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The Donaldson Geometric Flow

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To Wazy

Abstract

At the core of symplectic geometry lies the symplectic structure. This fundamental mathematical structure has it's origin in the theory of general mechanics and appears naturally in the study of the phase space of a dynamical system. It is defined on even dimensional manifolds and is characterized by a local and a global condition. The local condition says that the exterior product with itself is a volume form

$$\underbrace{\rho \wedge \ldots \wedge \rho}_{n} \neq 0,$$

where $2n$ equals the dimension of the manifold. The global structure says that it's exterior derivative vanishes,

$$d\rho = 0.$$

In two dimensions it is just the standart volume form. In higher even dimensional manifolds the structure captures a much more subtle information about the manifold. The space of symplectic structures is the set of all symplectic structures a manifold can carry. In two dimensions every two symplectic structures with the same orientation are connected by a path of symplectic structures. We say up to isotopy symplectic structures on two dimensional manifolds are unique. In higher dimensions the question of uniqueness of the symplectic structure turns out to be a delicate matter. Let $\mathscr{S}_a$ be the space of symplectic structures in a fixed cohomology class $a \in H^2(M;\mathbb{R})$ with a fixed orientation on a symplectic closed oriented manifold M. In dimension 6 there exists an example by McDuff [2] where this space has at least two connected components. In the four-dimensional case it is a completely open question how many connected components $\mathscr{S}_a$ can exist. An approach to answer this question is the Donaldson geometric flow, introduced by Simon Donaldson in [1].

This thesis consists of three papers laying the foundation for a geometric flow on $\mathscr{S}_a$, which we call the *Donaldson geometric flow*. It describes an evolution of symplectic structures that minimises an energy functional on $\mathscr{S}_a$, which we hope will always eventually converge to the same distinguished element in this space. The fundamental observation of Donaldson is that in the hyperKähler case higher critical points of the flow are not strictly stable. This gives rise to the hope that once long time existence and all analytic

questions of the flow are settled we can evolve a symplectic strucuture by the flow until it hits a critical point. Then a small pertubation should be enough to continue the flow and further decrease the associated energy of the sympletic structure until it reaches the absolute minimum, which is the only strictly stable critical point. This would provide a proof that any two symplectic structures in the same cohomology class with the same orientation are isotopic.

The origin of the Donaldson flow is a negative gradient flow of a moment map square functional on the space of diffeomorphisms on a closed, oriented Riemannian manifold M, where the group of symplectomorphisms to a fixed symplectic structure is acting by composition on the right. This setting can be translated by push forward of the fixed symplectic structure by the diffeomorphisms into an energy functional on $\mathscr{S}_a$ given by

$$\mathscr{E}(\rho) := 2\int_M \frac{|\rho^+|}{|\rho^+|^2 - |\rho^-|^2}\mathrm{dvol}, \tag{1}$$

a special metric, called the *Donaldson metric*, and the Donaldson geometric flow equation. In this setting the Donaldson flow becomes the negative gradient flow to the energy functional (1). The Donaldson flow equation is a partial differential equation on the space of two-forms on an oriented closed four-dimensional Riemannian manifold with Hodge star operator $*$ given by,

$$\frac{d}{dt}\rho_t = d *^{\rho_t} d\Theta^{\rho_t}, \tag{2}$$

where

$$\Theta^\rho := *\frac{\rho}{u} - \frac{1}{2}\left|\frac{\rho}{u}\right|^2\rho, \qquad \frac{1}{2}\rho\wedge\rho =: u\mathrm{dvol}, \qquad *^\rho\lambda := \frac{\rho\wedge *(\rho\wedge\lambda)}{\mathrm{u}}.$$

This equation is a fully nonlinear equation and in particular the question of existence needs to be addressed. Further, if a solution exists, we need to prove that it is smooth under reasonable assumptions on the initial conditions. Finally, we want to know that the solutions of the flow depend smoothly on the inital conditions.

The first paper, *The Donaldson geometric flow*, contains an exposition of the ideas of Simon Donaldson and describes how this flow can be seen as a negative gradient flow of an energy functional. It outlines the master plan to prove uniqueness of the symplectic structures on four-dimensional closed oriented manifolds. We develope the formulation of the flow as given in the previous equation. The introduction of the operator $*^\rho$ and a special metric g^ρ determined by $*^\rho$ and the fixed volume element dvol symplifies the flow equation greatly. It allows one to see that the highest order term of the linearization of the flow equation is essentially the Hodge Laplacian with respect to the metric g^ρ. This is the key to prove short time existence in the second paper. A highlight is the proof that the manifold $\mathbb{C}P^2$ doesn't contain any higher critical points.

The second paper, *The Donaldson flow is a local semiflow*, establishes regularity and short time existence of the Donaldson flow as well as smooth dependence on initial conditions. The regularity of the solutions does not follow directly from equation (2). The regularity proof relies on the observation that there exists a local Banach space diffeomorphism between the Banach space completions of the space $\mathscr{S}_a$ and a submanifold of $\Omega^0(M, \Lambda^+)$ of codimension b^+. This map allows one to reformulate the flow equation (2) into a parabolic equation on the space $\Omega^0(M, \Lambda^+)$ with coefficients depending on ρ that allows parabolic bootstrapping. Both, the existence and the regularity proof, use the theory of maximal regularity to understand parabolic regularity in the $L^p - L^2$ setting for $p \neq 2$ for parabolic operators with non-smooth coefficients. The natural space for this setting is the Besov space $B_2^{1,p}$.

The third paper, *Remarks on the Donaldson metric*, contains observations on the Donaldson metric. This is a metric on the space of symplectic structures naturally arising in the study of the flow as a negative gradient flow. We compute the Levi-Civita connection of this metric and the covariant Hessian of the energy functional of the Donaldson flow.

References

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Zusammenfassung

Im Zentrum der symplektischen Geometrie steht die symplektische Struktur. Das sind nicht-degenerierte geschlossene zwei Formen. Für zwei dimensionale Manigfaltigkeiten wissen wir, dass je zwei symplektische Strukturen mit derselben Orientierung durch einen Pfad symplektischer Strukturen verbunden werden können. Es stellt sich heraus, dass die Frage der Eindeutigkeit für symplektische Strukturen in höheren Dimensionen sehr schwer ist. Insbesondere ist die Frage für vier dimensionale geschlossen Manigfaltigkeiten komplett offen. In dieser Dissertation erarbeiten wir die Fundamente eines geometrischen Flusses auf dem Raum der symplektischen Strukturen mit fester Orientierung und Cohomologie Klasse. Die zentrale Idee dieses Flusses ist die Vermutung, dass er immer zu demselben ausgeprägten Element in diesem Raum konvergiert. Wir nennen diesen Fluss *Donaldson geometrischer Fluss* nach seinem Erfinder Simon Donaldson.

Die Dissertation besteht aus drei Publikationen. Die erste Publikation, *The Donaldson geometric flow*, enthält eine Übersicht der Fundamente des Flusses und der Ideen Donaldson's. Wir erklären, wie man den Donaldson Fluss als negativen Gradienten Fluss eines Energie Funktionals bezüglich einer speziellen Metrik namens *Donaldson Metrik* verstehen kann.

Die zweite Publikation, *The Donaldson flow is a smooth semiflow*, erarbeitet zentrale analytische Fragen, insbesondere die Existenz für kurze Zeiten, die Regularität und die glatte Abhängigkeit von den Anfangsbedingungen.

Die dritte Publikation enthält zusätzliche Beobachtungen über die Donaldson Metrik, die für fortführende Forschung von Bedeutung sein könnten.

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Contents

The Donaldson geometric flow for symplectic four-manifolds

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Abstract

This is an exposition of the Donaldson geometric flow on the space of symplectic forms on a closed smooth four-manifold, representing a fixed cohomology class. The original work appeared in [1].

1 Introduction

For any closed symplectic four-manifold (M, ω) it is an open question whether the space of symplectic forms on M representing the same cohomology class as ω is connected. By Moser isotopy a positive answer to this question is equivalent to the assertion that every symplectic form in the cohomology class of ω is diffeomorphic to ω via a diffeomorphism that is isotopic to the identity. In the case of the projective plane it follows from theorems of Gromov and Taubes that a positive answer is equivalent to the assertion that a diffeomorphism is isotopic to the identity if and only if it induces the identity on homology. In the case of the four-torus a positive answer is a longstanding conjecture in symplectic topology. This is part of the circle of questions around the uniqueness problem in symplectic topology as discussed in [5]. A remarkable geometric flow approach to the uniqueness problem in dimension four was explained by Donaldson in a lecture in Oxford in the spring of 1997 (attended by the second author) and written up in [1]. The purpose of this expository paper is to explain some of the details.

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The starting point of Donaldson's approach is the obervation that the space of diffeomorphism of a hyperKähler surface M can be viewed as an infinite-dimensional hyperKähler manifold, that the group of symplectomorphisms associated to a preferred symplectic structure ω acts on the right by hyperKähler isometries, and that this group action is generated by a hyperKähler moment map. In analogy to the finite-dimensional setting one can then study the gradient flow of the square of the hyperKähler moment map. Pushing ω forward under the diffeomorphisms of M one obtains a geometric flow on the space $\mathscr{S}_a$ of symplectic forms in the cohomology class $a := [\omega]$. It turns out that this geometric flow is well defined for each symplectic four-manifold (M, ω) equipped with a Riemannian metric g that is compatible with ω. It is the gradient flow of the energy functional

$$\mathscr{E}(\rho) := \int_M \frac{2|\rho^+|^2}{|\rho^+|^2 - |\rho^-|^2} \,\mathrm{dvol}, \qquad \rho \in \mathscr{S}_a, \tag{1}$$

with respect to a suitable metric on $\mathscr{S}_a$. To describe this metric, we first recall an observation of Donaldson [2] which asserts that for every positive rank-3 subbundle $\Lambda^+ \subset \Lambda^2 T^*M$ and every positive volume form $\mathrm{dvol} \in \Omega^4(M)$ there is a unique Riemannian metric on M with volume form dvol such that Λ^+ is the bundle of self-dual 2-forms (Theorem A.1). Second, every $\rho \in \mathscr{S}_a$ determines an involution $R^\rho : \Omega^2(M) \to \Omega^2(M)$ which sends ρ to $-\rho$ and acts as the identity on the orthogonal complement of ρ with respect to the exterior product; it is given by $R^\rho\tau := \tau - 2\frac{\tau\wedge\rho}{\rho\wedge\rho}\rho$ and preserves the pairing. Thus every $\rho \in \mathscr{S}_a$ determines a unique Riemannian metric g^ρ on M with the same volume form as g such that τ is self-dual with respect to g if and only if $R^\rho\tau$ is self-dual with respect to g^ρ (Theorem A.2). For $\rho \in \mathscr{S}_a$ denote by $*^\rho : \Omega^k(M) \to \Omega^{4-k}(M)$ the Hodge $*$-operator of g^ρ. Then the **Donaldson metric** on the infinite-dimensional manifold $\mathscr{S}_a$ is given by

$$\|\widehat{\rho}\|_\rho^2 := \int_M \lambda \wedge *^\rho \lambda, \qquad d\lambda = \widehat{\rho}, \qquad *^\rho\lambda \text{ is exact}, \tag{2}$$

for $\widehat{\rho} \in T_\rho\mathscr{S}_a = \mathrm{im}(d : \Omega^1(M) \to \Omega^2(M))$ (see Definition 3.2). Now the differential of the energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$ at a point $\rho \in \mathscr{S}_a$ is the linear map $\widehat{\rho} \mapsto \int_M \Theta^\rho \wedge \widehat{\rho}$ where the 2-form $\Theta^\rho \in \Omega^2(M)$ is given by

$$\Theta^\rho := *\frac{\rho}{u} - \frac{1}{2}\left|\frac{\rho}{u}\right|^2 \rho, \qquad u := \frac{\rho\wedge\rho}{2\mathrm{dvol}}. \tag{3}$$

This is the pointwise orthogonal projection of the 2-form $u^{-1} * \rho$ onto the orthogonal complement of ρ with respect to the exterior product.

The negative gradient flow of the energy functional $\mathscr{E}:\mathscr{S}_a\to\mathbb{R}$ in (1) with respect to the Donaldson metric (2) has the form

$$\partial_t\rho = d *^\rho d\Theta^\rho, \tag{4}$$

where $\Theta^\rho\in\Omega^2(M)$ is given by (3) (Proposition 3.4). This is the **Donaldson geometric flow**. The purpose of the present paper is to explain some of the geometric properties of this flow, and to give an exposition of the necessary background material. This includes a discussion of the Riemannian metrics g^ρ which is relegated to Appendix A. The Donaldson geometric flow in the original hyperKähler moment map setting is explained in Section 2, for general symplectic four-manifolds it is discussed in Section 3, and the Hessian of the energy functional is examined in Section 4.

The motivation for this study is the dream that the solutions of (4) can be used to settle the uniqueness problem for symplectic structures in dimension four in some favourable cases such as hyperKähler surfaces or the complex projective plane. This is backed up by the observations that the symplectic form ω is the unique absolute minimum of $\mathscr{E}$ (Corollary 3.5) and the Hessian of $\mathscr{E}$ at ω is positive definite (Corollary 4.4). For $M=\mathbb{C}\mathrm{P}^2$ we prove that the Fubini–Study form is the only critical point (Proposition 3.8). The present exposition also includes a proof of Donaldson's observation that higher critical points cannot be strictly stable in the hyperKähler setting (Theorem 4.5). Local existence and uniqueness and regularity for the solutions of (4) are estabished in the followup paper [3] for which the present paper provides the necessary background. Key problems for future research include long-time existence and to show that the solutions cannot *escape to infinity*.

Sign Conventions. Let (M,ω) be a symplectic manifold and let G be a Lie group with Lie algebra $\mathfrak{g}:=\mathrm{Lie}(\mathrm{G})$ that acts covariantly on M by symplectomorphisms. Denote the infinitesimal action by $\mathfrak{g}\to\mathrm{Vect}(M):\xi\mapsto X_\xi$. We use the sign convention $[X,Y]:=\nabla_Y X-\nabla_X Y$ for the Lie bracket of vector fields so the infinitesimal action is a Lie algebra homomorphism. We use the sign convention $\iota(X_H)\omega=dH$ for Hamiltonian vector fields so the map $C^\infty(M)\to\mathrm{Vect}(M):H\mapsto X_H$ is a Lie algebra homomorphism with respect to the Poisson bracket $\{F,G\}:=\omega(X_F,X_G)$. The group action is called *Hamiltonian* if there is a G-equivariant *moment map* $\mu:M\to\mathfrak{g}^*$ such that X_ξ is the Hamiltonian vector field of $H_\xi:=\langle\mu,\xi\rangle$ for $\xi\in\mathfrak{g}$. If $\mathfrak{g}$ is equipped with an invariant inner product it is convenient to write $\mu:M\to\mathfrak{g}$.

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2 The Moment Map Picture

Throughout this section M denotes a closed hyperKähler surface with symplectic forms $\omega_1, \omega_2, \omega_3$ and complex structures J_1, J_2, J_3. Thus each J_i is compatible with ω_i, the resulting Riemannian metric

$$\langle \cdot, \cdot \rangle := \omega_i(\cdot, J_i \cdot)$$

is independent of i, and the complex structures satisfy the quaternion relations $J_i J_j = -J_j J_i = J_k$ for every cyclic permutation i, j, k of $1, 2, 3$. Let (S, σ) be a symplectic four-manifold that is diffeomorphic to M and define

$$\mathscr{F} := \left\{ f : S \to M \,\middle|\, \begin{array}{l} f \text{ is a diffeomorphism and} \\ \text{the 2-form } f^*\omega_1 - \sigma \text{ is exact} \end{array} \right\}. \tag{5}$$

This space need not be connected. Assume it is nonempty. (Whether this implies that (S, σ) is symplectomorphic to (M, ω_1) is an open question.) Then the space $\mathscr{F}$ is a C^1 open set in the space of all smooth maps from S to M and can be viewed formally as an infinite-dimensional hyperKähler manifold. Its tangent space at $f \in \mathscr{F}$ is the space of vector fields along f and will be denoted by $T_f\mathscr{F} = \Omega^0(S, f^*TM)$. The three complex structures are given by $T_f\mathscr{F} \to T_f\mathscr{F} : \widehat{f} \mapsto J_i \widehat{f}$ and the three symplectic forms are given by

$$\Omega_i(\widehat{f}_1, \widehat{f}_2) := \int_S \omega_i(\widehat{f}_1, \widehat{f}_2)\, \mathrm{dvol}_\sigma, \qquad \mathrm{dvol}_\sigma := \frac{\sigma \wedge \sigma}{2} \tag{6}$$

for $\widehat{f}_1, \widehat{f}_2 \in T_f\mathscr{F}$. The group

$$\mathscr{G} := \mathrm{Symp}(S, \sigma) := \left\{ \varphi \in \mathrm{Diff}(S) \,\middle|\, \varphi^*\sigma = \sigma \right\} \tag{7}$$

of symplectomorphism of (S, σ) acts contravariantly on $\mathscr{F}$ by composition on the right. This group action preserves the hyperKähler structure of $\mathscr{F}$. The quotient space $\mathscr{F}/\mathscr{G}$ is homeomorphic to the space $\mathscr{S}$ of all symplectic forms on M that are cohomologous to ω_1 and diffeomorphic to σ via the homeomorphism $\mathscr{F}/\mathscr{G} \to \mathscr{S} : [f] \mapsto (f^{-1})^*\sigma$. The action of the subgroup

$$\mathscr{G}_0 := \mathrm{Ham}(S, \sigma) \tag{8}$$

of Hamiltonian symplectomorphisms is Hamiltonian for all three symplectic forms on $\mathscr{F}$. This is the content of the next proposition. We identify the Lie algebra of $\mathscr{G}_0$ with the space of smooth real valued functions on S with mean value zero and its dual space with the quotient $\Omega^0(S)/\mathbb{R}$ via the L^2 inner product associated to the volume form dvol_σ.

Proposition 2.1 (Moment Map). *The map*

$$\mu_i : \mathscr{F} \to \Omega^0(S), \qquad \mu_i(f) := \frac{f^*\omega_i \wedge \sigma}{\mathrm{dvol}_\sigma}, \tag{9}$$

is a moment map for the covariant action

$$\mathscr{G}_0 \times \mathscr{F} \to \mathscr{F} : (\varphi, f) \mapsto f \circ \varphi^{-1}$$

with respect to the symplectic form Ω_i.

Proof. The infinitesimal covariant action of a smooth function $H : S \to \mathbb{R}$ with mean value zero on $\mathscr{F}$ is given by the vector field on $\mathscr{F}$ which assigns to each $f \in \mathscr{F}$ the vector field $-df \circ X_H \in T_f\mathscr{F}$ along f. Here $X_H \in \mathrm{Vect}(S)$ is the Hamiltonian vector field on S associated to H and is determined by the equation $\iota(X_H)\sigma = dH$. The minus sign appears because composition on the right defines a contravariant action of $\mathscr{G}_0$ and the covariant action is given by composition with φ^{-1} on the right. By Cartan's formula, the differential of the map $\mu_i : \mathscr{F} \to \Omega^0(S)$ in (9) at f in the direction $\widehat{f} \in T_f\mathscr{F}$ is given by

$$d\mu_i(f)\widehat{f} = \frac{d\alpha_i \wedge \sigma}{\mathrm{dvol}_\sigma}, \qquad \alpha_i := \omega_i(\widehat{f}, df\cdot) \in \Omega^1(S). \tag{10}$$

Now contract the vector field $f \mapsto -df \circ X_H$ on $\mathscr{F}$ with the symplectic form Ω_i to obtain

$$\begin{aligned} \Omega_i(-df \circ X_H, \widehat{f}) &= \int_S \omega_i(\widehat{f}, df \circ X_H)\,\mathrm{dvol}_\sigma \\ &= \int_S (\iota(X_H)\alpha_i)\mathrm{dvol}_\sigma \\ &= \int_S \alpha_i \wedge \iota(X_H)\mathrm{dvol}_\sigma \\ &= \int_S \alpha_i \wedge dH \wedge \sigma \\ &= \int_S H d\alpha_i \wedge \sigma \\ &= \int_S H d\mu_i(f)\widehat{f}\,\mathrm{dvol}_\sigma. \end{aligned}$$

The last term is the differential of the function $\mathscr{F} \to \mathbb{R} : f \mapsto \langle \mu_i(f), H\rangle_{L^2}$ at f in the direction $\widehat{f}$. This proves Proposition 2.1. □

The norm squared of the moment map in the hyperKähler setting is the function $\mathscr{E} := \frac{1}{2}(\|\mu_1\|^2 + \|\mu_2\|^2 + \|\mu_3\|^2)$, where the norm on the (dual of the) Lie algebra is associated to an invariant inner product. In the case at hand the invariant inner product is the L^2 inner product on $\Omega^0(S)$ and the norm squared of the moment map is the energy functional $\mathscr{E} : \mathscr{F} \to \mathbb{R}$ given by

$$\mathscr{E}(f) := \frac{1}{2}\int_S \sum_{i=1}^{3} |H_i|^2 \,\mathrm{dvol}_\sigma, \qquad H_i := \frac{f^*\omega_i \wedge \sigma}{\mathrm{dvol}_\sigma}. \tag{11}$$

We next examine the negative gradient flow lines of $\mathscr{E}$ with respect to the hyperKähler metric on $\mathscr{F}$, given by

$$\left\langle \widehat{f}_1, \widehat{f}_2 \right\rangle_{L^2} := \int_S \langle \widehat{f}_1, \widehat{f}_2 \rangle \,\mathrm{dvol}_\sigma \qquad \text{for } \widehat{f}_1, \widehat{f}_2 \in T_f\mathscr{F}.$$

Proposition 2.2 (Gradient Flow). *An isotopy $\mathbb{R} \to \mathscr{F} : t \mapsto f_t$ is a negative L^2 gradient flow line of the energy functional $\mathscr{E} : \mathscr{F} \to \mathbb{R}$ in* (11) *if and only if it satisfies the partial differential equation*

$$\partial_t f_t = \sum_{i=1}^{3} J_i df_t \circ X_{H_{it}}, \qquad H_{it} := \frac{f_t^*\omega_i \wedge \sigma}{\mathrm{dvol}_\sigma}, \qquad \iota(X_{H_{it}})\sigma = dH_{it}. \tag{12}$$

Proof. The differential of the energy functional $\mathscr{E} : \mathscr{F} \to \mathbb{R}$ at $f \in \mathscr{F}$ in the direction $\widehat{f} \in T_f\mathscr{F}$ is given by

$$\begin{aligned} \delta\mathscr{E}(f)\widehat{f} &= \sum_{i=1}^{3} \left\langle d\mu_i(f)\widehat{f}, \mu_i(f) \right\rangle_{L^2} \\ &= \sum_{i=1}^{3} \Omega_i(-df \circ X_{H_i}, \widehat{f}) \\ &= -\sum_{i=1}^{3} \left\langle J_i df \circ X_{H_i}, \widehat{f} \right\rangle_{L^2}. \end{aligned}$$

Here $H_i := \mu_i(f) \in \Omega^0(S)$ is as in (11), the second equation follows from Proposition 2.1, and the third equation follows from the fact that $\omega_i = \langle J_i \cdot, \cdot \rangle$. Hence the L^2 gradient of $\mathscr{E}$ is given by

$$\mathrm{grad}\mathscr{E}(f) = -\sum_{i=1}^{3} J_i df \circ X_{H_i} \tag{13}$$

and this proves Proposition 2.2. □

The energy functional (11) and the L^2 metric on $\mathscr{F}$ are invariant under the action of the full group $\mathscr{G}$ of all symplectomorphisms of (S, σ) and so is the negative gradient flow (12). To eliminate the action of the infinite-dimensional symplectomorphism group it is convenient to replace the solutions $t \mapsto f_t$ of equation (12) by paths of symplectic forms $t \mapsto \rho_t$ on M obtained by pushing forward the symplectic form σ on S by the diffeomorphisms $f_t : S \to M$.

Proposition 2.3 (Pushforward Gradient Flow). *Let $\mathbb{R} \to \mathscr{F} : t \mapsto f_t$ be a solution of* (12) *and define the symplectic form $\rho_t \in \Omega^2(M)$ by*

$$\rho_t := (f_t^{-1})^* \sigma$$

for $t \in \mathbb{R}$. Then ρ_t is cohomologous to ω_1 for all t and the path $t \mapsto \rho_t$ satisfies the partial differential equation

$$\partial_t \rho_t = -\sum_{i=1}^{3} d(dK_i^{\rho_t} \circ J_i^{\rho_t}), \quad K_i^{\rho_t} := \frac{\omega_i \wedge \rho_t}{\mathrm{dvol}_{\rho_t}}, \quad \rho_t(J_i^{\rho_t} \cdot, \cdot) := \rho_t(\cdot, J_i \cdot). \tag{14}$$

Proof. Differentiate the equation $f_t^* \rho_t = \sigma$ using Cartan's formula to obtain

$$0 = f_t^* \partial_t \rho_t + d\beta_t, \qquad \beta_t := \rho_t(\partial_t f_t, df_t \cdot) \in \Omega^1(S). \tag{15}$$

Since f_t satisfies (12) it follows that

$$\begin{aligned} \beta_t &= \sum_{i=1}^{3} \rho_t(J_i df_t \circ X_{H_{it}}, df_t \cdot) \\ &= \sum_{i=1}^{3} \rho_t(df_t \circ X_{H_{it}}, J_i^{\rho_t} df_t \cdot) \\ &= \sum_{i=1}^{3} \sigma(X_{H_{it}}, f_t^* J_i^{\rho_t} \cdot) \\ &= \sum_{i=1}^{3} dH_{it} \circ f_t^* J_i^{\rho_t} \\ &= \sum_{i=1}^{3} f_t^* \big(dK_i^{\rho_t} \circ J_i^{\rho_t}\big). \end{aligned}$$

Here the last equation follows from the fact that $H_{it} = K_i^{\rho_t} \circ f_t = f_t^* K_i^{\rho_t}$. Now insert the formula $\beta_t = \sum_i f_t^*(dK_i^{\rho_t} \circ J_i^{\rho_t})$ into equation (15) to obtain (14). This proves Proposition 2.3. □

Equation (14) is the **Donaldson Geometric Flow** in the hyperKähler setting. It can be interpreted as the gradient flow of the pushforward energy functional on the space $\mathscr{S}_a$ of all symplectic forms on M representing the cohomology class $a := [\omega_1]$ with respect to a suitable Riemannian metric. (See Definition 3.2 below.) The energy functional and the Riemannian metric on $\mathscr{S}_a$ are independent of the choice of the symplectic four-manifold (S, σ).

Proposition 2.4 (Pushforward Energy). *Let $f \in \mathscr{F}$ and define*

$$\rho := (f^{-1})^*\sigma \in \Omega^2(M).$$

Then

$$\mathscr{E}(\rho) := \mathscr{E}(f) = \int_M \frac{2|\rho^+|^2}{|\rho^+|^2 - |\rho^-|^2} \,\mathrm{dvol}, \tag{16}$$

where $\rho^\pm := \frac{1}{2}(\rho \pm *\rho)$ are the self-dual and anti-self-dual parts of ρ and $\mathrm{dvol} := \frac{1}{2}\omega_i \wedge \omega_i$ *is the volume form of the hyperKähler metric.*

Proof. Define

$$u := \frac{\mathrm{dvol}_\rho}{\mathrm{dvol}}, \qquad \mathrm{dvol}_\rho := \frac{\rho \wedge \rho}{2}, \tag{17}$$

and

$$K_i^\rho := \frac{\omega_i \wedge \rho}{\mathrm{dvol}_\rho}, \qquad H_i := f^* K_i^\rho = \frac{f^*\omega_i \wedge \sigma}{\mathrm{dvol}_\sigma}$$

as in (14) and (11). Then

$$\rho^+ = \frac{u}{2}\sum_i K_i^\rho \omega_i, \qquad 2|\rho^+|^2 = u^2 \sum_i |K_i^\rho|^2, \qquad |\rho^+|^2 - |\rho^-|^2 = 2u. \tag{18}$$

Divide and integrate to obtain

$$\begin{aligned} \int_M \frac{2|\rho^+|^2}{|\rho^+|^2 - |\rho^-|^2}\mathrm{dvol} &= \int_M \frac{u}{2}\sum_i |K_i^\rho|^2 \,\mathrm{dvol} = \int_M \frac{1}{2}\sum_i |K_i^\rho|^2 \,\mathrm{dvol}_\rho \\ &= \frac{1}{2}\sum_{i=1}^3 \int_S |H_i|^2 \,\mathrm{dvol}_\sigma = \mathscr{E}(f). \end{aligned}$$

This proves Proposition 2.4. □

The energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$ in (16) is well defined for symplectic forms on any closed oriented Riemannian four-manifold M. Moreover, the space $\mathscr{S}_a$ carries a natural Riemannian metric which in the hyperKähler case agrees with the pushforward of the L^2 metric on $\mathscr{F}$. Thus the Donaldson geometric flow extends to the general setting as explained in the next section.

3 General Symplectic Four-Manifolds

Let M be a closed oriented Riemannian four-manifold. Denote by g the Riemannian metric on M, denote by $\mathrm{dvol} \in \Omega^4(M)$ the volume form of g, and let $* : \Omega^k(M) \to \Omega^{4-k}(M)$ be the Hodge $*$-operator associated to the metric and orientation. Fix a cohomology class $a \in H^2(M;\mathbb{R})$ such that $a^2 > 0$ and consider the space

$$\mathscr{S}_a := \left\{\rho \in \Omega^2(M)\,\middle|\, d\rho = 0,\, \rho\wedge\rho > 0,\, [\rho] = a\right\}$$

of symplectic forms on M representing the class a. This is an infinite-dimensional manifold and the tangent space of $\mathscr{S}_a$ at any element $\rho \in \mathscr{S}_a$ is the space of exact 2-forms on M. The next proposition is of preparatory nature. It summarizes the properties of a family of Riemannian metrics g^ρ on M, one for each nondegenerate 2-form ρ (and for each fixed background metric g). These Riemannian metrics will play a central role in our study of the Donaldson geometric flow.

Proposition 3.1 (Symplectic Forms and Riemannian Metrics). *Fix a nondegenerate 2-form $\rho \in \Omega^2(M)$ such that $\rho\wedge\rho > 0$ and define the function $u : M \to (0,\infty)$ by (17). Then there exists a unique Riemannian metric g^ρ on M that satisfies the following conditions.*

(i) *The volume form of g^ρ agrees with the volume form of g.*

(ii) *The Hodge $*$-operator $*^\rho : \Omega^1(M) \to \Omega^3(M)$ associated to g^ρ is given by*

$$*^\rho\lambda = \frac{\rho\wedge *(\rho\wedge\lambda)}{u} \tag{19}$$

for $\lambda \in \Omega^1(M)$ and by $*^\rho\iota(X)\rho = -\rho\wedge g(X,\cdot)$ for $X \in \mathrm{Vect}(M)$.

(iii) *The Hodge $*$-operator $*^\rho : \Omega^2(M) \to \Omega^2(M)$ associated to g^ρ is given by*

$$*^\rho\omega = R^\rho * R^\rho\omega, \qquad R^\rho\omega := \omega - \frac{\omega\wedge\rho}{\mathrm{dvol}_\rho}\rho, \tag{20}$$

for $\omega \in \Omega^2(M)$. The linear map $R^\rho : \Omega^2(M) \to \Omega^2(M)$ is an involution that preserves the exterior product, acts as the identity on the orthogonal complement of ρ with respect to the exterior product, and sends ρ to $-\rho$.

(iv) *Let $\omega \in \Omega^2(M)$ be a nondegenerate 2-form and let $J : TM \to TM$ be an almost complex structure such that $g = \omega(\cdot, J\cdot)$. Define the almost complex structure J^ρ by $\rho(J^\rho\cdot,\cdot) := \rho(\cdot, J\cdot)$ and define the 2-form $\omega^\rho \in \Omega^2(M)$ by $\omega^\rho := R^\rho\omega$. Then $g^\rho = \omega^\rho(\cdot, J^\rho\cdot)$ and so ω^ρ is self-dual with respect to g^ρ.*

Proof. See Theorem A.2. □

Definition 3.2. *Each nondegenerate 2-form $\rho \in \Omega^2(M)$ with $\rho^2 > 0$ determines an inner product $\langle\cdot,\cdot\rangle_\rho$ on the space of exact 2-forms defined by*

$$\langle\widehat{\rho}_1, \widehat{\rho}_2\rangle_\rho := \int_M \lambda_1 \wedge *^\rho \lambda_2, \qquad d\lambda_i = \widehat{\rho}_i, \qquad *^\rho\lambda_i \in \operatorname{im} d. \tag{21}$$

These inner products determine a Riemannian metric on the infinite-dimensional manifold $\mathscr{S}_a$ called the **Donaldson metric**.

The Donaldson geometric flow on a general symplectic four-manifold is the negative gradient flow of the energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$ in (16) with respect to the Donaldson metric in Definition 3.2. A central geometric ingredient in this flow is the following map $\Theta : \Omega^2_{\mathrm{ndg}}(M) \to \Omega^2(M)$. Its domain is the space $\Omega^2_{\mathrm{ndg}}(M) := \{\rho \in \Omega^2(M) \,|\, \rho\wedge\rho > 0\}$ of nondegenerate 2-forms compatible with the orientation and the map is given by

$$\Theta(\rho) := \Theta^\rho := *\frac{\rho}{u} - \frac{1}{2}\left|\frac{\rho}{u}\right|^2 \rho, \qquad u := \frac{\mathrm{dvol}_\rho}{\mathrm{dvol}}. \tag{22}$$

Proposition 3.3 (The Map Θ). *Let $\rho \in \Omega^2_{\mathrm{ndg}}(M)$ and define $u \in \Omega^0(M)$ and $\Theta^\rho \in \Omega^2(M)$ by (22). Then the following holds.*

(i) *Θ^ρ is the pointwise orthogonal projection of the 2-form $u^{-1} * \rho$ onto the orthogonal complement of ρ with respect to the wedge product. In particular*

$$\Theta^\rho \wedge \rho = 0. \tag{23}$$

(ii) *The 2-form Θ^ρ can be written as*

$$\Theta^\rho = \frac{2\rho^+}{u} - \left|\frac{\rho^+}{u}\right|^2 \rho = -\frac{|\rho^-|^2\rho^+ + |\rho^+|^2\rho^-}{u^2}. \tag{24}$$

(iii) *The square of Θ^ρ is given by*

$$\Theta^\rho \wedge \Theta^\rho = -\frac{2|\rho^+|^2|\rho^-|^2}{u^3}\mathrm{dvol}. \tag{25}$$

Thus $\Theta^\rho \wedge \Theta^\rho \le 0$ with equality if and only if ρ is self-dual.

(iv) *Let $\rho_t : \mathbb{R} \to \Omega^2_{\mathrm{ndg}}(M)$ be a smooth path with $\rho_0 = \rho$ and $\partial_t\rho_t|_{t=0} = \widehat{\rho}$. Then*

$$\widehat{\Theta} := \left.\frac{d}{dt}\right|_{t=0} \Theta^{\rho_t} = \frac{\widehat{\rho} + *^\rho\widehat{\rho}}{u} - \left|\frac{\rho^+}{u}\right|^2 \widehat{\rho}. \tag{26}$$

(v) *Assume the hyperKähler case. Then*

$$\Theta^\rho = \sum_{i=1}^3 \left(K_i^\rho \omega_i - \frac{1}{2}(K_i^\rho)^2 \rho \right), \qquad K_i^\rho := \frac{\omega_i \wedge \rho}{\mathrm{dvol}_\rho}, \tag{27}$$

and

$$d\Theta^\rho = *^\rho \sum_{i=1}^3 dK_i^\rho \circ J_i^\rho, \qquad \rho(J_i^\rho \cdot, \cdot) := \rho(\cdot, J_i \cdot). \tag{28}$$

Proof. It follows directly from the definition of Θ^ρ in (22) that $\Theta^\rho \wedge \rho = 0$ and this proves part (i).

To prove part (ii), denote by $\Pi : \Omega^2(M) \to \Omega^2(M)$ the pointwise orthogonal projection onto the orthogonal complement of ρ with respect to the wedge product. Thus

$$\Pi(\tau) = \tau - \frac{\tau \wedge \rho}{\rho \wedge \rho}\rho \qquad \text{for } \tau \in \Omega^2(M).$$

Since $*\rho = \rho^+ - \rho^- = 2\rho^+ - \rho$ it follows from part (i) that

$$\Theta^\rho = \Pi\left(*\frac{\rho}{u} \right) = \Pi\left(\frac{2\rho^+}{u} \right).$$

This proves the first equation in (24). The second equation in (24) follows by direct calculation, using the identity $2u = |\rho^+|^2 - |\rho^-|^2$. This proves part (ii).

To prove part (iii), use the last term in equation (24) to obtain

$$\Theta^\rho \wedge \Theta^\rho = \frac{|\rho^-|^4 \rho^+ \wedge \rho^+ + |\rho^+|^4 \rho^- \wedge \rho^-}{u^4} = -\frac{2|\rho^+|^2 |\rho^-|^2}{u^3}\mathrm{dvol}.$$

This proves equation (25) and part (iii).

To prove part (iv) choose a smooth path $\mathbb{R} \to \mathscr{S}_a : t \mapsto \rho_t$ such that $\rho_0 = \rho$ and $\partial_t \rho_t|_{t=0} = \widehat{\rho}$. Define

$$u_t := \frac{\rho_t \wedge \rho_t}{2\mathrm{dvol}}, \qquad \widehat{u} := \left.\frac{\partial}{\partial t}\right|_{t=0} u_t = \frac{\rho \wedge \widehat{\rho}}{\mathrm{dvol}}, \qquad \widehat{\Theta} := \left.\frac{\partial}{\partial t}\right|_{t=0} \Theta^{\rho_t}.$$

Then, by part (iii) of Proposition 3.1, we have

$$R^\rho \widehat{\rho} = \widehat{\rho} - \frac{\widehat{\rho} \wedge \rho}{\mathrm{dvol}_\rho}\rho = \widehat{\rho} - \frac{\widehat{u}}{u}\rho.$$

Hence

$$\begin{aligned}
*^{\rho}\widehat{\rho} &= R^{\rho} * R^{\rho}\widehat{\rho} \\
&= *R^{\rho}\widehat{\rho} - \frac{\rho \wedge *R^{\rho}\widehat{\rho}}{\mathrm{dvol}_{\rho}}\rho \\
&= *\widehat{\rho} - \frac{\widehat{u}}{u}*\rho - \frac{\rho \wedge *\widehat{\rho}}{\mathrm{dvol}_{\rho}}\rho + \frac{|\rho|^2\widehat{u}}{u^2}\rho.
\end{aligned} \tag{29}$$

This implies

$$\begin{aligned}
\widehat{\Theta} &= \frac{\partial}{\partial t}\bigg|_{t=0}\left(\frac{*\rho_t}{u_t} - \frac{1}{2}\left|\frac{\rho_t}{u_t}\right|^2\rho_t\right) \\
&= \frac{*\widehat{\rho}}{u} - \frac{\widehat{u}}{u}\frac{*\rho}{u} - \frac{\rho \wedge *\widehat{\rho}}{u^2\,\mathrm{dvol}}\rho + \frac{|\rho|^2\,\widehat{u}}{u^3}\rho - \frac{1}{2}\left|\frac{\rho}{u}\right|^2\widehat{\rho} \\
&= \frac{*^{\rho}\widehat{\rho}}{u} - \frac{1}{2}\left|\frac{\rho}{u}\right|^2\widehat{\rho} \\
&= \frac{\widehat{\rho} + *^{\rho}\widehat{\rho}}{u} - \left|\frac{\rho^+}{u}\right|^2\widehat{\rho}.
\end{aligned}$$

Here the third step follows from (29) and the last step uses the identities $|\rho|^2 = |\rho^+|^2 + |\rho^-|^2$ and $2u = |\rho^+|^2 - |\rho^-|^2$ in (18). This proves part (iv).

