11)) in [73] are fulfilled.

We proceed to show that if \(\Delta G(T) = 0\), then (4.13) and (4.22) for \(h = 0\) in [73] are satisfied. Indeed, the hypothesis \(\inf_{t \in [0,T]} y(t) > -1\) together with the fact that \(\int_0^T \kappa^G(u) \, du = \infty\) gives

\[
\int_0^T \left( \frac{a(u)}{f(u)} + 1 \right) \frac{G'(u)}{1 - G(u)} \, du = \int_0^T (1 + y(u)) \kappa^G(u) \, du = \infty.
\]

Next, we establish (4.14) and (4.15) in [73] if \(\Delta G(T) > 0\). Set \(A(T) := \int_0^T F'(u)y(u) \, du\), which is well defined by (B.4). Then \(\Delta A(T) = 0\) and so we have (4.14) in [73]. Moreover, the identity \(\frac{\gamma}{\gamma} = y\) (B.4) and the identity \(\int_0^T \kappa^G(u) \, du = -\log \Delta G(T) < \infty\) give

\[
\int_0^T \left| \frac{a(u)}{f(u)} \right| \frac{G'(u)}{1 - G(u)} \, du = \int_0^T |y(u)| \kappa^G(u) \, du < \infty.
\]

As \(\frac{1}{1 - G} \geq 1\) on \((0, T)\), the above yields \(\int_0^T \left| \frac{a(u)}{f(u)} \right| G'(u) \, du < \infty\), and we have condition (4.15) in [73].

Now, if we define \(\zeta\) by (4.16) in [73] for \(h = 0\), this simplifies to (B.5), and the assertion about \(M^G\zeta\) follows from [73, Theorem 4.2]. If we define \(H\) by (B.6), then (B.8) follows from the identity \(1 - G(t) = \exp \left( -\int_0^t \kappa^G(t) \, dt \right)\), \(t \in [0, T]\).

Formula (4.23) in [73] then shows that \(\gamma\) has distribution function \(H\) under \(Q^\gamma\).\(^{13}\)

Moreover (B.7) and (B.9) are straightforward, and (B.10) follows from assertion (4.24) in [73]. Finally, note that the hypothesis \(0 \leq F' \leq \kappa^G\) implies via (B.9) that \(0 \leq F'(1 + y) \leq \kappa^H\), and so Proposition B.3 with \(P = Q^\gamma\) and \(G = H\) yields that \(M^H(\int_0^T F'(u)(1 + y(u)) \, du)\) is a square-integrable \(Q^\gamma\)-martingale. \(\square\)

**Stochastic exponential and strict local martingales.** The next result gives a formula for the stochastic exponential of \(M^G F\).

**Proposition B.5.** Let \(F \in C^1[0, T]\) be such that \(F(0) = 0\) and \(0 \leq F' \leq \kappa^G\) \((0 \leq F' < \kappa^G)\). Then \(\int_0^T |A^G(\exp \circ F)(u)|G'(u) \, du < \infty\) and

\[
\mathcal{E} \left( M^G F \right) = M^G(\exp \circ F) \tag{B.11}
\]

is a nonnegative (positive) local martingale.

**Proof.** First, note that the assumptions \(0 \leq F' \leq \kappa^G\) \((0 \leq F' < \kappa^G)\) imply that \(\Delta M^G F \geq -1\) \((\Delta M^G F > -1)\). Therefore, by the formula for the stochastic exponential (see [138, Theorem II 37]), \(\mathcal{E} \left( M^G F \right)\) is nonnegative (positive). The identity (B.11) is an easy calculation. Finally, nonnegativity of \(M^G(\exp \circ F)\) implies that \(\int_0^T |A^G(\exp \circ F)(u)|G'(u) \, du < \infty\) by Proposition B.2 (a), which then also shows that \(\mathcal{E} \left( M^G F \right)\) is an integrable local martingale. \(\square\)

\(^{13}\)Note that \(H\) is called \(G^Q\) in [73] and that \(G^Q(T) = Q[\gamma \leq T] = P[\gamma \leq T] = 1\) as \(Q \approx P\).
Using the above results, we establish a condition under which $\mathcal{E}(\mathcal{M}^G F)$ is a strict local martingale, i.e., a local martingale which is not a martingale.

**Corollary B.6.** Suppose that $\Delta G(T) = 0$, $F(0) = 0$ and $0 \leq F'(t) \leq \kappa^G(t)$. Then $\mathcal{E}(\mathcal{M}^G F)$ is a strict local $P$-martingale if and only if

$$\int_0^T (\kappa^G(u) - F'(u)) \, du < \infty.$$  

Moreover, suppose that $y \in C^1[0, T]$ satisfies

$$\epsilon \leq 1 + y(t) \leq C + \frac{C}{F'(t)} 1_{\{\kappa^G(t) < CF'(t)\}}, \quad t \in [0, T), \quad (B.12)$$

for some constants $\epsilon \in (0, 1]$ and $C \geq 1$. Define $\zeta$, $H$ and $Q^\gamma$ as in Proposition B.4. Then $\mathcal{E}(\mathcal{M}^H (\int_0^T F'(u)(1 + y(u)) \, du))$ is a strict local $Q^\gamma$-martingale if and only if $\mathcal{E}(\mathcal{M}^G F)$ is a strict local $P$-martingale.

**Proof.** First, for $t \in [0, T)$, (2.1) and (B.9) give

$$1 - G(t) = \exp \left( - \int_0^t \kappa^G(t) \, dt \right), \quad (B.13)$$

$$1 - H(t) = \exp \left( - \int_0^t \kappa^H(t) \, dt \right) = \exp \left( - \int_0^t \kappa^G(t)(1 + y(t)) \, dt \right). \quad (B.14)$$

Now the first claim follows from Propositions B.5 and B.2 (b) (ii) and (B.13) because

$$(\exp \circ F)(t)(1 - G(t)) = \exp \left( - \int_0^t (\kappa^G(u) - F'(u)) \, du \right), \quad t \in [0, T).$$

For the second claim, note that integrability of $(\kappa^G(t) - F'(t))(1 + y(t))$ on $(0, T)$ is equivalent to integrability of $\kappa^G(t) - F'(t)$ on $(0, T)$ since by (B.12),

$$\epsilon (\kappa^G(t) - F'(t)) \leq (\kappa^G(t) - F'(t))(1 + y(t)) \leq C(\kappa^G(t) - F'(t)) + C(C - 1).$$

Now the second claim follows from the first one, Propositions B.5 and B.2 (b) (ii), and (B.14) because

$$\exp \left( \int_0^t F'(u)(1 + y(u)) \, du \right)(1 - H(t))$$

$$= \exp \left( - \int_0^t (\kappa^G(u) - F'(u))(1 + y(u)) \, du \right), \quad t \in [0, T).$$

\[\Box\]

**C Analytic results**

The main objective of this section is to show the existence and uniqueness of a solution to the integral equation (5.5).
An existence result for ODEs. Let \( y \in C[0, T] \) and
\[
U := \{(t, y) \in [0, T) \times \mathbb{R} : y > y(t)\}.
\]
Let \( f : U \to \mathbb{R} \) be a continuous function that is locally Lipschitz in its second variable. We consider the ordinary differential equation (ODE)
\[
y'(t) = f(t, y(t)), \quad t \in [0, T).
\] (C.1)
A function \( y \in C^1[0, T) \) with \( y > y \) is called a backward upper (lower) solution to (C.1) if
\[
y'(t) \leq (\geq) f(t, y(t)), \quad t \in [0, T).
\]
The function \( y \) is called a solution to (C.1) if it is both a backward upper and a backward lower solution.

Remark C.1. We define backward upper and lower solution without an initial condition. Moreover, note that what we call backward upper and lower solutions is called upper and lower solution to the left in [155]. Moreover, in [155] strict (as opposed to weak) inequalities are considered. But since we require \( f \) to be locally Lipschitz continuous in its second variable, all results hold also for the weak inequalities (see [155, Corollary VIII.9]).

