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The Donaldson Geometric Flow

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Abstract

At the core of symplectic geometry lies the symplectic structure. This fundamental mathematical structure has it's origin in the theory of general mechanics and appears naturally in the study of the phase space of a dynamical system. It is defined on even dimensional manifolds and is characterized by a local and a global condition. The local condition says that the exterior product with itself is a volume form

$$\underbrace{\rho \wedge \ldots \wedge \rho}_{n} \neq 0,$$

where $2n$ equals the dimension of the manifold. The global structure says that it's exterior derivative vanishes,

$$d\rho = 0.$$

In two dimensions it is just the standart volume form. In higher even dimensional manifolds the structure captures a much more subtle information about the manifold. The space of symplectic structures is the set of all symplectic structures a manifold can carry. In two dimensions every two symplectic structures with the same orientation are connected by a path of symplectic structures. We say up to isotopy symplectic structures on two dimensional manifolds are unique. In higher dimensions the question of uniqueness of the symplectic structure turns out to be a delicate matter. Let $\mathscr{S}_a$ be the space of symplectic structures in a fixed cohomology class $a \in H^2(M;\mathbb{R})$ with a fixed orientation on a symplectic closed oriented manifold M. In dimension 6 there exists an example by McDuff [2] where this space has at least two connected components. In the four-dimensional case it is a completely open question how many connected components $\mathscr{S}_a$ can exist. An approach to answer this question is the Donaldson geometric flow, introduced by Simon Donaldson in [1].

This thesis consists of three papers laying the foundation for a geometric flow on $\mathscr{S}_a$, which we call the *Donaldson geometric flow*. It describes an evolution of symplectic structures that minimises an energy functional on $\mathscr{S}_a$, which we hope will always eventually converge to the same distinguished element in this space. The fundamental observation of Donaldson is that in the hyperKähler case higher critical points of the flow are not strictly stable. This gives rise to the hope that once long time existence and all analytic

questions of the flow are settled we can evolve a symplectic strucuture by the flow until it hits a critical point. Then a small pertubation should be enough to continue the flow and further decrease the associated energy of the sympletic structure until it reaches the absolute minimum, which is the only strictly stable critical point. This would provide a proof that any two symplectic structures in the same cohomology class with the same orientation are isotopic.

The origin of the Donaldson flow is a negative gradient flow of a moment map square functional on the space of diffeomorphisms on a closed, oriented Riemannian manifold M, where the group of symplectomorphisms to a fixed symplectic structure is acting by composition on the right. This setting can be translated by push forward of the fixed symplectic structure by the diffeomorphisms into an energy functional on $\mathscr{S}_a$ given by

$$\mathscr{E}(\rho) := 2 \int_M \frac{|\rho^+|}{|\rho^+|^2 - |\rho^-|^2} \mathrm{dvol}, \tag{1}$$

a special metric, called the *Donaldson metric*, and the Donaldson geometric flow equation. In this setting the Donaldson flow becomes the negative gradient flow to the energy functional (1). The Donaldson flow equation is a partial differential equation on the space of two-forms on an oriented closed four-dimensional Riemannian manifold with Hodge star operator $*$ given by,

$$\frac{d}{dt}\rho_t = d *^{\rho_t} d\Theta^{\rho_t}, \tag{2}$$

where

$$\Theta^\rho := *\frac{\rho}{u} - \frac{1}{2}\left|\frac{\rho}{u}\right|^2 \rho, \qquad \frac{1}{2}\rho \wedge \rho =: u\mathrm{dvol}, \qquad *^\rho \lambda := \frac{\rho \wedge *(\rho \wedge \lambda)}{\mathrm{u}}.$$

This equation is a fully nonlinear equation and in particular the question of existence needs to be addressed. Further, if a solution exists, we need to prove that it is smooth under reasonable assumptions on the initial conditions. Finally, we want to know that the solutions of the flow depend smoothly on the inital conditions.