Equation (27) follows from (24) and the identities

$$\frac{\rho^+}{u} = \frac{1}{2}\sum_i K_i^{\rho}\omega_i, \qquad \left|\frac{\rho^+}{u}\right|^2 = \frac{1}{2}\sum_i (K_i^{\rho})^2.$$

To prove (28), define

$$\omega_i^{\rho} := \omega_i - K_i^{\rho}\rho, \qquad i = 1, 2, 3.$$

Then $\omega_i^{\rho}(\cdot, J_i^{\rho}\cdot) = g^{\rho}$ by part (iv) of Proposition 3.1 and hence

$$*^{\rho}(\lambda \circ J_i^{\rho}) = \lambda \wedge \omega_i^{\rho}$$

for all $\lambda \in \Omega^1(M)$ and all i. Take $\lambda = dK_i^{\rho}$ to obtain

$$*^{\rho}(dK_i^{\rho} \circ J_i^{\rho}) = dK_i^{\rho} \wedge \omega_i^{\rho} = d\left(K_i^{\rho}\omega_i - \frac{1}{2}(K_i^{\rho})^2\rho\right).$$

Take the sum over all i and use equation (27) to obtain (28). This proves part (v) and Proposition 3.3. $\square$

Proposition 3.4 (**The Gradient of the Energy).**
(i) *The differential of the energy functional* $\mathscr{E}:\mathscr{S}_a\to\mathbb{R}$ *at* $\rho\in\mathscr{S}_a$ *and its gradient with respect to the Donaldson metric are given by*

$$\begin{aligned}\delta\mathscr{E}(\rho)\widehat{\rho} &= \int_M \Theta^\rho\wedge\widehat{\rho},\\ \mathrm{grad}\mathscr{E}(\rho) &= -d*^\rho d\Theta^\rho.\end{aligned}\tag{30}$$

(ii) *Assume the hyperKähler case. Then*

$$\begin{aligned}\mathscr{E}(\rho) &= \frac{1}{2}\int_M\sum_{i=1}^3|K_i^\rho|^2, \qquad K_i^\rho := \frac{\omega_i\wedge\rho}{\mathrm{dvol}_\rho},\\ \delta\mathscr{E}(\rho)\widehat{\rho} &= \int_M\sum_{i=1}^3\left(K_i^\rho\omega_i-\frac{1}{2}(K_i^\rho)^2\rho\right)\wedge\widehat{\rho},\\ \mathrm{grad}\mathscr{E}(\rho) &= \sum_{i=1}^3 d\left(dK_i^\rho\circ J_i^\rho\right), \qquad \rho(J_i^\rho\cdot,\cdot):=\rho(\cdot,J_i\cdot).\end{aligned}\tag{31}$$

Proof. Let $\rho\in\mathscr{S}_a$ and define $u:=\frac{\mathrm{dvol}_\rho}{\mathrm{dvol}}$. Then, by equation (18),

$$\mathscr{E}(\rho)-\mathrm{Vol}(M)=\int_M\frac{|\rho^+|^2+|\rho^-|^2}{|\rho^+|^2-|\rho^-|^2}\mathrm{dvol}=\int_M\frac{\rho\wedge*\rho}{2u}.$$

Choose a path $\mathbb{R}\to\mathscr{S}_a:t\mapsto\rho_t$ and define

$$u_t:=\frac{\rho_t\wedge\rho_t}{2\mathrm{dvol}}, \qquad \widehat{u}_t:=\partial_t u_t=\frac{\rho_t\wedge\widehat{\rho}_t}{\mathrm{dvol}}, \qquad \widehat{\rho}_t:=\partial_t\rho_t.$$

Then

$$\begin{aligned}\frac{d}{dt}\mathscr{E}(\rho_t) &= \frac{d}{dt}\int_M\frac{\rho_t\wedge*\rho_t}{2u_t}=\int_M\frac{\rho_t\wedge*\widehat{\rho}_t}{u_t}-\int_M\frac{1}{2}\left|\frac{\rho_t}{u_t}\right|^2\widehat{u}_t\mathrm{dvol}\\ &= \int_M\frac{*\rho_t\wedge\widehat{\rho}_t}{u_t}-\int_M\frac{1}{2}\left|\frac{\rho_t}{u_t}\right|^2\rho_t\wedge\widehat{\rho}_t=\int_M\Theta^{\rho_t}\wedge\widehat{\rho}_t.\end{aligned}$$

This proves the formula for $\delta\mathscr{E}(\rho)$. Now let $\widehat{\rho}\in T_\rho\mathscr{S}_a$ and choose $\lambda\in\Omega^1(M)$ such that $d\lambda=\widehat{\rho}$ and $*^\rho\lambda\in\mathrm{im}\,d$. Then

$$\delta\mathscr{E}(\rho)\widehat{\rho}=\int_M\Theta^\rho\wedge d\lambda=-\int_M d\Theta^\rho\wedge\lambda=\left\langle -d*^\rho d\Theta^\rho,\widehat{\rho}\right\rangle_\rho.$$

This proves part (i). In part (ii) the first equation in (31) follows from Proposition 2.4, the second equation follows from (30) and (27) and the third equation follows from (30) and (28). This proves Proposition 3.4. □

By part (i) of Proposition 3.4 a smooth path $\mathbb{R} \to \mathscr{S}_a : t \mapsto \rho_t$ is a negative gradient flow line of $\mathscr{E}$ with respect to the Donaldson metric if and only if it satisfies the partial differential equation

$$\partial_t \rho_t = d *^{\rho_t} d\Theta_t, \qquad \Theta_t := \frac{2\rho_t^+}{u_t} - \left|\frac{\rho_t^+}{u_t}\right|^2 \rho_t, \qquad u_t := \frac{\mathrm{dvol}_{\rho_t}}{\mathrm{dvol}}. \tag{32}$$

Equation (32) is the **Donaldson Geometric Flow**. By part (ii) of Proposition 3.4 it agrees with the geometric flow (14) in the hyperKähler case.

Corollary 3.5. **(i)** *A symplectic form $\rho \in \mathscr{S}_a$ is a critical point of the energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$ in* (16) *if and only if the* 2*-form Θ^ρ is closed.*

(ii) *Suppose $\omega \in \mathscr{S}_a$ is compatible with the background metric g. Then ω is the unique absolute minimum of the energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$.*

(iii) *Assume the hyperKähler case. Then $\rho \in \mathscr{S}_a$ is a critical point of the energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$ if and only if $\sum_{i=1}^3 dK_i^\rho \circ J_i^\rho = 0$.*

Proof. Part (i) follows from equation (30) in Proposition 3.4. To prove (ii) observe that a symplectic form $\omega \in \mathscr{S}_a$ is compatible with the metric g if and only if it is self-dual. Moreover, every self-dual symplectic form is harmonic and the class a has a unique harmonic representative. Since

$$\frac{1}{2}\mathscr{E}(\rho) = \int_M \frac{|\rho^+|^2}{|\rho^+|^2 - |\rho^-|^2}\,\mathrm{dvol} \geq \int_M \mathrm{dvol} =: \mathrm{Vol}(M)$$

for all $\rho \in \mathscr{S}_a$, with equality if and only if $\rho^- = 0$, this proves (ii). Part (iii) follows from (i) and equation (28) in Proposition 3.3. □

The next proposition is an observation of Donaldson [1] which asserts that the energy controls the L^1 norm of ρ.

Proposition 3.6 **(Donaldson's L^1 Estimate).** *Every $\rho \in \mathscr{S}_a$ satisfies*

$$\|\rho\|_{L^1} \leq \sqrt{c(\mathscr{E}(\rho) - \mathrm{Vol}(M))}, \qquad c := \int_M \rho \wedge \rho = \langle a^2, [M]\rangle. \tag{33}$$

Proof. By the Cauchy–Schwarz inequality,

$$\begin{aligned}\left(\int_M |\rho|\,\mathrm{dvol}\right)^2 &\leq \left(\int_M \left(|\rho^+|^2 - |\rho^-|^2\right)\mathrm{dvol}\right)\int_M \frac{|\rho|^2}{|\rho^+|^2 - |\rho^-|^2}\,\mathrm{dvol} \\ &= \left(\int_M \rho \wedge \rho\right)\int_M \frac{|\rho^+|^2 + |\rho^-|^2}{|\rho^+|^2 - |\rho^-|^2}\,\mathrm{dvol} \\ &= c(\mathscr{E}(\rho) - \mathrm{Vol}(M)).\end{aligned}$$

This proves Proposition 3.6. □

Remark 3.7. **(i)** Donaldson's conjectural program involves a proof of long-time existence for all initial conditions, a proof that solutions cannot escape to infinity, and a proof that higher critical points can be *bypassed*, i.e. that they cannot be local minima. In those cases where this program can be carried out it would then follow that the space $\mathscr{S}_a$ is connected, which is an open question for all closed four-manifolds M and all cohomology classes a that can be represented by symplectic forms (see [5]). Short time existence and regularity, as well as long time existence for initial conditions sufficiently close to the absolute minimum, are established in [3].

(ii) In many situations (including certain Kähler classes on the K3-surface) a theorem of Seidel [6, 7, 8] asserts the existence of symplectomorphisms of (M,ω) that are smoothly, but not symplectically, isotopic to the identity. This implies the existence of noncontractible loops in $\mathscr{S}_a$. Hence, if the analytic difficulties in Donaldson's geometric flow approach can be settled, it would follow that in these cases the energy functional $\mathscr{E}:\mathscr{S}_a\to\mathbb{R}$ must have critical points of index one, assuming that they are nondegenerate. Many other examples of nontrivial cohomology classes in $\mathscr{S}_a$ of all degrees were found by Kronheimer [4] using Seiberg–Witten theory.

(iii) By an observation of Donaldson [1] higher critical points of $\mathscr{E}$ (not equal to the absolute minimum) cannot be strictly stable in the hyperKähler case. We include a proof of this result in Section 4 (Theorem 4.5).

Proposition 3.8. *Let $M=\mathbb{C}\mathrm{P}^2$ be the complex projective plane with its standard Kähler metric, let ω_{FS} be the Fubini–Study form, and define*

$$a := [\omega_{\mathrm{FS}}] \in H^2(M;\mathbb{R}).$$

Then ω_{FS} is the only critical point, and the absolute minimum, of the energy functional $\mathscr{E}:\mathscr{S}_a\to\mathbb{R}$ in (16).

Proof. That ω_{FS} is the unique absolute minimum of $\mathscr{E}$ follows from part (ii) of Corollary 3.5. Now let $\rho\in\mathscr{S}_a$ be any critical point of $\mathscr{E}$. Then Θ^ρ is closed by part (i) of Proposition 3.4 and $\Theta^\rho\wedge\rho=0$ by part (i) of Proposition 3.3. Since $H^2(M;\mathbb{R})$ is one-dimensional, this implies that Θ^ρ is exact. Hence it follows from part (iii) of Proposition 3.3 that

$$0=\int_{\mathbb{C}\mathrm{P}^2}\Theta^\rho\wedge\Theta^\rho=-\int_{\mathbb{C}\mathrm{P}^2}\frac{2|\rho^+|^2|\rho^-|^2}{u^3}\,\mathrm{dvol}_{\mathrm{FS}},\qquad u:=\frac{\mathrm{dvol}_\rho}{\mathrm{dvol}_{\mathrm{FS}}}.$$

This shows that $\rho^-=0$. Thus ρ is self-dual and hence is harmonic. Since $[\rho]=a=[\omega_{\mathrm{FS}}]$ and ω_{FS} is also a harmonic 2-form it follows that $\rho=\omega_{\mathrm{FS}}$. □

4 The Hessian

The infinite-dimensional *manifold* $\mathscr{S}_a$ is an open set in an affine space. Hence the Hessian of $\mathscr{E}$ is well defined for every $\rho \in \mathscr{S}_a$ as the second derivative $\mathscr{H}_\rho(\widehat{\rho}) := \frac{d^2}{dt^2}|_{t=0}\mathscr{E}(\rho_t)$ along a curve $\mathbb{R} \to \mathscr{S}_a : t \to \rho_t$ satisfying $\rho_0 = \rho$, $\frac{d}{dt}|_{t=0}\rho_t = \widehat{\rho}$, and $\frac{d^2}{dt^2}|_{t=0}\rho_t = 0$.

Theorem 4.1. *Let $\rho \in \mathscr{S}_a$. Then the following holds.*

(i) *The Hessian of $\mathscr{E}$ at ρ is the quadratic form $\mathscr{H}_\rho : T_\rho\mathscr{S}_a \to \mathbb{R}$ given by*

$$\mathscr{H}_\rho(\widehat{\rho}) = \int_M \widehat{\Theta} \wedge \widehat{\rho}, \qquad \widehat{\Theta} := \frac{\widehat{\rho} + *^\rho\widehat{\rho}}{u} - \left|\frac{\rho^+}{u}\right|^2 \widehat{\rho}. \tag{34}$$

As a linear operator the Hessian is the map $T_\rho\mathscr{S}_a \to T_\rho\mathscr{S}_a : \widehat{\rho} \mapsto -d *^\rho d\widehat{\Theta}$.

(ii) *Assume the hyperKähler case and define*

$$K_i := K_i^\rho = \frac{\omega_i \wedge \rho}{\mathrm{dvol}_\rho}, \qquad \omega_i^\rho := \omega_i - K_i\rho. \tag{35}$$

Let $\widehat{\rho} \in T_\rho\mathscr{S}_a$, choose $X \in \mathrm{Vect}(M)$ such that $-d\iota(X)\rho = \widehat{\rho}$, and define

$$\widehat{K}_i := \frac{(\omega_i - K_i\rho) \wedge \widehat{\rho}}{\mathrm{dvol}_\rho}, \qquad \widehat{H}_i := \frac{(d\iota(X)\omega_i) \wedge \rho}{\mathrm{dvol}_\rho}. \tag{36}$$

Then

$$\widehat{\Theta} = \sum_{i=1}^3 \widehat{K}_i\omega_i^\rho - \frac{1}{2}\sum_{i=1}^3 K_i^2\widehat{\rho}, \tag{37}$$

$$\mathscr{H}_\rho(\widehat{\rho}) = \int_M \sum_i \left(\widehat{K}_i^2\mathrm{dvol}_\rho - \frac{1}{2}K_i^2\widehat{\rho} \wedge \widehat{\rho}\right). \tag{38}$$

Moreover, if ρ is a critical point of $\mathscr{E}$, then

$$\begin{aligned} \mathscr{H}_\rho(\widehat{\rho}) &= \int_M \sum_{i=1}^3 \omega_i(X, X_{\widehat{H}_i} + \nabla_{X_{K_i}}X)\,\mathrm{dvol}_\rho \\ &= \int_M \sum_{i=1}^3 \left(\widehat{H}_i^2\mathrm{dvol}_\rho + \omega_i(X, \nabla_{X_{K_i}}X)\right)\,\mathrm{dvol}_\rho. \end{aligned} \tag{39}$$

Here ∇ denotes the Levi-Civita connection of the hyperKähler metric and X_F denotes the Hamiltonian vector field of a function $F : M \to \mathbb{R}$ with respect to ρ, i.e. $\iota(X_F)\rho = dF$.

Proof. See page 21 □

In [3] it is shown that, for every $\rho \in \mathscr{S}_a$, the quadratic form in (39) is the covariant Hessian of $\mathscr{E}$ with respect to the Donaldson metric in Definition 3.2. The proof of Theorem 4.1 relies on the following two lemmas.

Lemma 4.2. *Let ρ and ω be symplectic forms on M and define*

$$K := \frac{\omega \wedge \rho}{\mathrm{dvol}_\rho}, \qquad \omega^\rho := \omega - K\rho.$$

Then, for every vector field $X \in \mathrm{Vect}(M)$,

$$(\iota(X)\omega) \wedge \rho + \omega^\rho \wedge \iota(X)\rho = 0, \tag{40}$$

$$\frac{(d\iota(X)\omega) \wedge \rho}{\mathrm{dvol}_\rho} + \frac{\omega^\rho \wedge d\iota(X)\rho}{\mathrm{dvol}_\rho} = \mathcal{L}_X K. \tag{41}$$

If ω is self-dual and J is the almost complex structure such that $\omega(\cdot, J\cdot)$ is the background Riemannian metric on M then

$$(\iota(X)\omega) \wedge \rho = - *^\rho \iota(JX)\rho. \tag{42}$$

Proof. Equation (40) follows by direct computation, i.e.

$$\begin{aligned}
\big(\iota(X)\omega\big) \wedge \rho &= \iota(X)\big(\omega \wedge \rho\big) - \omega \wedge \iota(X)\rho \\
&= K\iota(X)\mathrm{dvol}_\rho - \omega \wedge \iota(X)\rho \\
&= -\big(\omega - K\rho\big) \wedge \iota(X)\rho.
\end{aligned}$$

Now differentiate equation (40) and use the identity $d\omega^\rho = -dK \wedge \rho$ to obtain

$$\begin{aligned}
0 &= d\Big(\big(\iota(X)\omega\big) \wedge \rho + \omega^\rho \wedge \iota(X)\rho\Big) \\
&= \big(d\iota(X)\omega\big) \wedge \rho + \omega^\rho \wedge d\iota(X)\rho - dK \wedge \rho \wedge \iota(X)\rho \\
&= \big(d\iota(X)\omega\big) \wedge \rho + \omega^\rho \wedge d\iota(X)\rho - dK \wedge \iota(X)\mathrm{dvol}_\rho \\
&= \big(d\iota(X)\omega\big) \wedge \rho + \omega^\rho \wedge d\iota(X)\rho - (\iota(X)dK)\mathrm{dvol}_\rho.
\end{aligned}$$

This proves (41). Now suppose ω is compatible with the almost complex structure J and $\omega(\cdot, J\cdot)$ is the background Riemannian metric. Define the almost complex structure J^ρ by $\rho(J^\rho\cdot, \cdot) := \rho(\cdot, J\cdot)$. Then $g^\rho = \omega^\rho(\cdot, J^\rho\cdot)$ by part (iv) of Theorem A.2. Hence it follows from (40) and Lemma A.4 that

$$(\iota(X)\omega) \wedge \rho = -\omega^\rho \wedge \iota(X)\rho = - *^\rho (\iota(X)\rho \circ J^\rho) = - *^\rho \iota(JX)\rho.$$

This proves equation (42) and Lemma 4.2 □

The identities in Lemma 4.2 are needed to establish the next result which relates equations (38) and (39) and is a key step in the proof of Theorem 4.1. It is shown in [3] that the last two integrals in equation (43) below arise from the Levi-Civita connection of the Donaldson metric in Definition 3.2 on the infinite-dimensional manifold $\mathscr{S}_a$. They vanish for critical points of $\mathscr{E}$. Both sides of equation (43) agree with the covariant Hessian of $\mathscr{E}$ at an arbitrary element $\rho \in \mathscr{S}_a$ (see [3]).

Lemma 4.3. *Let $\rho \in \mathscr{S}_a$, let $\widehat{\rho} \in T_\rho\mathscr{S}_a$ be an exact 2-form, let $K_i, \widehat{K}_i, \widehat{H}_i$ be as in equations (35) and (36) in Theorem 4.1, and let $X \in \mathrm{Vect}(M)$ be any vector field such that $-d\iota(X)\rho = \widehat{\rho}$. Then*

$$\begin{aligned}\int_M \sum_{i=1}^3 \Big(\widehat{H}_i^2 + \omega_i\big(X, \nabla_{X_{K_i}} X\big)\Big)\,\mathrm{dvol}_\rho &= \int_M \sum_{i=1}^3 \Big(\widehat{K}_i^2\mathrm{dvol}_\rho - \frac{1}{2}K_i^2\widehat{\rho}^2\Big) \\ + \int_M \sum_{i=1}^3 (\iota(X_{K_i})\omega_i) \wedge (\iota(X)\rho) \wedge \widehat{\rho} &+ \int_M \sum_{i=1}^3 \omega_i\big(X, \nabla_X X_{K_i}\big)\mathrm{dvol}_\rho.\end{aligned} \tag{43}$$

Here ∇ is the Levi-Civita connection of the Kähler metric and X_{K_i} is the Hamiltonian vector field of K_i associated to ρ so that $\iota(X_{K_i})\rho = dK_i$.

Proof. Equation (43) can be written in the form

$$\begin{aligned}\int_M \sum_{i=1}^3 \widehat{H}_i^2\mathrm{dvol}_\rho = &\int_M \sum_{i=1}^3 \Big(\widehat{K}_i^2\mathrm{dvol}_\rho - \frac{1}{2}K_i^2\widehat{\rho}\wedge\widehat{\rho}\Big) \\ &+ \int_M \sum_{i=1}^3 (\iota(X_{K_i})\omega_i) \wedge (\iota(X)\rho) \wedge \widehat{\rho} \\ &+ \int_M \sum_{i=1}^3 \omega_i\big(X, [X_{K_i}, X]\big)\mathrm{dvol}_\rho.\end{aligned} \tag{44}$$

To prove this formula we first observe that

$$\mathcal{L}_X K_i = \widehat{H}_i - \widehat{K}_i \tag{45}$$

for $i = 1, 2, 3$ by equation (41) in Lemma 4.2. This implies

$$\begin{aligned}\int_M \sum_{i=1}^3 \widehat{H}_i^2\mathrm{dvol}_\rho = &\int_M \sum_{i=1}^3 \widehat{K}_i^2\mathrm{dvol}_\rho - \int_M \sum_{i=1}^3 (\mathcal{L}_X K_i)^2\mathrm{dvol}_\rho \\ &+ 2\int_M \sum_{i=1}^3 \widehat{H}_i(\mathcal{L}_X K_i)\mathrm{dvol}_\rho.\end{aligned} \tag{46}$$

Define

$$\begin{aligned} A &:= -\frac{1}{2}\int_M \sum_{i=1}^3 K_i^2 \widehat{\rho}\wedge\widehat{\rho}, \\ B &:= \int_M \sum_{i=1}^3 (\iota(X_{K_i})\omega_i)\wedge(\iota(X)\rho)\wedge\widehat{\rho}, \\ C &:= \int_M \sum_{i=1}^3 \omega_i\big(X,[X_{K_i},X]\big)\mathrm{dvol}_\rho, \\ D &:= \int_M \sum_{i=1}^3 (\mathcal{L}_X K_i)^2\,\mathrm{dvol}_\rho, \\ E &:= \int_M \sum_{i=1}^3 \widehat{H}_i(\mathcal{L}_X K_i)\,\mathrm{dvol}_\rho. \end{aligned} \tag{47}$$

Then equation (46) shows that (44) is equivalent to the identity

$$A + B + C + D = 2E.$$

To prove this, we first observe that

$$\begin{aligned} A &= \int_M \sum_{i=1}^3 \frac{1}{2}K_i^2(d\iota(X)\rho)\wedge\widehat{\rho} \\ &= -\int_M \sum_{i=1}^3 K_i dK_i \wedge(\iota(X)\rho)\wedge\widehat{\rho} \\ &= -\int_M \sum_{i=1}^3 K_i(\iota(X_{K_i})\rho)\wedge(\iota(X)\rho)\wedge\widehat{\rho} \\ &= -B + \int_M \sum_{i=1}^3 (\iota(X_{K_i})(\omega_i - K_i\rho))\wedge(\iota(X)\rho)\wedge\widehat{\rho}. \end{aligned}$$

Hence $A + B = F + G$, where

$$\begin{aligned} F &:= -\int_M \sum_{i=1}^3 (\omega_i - K_i\rho)\wedge(\iota(X_{K_i})\iota(X)\rho)\wedge\widehat{\rho}, \\ G &:= \int_M \sum_{i=1}^3 (\omega_i - K_i\rho)\wedge(\iota(X)\rho)\wedge(\iota(X_{K_i})\widehat{\rho}). \end{aligned} \tag{48}$$

Since $\iota(X_{K_i})\iota(X)\rho = -\mathcal{L}_X K_i$ and $(\omega_i - K_i\rho)\wedge\widehat{\rho} = \widehat{K}_i \mathrm{dvol}_\rho$, we have

$$\begin{aligned} F &= \int_M \sum_{i=1}^3 \widehat{K}_i(\mathcal{L}_X K_i)\mathrm{dvol}_\rho \\ &= \int_M \sum_{i=1}^3 \Big(\widehat{H}_i(\mathcal{L}_X K_i) - (\mathcal{L}_X K_i)^2\Big)\,\mathrm{dvol}_\rho \\ &= E - D. \end{aligned}$$

Here we have used equation (45). To sum up, we have proved that

$$A + B + D = D + F + G = E + G.$$

Thus it remains to prove that $C = E - G$. To see this, observe that

$$\iota(\mathcal{L}_{X_{K_i}}X)\mathrm{dvol}_\rho = \rho\wedge\iota(\mathcal{L}_{X_{K_i}}X)\rho = \rho\wedge\mathcal{L}_{X_{K_i}}(\iota(X)\rho)$$

and, by Cartan's formula,

$$\mathcal{L}_{X_{K_i}}\big(\iota(X)\rho\big) = d\iota(X_{K_i})\iota(X)\rho + \iota(X_{K_i})d\iota(X)\rho = -d(\mathcal{L}_X K_i) - \iota(X_{K_i})\widehat{\rho}.$$

Since $\omega_i\big(X,[X_{K_i},X]\big)\mathrm{dvol}_\rho = -(\iota(X)\omega_i)\wedge\iota(\mathcal{L}_{X_{K_i}}X)\mathrm{dvol}_\rho$, this implies

$$\omega_i\big(X,[X_{K_i},X]\big)\mathrm{dvol}_\rho = (\iota(X)\omega_i)\wedge\rho\wedge\Big(d(\mathcal{L}_X K_i) + \iota(X_{K_i})\widehat{\rho}\Big)$$

for $i = 1,2,3$. Integrate over M and take the sum over i to obtain

$$\begin{aligned} C &= \int_M \sum_{i=1}^3 \omega_i\big(X,[X_{K_i},X]\big)\mathrm{dvol}_\rho \\ &= \int_M \sum_{i=1}^3 (\iota(X)\omega_i)\wedge\rho\wedge d(\mathcal{L}_X K_i) + \int_M \sum_{i=1}^3 (\iota(X)\omega_i)\wedge\rho\wedge\iota(X_{K_i})\widehat{\rho} \\ &= \int_M \sum_{i=1}^3 \big(d(\iota(X)\omega_i)\wedge\rho\big)\wedge\mathcal{L}_X K_i + \int_M \sum_{i=1}^3 (\iota(X)\omega_i)\wedge\rho\wedge\iota(X_{K_i})\widehat{\rho} \\ &= \int_M \sum_{i=1}^3 \widehat{H}_i(\mathcal{L}_X K_i)\mathrm{dvol}_\rho - \int_M \sum_{i=1}^3 (\omega_i - K_i\rho)\wedge(\iota(X)\rho)\wedge\iota(X_{K_i})\widehat{\rho} \\ &= E - G. \end{aligned}$$

Here the penultimate equation follows from the definition of $\widehat{H}_i$ and from equation (40) in Lemma 4.2. Thus $A + B + D = E + G$ and $C = E - G$, as claimed, and this completes the proof of Lemma 4.3. □

Proof of Theorem 4.1. Consider the map $\mathscr{S}_a \to \Omega^2(M) : \rho \mapsto \Theta^\rho$ in (22). By part (iv) of Proposition 3.3 its derivative at ρ in the direction $\widehat{\rho}$ is given by the 2-form $\widehat{\Theta}$ in (26). Hence equation (34) for the Hessian follows from the formula (30) for the differential of the energy functional $\mathscr{E}$ in (16).

Next we prove equation (37) and (38) in the hyperKähler case. The 2-forms $\omega_i^\rho = \omega_i - K_i\rho = R^\rho\omega_i$ span the space of self-dual 2-forms with respect to g^ρ by part (iii) of Proposition 3.1. Hence it follows from (36) that

$$\frac{\widehat{\rho} + *^\rho\widehat{\rho}}{u} = \sum_{i=1}^3 \widehat{K}_i\omega_i^\rho, \qquad \left|\frac{\rho^+}{u}\right|^2 = \frac{1}{2}\sum_{i=1}^3 K_i^2.$$

This proves (37) and (38). Alternatively, these two equations can be derived from the fact that Θ^ρ is given by equation (27) in the hyperkähler case. Namely, choose a smooth path $\rho_t \in \mathcal{S}_a$ such that $\rho_0 = \rho$ and $\frac{d}{dt}|_{t=0}\rho_t = \widehat{\rho}$. Then $\frac{d}{dt}|_{t=0}K_i^{\rho_t} = \widehat{K}_i$. Differentiate (27) to obtain that $\widehat{\Theta} = \frac{d}{dt}|_{t=0}\Theta^{\rho_t}$ is given by (37) and inserte this formula into (34) to obtain (38).

Next observe that the two integrals in (39) agree because

$$\begin{aligned}\int_M \widehat{H}_i^2\,\mathrm{dvol}_\rho &= \int_M \widehat{H}_i d(\iota(X)\omega_i) \wedge \rho = \int_M \iota(X)\omega_i \wedge d\widehat{H}_i \wedge \rho \\ &= \int_M \iota(X)\omega_i \wedge \iota(X_{\widehat{H}_i})\mathrm{dvol}_\rho = \int_M \omega_i(X, X_{\widehat{H}_i})\mathrm{dvol}_\rho.\end{aligned}$$

Here the first equation follows from the definition of the function $\widehat{H}_i$ in (36) and the third equation follows from the fact that $X_{\widehat{H}_i}$ is its Hamiltonian vector field with respect to ρ.

Now suppose that $\rho \in \mathscr{S}_a$ is a critical point of the energy functional $\mathscr{E}$ in (16). Then

$$\sum_{i=1}^3 \rho(J_iX_{K_i}, \cdot) = \sum_{i=1}^3 \rho(X_{K_i}, J_i^\rho\cdot) = \sum_{i=1}^3 dK_i \circ J_i^\rho = 0$$

by part (iii) of Corollary 3.5. Hence $\sum_{i=1}^3 J_iX_{K_i} = 0$ and this implies

$$\sum_{i=1}^3 J_i\nabla_X X_{K_i} = 0, \qquad \sum_{i=1}^3 \iota(X_{K_i})\omega_i = 0. \tag{49}$$

Thus the last two integrals in equation (43) vanish and so the right hand side of equation (38) agrees with the right hand side of equation (39) by Lemma 4.3. This proves Theorem 4.1. □

Corollary 4.4. *If $\rho = \omega \in \mathscr{S}_a$ is self-dual then*

$$\mathscr{H}_\omega(\widehat{\rho}) = \int_M |\widehat{\rho}|^2 \,\mathrm{dvol}$$

for all $\widehat{\rho} \in T_\omega \mathscr{S}$.

Proof. This follows from (34) with $u = 1$, $|\rho^+|^2 = |\omega|^2 = 2$, and $g^\rho = g$. □

Theorem 4.5 (Donaldson). *Assume the hyperKähler case and $a := [\omega_1]$. If $\rho \in \mathscr{S}_a$ is a critical point of $\mathscr{E}$ and $\rho \neq \omega_1$ then the Hessian $\mathscr{H}_\rho$ is not positive definite.*

Proof. The proof has four steps.

Step 1. *Let $\rho \in \Omega^2_{\mathrm{ndg}}(M)$ and define J_i^ρ by $\rho(J_i^\rho \cdot, \cdot) := \rho(\cdot, J_i \cdot)$ for $i = 1, 2, 3$. Then the first order differential operator $D : \Omega^1(M) \to \Omega^0(M, \mathbb{R}^4)$ defined by*

$$D\lambda := \left(\frac{d *^\rho \lambda}{\mathrm{dvol}}, \frac{d *^\rho (\lambda \circ J_1^\rho)}{\mathrm{dvol}}, \frac{d *^\rho (\lambda \circ J_2^\rho)}{\mathrm{dvol}}, \frac{d *^\rho (\lambda \circ J_3^\rho)}{\mathrm{dvol}} \right) \tag{50}$$

for $\lambda \in \Omega^1(M)$ is a Fredholm operator of Fredholm index $b_1 - 4$.

By part (iv) of Proposition 3.1 the 2-forms

$$\omega_i^\rho := \omega_i - \frac{\omega_i \wedge \rho}{\mathrm{dvol}_\rho} \rho, \qquad i = 1, 2, 3,$$

form a basis of the space of self-dual 2-forms with respect to g^ρ and they satisfy $\omega_i^\rho(\cdot, J_i^\rho \cdot) = g^\rho$ for $i = 1, 2, 3$. Hence, for $\lambda \in \Omega^1(M)$, twice the self-dual part of $d\lambda$ with respect to g^ρ is the 2-form

$$d\lambda + *^\rho d\lambda = \sum_{i=1}^3 \frac{\omega_i^\rho \wedge d\lambda}{\mathrm{dvol}} \omega_i^\rho.$$

Hence the self-duality operator

$$\Omega^1(M) \to \Omega^0(M) \oplus \Omega^{2,+}_{g^\rho}(M) : \lambda \mapsto (d^{*^\rho} \lambda, d\lambda + *^\rho d\lambda)$$

of g^ρ is isomorphic to the operator $D' : \Omega^1(M) \to \Omega^0(M, \mathbb{R}^4)$ given by

$$D'\lambda := \left(\frac{d *^\rho \lambda}{\mathrm{dvol}}, \frac{\omega_1^\rho \wedge d\lambda}{\mathrm{dvol}}, \frac{\omega_2^\rho \wedge d\lambda}{\mathrm{dvol}}, \frac{\omega_3^\rho \wedge d\lambda}{\mathrm{dvol}} \right).$$

Hence D' is a Fredholm operator of index $b_1 - 4$. Since $\omega_i^\rho \wedge \lambda = *^\rho(\lambda \circ J_i^\rho)$ by Lemma A.4, we have

$$d *^\rho (\lambda \circ J_i^\rho) - \omega_i^\rho \wedge d\lambda = (d\omega_i^\rho) \wedge \lambda.$$

Hence $D - D'$ is a zeroth order operator and therefore is a compact operator between the appropriate Sobolev completions. Hence D is a Fredholm operator of index $b_1 - 4$. This proves Step 1.

Step 2. *Let $\rho \in \mathscr{S}_a \setminus \{\omega_1\}$. Then at least one of the functions*

$$K_i^\rho = \frac{\omega_i \wedge \rho}{\mathrm{dvol}_\rho}, \qquad i = 1, 2, 3,$$

is nonconstant.

Suppose by contradiction that K_i^ρ is constant for $i = 1, 2, 3$. Since $\rho - \omega_1$ is exact, we have

$$\int_M K_i^\rho \mathrm{dvol}_\rho = \int_M \omega_i \wedge \rho = \int_M \omega_i \wedge \omega_1 = \begin{cases} 2\mathrm{Vol}(M), & \text{if } i = 1, \\ 0, & \text{if } i = 2, 3, \end{cases}$$

and hence $K_1^\rho = 2$ and $K_2^\rho = K_3^\rho = 0$. This implies

$$u_1 = 2u, \qquad u_2 = u_3 = 0, \qquad u := \frac{\mathrm{dvol}_\rho}{\mathrm{dvol}}, \qquad u_i := \frac{\omega_i \wedge \rho}{\mathrm{dvol}}.$$

Hence it follows from (18) that

$$\begin{aligned} 0 &\le |\rho^-|^2 \\ &= |\rho^+|^2 - 2u \\ &= \frac{1}{2}\sum_{i=1}^{3} u_i^2 - 2u \\ &= 2u(u-1). \end{aligned} \tag{51}$$

Hence $u \ge 1$ and

$$\int_M u\mathrm{dvol} = \int_M \mathrm{dvol}_\rho = \int_M \mathrm{dvol} = \mathrm{Vol}(M).$$

This shows that $u \equiv 1$, hence $\rho^- = 0$ by (51), and therefore $\rho = \omega_1$. This proves Step 2.

Step 3. *Let $\rho \in \mathscr{S}_a \setminus \{\omega_1\}$ be a critical point of $\mathscr{E}$. Then there exists a 1-form $\lambda \in \Omega^1(M)$ such that*

$$d *^\rho \lambda = 0, \qquad d *^\rho (\lambda \circ J_j^\rho) = 0, \qquad j = 1, 2, 3 \tag{52}$$

and the exact 2-forms

$$\widehat{\rho}_0 := d\lambda, \qquad \widehat{\rho}_j := d(\lambda \circ J_j^\rho), \qquad j = 1, 2, 3. \tag{53}$$

are linearly independent.

By part (iii) of Corollary 3.5, we have

$$\sum_{i=1}^{3} dK_i^\rho \circ J_i^\rho = 0.$$

Hence the function

$$h := (0, K_1^\rho, K_2^\rho, K_3^\rho) : M \to \mathbb{R}^4$$

is L^2 orthogonal to the image of the operator

$$D : \Omega^1(M) \to \Omega^0(M, \mathbb{R}^4)$$

in Step 1, i.e.

$$\langle h, D\lambda \rangle_{L^2} = -\int_M \sum_{i=1}^{3} dK_i^\rho \wedge *^\rho(\lambda \circ J_i^\rho) = \int_M \sum_{i=1}^{3} (dK_i^\rho \circ J_i^\rho) \wedge *^\rho \lambda = 0$$

for all $\lambda \in \Omega^1(M)$. Since h is nonconstant by Step 2, this shows that the cokernel of D has dimension greater than four. Since D is a Fredholm operator of index $b_1 - 4$ by Step 1, its kernel has dimension greater than b_1. The kernel of D is a quaternionic vector space and each nonzero element $\lambda \in \ker D$ determines a four-dimensional quaternionic subspace

$$V_\lambda := \mathrm{span}\,\{\lambda, \lambda \circ J_1^\rho, \lambda \circ J_2^\rho, \lambda \circ J_3^\rho\} \subset \ker D.$$

Denote by

$$H^1_{g^\rho}(M) := \left\{\lambda \in \Omega^1(M) \,|\, d\lambda = 0,\, d *^\rho \lambda = 0\right\}$$

the space of harmonic 1-forms with respect to g^ρ. This space has dimension zero when M is a $K3$-surface and dimension four when M is a four-torus. Since $\ker D$ is a quaternionic vector space of dimension $4k$ with $k \geq 2$, it has a four-dimensional quaternionic subspace that is transverse to $H^1_{g^\rho}(M)$. Thus there exists a nonzero element $\lambda \in \ker D$ such that $V_\lambda \cap H^1_{g^\rho}(M) = 0$. (See Lemma B.1.) This proves Step 3.