The following result gives the existence of a solution to the ODE via the existence of a backward lower and a backward upper solution. The proof for \( U = [0, \infty) \times \mathbb{R} \) can be found in [155, Theorem and Remark XIII.9], and it is straightforward to check that the argument carries over to our setting.

Lemma C.2. Let \( y_*, y^* \in C^1[0, T) \) with \( y_* \leq y^* \). Suppose that \( y_* \) is a backward lower and \( y^* \) a backward upper solution to (C.1). Then there exists a solution \( y \in C^1[0, T) \) to (C.1) with \( y_* \leq y \leq y^* \).

Properties of the auxiliary functions. We collect some analytic properties of the auxiliary functions \( a, b, m \) and \( n \) defined at the beginning of Section 5. If there is no danger of confusion, we drop the dependence on \( p \) in the notation.

It is easy to check that
\[
a, b, m, n \in C([0, T) \times [-1, \infty) \times (0, \infty)) \cap C^{1,2,1}([0, T) \times (-1, \infty) \times (0, \infty)).
\]

For further reference, we note the straightforward identities
\[
\frac{\partial}{\partial y} a(t, y) = \frac{1}{p^2 \sigma^2} \frac{\phi'(t)^2}{\kappa^G(t)} \geq 0, \tag{C.2}
\]
\[
\frac{\partial}{\partial y} m(t, y) = (1 + y)^{\frac{1}{2}} \left( \frac{1}{p} \frac{a(t, y, p)}{1 + y} + \frac{\partial}{\partial y} a(t, y, p) \right) \geq \frac{1}{p} \frac{m(t, y, p)}{1 + y}, \tag{C.3}
\]
\[
\frac{\partial}{\partial y} n(t, y) = \kappa^G(t) \left( \frac{1}{p} a(t, y, p) + (1 + y) \frac{\partial}{\partial y} a(t, y, p) \right) \geq \frac{1}{p} \kappa^G(t) a(t, y, p), \tag{C.4}
\]
\[
n(t, y) = \frac{1 - p}{2p^2 \sigma^2} \mu^2 + \frac{1 - p}{2p^2 \sigma^2} (\phi'(t)y - \mu)^2 + \frac{1}{p} \kappa^G(t) (b(t, y, 1) - 1). \tag{C.5}
\]
In view of the integral equation (5.5), we are interested in the domain where
the function \( m \) is positive. To this end, define the function \( y : [0, T) \to [-1, \infty) \)
by
\[
 y(t) := \begin{cases} 
 -1 & \text{if } \phi'(t) = 0, \\
 \max \left( -1, \frac{\mu}{\phi'(t)} - p\sigma^2 \kappa^G(t) \right) & \text{if } \phi'(t) > 0.
\end{cases}
\] (C.6)
Using that \( \kappa^G \) is continuous and positive on \([0, T)\), it is not difficult to check that
\( y \in C[0, T) \). Set
\[
 U = \{(t, y) \in [0, T) \times \mathbb{R} : y > y(t)\}. 
\] (C.7)
Then by the definition of \( y \), (C.3) and (C.4),
\[
a(t, y), m(t, y), \frac{\partial}{\partial y} m(t, y), \frac{\partial}{\partial y} n(t, y) > 0, \quad (t, y) \in U.
\] (C.8)

An implicit function result. The following inverse-function-type result is the
cornerstone of the subsequent analysis. Recall the definition of \( y \) in (C.6).

**Lemma C.3.** Fix \( p \in (0, \infty) \). Let \( f \in C^1[0, T) \) with \( f(t) > 0 \), \( t \in [0, T) \). Then
there exists a unique function \( y \in C^1[0, T) \) with \( y > \underline{y} \) such that
\[
m(t, y(t)) = f(t).
\] (C.9)
Moreover, if \( \lim_{t \uparrow T} f(t) = 1 \), then there exist constants \( \epsilon \in (0, 1] \) and \( C \geq 1 \) such
that
\[
\epsilon \leq 1 + y(t) \leq C + \frac{C}{\phi'(t)} \mathbf{1}_{\{\kappa^G(t) < C\phi'(t)\}}, \quad t \in [0, T).
\] (C.10)
In this case, if in addition \( \int_0^T |\phi'(u)y(u)| \, du < \infty \), then
\[
\int_0^T \mathbf{1}_{\{\Delta G(t) > 0\}} \kappa^G(u)(1 + y(u)) \, du < \infty.
\] (C.11)

**Proof.** First, for fixed \( t \in [0, T) \), by (5.3) and (C.3), \( y \mapsto m(t, y) \) is continuous
and increasing\(^{14}\) on \([y(t), \infty)\) with \( m(t, y(t)) = 0 \) and \( \lim_{y \to \infty} m(t, y) = +\infty \).
Thus, there exists a unique function \( y : [0, T) \to \mathbb{R} \) with \( y > \underline{y} \) satisfying (C.9).
Moreover, \( y \in C^1(0, T) \) by the implicit function theorem.

Second, for fixed \( t \in [0, T) \), we claim that
\[
 y(t) \leq (2f(t))^p \quad \text{if } \phi'(t) \leq \frac{p\sigma^2}{2\mu} \kappa^G(t), \quad \text{ (C.12)}
\]
\[
 y(t) \leq \max \left( (f(t))^p, \frac{\mu}{\phi'(t)} \right) \quad \text{if } \phi'(t) > 0. \quad \text{ (C.13)}
\]
\(^{14}\)We emphasise that we use qualifiers like “increasing”, “decreasing”, “positive”, “negative”
in the strict sense; the corresponding wide-sense notions are “nondecreasing”, “nonincreasing”,
“nonnegative”, “nonpositive”.


Indeed, fix $t \in [0, T)$. If $\phi'(t) \leq \frac{\rho^2}{2\mu}\kappa^G(t)$, then $a(t, 0) = 1 - \frac{\mu}{\rho^2} \phi'(t) \geq \frac{1}{2}$. Seeking a contradiction, suppose that $y(t) > (2f(t))^p > 0$. Then by the definitions of $m$ and $y(t)$ and monotonicity of $a$ in the second variable,

$$f(t) = m(t, y(t)) = (1 + y(t))\hat{a}(t, y(t)) > (y(t))\hat{a}(t, 0) \geq f(t),$$

which is absurd. If $\phi'(t) > 0$, then $a\left(t, \frac{\mu}{\sigma(t)}\right) = 1$, and (C.13) follows from a similar argument.

Third, by the implicit function theorem and (C.3), for $t \in (0, T)$,

$$|y'(t)| = \left| \frac{f'(t) - \frac{\partial}{\partial y} m(t, y(t))}{\frac{\partial}{\partial y} m(t, y(t))} \right| \leq p(1 + y(t)) \left| \frac{f'(t) - \frac{\partial}{\partial y} m(t, y(t))}{f(t)} \right|.$$

Now, fix $t_0 \in (0, T)$ and let $C > 0$ be such that $y(t) \leq C$ for all $t \in [0, t_0]$. (This is possible by (C.12), (C.13) and the facts that $f$ is continuous and $\kappa^G$ is continuous and positive). Then positivity and continuity of $f$ in $[0, t_0]$, continuity of $f'$ in $[0, t_0]$ and continuity of $\frac{\partial}{\partial y} m(t, y)$ in $[0, t_0] \times [-1, C]$ together with the fact that $-1 \leq y(t) \leq C$ for $t \in [0, t_0]$ show that $y'$ is uniformly bounded in $(0, t_0]$. Moreover, by the fundamental theorem of calculus and the fact that $y \in C^1(0, T)$,

$$y(t) = y(t_0) - \int_t^{t_0} y'(u) \, du, \quad t \in (0, t_0].$$

Thus, by dominated convergence, $\lim_{t \uparrow t_0} y(t)$ exists in $\mathbb{R}$. Continuity of $f$ and $m$ and (C.9) give

$$m(0, y(0)) = f(0) = \lim_{t \uparrow t_0} f(t) = \lim_{t \uparrow t_0} m(t, y(t)) = m(0, \lim_{t \uparrow t_0} y(t)),$$

and so by uniqueness of $y$ on $[0, T)$, $\lim_{t \uparrow t_0} y(t) = y(0) > 2y(0) \geq -1$. This together with continuity of $\frac{\partial}{\partial y} m$ and $\frac{\partial}{\partial y} m$ on $[0, T) \times (-1, \infty)$, continuity of $f'$ on $[0, T)$ and the identity $y'(t) = \left( f'(t) - \frac{\partial}{\partial y} m(t, y(t)) \right) / \frac{\partial}{\partial y} m(t, y(t))$ for $t \in (0, T)$ (by the implicit function theorem) shows that the limit $\lim_{t \uparrow t_0} y'(t)$ exists in $\mathbb{R}$. So $y \in C^1[0, T]$.