The first paper, *The Donaldson geometric flow*, contains an exposition of the ideas of Simon Donaldson and describes how this flow can be seen as a negative gradient flow of an energy functional. It outlines the master plan to prove uniqueness of the symplectic structures on four-dimensional closed oriented manifolds. We develope the formulation of the flow as given in the previous equation. The introduction of the operator $*^\rho$ and a special metric g^ρ determined by $*^\rho$ and the fixed volume element dvol symplifies the flow equation greatly. It allows one to see that the highest order term of the linearization of the flow equation is essentially the Hodge Laplacian with respect to the metric g^ρ. This is the key to prove short time existence in the second paper. A highlight is the proof that the manifold $\mathbb{C}P^2$ doesn't contain any higher critical points.

The second paper, *The Donaldson flow is a local semiflow*, establishes regularity and short time existence of the Donaldson flow as well as smooth dependence on initial conditions. The regularity of the solutions does not follow directly from equation (2). The regularity proof relies on the observation that there exists a local Banach space diffeomorphism between the Banach space completions of the space $\mathscr{S}_a$ and a submanifold of $\Omega^0(M, \Lambda^+)$ of codimension b^+. This map allows one to reformulate the flow equation (2) into a parabolic equation on the space $\Omega^0(M, \Lambda^+)$ with coefficients depending on ρ that allows parabolic bootstrapping. Both, the existence and the regularity proof, use the theory of maximal regularity to understand parabolic regularity in the $L^p - L^2$ setting for $p \neq 2$ for parabolic operators with non-smooth coefficients. The natural space for this setting is the Besov space $B_2^{1,p}$.

The third paper, *Remarks on the Donaldson metric*, contains observations on the Donaldson metric. This is a metric on the space of symplectic structures naturally arising in the study of the flow as a negative gradient flow. We compute the Levi-Civita connection of this metric and the covariant Hessian of the energy functional of the Donaldson flow.

References

[1] S.K. Donaldson, Moment Maps and Diffeomorphisms. *Asian J. Math.* **3** (1999), 1–16. `http://bogomolov-lab.ru/G-sem/AJM-3-1-001-016.pdf`

[2] D. McDuff, Examples of symplectic structures. *Invent. Math.* **89** (1987), 353–366.

Zusammenfassung

Im Zentrum der symplektischen Geometrie steht die symplektische Struktur. Das sind nicht-degenerierte geschlossene zwei Formen. Für zwei dimensionale Manigfaltigkeiten wissen wir, dass je zwei symplektische Strukturen mit derselben Orientierung durch einen Pfad symplektischer Strukturen verbunden werden können. Es stellt sich heraus, dass die Frage der Eindeutigkeit für symplektische Strukturen in höheren Dimensionen sehr schwer ist. Insbesondere ist die Frage für vier dimensionale geschlossen Manigfaltigkeiten komplett offen. In dieser Dissertation erarbeiten wir die Fundamente eines geometrischen Flusses auf dem Raum der symplektischen Strukturen mit fester Orientierung und Cohomologie Klasse. Die zentrale Idee dieses Flusses ist die Vermutung, dass er immer zu demselben ausgeprägten Element in diesem Raum konvergiert. Wir nennen diesen Fluss *Donaldson geometrischer Fluss* nach seinem Erfinder Simon Donaldson.

Die Dissertation besteht aus drei Publikationen. Die erste Publikation, *The Donaldson geometric flow*, enthält eine Übersicht der Fundamente des Flusses und der Ideen Donaldson's. Wir erklären, wie man den Donaldson Fluss als negativen Gradienten Fluss eines Energie Funktionals bezüglich einer speziellen Metrik namens *Donaldson Metrik* verstehen kann.

Die zweite Publikation, *The Donaldson flow is a smooth semiflow*, erarbeitet zentrale analytische Fragen, insbesondere die Existenz für kurze Zeiten, die Regularität und die glatte Abhängigkeit von den Anfangsbedingungen.

Die dritte Publikation enthält zusätzliche Beobachtungen über die Donaldson Metrik, die für fortführende Forschung von Bedeutung sein könnten.