Step 4. *Let $\rho \in \mathscr{S}_a \setminus \{\omega_1\}$ be a critical point of $\mathscr{E}$ and let $\lambda \in \Omega^1(M)$ and $\widehat{\rho}_j$ for $j = 0, 1, 2, 3$ be as in Step 3. Then*

$$\sum_{j=0}^{3} \mathscr{H}_\rho(\widehat{\rho}_j) = 0.$$

Choose $X \in \mathrm{Vect}(M)$ such that $\iota(X)\rho = -\lambda$. Then

$$\widehat{\rho}_0 = -d\iota(X)\rho, \qquad \widehat{\rho}_j = -d(\iota(X)\rho \circ J_j^\rho) = -d\iota(J_jX)\rho, \qquad j = 1, 2, 3.$$

For $i, j = 1, 2, 3$ define

$$\widehat{H}_{0i} := \frac{(d\iota(X)\omega_i) \wedge \rho}{\mathrm{dvol}_\rho}, \qquad \widehat{H}_{ji} := \frac{(d\iota(J_jX)\omega_i) \wedge \rho}{\mathrm{dvol}_\rho}.$$

By equation (40) in Lemma 4.2, we have

$$(d\iota(Y)\omega_i) \wedge \rho = -d(\omega_i^\rho \wedge \iota(Y)\rho) = -d *^\rho ((\iota(Y)\rho) \circ J_i^\rho)$$

for every vector field $Y \in \mathrm{Vect}(M)$. Apply this formula to the vector fields $Y = X$ and $Y = J_jX$ and use Step 3 to obtain $\widehat{H}_{ji} = 0$ for $j = 0, 1, 2, 3$ and $i = 1, 2, 3$. Hence, by equation (39) in Theorem 4.1, we have

$$\mathcal{H}_\rho(\widehat{\rho}_0) = \sum_{i=1}^{3} \int_M \omega_i(X, \nabla_{X_{K_i^\rho}} X)\mathrm{dvol}_\rho,$$
$$\mathcal{H}_\rho(\widehat{\rho}_j) = \sum_{i=1}^{3} \int_M \omega_i(J_jX, \nabla_{X_{K_i^\rho}} (J_jX))\mathrm{dvol}_\rho$$

for $j = 1, 2, 3$. Hence

$$\begin{aligned}\sum_{j=1}^{3} \mathcal{H}_\rho(\widehat{\rho}_j) &= \sum_{i,j=1}^{3} \int_M \langle X, J_jJ_iJ_j\nabla_{X_{K_i^\rho}} X\rangle \mathrm{dvol}_\rho \\ &= \sum_{i=1}^{3} \int_M \langle X, J_i\nabla_{X_{K_i^\rho}} X\rangle \mathrm{dvol}_\rho \\ &= -\mathscr{H}_\rho(\widehat{\rho}_0)\end{aligned}$$

This proves Step 4 and Theorem 4.5. □

Theorem 4.5 is an infinite-dimensional analogue of a general observation about finite-dimensional hyperKähler moment maps. Let (M, ω_i, J_i) be a hyperKähler manifold, let G be a compact Lie group that acts on M by hyperKähler isometries, and for $x \in M$ let $L_x : \mathfrak{g} \to T_xM$ denote the infinitesimal action of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$. Suppose $\mathfrak{g}$ is equipped with an invariant inner product and the group action is Hamiltonian for each ω_i. For $i = 1, 2, 3$ let $\mu_i : M \to \mathfrak{g}$ be an equivariant moment map so that $\langle d\mu_i(x)\widehat{x}, \xi\rangle = \omega_i(L_x\xi, \widehat{x})$ for $\xi \in \mathfrak{g}$ and $\widehat{x} \in T_xM$. Then the gradient of the function $\mathcal{E} := \frac{1}{2}\sum_i \|\mu_i\|^2$ is given by $\mathrm{grad}\mathcal{E}(x) = \sum_i J_i L_x \mu_i(x)$. Assume $\dim M \geq 4 \dim \mathrm{G}$ and let $x \in M$ be a critical point of $\mathcal{E}$ such that $\mathcal{E}(x) \neq 0$. Then the linear map $\mathfrak{g}^4 \to T_xM : (\xi_0, \xi_1, \xi_2, \xi_3) \mapsto L_x\xi_0 + \sum_i J_i L_x \xi_i$ is not injective and hence not surjective. Thus there exists a vector $\widehat{x} \in T_xM$ such that $L_x^*\widehat{x} = 0$ and $L_x^* J_i \widehat{x} = 0$ for all i. Denote by $\mathcal{H}_x : T_xM \to \mathbb{R}$ the Hessian of $\mathcal{E}$ at x. Then a calculation shows that $\mathcal{H}_x(\widehat{x}) + \sum_i \mathcal{H}_x(J_i\widehat{x}) = 0$. (See Donaldson [1, Proposition 6].) In the case at hand it would be interesting to find an exact 2-form $\widehat{\rho}$ such that $\mathcal{H}_\rho(\widehat{\rho}) < 0$.

A Four-Dimensional Linear Algebra

Let V be a 4-dimensional oriented real vector space and let $V^* := \mathrm{Hom}(V, \mathbb{R})$ be the dual space. Associated to an inner product $g : V \times V \to \mathbb{R}$ is the Hodge $*$-operator $*_g : \Lambda^k V^* \to \Lambda^{4-k} V^*$, the volume form $\mathrm{dvol}_g = *_g 1 \in \Lambda^4 V^*$, and the space

$$\Lambda_g^+ := \left\{\omega \in \Lambda^2 V^* \,\middle|\, \omega = *_g \omega\right\}$$

of self-dual 2-forms. By an observation of Donaldson [2], the inner product g is uniquely determined by dvol_g and Λ_g^+. This is the content of Theorem A.1 below. Call a linear subspace $\Lambda \subset \Lambda^2 V^*$ **positive** if the quadratic form $\Lambda \times \Lambda \to \mathbb{R} : (\omega, \tau) \mapsto \frac{\omega \wedge \tau}{\mathrm{dvol}}$ is positive definite for some (and hence every) positive volume form $\mathrm{dvol} \in \Lambda^4 V^*$. Denote by $\mathcal{G}(V)$ the space of all inner products $g : V \times V \to V$, by $\mathcal{S}(V)$ the space of 2-forms $\rho \in \Lambda^2 V^*$ such that $\rho \wedge \rho > 0$, and by $\mathcal{J}(V)$ the set of linear complex structures $J : V \to V$ that are compatible with the orientation.

Theorem A.1 (Donaldson). *For every positive volume form* $\mathrm{dvol} \in \Lambda^4 V^*$ *and every three-dimensional positive linear subspace* $\Lambda^+ \subset \Lambda^2 V^*$ *there exists a unique inner product* g *on* V *such that* $\mathrm{dvol}_g = \mathrm{dvol}$ *and* $\Lambda_g^+ = \Lambda^+$*.*

Proof. See page 32. □

Theorem A.2. *Let $g \in \mathcal{G}(V)$, $\rho \in \mathcal{S}(V)$, define $u > 0$ and $A \in \mathrm{GL}(V)$ by*

$$u := \frac{\mathrm{dvol}_\rho}{\mathrm{dvol}_g}, \qquad \mathrm{dvol}_\rho := \frac{\rho \wedge \rho}{2}, \qquad g(A\cdot, \cdot) := \rho, \tag{54}$$

and define the linear map $R : \Lambda^2 V^* \to \Lambda^2 V^*$ by

$$R\omega := \omega - \frac{\omega \wedge \rho}{\mathrm{dvol}_\rho}\rho \qquad \text{for all } \omega \in \Lambda^2 V^*. \tag{55}$$

Then R is an involution that preserves the exterior product, acts as the identity on the orthogonal complement of ρ with respect to the exterior product, and $R\rho = -\rho$. Moreover, for every $\widetilde{g} \in \mathcal{G}(V)$, the following are equivalent.

(i) $\widetilde{g}(v, w) = u^{-1} g(Av, Aw)$ *for all $v, w \in V$.*

(ii) $\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}_g$ *and* $*_{\widetilde{g}}\lambda = u^{-1}\rho \wedge *_g(\rho \wedge \lambda)$ *for all $\lambda \in V^*$.*

(iii) $\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}_g$ *and* $*_{\widetilde{g}}\iota(v)\rho = -\rho \wedge g(v, \cdot)$ *for all $v \in V$.*

(iv) *Suppose $\omega \in \mathcal{S}(V)$ and $J \in \mathcal{J}(V)$ satisfy $g = \omega(\cdot, J\cdot)$. Define $\widetilde{\omega} \in \Lambda^2 V^*$ and $\widetilde{J} \in \mathcal{J}(V)$ by $\widetilde{\omega} := R\omega$ and $\rho(\widetilde{J}\cdot, \cdot) := \rho(\cdot, J\cdot)$. Then $\widetilde{g} = \widetilde{\omega}(\cdot, \widetilde{J}\cdot)$.*

(v) $\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}_g$ *and* $\Lambda^+_{\widetilde{g}} = R\Lambda^+_g$.

(vi) $\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}_g$ *and* $*_{\widetilde{g}}\omega = R *_g R\omega$ *for all $\omega \in \Lambda^2 V^*$.*

Proof. See page 32. □

The proofs of both theorems are based on the following six lemmas.

Lemma A.3. *For every $g \in \mathcal{G}(V)$ and every $v \in V$*

$$*_g \iota(v)\mathrm{dvol}_g = -g(v, \cdot), \qquad *_g g(v, \cdot) = \iota(v)\mathrm{dvol}_g.$$

Proof. Direct verification for the standard structures on $V = \mathbb{R}^4$. □

Lemma A.4. *Let $\omega \in \mathcal{S}(V)$, $g \in \mathcal{G}(V)$, $J \in \mathcal{J}(V)$. The following are equivalent.*

(i) $\omega(v, Jw) = g(v, w)$ *for all $v, w \in V$.*

(ii) $\mathrm{dvol}_\omega = \mathrm{dvol}_g$ *and* $*_g(\omega \wedge \lambda) = -\lambda \circ J$ *for all $\lambda \in V^*$.*

Proof. That (i) implies (ii) follows by direct verification for the standard structures on $V = \mathbb{C}^2$. We prove that (ii) implies (i). Assume ω, g, J satisfy (ii) and let $v \in V$. Then, by Lemma A.3 and (ii),

$$g(v, \cdot) = - *_g \iota(v)\mathrm{dvol}_\omega = - *_g \big(\omega \wedge \iota(v)\omega\big) = \iota(v)\omega \circ J = \omega(v, J\cdot).$$

Hence ω, g, J satisfy (i). This proves Lemma A.4. □

A symplectic form $\omega \in \mathcal{S}(V)$ is called **compatible with the inner product** $g \in \mathcal{G}(V)$ (respectively **compatible with the complex structure** $J \in \mathcal{J}(V)$) if there exists a $J \in \mathcal{J}(V)$ (respectively a $g \in \mathcal{G}(V)$) such that the equivalent conditions (i) and (ii) in Lemma A.4 are satisfied.

Lemma A.5. *Let $\omega \in \mathcal{S}(V)$ and $g \in \mathcal{G}(V)$. The following are equivalent.*

(i) *ω is compatible with g.*

(ii) $\mathrm{dvol}_\omega = \mathrm{dvol}_g$ *and* $\omega \in \Lambda_g^+$.

Proof. That (i) implies (ii) follows by direct verification for the standard structures on $V = \mathbb{C}^2$. To prove the converse, consider the standard inner product and orientation on the quaternions $V = \mathbb{H}$ with coordinates $x = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$. Define

$$\omega_i := dx_0 \wedge dx_i + dx_j \wedge dx_k$$

for $i = 1, 2, 3$ and i, j, k a cyclic permutation of $1, 2, 3$. If ω satisfies (ii) then

$$\omega = \sum_i t_i \omega_i, \qquad t_i \in \mathbb{R}, \qquad \sum_i t_i^2 = 1.$$

Hence ω is compatible with the inner product and the complex structure

$$J := t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k}$$

(acting on $\mathbb{H}$ on the left). This proves Lemma A.5. □

Lemma A.6. *Let $\rho \in \mathcal{S}(V)$ and $g \in \mathcal{G}(V)$. If u and A are defined by* (54) *then*

$$\det(A) = u^2.$$

Proof. Assume $V = \mathbb{R}^4$ with the standard inner product and standard orientation. Denote the coordinates on $\mathbb{R}^4$ by $x = (x_0, x_1, x_2, x_3)$ and write

$$\rho = \sum_{i<j} \rho_{ij} dx_i \wedge dx_j, \qquad \rho_{ij} + \rho_{ji} = 0.$$

The nondegeneracy and orientation condition on ρ asserts that

$$u = \rho_{01}\rho_{23} + \rho_{02}\rho_{31} + \rho_{03}\rho_{12} > 0. \tag{56}$$

In the standard basis of $\mathbb{R}^4$ the linear operator A is represented by the matrix

$$A = \begin{pmatrix} 0 & \rho_{01} & \rho_{02} & \rho_{03} \\ -\rho_{01} & 0 & \rho_{12} & -\rho_{31} \\ -\rho_{02} & -\rho_{12} & 0 & \rho_{23} \\ -\rho_{03} & \rho_{31} & -\rho_{23} & 0 \end{pmatrix}. \tag{57}$$

It follows from equations (56) and (57) that

$$\begin{aligned} \det(A) &= \rho_{01} \det \begin{pmatrix} \rho_{01} & \rho_{02} & \rho_{03} \\ -\rho_{12} & 0 & \rho_{23} \\ \rho_{31} & -\rho_{23} & 0 \end{pmatrix} \\ &\quad - \rho_{02} \det \begin{pmatrix} \rho_{01} & \rho_{02} & \rho_{03} \\ 0 & \rho_{12} & -\rho_{31} \\ \rho_{31} & -\rho_{23} & 0 \end{pmatrix} \\ &\quad + \rho_{03} \det \begin{pmatrix} \rho_{01} & \rho_{02} & \rho_{03} \\ 0 & \rho_{12} & -\rho_{31} \\ -\rho_{12} & 0 & \rho_{23} \end{pmatrix} \\ &= \rho_{01} \left(\rho_{02}\rho_{23}\rho_{31} + \rho_{03}\rho_{12}\rho_{23} + \rho_{01}\rho_{23}^2 \right) \\ &\quad + \rho_{02} \left(\rho_{03}\rho_{12}\rho_{31} + \rho_{01}\rho_{23}\rho_{31} + \rho_{02}\rho_{31}^2 \right) \\ &\quad + \rho_{03} \left(\rho_{01}\rho_{23}\rho_{12} + \rho_{02}\rho_{31}\rho_{12} + \rho_{03}\rho_{12}^2 \right). \end{aligned}$$

Thus $\det(A) = u^2$ and this proves Lemma A.6. □

Lemma A.7. *Let $\omega_1, \omega_2, \omega_3 \in \mathcal{S}(V)$ and $J_1, J_2, J_3 \in \mathrm{GL}(V)$ such that*

$$\omega_2(\cdot, J_3\cdot) := \omega_1, \qquad \omega_3(\cdot, J_1\cdot) := \omega_2, \qquad \omega_1(\cdot, J_2\cdot) := \omega_3. \tag{58}$$

Then

$$\omega_i(J_j v, w) = \omega_i(v, J_j w) = \omega_k(v, w) \tag{59}$$

for every cyclic permutation i, j, k of $1, 2, 3$ and all $v, w \in V$. Moreover, the following are equivalent.

(i) *$\omega_i \wedge \omega_j = 0$ and $\omega_i \wedge \omega_i = \omega_j \wedge \omega_j$ for $1 \le i < j \le 3$.*

(ii) *$J_i^2 = -\mathbb{1}$ and $J_j J_k = -J_k J_j = J_i$ for cyclic permutations i, j, k of $1, 2, 3$.*

If these equivalent conditions are satisfied then the following holds.

(a) *The vectors $v, J_1 v, J_2 v, J_3 v$ form a basis of V for every $v \in V \setminus \{0\}$.*

(b) *$\omega_1(v, J_1 w) = \omega_2(v, J_2 w) = \omega_3(v, J_3 w)$ for $v, w \in V$.*

(c) *$\omega_i(w, J_i v) = \omega_i(v, J_i w)$ for $i = 1, 2, 3$ and $v, w \in V$.*

(d) *$\omega_i(v, J_i v) \neq 0$ for $i = 1, 2, 3$ and $v \in V \setminus \{0\}$.*

Proof. That (58) implies (59) follows from the skew-symmetry of the ω_i.

(i) implies (ii). Since $\iota(J_j v)\omega_i = \iota(v)\omega_k$, it follows from (i) that $\omega_i \wedge \iota(v)\omega_i = \omega_k \wedge \iota(v)\omega_k = \omega_k \wedge \iota(J_j v)\omega_i = -\omega_i \wedge \iota(J_j v)\omega_k$ for $v \in V$ and every cyclic permutation i, j, k of $1, 2, 3$. Hence

$$\omega_k(J_j v, w) = \omega_k(v, J_j w) = -\omega_i(v, w). \tag{60}$$

Second, $\omega_2(\cdot, J_3 J_2 J_1 \cdot) = \omega_1(\cdot, J_2 J_1 \cdot) = \omega_3(\cdot, J_1 \cdot) = \omega_2$, by equation (59), and $\omega_2(\cdot, J_1 J_2 J_3 \cdot) = -\omega_3(\cdot, J_2 J_3 \cdot) = \omega_1(\cdot, J_3 \cdot) = -\omega_2$, by equation (60). Hence

$$J_3 J_2 J_1 = \mathbb{1} = -J_1 J_2 J_3. \tag{61}$$

Third, by (59) and (60), $\omega_j(\cdot, J_i^2 \cdot) = -\omega_k(\cdot, J_i \cdot) = -\omega_j$ and hence

$$J_1^2 = J_2^3 = J_3^2 = -\mathbb{1}. \tag{62}$$

Fourth, $J_2 J_1 = J_3^{-1} = -J_1 J_2$, by (61), and hence $J_2 J_1 = -J_3 = -J_1 J_2$, by (62). Multiply this equation by J_1 and J_2 on the left and right to obtain the quaternion relations $J_i J_j = -J_j J_i = J_k$ for i, j, k cyclic. This shows that (i) implies (ii).

(ii) implies (a). Let $v \in V \setminus \{0\}$ and $x_i \in \mathbb{R}$ such that $x_0 v + \sum_i x_i J_i v = 0$. Then

$$0 = \left(x_0 \mathbb{1} - \sum_{i=1}^{3} x_i J_i \right) \left(x_0 v + \sum_{i=1}^{3} x_i J_i v \right) = \left(\sum_{i=0}^{3} x_i^2 \right) v$$

and hence $x_0 = x_1 = x_2 = x_3 = 0$.

(ii) implies (b). It follows from equation (59) that, for i, j, k cyclic, $\omega_i(v, J_i w) = \omega_j(J_k v, J_i w) = \omega_j(v, J_k J_i w) = \omega_j(v, J_j w)$.

(ii) implies (60)**.** It follows from equation (59) that, for i, j, k cyclic, $\omega_k(J_j v, w) = \omega_i(J_j J_j v, w) = -\omega_i(v, w)$.

(ii) implies (c). It follows from equation (60) that, for i, j, k cyclic, $\omega_i(w, J_i v) = \omega_i(w, J_j J_k v) = \omega_i(J_k J_j w, v) = \omega_i(-J_i w, v) = \omega_i(v, J_i w)$.

(ii) implies (d). Fix a nonzero vector $v \in V$. Then $\omega_1(v, v) = 0$ and, by (59) and (60), $\omega_1(v, J_2 v) = \omega_3(v, v) = 0$ and $\omega_1(v, J_3 v) = -\omega_2(v, v) = 0$. Since ω_1 is nondegenerate, it follows from (a) that $\omega_1(v, J_1 v) \neq 0$.

(ii) implies (i). Fix a nonzero vector $v \in V$ and define $\Phi : \mathbb{H} \to V$ by $\Phi(x) := x_0 v + \sum_i x_i J_i v$. By (a) this is an isomorphism. By (b) and (d),

$$\lambda := \omega_1(v, J_1 v) = \omega_2(v, J_2 v) = \omega_3(v, J_3 v) \neq 0.$$

By (59) and (60), we have $\Phi^* \omega_i = \lambda(dx_0 \wedge dx_i + dx_j \wedge dx_k)$ for $i = 1, 2, 3$ and i, j, k a cyclic permutation of $1, 2, 3$. This shows that (ii) implies (i). $\square$

Lemma A.8. *If $J_1, J_2, J_3 \in \mathcal{J}(V)$ are compatible with $g \in \mathcal{G}(V)$ and satisfy $J_iJ_j + J_jJ_i = 0$ for $i \neq j$ then $J_3 = \pm J_1J_2$.*

Proof. Fix a unit vector v. Then $g(J_iv, J_jv) = g(v, J_jJ_iv) = -g(J_jv, J_iv)$. Hence v, J_1v, J_2v, J_3v form an orthonormal basis of V and J_1J_2v is orthogonal to v, J_1v, J_2v. Hence $J_1J_2v = \pm J_3v$. It follows that $J_1J_2 = \pm J_3$. □

Proof of Theorem A.1. **Existence.** Fix a basis $\omega_1, \omega_2, \omega_3$ of Λ^+ such that

$$\omega_i \wedge \omega_j = 2\delta_{ij}\mathrm{dvol}.$$

Choose $J_i \in \mathrm{GL}(V)$ such that (58) holds. By Lemma A.7, the bilinear map

$$V \times V \to \mathbb{R} : (v, w) \mapsto \omega_i(v, J_iw)$$

is independent of i, symmetric, and definite. Assume without loss of generality that $\omega_i(v, J_iv) > 0$ for all $v \in V \setminus \{0\}$. (Otherwise, replace the triple J_1, J_2, ω_3 by $-J_1, -J_2, -\omega_3$.) Then the inner product $g(v, w) := \omega_i(v, J_iw)$ is compatible with ω_i. Hence it follows from Lemma A.5 that $\mathrm{dvol}_g = \mathrm{dvol}_{\omega_i}$ and $\omega_i \in \Lambda^+_g$ for $i = 1, 2, 3$. Thus $\mathrm{dvol}_g = \mathrm{dvol}$ and $\Lambda^+_g = \Lambda^+$.

Uniqueness. Let $\widetilde{g} \in \mathcal{G}(V)$ such that $\Lambda^+_{\widetilde{g}} = \Lambda^+$ and $\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}$. By Lemma A.5, the symplectic forms $\omega_1, \omega_2, \omega_3$ are compatible with $\widetilde{g}$. Hence there exist complex structures $\widetilde{J}_1, \widetilde{J}_2, \widetilde{J}_3 \in \mathcal{J}(V)$ such that

$$\omega_i(\cdot, \widetilde{J}_i\cdot) = \widetilde{g}(\cdot, \cdot).$$

Thus

$$\omega_j(\cdot, \widetilde{J}_j\widetilde{J}_k\cdot) = \widetilde{g}(\cdot, \widetilde{J}_k\cdot) = -\omega_k(\cdot, \cdot) = \omega_j(\cdot, J_i\cdot)$$

and so $\widetilde{J}_j\widetilde{J}_k = J_i$ for i, j, k cyclic. Hence

$$\widetilde{J}_j\widetilde{J}_k + \widetilde{J}_k\widetilde{J}_j = J_i - \widetilde{J}_k\widetilde{J}_i\widetilde{J}_i\widetilde{J}_j = J_i - J_jJ_k = 0$$

for i, j, k cyclic. By Lemma A.8,

$$\widetilde{J}_3 = \pm\widetilde{J}_1\widetilde{J}_2 = \pm J_3.$$

Since $\omega_3(v, J_3v) > 0$ and $\omega_3(v, \widetilde{J}_3v) > 0$ for $v \neq 0$, we have $\widetilde{J}_3 = J_3$. Hence

$$\widetilde{g} = \omega_3(\cdot, \widetilde{J}_3\cdot) = \omega_3(\cdot, J_3\cdot) = g.$$

This proves Theorem A.1. □

Proof of Theorem A.2. That the linear map $R : \Lambda^2 V^* \to \Lambda^2 V^*$ in (55) has the required properties follows by direct calculation.

We prove that (i) implies (ii). By Lemma A.6, $\det(-A^2) = \det(A)^2 = u^4$ and hence the inner product $\widetilde{g}(v, w) := u^{-1} g(Av, Aw)$ has the volume form

$$\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}_g = u^{-1}\mathrm{dvol}_\rho.$$

Now let $\lambda \in V^*$ and choose $v \in V$ such that $\widetilde{g}(v, \cdot) = \lambda$. Then, by Lemma A.3,

$$*_{\widetilde{g}}\lambda = *_{\widetilde{g}}\widetilde{g}(v, \cdot) = \iota(v)\mathrm{dvol}_{\widetilde{g}} = u^{-1}\iota(v)\mathrm{dvol}_\rho = u^{-1}\rho \wedge \iota(v)\rho. \tag{63}$$

Since $\iota(v)\rho = g(Av, \cdot)$ it follows also from Lemma A.3 that

$$\begin{aligned} *_g\iota(v)\rho &= \iota(Av)\mathrm{dvol}_g = u^{-1}\iota(Av)\mathrm{dvol}_\rho = u^{-1}\rho \wedge \iota(Av)\rho \\ &= u^{-1}\rho \wedge g(A^2 v, \cdot) = -\rho \wedge \widetilde{g}(v, \cdot) = -\rho \wedge \lambda. \end{aligned}$$

Thus $\iota(v)\rho = *_g(\rho \wedge \lambda)$ and so $*_{\widetilde{g}}\lambda = u^{-1}\rho \wedge *_g(\rho \wedge \lambda)$ by (63). This shows that $\widetilde{g}$ satisfies (ii).

We prove that (ii) implies (iii). Assume $\widetilde{g}$ satisfies (ii) and let $v \in V$. Use the equation $u^{-1}(\rho \wedge \iota(v)\rho) = u^{-1}\iota(v)\mathrm{dvol}_\rho = \iota(v)\mathrm{dvol}_g$ to obtain

$$*_{\widetilde{g}}\iota(v)\rho = u^{-1}\rho \wedge *_g(\rho \wedge \iota(v)\rho) = \rho \wedge *_g\iota(v)\mathrm{dvol}_g = -\rho \wedge g(v, \cdot).$$

Here the last step follows from Lemma A.3. This shows that $\widetilde{g}$ satisfies (iii).

We prove that (iii) implies (iv). Assume $\widetilde{g}$ satisfies (iii). Let $\omega \in \mathcal{S}(V)$ and $J \in \mathcal{J}(V)$ such that $\omega(\cdot, J\cdot) = g$. Then, by Lemma A.4,

$$\mathrm{dvol}_\omega = \mathrm{dvol}_g, \qquad *_g(\omega \wedge \lambda) = -\lambda \circ J \tag{64}$$

for every $\lambda \in V^*$. Define $\widetilde{\omega}$ and $\widetilde{J}$ by $\widetilde{\omega} := R\omega$ and $\rho(\widetilde{J}\cdot, \cdot) := \rho(\cdot, J\cdot)$. Then $\widetilde{\omega} \wedge \widetilde{\omega} = \omega \wedge \omega$ and so $\mathrm{dvol}_{\widetilde{\omega}} = \mathrm{dvol}_\omega = \mathrm{dvol}_g = \mathrm{dvol}_{\widetilde{g}}$ by (iii). Now let $\lambda \in V^*$ and choose $v \in V$ such that $\iota(v)\rho = \lambda$. Then

$$\lambda \circ \widetilde{J} = \iota(v)\rho \circ \widetilde{J} = \rho(v, \widetilde{J}\cdot) = \rho(Jv, \cdot) = \iota(Jv)\rho.$$

Abbreviate $K := \frac{\omega \wedge \rho}{\mathrm{dvol}_\rho}$. Then $\widetilde{\omega} = \omega - K\rho$ and so

$$\widetilde{\omega} \wedge \lambda = \omega \wedge \iota(v)\rho - K\iota(v)\mathrm{dvol}_\rho = \omega \wedge \iota(v)\rho - \iota(v)(\omega \wedge \rho) = -(\iota(v)\omega) \wedge \rho.$$

This implies

$$*_{\widetilde{g}}(\widetilde{\omega} \wedge \lambda) = - *_{\widetilde{g}} (\rho \wedge (\iota(v)\omega) = - *_{\widetilde{g}} (\rho \wedge g(Jv, \cdot)) = \iota(Jv)\rho = \lambda \circ \widetilde{J}.$$

Hence $\widetilde{\omega}(\cdot, \widetilde{J}\cdot) = \widetilde{g}$ by Lemma A.4. This shows that $\widetilde{g}$ satisfies (iv).

We prove that (iv) implies (v). Assume $\widetilde{g}$ satisfies (iv) and choose a symplectic form $\omega \in \mathcal{S}(V)$ that is compatible with g. Then $\widetilde{\omega} := R\omega$ is compatible with $\widetilde{g}$ by (iv), and hence

$$\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}_{\widetilde{\omega}} = \mathrm{dvol}_{\omega} = \mathrm{dvol}_g$$

by Lemma A.5. If $\omega \in \Lambda^+_g \setminus \{0\}$ then, by Lemma A.5, there is a $c > 0$ such that $c\omega$ is compatible with g, hence $cR\omega$ is compatible with $\widetilde{g}$ by (iv), and hence $c\widetilde{\omega} \in \Lambda^+_{\widetilde{g}}$ by Lemma A.5. This shows that $R\Lambda^+_g \subset \Lambda^+_{\widetilde{g}}$. Since R is an involution of $\Lambda^2 V^*$, the subspace $R\Lambda^+_g$ has dimension three and hence agrees with $\Lambda^+_{\widetilde{g}}$. This shows that $\widetilde{g}$ satsfies (v).

We prove that (v) implies (vi). The map $R : \Lambda^2 V^* \to \Lambda^2 V^*$ in (55) is an involution and preserves the exterior product, i.e.

$$R \circ R = \mathrm{id}, \qquad R\omega \wedge R\tau = \omega \wedge \tau$$

for all $\omega, \tau \in \Lambda^2 V^*$. By (v) it also satisfies

$$R\Lambda^+_g = \Lambda^+_{\widetilde{g}}.$$

If $\tau \in \Lambda^-_g$ then $R\tau \wedge R\omega = \tau \wedge \omega = 0$ for all $\omega \in \Lambda^+_g$, hence $R\tau \wedge \widetilde{\omega} = 0$ for every $\widetilde{\omega} \in \Lambda^+_{\widetilde{g}}$, and hence $R\tau \in \Lambda^-_{\widetilde{g}}$. Thus $R\Lambda^-_g = \Lambda^-_{\widetilde{g}}$. It follows that

$$R *_g \omega = R\omega = *_{\widetilde{g}} R\omega, \qquad R *_g \tau = -R\tau = *_{\widetilde{g}} R\tau$$

for all $\omega \in \Lambda^+_g$ and all $\tau \in \Lambda^-_g$. This shows that $R *_g = *_{\widetilde{g}} R$ on $\Lambda^2 V^*$ and hence $\widetilde{g}$ satisfies (vi).

We prove that (vi) implies (i). Let $\widetilde{g} \in \mathcal{G}(V)$ be any inner product that satisfies (vi) and let $h \in \mathcal{G}(V)$ be the inner product defined by the formula

$$h(v, w) := u^{-1} g(Av, Aw)$$

in (i). Since we have already proved that (i) implies (vi), the inner products $\widetilde{g}$ and h both satisfy (vi). Thus they have the same volume form and the same Hodge $*$-operator on 2-forms. Hence

$$\mathrm{dvol}_{\widetilde{g}} = \mathrm{dvol}_h, \qquad \Lambda^+_{\widetilde{g}} = \Lambda^+_h$$

and so $\widetilde{g} = h$ by Theorem A.1. In other words, every inner product $\widetilde{g} \in \mathcal{G}(V)$ that satisfies (vi) is given by $\widetilde{g}(v, w) = u^{-1} g(Av, Aw)$. This completes the proof of Theorem A.2. □

B Quaternionic Subspaces

Denote by $\mathbb{H} \cong \mathbb{R}^4$ the quaternions and by $\mathrm{Sp}(1) \cong S^3$ the unit quaternions. For $\lambda \in \mathbb{H}$ define $V_\lambda := \{(x, x\lambda) \,|\, x \in \mathbb{H}\}$. Thus V_λ is the unique quaternionic subspace of $\mathbb{H}^2$ of real dimension four that contains the pair $(1, \lambda)$.

Lemma B.1. *Let $W \subset \mathbb{H}^2$ be a real linear subspace of real dimension $\dim^{\mathbb{R}} W \leq 4$. Then there exists an element $\lambda \in \mathbb{H}$ such that $V_\lambda \cap W = 0$.*

Proof. The proof is a standard transversality argument and has two steps.

Step 1. *Define $f : \mathrm{Sp}(1) \times \mathbb{H} \to \mathbb{H}^2$ by $f(x, \lambda) := (x, x\lambda)$ for $x \in \mathrm{Sp}(1)$ and $\lambda \in \mathbb{H}$. Then f is transverse to every real linear subspace of $\mathbb{H}^2$.*

Let $W \subset \mathbb{H}^2$ be a real linear subspace and let $(x, \lambda) \in \mathrm{Sp}(1) \times \mathbb{H}$ such that $f(x, \lambda) \in W$. We must prove that $\mathrm{im} df(x, \lambda) + W = \mathbb{H}^2$. To see this, fix any pair $(\xi, \eta) \in \mathbb{H}^2$ and define $\widehat{x} := \xi - \langle \xi, x \rangle x$ and $\widehat{\lambda} := x^{-1}(\eta - \xi\lambda)$. Then

$$df(x, \lambda)(\widehat{x}, \widehat{\lambda}) - (\xi, \eta) = (\widehat{x} - \xi, \widehat{x}\lambda + x\widehat{\lambda} - \eta) = -\langle \xi, x \rangle (x, x\lambda) \in W.$$

This proves Step 1.

Step 2. *We prove the lemma.*

Let $W \subset \mathbb{H}^2$ be a real linear subspace of real dimension at most four. Then the set $\mathcal{M} := f^{-1}(W) = \{(x, \lambda) \in \mathrm{Sp}(1) \times \mathbb{H} \,|\, (x, x\lambda) \in W\}$ is a smooth submanifold of $\mathrm{Sp}(1) \times \mathbb{H}$ of (real) dimension at most three by Step 1. Hence the projection $\mathcal{M} \to \mathbb{H} : (x, \lambda) \mapsto \lambda$ is not surjective by Sard's theorem. Hence there exists an element $\lambda \in \mathbb{H}$ such that $\mathcal{M} \cap (\mathrm{Sp}(1) \times \{\lambda\}) = \emptyset$ and so $V_\lambda \cap W = 0$. This proves Lemma B.1. $\square$

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The Donaldson geometric flow is a local smooth semiflow

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Abstract

We prove the local existence of unique smooth solutions of the Donaldson geometric flow on the space of symplectic forms on a closed smooth four-manifold, representing a fixed cohomology class. It is a semiflow on the Besov space $B_2^{1,p}(M, \Lambda^2)$ for $p > 4$. The Donaldson geometric flow was introduced by Simon Donaldson in [2]. For a detailed exposition see [6].

Contents

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1 Introduction

Let M be a smooth closed Riemannian four-manifold. Denote by g the Riemannian metric, denote by dvol $\in \Omega^4(M)$ the volume form of g and let $* : \Omega^k(M) \to \Omega^{n-k}(M)$ be the Hodge $*$-operator associated to the metric and orientation. Let ω be a symplectic form on M compatible with the metric and let $\mathscr{S}_a$ be the space of symplectic forms representing the cohomology class $a = [w] \in H^2(M, \mathbb{R})$. This is formally an infinite dimensional manifold and the tangent space at any element $\rho \in \mathscr{S}_a$ is the space of exact two-forms. The Donaldson geometric flow on $\mathscr{S}_a$ is given by the evolution equation

$$\frac{d}{dt}\rho_t = d *^{\rho_t} d\Theta^{\rho_t}, \tag{1}$$

where

$$\Theta^\rho := *\frac{\rho}{u} - \frac{1}{2}\left|\frac{\rho}{u}\right|^2 \rho, \qquad \frac{1}{2}\rho \wedge \rho =: u\mathrm{dvol}, \qquad *^\rho \lambda := \frac{\rho \wedge *(\rho \wedge \lambda)}{u} \tag{2}$$

for all one-forms $\lambda \in \Omega^1(M)$. The operator $*^\rho$ is the Hodge star operator for the metric g^ρ which is uniquely determined by the conditions

$$\mathrm{dvol}_{g^\rho} = \mathrm{dvol}, \qquad *^\rho\left(\omega - \frac{\omega \wedge \rho}{\mathrm{dvol}_\rho}\rho\right) = \left(\omega - \frac{\omega \wedge \rho}{\mathrm{dvol}_\rho}\rho\right) \Leftrightarrow *\omega = \omega$$

for all $\omega \in \Lambda^2 T^*M$. Each $\rho \in \Omega^2$ with $\rho^2 > 0$ determines an inner product $\langle \cdot, \cdot \rangle_\rho$ on the space of exact two-forms defined by

$$\langle \widehat{\rho}_1, \widehat{\rho}_2 \rangle_\rho := \int_M \lambda_1 \wedge *^\rho \lambda_2, \qquad d\lambda_i = \widehat{\rho}_i, \qquad *^\rho \lambda_i \text{ is exact} \tag{3}$$

These inner products determine a metric on the infinite dimensional space $\mathscr{S}_a$ called the Donaldson metric. The Donaldson geometric flow is the negative gradient flow with respect to the Donaldson metric of the energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$ defined by

$$\mathscr{E}(\rho) := \int_M \frac{2\left|\rho^+\right|^2}{\left|\rho^+\right|^2 - \left|\rho^-\right|^2}, \qquad \rho \in \mathscr{S}_a. \tag{4}$$

The Donaldson flow has a beautiful geometric origin laid out in Donaldson's paper [2]. The key idea is that the space of diffeomorphisms of a hyperKähler surface has the structure of an infinite dimensional hyperKähler manifold. The group of symplectomorphisms with respect to a preferred symplectic structure ω then acts by composition on the right and this group action is generated by a hyperKähler moment map. In analogy with the finite dimensional case one then studies the negative gradient flow to the moment map square functional with respect to the L^2-inner product. If we push the preferred symplectic structure ω forward by the diffeomorpisms of M to the space of symplectic structures in a fixed cohomology class, we obtain the Donaldson flow (1). If we push the moment map square forward we obtain the energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$ in (4) and if we push the L^2-metric forward we obtain the Donaldson metric (3). The Donaldson flow remains the negative gradient flow to this energy functional with respect to the Donaldson metric (3) on the space of symplectic structures in a fixed cohomology class. The Donaldson flow remains well defined for a general symplectic 4-manifold (M, ω) equipped with a compatible Riemannian metric g.

The motivation to study the Donaldson flow on the space of symplectic structures comes from the longstanding open uniqueness problem for symplectic structures on closed four-manifolds (see [9] for an exposition). The hope is that the Donaldson flow provides a tool to settle this question at least in some favorable cases, such as the hyperKähler surface and the complex projective plane $\mathbb{C}\mathrm{P}^2$. This hope is strengthened by the observation that the preferred symplectic structure ω is the unique absolute minimum and that the Hessian of the energy functional $\mathscr{E}$ is positive definite at the absolute minimum (see [6]). In the case of $M = \mathbb{C}\mathrm{P}^2$ we can further show that the Fubini-Study form is the only critical point of the flow. Thus, if longtime existence and convergence of the Donaldson flow can be established, the Donaldson flow would provide a proof for the uniqueness of the symplectic structures on $\mathbb{C}\mathrm{P}^2$ of a given cohomolgy class up to isotopy. In the case of the hyperKähler surface Donaldson [2] proved that the higher critical points are not strictly stable. The idea would then be to perturb the flow near the critical points such that eventually the perturbed flow converges towards the unique absolute mimimum.

The main results of this paper are regularity for the critical points and the regularity for the flow (section 4), short time existence (section 5) and the semiflow property (section 6). These results lay the foundation for future studies focusing on longtime existence and the problem of solutions *escaping to infinity*. Section 2 contains the main geometric ideas, in particular a result by Donaldson [3] that shows that the map $\rho \mapsto \frac{\rho^+}{u}$ from the space of symplectic forms representing a cohomology class in a fixed affine space in $H^2(M;\mathbb{R})$ with dimension equal to b_2^+ to the space of self-dual two-forms is locally a Banach space diffeomorphism. In section 3 we then study the evolution of $\frac{\rho^+}{u}$ if ρ evolves by the Donaldson flow. This evolution equation is the key to prove regularity of the Donaldson flow. Sections 5 and 6 are independent of the previous ones on regularity and can be read separately. In section 5 we use a standard argument involving Banach's fixed point theorem to prove short time existence in an appropriate Sobolev completion of the space of two-forms. In section 6 we prove the semiflow property of the flow on the related interpolation space. In particular we can use the local stable manifold theorem to show that there exists an attracting neighborhood around the absolute minimum in the topology of this space. The appendix deals with products and compositions of functions in Sobolev spaces.