Fourth, assume $\lim_{t \uparrow T} f(t) = 1$. Set

$$C := 1 + \max \left( \sup_{t \in [0, T)} (2f(t))^p, \mu, \frac{2\mu}{\rho^2} \right). \quad \text{(C.14)}$$

Then $1 \leq C < \infty$. Fix $t \in [0, T)$. If $\kappa^G(t) \geq C\phi'(t)$, then $\phi'(t) \leq \frac{1}{4} \kappa^G(t) \leq \frac{\rho^2}{2\mu}\kappa^G(t)$, and so $1 + y(t) \leq C$ by (C.12) and the definition of $C$. Otherwise, if $\kappa^G(t) < C\phi'(t)$, then $\phi'(t) > 0$, and so $1 + y(t) \leq C + \frac{C}{\phi'(t)}$ by (C.13) and the definition of $C$. For the left inequality in (C.10), by continuity of $y$ in $[0, T)$ and
the fact that \( y > y \geq -1 \) on \([0, T]\), it suffices to show that \( \liminf_{t \uparrow T} y(t) > -1 \).

Seeking a contradiction, suppose there is a sequence \((t_n)_{n \in \mathbb{N}} \subset [0, T)\) increasing to \( T \) such that \( \lim_{n \to \infty} y(t_n) = -1 \). Passing to a subsequence if necessary, we may assume that \( y(t_n) \leq 0 \) for all \( n \in \mathbb{N} \). As \( \phi' \geq 0 \) by (2.5), the definition of \( a \) in (5.1) gives \( a(t_n, y(t_n)) \leq 1 \) for all \( n \in \mathbb{N} \). Now, using the definition of \( m \) in (5.3), we arrive at the contradiction

\[
1 = \lim_{n \to \infty} f(t_n) = \lim_{n \to \infty} m(t_n, y(t_n)) \leq \limsup_{n \to \infty} (1 + y(t_n))^{\frac{1}{p}} = 0.
\]

Finally, assume that in addition \( \Delta G(T) > 0 \) and \( \int_0^T |\phi'(u)y(u)| \, du < \infty \). Then by (2.1),

\[
\int_0^T \kappa^G(u) \, du = -\log(\Delta G(T)) < \infty.
\]

Define \( C \) as in (C.14), and set \( A := \{u \in [0, T) : \phi'(u) \leq \frac{\nu R^2}{2\mu} \kappa^G(u)\} \). Then \( y(u) \leq C \) for \( u \in A \) by (C.12), and \( \kappa^G(u) < \frac{2\mu}{p\sigma^2} \phi'(u) \) for \( u \in A^C \). This together with the above yields (C.11) via

\[
\int_0^T (1 + y(u)) \kappa^G(u) \, du = \int_0^T \left( \kappa^G(u) + \mathbf{1}_A(u)y(u)\kappa^G(u) + \mathbf{1}_{A^C}(u)y(u)\kappa^G(u) \right) \, du \\
\leq \int_0^T \left( (1 + C)\kappa^G(u) + \frac{2\mu}{p\sigma^2} \phi'(u)|y(u)| \right) \, du < \infty. \quad \square
\]

**Corollary C.4.** Fix \( p \in (0, \infty) \) and let \( y^1 \in C^1[0, T) \) with \( y^1 > y \) be the unique function from Lemma C.3 satisfying \( m(t, y^1(t)) = 1 \). Then \( y^1 \) is nonnegative and for each \( t \in [0, T) \), \( y^1(t) = 0 \) if and only if \( \phi'(t) = 0 \).

**Proof.** Fix \( t \in [0, T) \). By the definition of \( m \) and \( a \) in (5.3) and (5.1),

\[
1 = m(t, y^1(t)) = (1 + y^1(t))^{\frac{1}{p}} a(t, y^1(t)) \\
= (1 + y^1(t))^{\frac{1}{p}} \left( 1 - \frac{1}{p\sigma^2} \phi'(t)(\mu - \phi'(t)y^1(t)) \right).
\]

If \( y^1(t) < 0 \), then the right-hand side of (C.15) is strictly smaller than 1, which is absurd. So \( y^1 \) is nonnegative. Next, assume that \( y^1(t) = 0 \). Then (C.15) simplifies to \( \phi'(t) = 0 \). Conversely, if \( \phi'(t) = 0 \), then (C.15) simplifies to \( y^1(t) = 0 \). \( \square \)

**Corollary C.5.** Fix \( p \in (0, \infty) \). Let \( f, g \in C^1[0, T) \) be such that \( g(t) > f(t) > 0 \), \( t \in [0, T) \), and \( \lim_{t \uparrow T} g(t) = \lim_{t \uparrow T} f(t) = 1 \). Let \( y^0, y^f \in C^1[0, T) \) with \( y^0, y^f > y \) be the unique functions from Lemma C.3 satisfying \( m(t, y^0(t)) = f(t) \) and \( m(t, y^f(t)) = g(t) \). Assume that \( \Delta G(T) > 0 \) and \( \limsup_{t \uparrow T} G'(t) < \infty \). Then

\[
\lim_{t \uparrow T} \phi'(t)(y^0(t) - y^f(t)) = 0.
\]
Proof. By Lemma C.3, there are constants \( C^g, \epsilon^f > 0 \) such that \( 1 + y^g(t) \leq C^g + \frac{C^g}{\phi(t)}1_{\{\phi(t) > 0\}} \) and \( 1 + y^f(t) \geq \epsilon^f \), \( t \in [0,T) \). Since \( \Delta G(T) > 0 \) and \( \limsup_{t \uparrow T} G^f(t) < \infty \), there exists a constant \( C^f > 0 \) such that \( \frac{1}{\kappa(t)} = \frac{1 - G(t)}{G(t)} \geq C^f \), \( t \in [0,T) \). Next, since \( f \in C^1[0,T), f > 0 \) on \( [0,T) \) and \( \lim_{t \uparrow T} f(t) = 1 \), there exists a constant \( C^f > 0 \) such that \( f(t) \geq C^f \), \( t \in [0,T) \). Finally, set \( C = \min \left( \frac{C^f}{2C^g}, \frac{(\epsilon^f)^2 C^f}{\sigma^2} \right) > 0 \). Fix \( t \in [0,T) \). Then \( y^g(t) \geq y^f(t) \) by monotonicity of \( m \) in the second variable. Using successively (C.9), the mean value theorem, (C.3), (5.3) and (C.2), monotonicity of \( m \) in the second variable, (C.9) and the choices of \( C^g, \epsilon^f \) and \( C^f \), the choice of \( C^g \), and finally the choice of \( C \) (distinguishing the cases \( \phi'(t) \geq 1 \) and \( \phi'(t) < 1 \)) yields, for some \( \bar{y} \in [y^f(t), y^g(t)] \),

\[
g(t) - f(t) = m(t, y^g(t)) - m(t, y^f(t)) = (y^g(t) - y^f(t)) \frac{\partial}{\partial y} m(t, \bar{y})
\]

\[
= (y^g(t) - y^f(t))(1 + \bar{y}) \frac{1}{p} a(t, \bar{y}) \frac{1}{1 + \bar{y}} + \frac{\partial}{\partial y} a(t, \bar{y})
\]

\[
= \frac{1}{p} (y^g(t) - y^f(t)) \left( \frac{m(t, y^g(t))}{1 + y^g(t)} + \frac{1}{\sigma^2} (1 + \bar{y}) \frac{1}{\kappa(t)} \phi'(t)^2 \right)
\]

\[
\geq \frac{1}{p} (y^g(t) - y^f(t)) \left( \frac{m(t, y^f(t))}{1 + y^g(t)} + \frac{1}{\sigma^2} (1 + y^f(t)) \frac{1}{\kappa(t)} \phi'(t)^2 \right)
\]

\[
\geq \frac{1}{p} (y^g(t) - y^f(t)) \left( \frac{f(t)}{C^g + \frac{C^g}{\phi(t)}1_{\{\phi(t) > 0\}}} + \frac{(\epsilon^f)^2 C^f}{\sigma^2} \phi'(t)^2 \right)
\]

\[
\geq \frac{1}{p} \phi'(t) (y^g(t) - y^f(t)) \left( \frac{C^f}{C^g \phi'(t)} + C^g + \frac{(\epsilon^f)^2 C^f}{\sigma^2} \phi'(t) \right)
\]

\[
\geq \frac{C}{p} \phi'(t) (y^g(t) - y^f(t)).
\]

Now the claim follows from letting \( t \uparrow T \).

Existence and uniqueness of a solution to the integral equation. We are now in a position to prove the main existence and uniqueness result for the integral equation (5.5). Recall the definition of \( \bar{y} \) in (C.6).

Theorem C.6. Fix \( p \in (0, \infty) \). Then there exists a unique solution \( \bar{y} \in C^1[0,T) \) with \( \bar{y} > y \) to the integral equation

\[
m(t, y(t), p) = \exp \left( - \int^T_t n(u, y(u), p) \, du \right), \quad t \in [0,T), \quad \text{(C.16)}
\]

satisfying (C.10) and (C.11) (with \( y \) replaced by \( \bar{y} \)) as well as

\[
\int^T_0 |n(u, \bar{y}(u), p)| \, du < \infty \quad \text{and} \quad \int^T_0 (\phi'(u)\bar{y}(u))^2 \, du < \infty. \quad \text{(C.17)}
\]
Moreover, \( y_s \leq \hat{y} \leq y^* \) on \([0, T)\), where \( y_s, y^* \in C^1[0, T)\) are the unique functions from Lemma C.3 satisfying \( y_s, y^* > y \) and

\[
m(t, y^*(t), p) = \begin{cases} 
\exp\left(\frac{1-p}{2p^2\sigma^2}\mu^2(T-t)\right) & \text{if } p < 1, \\
1 & \text{if } p \geq 1,
\end{cases}
\]

(C.18)

\[
m(t, y_s(t), p) = \begin{cases} 
1 & \text{if } p < 1, \\
\exp\left(\frac{1-p}{2p^2\sigma^2}\mu^2(T-t)\right) & \text{if } p \geq 1.
\end{cases}
\]

(C.19)

Note that (C.10) and (C.11) (with \( y \) replaced by \( \hat{y} \)) as well as (C.17) imply in particular that (3.2) and (3.9) (with \( y \) replaced by \( \hat{y} \)) are fulfilled.

**Proof.** First, we transform the integral equation (C.16) into an ODE. Taking logarithms on both sides of (C.16) and differentiating shows that a solution \( y \in C^1[0, T)\) to (5.5) solves

\[
\frac{d}{dt} \log(m(t, y(t)), p) = n(t, y(t), p).
\]

(C.20)

An easy calculation using (5.3) and (C.3) gives

\[
\frac{d}{dt} \log(m(t, y(t)), p) = \frac{y'(t)\left(\frac{1}{p} \frac{a(t, y(t), p)}{1+y(t)} + \frac{\partial}{\partial y} a(t, y(t), p)\right) + \frac{\partial}{\partial t} a(t, y(t), p)}{a(t, y(t), p)}.
\]

Rearranging shows that \( y \) solves the ODE

\[
y'(t) = f(t, y(t), p), \quad t \in [0, T),
\]

(C.21)

where the function \( f : U \times (0, \infty) \to (0, \infty) \) is given by

\[
f(t, y, p) = \frac{a(t, y, p)n(t, y, p) - \frac{\partial}{\partial y} a(t, y, p)}{\frac{1}{p} \frac{a(t, y, p)}{1+y} + \frac{\partial}{\partial y} a(t, y, p)}
\]

and \( U \) is defined in (C.7). Clearly, \( f \in C^{0,1,1}(U \times (0, \infty)) \). Note that positivity of the denominator is ensured by positivity of \( a \) in \( U \times (0, \infty) \) by (C.8) and (C.2). Moreover, (C.16) gives the implicit “terminal condition”

\[
\lim_{t \uparrow T} m(t, y(t), p) = 1.
\]

Second, we establish uniqueness of \( \hat{y} \). Assume that \( \hat{y}^1, \hat{y}^2 \in C^1[0, T) \) are two solutions of (C.16). Then \( \hat{y}^1, \hat{y}^2 > y \) and both functions are solutions to the ODE (C.21). Assume without loss of generality that \( \hat{y}^2(0) \geq \hat{y}^1(0) \). Since \( f \) is locally Lipschitz in the second variable on \( U \), it follows by the standard local existence and uniqueness theorem for ODEs that either \( \hat{y}^1 = \hat{y}^2 \) or \( \hat{y}^2 > \hat{y}^1 \). Seeking a contradiction, assume the second case. Then by strict monotonicity of \( m \) and
n in the second variable (by (C.8)) and the fact that \( \hat{y}^1 \) and \( \hat{y}^2 \) are solutions to (5.5),

\[
m(0, \hat{y}^2(0)) > m(0, \hat{y}^1(0)) = \exp \left( - \int_0^T n(u, \hat{y}^1(u)) \, du \right) > \exp \left( - \int_0^T n(u, \hat{y}^2(u)) \, du \right) = m(0, \hat{y}^2(0)),
\]

which is absurd. So \( \hat{y}^1 = \hat{y}^2 \).

Third, we use Lemma C.2 to show the existence of a solution to (C.21). To this end, we show that \( y^* \) and \( y_* \) are backward upper and backward lower solutions, respectively. The existence and uniqueness of the functions \( y_* \) and \( y^* \) satisfying (C.18) and (C.19) follows from Lemma C.3. Note that \( y \in C^1[0,T] \) with \( y > y_* \) is a backward upper (lower) solution to (C.21) if and only if

\[
\frac{d}{dt} \log(m(t, y(t), p)) \leq (\geq) \ n(t, y(t), p), \quad t \in [0, T];
\]

this follows from the same rearrangement that led from (C.20) to (C.21) using that \( \frac{1}{p} \hat{y}^1 + \frac{1}{y^2} a \) and \( a \) are positive in \( U \times (0, \infty) \) by (C.8) and (C.2).