Notation and Conventions. Let (M, g) be a closed Riemannian oriented four-manifold. Let $\pi : E \to M$ be a natural k-dimensional vector bundle over M. We denote by $W^{\ell,p}(M, E)$ the Sobolev completion of the space of sections $\Omega^0(M, E)$. If the bundle in question is clear from the context we will just write $W^{\ell,p}$ instead of $W^{\ell,p}(M, E)$. We will suppress the constants that solely depend on the parameters $\dim(M), \mathrm{vol}(M), k, p$ from the notation when we make estimates in Sobolev norms. We denote by $\mathscr{S}$ the space of smooth symplectic structures on M compatible with the orientation. We write $\mathscr{S}_a$ for $\rho \in \mathscr{S}$ that represent a fixed cohomology class $a \in H^2(M;\mathbb{R})$. Let $L \in H^2(M;\mathbb{R})$ be an affine subspace. Then $\mathscr{S}_L$ denotes the subset of $\mathscr{S}$ such that the symplectic structures represent cohomology classes in L. We define

$$\mathscr{S}^{k,p} := \{\rho \in W^{k,p}(M, \Lambda^2) |\, \rho \wedge \rho > 0,\ d\rho = 0\}.$$

A cohomology class or an affine subspace of cohomology classes as subscript has the same meaning as in the smooth counterpart.

2 The Map K

In this section we study the map

$$\mathscr{S} \to \Omega^+ : \qquad \rho \to \frac{\rho^+}{u}, \qquad u := \frac{\rho \wedge \rho}{2\mathrm{dvol}}$$

from the space of symplectic forms to the space of self-dual two-forms. We apply a theory developed by Donaldson [3] on elliptic problems for closed two-forms on four dimensional closed manifolds that fullfill a point wise constraint with 'negative tangents'. The main insight is that this map is a local Banach space diffeomorphism between appropriatly choosen spaces and the regularity of $\frac{\rho^+}{u}$ determines the regularity of ρ. The precise statement is given in Theorem 2.1.

We will use the construction of a Riemannian metric determined by a nondegenerate two-form and a volume form explained in [6]. Here is a brief overview. Fix a nondegenerate two-form $\rho \in \Omega^2(M)$ such that $\rho \wedge \rho > 0$. There exists a Riemannian metric g^ρ such that its volume form agrees with dvol of (M, g). The associated Hodge star operator $*^\rho : \Omega^1(M) \to \Omega^3(M)$ is given by

$$*^\rho \lambda = \frac{\rho \wedge *(\rho \wedge \lambda)}{u}.$$

If $X \in \mathrm{Vect}(M)$ is a vector field then

$$*^\rho \rho(X, \cdot) = -\rho \wedge g(X, \cdot).$$

The map

$$R^\rho : \Omega^2(M) \to \Omega^2(M), \qquad R^\rho \omega := \omega - \frac{\omega \wedge \rho}{\mathrm{dvol}_\rho} \rho \tag{5}$$

is an involution that preserves the exterior product, acts as the identity on the orthogonal complement of ρ with respect to the exterior product and it sends ρ to $-\rho$. Moreover it maps Ω^+ to Ω^{+^ρ}, the self-dual forms with respect to the metric g^ρ. The Hodge star operator $*^\rho : \Omega^2(M) \to \Omega^2(M)$ associated to g^ρ is given by

$$*^\rho \omega = R^\rho * R^\rho \omega.$$

Let $\omega \in \Omega^2(M)$ be a self-dual two-form and let $J : TM \to TM$ be an almost complex structure such that $g = \omega(\cdot, J\cdot)$. We define the almost complex structure J^ρ by

$$\rho(J^\rho \cdot, \cdot) := \rho(\cdot, J\cdot) \tag{6}$$

and a self-dual two-form ω^ρ with respect to g^ρ by

$$\omega^\rho := R^\rho \omega. \tag{7}$$

Then

$$g^\rho = \omega^\rho(\cdot, J^\rho \cdot)$$

and

$$*^\rho(\lambda \wedge \omega^\rho) = \lambda \circ J^\rho \tag{8}$$

for all one-forms $\lambda \in \Omega^1(M)$.

Fix a symplectic form $\rho \in \mathscr{S}_a$. Let $\mathscr{S}_{a+\mathcal{H}^\rho}$ be the space of symplectic forms representing a cohomology class in the affine space $a+\mathcal{H}^\rho \subset H^2(M,\mathbb{R})$, where $\mathcal{H}^\rho$ are the harmonic self-dual forms with respect to the Hodge star operator $*^\rho$. Define the map $K : \mathscr{S}_{a+\mathcal{H}^\rho} \to \Omega^+$ by

$$K(\rho) := \frac{\rho + *\rho}{2u}, \qquad u = \frac{\rho \wedge \rho}{2\mathrm{dvol}}.$$

Denote the extension of this map to the Sobolev space $\mathscr{S}^{k,p}_{a+\mathcal{H}_\rho}$ also by K.

Theorem 2.1 (**The Map** K)**.** *Let* $k - \frac{4}{p} > 0$.

(i) *For every* $\rho \in \mathscr{S}^{k,p}_{a+\mathcal{H}_\rho}$ *there exists a* $W^{k,p}$*-neighborhood of* ρ *such that* K *restricted to this neighborhood is a diffeomorphism of Banach spaces.*

(ii) *Let* $\mathcal{H} \in H^2(M;\mathbb{R})$ *be a positive linear subspace and* $a \in H^2(M;\mathbb{R})$. *The map* $K : \mathscr{S}_{a+\mathcal{H}} \to \Omega^+$ *is injective.*

(iii) *There exists polynomials* $\mathfrak{p}_1, \mathfrak{p}_2$ *with positive coefficients with the following significance. If* $\rho \in \mathscr{S}^{k,p}$ *and* $K(\rho) \in W^{k+1,p}(M,\Lambda^+)$ *with* $\frac{1}{u} \leq C < \infty$, *then* $\rho \in \mathscr{S}^{k+1,p}$ *and*

$$\|\rho\|_{W^{k+1,p}} \leq \mathfrak{p}_1\left(C, \|\rho\|_{L^\infty}\right) \|\frac{\rho^+}{u}\|_{W^{k+1,p}} + \mathfrak{p}_2\left(C, \|\rho\|_{L^\infty}, \|\rho\|_{W^{k-1,p}}, \|\frac{\rho^+}{u}\|_{W^{k,p}}\right) \|\rho\|_{W^{k,p}}$$

Proof. See page 44. □

We will need the following three lemmas to prove Theorem 2.1.

Lemma 2.2 (**Negative chords**)**.** *Let* V *be a real four-dimensional vector space equipped with the standart metric,* $\theta \in \Lambda^+V$ *with* $\theta^2 = \mathrm{dvol}$ *and* $\rho_1, \rho_2 \in \Lambda^2V$ *such that*

$$\rho_i \wedge \rho_i = \mathrm{dvol}, \qquad \rho_i^+ = \lambda_i\theta, \qquad \lambda_i \geq 1, \qquad i = 1, 2.$$

Then

$$(\rho_1 - \rho_2)^2 \leq 0,$$

and

$$(\rho_1 - \rho_2)^2 = 0 \Leftrightarrow \rho_1 = \rho_2.$$

Proof. Since $\rho_1^+ = \lambda_1\theta$ and $\rho_2^+ = \lambda_2\theta$ for $\lambda_1, \lambda_2 \geq 1$ we find

$$\begin{aligned}(\rho_1 - \rho_2)^2 &= \left(\rho_1^+ - \rho_2^+ + \rho_1^- - \rho_2^-\right)^2 \\ &= \left((\lambda_1 - \lambda_2)\,\theta + \rho_1^- - \rho_2^-\right)^2 \\ &= (\lambda_1 - \lambda_2)^2 \,\mathrm{dvol} + \left(\rho_1^- - \rho_2^-\right)^2 \\ &= (\lambda_1 - \lambda_2)^2 \,\mathrm{dvol} - \left|\rho_1^-\right|^2 \mathrm{dvol} - \left|\rho_2^-\right|^2 \mathrm{dvol} - 2\rho_1^- \wedge \rho_2^-.\end{aligned}$$

Thus,

$$\frac{(\rho_1 - \rho_2)^2}{\mathrm{dvol}} = (\lambda_1 - \lambda_2)^2 - \left|\rho_1^-\right|^2 - \left|\rho_2^-\right|^2 + 2\left\langle \rho_1^-, \rho_2^-\right\rangle.$$

Since

$$\mathrm{dvol} = \rho_i \wedge \rho_i = \lambda_i^2 \theta \wedge \theta - \rho_i^- \wedge \rho_i^- = \left(\lambda_i^2 - \left|\rho_i^-\right|^2\right) \mathrm{dvol}$$

we have $\left|\rho_i^-\right|^2 = \lambda_i^2 - 1$ and hence

$$\begin{aligned}\frac{(\rho_1 - \rho_2)^2}{\mathrm{dvol}} &= \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 - \lambda_1^2 + 1 - \lambda_2^2 + 1 + 2\left\langle \rho_1^-, \rho_2^-\right\rangle \\ &= -2\lambda_1\lambda_2 + 2 + 2\left\langle \rho_1^-, \rho_2^-\right\rangle.\end{aligned}$$

By the Cauchy-Schwarz inequality $\left\langle \rho_1^-, \rho_2^-\right\rangle \leq \left|\rho_1^-\right|\left|\rho_2^-\right|$ and equality holds if and only if ρ_1^- and ρ_2^- are colinear. Thus,

$$\begin{aligned}\frac{(\rho_1 - \rho_2)^2}{\mathrm{dvol}} &\leq -2\lambda_1\lambda_2 + 2 + 2\left|\rho_1^-\right|\left|\rho_2^-\right| \\ &= -2\lambda_1\lambda_2 + 2 + 2\sqrt{\lambda_1^2 - 1}\sqrt{\lambda_2^2 - 1}\end{aligned} \tag{9}$$

and equality holds only if ρ_1^- and ρ_2^- are colinear. Suppose $(\rho_1 - \rho_2)^2 \geq 0$. Then

$$\lambda_1\lambda_2 - 1 \leq \sqrt{\lambda_1^2 - 1}\sqrt{\lambda_2^2 - 1}.$$

Squaring both sides of the inequality yields

$$\lambda_1^2\lambda_2^2 - 2\lambda_1\lambda_2 + 1 \leq (\lambda_1^2 - 1)(\lambda_2^2 - 1) = \lambda_1^2\lambda_2^2 - \lambda_1^2 - \lambda_2^2 + 1.$$

Hence we find

$$\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \leq 0$$

with equality if and only if ρ_1^- and ρ_2^- are colinear. But clearly

$$\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 = (\lambda_1 - \lambda_2)^2 \geq 0$$

and it follows that $\lambda_1 = \lambda_2$ and ρ_1^- and ρ_2^- are colinear! From $\rho_i^+ = \lambda_i\theta$ we get $\rho_1^+ = \rho_2^+$. Further, since $\lambda_1 = \lambda_2$ and $\left|\rho_i^-\right|^2 = \lambda_i^2 - 1$ we get $\left|\rho_1^-\right| = \left|\rho_2^-\right|$ and therefore $\rho_1^- = \pm\rho_2^-$. However, if $\rho_2 = \rho_1^+ - \rho_1^-$ and $\left|\rho_1^-\right| > 0$, then $(\rho_1 - \rho_2)^2 \leq 0$ in violation to our assumption. We conclude that $\rho_1 = \rho_2$. □

Remark 2.3. *The set*

$$\mathcal{P}_{\mathrm{dvol},\theta} := \left\{ \rho \in \Lambda^2 V \,\middle|\, \rho \wedge \rho = \mathrm{dvol}, \rho^+ = \lambda\theta \textit{ for } |\lambda| \geq 1 \right\}. \tag{10}$$

for a fixed volume form dvol *and a self-dual two-form* θ *was consider by Donaldson in [3] in the context of the following problem: How many symplectic structures* ρ *exist, such that* $\rho \wedge \rho = \mathrm{dvol}$ *for a prescribed volume form* dvol, ρ *is compatible with a prescribed almost complex structure and* ρ *lays in a given positive affine subspace of* $H^2(M, \mathbb{R})$*? The answer is that it is unique. The set* $\mathcal{P}_{\mathrm{dvol},\theta}$ *is a three-dimensional submanifold of* $\Lambda^2 V$ *with two components. The key property of this manifold established in Lemma 2.2 is called 'negative chords'.*

Lemma 2.4 (**The Linearization of** K)**.** *Let* ρ_s *be a path of nondegenerate 2-forms and* $\widehat{\rho} = \frac{d}{ds}\big|_{s=0} \rho_s$, $\rho = \rho_0$*. Then*

$$\frac{d}{ds}\bigg|_{s=0} \left(\frac{\rho_s^+}{u_s}\right) = \frac{1}{u} R^\rho \widehat{\rho}^{+^\rho}, \qquad u_s = \frac{\rho_s \wedge \rho_s}{2\mathrm{dvol}}, \qquad R^\rho w = w - \frac{w \wedge \rho}{\mathrm{dvol}_\rho}\rho. \tag{11}$$

In particular,

$$\widehat{\rho}^{+^\rho} = u R^\rho \frac{d}{ds}\bigg|_{s=0} \left(\frac{\rho_s^+}{u_s}\right).$$

Proof. We compute

$$\frac{d}{ds}\bigg|_{s=0} \left(\frac{\rho_s^+}{u_s}\right) = \frac{\widehat{\rho}^+}{u} - \frac{\widehat{\rho} \wedge \rho}{\mathrm{dvol}_\rho} \frac{\rho^+}{u} = \frac{1}{u} \left(R^\rho \widehat{\rho}\right)^+ = \frac{1}{u} R^\rho \widehat{\rho}^{+_\rho}.$$

For the last equality we used that the linear map $R^\rho : \Lambda^2 V \to \Lambda^2 V$ is an involution on $\Lambda^2 V$ for a 4-dimensional real vector space V and it maps Λ^+ to Λ^{+^ρ} and vice versa, where Λ^{+^ρ} is the space of self-dual 2-forms for the metric g^ρ (see [6] for a proof of these facts). □

Lemma 2.5 (Lie Derivative). *Let X be a vector field on M and ρ a non-degenerate two-form. Then*

$$(\mathcal{L}_X\rho)^{+^\rho} = uR^\rho\left(\mathcal{L}_X\frac{\rho^+}{u} - \frac{\mathcal{L}_X\mathrm{dvol}}{\mathrm{dvol}}\frac{\rho^+}{u} - \frac{1}{2}\left(\mathcal{L}_X *\right)\frac{\rho}{u}\right).$$

Proof. Let ψ_s, $s \geq 0$ be the family of diffeomorphisms on M generated by the vector field X. Then

$$\begin{aligned}
2\psi_s^*\frac{\rho^+}{u} &= \psi_s^*\left((\rho + *\rho)\frac{2\mathrm{dvol}}{\rho\wedge\rho}\right)\\
&= \left(\psi_s^*\rho + (\psi_s^* *)\left(\psi_s^*\rho\right)\right)\frac{2\psi_s^*\mathrm{dvol}}{\psi_s^*(\rho\wedge\rho)}\\
&= \left(\psi_s^*\rho + *\left(\psi_s^*\rho\right)\right)\frac{2\mathrm{dvol}}{\psi_s^*(\rho\wedge\rho)}\frac{2\psi_s^*\mathrm{dvol}}{2\mathrm{dvol}}\\
&\quad + \left((\psi_s^* *)\left(\psi_s^*\rho\right) - *\left(\psi_s^*\rho\right)\right)\frac{2\psi_s^*\mathrm{dvol}}{\psi_s^*(\rho\wedge\rho)}
\end{aligned}$$

Thus

$$\begin{aligned}
\left.\frac{d}{ds}\right|_{s=0}\left(2\psi_s^*\frac{\rho^+}{u}\right) &= \left.\frac{d}{ds}\right|_{s=0}\left(2\left(\psi_s^*\rho\right)^+\frac{2\mathrm{dvol}}{\psi_s^*\left(\rho\wedge\rho\right)}\right) + \frac{2\rho^+}{u}\left.\frac{d}{ds}\right|_{s=0}\frac{\psi_s^*\mathrm{dvol}}{\mathrm{dvol}}\\
&\quad + \left.\frac{d}{ds}\right|_{s=0}\left(\psi_s^* *\right)\frac{\rho}{u}.
\end{aligned}$$

Using Lemma 2.4 we then compute

$$\begin{aligned}
2\left(\mathcal{L}_X\rho\right)^{+^\rho} &= 2\left(\left.\frac{d}{ds}\right|_{s=0}\left({\psi_s}^*\rho\right)\right)^{+^\rho}\\
&= uR^\rho\left.\frac{d}{ds}\right|_{s=0}\left(2(\psi_s^*\rho)^+\frac{2\mathrm{dvol}}{\psi_s^*(\rho\wedge\rho)}\right)\\
&= uR^\rho\left(2\mathcal{L}_X\frac{\rho^+}{u} - \frac{2\rho^+}{u}\frac{\mathcal{L}_X\mathrm{dvol}}{\mathrm{dvol}} - \left(\mathcal{L}_X *\right)\frac{\rho}{u}\right)
\end{aligned}$$

This proves the lemma. □

Proof of Theorem 2.1. We prove (i). By Lemma 2.4 the linearization of K is given by

$$\widehat{K} : T_\rho\mathscr{S}^{k,p}_{a+\mathcal{H}_\rho} \to W^{k,p}(M,\Lambda^+) : \qquad \widehat{\rho} \mapsto \frac{1}{u}R^\rho\widehat{\rho}^{+^\rho}$$

We claim that this is an isomorphism. Then (i) follows from the inverse function theorem for Banach spaces. By Hodge theory for the operator $d^{*^\rho}d+dd^{*^\rho}$ every $\sigma \in \mathscr{S}^{k,p}_{a+\mathcal{H}^\rho}$ can be written as a sum $\sigma = \rho + d\lambda + h$ for a unique $\lambda \in W^{k+1,p}(M, T^*M)$ and a $h \in \mathcal{H}^\rho$ such that $\int_M d\lambda \wedge *^\rho h = 0$ and $d^{*^\rho}\lambda = 0$. Hence

$$\widehat{\rho} = d\lambda + h$$

for such unique λ and h. Now suppose $\widehat{K}\widehat{\rho} = 0$. It follows that

$$d^{(+\rho)}\lambda + h = \frac{1}{2}(d\lambda + *^\rho d\lambda) + h = 0.$$

Since h is closed, $dd^{(+\rho)}\lambda = 0$, and thus

$$0 = \int_M dd^{+^\rho}\lambda \wedge \lambda = -\int_M d^{(+\rho)}\lambda \wedge d\lambda = -\int_M \left|d^{(+\rho)}\lambda\right| \mathrm{dvol}.$$

Since $0 = \int_M d\lambda \wedge d\lambda = \int_M \left|d^{(+\rho)}\lambda\right| - \left|d^{(-\rho)}\lambda\right| \mathrm{dvol}$ it follows that

$$d\lambda = 0.$$

Together with $d^{*^\rho}\lambda = 0$ this implies that λ is harmonic and $h = 0$. This shows that $\widehat{K}$ is injective. Now let $\eta \in W^{k,p}(M, \Lambda^+)$. Since $W^{k,p}$ is closed under products and composition with smooth functions for $k - \frac{4}{p} > 0$, $u_0 R^\rho \eta$ is in $W^{k,p}(M, \Lambda^{+^\rho})$ and by Hodge theory there exists a unique one-form $\lambda \in W^{k+1,p}(M, T^*M)$ and a harmonic two-form h which is self-dual with respect to $*^\rho$ such that $u_0 R^\rho \eta = d\lambda + h$. Then $\widehat{K}(d\lambda + h) = \widehat{K}(u_0 R^\rho \eta) = \eta$ and this shows that $\widehat{K}$ is surjective. This proves (i).

We prove (ii). Let $\mathcal{H} \subset H^2(M; \mathbb{R})$ be a positive linear subspace. Let ρ_1, ρ_2 be two elements of $\mathscr{S}_{a+\mathcal{H}}$ such that

$$\eta := K(\rho_1) = K(\rho_2).$$

Define

$$\theta := \frac{\eta}{|\eta|^2} \in \Omega^+.$$

Then

$$\theta \wedge \theta = \frac{1}{|\eta|^2}\mathrm{dvol}, \qquad \rho_i^+ = \lambda_i \theta, \qquad \lambda_i := |\eta|^2 u_i, \qquad u_i = \frac{\rho_i \wedge \rho_i}{2\mathrm{dvol}}$$

and

$$\begin{aligned}(\rho_i - \theta)^2 &= \rho_i \wedge \rho_i - 2\rho_i^+ \wedge \theta + \theta^2 \\ &= 2u_i \mathrm{dvol} - 2\lambda_i \theta^2 + \theta^2 \\ &= 2u_i \mathrm{dvol} - 2\left|\eta\right|^2 u_i \frac{1}{\left|\eta\right|^2} \mathrm{dvol} + \theta^2 \\ &= \theta^2\end{aligned}$$

for $i = 1, 2$. It now follows from Lemma 2.2 that

$$(\rho_1 - \rho_2)^2 = ((\rho_1 - \theta) - (\rho_2 - \theta))^2 \leq 0$$

point wise. On the other hand, since $[\rho_1], [\rho_2] \in a + \mathcal{H}$ and $\mathcal{H}$ is a positive subspace of $H^2(M)$, we have

$$\int_M (\rho_1 - \rho_2)^2 \geq 0$$

and therefore $(\rho_1 - \rho_2)^2 = 0$. We conclude with Lemma 2.2 that $\rho_1 = \rho_2$. This proves (ii).

We prove (iii). Let $\rho \in \mathscr{S}^{k,p}$ and let $\mathcal{L}\rho$ be a Lie derivative of ρ in an arbitrary direction. By Cartan's formula the Lie derivative of ρ is exact and by Lemma 2.5

$$\begin{aligned}(d + d^{*^\rho})\mathcal{L}\rho &= d^{*^\rho}\mathcal{L}\rho \\ &= - *^\rho d *^\rho \mathcal{L}\rho \\ &= - *^\rho d \left((\mathcal{L}\rho)^{+^\rho} - (\mathcal{L}\rho)^{-^\rho}\right) \\ &= -2 *^\rho d(\mathcal{L}\rho)^{+^\rho} \\ &= - *^\rho d \left(uR^\rho \left(\mathcal{L}\frac{\rho^+}{u} - \frac{\mathcal{L}\mathrm{dvol}}{\mathrm{dvol}}\frac{\rho^+}{u} - \frac{1}{2}\frac{(\mathcal{L}*)\rho}{u}\right)\right).\end{aligned}$$

The right hand side is a term of the form

$$P_1(\frac{1}{u}, \rho)\partial^2 \frac{\rho^+}{u} + P_2(\frac{1}{u}, \rho)\partial\rho\partial\frac{\rho^+}{u} + P_3(\frac{1}{u}, \rho)\partial\rho$$

for polynoms P_1, P_2, P_3 with smooth coefficient functions in the indicated variables. It follows from elliptic regularity theory and the product estimates for Sobolev spaces of Lemma A.2 that $\mathcal{L}\rho \in W^{k,p}$ and that there exists polynomials $\mathfrak{p}_1, \mathfrak{p}_2$ independent of ρ with the following significance

$$\|\mathcal{L}\rho\|_{W^{k,p}} \leq \mathfrak{p}_1\left(C, \|\rho\|_{L^\infty}\right) \|\frac{\rho^+}{u}\|_{W^{k+1,p}} + \mathfrak{p}_2\left(C, \|\rho\|_{L^\infty}, \|\rho\|_{W^{k-1,p}}, \|\frac{\rho^+}{u}\|_{W^{k,p}}\right) \|\rho\|_{W^{k,p}}$$

for $C = \sup_{x\in M} \frac{1}{u(x)}$. Since the Lie derivative $\mathcal{L}\rho$ was arbitrary, the result follows. □

3 The Evolution Equation for $K(\rho)$

In view of Theorem 2.1 the Donaldson flow has an equivalent description on the space of self-dual two-forms, given by the evolution of $K(\rho) = \frac{\rho^+}{u}$. This evolution equation exposes the parabolic nature of the Donaldson flow and it is the key for the regularity theorems we will prove in the later sections.

To obtain a global formula we introduce the operator S^ρ,

$$S^\rho : \Omega^1 \to \Omega^+, \qquad S^\rho \lambda := -R^\rho d^{+^\rho} \lambda + u \nabla_{X_\lambda} \frac{\rho^+}{u}, \tag{12}$$

where $\lambda \in \Omega^1$, R^ρ is defined by (5) and $\rho(X_\lambda, \cdot) := \lambda$. We say $\omega_1, \omega_2, \omega_3 \in \Omega^+$ form a standart local frame of Ω^+ if and only if locally

$$\omega_i \wedge \omega_j = \begin{cases} 2\mathrm{dvol} & i = j \\ 0 & i \neq j \end{cases}.$$

Theorem 3.1 (**The Evolution of** $\frac{\rho^+}{u}$). **(i)** *Suppose ρ is a smooth solution to the Donaldson flow. Then the evolution of the 2-form $\frac{\rho^+}{u}$ is given by the equation*

$$\partial_t \frac{\rho^+}{u} = \frac{1}{u} R^\rho d^{+^\rho} *^\rho d\theta^\rho, \qquad u = \frac{\rho \wedge \rho}{2\mathrm{dvol}}, \qquad \theta^\rho = \frac{2\rho^+}{u} - \left|\frac{\rho^+}{u}\right| \rho. \tag{13}$$

Here R^ρ is defined by equation (5).

(ii) *Equation* (13) *is the same as*

$$\partial_t \frac{\rho^+}{u} = -\frac{2}{u} S^\rho \left(S^\rho\right)^{*^\rho} \frac{\rho^+}{u} + \nabla_{X_{(*^\rho d\theta^\rho)}} \frac{\rho^+}{u},$$

where S^ρ *is defined by* (12), $(S^\rho)^{*^\rho}$ *is the adjoint of* S^ρ *with respect to the inner products* $\int_M \langle \cdot, \cdot \rangle_{g^\rho} \mathrm{dvol}$ *on* Ω^1 *and* $\int_M \langle \cdot, \cdot \rangle_g \mathrm{dvol}$ *on* Ω^+ *and*

$$\rho\left(X_{(*^\rho d\theta^\rho)}, \cdot\right) := *^\rho d\theta^\rho.$$

(iii) *Let* $\omega_1, \omega_2, \omega_3$ *form a local standart frame of* Ω^+*. Then the evolution of the functions* $K_i := \frac{\rho \wedge \omega_i}{\mathrm{dvol}_\rho}$ *is given by*

$$\begin{aligned} \partial_t \sum_i K_i \omega_i = &- \sum_{i,j,k \text{ cyclic}} \left(\frac{1}{u} d^{*^\rho} dK_i - 2\{K_j, K_k\}_\rho - \rho\left(X_{K_i}, \sum_\ell J_\ell X_{K_\ell}\right)\right)\omega_i \\ &- \frac{1}{u} E^\rho_\omega K + E'^\rho_\omega K. \end{aligned} \tag{14}$$

Here X_H *denotes the Hamiltonian vector field of the function* H*. The bracket* $\{\cdot, \cdot\}_\rho$ *denotes the Poisson bracket with respect to the symplectic structure* ρ*.* E^ρ_ω *and* E'^ρ_ω *are error terms depending on the frame* $\omega_1, \omega_2, \omega_3$ *that vanish whenever* $\nabla \omega_i = 0$ *and are given by*

$$\begin{aligned} E^\rho_w f &:= \sum_{i,j} \langle (S^\rho)^{*^\rho} \omega_i, df_j \circ J^\rho_j \rangle_\rho \omega_i + 2 S^\rho \sum_j f_j \left(S^\rho \omega_j\right)^{*^\rho} \\ &\quad + \sum_{i,j,k \text{ cyclic}} *^\rho \left(df_j \wedge d\omega_k - df_k \wedge d\omega_j\right)\omega_i \\ E'^\rho_\omega f &:= \sum_{i,j} K_j df_i (X_{(S^\rho)^{*^\rho}\omega_j})\omega_i + \sum_i f_i \nabla_{X_{*^\rho d\theta^\rho}} \omega_i. \end{aligned} \tag{15}$$

for $f = (f_1, f_2, f_3) \in \Omega^0(M, \mathbb{R}^3)$*.*

(iv) *Assume the hyperKähler case. Then the evolution of the functions* $K_i = \frac{\rho \wedge \omega_i}{\mathrm{dvol}_\rho}$ *is given by*

$$\begin{aligned} &\partial_t \sum_i K_i \omega_i \\ &\qquad = - \sum_{i,j,k \text{ cyclic}} \left(\frac{1}{u} d^{*^\rho} dK_i - 2\{K_j, K_k\}_\rho - \rho\left(X_{K_i}, \sum_\ell J_\ell X_{K_\ell}\right)\right)\omega_i. \end{aligned}$$

Proof. See page 53. □

We need the following lemma on the properties of S^ρ and its adjoint $(S^\rho)^{*^\rho}$.

Lemma 3.2 (**The Operators** S^ρ**,** $(S^\rho)^{*^\rho}$**).** *Let* $\rho \in \mathscr{S}_a$*.*

(i) *The adjoint of* $S^\rho : \Omega^1(M) \to \Omega^+(M)$ *with respect to the inner product* $\int_M \langle \cdot, \cdot \rangle_{g^\rho} \mathrm{dvol}$ *on* Ω^1 *and* $\int_M \langle \cdot, \cdot \rangle_g \mathrm{dvol}$ *on* Ω^+ *is given by*

$$(S^\rho)^{*^\rho} \xi := -d^{*^\rho} (R^\rho \xi) + *^\rho \left(g \left(\nabla \frac{\rho^+}{u}, \xi \right) \wedge \rho \right), \tag{16}$$

where $g\left(\nabla \frac{\rho^+}{u}, \xi\right)$ *is the 1-form given by* $X \mapsto g\left(\nabla_X \frac{\rho^+}{u}, \xi\right)$ *for a vector field* X*.*

(ii) $(S^\rho)^{*^\rho} \frac{2\rho^+}{u} = *^\rho d\theta^\rho$*, where* $\theta^\rho = \frac{2\rho^+}{u} + \left|\frac{\rho^+}{u}\right|^2 \rho$*.*

(iii) *Let* $\omega \in \Omega^+$*. Then*

$$(S^\rho)^{*^\rho} \omega = *^\rho \left(d\omega - g \left(\nabla \omega, \frac{\rho^+}{u} \right) \wedge \rho \right)$$

In particular $(S^\rho)^{*^\rho} \omega$ *is independent of derivatives of* ρ *and if* $\nabla \omega = 0$*, then* $(S^\rho)^{*^\rho} \omega = 0$*.*

(iv) *The operator* $(S^\rho)^{*^\rho}$ *satisfies the following Leipniz rule,*

$$(S^\rho)^{*^\rho}(f\xi) = *^\rho(df \wedge R^\rho \xi) + f (S^\rho)^{*^\rho} \xi$$

for all $\xi \in \Omega^+$ *and* $f \in \Omega^0(M, \mathbb{R})$*.*

(v) *Let* $\omega_1, \omega_2, \omega_3$ *be a standard frame for* Ω^+ *with respect to the background metric* g *and* $f_i \in \Omega^0(M, \mathbb{R})$*, then*

$$(S^\rho)^{*^\rho} \left(\sum_i f_i \omega_i \right) = \sum_i \left(df_i \circ J_i^\rho + f_i (S^\rho)^{*^\rho} \omega_i \right).$$

(vi) *In the hyperKähler case,*

$$(S^\rho)^{*^\rho} \left(\sum_i f_i \omega_i \right) = \sum_i df_i \circ J_i^\rho.$$

(vii) *Let $\omega_1, \omega_2, \omega_3$ be a standard frame for Ω^+ with respect to the background metric g and $f_i \in \Omega^0(M, \mathbb{R})$, then*

$$2S^\rho (S^\rho)^{*^\rho} \left(\sum_j f_j \omega_j \right) \\ = \sum_{i,j,k \text{ cyclic}} \left(d^{*^\rho} df_i - u \left(\{f_j, K_k^\rho\}_\rho - \{K_j^\rho, f_k\}_\rho \right) \right) \omega_i + E_w^\rho f,$$

where $K_i^\rho := \frac{\omega_i \wedge \rho}{\mathrm{dvol}_\rho}$ and $E_\omega^\rho : \Omega^0(M, \mathbb{R}^3) \to \Omega^+$ is the linear first order differential operator given by (15)*. It vanishes whenever $\nabla \omega_i = 0$ for $i = 1, 2, 3$.*

Proof. We prove (i). To compute the adjoint of S^ρ let λ be a 1-form and $\xi \in \Omega^+$, then

$$\int_M \langle -d^{*^\rho} (R^\rho \xi) + *^\rho \left(g \left(\nabla \frac{\rho^+}{u}, \xi \right) \wedge \rho \right), \lambda \rangle_{g^\rho} \mathrm{dvol} \\ = \int_M \langle \xi, -R^\rho d^{+^\rho} \lambda \rangle_g \mathrm{dvol} + \int_M \langle \rho \left(Y_{g\left(\nabla \frac{\rho^+}{u}, \xi\right)}, \cdot \right), \rho(X_\lambda, \cdot) \rangle_{g^\rho} \mathrm{dvol},$$

where

$$g \left(Y_{g\left(\nabla \frac{\rho^+}{u}, \xi\right)}, \cdot \right) := g \left(\nabla \frac{\rho^+}{u}, \xi \right), \qquad \rho(X_\lambda, \cdot) := \lambda.$$

Here we used the identity

$$*^\rho \iota(X) \rho = -\rho \wedge g(X, \cdot)$$

in the last equation. By the definition of the Donaldson metric

$$\int_M \langle \rho \left(Y_{g\left(\nabla \frac{\rho^+}{u}, \xi\right)}, \cdot \right), \rho(X_\lambda, \cdot) \rangle_{g^\rho} \mathrm{dvol} = \int_M g \left(Y_{g\left(\nabla \frac{\rho^+}{u}, \xi\right)}, X_\lambda \right) \mathrm{dvol}_\rho \\ = \int_M g \left(\nabla_{X_\lambda} \frac{\rho^+}{u}, \xi \right) \mathrm{dvol}_\rho$$

Thus we have proved that

$$\int_M \langle -d^{*^\rho} (R^\rho \xi) + *^\rho \left(g \left(\nabla \frac{\rho^+}{u}, \xi \right) \wedge \rho \right), \lambda \rangle_{g^\rho} \mathrm{dvol} \\ = \int_M \langle \xi, -R^\rho d^{+^\rho} \lambda + u \nabla_{X_\lambda} \frac{\rho^+}{u} \rangle_g \mathrm{dvol}$$

for all self-dual two-forms $\xi \in \Omega^+(M)$ and one-forms $\lambda \in \Omega^1(M)$. This proves (i).

Part (ii) follows from the computation

$$\begin{aligned}
(S^\rho)^{*_\rho}\frac{2\rho^+}{u} &= *^\rho dR^\rho \frac{2\rho^+}{u} + *^\rho\left(2g\left(\nabla\frac{\rho^+}{u},\frac{\rho^+}{u}\right)\wedge\rho\right) \\
&= *^\rho d\left(R^\rho\frac{2\rho^+}{u} + \left|\frac{\rho^+}{u}\right|^2\rho\right) \\
&= *^\rho d\left(\frac{2\rho^+}{u} - \frac{\rho\wedge\frac{2\rho^+}{u}}{\mathrm{dvol}_\rho}\rho + \left|\frac{\rho^+}{u}\right|^2\rho\right) \\
&= *^\rho d\left(\frac{2\rho^+}{u} - \left|\frac{\rho^+}{u}\right|^2\rho\right) \\
&= *^\rho d\theta^\rho,
\end{aligned}$$

where we used that $*^\rho R^\rho\frac{2\rho^+}{u} = R^\rho\frac{2\rho^+}{u}$ in the first equation. This proves (ii).

We prove (iii). Let $\omega \in \Omega^+$. Observe that

$$\begin{aligned}
d\left(R^\rho\omega\right) &= d\left(\omega - \frac{\omega\wedge\rho}{\mathrm{dvol}_\rho}\rho\right) \\
&= d\omega - \left(g\left(\nabla\omega,\frac{\rho^+}{u}\right) + g\left(\omega,\nabla\frac{\rho^+}{u}\right)\right)\wedge\rho.
\end{aligned}$$

and hence

$$\begin{aligned}
(S^\rho)^{*_\rho}\omega &= *^\rho d\left(R^\rho\omega\right) + *^\rho\left(g\left(\nabla\frac{\rho^+}{u},\omega\right)\wedge\rho\right) \\
&= *^\rho\left(d\omega - \left(g\left(\nabla\omega,\frac{\rho^+}{u}\right)\wedge\rho\right)\right)
\end{aligned}$$

Since $\nabla\omega = 0$ implies $d\omega = 0$ it follows from the last equation that $(S^\rho)^{*_\rho}\omega = 0$. This proves (iii).

We prove (iv). Let $\xi \in \Omega^+$ and $f \in \Omega^0(M,\mathbb{R})$. Then

$$\begin{aligned}
(S^\rho)^{*_\rho}(f\xi) &= *^\rho d\left(R^\rho f\xi\right) + *^\rho\left(g\left(\nabla\frac{\rho^+}{u},f\xi\right)\wedge\rho\right) \\
&= *^\rho\left(df\wedge R^\rho\xi\right) + f *^\rho d\left(R^\rho\xi\right) + f *^\rho\left(g\left(\nabla\frac{\rho^+}{u},\xi\right)\wedge\rho\right) \\
&= *^\rho\left(df\wedge R^\rho\xi\right) + f(S^\rho)^{*_\rho}\xi.
\end{aligned}$$

This proves (iv).

We prove (v). It follows from (iv) that

$$\begin{aligned}(S^\rho)^{*^\rho}(\sum_i f_i\omega_i) &= \sum_i \left(*^\rho (df_i \wedge R^\rho \omega_i) + f_i (S^\rho)^{*^\rho}\omega_i\right)\\ &= \sum_i \left(df_i \circ J_i^\rho + f_i(S^\rho)^{*^\rho}\omega_i\right),\end{aligned}$$

where the last equality follows from identity (8). This proves (v).

(vi) follows directly from (v) since $(S^\rho)^{*^\rho}\omega_i = 0$ by (iii) for the hyperKähler structures $\omega_1, \omega_2, \omega_3$.