We only consider the case \( p < 1 \), the case \( p \geq 1 \) follows from a similar argument, basically reversing all inequalities. Bernoulli’s inequality, (5.2) and (5.3) yield

\[
b(t, y, p) \leq m(t, y, p) \leq b(t, y, 1)^{1/p}, \quad (t, y) \in U.
\]

To establish that \( y^* \) is a backward upper solution, we note from (C.18) that \( m(t, y^*(t), p) \geq 1 \) for \( t \in [0, T] \). Thus, \( b(t, y^*(t), 1) \geq 1 \) for \( t \in [0, T] \) by (C.23), and so \( n(t, y^*(t), p) \geq -\frac{1-p}{2y^{p+2}} \mu^2 \) for \( t \in [0, T] \) by (C.5). Now, taking logarithms in (C.18) and differentiating shows that \( y^* \) fulfills (C.22) with “\( \leq \)”, and so \( y^* \) is a backward upper solution. To establish that \( y_* \) is a backward lower solution, we note from (C.19) that \( m(t, y_*(t), p) = 1 \) for \( t \in [0, T] \), and so \( b(t, y_*(t), p) \leq 1 \) for \( t \in [0, T] \) by (C.23). Thus, \( n(t, y_*(t), p) \leq 0 \) by (5.4), and the claim follows as above by taking logarithms in (C.19) and differentiating. Clearly, \( y_* \leq y^* \) by monotonicity of \( m \) in the second variable, and \( \lim_{t \uparrow T} m(t, y_*(t), p) = \lim_{t \uparrow T} m(t, y^*(t), p) = 1 \) by construction. So by Lemma C.2, there exists a solution \( \hat{y} \in C^1[0,T) \) of (C.21) with \( y_* \leq \hat{y} \leq y^* \).

Fourth, \( \hat{y} > y_* \) because \( \hat{y} \geq y_* > y \) by construction. Monotonicity of \( m \) in the second variable and the fact that \( \lim_{t \uparrow T} m(t, y^*(t), p) = 1 = \lim_{t \uparrow T} m(t, y_*(t), p) \) by (C.18) and (C.19) yield \( \lim_{t \uparrow T} m(t, \hat{y}(t), p) = 1 \). Moreover, as \( \hat{y} \) satisfies (C.20), the fundamental theorem of calculus shows that there exists a constant \( c > 0 \) such that \( \hat{y} \) satisfies the integral equation

\[
m(t, \hat{y}(t), p) = c \exp \left( \int_0^t n(u, \hat{y}(u), p) \, du \right), \quad t \in [0, T).
\]

Now, we have to distinguish the cases \( p < 1 \) and \( p \geq 1 \). We only consider the first one; the second one follows from a similar argument, basically reversing all
inequalities. So let $p \in (0, 1)$. Then $m(t, \hat{y}(t), p) \geq m(t, y_*(t), p) = 1$ by monotonicity of $m$ in the second variable and (C.19). Thus, (C.23) gives $b(t, \hat{y}(t), 1) \geq 1$, and so $n(t, \hat{y}(t), p) \geq -\frac{1-p}{2\sigma^2 t \mu^2}$ by (C.5). Taking the limit $t \uparrow T$ in (C.24), by monotone convergence and the fact that $\lim_{t \uparrow T} m(t, \hat{y}(t), p) = 1$, we deduce that

$$c = \exp \left( -\int_0^T n(u, \hat{y}(u), p) \, du \right). \quad \text{(C.25)}$$

Plugging this back into (C.24) shows that $\hat{y}$ is a solution to (C.16). Moreover, since $n(t, \hat{y}(t), p)$ is bounded from below and $c > 0$, (C.25) implies that the first condition in (C.17) is satisfied. This together with the representation of $n$ in (C.5) and $b(t, \hat{y}(t), 1) \geq 1$ (from above) then also establishes the second condition in (C.17). Finally, define $\hat{f}(t)$ by the right-hand side of (C.16) (with $y$ replaced by $\hat{y}$). Then $\hat{y}$ is trivially a solution to $m(t, \hat{y}(t)) = \hat{f}(t)$, $t \in [0, T]$, and $\lim_{t \uparrow T} \hat{f}(t) = 1$. Hence, Lemma C.3 gives (C.10) and (C.11) for $\hat{y}$ (note that the condition $\int_0^T |\phi'(u)\hat{y}(u)| \, du < \infty$ follows from (C.17)).

\[ \square \]

**D Verification and technical details**

**Verification.** Here, we collect the technical parts of Steps 2 and 3 of the proof of Theorem 5.1. The first result identifies the wealth process corresponding to the strategy $\hat{\pi}$ and shows that it remains positive. The second and third result verify (OC1) and (OC2) for the candidate triplet $(\hat{\pi}, \hat{Q}, \hat{z})$.

**Lemma D.1.** Let $(\hat{\pi}, \hat{Q}, \hat{z})$ be the triplet defined in (the proof of) Theorem 5.1. Denote by $W^\hat{Q}$ the $\hat{Q}$-Brownian motion given by Theorem 3.2 (with $y = \hat{y}$) and let $H$ be the distribution function of $\gamma$ under $\hat{Q}$. Define $\xi \in C^1[0, T)$ by

$$\hat{\xi}(t) = \exp \left( \int_0^t \phi'(u) \frac{1}{\sigma^2} (\mu - \phi'(u)\hat{y}(u))(1 + \hat{y}(u)) \, du \right) \quad \text{(D.1)}$$

and set $\hat{X} := \mathcal{E} \left( \sigma \int_0^\cdot \hat{\pi}_t \, dW_t^\hat{Q} \right) \mathcal{M}^H \xi$. Then $X^\ast = x\hat{X}$ is the wealth process corresponding to the strategy $\hat{\pi}$ and initial capital $x$. Moreover, $\mathcal{M}^H \xi$ and $\hat{X}$ are positive and thus $\hat{\pi}$ is admissible.

**Proof.** For the first claim, it suffices to show that $\hat{X}$ satisfies the SDE (2.10) with initial condition $\hat{X}_0 = 1$. Set $M := \mathcal{E} \left( \sigma \int_0^\cdot \hat{\pi}_t \, dW_t^\hat{Q} \right)$ and $N := \mathcal{M}^H \hat{\xi}$ for brevity and note from (5.6) that $\hat{\pi}_t = \bar{\pi}(t, \gamma)$, $t \in [0, T]$, where

$$\bar{\pi}(t, v) := \frac{1}{\rho \sigma^2} (\mu - \phi'(t)\hat{y}(t)1_{t \leq v, t < T}) , \quad (t, v) \in [0, T]^2. \quad \text{(D.2)}$$

With this notation, by the definition of $\hat{\xi}$, we obtain

$$\hat{\xi}(t) = \hat{\xi}(t)\phi'(t)\bar{\pi}(t, t)(1 + \hat{y}(t)), \quad t \in [0, T). \quad \text{(D.3)}$$
Fix $t \in [0,T]$. By using successively that $M$ is continuous and $N$ is purely discontinuous, that $\Delta M^H \dot{\xi}_{\gamma} = -\frac{\xi'(\gamma)}{\kappa_H(\gamma)} 1_{\gamma<T}$ by the definitions of $M^H \dot{\xi}$ (cf. (2.2)) and $A^H \dot{\xi}$ (cf. (2.3)), (D.3), that $\dot{\xi}(s) = N_s$ and $\bar{\xi}(s,s) = \bar{\pi}(s,\gamma) = \hat{\pi}_s$ on $\{s \leq \gamma\}$, and finally the dynamics of $S$ in (3.6) (for $y = \hat{y}$ etc.),

$$
\hat{X}_t - 1 = M_t N_t - M_0 N_0 = \int_0^t N_s - dM_s + \int_0^t M_s - dN_s \\
= \sigma \int_0^t N_s - M_s - \hat{\pi}_s dW^Q_s + \int_0^t M_s - dM^H \dot{\xi}_s \\
= \sigma \int_0^t \hat{X}_s - \hat{\pi}_s dW^Q_s + \int_0^{\gamma} M_s - \Delta M^H \dot{\xi}_s \, ds + M_T - \Delta M^H \dot{\xi}_s 1_{\gamma<T} \\
= \sigma \int_0^t \hat{X}_s - \hat{\pi}_s dW^Q_s + \int_0^{\gamma} M_s - \dot{\xi}(s) \bar{\xi}(s,s) \phi'(s) \big((1 + \hat{y}(s)) - M_T - \Delta M^H \dot{\xi}_s \big) \\
= \int_0^t \hat{X}_s \hat{\pi}_s \bigg[ \sigma dW^Q_s + dM^H \Big( \int_0^\gamma \phi'(u)(1 + \hat{y}(u)) \, du \Big)_s \bigg] \\
= \int_0^t \hat{X}_s \hat{\pi}_s \frac{dS_s}{S_s} \quad P\text{-a.s.}
$$

For the second claim, since $M$ and $\dot{\xi}$ are positive, it suffices to show that also $A^H \dot{\xi}$ is positive. Indeed, using the definition of $A^H \dot{\xi}$ (cf. (2.3)), (D.3), (B.9), the fact that $a(v, \hat{y}(v), p) = 1 - \frac{\phi'(v)}{\kappa^H(v)} \bar{\pi}(v,v)$ by the definitions of $\bar{\pi}$ and $a$ in (D.2) and (5.1), and (C.8),

$$
A^H \dot{\xi}(v) = \dot{\xi}(v) - \frac{\dot{\xi}'(v)}{\kappa^H(v)} = \dot{\xi}(v) \left(1 - \frac{\phi'(v)}{\kappa^H(v)} \right) \\
= \dot{\xi}(v) a(v, \hat{y}(v), p) > 0, \quad v \in (0,T).
$$

\[\square\]

**Lemma D.2.** The triplet $(\hat{\pi}, \hat{Q}, \hat{\zeta})$ defined in the proof of Theorem 5.1 satisfies

$$
U'(X_T^\pi) = \frac{d\hat{Q}}{d\hat{P}}.
$$

**Proof.** By Lemma D.1 and the fact that $U'(x) = x^{-p}$,

$$
U'(X_T^\pi) = x^{-p} \hat{X}_T^{-p} = x^{-p} \mathcal{E}_T \left( \sigma \int_0^t \hat{\pi}_s dW^Q_s \right)^{-p} \left( M^H_t \dot{\xi} \right)^{-p}. \quad (D.5)
$$
First, a standard calculation gives
\[
E_T \left( \sigma \int_0^T \tilde{\pi}_t dW_t^Q \right)^{-p} = E_T \left( -p\sigma \int_0^T \tilde{\pi}_t dW_t \right) \exp \left( (1 - p) \frac{p\sigma^2}{2} \int_0^T \tilde{\pi}_t^2 dt \right). \tag{D.6}
\]

To compute the second factor, we claim that for \( v \in [0, T) \),
\[
\hat{\xi}(v)m(v, \hat{y}(v), p) = x^{-1} \hat{z}^{-\frac{1}{p}} \exp \left( (1 - p) \frac{\sigma^2}{2} \int_0^T \tilde{\pi}(u, u)^2 du \right) \hat{\zeta}(v)^{-\frac{1}{p}}, \tag{D.7}
\]
where \( \tilde{\pi} \) is defined in (D.2) and \( \hat{\zeta} \) is given in (3.3) (with \( y \) replaced by \( \hat{y} \)). Moreover, in the case \( \Delta G(T) > 0 \), we claim that
\[
\hat{\xi}(T-) = x^{-1} \hat{z}^{-\frac{1}{p}} \exp \left( (1 - p) \frac{\sigma^2}{2} \int_0^T \tilde{\pi}(u, u)^2 du \right) \hat{\zeta}(T-)^{-\frac{1}{p}}. \tag{D.8}
\]

Then, by (D.4), the definition of \( m \) in (5.3), (D.7) and (B.7), on \( \{ \gamma < T \} \),
\[
\left( \mathcal{M}_T^{\beta}(\hat{\xi}) \right)^{-p} = \left( A^{\beta}(\hat{\xi}(\gamma)) \right)^{-p} = \hat{\xi}(\gamma)^{-p} a(\gamma, \hat{y}(\gamma), p)^{-p}
= \left( \hat{\xi}(\gamma)m(\gamma, \hat{y}(\gamma), p) \right)^{-p} (1 + \hat{y}(\gamma))
= x^p \hat{z} \exp \left( (1 - p) \frac{\sigma^2}{2} \int_0^T \tilde{\pi}_t^2 dt \right)^{-p} \hat{\zeta}(\gamma)(1 + \hat{y}(\gamma))
= x^p \hat{z} \exp \left( -(1 - p) \frac{p\sigma^2}{2} \int_0^T \tilde{\pi}_t^2 dt \right) A^G \hat{\zeta}(\gamma). \tag{D.9}
\]

If \( \Delta G(T) > 0 \), then \( A^{\beta}(\hat{\xi}(\gamma)) = \hat{\xi}(T-) \) and \( A^G \hat{\zeta}(\gamma) = \hat{\zeta}(T-) \) on \( \{ \gamma = T \} \). This together with (D.8) shows that (D.9) holds on \( \{ \gamma = T \} \), too.

Finally, plugging (D.6) and (D.9) into (D.5) yields by the definitions of \( \tilde{\pi} \) in (5.6) and \( \frac{dQ}{dP} \) in Theorem 3.2 (cf. (3.5)) that
\[
U'(X_T^\beta) = \hat{z}E_T \left( -p\sigma \int_0^T \tilde{\pi}_t dW_t^Q \right) A^G \hat{\zeta}(\gamma) = \hat{z}E_T \left( -p\sigma \int_0^T \tilde{\pi}_t dW_t \right) \mathcal{M}_T^G \hat{\zeta}
= \hat{z}E_T \left( - \int_0^T \frac{1}{\sigma} \left( \mu - \phi'(t) \hat{y}(t) \mathbf{1}_{t \leq \gamma < T} \right) dt \right) \mathcal{M}_T^G \hat{\zeta} = \hat{z} \frac{dQ}{dP}. \]

It remains to show (D.7) and (D.8). First, an easy but tedious calculation using the definitions of \( \tilde{\pi} \) and \( \phi \) in (D.2) and (D.4) shows that for \( u \in [0, T) \),
\[
\phi'(u)\tilde{\pi}(u, u)(1 + \hat{y}(u)) + \phi(u, \hat{y}(u), p)
= \frac{1 - p}{2p^2\sigma^2} \phi'(u)\hat{y}(u) (\phi'(u)\hat{y}(u) - 2\mu) + \frac{1}{p} \kappa^G(u)\hat{y}(u). \tag{D.10}
\]
Next, fix \( v \in [0, T] \). Using first that \( \hat{y} \) is a solution to the integral equation (5.5) and the definitions of \( \check{\xi} \) in (D.1) and \( \check{\pi} \) in (D.2), then (D.10), and finally again the definition of \( \check{\pi} \) and the definition of \( \hat{\xi} \) in (3.3) (with \( y \) replaced by \( \hat{y} \)),

\[
\check{\xi}(v) \frac{m(v, \hat{y}(v), p)}{m(0, \hat{y}(0), p)} = \exp \left( \int_0^v (\phi'(u)\check{\pi}(u, u)(1 + \hat{y}(u)) + n(u, \hat{y}(u), p)) \, du \right) \\
= \exp \left( \int_0^v \left( \frac{1 - p}{2p^2\sigma^2}\phi'(u)\hat{y}(u) (\phi'(u)\hat{y}(u) - 2\mu) + \frac{1}{p}G(u)\hat{y}(u) \right) \, du \right) \\
= \exp \left( -\frac{1 - p}{2p^2\sigma^2}\mu^2 T + (1 - p)\frac{\sigma^2}{2} \int_0^T \check{\pi}(u, v)^2 \, du \right) \hat{\xi}(v)^{-\frac{1}{\mu}},
\]

and using the definition of \( \hat{z} \) in (5.9) gives (D.7).