We prove (vii). It follows from (v) that in an standard frame $\omega_1, \omega_2, \omega_3$ for Ω^+, S^ρ is given by

$$S^\rho \lambda = \frac{1}{2}\sum_i \left(-d^{*^\rho}\left(\lambda \circ J_i^\rho\right) + \langle (S^\rho)^{*^\rho}\omega_i, \lambda\rangle_\rho\right)\omega_i.$$

for a 1-form λ. Again by (v)

$$\begin{aligned}2S^\rho\left(S^\rho\right)^{*^\rho}\sum_j f_j\omega_j &= 2S^\rho(\sum_j df_j \circ J_j^\rho + f_j(S^\rho)^{*^\rho}\omega_j)\\ &= \sum_{i,j} -d^{*^\rho}(df_j \circ J_j^\rho \circ J_i^\rho)\omega_i\\ &\quad + \sum_i \langle (S^\rho)^{*^\rho}\omega_i, \sum_j df_j \circ J_j^\rho\rangle_\rho \omega_i + 2S^\rho \sum_j f_j\left(S^\rho\right)^{*^\rho}\omega_j.\end{aligned}$$

Observe that

$$d\left(R^\rho\omega_i\right) = d\left(\omega_i - K_i^\rho \rho\right) = d\omega_i - dK_i^\rho \wedge \rho.$$

Then by identity (8) and the hyperKähler relations for J_i^ρ

$$\begin{aligned}&d*^\rho(\sum_j df_j \circ J_j^\rho \circ J_1^\rho) = d*^\rho\left(-df_1 - df_2 \circ J_3^\rho + df_3 \circ J_2^\rho\right)\\ &\quad = -d*^\rho df_1 + d\left(df_2 \wedge R^\rho\omega_3 - df_3 \wedge R^\rho\omega_2\right)\\ &\quad = -d*^\rho df_1 - df_2 \wedge dK_3^\rho \wedge \rho + df_3 \wedge dK_2^\rho \wedge \rho + df_2 \wedge d\omega_3 - df_3 \wedge d\omega_2\\ &\quad = -d*^\rho df_1 - \iota\left(X_{f_2}\right)\rho \wedge \iota\left(X_{K_3^\rho}\right)\rho \wedge \rho + \iota\left(X_{f_3}\right)\rho \wedge \iota\left(X_{K_2^\rho}\right)\rho \wedge \rho\\ &\qquad\quad + df_2 \wedge d\omega_3 - df_3 \wedge d\omega_2\\ &\quad = -d*^\rho df_1 - \{f_2, K_3^\rho\}_\rho \mathrm{dvol}_\rho - \{K_2^\rho, f_3\}_\rho \mathrm{dvol}_\rho + df_2 \wedge d\omega_3 - df_3 \wedge d\omega_2\end{aligned}$$

and hence

$$\sum_{i,j} -d^{*^\rho}(df_j \circ J_j^\rho \circ J_i^\rho)\omega_i = \sum_{i,j,k\,\mathrm{cyclic}} \left(d^{*^\rho} df_i - u\left(\{f_j, K_k^\rho\}_\rho - \{K_j^\rho, f_k\}_\rho\right)\right)\omega_i + \sum_{i,j,k\,\mathrm{cyclic}} *^\rho\left(df_j \wedge d\omega_k - df_k \wedge d\omega_j\right)\omega_i.$$

This proves (vii). □

We end the section with a proof of Theorem 3.1.

Proof of Theorem 3.1. We prove (i). Let ρ be a smooth solution to the Donaldson flow. Recall that the Donaldson flow equation is $\partial_t \rho = d *^\rho d\theta^\rho$. Then by equation (11)

$$\partial_t \frac{\rho^+}{u} = \frac{1}{u}(R^\rho \partial_t \rho)^+ = \frac{1}{u} R^\rho (\partial_t \rho)^{+^\rho} = \frac{1}{u} R^\rho d^{+^\rho} *^\rho d\theta^\rho.$$

Here we use that R^ρ maps Λ^+ to Λ^{+^ρ} and vise versa in the second equation. This proves (i).

We prove (ii). By the definition of the operator S^ρ given by (12) and Lemma 3.2 (ii),

$$-\frac{2}{u} S^\rho \left(S^\rho\right)^{*^\rho} \frac{\rho^+}{u} = -\frac{1}{u} S^\rho\left(*^\rho d\theta^\rho\right) = \frac{1}{u} R^\rho d^{+^\rho} *^\rho d\theta^\rho - \nabla_{X_{(*^\rho d\theta^\rho)}} \frac{\rho^+}{u}.$$

Together with (i), this proves (ii).

We prove (iii). First note that in a local standard frame $\omega_1, \omega_2, \omega_3$ for Ω^+ we have $\frac{2\rho^+}{u} = \sum_i K_i \omega_i$ and by Lemma 3.2 (ii) and (v)

$$*^\rho d\theta^\rho = \left(S^\rho\right)^{*^\rho} \frac{2\rho^+}{u} = \sum_i \left(dK_i \circ J_i^\rho + K_i (S^\rho)^{*^\rho} \omega_i\right).$$

Since

$$\rho(J_i X_{K_i}, \cdot) = \rho(X_{K_i}, J_i^\rho \cdot)$$

the vector field $X_{(*^\rho d\theta^\rho)}$ that satisfies $\rho(X_{(*^\rho d\theta^\rho)}, \cdot) = *^\rho d\theta^\rho$ is given by

$$\sum_j \left(J_j X_{K_j} + K_j X_{(S^\rho)^{*^\rho} \omega_j}\right).$$

Hence,

$$\begin{aligned}
\nabla_{X_{*^\rho d\theta^\rho}} \frac{2\rho^+}{u} &= \nabla_{X_{*^\rho d\theta^\rho}} \sum_i K_i \omega_i \\
&= \sum_i \left(dK_i(\sum_j J_j X_{K_j}) + \sum_j K_j dK_i(X_{(S^\rho)^{*\rho}\omega_j}) \right) \omega_i \\
&\quad + \sum_i K_i \nabla_{X_{*^\rho d\theta^\rho}} \omega_i \\
&= \sum_i \rho \left(X_{K_i}, \sum_j J_j X_{K_i} \right) \omega_i + \sum_{i,j} K_j dK_i(X_{(S^\rho)^{*\rho}\omega_j}) \omega_i \\
&\quad + \sum_i K_i \nabla_{X_{*^\rho d\theta^\rho}} \omega_i \\
&= \sum_i \rho \left(X_{K_i}, \sum_j J_j X_{K_i} \right) \omega_i + E'^{\rho}_{\omega} K
\end{aligned}$$

for $K = (K_1, K_2, K_3)$. By Lemma 3.2 (vii)

$$\begin{aligned}
\frac{2}{u} S^\rho \left(S^\rho\right)^{*\rho} \frac{2\rho^+}{u} &= \frac{2}{u} S^\rho \left(S^\rho\right)^{*\rho} \left(\sum_j K_j \omega_j \right) \\
&= \sum_{i,j,k \,\text{cyclic}} \left(\frac{1}{u} d^{*\rho} dK_i - 2\{K_j, K_k\}_\rho \right) \omega_i + \frac{1}{u} E^\rho_\omega K.
\end{aligned}$$

Thus, from

$$\partial_t \frac{2\rho^+}{u} = -\frac{2}{u} \left(S^\rho\right)^{*\rho} S^\rho \frac{2\rho^+}{u} + \nabla_{X^\rho_* d\theta^\rho} \frac{2\rho^+}{u}$$

we find that

$$\begin{aligned}
\partial_t \sum_i K_i \omega_i = &- \sum_{i,j,k \,\text{cyclic}} \left(\frac{1}{u} d^{*\rho} dK_i - 2\{K_j, K_k\}_\rho - \rho \left(X_{K_i}, \sum_\ell J_\ell X_{K_\ell} \right) \right) \omega_i \\
&- \frac{1}{u} E^\rho_\omega K + E'^{\rho}_{\omega}(K).
\end{aligned}$$

Finally, (iv) follows from (v), since $\nabla \omega_i = 0$ for the three hyperKähler structures $\omega_1, \omega_2, \omega_3$ and therefore the error terms E^ρ_ω and E'_ω vanish. $\square$

4 Regularity

In this section we prove that a solution to the Donaldson flow that is element of $L^2(I, W^{2,p}) \cap W^{1,2}(I, L^p)$ for $p > 4$ is as smooth as it's initial condition allows. In particular it is smooth if its initial conditions are smooth. The proof combines two insights. First, the regularity of $\frac{\rho^+}{u}$ determines the regularity of ρ. This is the content of theorem 2.1 (iii). Second, the evolution of $\frac{\rho^+}{u}$ is given by a parabolic operator, where the right hand side of the equation is essentially a product of two derivatives. This is the content of Theorem 3.1 (iii). This allows bootstrapping. The details are given in the next two theorems. The first theorem illustrates the ideas in the simpler case of a critical point.

Theorem 4.1 (Critical Point). *Let $p > 4$ and let $\rho \in W^{1,p}(M, \Lambda^2)$ be a critical point of the Donaldson flow. Then ρ is smooth.*

The next lemma will be needed in the bootstrapping process.

Lemma 4.2. *Let $p' > p > 4$. If $\rho \in W^{1,p}$ is a symplectic form with $\frac{\rho^+}{u} \in W^{1,p'}$ then $\rho \in W^{1,p'}$ as well.*

Proof. Let $\mathcal{L}\rho$ be a Lie derivative of ρ in an arbitrary direction. Let ρ_0 be a smooth nondegenerate form such that $\|\rho - \rho_0\|_{L^\infty} < \delta$ for a small $\delta > 0$. Then there exists a unique 1-form $\lambda \in W^{1,p}$ such that $d\lambda = \mathcal{L}\rho$ and $d^{*^{\rho_0}}\lambda = 0$. By elliptic regularity theory for the operator $d^{+^{\rho_0}} + d^{*^{\rho_0}}$ there exists a constant $c > 0$ such that

$$\|\lambda\|_{W^{1,p'}} \leq c\left(\|(d\lambda)^{+^{\rho_0}}\|_{L^{p'}} + \|d\lambda\|_{L^2}\right).$$

Since

$$\|(*^\rho - *^{\rho_0})d\lambda\|_{L^{p'}} \leq \|\rho - \rho_0\|_{L^\infty}\|d\lambda\|_{L^{p'}}$$

we have

$$\|(d\lambda)^{+^{\rho_0}}\|_{L^{p'}} \leq \|(d\lambda)^{+^\rho}\|_{L^{p'}} + \delta\|d\lambda\|_{L^{p'}}$$

Further, since

$$0 = \int_M d\lambda \wedge d\lambda = \int_M |d\lambda|^{+^\rho} - |d\lambda|^{-^\rho}.$$

we have

$$\|d\lambda\|_{L^2} \leq 2\|(d\lambda)^{+^\rho}\|_{L^2}$$

By Lemma 2.5,

$$(\mathcal{L}\rho)^{+^\rho} = uR^\rho\left(\mathcal{L}\frac{\rho^+}{u} - \frac{\mathcal{L}_X\mathrm{dvol}}{\mathrm{dvol}}\frac{\rho^+}{u} - \frac{1}{2}\frac{(\mathcal{L}*)\rho}{u}\right).$$

Therefore,

$$\begin{aligned}\|\lambda\|_{W^{1,p'}} &\le c\left(\|(\mathcal{L}\lambda)^{+^\rho}\|_{L^{p'}} + \delta\|d\lambda\|_{L^{p'}} + 2\|(d\lambda)^{+^\rho}\|_{L^2}\right)\\ &\le \mathfrak{p}_1(C,\|\rho\|_{L^\infty})\|\mathcal{L}\frac{\rho^+}{u}\|_{L^{p'}} + \mathfrak{p}_2(C,\|\rho\|_{L^\infty}) + c\delta\|\lambda\|_{W^{1,p'}}\end{aligned}$$

for

$$C := \sup_{x\in M}\frac{1}{u(x,t)}, \qquad u := \frac{\rho\wedge\rho}{2\mathrm{dvol}}.$$

Since this is true for an arbitrary Lie derivative of ρ and $\delta > 0$, it follows that

$$\rho \in W^{1,p'}.$$

This proves the lemma. □

We will need the following lemma on elliptic regularity of the operator $d^{*^\rho}\frac{d}{u} : C^\infty(M) \to C^\infty(M)$ in the case that ρ and thus the coefficients are not smooth.

Lemma 4.3 (**Elliptic Regularity).** *Let $p > 4$, $q > 1$ and $k \ge 0$. Let $\rho \in W^{k+1,p}(M,\Lambda^2)$ such that $\rho\wedge\rho > 0$. Assume there exists a constant $c_0 > 0$ such that for all $v, w \in C^\infty(M)$ and all $\epsilon > 0$ we can estimate*

$$\|\partial v\partial w\|_{W^{k,q}} \le c_0\|v\|_{W^{k+1,p}}\left(\frac{1}{\epsilon}\|w\|_{L^q} + \epsilon\|w\|_{W^{k+2,q}}\right). \tag{17}$$

Let $v \in W^{1,q}(M)$ and $f \in W^{k,q}(M)$ such that

$$\int_M \frac{1}{u}dv\wedge *^\rho d\varphi = \int_M f\varphi\mathrm{dvol}$$

for all $\varphi \in C^1(M)$. Then $v \in W^{k+2,q}(M)$ and there exists a constant $c = c(q,k,M,\|\rho\|_{W^{1,p}})$ such that

$$\|v\|_{W^{k+2,q}} \le c\left(\|f\|_{W^{k,q}} + \|v\|_{L^q}\right).$$

Proof. We only proof the case $k = 0$, the general case follows by induction over k. Choose coordinate charts for M and a subordinate partition of unity of M. Let $\psi \in C_0^\infty(M)$ be a cutoff function. Then we have

$$\int_M \frac{1}{u} d(\psi v) \wedge *^\rho d\varphi = \int_M f' \varphi \mathrm{dvol}$$

for all $\varphi \in C^1(M)$ and

$$f' := \psi f - \left(d^{*^\rho} \frac{d}{u} \psi \right) v + *^\rho \left(dv \wedge *^\rho \frac{d}{u} \psi \right).$$

Let $B \subset \mathbb{R}^4$ be a ball around zero. Let $\triangle$ be the Hodge laplacian on $\mathbb{R}^4$ with respect to the standart metric. We know from ellpitic regularity if $v \in W_0^{1,p}(B)$ is a weak solution to the equation

$$\triangle v = f$$

in the sense that for every $\varphi \in C_0^1(B)$ we have

$$\int_B dv \wedge * d\varphi = \int_B f \varphi \mathrm{dvol}$$

for an $f \in L^q(B)$ and $q > 1$, then $v \in W^{2,q}(B)$ and there exists a constant $c_1(q, B)$ such that

$$\|v\|_{W^{2,q}(B)} \leq c_1 \left(\|f\|_{L^q(B)} + \|v\|_{L^q(B)} \right).$$

We choose a coordinate chart such that the image of the support of ψ is contained in B. We can assume that the push forward of ρ under this coordinate chart equals the standart symplectic structure at $0 \in B$. If not we can always achieve this by a change of coordinates. Let us denote the pushforward of ρ under this coordinate by ρ_α. Further we denote by $\triangle^{\rho_\alpha}$ the operator $d^{*^\rho} \frac{d}{u}$ expressed in this chart, by v_α the function ψv expressed in this chart and by f'_α the function f' expressed in this chart. By estimate (17) there exists a polynomial with positive coefficents $\mathfrak{p}$ such that

$$\begin{aligned} &\|(\triangle^{\rho_\alpha} - \triangle)\, v_\alpha\|_{L^q} \\ &\quad \leq \mathfrak{p}(\|\rho_\alpha\|_{L^\infty}) \left(\|\rho_\alpha - \rho(0)\|_{L^\infty} \|v_\alpha\|_{W^{2,q}} + \|\partial \rho_\alpha \partial v_\alpha\|_{L^q} \right) \\ &\quad \leq \mathfrak{p}(\|\rho_\alpha\|_{L^\infty}) \Bigg(\|\rho_\alpha - \rho(0)\|_{L^\infty} \|v_\alpha\|_{W^{2,q}} \\ &\qquad\qquad + \|\rho_\alpha\|_{W^{1,p}} \left(\frac{1}{\epsilon} \|v_\alpha\|_{L^q} + \epsilon \|v_\alpha\|_{W^{2,q}} \right) \Bigg) \end{aligned} \tag{18}$$

Further, by interpolating $\|\partial v_\alpha\|_{L^q} \leq c\|v_\alpha\|_{L^q}^{\frac{1}{2}}\|v_\alpha\|_{W^{2,q}}^{\frac{1}{2}}$ with the Galgliardo-Nirenberg interpolation inequality A.1 we find

$$\|f'_\alpha\|_{L^q} \leq c_2\|f\|_{L^q} + c_3\left(\frac{1}{\epsilon}\|v_\alpha\|_{L^q} + \epsilon\|v_\alpha\|_{W^{2,q}}\right). \tag{19}$$

If v_α is a smooth solution to

$$\triangle^{\rho_\alpha} v_\alpha = f'_\alpha$$

then it also solves

$$\triangle v_\alpha = f'_\alpha + \left(\triangle - \triangle^{\rho_\alpha}\right) v_\alpha.$$

By choosing ϵ and the ball B small we see that there exits a constant $c_4 = c_4(q, \|\rho\|_{W^{1,p}})$ such that

$$\|v\|_{W^{2,q}} \leq c_4\left(\|f\|_{L^q} + \|v\|_{L^q}\right). \tag{20}$$

Here we are using that $W^{1,p}(M) \hookrightarrow C^0(M)$ for $p > 4$ and therefore $\|\rho_\alpha - \rho_0\|_{L^\infty(B)}$ is small for a small ball. If $v \in W^{1,q}(M)$ is a weak solution to $d^{*\rho}\frac{d}{u}v = f$ in the sense that

$$\int_M \frac{1}{u} dv \wedge *^\rho d\varphi = \int_M f\varphi \mathrm{dvol}$$

for all $\varphi \in C^1(M)$ then we approximate f and ρ by smooth functions f_k respectively smooth nondegenerate two-forms ρ_k, such that

$$\lim_{k\to\infty}\|f_k - f\|_{L^q} = 0, \qquad \lim_{k\to\infty}\|\rho - \rho_k\|_{W^{1,p}} = 0.$$

For each pair (f_k, ρ_k) we find by standart L^2-theory for elliptic operators a smooth function v_k that solves

$$d^{*\rho_k}\frac{d}{u_k}v_k = f_k.$$

The constant c_4 in (20) can be choosen uniform in k for big k and it follows that $\{v_k\}_{k\in\mathbb{N}}$ has a weakly convergent subsequence with limit $\bar{v} \in W^{2,q}$ that satisfies the estimate (20) and $d^{*\rho}\frac{d}{u}\bar{v} = f$. Hence

$$\int_M \frac{1}{u} d(v - \bar{v}) \wedge *^\rho d\varphi = 0$$

for all $\varphi \in C^1(M)$. By choosing a sequnce $\varphi_k \in C^1(M)$ such that φ_k converges to $(v - \bar{v})$ in $W^{1,2}(M)$ we see that $\bar{v} = v$. This proves the lemma. $\square$

Proof of Theorem 4.1. If ρ is a critical point of the Donaldson flow then it follows from Theorem 3.1 (iii) that for a local standard frame $\omega_1, \omega_2, \omega_3$ and

$$K_i = \frac{\rho \wedge \omega_i}{\mathrm{dvol}_\rho}$$

we have the equation

$$\frac{1}{u} d^{*^\rho} dK_i = -2\{K_j, K_k\}_\rho - \rho\left(X_{K_i}, \sum_\ell J_\ell X_{K_\ell}\right) + \frac{\left(\frac{1}{u}E_\omega^\rho - {E'_\omega}^\rho\right) \wedge \omega_i}{2\mathrm{dvol}}$$

and cyclic permutations of i, j, k. The right hand side of this equation consists of products of derivatives of the K_i functions times a polynomial in the ρ and $\frac{1}{u}$ variables plus lower order terms in the K_i functions times another polynomial of the same form. Thus, schematically we may write

$$d^{*^\rho} dK = P_1(\frac{1}{u}, \rho)\partial K \cdot \partial K + P_2(\frac{1}{u}, \rho)\partial K \tag{21}$$

Since $1 - \frac{4}{p} > 0$, $\rho \in C^0$ and since we assume that ρ is a symplectic structure we have $\sup_M \frac{1}{u} < \infty$. It follows that the L^∞-norms of P_1, P_2 are bounded. Using Hölder's inequality we see that the right hand side is element of $L^{\frac{p}{2}}$. For two functions $v, w \in C^\infty(M)$ we have the estimate

$$\|\partial v \partial w\|_{L^{\frac{p}{2}}} \leq \|\partial v\|_{L^p} \|\partial w\|_{L^p} \leq \|v\|_{W^{1,p}} \left(\frac{1}{\epsilon}\|w\|_{L^p} + \epsilon \|w\|_{W^{2,p}}\right) \tag{22}$$

where we used Hölders inequality in the first inequality and the Gagliardo-Nirenberg interpolation inequality A.1 in the second. It then follows from Lemma 4.3 that K is element of $W^{2,\frac{p}{2}}$. By Rellich's embedding theorem $W^{1,\frac{p}{2}} \hookrightarrow L^{p'}$ where

$$p' = \frac{4p}{8-p}$$

For $4 < p < 8$ we have

$$p' > p.$$

Thus $K \in W^{1,p'}$ and by Lemma 4.2 $\rho \in W^{1,p'}$. If we repeat this argument with p replaced by p', we find that $K \in W^{1,\frac{p''}{2}}$, $p'' > p'$ and

$$p'' - p' > p' - p.$$

Hence, eventually we find that $K \in W^{2,q}$ and $\rho \in W^{1,q}$ for all $q \geq p$. Now we can use Theorem 2.1 to see that $\rho \in W^{2,q}$ as well. This implies that the right hand side of equation (21) is element of $W^{1,q}$ and elliptic regularity gives us that $K \in W^{3,q}$. Now an obvious iteration of these arguments using elliptic regularity and Theorem 2.1, to deduce the regularity of ρ from the regularity of K, proves the theorem. $\square$

Let $T > 0$ and
$$I := [0, T) \subset \mathbb{R}.$$
For $p > 4$, and integers $k \geq 2$, $0 \leq r \leq \lfloor \frac{k}{2} \rfloor$ define the Sobolev space $W^{r,k,p}$ of functions on $I \times M$ such that the weak derivatives $\partial_t^s \partial^\ell$ exist and are bounded in the $L^p - L^2$-norm for $2s + \ell \leq k$, $s \leq r$,

$$W^{r,k,p}(M_I) := \\ L^2\left(I, W^{k,p}(M)\right) \cap W^{1,2}\left(I, W^{k-2,p}(M)\right) \cap \cdots \cap W^{r,2}\left(I, W^{k-2r,p}(M)\right).$$

This definition extends in an obvious way to functions from I to the space of sections of a vector bundle over M. In the case at hand the relevant vector bundle is the bundle of two-forms over M and the corresponding space will be denoted by
$$W^{r,k,p}(I, \Lambda^2).$$
If the vector bundle in question is clear, we will simply write $W^{r,k,p}$. The norm on this space is given by

$$\|u\|_{W^{r,k,p}} := \sum_{\substack{2s+\ell \leq k \\ s \leq r}} \left(\int_M \left(\|\partial_t^s u\|_{W^{\ell,p}} \right)^2 dt \right)^{\frac{1}{2}}. \tag{23}$$

Remark 4.4 (Besov Spaces). *1) The restriction map $\rho \mapsto \rho(t = 0, \cdot)$ extends to a bounded linear operator*

$$W^{r,k,p}(M_I, \Lambda^2) \to B_2^{k-1,p}(M, \Lambda^2),$$

where $B_2^{k-1,p}(M, \Lambda^2)$ denotes the Besov space (see [5]) with exponents p and $q = 2$.

2) This restriction map is surjective and it has a bounded right inverse.

3) The identity operator on real smooth functions on I to sections in the vector bundle Λ^2 over M with compact support extends to a bounded linear operator

$$W^{r,k,p}(M_I, \Lambda^2) \to C^0(I, B_2^{k-1,p}(M, \Lambda^2))$$

4) It holds

$$W^{k,2}(M, \Lambda^2) = B_2^{k,2}(M, \Lambda^2).$$

Theorem 4.5 (Flow Lines). *Let $\rho \in W^{1,2}(I, L^p) \cap L^2(I, W^{2,p})$ be a solution to the Donaldson flow for $p > 4$ with initial condition $\rho(t = 0, \cdot) = \rho_0$. For every integer $k \geq 1$ and all $4 < p' < p$ the following are equivalent:*

1) $\rho_0 \in B_2^{k,p'}(M, \Lambda^2)$.

2) $\rho \in W^{\frac{k+1}{2}, k+1, p'}(M, \Lambda^2)$.

The proof of this theorem uses parabolic regularity theory. In particular we use the 'maximal regularity' property of parabolic operators in divergence form. We refer to [7] for these results. The maximal regularity property is usually formulated for operators with time independent smooth coefficients for operators in divergence form, the Hodge laplacian being the archetypal example. The next lemma assures that the operator

$$d^{*_\rho} \frac{d}{u} : C^\infty(M) \to C^\infty(M)$$

has the maximal regularity property as well, even though it's coefficients depend on time and are non-smooth in our use case.

Lemma 4.6 (Maximal Regularity). *Let $p > 4$, $q > 1$. Let*

$$\rho \in W^{1,2}(\mathbb{R}, L^p(M, \Lambda^2)) \cap L^2(\mathbb{R}, W^{2,p}(M, \Lambda^2))$$

be a path of nondegenerate forms. Assume that there exists a constant $c_0 > 0$ such that for all $v, w \in C^\infty(M)$ and all $\epsilon > 0$ we can estimate

$$\|\partial v \partial w\|_{L^q} \leq c_0 \|v\|_{W^{1,p}} \left(\epsilon \|w\|_{W^{2,q}} + \frac{1}{\epsilon} \|w\|_{L^q} \right). \tag{24}$$

Then for all $v \in C_0^\infty(\mathbb{R}, C_0^\infty(M))$ there exists a constant $c(q, \|\rho\|_{L^\infty(\mathbb{R}, W^{1,p})}) > 0$ such that

$$\|\partial_t v\|_{L^2(\mathbb{R}, L^q)} \leq c \|\partial_t v + d^{*_\rho} \frac{d}{u} v\|_{L^2(\mathbb{R}, L^q)} + \|v\|_{L^2(\mathbb{R}, L^q)}.$$

Proof. Choose coordinate charts for $\mathbb{R} \times M$ and a subordinate partition of unity. Let $\psi \in C^\infty(\mathbb{R} \times M)$ be a cutoff function. Let $I \times B \in \mathbb{R} \times \mathbb{R}^4$ be a ball around $(0,0)$. Choose a coordinate chart such that the support of ψ is mapped into $I \times B$ and such that the pushforward of ρ under this coordinate chart at $(0,0)$ equals the standart symplectic structure on $\mathbb{R}^4$. We can always achieve this by a change of coordinates. Let us denote the pushforward of ρ under this coordinate chart by ρ_α and the operator $d^{*^\rho}\frac{d}{u}$ expresed in this chart by $\triangle^{\rho_\alpha}$. Further we denote by v_α the function ψv expressed in this chart. From maximal regularity for the standart Laplace operator $\triangle$ on $\mathbb{R} \times \mathbb{R}^4$ there exists a constant $c_1 = c_1(q) > 0$ such that

$$\begin{aligned}\|\partial_t v_\alpha\|_{L^2(\mathbb{R},L^q)} &\leq c_1\|\partial_t v_\alpha + \triangle v_\alpha\|_{L^2(\mathbb{R},L^q)} \\ &\leq c_1\|\partial_t v_\alpha + \triangle^{\rho_\alpha} v_\alpha\|_{L^2(\mathbb{R},L^q)} + \|(\triangle - \triangle^{\rho_\alpha})v_\alpha\|_{L^2(\mathbb{R},L^q)}\end{aligned}$$

By choosing the partition of unity such that ψ has small enough support and using the estimate (18) we find that there exists a constant

$$c_2 = c_2(q, \|\rho\|_{L^\infty(\mathbb{R},W^{1,p})})$$

such that

$$\|\partial_t v_\alpha\|_{L^2(\mathbb{R},L^q)} \leq c_2\|\partial_t v_\alpha + \triangle^{\rho_\alpha} v_\alpha\|_{L^2(\mathbb{R},L^q)} + \|v_\alpha\|_{L^2(\mathbb{R},L^q)}.$$

From this the global estimate

$$\|\partial_t v\|_{L^2(\mathbb{R},L^q)} \leq c_3\|\partial_t v + d^{*^\rho}\frac{d}{u}v\|_{L^2(\mathbb{R},L^2)} + \|v\|_{L^2(\mathbb{R},L^q)}$$

folows by choosing a partition of unity such that the previous estimates holds for all chart domains and by estimating additional first order terms appearing from the multiplication of v with the cutoff functions with the Gagliardo-Nirenberg interpolation inequality. This proves the lemma. □

Proof of Theorem 4.5. Let $\rho(t=0,\cdot) = \rho_0 \in B_2^{2,p}$ and let

$$\widetilde{\rho}_0 \in W^{1,2}(I, W^{1,p}) \cap L^2(I, W^{3,p})$$

be an extension of ρ_0 with $\widetilde{\rho}_0(t=0,\cdot) = \rho_0$. From Theorem 3.1 (iii) it follows that the evolution of the functions $K_i = \frac{\rho\wedge\omega_i}{\mathrm{dvol}_\rho}$ in a local standard frame $\omega_1, \omega_2, \omega_3$ is given by

$$\partial_t K_i + \frac{1}{u}d^{*^\rho}dK_i = P_1(\frac{1}{u},\rho)\partial K \cdot \partial K + P_2(\frac{1}{u},\rho)\partial K,$$

where $K = K_k$ for an arbitrary $k \in \{1,2,3\}$, ∂ is an arbitrary space derivative and $P_{1,2}(\frac{1}{u},\rho)$ are polynomials in the variables $\frac{1}{u}$ and ρ with coefficients that are smooth functions on the manifold. Let

$$\widetilde{K}_{i,0} := \frac{\omega_i \wedge \widetilde{\rho}_0}{\mathrm{dvol}_{\widetilde{\rho}_0}}, \qquad \widehat{K}_i := K_i - \widetilde{K}_{i,0}$$

The evolution of the functions $\widehat{K}_i$ is then given by the equations

$$\begin{aligned} \partial_t \widehat{K}_i + d^{*\rho}\frac{d}{u}\widehat{K}_i &= P_3(\frac{1}{u},\rho)\partial\rho\cdot\partial K + P_2(\frac{1}{u},\rho)\partial K \\ &\quad - (\partial_t + \frac{1}{u}d^{*\rho}d)\widetilde{K}_{0,i} \\ \widehat{K}_i(t=0,\cdot) &= 0, \end{aligned} \tag{25}$$

for a polynom P_3 with smooth coefficient functions. By assumption ρ is element of $W^{1,2}(I,L^p)\cap L^2(I,W^{2,p})$ and therefore $\rho, K \in C^0(I,W^{1,p})$. It follows with Hölder's inequality that

$$\partial\rho\cdot\partial K \in L^{\frac{q}{2}}(I,L^{\frac{p}{2}}) \qquad \forall q \geq 0.$$

Since $W^{1,p}\subset C^0$ for $p>4$, we have

$$\|P_{2,3}\|_{L^\infty(I,L^\infty)} < \infty.$$

The last term on the right handside of (25) is in $L^2(I,W^{1,p}) \subseteq L^2(I,L^\infty)$, hence the right hand side of (25) is element of $L^2(I,L^{\frac{p}{2}})$. As in the critical point case the estimate (22) is valid for any two functions $v,w \in C^\infty(M)$. Then by the ellipitic regularity Lemma 4.3 and the maximal regularity Lemma 4.6 we have,

$$\widehat{K} \in W^{1,2}\left(I,L^{\frac{p}{2}}\right) \cap L^2\left(I,W^{2,\frac{p}{2}}\right).$$

And by Rellich's theorem,

$$\widehat{K} \in C^0(I,W^{1,p'}), \qquad p' = \frac{4p}{8-p}.$$

For $4<p<8$ we have

$$p' > p.$$

Since $\widetilde{K}_0 \in C^0(I,W^{2,p}) \subseteq C^0(I,C^1)$ it follows that $K \in C^0(I,W^{1,p'})$. Then with Lemma 4.2 we find

$$\rho \in C^0(I,W^{1,p'}).$$

If we now repeat these arguments with p replaced by p' we find that $\widehat{K} \in C^0(I, W^{1,p''})$ with $p'' > p'$ and

$$p'' - p' > p' - p.$$

Hence, we eventually find that

$$\widehat{K}, K \in C^0(I, W^{1,q})$$

for all $q \geq 1$. In particular, the right hand side of (25) is element of $L^2(I, W^{1,p'''})$ for a $4 < p''' < p$ and

$$K \in W^{1,2}(I, W^{1,p'''}) \cap L^2(I, W^{3,p'''}).$$

We claim that the following implications hold for $k \geq 3$, $p > 4$,

$$K \in W^{\lfloor \frac{k}{2} \rfloor, k, p}, \rho \in W^{\lfloor \frac{k-1}{2} \rfloor, k-1, p} \Rightarrow \rho \in W^{\lfloor \frac{k}{2} \rfloor, k, p},$$
$$\rho, K \in W^{\lfloor \frac{k}{2} \rfloor, k, p} \Rightarrow P_1(\frac{1}{u}, \rho)\partial K \cdot \partial K + P_2(\frac{1}{u}, \rho)\partial K \in W^{\lfloor \frac{k-1}{2} \rfloor, k-1, p}$$
$$P_1(\frac{1}{u}, \rho)\partial K \cdot \partial K + P_2(\frac{1}{u}, \rho)\partial K \in W^{\lfloor \frac{k-1}{2} \rfloor, k-1, p},\ \rho \in W^{\lfloor \frac{k}{2} \rfloor, k, p},\ \rho_0 \in B_2^{k,p}$$
$$\Rightarrow K \in W^{\lfloor \frac{k+1}{2} \rfloor, k+1, p}.$$

The statement of the theorem then follows by induction over $k \geq 3$ and $p = p'''$. We prove the first implication. It follows from Theorem 2.1 that $\rho \in L^2(I, W^{k,p})$ under the stated assertions. Then from the Donaldson flow equation

$$\|\partial_t \rho\|_{W^{\lfloor \frac{k}{2} \rfloor - 1, k-2, p}} = \|d *^\rho d\theta^\rho\|_{W^{\lfloor \frac{k}{2} \rfloor - 1, k-2, p}}.$$

This expression is bounded because of the product estimates of Lemma A.2 in the case $k = 3$ and Corollary A.3 for higher k, since the expressions $*^\rho, \theta^\rho$ are just given by polynomials in the variables $\rho, \frac{1}{u}$. The second implication follows also by these product estimates in Sobolev spaces. To see the third implication, note that there exists an extension of ρ_0 to an element $\widetilde{\rho}_0$ such that $\widetilde{K}_0 \in W^{\lfloor \frac{k+1}{2} \rfloor, k+1, p}$. It follows from equation (25) and maximal regularity for the operator $\frac{1}{u} d^{*^\rho} d$ that $\widehat{K} \in W^{\lfloor \frac{k+1}{2} \rfloor, k+1, p}$ and hence so is K. This proves the theorem. □

The following is an immidiate corollary to Theorem 4.5.

Corollary 4.7 (Instant Regularity)**.** *Let $\rho \in L^2(I, W^{2,p}) \cap W^{1,2}(I, L^p)$ be a solution to the Donaldson flow. For all $4 < p' < p$ the following holds true. If $\rho(t = 0, \cdot) \in B_2^{k,p'}$ then the map $t \mapsto \rho(t, \cdot)$ is a continous map from (0,T) to $B_2^{k+1,p'}$. In particular, $\rho(t, \cdot)$ is smooth for all $t \in (0, T)$.*

Remark 4.8. *Theorem 4.5 can be refined using the more sophisticated machinery of maximal $L^p - L^q$-regularity and the theory of Besov spaces [5]. The correct space for the initial condition for a solution in the space*

$$L^q(I, W^{k,p}) \cap W^{1,q}(I, W^{k-2,p}) \cap \ldots \cap W^{k,q}(I, L^p)$$

is the Besov space $B_q^{s,p}$ for

$$s := k - \frac{2}{q}.$$

For $sp > 4$ a solution is continuous in space and time and this bound is sharp. The argument of Theorem 4.5 can be used to show that a solution in this space but initial conditions in $B_q^{s+1,p}$ is in

$$L^q(I, W^{k+1,p}) \cap W^{1,q}(I, W^{k-1,p}) \cap \ldots \cap W^{k,q}(I, W^{1,p}).$$

5 Local Existence

In this section we show that we can find solutions to the Donaldson flow that exist for a short time. The method of proof is essentially to apply Banach's fixed point theorem to the linearized flow equation and show that the difference between the linear and the non-linear operator has quadratic growth. We will find a solution in the set $\mathscr{S}_a^{1,2,p}$ for a $p > 4$ with initial condition $\rho(t = 0, \cdot)$ in the Besov space $B_2^{1,p}$ (see [5] for a definition). The set $\mathscr{S}_a^{1,2,p}$ is the set of paths of sections $\rho \in W^{1,2,p}(M_T, \Lambda^2)$, such that $\rho(t, \cdot)$ is a symplectic form and represents the cohomolgy class $a \in H^2$ for all $t \in [0, T)$, where $W^{1,2,p}(M_T, \Lambda^2)$ is the Banach space

$$W^{1,2,p}(M_T, \Lambda^2) := W^{1,2}(I, L^p(M, \Lambda^2)) \cap L^2(I, W^{2,p}(M, \Lambda^2)), \quad I := [0, T).$$

Theorem 5.1 (Short Time Existence). *Let $p > 4$. For all $\rho_0 \in B_2^{1,p}$ there exists a $T > 0$ such that there exists a unique solution $\rho \in \mathscr{S}_a^{1,2,p}$ on the interval $[0, T)$ to the Donaldson flow with $\rho(0, \cdot) = \rho_0$.*

Proof. See page 75 □

Consider the operator $d\frac{d^{*\rho_0}}{u_0} : d\Omega^1 \to d\Omega^1$. It is self-adjoint with respect to the inner product

$$\langle \widehat{\rho}_1, \widehat{\rho}_2 \rangle = \int_M \langle \widehat{\rho}_1, \widehat{\rho}_2 \rangle_{g^{\rho_0}} \mathrm{dvol}$$

on $d\Omega^1$ and the inner product defined by

$$\langle \xi_1, \xi_2 \rangle = \int_M \langle \xi_1, \xi_2 \rangle_{g^{\rho_0}} \mathrm{dvol}_{\rho_0}$$

on Ω^1. It generates a strongly continuous semigroup on $d\,(W^{1,p}(M, \Lambda^1))$ with domain $d\,(W^{3,p}(\Lambda^1))$ and it has the 'maximal regularity' property. In particular, there exists a unique solution

$$\widetilde{\rho}_0 \in L^2(I, W^{2,p}(M, \Lambda^2)) \cap W^{1,2}(I, L^p(M, \Lambda^2))$$

to the homogeneous Cauchy problem

$$\partial_t \widetilde{\rho}_0 + d\frac{d^{*\rho_0}}{u_0} \widetilde{\rho}_0 = 0, \qquad \widetilde{\rho}(t = 0, \cdot) = \rho_0. \tag{26}$$

Further there exists a unique solution

$$\widehat{\rho} \in L^2(I, W^{2,p}(M, \Lambda^2)) \cap W^{1,2}(I, L^p(M, \Lambda^2))$$

to the Cauchy problem

$$\partial_t \widehat{\rho} + d\frac{d^{*\rho_0}}{u_0} \widehat{\rho} = f, \qquad \widehat{\rho}(t = 0, \cdot) = 0 \tag{27}$$

for $f \in d(L^2(I, W^{1,p}))$. Let

$$B_r \subset d\left\{\lambda \in W^{1,3,p}(M, \Lambda^1) |\, \lambda(t = 0, \cdot) = 0\right\}$$

be a ball of radius $r > 0$ centered at zero. Given $\widehat{\rho} \in B_r$, let $\widehat{\rho}'$ be the unique solution to the equation

$$\partial_t \widehat{\rho}' + d\frac{d^{*\rho_0}}{u_0} \widehat{\rho}' = f(\widehat{\rho}), \qquad \widehat{\rho}(t = 0, \cdot) = 0$$

$$f(\widehat{\rho}) := \left(d *^{\rho} d\theta^{\rho} - d *^{\widetilde{\rho}_0} d\theta^{\widetilde{\rho}_0} + L_{\widetilde{\rho}_0} \widehat{\rho}\right) + d^{*\widetilde{\rho}_0} d\theta^{\widetilde{\rho}_0}$$
$$+ \left(-A^{\widetilde{\rho}_0} + d\frac{d^{*\rho_0}}{u_0} - d\frac{d^{*\widetilde{\rho}_0}}{\tilde{u}_0}\right) \widehat{\rho}$$

$$\rho := \widetilde{\rho}_0 + \widehat{\rho}, \qquad u_0 = \frac{\rho_0 \wedge \rho_0}{2\mathrm{dvol}}, \qquad \tilde{u}_0 := \frac{\widetilde{\rho}_0 \wedge \widetilde{\rho}_0}{2\mathrm{dvol}}, \qquad \theta^{\rho} = \frac{2\rho^+}{u} - \left|\frac{\rho^+}{u}\right|^2 \rho. \tag{28}$$

Here, $L_{\widetilde{\rho}_0}$ is the linearized gradient operator and $A^{\widetilde{\rho}_0}$ is the lower order part of this operator. Both operators are given in the following Lemma 5.2 . This defines us a map $\mathscr{R}$,

$$\mathscr{R} : dW^{1,3,p}(M,\Lambda^1) \to dW^{1,3,p}(M,\Lambda^1) : \qquad \widehat{\rho} \mapsto \widehat{\rho}'.$$

If $\widehat{\rho}$ is a fixed point of $\mathscr{R}$, then $\rho = \rho_0 + \widehat{\rho}$ is a solution to the Donaldson flow. We will show that $\mathscr{R}(B_r) \subseteq B_r$ for a small $r > 0$ and that $\mathscr{R}$ is a contraction on B_r. Theorem 5.1 will follow from these two claims by the Banach fixed point theorem.