Finally, assume that \( \Delta G(T) > 0 \). Then also \( \Delta \hat{H}(T) > 0 \) since \( \hat{Q} \approx P \). Moreover, by the proof of Theorem 3.2, \( M^G\check{\xi} \) is positive, and by Lemma D.1, \( M^H\hat{\xi} \) is positive. This together with Proposition B.2 (a) and (b) (i) implies that the limits \( \check{\xi}(T-) \) and \( \check{\pi}(T-) \) exist in \( \mathbb{R} \). Moreover, \( \lim_{v \uparrow T} m(v, \hat{y}(v), p) = 1 \) by (5.5) (recall that Theorem C.6 shows that \( \hat{y} \) is a solution to the integral equation) and thus (D.8) follows from taking the limit \( v \uparrow T \) in (D.7); the exchange of limit and integration on the right-hand side is justified by dominated convergence using \( |\check{\pi}(u, v)| \leq \frac{1}{\rho\sigma^2}(\mu + |\phi'(u)\hat{y}(u)|) \) and (C.17); also note that for \( u \in [0, T) \), \( \lim_{v \uparrow T} \check{\pi}(u, v) = \frac{1}{\rho\sigma^2}(\mu - \phi'(u)\hat{y}(u)) = \check{\pi}(u, u) \). \( \square \)

**Lemma D.3.** The triplet \((\check{\pi}, \hat{Q}, \hat{z})\) defined in the proof of Theorem 5.1 satisfies

\[
E^{\hat{Q}} \left[ X_T^{\hat{\xi}} \right] = x.
\]

**Proof.** It suffices to show that \( X^{\hat{\xi}} \) is a \( \hat{Q} \)-martingale. Lemma D.1 shows that \( X^{\hat{\xi}} \) is of the form (3.11). Therefore, by Corollary 3.6, it suffices to prove that \( \int_0^T |M^H\hat{\xi}(u)|\hat{H}'(u) \, du < \infty \) and that \( M^H\hat{\xi} \) is a \( \hat{Q} \)-martingale. The first assertion follows directly from Proposition B.2 (a) noting that \( M^H\hat{\xi} \) is positive by Lemma D.1. For the second assertion, we note that by Lemma A.1 (b) (ii), it is enough to show that \( M^{\hat{H}}\hat{\xi} \) is a \((Q^\gamma, \mathbb{F}^\gamma)\)-martingale. To this end, by Proposition B.2 (a) and (b), we may assume that \( \Delta G(T) = 0 \) (using that \( \hat{Q} \approx P \)) and have to check that \( \lim_{t \uparrow T} \hat{\xi}(t)(1 - \hat{H}(t)) = 0 \). We distinguish two cases.

First, let \( p \geq 1 \) and fix \( t \in [0, T) \). Then as \( 1 - \hat{H}(t) \leq 1, \)

\[
0 \leq \hat{\xi}(t)(1 - \hat{H}(t)) \leq \hat{\xi}(t)(1 - \hat{H}(t))^{1/p}
\]

and it suffices to show that the right-hand side converges to 0 as \( t \uparrow T \). Using first the definitions of \( \hat{\xi} \) and \( \hat{H} \) in (D.1) and (3.4), then the definition of \( a(\cdot, \cdot, 1) \)
in (5.1), and finally the definition of $b(\cdot, \cdot, 1)$ in (5.2),

$$\dot{\xi}(t)(1 - \dot{H}(t))^{1/p}$$

$$= \exp \left( \int_0^t \left( \phi'(u) \frac{1}{p\sigma^2} (\mu - \phi'(u)\dot{y}(u))(1 + \dot{y}(u)) - \frac{\kappa^G(u)}{p} (1 + \dot{y}(u)) \right) \, du \right)$$

$$= \exp \left( - \int_0^t \frac{\kappa^G(u)}{p}(1 + \dot{y}(u)) \left( 1 - \frac{\phi'(u)}{\kappa^G(u) \sigma^2} (\mu - \phi'(u)\dot{y}(u)) \right) \, du \right)$$

$$= \exp \left( - \int_0^t \frac{\kappa^G(u)}{p} b(u, \dot{y}(u), 1) \, du \right). \quad \text{(D.11)}$$

By the representation of $n$ in (C.5), we have for $u \in [0, T)$,

$$n(u, \dot{y}(u), p) = -\frac{1 - p}{2p^2\sigma^2} \mu^2 + \frac{1 - p}{2p^2\sigma^2} \phi'(u)\dot{y}(u) - \mu^2 + \frac{1}{p} \kappa^G(u)b(u, \dot{y}(u), 1) - 1.$$  

Since the left-hand side as well as the first two summands on the right-hand side are integrable on $(0, T)$ by (C.17), we infer that $\int_0^T \kappa^G(u)b(u, \dot{y}(u), 1) - 1 \, du < \infty$. But $\Delta G(T) = 0$ implies that

$$\int_0^T \kappa^G(u) \, du = -\log(\Delta G(T)) = \infty, \quad \text{(D.12)}$$

and so the right-hand side of (D.11) converges to 0 as $t \uparrow T$.

Second, let $p < 1$ and fix $t \in [0, T)$. Using first the definitions of $\dot{\xi}$ and $\dot{H}$ in (D.1) and (3.4), and then the definition of $a$ in (5.1),

$$\dot{\xi}(t)(1 - \dot{H}(t))$$

$$= \exp \left( \int_0^t \left( \phi'(u) \frac{1}{p\sigma^2} (\mu - \phi'(u)\dot{y}(u))(1 + \dot{y}(u)) - \frac{\kappa^G(u)}{p} (1 + \dot{y}(u)) \right) \, du \right)$$

$$= \exp \left( - \int_0^t \frac{\kappa^G(u)}{p}(1 + \dot{y}(u)) \left( 1 - \frac{\phi'(u)}{\kappa^G(u) \sigma^2} (\mu - \phi'(u)\dot{y}(u)) \right) \, du \right)$$

$$= \exp \left( - \int_0^t \frac{\kappa^G(u)}{p} a(u, \dot{y}(u), p) \, du \right).$$

Using the estimate

$$(1 + \dot{y}(u))a(u, \dot{y}(u), p) \geq p \left( 1 + \frac{1}{p}\dot{y}(u) \right) a(u, \dot{y}(u), p) = pb(u, \dot{y}(u), p), \quad u \in [0, T),$$

we obtain

$$0 \leq \dot{\xi}(t)(1 - \dot{H}(t)) \leq \exp \left( -p \int_0^t \kappa^G(u)b(u, \dot{y}(u), p) \, du \right). \quad \text{(D.13)}$$
By the definition of $n$ in (5.4), we have for $u \in [0,T)$,
\[ n(u, \check{y}(u), p) = -\frac{1-p}{2p^2\sigma^2} (\phi'(u)\check{y}(u))^2 + \kappa^G(u)(b(u, \check{y}(u), p) - 1). \]
Since the left-hand side as well as the first summand on the right-hand side are integrable on $(0, T)$ by (C.17), we infer that \( \int_0^T \kappa^{G}(u)[b(u, \check{y}(u), p) - 1] du < \infty. \) Combining this with (D.12) shows that the right-hand side of (D.13) converges to 0 as $t \uparrow T$. \hfill \Box

**Technical details for the remaining results.**

*Proof of (5.17).* We only consider the case $p < 1$ and $y \in (-1,0) \cup (0,\infty)$; the other cases are analogous or trivial. Bernoulli’s inequality gives $p((1+y)^{\frac{1}{p}} - 1) \geq y$ and both sides have the same sign; so we get $h(y, p) \leq \frac{1}{p}$ from (5.16). Similarly, setting $\check{y} := (1+y)^{-\frac{1}{p}} - 1$ and using Bernoulli's inequality,
\[ (1+y)^{-1} - 1 = (1 + (1+y)^{-\frac{1}{p}} - 1)^p - 1 = (1 + \check{y})^p - 1 \leq p\check{y} = p((1+y)^{-\frac{1}{p}} - 1) \]
and again both sides have the same sign. This yields the lower bound $h(y, p) \geq 1$ via the identity
\[ h(y, p) = 1 + \frac{1}{(1+y)^{-1} - 1} - \frac{1}{p((1+y)^{-\frac{1}{p}} - 1)} \]
for $p \in (0, \infty)$ and $y \in (-1,\infty) \setminus \{0\}$. \hfill \Box

*Proof of Theorem 5.4.* First, we establish (5.18). This is trivial on the event \( \{t > \gamma\} \cup \{t = T\} \), so we prove the equality on \( \{t \leq \gamma, t < T\} \), i.e., we show that
\[ \frac{1}{p\sigma^2} (\mu - \phi'(t)y^m(t)) = \frac{1}{p \sigma^2 + \frac{\phi'(t)^2}{\kappa^{G}(t)}} \frac{\mu}{1 + y^m(t)h(y^m(t), p)), \quad \in [0, T). \] \]
Now if $\phi'(t) = 0$, then $y^m(t) = 0$ by Corollary C.4, and (D.14) is trivially satisfied. So assume that $\phi'(t) > 0$. Then $y^m(t) > 0$ by Corollary C.4. Rearranging (D.14) and dividing by $\phi'(t)$ gives
\[ p (1 + y^m(t)h(y^m(t), p)) \frac{1}{p\sigma^2 \kappa^{G}(t)} \frac{\phi'(t)}{\mu} (\mu - \phi'(t)y^m(t)) = y^m(t), \quad \in [0, T). \]

The definitions of $a$ and $m$ in (5.1) and (5.3) and the fact that $m(t, y^m(t), p) = 1$ by (5.10) give
\[ \frac{1}{p\sigma^2 \kappa^{G}(t)} (\mu - \phi'(t)y^m(t)) = 1 - a(t, y^m(t), p) = 1 - (1 + y^m(t))^{-\frac{1}{p}}, \]
and the definition of $h$ in (5.16) yields
\[
p(1 + y^m(t)h(y^m(t), p)) = y^m(t) \left(1 + \frac{1}{(1 + y^m(t))p - 1}\right)
\]
which taken together give (D.15).

We proceed to show that $\tilde{y}^m$ satisfies (3.2) and (3.9). By (5.15), $\tilde{y}^m(t) = y^m(t)h(y^m(t), p)$ and $h$ is uniformly bounded from above and from below by (5.17). Thus, it suffices to check that $y^m$ satisfies (3.2) and (3.9). Since the right-hand side in (D.14) is positive and $y^m$ is nonnegative by Theorem 5.3, it follows that
\[
0 \leq \phi'(t)y^m(t) = \frac{1}{(1 + y^m(t))p - 1}y^m(t)\frac{1}{1 - (1 + y^m(t))p - 1},
\]
which taken together give (D.15).

**Proof of Theorem 5.5.** First, assume that $p = 1$. By definition of the wealth process and the fact that $(\mu t + \sigma W_t)_{t \in [0,T]}$ is a continuous semimartingale and $\mathcal{M}^G \phi$ a purely discontinuous martingale,
\[
X_T^\pi = x \mathcal{E}_T \left( \int_0^T \tilde{\pi}_u dR_u \right) = x \mathcal{E}_T \left( \int_0^T \tilde{\pi}_u d(\mu u + \sigma W_u) \right) \mathcal{E}_T \left( \int_0^T \tilde{\pi}_u d\mathcal{M}^G \phi_u \right) \text{ P-a.s.}
\]
We first compute the expected value of the logarithm of the first factor on the right-hand side of (D.16); this corresponds exactly to the utility an investor obtains from employing the strategy $\tilde{\pi}$ in the standard Black–Scholes model. Since $\int_0^T \alpha d\tilde{\pi}_u dW_u$ is a square-integrable martingale by the definition of $\tilde{\pi}$ in (5.6) and (C.17), a standard calculation gives
\[
E \left[ \log \left( \mathcal{E}_T \left( \int_0^T \tilde{\pi}_u d(\mu u + \sigma W_u) \right) \right) \right]
= E \left[ \frac{1}{2\sigma^2} \int_0^T (\mu^2 - \phi'(u)^2 \bar{y}(u)^2 1_{(u \leq \gamma, u < T)}) du \right]
= \frac{\mu^2}{2\sigma^2} T - \int_0^T \phi'(u)^2 \bar{y}(u)^2 (1 - G(u)) du.
\]
To compute the expected value of the logarithm of the second factor, we first note that by the dynamics of $\mathcal{M}^G \phi$ (cf. the discussion before Remark 2.2),
\[
\int_0^T \tilde{\pi}_u d\mathcal{M}^G \phi_u = \int_0^\gamma \bar{\pi}(u, u) \phi'(u) du - \bar{\pi}(\gamma, \gamma) \delta(\gamma) 1_{(\gamma < T)} \text{ P-a.s.,}
\]
and so by the formula for the stochastic exponential,
\[
\mathcal{E}_T \left( \int_0^T \tilde{\pi}_u d\mathcal{M}^G \phi_u \right)
= \exp \left( \int_0^T 1_{(u < \gamma)} \bar{\pi}(u, u) \phi'(u) du \right) (1 - \bar{\pi}(\gamma, \gamma) \delta(\gamma) 1_{(\gamma < T)}) \text{ P-a.s.}
\]
Thus, using the definitions of $\bar{\pi}$, $\delta$, $a$ and $m$ in (D.2), (2.6), (5.1) and (5.3) and the fact that $m(t, \hat{y}(t), 1) \equiv 1$ by (5.10) (since $\hat{\pi} = \pi^m$ for $p = 1$ by Theorem 5.3 (b)), for $v \in [0, T)$,

$$1 - \bar{\pi}(v, v) \delta(v) = 1 - \bar{\pi}(v, v) \frac{\phi'(v)}{\kappa^G(v)} = a(v, \hat{y}(v), 1) = \frac{m(v, \hat{y}(v), 1)}{1 + \hat{y}(v)} = \frac{1}{1 + \hat{y}(v)}.$$ 

The above together with the definition of $\kappa^G$ in (2.1) and the fact that

$$\log \left( 1 - \bar{\pi}(\gamma, \gamma) \delta(\gamma) \mathbf{1}_{\{\gamma < T\}} \right) = \log \left( 1 - \bar{\pi}(\gamma, \gamma) \delta(\gamma) \right) \mathbf{1}_{\{\gamma < T\}}$$

gives

$$E \left[ \log \left( \mathcal{E}_T \left( \int_0^T \bar{\pi}_u d\mathcal{M} \phi_u \right) \right) \right] = \int_0^T \bar{\pi}(u, u) \phi'(u) (1 - G(u)) du + \int_0^T \log \left( \frac{1}{1 + \hat{y}(u)} \right) G'(u) du$$

$$= \int_0^T \left( \bar{\pi}(u, u) \frac{\phi'(u)}{\kappa^G(u)} + \log \left( \frac{1}{1 + \hat{y}(u)} \right) \right) G'(u) du$$

$$= - \int_0^T \left( \log(1 + \hat{y}(u)) - \frac{\hat{y}(u)}{1 + \hat{y}(u)} \right) G'(u) du.$$ 

Putting everything together establishes (5.19).

Now assume that $p \neq 1$. Then by the optimality conditions (OC1) and (OC2) and the definition of $\hat{z}$ in (5.9),

$$E \left[ U \left( X^\hat{\pi}_T \right) \right] = \frac{1}{1 - p} E \left[ (X^\hat{\pi}_T)^{1-p} \right] = \frac{1}{1 - p} E \left[ X^\hat{\pi}_TU' \left( X^\hat{\pi}_T \right) \right]$$

$$= \frac{1}{1 - p} E \left[ X^\hat{\pi}_T \hat{z} d\hat{Q} \right] = \hat{z} \frac{1}{1 - p} E^{\hat{Q}} \left[ X^\hat{\pi}_T \right] = \hat{z} \frac{1}{1 - p} x$$

$$= \frac{x^{1-p}}{1 - p} \exp \left( (1 - p) \frac{\mu^2}{2p\sigma^2} T \right) m(0, \hat{y}(0), p)^{-p}. \quad \Box$$
Bibliography