In preparation for the proof of Theorem 5.1 we need to following four lemmas. The first is the computation of the linearized gradient operator. Recall that the gradient operator is given by

$$\mathscr{S}_a \to T\mathscr{S}_a : \qquad \rho \to d *^{\rho} d\theta^{\rho}.$$

Lemma 5.2 (The linearized Operator). *Let* $\rho \in \mathscr{S}_a$ *and* $\rho_s : (-1,1) \to \mathscr{S}_a$ *be a smooth path such that* $\rho_0 = \rho$ *and* $\frac{d}{ds}\big|_{s=0} \rho_s = \widehat{\rho}$. *Then*

$$\frac{d}{ds}\bigg|_{s=0} - d{*^{\rho_s}} d\theta^{\rho_s} =: L_\rho \widehat{\rho} = d\frac{d^{*^{\rho}}}{u}\widehat{\rho} + A^{\rho}\widehat{\rho} \tag{29}$$

where

$$A^{\rho}\widehat{\rho} := d *^{\rho} \left(\left(\frac{du}{u^2} \right) \wedge (\widehat{\rho} + *^{\rho}\widehat{\rho}) \right) + d *^{\rho} \left(d \left(\left| \frac{\rho^+}{u} \right|^2 \right) \wedge \widehat{\rho} \right) - d\hat{*}^{\rho,\widehat{\rho}} d\theta^{\rho} \tag{30}$$

and $\hat{*}^{\rho,\widehat{\rho}} : \Lambda^3 \to \Lambda^1$ *is the linearization of the the Hodge star operator* $*^{\rho_s}$ *at* ρ.

Proof. We proved in [6] that

$$\frac{d}{ds}\bigg|_{s=0} \theta^{\rho_s} = \frac{\widehat{\rho} + *^{\rho}\widehat{\rho}}{u} - \left| \frac{\rho^+}{u} \right|^2 \widehat{\rho}.$$

Since $d\widehat{\rho} = 0$,

$$\begin{aligned}
L_\rho \widehat{\rho} &= -d *^{\rho} d \left(\frac{d}{ds}\bigg|_{s=0} \theta^{\rho_s} \right) - d\hat{*}^{\rho,\widehat{\rho}} d\theta^{\rho} \\
&= -d *^{\rho} \frac{1}{u} d *^{\rho} \widehat{\rho} + d *^{\rho} \left(\left(\frac{du}{u^2} \right) \wedge (\widehat{\rho} + *^{\rho}\widehat{\rho}) \right) + d *^{\rho} \left(d \left(\left| \frac{\rho^+}{u} \right|^2 \right) \wedge \widehat{\rho} \right) \\
&\quad - d\hat{*}^{\rho,\widehat{\rho}} d\theta^{\rho} \\
&= d\frac{d^{*^{\rho}}}{u}\widehat{\rho} + A^{\rho}\widehat{\rho}.
\end{aligned}$$

This proves the lemma. $\square$

The content of the next lemma are the quadratic estimates necessary to proof existence of a solution to the Donaldson flow by the Banach fixed point theorem. We establish these estimates in terms of polynomials with positive real coefficients in several variables. We denote such a polynom by

$$\mathfrak{p}(x_1, \ldots, x_\ell) = \sum_{|\alpha| \le m} a_\alpha x^\alpha$$

where the sum runs over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{N}_0^\ell$ with $|\alpha| = \sum_{i=1}^\ell \alpha_i \le m$ and $a_\alpha \ge 0$ for all α.

Lemma 5.3 (**Quadratic Estimates**). *Let $p > 4$.*
(i) *There exist two polynomials $\mathfrak{p}_1, \mathfrak{p}_2$ with positive coefficients with the following significance. For all $\rho \in \mathscr{S}_a$ there exists a $\delta > 0$ such that for all $\widehat{\rho} \in d\Omega^1$ with $\|\widehat{\rho}\|_{L^\infty} < \delta$ and $\rho' := \rho + \widehat{\rho}$*

$$\begin{aligned}\|d *^{\rho'} d\theta^{\rho'} - d*^\rho d\theta^\rho - L_\rho \widehat{\rho}\|_{L^p} \le{}& \mathfrak{p}_1 (C, \|\rho\|_{W^{1,p}}, \|\widehat{\rho}\|_{W^{1,p}}) \|\widehat{\rho}\|_{W^{1,p}} \|\widehat{\rho}\|_{W^{2,p}} \\ &+ \mathfrak{p}_2 (C, \|\rho\|_{W^{1,p}}, \|\widehat{\rho}\|_{W^{1,p}}) \|\rho\|_{W^{2,p}} \|\widehat{\rho}\|^2_{W^{1,p}}.\end{aligned}$$

where

$$C := \sup_{\substack{x \in M \\ 0 \le s \le 1}} \frac{1}{u_{\rho(x) + s\widehat{\rho}(x)}}, \qquad u_\rho := \frac{\rho \wedge \rho}{2\mathrm{dvol}}.$$

(ii) *There exist two polynomials $\mathfrak{p}_3$, $\mathfrak{p}_4$ with positive coefficients with the following significance. For all $\rho \in \mathscr{S}_a$ there exists a $\delta > 0$ such that if $\rho_1 := \rho + \widehat{\rho}_1$, $\rho_2 := \rho + \widehat{\rho}_2$ for $\widehat{\rho}_1, \widehat{\rho}_2 \in d\Omega^1$ with $\|\widehat{\rho}_1\|_{L^\infty}, \|\widehat{\rho}_2\|_{L^\infty} < \delta$, then*

$$\begin{aligned}&\|d *^{\rho_1} d\theta^{\rho_1} - d *^{\rho_2} d\theta^{\rho_2} + L_\rho (\widehat{\rho}_1 - \widehat{\rho}_2)\|_{L^p} \\ &\le \mathfrak{p}_3 (C, \|\rho\|_{W^{1,p}}, \|\widehat{\rho}_1\|_{W^{1,p}}, \|\widehat{\rho}_2\|_{W^{1,p}}) (\|\widehat{\rho}_1\|_{W^{1,p}} + \|\widehat{\rho}_2\|_{W^{1,p}}) \|\widehat{\rho}_1 - \widehat{\rho}_2\|_{W^{2,p}} \\ &\quad + \mathfrak{p}_4 (C, \|\rho\|_{W^{1,p}}, \|\widehat{\rho}_1\|_{W^{1,p}}, \|\widehat{\rho}_2\|_{W^{1,p}}) (\|\rho\|_{W^{2,p}} + \|\widehat{\rho}_1\|_{W^{2,p}} + \|\widehat{\rho}_2\|_{W^{2,p}}) \\ &\qquad \cdot (\|\widehat{\rho}_1\|_{W^{1,p}} + \|\widehat{\rho}_2\|_{W^{1,p}}) \|\widehat{\rho}_2 - \widehat{\rho}_2\|_{W^{1,p}}.\end{aligned}$$

where

$$C := \sup_{\substack{x \in M \\ 0 \le s, s' \le 1}} \frac{1}{u_{\rho(x) + s(s'\widehat{\rho}_1(x) + (1-s')\widehat{\rho}_2(x))}}.$$

Proof. Let

$$\rho_s = \rho + s\widehat{\rho}.$$

Then,

$$\begin{aligned} d *^{\rho'} d\theta^{\rho'} - d *^{\rho} d\theta^{\rho} + L_\rho \widehat{\rho} &= \int_0^1 \frac{d}{ds}\left(d *^{\rho_s} d\theta^{\rho_s}\right) ds - \left.\frac{d}{ds}\right|_{s=0} d *^{\rho_s} d\theta^{\rho_s} \\ &= \int_0^1 \int_0^s \left(\frac{d^2}{d^2 s'} d *^{\rho_{s'}} d\theta^{\rho_{s'}}\right) ds' ds. \end{aligned} \tag{31}$$

The integral in this equation is to be understood point wise in each fibre of the vector bundle $\Lambda^2 T^* M$. By Minkowski's integral inequality, it is enough to show that $\|\frac{d^2}{d^2 s} d *^{\rho_s} d\theta^{\rho_s}\|_{L^p}$ is bounded in the specified way. We compute,

$$\begin{aligned} \frac{d^2}{d^2 s} d *^{\rho_s} d\theta^{\rho_s} &= \frac{d}{ds}\left(d\widehat{*}^{\rho_s,\widehat{\rho}} d\theta^{\rho_s} + d *^{\rho_s} d\left(\widehat{\theta}(\rho_s)\widehat{\rho}\right)\right) \\ &= d\widehat{\widehat{*}}^{\rho_s,\widehat{\rho},\widehat{\rho}} d\theta^{\rho_s} + 2 d\widehat{*}^{\rho_s,\widehat{\rho}} d\left(\widehat{\theta}(\rho_s)\widehat{\rho}\right) + d *^{\rho_s} d\left(\widehat{\widehat{\theta}}(\rho_s)\left(\widehat{\rho},\widehat{\rho}\right)\right), \end{aligned} \tag{32}$$

where

$$\begin{aligned} \widehat{*}^{\rho_s,\widehat{\rho}} &:= \frac{d}{ds} *^{\rho_s}, & \widehat{\widehat{*}}^{\rho_s,\widehat{\rho},\widehat{\rho}} &:= \frac{d^2}{d^2 s} *^{\rho_s}, \\ \widehat{\theta}(\rho_s)\widehat{\rho} &:= \frac{d}{ds}\theta^{\rho_s}, & \widehat{\widehat{\theta}}(\rho_s)\left(\widehat{\rho},\widehat{\rho}\right) &:= \frac{d^2}{d^2 s}\theta^{\rho_s}. \end{aligned}$$

In what follows we suppress the notation of constants appearing in inequalities. We estimate,

$$\begin{aligned} \|\frac{d^2}{d^2 s} d *^{\rho_s} d\theta^{\rho_s}\|_{L^p} &\le \|\widehat{\widehat{*}}^{\rho_s,\widehat{\rho},\widehat{\rho}} d\theta^{\rho_s}\|_{W^{1,p}} + \|\widehat{*}^{\rho_s,\widehat{\rho}} d\left(\widehat{\theta}(\rho_s)\widehat{\rho}\right)\|_{W^{1,p}} \\ &\quad + \|*^{\rho_s} d\left(\widehat{\widehat{\theta}}(\rho_s)\left(\widehat{\rho},\widehat{\rho}\right)\right)\|_{W^{1,p}} \\ &\le \|\widehat{\widehat{*}}^{\rho_s,\widehat{\rho},\widehat{\rho}}\|_{W^{1,p}} \|\theta^{\rho_s}\|_{W^{2,p}} + \|\widehat{*}^{\rho_s,\widehat{\rho}}\|_{W^{1,p}} \|\left(\widehat{\theta}(\rho_s)\widehat{\rho}\right)\|_{W^{2,p}} \\ &\quad + \|*^{\rho_s}\|_{W^{1,p}} \|\left(\widehat{\widehat{\theta}}(\rho_s)\left(\widehat{\rho},\widehat{\rho}\right)\right)\|_{W^{2,p}}, \end{aligned} \tag{33}$$

For ease of notation we don't number the different polynomials appearing in the next estimates. We estimate the first of these summands by

$$\begin{aligned}
\|\widehat{\ast}^{\rho_s,\widehat{\rho},\widehat{\rho}}\|_{W^{1,p}}\|\theta^{\rho_s}\|_{W^{2,p}} &\leq \mathfrak{p}\left(C,\|\rho\|_{W^{1,p}},\|\widehat{\rho}\|_{W^{1,p}}\right)\|\widehat{\rho}\|^2_{W^{1,p}}\|\theta^{\rho_s}\|_{W^{2,p}}\\
&\leq \mathfrak{p}\left(C,\|\rho\|_{W^{1,p}},\|\widehat{\rho}\|_{W^{1,p}}\right)\|\widehat{\rho}\|^2_{W^{1,p}}\\
&\qquad\cdot\mathfrak{p}(C,\|\rho\|_{W^{1,p}},\|\widehat{\rho}\|_{W^{1,p}})(1+\|\rho_s\|_{W^{2,p}}(1+\|\rho_s\|_{L^\infty}))\\
&\leq \mathfrak{p}\left(C,\|\rho\|_{W^{1,p}},\|\widehat{\rho}\|_{W^{1,p}}\right)\|\widehat{\rho}\|_{W^{1,p}}\|\widehat{\rho}\|_{W^{2,p}}\\
&\qquad+\mathfrak{p}(C,\|\rho\|_{W^{1,p}},\|\widehat{\rho}\|_{W^{1,p}})\|\rho\|_{W^{2,p}}\|\widehat{\rho}\|^2_{W^{1,p}}.
\end{aligned}$$

Here the first inequality uses the product estimates of Lemma A.2 (i) and the fact that $\widehat{\ast}^{\rho_s,\widehat{\rho},\widehat{\rho}}$ is quadratic in $\widehat{\rho}$. Further, by the definition of $\ast^{\rho_s}$, $\widehat{\ast}^{\rho_s}$ is a quadratic form given by a polynomial in the variables $\frac{1}{u_s}$ and ρ_s, where $u_s = \frac{\rho_s\wedge\rho_s}{2\mathrm{dvol}}$. Therefore we can estimate it's $W^{1,p}$-norm by a polynomial $\mathfrak{p}(C,\|\rho_s\|_{W^{1,p}})$ using Lemma A.2 (ii). The second inequality follows again from Lemma A.2 (ii) now using that θ^{ρ_s} is a polynomial in the variables $\frac{1}{u_s}$ and ρ_s. The third inequality follows by grouping the appearing summands accordingly and using $\|\widehat{\rho}\|_{L^\infty}\leq\|\widehat{\rho}\|_{W^{1,p}}$. With the same arguments we estimate

$$\begin{aligned}
\|\widehat{\ast}^{\rho_s,\widehat{\rho}}\|_{W^{1,p}}\|\left(\widehat{\theta}(\rho_s)\widehat{\rho}\right)\|_{W^{2,p}} &\leq \mathfrak{p}(C,\|\rho\|_{W^{1,p}},\|\widehat{\rho}\|_{W^{1,p}})\|\widehat{\rho}\|_{W^{1,p}}\|\left(\widehat{\theta}(\rho_s)\widehat{\rho}\right)\|_{W^{2,p}}\\
&\leq \mathfrak{p}(C,\|\rho\|_{W^{1,p}},\|\widehat{\rho}\|_{W^{1,p}})\|\widehat{\rho}\|_{W^{1,p}}\\
&\qquad\cdot\left(\|\widehat{\theta}(\rho_s)\|_{L^\infty}\|\widehat{\rho}\|_{W^{2,p}}+\|\widehat{\theta}(\rho_s)\|_{W^{2,p}}\|\widehat{\rho}\|_{L^\infty}\right)\\
&\leq \mathfrak{p}(C,\|\rho\|_{W^{1,p}},\|\widehat{\rho}\|_{W^{1,p}})\|\widehat{\rho}\|_{W^{1,p}}\|\widehat{\rho}\|_{W^{2,p}}\\
&\qquad+\mathfrak{p}(C,\|\rho\|_{W^{1,p}},\|\widehat{\rho}\|_{W^{1,p}})\|\widehat{\rho}\|_{W^{1,p}}\|\widehat{\rho}\|_{L^\infty}(1+\|\rho_s\|_{W^{2,p}}(1+\|\rho_s\|_{L^\infty}))\\
&\leq \mathfrak{p}\left(C,\|\rho\|_{W^{1,p}},\|\widehat{\rho}\|_{W^{1,p}}\right)\|\widehat{\rho}\|_{W^{1,p}}\|\widehat{\rho}\|_{W^{2,p}}\\
&\qquad+\mathfrak{p}(C,\|\rho\|_{W^{1,p}},\|\widehat{\rho}\|_{W^{1,p}})\|\rho\|_{W^{2,p}}\|\widehat{\rho}\|^2_{W^{1,p}}
\end{aligned}$$

and

$$
\begin{aligned}
\|*^{\rho_s}\|_{W^{1,p}} \| \left(\widehat{\widehat{\theta}}(\rho_s)\,(\widehat{\rho},\widehat{\rho}) \right) \|_{W^{2,p}} &\leq \mathfrak{p}(C, \|\rho\|_{W^{1,p}}, \|\widehat{\rho}\|_{W^{1,p}}) \\
&\cdot \left(\|\widehat{\widehat{\theta}}(\rho_s)\|_{W^{2,p}} \|\widehat{\rho}\|^2_{L^\infty} + \|\widehat{\widehat{\theta}}(\rho_s)\|_{L^\infty} \|\widehat{\rho}\|_{W^{2,p}} \|\widehat{\rho}\|_{L^\infty} \right) \\
&\leq \mathfrak{p}(C, \|\rho\|_{W^{1,p}}, \|\widehat{\rho}\|_{W^{1,p}}) \|\widehat{\rho}\|_{W^{1,p}} \|\widehat{\rho}\|_{W^{2,p}} \\
&\quad + \mathfrak{p}(C, \|\rho\|_{W^{1,p}}, \|\widehat{\rho}\|_{W^{1,p}}) \|\widehat{\rho}\|^2_{L^\infty} (1 + \|\rho_s\|_{W^{2,p}} (1 + \|\rho_s\|_{L^\infty})) \\
&\leq \mathfrak{p}\left(C, \|\rho\|_{W^{1,p}}, \|\widehat{\rho}\|_{W^{1,p}}\right) \|\widehat{\rho}\|_{W^{1,p}} \|\widehat{\rho}\|_{W^{2,p}} \\
&\quad + \mathfrak{p}(C, \|\rho\|_{W^{1,p}}, \|\widehat{\rho}\|_{W^{1,p}}) \|\rho\|_{W^{2,p}} \|\widehat{\rho}\|^2_{W^{1,p}}.
\end{aligned}
$$

This proves part (i).

We prove statement (ii). By (31),

$$
\begin{aligned}
&d *^{\rho_1} d\theta^{\rho_1} - d *^{\rho_2} d\theta^{\rho_2} - L_\rho \left(\widehat{\rho}_1 - \widehat{\rho}_2\right) \\
&\quad = d *^{\rho_1} d\theta^{\rho_1} - d *^{\rho} d\theta^{\rho} - L_\rho \widehat{\rho}_1 - (d *^{\rho_2} d\theta^{\rho_2} - d *^{\rho} d\theta^{\rho} - L_\rho \widehat{\rho}_2) \\
&\quad = \int_0^1 \int_0^s \left(\frac{d^2}{d^2 s'} d *^{\rho_{1,s'}} d\theta^{\rho_{1,s'}} - \frac{d^2}{d^2 s'} d *^{\rho_{2,s'}} d\theta^{\rho_{2,s'}} \right) ds' ds,
\end{aligned}
$$

where

$$\rho_{i,s'} := \rho + s'\widehat{\rho}_i$$

for $i = 1, 2$. It follows from equation (32) that we need to estimate the three terms

$$\|\widehat{\widehat{*}}^{\rho_{1,s},\widehat{\rho}_1,\widehat{\rho}_1} d\theta^{\rho_{1,s}} - \widehat{\widehat{*}}^{\rho_{2,s},\widehat{\rho}_2,\widehat{\rho}_2} d\theta^{\rho_{2,s}}\|_{W^{1,p}}, \tag{34}$$

$$\|\widehat{*}^{\rho_{1,s},\widehat{\rho}_1} d\left(\widehat{\theta}(\rho_{2,s})\widehat{\rho}_2\right) - \widehat{*}^{\rho_{2,s},\widehat{\rho}_2} d\left(\widehat{\theta}(\rho_{2,s})\widehat{\rho}_2\right)\|_{W^{1,p}}, \tag{35}$$

$$\|*^{\rho_{1,s}} d\left(\widehat{\widehat{\theta}}(\rho_{1,s})\,(\widehat{\rho}_1,\widehat{\rho}_2)\right) - *^{\rho_{2,s}} d\left(\widehat{\widehat{\theta}}(\rho_{2,s})\,(\widehat{\rho}_2,\widehat{\rho}_2)\right)\|_{W^{1,p}} \tag{36}$$

in the specified way. We estimate the first term by

$$
\begin{aligned}
&\|\widehat{\widehat{*}}^{\rho_{1,s},\widehat{\rho}_1,\widehat{\rho}_1} d\theta^{\rho_{1,s}} - \widehat{\widehat{*}}^{\rho_{2,s},\widehat{\rho}_2,\widehat{\rho}_2} d\theta^{\rho_{2,s}}\|_{W^{1,p}} \\
&\quad \leq \|\widehat{\widehat{*}}^{\rho_{1,s},\widehat{\rho}_1,\widehat{\rho}_1} d\theta^{\rho_{1,s}} - \widehat{\widehat{*}}^{\rho_{2,s},\widehat{\rho}_2,\widehat{\rho}_2} d\theta^{\rho_{1,s}}\|_{W^{1,p}} \\
&\qquad + \|\widehat{\widehat{*}}^{\rho_{2,s},\widehat{\rho}_2,\widehat{\rho}_2} d\theta^{\rho_{1,s}} - \widehat{\widehat{*}}^{\rho_{2,s},\widehat{\rho}_2,\widehat{\rho}_2} d\theta^{\rho_{2,s}}\|_{W^{1,p}} \\
&\quad \leq \|\widehat{\widehat{*}}^{\rho_{1,s},\widehat{\rho}_1,\widehat{\rho}_1} - \widehat{\widehat{*}}^{\rho_{2,s},\widehat{\rho}_2,\widehat{\rho}_2}\|_{W^{1,p}} \|\theta^{\rho_{1,s}}\|_{W^{2,p}} + \|\widehat{\widehat{*}}^{\rho_{2,s},\widehat{\rho}_2,\widehat{\rho}_2}\|_{W^{1,p}} \|\theta^{\rho_{1,s}} - \theta^{\rho_{2,s}}\|_{W^{2,p}}
\end{aligned}
$$

Then,

$$\|\widehat{\widehat{*}}^{\rho_{1,s},\widehat{\rho}_1,\widehat{\rho}_1} - \widehat{\widehat{*}}^{\rho_{2,s},\widehat{\rho}_2,\widehat{\rho}_2}\|_{W^{1,p}}$$
$$\leq \|\widehat{\widehat{*}}^{\rho_{1,s},\widehat{\rho}_1,\widehat{\rho}_1} - \widehat{\widehat{*}}^{\rho_{2,s},\widehat{\rho}_1,\widehat{\rho}_1} + \widehat{\widehat{*}}^{\rho_{2,s},\widehat{\rho}_1,\widehat{\rho}_1} - \widehat{\widehat{*}}^{\rho_{2,s},\widehat{\rho}_2,\widehat{\rho}_1} + \widehat{\widehat{*}}^{\rho_{2,s},\widehat{\rho}_2,\widehat{\rho}_1} - \widehat{\widehat{*}}^{\rho_{2,s},\widehat{\rho}_2,\widehat{\rho}_2}\|_{W^{1,p}}$$
$$\leq \|\widehat{\widehat{*}}^{\rho_{1,s}} - \widehat{\widehat{*}}^{\rho_{2,s}}\|_{W^{1,p}}\|\widehat{\rho}_1\|^2_{W^{1,p}}$$
$$+ \mathfrak{p}\left(C, \|\rho\|_{W^{1,p}}, \|\widehat{\rho}_1\|_{W^{1,p}}, \|\widehat{\rho}_2\|_{W^{1,p}}\right)\left(\|\widehat{\rho}_1\|_{W^{1,p}} + \|\widehat{\rho}_2\|_{W^{1,p}}\right)\|\widehat{\rho}_1 - \widehat{\rho}_2\|_{W^{1,p}}$$
$$\leq \mathfrak{p}\left(C, \|\rho\|_{W^{1,p}}, \|\widehat{\rho}_1\|_{W^{1,p}}, \|\widehat{\rho}_2\|_{W^{1,p}}\right)\left(\|\widehat{\rho}_1\|_{W^{1,p}} + \|\widehat{\rho}_2\|_{W^{1,p}}\right)\|\widehat{\rho}_1 - \widehat{\rho}_2\|_{W^{1,p}},$$

using

$$\|\widehat{\widehat{*}}^{\rho_{1,s}} - \widehat{\widehat{*}}^{\rho_{2,s}}\|_{W^{1,p}} \leq \|\int_0^1 \frac{d}{ds'}\widehat{\widehat{*}}^{(s'\rho_{1,s}+(1-s')\rho_{2,s})}\, ds'\|_{W^{1,p}}$$
$$\leq \mathfrak{p}\left(C, \|\rho\|_{W^{1,p}}, \|\widehat{\rho}_1\|_{W^{1,p}}, \|\widehat{\rho}_2\|_{W^{1,p}}\right)\|\widehat{\rho}_1 - \widehat{\rho}_2\|_{W^{1,p}}.$$

Further,

$$\|\theta^{\rho_{1,s}}\|_{W^{2,p}} \leq \mathfrak{p}\left(C, \|\rho\|_{L^\infty}, \|\widehat{\rho}_1\|_{L^\infty}\right)\left(1 + \|\rho_{1,s}\|_{W^{2,p}}\left(1 + \|\rho_{1,s}\|_{L^\infty}\right)\right)$$
$$\leq \mathfrak{p}\left(C, \|\rho\|_{L^\infty}, \|\widehat{\rho}_1\|_{L^\infty}\right)$$
$$+ \mathfrak{p}\left(, \|\rho\|_{L^\infty}, \|\widehat{\rho}_1\|_{L^\infty}\right)\left(\|\rho\|_{W^{2,p}} + \|\widehat{\rho}_1\|_{W^{2,p}}\right)$$

and

$$\|\widehat{\widehat{*}}^{\rho_{2,s},\widehat{\rho}_2,\widehat{\rho}_2}\|_{W^{1,p}} \leq \mathfrak{p}\left(C, \|\rho\|_{W^{1,p}}, \|\widehat{\rho}_2\|_{W^{1,p}}\right)\|\widehat{\rho}_2\|^2_{W^{1,p}}.$$

Further,

$$\|\theta^{\rho_{1,s}} - \theta^{\rho_{2,s}}\|_{W^{2,p}} \leq \|\int_0^1 \frac{d}{ds'}\theta^{s'\rho_{1,s}+(1-s')\rho_{2,s}}\, ds'\|_{W^{2,p}}$$
$$\leq \sup_{s'\in[0,1]}\Big(\|\widehat{\theta}(s'\rho_{1,s} + (1-s')\rho_{2,s})\|_{L^\infty}\|\widehat{\rho}_1 - \widehat{\rho}_2\|_{W^{2,p}}$$
$$+ \|\widehat{\theta}(s'\rho_{1,s} + (1-s')\rho_{2,s})\|_{W^{2,p}}\|\widehat{\rho}_1 - \widehat{\rho}_2\|_{L^\infty}\Big)$$
$$\leq \mathfrak{p}\left(C, \|\rho\|_{L^\infty}, \|\widehat{\rho}_1\|_{L^\infty}, \|\widehat{\rho}_2\|_{L^\infty}\right)\|\widehat{\rho}_1 - \widehat{\rho}_2\|_{W^{2,p}}$$
$$+ \mathfrak{p}\left(C, \|\rho\|_{L^\infty}, \|\widehat{\rho}_1\|_{L^\infty}, \|\widehat{\rho}_2\|_{L^\infty}\right)\|\widehat{\rho}_1 - \widehat{\rho}_2\|_{L^\infty}$$
$$\cdot \sup_{s'\in[0,1]}\left(1 + \|s'\rho_{1,s} + (1-s')\rho_{2,s}\|_{W^{2,p}}\left(1 + \|s'\rho_{1,s} + (1-s')\rho_{2,s}\|_{L^\infty}\right)\right)$$
$$\leq \mathfrak{p}\left(C, \|\rho\|_{L^\infty}, \|\widehat{\rho}_1\|_{L^\infty}, \|\widehat{\rho}_2\|_{L^\infty}\right)\|\widehat{\rho}_1 - \widehat{\rho}_2\|_{W^{2,p}}$$
$$+ \mathfrak{p}\left(C, \|\rho\|_{L^\infty}, \|\widehat{\rho}_1\|_{L^\infty}, \|\widehat{\rho}_2\|_{L^\infty}\right)\|\widehat{\rho}_1 - \widehat{\rho}_2\|_{L^\infty}$$
$$\cdot\left(\|\rho\|_{W^{2,p}} + \|\widehat{\rho}_1\|_{W^{2,p}} + \|\widehat{\rho}_2\|_{W^{2,p}}\right)$$

Combining these inequalities yields

$$\begin{aligned}
&\|\widehat{*}^{\rho_{1,s},\widehat{\rho}_1,\widehat{\rho}_1} d\theta^{\rho_{1,s}} - \widehat{*}^{\rho_{2,s},\widehat{\rho}_2,\widehat{\rho}_2} d\theta^{\rho_{2,s}}\|_{W^{1,p}} \\
&\qquad \leq \mathfrak{p}\left(C, \|\rho\|_{W^{1,p}}, \|\widehat{\rho}_1\|_{W^{1,p}}, \|\widehat{\rho}_2\|_{W^{1,p}}\right)\left(\|\widehat{\rho}_1\|_{W^{1,p}} + \|\widehat{\rho}_2\|_{W^{1,p}}\right)\|\widehat{\rho}_1 - \widehat{\rho}_2\|_{W^{2,p}} \\
&\qquad\quad + \mathfrak{p}\left(C, \|\rho\|_{W^{1,p}}, \|\widehat{\rho}_1\|_{W^{1,p}}, \|\widehat{\rho}_2\|_{W^{1,p}}\right)\left(\|\rho\|_{W^{2,p}} + \|\widehat{\rho}_1\|_{W^{2,p}} + \|\widehat{\rho}_2\|_{W^{2,p}}\right) \\
&\qquad\qquad\quad \cdot \left(\|\widehat{\rho}_1\|_{W^{1,p}} + \|\widehat{\rho}_2\|_{W^{1,p}}\right)\|\widehat{\rho}_1 - \widehat{\rho}_2\|_{W^{1,p}}
\end{aligned}$$

The terms (35) and (36) can be estimated with the same techniques. This proves statement (ii) and the lemma. □

We will need the following lemma to use the technique of freezing the coefficients for the operator $d\frac{d^{*\rho}}{u} : d\Omega^1 \to d\Omega^1$.

Lemma 5.4 (**Freezing the Coefficients**). *Let $p > 4$. There exists a polynomial with positive coefficients $\mathfrak{p}$ with the following significance. Let $\rho_0 \in B_2^{1,p}(M, \Lambda^2)$ be a nondegenerate 2-form and let*

$$\widetilde{\rho}_0 \in W^{1,2}(I, L^p(M, \Lambda^2)) \cap L^2(I, W^{2,p}(M, \Lambda^2))$$

be a path of nondegenerate 2-forms such that it is a continous extension of ρ_0 with $\widetilde{\rho}_0(t = 0, \cdot) = \rho_0$. Then

$$\begin{aligned}
&\|(d\frac{d^{*\widetilde{\rho}_0}}{\tilde{u}_0} - d\frac{d^{*\rho_0}}{u_0})\widehat{\rho}\|_{L^2(I,L^p)} \\
&\qquad \leq \mathfrak{p}(C, \|\widetilde{\rho}_0\|_{L^\infty(I,W^{1,p})}, \|\rho_0\|_{W^{1,p}}, T)\|\widehat{\rho}\|_{L^2(I,W^{2,p})}\|\widetilde{\rho}_0 - \rho_0\|_{L^\infty(I,W^{1,p})},
\end{aligned}$$

where

$$C := \sup_{\substack{(t,x)\in I\times M \\ s\in[0,1]}} \frac{1}{u_{s\widetilde{\rho}_0(t,x)+(1-s)\rho_0}}, \qquad I := [0, T).$$

Proof. For ease of notation we don't number the different polynomials appearing in the following estimates. Let $\widehat{\rho} \in d\Omega^2(M)$. In a local coordinate chart the expression $d\frac{d^{*\rho_0}}{u_0}\widehat{\rho}$ is given by

$$\frac{1}{\sqrt{|\det g^{\rho_0}|}} \sum_{a<b} \sum_{ij} \partial_i \left(\frac{\sqrt{|\det g^{\rho_0}|}}{u_0} (g^{\rho_0})^{ij}_{ab} \partial_j \widehat{\rho}_{ab} \right) dx^a \wedge dx^b,$$

where $(g^{\rho_0})^{cd}_{ab}$ are the coefficients of the inverse of the metric on the two-forms induced by g^{ρ_0}. Let

$$f_{\rho_0} := \frac{1}{\sqrt{|\det g^{\rho_0}|}}, \qquad h_{\rho_0} := \frac{\sqrt{|\det g^{\rho_0}|}}{u_0}(g^{\rho_0})^{ij}_{ab}.$$

Let $\widetilde{\rho}_0$ be a continuous extension of ρ_0 in $W^{1,2}(I, L^p) \cap L^2(I, W^{2,p})$. For a fixed $t \in I$ we estimate

$$\begin{aligned}
&\|f_{\widetilde{\rho}_0}\partial_i\left(h_{\widetilde{\rho}_0}\partial_j\widehat{\rho}_{ab}\right) - f_{\rho_0}\partial_i\left(h_{\rho_0}\partial_j\widehat{\rho}_{ab}\right)\|_{L^p} \\
&\quad \le \|(f_{\widetilde{\rho}_0} - f_{\rho_0})\,\partial_i\left(h_{\widetilde{\rho}_0}\partial_j\widehat{\rho}_{ab}\right)\|_{L^p} + \|f_{\rho_0}\left(\partial_i\left(h_{\widetilde{\rho}_0}\partial_j\widehat{\rho}_{ab}\right) - \partial_i\left(h_{\rho_0}\partial_j\widehat{\rho}_{ab}\right)\right)\|_{L^p}
\end{aligned}$$

Then

$$\begin{aligned}
&\|(f_{\widetilde{\rho}_0} - f_{\rho_0})\,\partial_i\left(h_{\widetilde{\rho}_0}\partial_j\widehat{\rho}_{ab}\right)\|_{L^p} \\
&\quad \le \|f_{\widetilde{\rho}_0} - f_{\rho_0}\|_{L^\infty}\|(h_{\widetilde{\rho}_0}\partial_j\widehat{\rho}_{ab})\|_{W^{1,p}} \\
&\quad \le \|f_{\widetilde{\rho}_0} - f_{\rho_0}\|_{L^\infty}\|h_{\widetilde{\rho}_0}\|_{W^{1,p}}\|\widehat{\rho}_{ab}\|_{W^{2,p}} \\
&\quad \le \mathfrak{p}\left(C, \|\widetilde{\rho}_0\|_{L^\infty}, \|\rho_0\|_{L^\infty}\right)\|\widetilde{\rho}_0 - \rho_0\|_{L^\infty}\|\widetilde{\rho}_0\|_{W^{1,p}}\|\widehat{\rho}\|_{W^{2,p}}
\end{aligned}$$

where

$$C := \sup_{\substack{(t,x)\in[0,T)\times M \\ s\in[0,1]}} \frac{1}{u_{s\widetilde{\rho}_0(t,x)+(1-s)\rho_0}}$$

and $\mathfrak{p}$ is a polynomial with positive coefficients in the indicated variables. In the same way we find that

$$\begin{aligned}
&\|f_{\rho_0}\left(\partial_i\left(h_{\widetilde{\rho}_0}\partial_j\widehat{\rho}_{ab}\right) - \partial_i\left(h_{\rho_0}\partial_j\widehat{\rho}_{ab}\right)\right)\|_{L^p} \\
&\qquad\qquad \le \mathfrak{p}\left(C, \|\widetilde{\rho}_0\|_{W^{1,p}}, \|\rho_0\|_{W^{1,p}}\right)\|h_{\widetilde{\rho}_0} - h_{\rho_0}\|_{W^{1,p}}\|\widehat{\rho}\|_{W^{2,p}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|f_{\widetilde{\rho}_0}\partial_i\left(h_{\widetilde{\rho}_0}\partial_j\widehat{\rho}_{ab}\right) - f_{\rho_0}\partial_i\left(h_{\rho_0}\partial_j\widehat{\rho}_{ab}\right)\|_{L^2(I,L^p)} \\
&\quad \le \mathfrak{p}\left(C, \|\widetilde{\rho}_0\|_{L^\infty(I,W^{1,p})}, \|\rho_0\|_{W^{1,p}}, T\right) \\
&\qquad \cdot \|\widehat{\rho}\|_{L^2(I,W^{2,p})}\left(\|f_{\widetilde{\rho}_0} - f_{\rho_0}\|_{L^\infty(I,W^{1,p})} + \|h_{\widetilde{\rho}_0} - h_{\rho_0}\|_{L^\infty(I,W^{1,p})}\right).
\end{aligned}$$

The functions f_ρ and h_ρ depend smoothly on the point wise values of ρ, in fact they are polynomials in the variables ρ and $\frac{1}{u}$. Therefore there exists a polynomial $\mathfrak{p}$ such that

$$\begin{aligned}
&\|f_{\widetilde{\rho}_0} - f_{\rho_0}\|_{L^\infty(I,W^{1,p})} + \|h_{\widetilde{\rho}_0} - h_{\rho_0}\|_{L^\infty(I,W^{1,p})} \\
&\qquad\qquad \le \mathfrak{p}\left(C, \|\rho_0\|_{W^{1,p}}\|\widetilde{\rho}_0\|_{L^\infty(I,W^{1,p})}\right)\|\widetilde{\rho}_0 - \rho_0\|_{L^\infty(I,W^{1,p})}.
\end{aligned}$$

This proves the lemma. □

The following lemma gives an estimate for the lower order term appearing in the linearization of the gradient.

Lemma 5.5 (**Lower Order Terms**). *Let $p > 4$. There exist polynomials $\mathfrak{p}_1, \mathfrak{p}_2$ with positive coefficients with the following significance. Let $\rho \in W^{1,2}(I, L^p(M, \Lambda^2)) \cap L^2(I, W^{2,p}(M, \Lambda^2))$ be a path of nondegenerate two-forms and let $A^\rho : d\Omega^1 \to d\Omega^1$ be the operator defined by* (30). *Then for all $\widehat{\rho} \in W^{1,2}(I, L^p(M, \Lambda^2)) \cap L^2(I, W^{2,p}(M, \Lambda^2))$,*

$$\begin{aligned}\|A^\rho \widehat{\rho}\|_{L^2(I,L^p)} \leq \Big(& T^{\frac{1}{2}} \mathfrak{p}_1 \left(C, \|\rho\|_{L^\infty(I,W^{1,p})}\right) \\ & + \mathfrak{p}_2 \left(C, \|\rho\|_{L^\infty(I,W^{1,p})}\right) \|\rho\|_{L^2(I,W^{2,p})} \Big) \|\widehat{\rho}\|_{L^\infty(I,W^{1,p})},\end{aligned}$$

where

$$I := [0, T), \qquad C := \sup_{(t,x) \in I \times M} \frac{1}{u(t,x)}, \qquad u := \frac{\rho \wedge \rho}{2\mathrm{dvol}}.$$

Proof. For a fixed $t \in I$ we can estimate

$$\begin{aligned}\|A^\rho \widehat{\rho}\|_{L^p} &\leq \|*^\rho \left(\left(\frac{du}{u^2} \right) \wedge (\widehat{\rho} + *^\rho \widehat{\rho}) \right) \|_{W^{1,p}} + \|*^\rho \left(d \left(\left| \frac{\rho^+}{u} \right|^2 \right) \wedge \widehat{\rho} \right) \|_{W^{1,p}} \\ &\quad + \|\hat{*}^{\rho, \widehat{\rho}} d\theta^\rho \|_{W^{1,p}} \\ &\leq \mathfrak{p} \left(C, \|\rho\|_{W^{1,p}}\right) \left(1 + \|\rho\|_{W^{2,p}} \left(1 + \|\rho\|_{L^\infty}\right)\right) \|\widehat{\rho}\|_{W^{1,p}}\end{aligned}$$

using Lemma A.2. Hence,

$$\begin{aligned}\|A^\rho \widehat{\rho}\|_{L^2(I,L^p)} \leq \Big(& T^{\frac{1}{2}} \mathfrak{p}_1 \left(C, \|\rho\|_{L^\infty(I,W^{1,p})}\right) \\ & + \mathfrak{p}_2 \left(C, \|\rho\|_{L^\infty(I,W^{1,p})}\right) \|\rho\|_{L^2(I,W^{2,p})} \Big) \|\widehat{\rho}\|_{L^\infty(I,W^{1,p})},\end{aligned}$$

for two polynomials $\mathfrak{p}_1$, $\mathfrak{p}_2$ with positive coefficients. This proves the lemma. □

We are now ready to give a proof of the local existence Theorem 5.1.

Proof of Theorem 5.1. Define $\widetilde{\rho}_0$ by (26) and f by (28). It follows with Lemma A.2 that for a fixed $t \in I$, there exists a polynomial with positive coefficients $\mathfrak{p}$ such that

$$\begin{aligned}\|d *^{\widetilde{\rho}_0} d\theta^{\widetilde{\rho}_0}\|_{L^p} &\le \|*^{\widetilde{\rho}_0} d\theta^{\widetilde{\rho}_0}\|_{W^{1,p}} \\ &\le \mathfrak{p}\left(C, \|\widetilde{\rho}_0\|_{W^{1,p}}\right)\left(1 + \|\widetilde{\rho}_0\|_{W^{2,p}}\left(1 + \|\widetilde{\rho}_0\|_{L^\infty}\right)\right)\end{aligned}$$

where

$$C := \sup_{(t,x)\in I\times M} \frac{1}{\tilde{u}_0(t,x)}, \qquad \tilde{u}_0 = \frac{\widetilde{\rho}_0 \wedge \widetilde{\rho}_0}{2\mathrm{dvol}}.$$

Hence,

$$\begin{aligned}&\|d *^{\widetilde{\rho}_0} d\theta^{\widetilde{\rho}_0}\|_{L^2(I,L^p)} \\ &\quad\le T^{\frac{1}{2}}\mathfrak{p}_1\left(C, \|\widetilde{\rho}_0\|_{L^\infty(I,W^{1,p})}\right) + \mathfrak{p}_2\left(C, \|\widetilde{\rho}_0\|_{L^\infty(I,W^{1,p})}\right)\|\widetilde{\rho}_0\|_{L^2(I,W^{2,p})}.\end{aligned}$$

Together with the quadtratic estimates of Lemma 5.3 and Lemmas 5.4 and 5.5 it follows that there exists polynomials with positive coefficients $\mathfrak{p}_1 \dots \mathfrak{p}_7$ with the following significance,

$$\begin{aligned}&\|f\|_{L^2(I,L^p)} \le \\ &\quad \mathfrak{p}_1\left(C, \|\widetilde{\rho}_0\|_{L^\infty(I,W^{1,p})}, \|\widehat{\rho}\|_{L^\infty(I,W^{1,p})}\right)\|\widehat{\rho}\|_{L^\infty(I,W^{1,p})}\|\widehat{\rho}\|_{L^2(I,W^{2,p})} \\ &\quad + \mathfrak{p}_2\left(C, \|\widetilde{\rho}_0\|_{L^\infty(I,W^{1,p})}, \|\widehat{\rho}\|_{L^\infty(I,W^{1,p})}\right)\|\widetilde{\rho}_0\|_{L^2(I,W^{2,p})}\|\widehat{\rho}\|^2_{L^\infty(I,W^{1,p})} \\ &\quad + T^{\frac{1}{2}}\mathfrak{p}_3\left(C, \|\widetilde{\rho}_0\|_{L^\infty(I,W^{1,p})}\right) + \mathfrak{p}_4(C, \|\widetilde{\rho}_0\|_{L^\infty(I,W^{1,p})})\|\widetilde{\rho}_0\|_{L^2(I,W^{2,p})} \\ &\quad + \left(T^{\frac{1}{2}}\mathfrak{p}_5\left(C, \|\widetilde{\rho}_0\|_{L^\infty(I,W^{1,p})}\right) + \mathfrak{p}_6\left(C, \|\widetilde{\rho}_0\|_{L^\infty(I,W^{1,p})}\right)\|\widetilde{\rho}_0\|_{L^2(I,W^{2,p})}\right) \\ &\qquad\quad \cdot \|\widehat{\rho}\|_{L^\infty(I,W^{1,p})} \\ &\quad + \mathfrak{p}_7\left(C, \|\widetilde{\rho}_0\|_{L^\infty(I,W^{1,p})}, \|\rho_0\|_{W^{1,p}}, T\right)\|\widehat{\rho}\|_{L^2(I,W^{2,p})}\|\widetilde{\rho}_0 - \rho_0\|_{L^\infty(I,W^{1,p})},\end{aligned} \tag{37}$$

where

$$C := \sup_{\substack{(t,x)\in I\times M\\ 0\le s\le 1}} \frac{1}{u_{\widetilde{\rho}_0(t,x)+s\widehat{\rho}(t,x)}}, \qquad u_\rho := \frac{\rho\wedge\rho}{2\mathrm{dvol}}.$$

Since $\widetilde{\rho}_0$ solves the homogenous Cauchy problem (26),

$$\|\widetilde{\rho}_0\|_{L^2(I,W^{2,p})} \le \|\rho_0\|_{B^{1,p}_2}$$

and by choosing T small we can get $\|\widetilde{\rho}_0\|_{L^2(I,W^{2,p})}$ arbitrary small. Also by choosing T small enough, we can garantee that $\|\widetilde{\rho}_0 - \rho_0\|_{L^\infty(I,W^{1,p})}$ is small and that $\widetilde{\rho}_0$ is nondegenerate for all $t \in I$. By choosing $r > 0$ small, we can garantee that $C < \infty$, since $\|\widehat{\rho}\|_{L^\infty(I,L^\infty)} \leq r$. It then follows from (37) that by choosing $r > 0, T > 0$ small we get that

$$\|\widehat{\rho}'\|_{L^2(I,W^{2,p})} + \|\widehat{\rho}'\|_{L^\infty(I,W^{1,p})} + \|\partial_t \widehat{\rho}'\|_{L^2(I,L^p)} \leq \|f\|_{L^2(I,L^p)} \leq r$$

and in particular $\mathscr{R}(B_r) \subseteq B_r$.

Now let $\widehat{\rho}_1, \widehat{\rho}_2 \in B_r$ and

$$\widehat{\rho}_1' := \mathscr{R}\left(\widehat{\rho}_1\right), \qquad \widehat{\rho}_2' := \mathscr{R}\left(\widehat{\rho}_2\right).$$

Then

$$\partial_t \left(\widehat{\rho}_1' - \widehat{\rho}_2'\right) + d\frac{d^{*\rho_0}}{u_0}\left(\widehat{\rho}_1' - \widehat{\rho}_2'\right) = f\left(\widehat{\rho}_1\right) - f\left(\widehat{\rho}_2\right).$$

where

$$\begin{aligned} f\left(\widehat{\rho}_1\right) - f\left(\widehat{\rho}_2\right) = &\left(d *^{\rho_1} d\theta^{\rho_1} - d *^{\rho_2} d\theta^{\rho_2} + L_{\widetilde{\rho}_0}\left(\widehat{\rho}_1 - \widehat{\rho}_2\right)\right) \\ &+ \left(-A^{\widetilde{\rho}_0} + d\frac{d^{*\rho_0}}{u_0} - d\frac{d^{*\widetilde{\rho}_0}}{\tilde{u}_0}\right)\left(\widehat{\rho}_1 - \widehat{\rho}_2\right) \end{aligned}$$

and

$$\rho_1 := \widetilde{\rho}_0 + \widehat{\rho}_1, \qquad \rho_2 := \widetilde{\rho}_0 + \widehat{\rho}_2, \qquad u_0 := \frac{\rho_0 \wedge \rho_0}{2\mathrm{dvol}}, \qquad \tilde{u}_0 := \frac{\widetilde{\rho}_0 \wedge \widetilde{\rho}_0}{2\mathrm{dvol}}.$$

It follows from Lemma 5.3 (ii), Lemma 5.5 and Lemma 5.4 that there exist polynomials with positive coefficients $\mathfrak{p}_1, \ldots, \mathfrak{p}_5$ with the following significance,

$$\begin{aligned} &\|f\left(\widehat{\rho}_1\right) - f\left(\widehat{\rho}_2\right)\|_{L^2(I,L^p)} \\ &\quad \leq \mathfrak{p}_1\Big(C, \|\widehat{\rho}_1\|_{L^\infty(I,W^{1,p})}, \|\widehat{\rho}_2\|_{L^\infty(I,W^{1,p})}, \|\widehat{\rho}_1\|_{L^2(I,W^{2,p})}, \|\widehat{\rho}_2\|_{L^2(I,W^{2,p})}, \\ &\qquad\qquad \|\widetilde{\rho}_0\|_{L^2(I,W^{2,p})}\Big) \cdot r \cdot \left(\|\widehat{\rho}_1 - \widehat{\rho}_2\|_{L^\infty(I,W^{1,p})} + \|\widehat{\rho}_1 - \widehat{\rho}_2\|_{L^2(I,W^{2,p})}\right) \\ &\quad + \mathfrak{p}_3\left(C, \|\widetilde{\rho}_0\|_{L^\infty(I,W^{1,p})}, \|\rho_0\|_{W^{1,p}}, T\right)\|\widetilde{\rho}_0 - \rho_0\|_{L^\infty(I,W^{1,p})}\|\widehat{\rho}_1 - \widehat{\rho}_2\|_{L^2(I,W^{2,p})} \\ &\quad + \Big(T^{\frac{1}{2}}\mathfrak{p}_4(C, \|\widetilde{\rho}_0\|_{L^\infty(I,W^{1,p})} + \mathfrak{p}_5\left(C, \|\widetilde{\rho}_0\|_{L^\infty(I,W^{1,p})}\right)\|\widetilde{\rho}_0\|_{L^2(I,W^{2,p})}\Big) \\ &\qquad\qquad \cdot \|\widehat{\rho}_1 - \widehat{\rho}_2\|_{L^\infty(I,W^{1,p})}, \end{aligned}$$

where

$$C := \sup_{(t,x)\in I\times M} \frac{1}{\tilde{u}_0(t,x)} + \sup_{(t,x)\in I\times M} \frac{1}{u_{\rho_0}(x) + s\left(s'\widehat{\rho}_1(t,x) + (1-s')\,\widehat{\rho}_2(t,x)\right)}$$

Therefore we can choose r and T small such that

$$\begin{aligned}\|\widehat{\rho}_1' - \widehat{\rho}_2'\|_{L^2(I,W^{2,p})} + \|\partial_t(\widehat{\rho}_1' - \widehat{\rho}_2')\|_{L^2(I,L^p)}& \\ \leq \frac{1}{2}\left(\|\widehat{\rho}_1 - \widehat{\rho}_2\|_{L^\infty(I,W^{1,p})} + \|\widehat{\rho}_1 - \widehat{\rho}_2\|_{L^2(I,W^{2,p})}\right)&\end{aligned}$$

Thus, $\mathscr{R}$ is a contraction on B_r. The theorem now follows from Banach's fixed point theorem. □

Remark 5.6. *If one would use the more heavy machinery of parabolic $L^p - L^q$ theory, one would find with the same line of arguments that there exists a solution to the Donaldson flow for short times in the space*

$$L^q(I, W^{2,p}) \cap W^{1,q}(I, L^p)$$

for the initial condition

$$\rho(t=0,\cdot) \in B_q^{s,p}, \quad s := 2 - \frac{2}{q}$$

for $sp > 4$. Here, $B_q^{s,p}$ denotes the Besov space [5]. This is the lowest possible condition at the regularity of the initial condition. It asserts that solutions are continuous in time and space.

6 Semiflow

We prove that the Donaldson flow is a local semiflow on the Besov space $B_2^{1,p}(M,\Lambda^2)$ restricted to symplectic forms representing a given cohomology class, i.e. for every symplectic form in $B_2^{1,p}(M,\Lambda^2)$ representing the cohomology class $a \in H^2(M;\mathbb{R})$ there is an open neighborhood $\mathscr{U} \subset B_{2,a}^{1,p}(M,\Lambda^2)$ of this element and a $T > 0$ such that the map

$$[0,T] \times \mathscr{U} \to \mathscr{S}_{a,2}^{1,p}(M,\Lambda^2): \qquad (t,\rho_0) \mapsto \rho(t,\cdot)$$

is smooth, where

$$\mathscr{S}_{a,2}^{1,p} := \left\{\rho \in B_2^{1,p}(M,\Lambda^2) | d\rho = 0,\ \rho\wedge\rho > 0,\ [\rho] = a\right\}.$$

and ρ is the unique solution to the Donaldson flow with initial condition $\rho(t=0,\cdot)=\rho_0$. Our argumentation for this result follows the one given in [1], section 2.2.

Theorem 6.1 (Semiflow). *The Donaldson flow is a local semiflow on* $\mathscr{S}^{1,p}_{2,a}(M,\Lambda^2)$.

Proof. See page 85. □

For the proof of the theorem we need the following two lemmas. Let $p>4$. Let $I=[0,T]$ be a closed interval of the reals. Let

$$\bar\rho \in L^2(I, W^{2,p}(M,\Lambda^2)) \cap W^{1,2}(I, L^p(M,\Lambda^2))$$

be a path of nondegenerate two-forms and let $L_{\bar\rho}$ be the operator defined by equation (29). Define the spaces

$$\widehat{\mathscr{X}} := \{\widehat\rho \in L^2(I, W^{2,p}(M,\Lambda^2)) \cap W^{1,2}(I, L^p(M,\Lambda^2))|\ \forall t : \widehat\rho(t,\cdot) \text{ exact}\}.$$

and

$$\widehat{\mathscr{Y}} := \{\widehat\eta \in L^2(I, L^p(M,\Lambda^2))|\ \forall t : \widehat\eta(t,\cdot) \text{ exact}\}$$

and consider the operator

$$\partial_t + L_{\bar\rho} : \widehat{\mathscr{X}} \to \widehat{\mathscr{Y}}.$$

We'll also need the following L^2 versions of the spaces $\widehat{\mathscr{X}}, \widehat{\mathscr{Y}}$,

$$\widehat{\mathscr{X}_2} := \{\widehat\rho \in L^2(I, W^{2,2}(M,\Lambda^2)) \cap W^{1,2}(I, L^2(M,\Lambda^2))|\ \forall t : \widehat\rho(t,\cdot) \text{ exact}\}.$$

and

$$\widehat{\mathscr{Y}_2} := \{\widehat\eta \in L^2(I, L^2(M,\Lambda^2))|\ \forall t : \widehat\eta(t,\cdot) \text{ exact}\}$$

The next lemma shows that the image of the operator $\partial_t + L_{\bar\rho}$ for a smooth $\bar\rho$ restricted to $\widehat{\mathscr{X}_2}$ is dense in $\widehat{\mathscr{Y}_2}$.

Lemma 6.2 (Dense Image). *Let $\bar\rho$ be a smooth path of smooth nondegenerate two-forms. There exists a constant $c_0>0$ such that for all $c>c_0$ the following holds. Assume $\widehat\eta \in \widehat{\mathscr{Y}_2}$ such that*

$$\int_I \int_M g^{\bar\rho}\left(\partial_t\widehat\varphi + L_{\bar\rho}\widehat\varphi + c\widehat\varphi, \widehat\eta\right) \mathrm{dvol} dt = 0 \tag{38}$$

for all exact $\widehat\varphi \in C_0^\infty(I, C_0^\infty(M,\Lambda^2))$. Then $\widehat\eta = 0$.

Proof. We follow the proof of Theorem 3.10 in [8]. The proof has 5 steps. Let $B^{\bar\rho_t} \in \Gamma(M, \mathrm{Aut}(\Lambda^2))$ be the automorphism valued section of M defined by the equation

$$g^{\bar\rho_t}(\eta,\sigma) =: g(\eta, B^{\bar\rho_t}\sigma)$$

for each fixed $t \in I$ and two-forms η, σ, where g is the background metric of M and $g^{\bar\rho}$ is as described in the introduction.

Step 1. $\widehat\eta \in W^{1,2}(I, (W^{2,2}(\Lambda^2))^*)$ and

$$\partial_t\widehat\eta - L^{*\bar\rho}_{\bar\rho}\widehat\eta - c\widehat\eta + (B^{\bar\rho})^{-1}(\partial_t B^{\bar\rho})\widehat\eta = 0, \tag{39}$$

where $L^{*\bar\rho}_{\bar\rho} : \widehat{\mathscr{Y}_2} \to L^2(I, (W^{2,2}(M,\Lambda^2))^*)$ is the formal adjoint of $L_{\bar\rho}$ with respect to the inner product (3).

For $\widehat\varphi \in C_0^\infty(I, C_0^\infty(M,\Lambda^2))$

$$\begin{aligned}
&\int_I\int_M g(\partial_t\widehat\varphi, B^{\bar\rho}\widehat\eta)\mathrm{dvol}dt\\
&= \int_I\int_M g^{\bar\rho}(\partial_t\widehat\varphi, \widehat\eta)\mathrm{dvol}dt\\
&= -\int_I\int_M g^{\bar\rho}(L_{\bar\rho}\widehat\varphi + c\widehat\varphi, \widehat\eta)\mathrm{dvol}dt\\
&= -\int_I\int_M g^{\bar\rho}(\widehat\varphi, L^{*\bar\rho}_{\bar\rho}\widehat\eta + c\widehat\eta)\mathrm{dvol}dt\\
&= -\int_I\int_M g\left(\widehat\varphi, B^{\bar\rho}\left(L^{*\bar\rho}_{\bar\rho}\widehat\eta + c\widehat\eta\right)\right)\mathrm{dvol}dt\\
&= -\int_I\int_M g\left(\int_0^t \partial_s\widehat\varphi(s,\cdot)\,ds, B^{\bar\rho}\left(L^{*\bar\rho}_{\bar\rho}\widehat\eta + c\widehat\eta\right)\right)\mathrm{dvol}\,dt\\
&= -\int_I\int_t^T\int_M g\left(\partial_s\widehat\varphi(s,\cdot), \left(B^{\bar\rho}\left(L^{*\bar\rho}_{\bar\rho}\widehat\eta + c\widehat\eta\right)\right)(t)\right)\mathrm{dvol}\,dtds\\
&= -\int_I\int_M g\left(\partial_t\widehat\varphi, \int_t^T\left(B^{\bar\rho}\left(L^{*\bar\rho}_{\bar\rho}\widehat\eta + c\widehat\eta\right)\right)(s)ds\right)\mathrm{dvol}\,dt.
\end{aligned}$$

Since the time derivatives of test functions $\varphi \in C_0^\infty(I, C_0^\infty(M,\Lambda^2))$ are dense in $L^2(I, L^2(M,\Lambda^2))$ it follows that

$$B^{\bar\rho}\widehat\eta = -\int_t^T\left(B^{\bar\rho}\left(L^{*\bar\rho}_{\bar\rho}\widehat\eta + c\widehat\eta\right)\right)(s)ds$$

and therefore

$$(\partial_t B^{\bar\rho})\widehat\eta + B^{\bar\rho}\partial_t\widehat\eta = B^{\bar\rho}\left(L^{*\bar\rho}_{\bar\rho}\widehat\eta + c\widehat\eta\right).$$

This proves Step 1.

For ease of notation we define

$$L'_{\bar\rho} := L^{*\bar\rho}_{\bar\rho} + c\,\mathrm{id} - (B^{\bar\rho})^{-1}(\partial_t B^{\bar\rho})$$
$$\widetilde L_{\bar\rho}(t,\cdot) := L'_{\bar\rho(T-t,\cdot)}.$$

Now choose a smooth cutoff function $\theta : \mathbb{R} \to \mathbb{R}$ such that $\theta(t) = 0$ for $t \in \mathbb{R} \setminus [0,1]$, $\theta(t) \geq 0$ for all $t \in \mathbb{R}$ and $\int \theta = 1$. For $\delta > 0$ define $\theta_\delta(t) := \delta^{-1}\theta(\delta^{-1}t)$.

Step 2. Extend $\widehat\eta$ with zero outside of I and define $\widetilde\eta(t,\cdot) := \widehat\eta(T-t,\cdot)$. For $\delta > 0$ sufficiently small we have

$$\widetilde\eta_\delta := \theta_\delta * \widetilde\eta \in L^2(\mathbb{R}, W^{2,2}(M,\Lambda^2)) \cap W^{1,2}(\mathbb{R}, L^2(M,\Lambda^2)).$$

First note that the highest order term of $\widetilde L_{\bar\rho_t}$ equals

$$d\frac{d^{*\bar\rho_{T-t}}}{u_{T-t}} : \{\widehat\rho \in W^{2,2}(M,\Lambda^2) \,|\, \widehat\rho \text{ exact}\} \to \{\widehat\eta \in L^2(M,\Lambda^2) \,|\, \widehat\eta \text{ exact}\}$$

for a fixed t (where $\bar\rho_t$ is defined). If $c > c_0 > 0$ is big enough, $\widetilde L_{\bar\rho}$ is positive definite on the exact two-forms in $W^{1,2}(M,\Lambda^2)$ and it follows from the Lax-Milgram theorem and the elliptic regularity Lemma 4.3 that this operator is invertible on the exact two-forms in $L^2(M,\Lambda^2)$. By Step 1,

$$\partial_t\widetilde\eta(t,\cdot) = -\partial_t\widehat\eta(T-t,\cdot) = -(L'_{\bar\rho}\widehat\eta)(T-t,\cdot) = -\widetilde L_{\bar\rho}(t,\cdot)\widetilde\eta(t,\cdot).$$

We multiply this equation with $-(\widetilde L_{\bar\rho})^{-1}$ to find

$$\widetilde\eta = -(\widetilde L_{\bar\rho})^{-1}\partial_t\widetilde\eta.$$

The convolution with θ_δ is then given by

$$\begin{aligned}
\widetilde\eta_\delta &= -\theta_\delta * \left((\widetilde L_{\bar\rho})^{-1}\partial_t\widetilde\eta\right)\\
&= -\,(\partial_t\theta_\delta) * \left((\widetilde L_{\bar\rho})^{-1}\widetilde\eta\right) - \theta_\delta * \left((\widetilde L_{\bar\rho})^{-1}(\partial_t\widetilde L_{\bar\rho})(\widetilde L_{\bar\rho})^{-1}\widetilde\eta\right)\\
&= -\,(\partial_t\theta_\delta) * \left((\widetilde L_{\bar\rho})^{-1}\widetilde\eta\right) + \theta_\delta * \left((\widetilde L_{\bar\rho})^{-1}\widetilde\zeta\right)
\end{aligned}$$

where

$$\widetilde{\zeta} := -(\widetilde{L}_{\bar\rho})^{-1}(\partial_t \widetilde{L}_{\bar\rho})(\widetilde{L}_{\bar\rho})^{-1}\widetilde{\eta}$$

and we're using the identity $\theta * (u\partial_t v) = (\partial_t\theta) * (uv) - \theta * ((\partial_t u)v)$. This proves Step 2.

Step 3. There exists a constant $c > 0$ independent of δ such that

$$\|\partial_t\widetilde{\eta}_\delta + \widetilde{L}_{\bar\rho}\widetilde{\eta}_\delta\|_{L^2(\mathbb{R},L^2)} \le c$$

for all sufficiently small $\delta > 0$.

With Step 2 and the identity $\partial_t\widetilde{\eta}_\delta = (\partial_t\theta_\delta) * \widetilde{\eta}$ it follows that

$$\begin{aligned}
&\partial_t\widetilde{\eta}_\delta + \widetilde{L}_{\bar\rho}\widetilde{\eta}_\delta \\
&\quad = (\partial_t\theta_\delta) * \widetilde{\eta} - \widetilde{L}_{\bar\rho}\,(\partial_t\theta_\delta) * \left((\widetilde{L}_{\bar\rho})^{-1}\widetilde{\eta}\right) + \widetilde{L}_{\bar\rho}\theta_\delta * \left((\widetilde{L}_{\bar\rho})^{-1}\widetilde{\zeta}\right) \\
&\quad = -\widetilde{L}_{\bar\rho}\left(-\widetilde{L}_{\bar\rho}^{-1}(\partial_t\theta_\delta) * \widetilde{\eta} + (\partial_t\theta_\delta) * \left((\widetilde{L}_{\bar\rho})^{-1}\widetilde{\eta}\right)\right) + \widetilde{L}_{\bar\rho}\theta_\delta * \left((\widetilde{L}_{\bar\rho})^{-1}\widetilde{\zeta}\right).
\end{aligned}$$

Since

$$\bar\rho \in L^2(I, W^{2,p}(M,\Lambda^2)) \cap W^{1,2}(I, L^p(M,\Lambda^2)) \hookrightarrow C^0(I, W^{1,p}(M,\Lambda^2)$$

the operator norms of $L_{\bar\rho}, \widetilde{L}_{\bar\rho}$ are bounded on I and so is the norm of $\widehat{\zeta}$. Therefore the second term on the right is bounded in $L^2(\mathbb{R}, L^2(M,\Lambda^2))$, uniformly in δ. For the first term on the right hand side we have the estimate

$$\begin{aligned}
&\|\left(\widetilde{L}_{\bar\rho}^{-1}(\partial_t\theta_\delta) * \widetilde{\eta} - ((\partial_t\theta_\delta) * \left((\widetilde{L}_{\bar\rho})^{-1}\widetilde{\eta}\right)\right)(t)\|_{W^{2,2}} \\
&\qquad \le \|\int_{t-\delta}^{t+\delta} \frac{1}{\delta}\dot\theta\left(\frac{t-s}{\delta}\right)\frac{\widetilde{L}_{\bar\rho}^{-1}(t) - \widetilde{L}_{\bar\rho}^{-1}(s)}{\delta}\widetilde{\eta}(s)\,ds\|_{W^{2,2}} \\
&\qquad \le c\int_{\mathbb{R}}\left|\frac{1}{\delta}\dot\theta\left(\frac{t-s}{\delta}\right)\right|\|\widetilde{\eta}(s)\|_{L^2}\,ds
\end{aligned}$$

Here we're using that $\widetilde{L}_{\bar\rho}^{-1}(s) = L'^{-1}_{\bar\rho(T-s)}$ depends smoothly on $\bar\rho(T-s)$. Then it follows with Young's inequality that

$$\begin{aligned}
&\|-\widetilde{L}_{\bar\rho}\left(-\widetilde{L}_{\bar\rho}^{-1}(\partial_t\theta_\delta) * \widetilde{\eta} + ((\partial_t\theta_\delta) * \left((\widetilde{L}_{\bar\rho})^{-1}\widetilde{\eta}\right)\right)\|_{L^2(\mathbb{R},L^2)} \\
&\qquad\qquad \le c\|\partial_t\theta\|_{L^1(\mathbb{R})}\|\widetilde{\eta}\|_{L^2(\mathbb{R},L^2)}.
\end{aligned}$$

This proves Step 3.

Step 4. $\widehat{\eta} \in L^2(I, W^{2,2}(M, \Lambda^2)) \cap W^{1,2}(I, L^2(M, \Lambda^2))$ and $\partial_t \widehat{\eta} - L'_{\bar{\rho}} \widehat{\eta} = 0$.

As in Lemma 4.6 we can see that the operator $\partial_t + \widetilde{L}_{\bar{\rho}}$ has the maximal regularity property. Then it follows from this and Step 2 that there exists a weakly converging subsequence $\widetilde{\eta}_\delta$ in $L^2(I, W^{2,2}(M, \Lambda^2)) \cap W^{1,2}(I, L^2(M, \Lambda^2))$ that converges to a limit $\widetilde{\eta}_*$ as δ goes to zero. At the same time $\widehat{\eta}_\delta$ converges strongly to $\widetilde{\eta}$ in $L^2(I, \Lambda^2(I, \Lambda^2))$ and therefore

$$\widetilde{\eta}_* = \widetilde{\eta} \in L^2(I, W^{2,2}(M, \Lambda^2)) \cap W^{1,2}(I, L^2(M, \Lambda^2))$$

and

$$\widehat{\eta} \in L^2(I, W^{2,2}(M, \Lambda^2)) \cap W^{1,2}(I, L^2(M, \Lambda^2)).$$

That $\partial_t \widehat{\eta} - L'_{\bar{\rho}} \widehat{\eta} = 0$ then follows from equation (39). This proves Step 4.

Step 5. $\widehat{\eta} = 0$.

By Step 4 we have $\widehat{\eta} \in C^0(I, W^{1,2}(M, \Lambda^2))$ and therefore

$$\int_M g^{\bar{\rho}_T} (L_{\bar{\rho}} \widehat{\varphi} + c\widehat{\varphi}, \widehat{\eta}_T) \mathrm{dvol} = 0$$

for all $\widehat{\varphi} \in C_0^\infty(M, \Lambda)$. It follows that $\widehat{\eta}_T \in W^{1,2}(M, \Lambda^2)$ is a weak solution to the equation

$$L_{\bar{\rho}}^{*^{\bar{\rho}}} \widehat{\eta}_T + c\widehat{\eta}_T = 0.$$

Since $L_{\bar{\rho}}^{*^{\bar{\rho}}} + c\,\mathrm{id}$ is injective for $c > c_0$ big enough, this implies that $\widehat{\eta}_T = 0$. But this means that $\widetilde{\eta}$ is a solution to the Cauchy problem

$$\partial_t \widetilde{\eta} + \widetilde{L}_{\bar{\rho}} \widetilde{\eta} = 0, \qquad \widetilde{\eta}(0, \cdot) = 0.$$

The standart L^2 parabolic estimate shows that such a solution is unique for $c > 0$ big enough and hence $\widetilde{\eta} = \widehat{\eta} = 0$. This proves the lemma. □

Lemma 6.3 (Cauchy Problem). *For every* $\widehat{f} \in \widehat{\mathscr{Y}}$ *there exists a unique solution* $\widehat{\rho} \in \widehat{\mathscr{X}}$ *to the Cauchy problem*

$$\partial_t \widehat{\rho} + L_{\bar{\rho}} \widehat{\rho} = \widehat{f}, \qquad \widehat{\rho}(t = 0, \cdot) = 0. \tag{40}$$

Proof. Let $\widehat{\rho}$ and $\widehat{f}$ be smooth path of smooth exact two-forms with support on a compact subset of $I \times \mathbb{R}^4$ satisfying

$$\partial_t \widehat{\rho} + \triangle \widehat{\rho} = \widehat{f}, \qquad \widehat{\rho}(t = 0, \cdot) = 0,$$

where $\triangle$ denotes the standard Laplace operator of the standard metric of $\mathbb{R}^4$. Then we have from standard parabolic L^2-theory the estimate

$$\|\widehat{\rho}\|_{L^2(I,W^{2,2})} + \sup_{t\in I}\|\widehat{\rho}\|_{W^{1,2}} + \|\partial_t\widehat{\rho}\|_{L^2(I,L^2)} \leq c\|f\|_{L^2(I,L^2)}.$$

Choosing charts and a subordinate partition of unity for $I \times \mathbb{R}^4$ and using the estimate (18) with $q = 2$ shows that there exists a polynomial with positive coefficients $\mathfrak{p}(\|\bar{\rho}\|_{L^\infty(I,W^{1,p})})$ such that for all exact $\widehat{\rho} \in L^2(I, W^{2,2}) \cap W^{1,2}(I, L^2)$ solving (40)

$$\|\partial_t\widehat{\rho}\|_{L^2(I,L^2)} + \sup_{t\in I}\|\widehat{\rho}\|_{L^2(M,\Lambda^2)} + \|\widehat{\rho}\|_{L^2(I,W^{1,2}(M,\Lambda^2))} \leq \mathfrak{p}\left(\|\widehat{f}\|_{L^2(I,L^2)} + \|\widehat{\rho}\|_{L^2(I,L^2)}\right).$$

This estimate shows that the operator $\partial_t + L_{\bar{\rho}} + c\,\mathrm{id}$ from the exact two-forms in $L^2(I, W^{2,2}) \cap W^{1,2}(I, L^2)$ with initial condition $\widehat{\rho}(t = 0, \cdot) = 0$ to the exact two-forms in $L^2(I, L^2)$ is injective and has closed image, if the constant $c > 0$ is choosen suitably big. If $\bar{\rho}$ is a smooth path of nondegenerate two-forms by Lemma 6.2 it's image is dense and hence this operator is surjective. Therefore for $\bar{\rho}$ smooth

$$\partial_t\widehat{\rho} + L_{\bar{\rho}}\widehat{\rho} + c\widehat{\rho} = \widehat{f}, \qquad \widehat{\rho}(t = 0, \cdot) = 0$$

has a unique exact solution in $L^2(I, W^{2,2}) \cap W^{1,2}(I, L^2)$ for every exact two-form $\widehat{f} \in L^2(I, L^2)$. From standard parabolic regularity theory it follows that this solution is smooth when $\bar{\rho}$ and $\widehat{f}$ are smooth. Then $e^{ct}\widehat{\rho}$ is the unique smooth solution to (40).

We argue exactly as in the maximal regularity Lemma 4.6 for the operator $d^{*\rho}\frac{d}{u}$ that there exists a constant $c = c(\|\bar{\rho}\|_{L^\infty(I,W^{1,p})}) > 0$ such that for all exact $\widehat{\rho} \in C^\infty(I, C^\infty(M, \Lambda^2))$,

$$\|\partial_t\widehat{\rho}\|_{L^2(I,L^p)} \leq c(\|\partial_t\widehat{\rho} + L_{\bar{\rho}}\widehat{\rho}\|_{L^2(I,L^p)} + \|\widehat{\rho}\|_{L^2(I,L^p)}). \tag{41}$$

The assumption (24) can easily be seen to be satisfied for $p > 4$. The additional lower order terms introduced by $A^{\bar{\rho}}$ can be controlled by the Gagliardo-Nirenberg interpolation inequality. To find a solution to (40) in $\widehat{\mathscr{X}}$ we choose a sequence of smooth paths of smooth exact two-forms $\{\widehat{f}_k\}_{k\in\mathbb{N}}$ and a sequence of smooth paths of smooth nondegenerate two-forms $\{\bar{\rho}_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}\|\widehat{f} - \widehat{f}_k\|_{L^2(I,L^p)} = 0, \quad \lim_{k\to\infty}\|\bar{\rho} - \bar{\rho}_k\|_{C^0(I,W^{1,p})} = 0.$$

Then for every $k \geq 0$ there exists a smooth exact solution $\widehat{\rho}_k$ to

$$\partial_t \widehat{\rho}_k + L_{\bar{\rho}_k} \widehat{\rho}_k = \widehat{f}_k, \qquad \widehat{\rho}_k(t=0,\cdot) = 0.$$

The maximal regularity estimate (41) holds with a uniform constant in a neighbourhood of $\bar{\rho}$. Therefore there exists a weakly converging subsequence converging to a solution $\widehat{\rho} \in \mathscr{X}$ of (40). This proves the lemma. □

Proof of Theorem 6.1. We need to show that the Donaldson flow depends smoothly on the initial conditions. Let $\rho_0 \in \mathscr{S}_2^{1,p}(M,\Lambda^2)$ and $\bar{\rho}$ be the solution to the Donaldson flow in $W^{1,2,p}(M_I,\Lambda^2)$ for the initial condition $\rho_0 \in \mathscr{S}_2^{1,p}(M,\Lambda^2)$ on the time interval $I = [0,T]$. Let us define the following spaces.

$$\begin{aligned}\mathscr{X}_0 := \big\{\rho \in & L^2(I, W^{2,p}(M,\Lambda^2)) \cap W^{1,2}(I, L^p(M,\Lambda^2)) \\ & \mid \forall t : d\rho(t,\cdot) = 0,\ \rho(t,\cdot)\wedge\rho(t,\cdot) > 0,\ [\rho(t,\cdot)] = a, \\ & \rho(t=0,\cdot) = \rho_0\big\}.\end{aligned}$$

This is the space of symplectic forms in $W^{1,2,p}(M_I,\Lambda^2)$ representing the fixed cohomology class $a \in H^2(M;\mathbb{R})$ and have the fixed initial condition ρ_0. The formal tangent space to $\mathscr{X}_0$ at ρ is given by

$$\begin{aligned}\widehat{\mathscr{X}_0} := \{\widehat{\rho} \in & L^2(I, W^{2,p}(M,\Lambda^2)) \cap W^{1,2}(I, L^p(M,\Lambda^2)) \\ & \mid \forall t : \widehat{\rho}(t,\cdot) \text{ exact},\ \widehat{\rho}(t=0,\cdot) = 0\}.\end{aligned}$$

The space $\mathscr{S}_{2,a}^{1,p}(M,\Lambda^2)$ is an open subset of an affine space, the corresponding vector space is

$$\widehat{\mathscr{Z}} := \{\widehat{\tau} \in B_2^{1,p}(M,\Lambda^2) \mid \widehat{\tau} \text{ exact}\}.$$

There exists a bounded linear extension operator

$$\mathscr{T} : \widehat{\mathscr{Z}} \to \widehat{\mathscr{X}}$$

such that

$$(\mathscr{T}\widehat{\tau})(0) = \widehat{\tau}.$$

For example we can define $\mathscr{T}$ as follows. Let $T(t)$ be the semigroup created by the negative Hodge laplacian with respect to the background metric g on the exact two-forms in $L^p(M,\Lambda^2)$ with domain the exact two-forms in $W^{2,p}(M,\Lambda^2)$. The theory developed in [5] asserts that $t \mapsto T(t)\widehat{\tau}$ is a continous map from I to $\widehat{\mathscr{Z}}$. We define

$$(\mathscr{T}\widehat{\tau})(t) := T(t)\widehat{\tau}.$$

Now consider the map

$$\mathscr{F} : \mathscr{X}_0 \times \widehat{\mathscr{Z}} \to \widehat{\mathscr{Y}}, \qquad \mathscr{F}(\rho,\widehat{\tau}) := \partial_t \rho + \partial_t(\mathscr{T}\widehat{\tau}) - d *^{\rho+\mathscr{T}\widehat{\tau}} d\theta^{\rho+\mathscr{T}\widehat{\tau}}.$$

Then,

$$\mathscr{F}(\bar{\rho},0) = 0$$

and this map is clearly infinitely Fréchet differentiable. We compute the linearization in the first factor at $(\bar{\rho},0)$,

$$d\mathscr{F}(\bar{\rho},0) : \widehat{\mathscr{X}} \to \widehat{\mathscr{Y}}, \qquad d\mathscr{F}(\bar{\rho},0)\widehat{\rho} := \left.\frac{d}{ds}\right|_{s=0} \mathscr{F}(\bar{\rho}+s\widehat{\rho},\widehat{\tau}) = \partial_t\widehat{\rho} + L_{\bar{\rho}}\widehat{\rho},$$

where $L_{\bar{\rho}}$ is given by equation (29). By Lemma 6.3 the Cauchy problem

$$\partial_t\widehat{\rho} + L_{\bar{\rho}}\widehat{\rho} = \widehat{f}, \qquad \widehat{\rho}(t=0,\cdot) = 0$$

has a unique solution $\widehat{\rho} \in \widehat{\mathscr{X}}$ for every $\widehat{f} \in \widehat{\mathscr{Y}}$. Therefore $d\mathscr{F}(\bar{\rho},0)$ is bijectiv. It follows from the implicit function theorem for Banach spaces that there exists an open neighborhood $0 \in \mathscr{U} \subseteq \widehat{\mathscr{Z}}$ and a smooth map

$$\Phi : \mathscr{U} \to \mathscr{X}_0$$

with

$$\mathscr{F}(\Phi(\widehat{\tau}),\widehat{\tau}) = 0, \qquad \Phi(0) = \bar{\rho}.$$

Therefore,

$$\partial_t(\Phi(\widehat{\tau}) + \mathscr{T}\widehat{\tau}) = d *^{\Phi(\widehat{\tau})+\mathscr{T}\widehat{\tau}} d\theta^{\Phi(\widehat{\tau})+\mathscr{T}\widehat{\tau}}, \qquad (\Phi(\widehat{\tau}) + \mathscr{T}\widehat{\tau})(t=0,\cdot) = \rho_0 + \widehat{\tau}.$$

In particular the map

$$\widehat{\tau} \mapsto \Phi(\widehat{\tau}) + \mathscr{T}\widehat{\tau}$$

is a smooth map from initial conditions in $\mathscr{U}$ to solutions of the Donaldson flow in $W^{1,2,p}(M_I,\Lambda^2)$ with these initial conditions. This proves the theorem.

□

The semiflow property of the Donaldson flow allows one to apply classic stability analysis results of dynamical systems to the time one map of the Donaldson flow. In particular we can prove that there is a neighborhood around the absolute minimum in $\mathscr{S}_a$ in the Besov space topology which is a local stable manifold for the absolute minimum. Every solution to the Donaldson flow whose initial conditions lay within that neighborhood converges to the absolute minimum.

Corollary 6.4. *There exists an open set in the topology of the Besov space $B_2^{1,p}(M,\Lambda^2)$ which is a local stable manifold around the absolute minimum of $\mathscr{S}_a$, i.e. the iterates of the time one map of the Donaldson flow converge to the absolute minimum for every point in this neighborhood.*

Proof. By Theorem 6.1 there exists an open set around the absolute minimum ω in $\mathscr{S}_{2,a}^{1,p}$ such that every solution to the Donaldson flow with initial condition in this neighborhood exists on the interval $[0,1]$. Let $\varphi^1 : B_\omega \to \mathscr{S}_{2,a}^{1,p}$ be the time one map of the Donaldson flow restricted to a small ball centered at ω within that neighborhood. We denote

$$\widehat{\mathscr{X}} := T_\omega \mathscr{S}_{2,a}^{1,p} = \{\widehat{\tau} \in B_2^{1,p}(M,\Lambda^2) |\ \widehat{\tau} \text{ exact}\}.$$

Let $\widehat{\varphi} : \widehat{\mathscr{X}} \to \widehat{\mathscr{X}}$ be the linearization of φ^1 at ω. It is given by the solution to the linearized Donaldson flow equation,

$$\partial_t \widehat{\rho} + L_\omega \widehat{\rho} = 0, \qquad \widehat{\rho}(t=0,\cdot) = \widehat{\tau} \in \widehat{\mathscr{X}}, \tag{42}$$

where $\omega \in \mathscr{S}_a$ is the unique minimum of the energy functional

$$\mathscr{E}(\rho) = \int_M \frac{2\left|\rho^+\right|^2}{\left|\rho^+\right|^2 - \left|\rho^-\right|^2} \mathrm{dvol}, \qquad \rho \in \mathscr{S}_a$$

and L_ω is given by (29). Recall that ω is a symplectic form on M compatible with the background metric and $[w] = a \in H^2(M;\mathbb{R})$. In particular ω is the unique self-dual symplectic form on M representing the cohomology class a. Since ω is compatible with the background metric, $*^\omega$ coincides with the Hodge star operator of the background metric, $u_\omega = \frac{\mathrm{dvol}_\omega}{\mathrm{dvol}} = 1$ and

$$\theta^\omega = \omega - \frac{1}{2}\left|\omega\right|^2 \omega = \omega - \omega = 0.$$

It follows that the linearized gradient operator L_ω is the Hodge laplacian,

$$L_\omega = d\frac{d^{*^\omega}}{u_\omega} + A^\omega = dd^*.$$

Thus the solution to (42) is the heat flow on $\widehat{\mathscr{Z}}$. This is well know to be a contraction and

$$\|\widehat{\varphi}\|_{L(\widehat{\mathscr{Z}},\widehat{\mathscr{Z}})} < 1,$$

where $\|\cdot\|_{L(\widehat{\mathscr{Z}},\widehat{\mathscr{Z}})}$ denotes the operator norm for operators on $\widehat{\mathscr{Z}}$. The map φ^1 is C^∞-Fréchet differentiable and from the previous inequality we know that it's derivative is a contraction on an open ball around ω in $\mathscr{S}^{1,p}_{2,a}$. It follows from the mean value theorem that φ^1 is a contraction on this ball. The Banach fixed point theorem now implies that the iteration of φ^1 on this ball converges to a unique fixed point and this clearly is ω. This proves the theorem. □

Remark 6.5. *Alternatively, we can prove the existence of an open neighborhood around the absolute minimum of the energy functional in* $\mathscr{S}^{1,p}_{2,a}(M,\Lambda^2)$ *such that the Donaldson flow converges exponentially fast to the absolute minimum for every initial condition laying in this neighborhood by repeating the proof of the short time existence theorem 5.1, where one applies the Banach fixed point theorem to the analogous iteration on a small ball around zero in the space*

$$\begin{aligned}\mathscr{X}_{\delta,\epsilon,\rho_0} := \{\widehat{\rho} \in W^{1,2,p}(M_{\mathbb{R}_+},\Lambda^2) |\ & \widehat{\rho} \text{ exact}, \\ & \widehat{\rho}(t=0,\cdot) = 0, \\ & \|e^{\delta t}(\widetilde{\rho}_0 + \widehat{\rho} - \omega\|_{W^{1,2,p}} < \epsilon\}.\end{aligned}$$

Here $\widetilde{\rho}_0$ *is a extension of the initial condition* $\rho_0 \in \mathscr{S}^{1,p}_2$ *to* $W^{1,2,p}(M,\Lambda^2)$, ω *is the absolute minimum of the energy functional and* $\delta,\epsilon > 0$ *are real constants that have to be choosen sufficiently small. The norm* $\|\cdot\|_{W^{1,2,p}}$ *is defined by* (23).

A Products in Sobolev Spaces.

A proper open connected subset $\Omega \subset \mathbb{R}^n$ is called a *smooth domain* if for every $x \in \partial\Omega$ there is a ball $B = B(x)$ and a smooth diffeomorphism ψ of B

onto $D \subset \mathbb{R}^n$ such that

$$\psi(B \cap \Omega) \subset \mathbb{R}^n_+, \qquad \psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+.$$

If ψ is not smooth but of class C^k then the domain is said to have C^k boundary.

Proposition A.1 (**Gagliardo-Nirenberg**). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with C^k boundary. Suppose that $j, k \geq 0$ are integers with $j < k$ and $1 \leq p, q, r \leq \infty$ with $k - n/q + n/r \geq 0$ and*

$$j - \frac{n}{p} = \lambda \left(k - \frac{n}{q}\right) + (1-\lambda)\left(-\frac{n}{r}\right), \qquad \frac{j}{k} \leq \lambda \leq 1.$$

If $(k-j)q = n$ assume also that $\lambda \neq 1$. Then there exists a constant $c > 0$ such that

$$\|\partial^j u\|_{L^p} \leq c \|\partial^k u\|^{\lambda}_{L^q} \|u\|^{1-\lambda}_{L^r}$$

for $u \in W^{k,q}(\Omega)$.

Proof. A proof of this proposition can be found in [4]. □

Lemma A.2 (**Products in Sobolev Spaces**). **(i)** *Let M be a closed manifold of dimension n. Let $f, g \in C^\infty(M, \mathbb{R})$. Let $k, p \in \mathbb{N}$ such that $k - \frac{n}{p} > 0$. There exists a constant $c > 0$ such that*

$$\|fg\|_{W^{k,p}} \leq c \left(\|f\|_{L^\infty} \|g\|_{W^{k,p}} + \|f\|_{W^{k,p}} \|g\|_{L^\infty} \right).$$

(ii) *Let $E \to M$ be a smooth vector bundle over M. Let $k - \frac{n}{p} > 0$. Let $f \in C^\infty(E, \mathbb{R})$ and $u \in C^\infty(M, E)$. Let $U \subset E$ be an open set containing the image of u. Then there exists a constant $c > 0$ such that*

$$\|f \circ u\|_{W^{k,p}} \leq c \, |f|_{C^k(U)} \left(1 + \|u\|_{W^{k,p}} \left(1 + \|u\|_{L^\infty}\right)^{k-1}\right).$$

Proof. It is enough to prove these statements over an open domain of $\mathbb{R}^n$ with smooth boundary. We prove (i). Let $f, g \in C^\infty(\Omega, \mathbb{R})$. Let $0 \leq i \leq k$ and let $\partial^i = \partial^\alpha$ for an arbitrary multi-indice $\alpha \in \mathbb{N}^n$ with $|\alpha| = i$. By the product rule

$$\partial^i(fg) = \sum_{\ell=0}^{i} (\partial^\ell f)(\partial^{i-\ell} g) = f(\partial^i g) + (\partial^i f) g + \sum_{\ell=1}^{i-1} (\partial^\ell f)(\partial^{i-\ell} g).$$

Clearly,

$$\|f(\partial^i g) + (\partial^i f)g\|_{L^p} \le \|f\|_{L^\infty}\|g\|_{W^{k,p}} + \|f\|_{W^{k,p}}\|g\|_{L^\infty}.$$

Let $s = \frac{pk}{\ell}, t = \frac{pk}{k-\ell}$. Then by the Hölder inequality and the Gagliardo-Nirenberg interpolation inequality A.1

$$\begin{aligned}\|(\partial^\ell f)(\partial^{i-\ell} g)\|_{L^p} &\le \|\partial^\ell f\|_{L^s}\|\partial^{i-\ell} g\|_{L^t}\\ &\le c_1\|f\|_{W^{k,p}}^{\lambda_1}\|f\|_{L^\infty}^{1-\lambda_1}\|g\|_{W^{k,p}}^{\lambda_2}\|g\|_{L^\infty}^{1-\lambda_2}\end{aligned}$$

for

$$\lambda_1 = \frac{\ell - \frac{n}{s}}{k - \frac{n}{p}} = \frac{\ell}{k}, \qquad \lambda_2 = \frac{k - \ell - \frac{n}{t}}{k - \frac{n}{p}} = \frac{k-\ell}{k}, \qquad 1 \le \ell \le i-1.$$

By Young's inequality and $\lambda_1 + \lambda_2 = 1$

$$\begin{aligned}\|(\partial^\ell f)(\partial^{i-\ell} g)\|_{L^p} &\le c_1\|f\|_{W^{k,p}}^{\lambda_1}\|g\|_{L^\infty}^{\lambda_1}\|f\|_{L^\infty}^{\lambda_2}\|g\|_{W^{k,p}}^{\lambda_2}\\ &\le c_2(\|f\|_{L^\infty}\|g\|_{W^{k,p}} + \|f\|_{W^{k,p}}\|g\|_{L^\infty}).\end{aligned}$$

This proves (i).

We prove (ii). Let $U \subset E$ be an open set such that $u(M) \subseteq U$. Clearly

$$\|f \circ u\|_{L^p} \le (\mathrm{Vol}(M))^{\frac{1}{p}} \left|f\right|_{C^0(U)}.$$

Let $\partial^i = \partial^\alpha$ for an arbitrary multi-indice $\alpha \in \mathbb{N}^n$ with $|\alpha| = i$. We claim that $\partial^i f(u)$ equals the sum of terms of the form

$$\partial^{\ell_0} f(u) \cdot (\partial^1 u)^{\ell_1} \cdot (\partial^2 u)^{\ell_2} \cdots (\partial^i u)^{\ell_i} \tag{43}$$

for $i \ge 1$, $1 \le \ell_0 \le i$ and $\sum_{j=1}^i j\ell_j = i$. By the chain rule $\partial f(u) = \partial f(u)\partial u$ and hence the claim holds true for $i = 1$. Suppose the claim is true for an $i \ge 1$. Then

$$\begin{aligned}\partial^{i+1}(f(u)) &= \partial\partial^i(f(u))\\ &= \partial\left(\partial^{\ell_0} f(u)\left((\partial^1 u)^{\ell_1}(\partial^2 u)^{\ell_2}\cdots(\partial^i u)^{\ell_i}\right)\right)\\ &= \partial^{\ell_0+1} f(u)\left((\partial^1 u)^{\ell_1+1}(\partial^2 u)^{\ell_2}\cdots(\partial^i u)^{\ell_i}\right)\\ &\quad + \partial^{\ell_0} f(u)\partial\left((\partial^1 u)^{\ell_1}(\partial^2 u)^{\ell_2}\cdots(\partial^i u)^{\ell_i}\right)\end{aligned}$$

and the summands of the last expression are again of the claimed form. This proves the claim. By the Hölder inequality

$$\|\partial^{\ell_0} f(u)(\partial^1 u)^{\ell_1}(\partial^2 u)^{\ell_2}\cdots(\partial^i u)^{\ell_i}\|_{L^p} \leq |f|_{C^{\ell_0}(U)} \|\partial^1 u\|_{L^{s_1}}^{\ell_1}\|\partial^2 u\|_{L^{s_2}}^{\ell_2}\cdots\|\partial^i u\|_{L^{s_i}}^{\ell_i}$$

for $s_j = \frac{p \cdot i}{j}$. Let

$$\lambda_j = \frac{j - \frac{n}{s_j}}{i - \frac{n}{p}} = \frac{j}{i}.$$

By the Gagliardo-Nirenberg interpolation inequality

$$\begin{aligned}\|\partial^1 u\|_{L^{s_1}}^{\ell_1}\|\partial^2 u\|_{L^{s_2}}^{\ell_2}\cdots\|\partial^i u\|_{L^{s_i}}^{\ell_i} &\leq c_3\|u\|_{W^{i,p}}^{\ell_1\lambda_1}\|u\|_{L^\infty}^{\ell_1(1-\lambda_1)}\cdots\|u\|_{W^{i,p}}^{\ell_i\lambda_i}\|u\|_{L^\infty}^{\ell_i(1-\lambda_i)} \\ &\leq c_3\|u\|_{W^{k,p}}\|u\|_{L^\infty}^{(\sum_{j=1}^i \ell_j)-1}.\end{aligned}$$

Since $\sum_{j=1}^i \ell_j \leq i$, $i \leq k$ and $\ell_0 \leq k$ this shows that

$$\begin{aligned}\|f\circ u\|_{W^{k,p}} \leq c_4\,|f|_{C^k(U)}\big(&1 + \|u\|_{W^{k,p}} + \|u\|_{W^{k,p}}\|u\|_{L^\infty} + \|u\|_{W^{k,p}}\|u\|_{L^\infty}^2 + \\ &\cdots + \|u\|_{W^{k,p}}\|u\|_{L^\infty}^{k-1}\big)\end{aligned}$$

for a constant $c_4 > 0$. This proves (ii). □

Recall that we defined

$$\|u\|_{W^{r,k,p}(M_I)} := \sum_{\substack{2s+\ell\leq k \\ s\leq r}} \|\partial_t^s \partial^\ell u\|_{L^{p,2}(M_I)}.$$

for a function $u \in C^\infty(M_I, \mathbb{R})$ and an open interval $I \subseteq \mathbb{R}$.

Corollary A.3. *Let $1 - \frac{n}{p} > 0$, $p \geq 2$, $k \geq 2$, $s \geq 1$. Let $u, v \in C^\infty(M_I, \mathbb{R})$ for an open interval $I \subseteq \mathbb{R}$. Then*

$$\|uv\|_{W^{s,k,p}} \leq \|u\|_{W^{s,k,p}}\|v\|_{W^{s,k,p}}$$

Proof. For all $1 \leq r \leq s$ and a fixed $t \in I$

$$\|\partial_t^r(uv)\|_{W^{k-2r,p}} \leq \|(\partial_t^r u)v\|_{W^{k-2r,p}} + \|u\partial_t^r v\|_{W^{k-2r,p}} + \sum_{i=1}^{r-1}\|(\partial_t^i u)\partial_t^{r-i} v\|_{W^{k-2r,p}}.$$

If $k-2r=0$, then

$$\begin{aligned}&\|(\partial_t^r u)v\|_{L^2(I,L^p)}+\|u\partial_t^r v\|_{L^2(I,L^p)}\\&\qquad\leq\|\partial_t^r u\|_{L^2(I,L^p)}\|v\|_{L^\infty(I,L^\infty)}+\|u\|_{L^\infty(I,L^\infty)}\|\partial_t^r v\|_{L^2(I,L^p)}\\&\qquad\leq\|u\|_{W^{s,k,p}}\|v\|_{W^{s,k,p}}.\end{aligned}$$

and for $1\leq i\leq r-1$

$$\|(\partial_t^i u)\partial_t^{r-i}v\|_{L^2(I,L^p)}=\|\partial_t^i u\|_{L^\infty(I,L^\infty)}\|\partial_t^{r-i}v\|_{L^2(I,L^p)}\leq\|u\|_{W^{s,k,p}}\|v\|_{W^{s,k,p}}.$$

Here we use that for $p\geq 2$ the identity operator on real smooth functions on $I\times M$ with compact support extends to a bounded linear operator

$$L^2(I,W^{2,p})\cap W^{1,2}(I,L^p)\to C(I,W^{1,p})$$

and hence for $1-\frac{n}{p}>0$ and $s\geq 1$

$$\|u\|_{L^\infty(I,L^\infty)}\leq\|u\|_{L^\infty(I,W^{1,p})}\leq\|u\|_{W^{s,k,p}}.$$

and likewise for $1\leq i\leq r-1$

$$\|\partial_t^i u\|_{L^\infty(I,L^\infty)}\leq\|\partial_t^i u\|_{L^\infty(I,W^{1,p})}\leq\|u\|_{W^{s,k,p}}.$$

If $1\leq k-2r<k$ then

$$\begin{aligned}&\|(\partial_t^r u)v\|_{W^{k-2r,p}}+\|u\partial_t^r v\|_{W^{k-2r,p}}\\&\qquad\leq\|\partial_t^r u\|_{W^{k-2r,p}}\|v\|_{L^\infty}+\|\partial_t^r u\|_{L^\infty}\|v\|_{W^{k-2r,p}}\\&\qquad\quad+\|u\|_{W^{k-2r,p}}\|\partial_t^r v\|_{L^\infty}+\|u\|_{L^\infty}\|\partial_t^r v\|_{W^{k-2r,p}}\end{aligned}$$

and

$$\begin{aligned}&\|(\partial_t^r u)v\|_{L^2(I,W^{k-2r,p})}+\|u\partial_t^r v\|_{L^2(I,W^{k-2r,p})}\\&\qquad\leq\|\partial_t^r u\|_{L^2(I,W^{k-2r,p})}\|v\|_{L^\infty(I,L^\infty)}+\|\partial_t^r u\|_{L^2(I,W^{1,p})}\|v\|_{L^\infty(I,W^{k-2r,p})}\\&\qquad\quad+\|u\|_{L^\infty(I,W^{k-2r,p})}\|\partial_t^r v\|_{L^2(I,W^{1,p})}+\|u\|_{L^\infty(I,L^\infty)}\|\partial_t^r v\|_{L^2(I,W^{k-2r,p})}\\&\qquad\leq\|u\|_{W^{s,k,p}}\|v\|_{W^{s,k,p}}.\end{aligned}$$

Further, for $1\leq i\leq r-1$

$$\|(\partial_t^i u)\partial_t^{r-i}v\|_{W^{k-2r,p}}\leq\|\partial_t^i u\|_{L^\infty}\|\partial_t^{r-i}v\|_{W^{k-2r,p}}+\|\partial_t^i u\|_{W^{k-2r,p}}\|\partial_t^{r-i}v\|_{L^\infty}$$

and

$$\begin{aligned}
&\|(\partial_t^i u)\partial_t^{r-i} v\|_{L^2(I,W^{k-2r,p})} \\
&\quad \le \|\partial_t^i u\|_{L^\infty(I,W^{1,p})}\|\partial_t^{r-i} v\|_{L^2(I,W^{k-2r,p})} + \|\partial_t^i u\|_{L^2(I,W^{k-2r,p})}\|\partial_t^{r-i} v\|_{L^\infty(I,W^{1,p})} \\
&\quad \le \|u\|_{W^{s,k,p}}\|v\|_{W^{s,k,p}}.
\end{aligned}$$

Finally, if $r = 0$, then

$$\|uv\|_{W^{k,p}} \le \|u\|_{L^\infty}\|v\|_{W^{k,p}} + \|u\|_{W^{k,p}}\|v\|_{L^\infty}$$

and

$$\begin{aligned}
\|uv\|_{L^2(I,W^{k,p})} &\le \|u\|_{L^\infty(I,L^\infty)}\|v\|_{L^2(I,W^{k,p})} + \|u\|_{L^2(I,W^{k,p})}\|v\|_{L^\infty(I,L^\infty)} \\
&\le \|u\|_{W^{s,k,p}}\|v\|_{W^{s,k,p}}.
\end{aligned}$$

This proves the corollary. □

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Remarks on the Donaldson metric

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Abstract

The Donaldson metric is a metric on the space of symplectic two-forms in a fixed cohomology class. It was introduced in [2]. We compute the associated Levi-Civita connection, describe it's geodesics and compute the formula for the covariant Hessian of an energy functional on the space of symplectic structures in a fixed cohomology class, introduced by S. Donaldson in [1].

Let M be a closed oriented Riemannian four-manifold. Denote by g the Riemannian metric on M, denote by $\mathrm{dvol} \in \Omega^4(M)$ the volume form of g, and let $* : \Omega^k(M) \to \Omega^{4-k}(M)$ be the Hodge $*$-operator associated to the metric and orientation. Fix a cohomology class $a \in H^2(M;\mathbb{R})$ such that $a^2 > 0$ and consider the space

$$\mathscr{S}_a := \left\{\rho \in \Omega^2(M) \,\middle|\, d\rho = 0,\, \rho\wedge\rho > 0,\, [\rho] = a\right\}$$

of symplectic forms on M representing the class a. This is an infinite-dimensional manifold and the tangent space of $\mathscr{S}_a$ at any element $\rho \in \mathscr{S}_a$ is the space of exact 2-forms on M. The next proposition is proved in [2]. It summarizes the properties of a family of Riemannian metrics g^ρ on M, one for each nondegenerate 2-form ρ (and for each fixed background metric g). For each nondegenerate 2-form define the function u by the equation

$$2u\mathrm{dvol} = \rho \wedge \rho \tag{1}$$

Proposition 0.4 (Symplectic Forms and Riemannian Metrics). *Fix a nondegenerate 2-form $\rho \in \Omega^2(M)$ such that $\rho \wedge \rho > 0$ and define the function $u : M \to (0,\infty)$ by (1). Then there exists a unique Riemannian metric g^ρ on M that satisfies the following conditions.*

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(i) *The volume form of* g^ρ *agrees with the volume form of* g.

(ii) *The Hodge* $*$*-operator* $*^\rho : \Omega^1(M) \to \Omega^3(M)$ *associated to* g^ρ *is given by*

$$*^\rho \lambda = \frac{\rho \wedge *(\rho \wedge \lambda)}{u} \tag{2}$$

for $\lambda \in \Omega^1(M)$ *and by* $*^\rho \iota(X)\rho = -\rho \wedge g(X, \cdot)$ *for* $X \in \mathrm{Vect}(M)$.

(iii) *The Hodge* $*$*-operator* $*^\rho : \Omega^2(M) \to \Omega^2(M)$ *associated to* g^ρ *is given by*

$$*^\rho \omega = R^\rho * R^\rho \omega, \qquad R^\rho \omega := \omega - \frac{\omega \wedge \rho}{\mathrm{dvol}_\rho}\rho, \tag{3}$$

for $\omega \in \Omega^2(M)$. *The linear map* $R^\rho : \Omega^2(M) \to \Omega^2(M)$ *is an involution that preserves the exterior product, acts as the identity on the orthogonal complement of* ρ *with respect to the exterior product, and sends* ρ *to* $-\rho$.

(iv) *Let* $\omega \in \Omega^2(M)$ *be a nondegenerate* 2*-form and let* $J : TM \to TM$ *be an almost complex structure such that* $g = \omega(\cdot, J\cdot)$. *Define the almost complex structure* J^ρ *by* $\rho(J^\rho\cdot, \cdot) := \rho(\cdot, J\cdot)$ *and define the* 2*-form* $\omega^\rho \in \Omega^2(M)$ *by* $\omega^\rho := R^\rho\omega$. *Then* $g^\rho = \omega^\rho(\cdot, J^\rho\cdot)$ *and so* ω^ρ *is self-dual with respect to* g^ρ.

The following is the central object of this paper.

Definition 0.5. *Each nondegenerate* 2*-form* $\rho \in \Omega^2(M)$ *with* $\rho^2 > 0$ *determines an inner product* $\langle \cdot, \cdot \rangle_\rho$ *on the space of exact* 2*-forms defined by*

$$\langle \widehat{\rho}_1, \widehat{\rho}_2 \rangle_\rho := \int_M \lambda_1 \wedge *^\rho \lambda_2, \qquad d\lambda_i = \widehat{\rho}_i, \qquad *^\rho \lambda_i \in \mathrm{im}\, d. \tag{4}$$

These inner products determine a Riemannian metric on the infinite-dimensional manifold $\mathscr{S}_a$ *called the* **Donaldson metric**.

Definition 0.6. *A vector field* $X_{\widehat{\rho}}$ *is* **associated** *to an exact 2-form* $\widehat{\rho} \in T_\rho \mathscr{S}_a$ *if it is the unique vector field satisfing*

$$-d\iota(X_{\widehat{\rho}})\rho = \widehat{\rho}, \qquad *^\rho \iota(X_{\widehat{\rho}})\rho \in \mathrm{im} d. \tag{5}$$

Every metric has an associated Levi-Civita connection. This is the unique torsion free and Riemannian connection with respect to the metric. The formula and computation is the content of the following theorem.

Theorem 0.7 (Levi Civita Connection). *Let $\rho_t : \mathbb{R} \to \mathscr{S}_a$ be a smooth path of symplectic forms with $\rho := \rho_0$ and $\widehat{\rho} := \partial_t|_{t=0} \rho_t$. Let X be the associated vector field of $\widehat{\rho}$. Let $Y_t : \mathbb{R} \to \mathrm{Vect}(M)$ be a smooth path of vector fields such that $*^\rho \iota(Y_t)\rho_t$ is exact and define*

$$\widehat{\sigma}_t := -d\iota(Y_t)\rho_t, \quad \widehat{\sigma} := \widehat{\sigma}_0, \quad Y := Y_0.$$

The unique Levi-Civita connection associated to the Donaldson metric is given by

$$\nabla^\rho_{\widehat{\rho}}\widehat{\sigma} = \frac{d}{dt}\bigg|_{t=0} \widehat{\sigma}_t + \frac{1}{2}d\iota(Y)\widehat{\rho} + \frac{1}{2}d\iota(X)\widehat{\sigma} - \frac{1}{2}d\iota\left(\nabla_Y X + \nabla_X Y\right)\rho. \qquad (6)$$

Here $\nabla_X Y$ denotes the covariant derivative of the Levi-Civita connection of the metric g for two vector fields X and Y.

Proof. The Levi-Civita connection of the Donaldson metric is the unique connection that is torsion free and Riemannian with respect to the Donaldson metric. Since the Christoffel symbol given by (6) is symmetric in $\widehat{\rho}$ and $\widehat{\sigma}$, the torsion of the connection ∇^ρ vanishes. It remains to show that it is Riemannian. Let Z_t be a smooth path of vector fields such that $*^{\rho_t}\iota(Z_t)\rho_t$ is exact. Denote $\widehat{\tau}_t := -d\iota(Z_t)\rho_t$, $\widehat{\tau} := \widehat{\tau}_0$. We claim that

$$\frac{d}{dt}\bigg|_{t=0} \langle \widehat{\sigma}_t, \widehat{\tau}_t\rangle_{\rho_t} = \langle \nabla_{\widehat{\rho}}\widehat{\sigma}, \widehat{\tau}\rangle_\rho + \langle \widehat{\sigma}, \nabla_{\widehat{\rho}}\widehat{\tau}\rangle_\rho.$$

By the definition of the Donaldson metric and since $*^\rho\iota(X)\rho = -\rho \wedge \iota(X)g$ by Proposition 3.1 ii) in [2],

$$\langle \widehat{\sigma}_t, \widehat{\tau}_t\rangle_{\widehat{\rho}_t} = \int_M (\iota(Y_t)\rho_t) \wedge *^\rho\iota(Z_t)\rho_t = -\int_M (\iota(Y_t)\rho_t) \wedge \rho_t \wedge \iota(Z_t)g$$

and

$$\begin{aligned}\langle \frac{d}{dt}\bigg|_{t=0} \widehat{\sigma}_t, \widehat{\tau}\rangle_\rho &= -\int_M (\iota(\widehat{Y})\rho) \wedge \rho \wedge \iota(Z)g - \int_M (\iota(Y)\widehat{\rho}) \wedge \rho \wedge \iota(Z)g \\ \langle \widehat{\sigma}, \frac{d}{dt}\bigg|_{t=0} \widehat{\tau}_t\rangle_\rho &= -\int_M (\iota(\widehat{Z})\rho) \wedge \rho \wedge \iota(Y)g - \int_M (\iota(Z)\widehat{\rho}) \wedge \rho \wedge \iota(Y)g,\end{aligned}$$

where $\widehat{Y} := \frac{d}{dt}\Big|_{t=0} Y_t$, $\widehat{Z} := \frac{d}{dt}\Big|_{t=0} Z_t$. Hence,

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} \langle \widehat{\sigma}_t, \widehat{\tau}_t \rangle_{\widehat{\rho}_t} \\
&= \langle \frac{d}{dt}\Big|_{t=0} \widehat{\sigma}_t, \widehat{\tau} \rangle_\rho - \int_M (\iota(Y)\rho) \wedge \widehat{\rho} \wedge \iota(Z) g - \int_M (\iota(Y)\rho) \wedge \rho \wedge \iota(\widehat{Z}) g \\
&= \langle \frac{d}{dt}\Big|_{t=0} \widehat{\sigma}_t, \widehat{\tau} \rangle_\rho - \int_M (\iota(Y)\rho) \wedge \widehat{\rho} \wedge \iota(Z) g - \int_M (\iota(\widehat{Z})\rho) \wedge \rho \wedge \iota(Y) g \\
&= \langle \frac{d}{dt}\Big|_{t=0} \widehat{\sigma}_t, \widehat{\tau} \rangle_\rho + \langle \widehat{\sigma}, \frac{d}{dt}\Big|_{t=0} \widehat{\tau}_t \rangle \\
&\qquad - \int_M (\iota(Y)\rho) \wedge \widehat{\rho} \wedge \iota(Z) g + \int_M (\iota(Z)\widehat{\rho}) \wedge \rho \wedge \iota(Y) g.
\end{aligned}$$

Define the Christoffel symbols $\Gamma_{\widehat{\sigma}\widehat{\tau}\widehat{\rho}}$ by

$$\begin{aligned}
2\Gamma_{\widehat{\sigma}\widehat{\tau}\widehat{\rho}} := &(\iota(Y)\rho) \wedge (d\iota(X)\rho) \wedge \iota(Z) g - (\iota(Z) d\iota(X)\rho) \wedge \rho \wedge \iota(Y) g \\
&+ (\iota(Y)\rho) \wedge (d\iota(Z)\rho) \wedge \iota(X) g - (\iota(X) d\iota(Z)\rho) \wedge \rho \wedge \iota(Y) g \\
&- (\iota(Z)\rho) \wedge (d\iota(Y)\rho) \wedge \iota(X) g + (\iota(X) d\iota(Y)\rho) \wedge \rho \wedge \iota(Z) g.
\end{aligned}$$

Then

$$\begin{aligned}
\Gamma_{\widehat{\sigma}\widehat{\tau}\widehat{\rho}} + \Gamma_{\widehat{\tau}\widehat{\sigma}\widehat{\rho}} &= (\iota(Y)\rho) \wedge d\iota(X)\rho \wedge \iota(Z) g - (\iota(Z) d\iota(X)\rho) \wedge \rho \wedge \iota(Y) g \\
&= (-\iota(Y)\rho) \wedge \widehat{\rho} \wedge \iota(Z) g + (\iota(Z)\widehat{\rho}) \wedge \rho \wedge \iota(Y) g.
\end{aligned}$$

Hence, it remains to show that

$$\int_M \Gamma_{\widehat{\sigma}\widehat{\tau}\widehat{\rho}} = \langle \widehat{\sigma}, \frac{1}{2} d\iota(Z)\widehat{\rho} + \frac{1}{2} d\iota(X)\widehat{\tau} - \frac{1}{2} d\iota(\nabla_Z X + \nabla_X Z)\rho \rangle_\rho.$$

Let

$$\begin{aligned}
A &:= (-\iota(Z) d\iota(X)\rho) \wedge \rho \wedge \iota(Y) g - (\iota(X) d\iota(Z)\rho) \wedge \rho \wedge \iota(Y) g \\
B &:= 2\Gamma_{\widehat{\sigma}\widehat{\tau}\widehat{\rho}} - A \\
&= (\iota(Y)\rho) \wedge (d\iota(X)\rho) \wedge \iota(Z) g + (\iota(Y)\rho) \wedge (d\iota(Z)\rho) \wedge \iota(X) g \\
&\qquad - (\iota(Z)\rho) \wedge (d\iota(Y)\rho) \wedge \iota(X) g + (\iota(X) d\iota(Y)\rho) \wedge \rho \wedge \iota(Z) g.
\end{aligned}$$

Since $*^\rho \iota(X)\rho = -\rho \wedge \iota(X) g$ for any vector field X,

$$\begin{aligned}
\int_M A &= -\int_M \iota(Z) d\iota(X)\rho \wedge \rho \wedge \iota(Y) g - \int_M \iota(X) d\iota(Z)\rho \wedge \rho \wedge \iota(Y) g \\
&= \langle \widehat{\sigma}, d\iota(Z)\widehat{\rho} + d\iota(X)\widehat{\tau} \rangle_\rho.
\end{aligned}$$

Since

$$\begin{aligned} &Zg(X,Y)+Xg(Y,Z)-Yg(X,Z)\\ &\qquad\qquad\qquad = g(\nabla_Z X+\nabla_X Z, Y)+g([Y,Z],X)+g([Y,X],Z) \end{aligned}$$

(see our sign convention for the Lie bracket), we have

$$\begin{aligned} \langle\widehat{\sigma}, -d\iota(\nabla_Z X+\nabla_X Z)\rho\rangle_\rho &= \int_M g(Y,\nabla_Z X+\nabla_X Z)\mathrm{dvol}_\rho\\ &= \int_M \big(Zg(X,Y)+Xg(Y,Z)-Yg(X,Z)\\ &\qquad - g([Y,Z],X)-g([Y,X],Z)\big)\mathrm{dvol}_\rho. \end{aligned}$$

Here we used that $\langle -d\iota(X)\rho, -d\iota(Y)\rho\rangle_\rho = \int_M g(X,Y)\mathrm{dvol}_\rho$ for any vector field X and a vector field Y such that $*^\rho\iota(Y)\rho$ is exact. Then

$$\begin{aligned} &-\int_M g([Y,Z],X)+g([Y,X],Z)\mathrm{dvol}_\rho\\ &= -\langle d\iota([Y,Z])\rho, d\iota(X)\rho\rangle_\rho - \langle d\iota([Y,X])\rho, d\iota(Z)\rho\rangle_\rho\\ &= -\langle \mathcal{L}_{[Y,Z]}\rho, d\iota(X)\rho\rangle_\rho - \langle \mathcal{L}_{[Y,X]}\rho, d\iota(Z)\rho\rangle_\rho\\ &= \langle [\mathcal{L}_Y,\mathcal{L}_Z]\rho, d\iota(X)\rho\rangle_\rho + \langle [\mathcal{L}_Y,\mathcal{L}_X]\rho, d\iota(Z)\rho\rangle_\rho\\ &= -\int_M (\iota(Y)d\iota(Z)\rho)\wedge\rho\wedge\iota(X)g + \int_M (\iota(Z)d\iota(Y)\rho)\wedge\rho\wedge\iota(X)g\\ &\quad -\int_M (\iota(Y)d\iota(X)\rho)\wedge\rho\wedge\iota(Z)g + \int_M (\iota(X)d\iota(Y)\rho)\wedge\rho\wedge\iota(Z)g. \end{aligned}$$

Here we used the identity $\mathcal{L}_{[X,Y]} = -[\mathcal{L}_X,\mathcal{L}_Y]$ for all vector fields X, Y in the third equality. Using the Leibniz rule for the interior product for the first three terms yields,

$$\begin{aligned} &-\int_M g([Y,Z],X)+g([Y,X],Z)\mathrm{dvol}_\rho\\ &= \int_M (d\iota(Z)\rho)\wedge(\iota(Y)\rho)\wedge\iota(X)g + \int_M g(X,Y)(d\iota(Z)\rho)\wedge\rho\\ &\quad -\int_M (d\iota(Y)\rho)\wedge(\iota(Z)\rho)\wedge\iota(X)g - \int_M g(X,Z)(d\iota(Y)\rho)\wedge\rho\\ &\quad +\int_M (d\iota(X)\rho)\wedge(\iota(Y)\rho)\wedge\iota(Z)g + \int_M g(Y,Z)(d\iota(X)\rho)\wedge\rho\\ &\quad +\int_M (\iota(X)d\iota(Y)\rho)\wedge\rho\wedge\iota(Z)g. \end{aligned}$$

Since

$$\begin{aligned}\int_M Xg(Y,Z)\mathrm{dvol}_\rho &= \int_M (\iota(X)dg(Y,Z))\mathrm{dvol}_\rho\\ &= \int_M (dg(Y,Z))(\iota(X)\rho)\wedge\rho\\ &= -\int_M g(Y,Z)(d\iota(X)\rho)\wedge\rho\end{aligned}$$

for all vector fields X, Y, Z we find

$$\begin{aligned}&\langle\widehat{\sigma}, -d\iota(\nabla_Z X + \nabla_X Z)\rho\rangle_\rho\\ &= \int_M (d\iota(Z)\rho)\wedge(\iota(Y)\rho)\wedge\iota(X)g - \int_M (d\iota(Y)\rho)\wedge(\iota(Z)\rho)\wedge\iota(X)g\\ &\quad + \int_M (d\iota(X)\rho)\wedge(\iota(Y)\rho)\wedge\iota(Z)g + \int_M (\iota(X)d\iota(Y)\rho)\wedge\rho\wedge\iota(Z)g\\ &= \int_M B.\end{aligned}$$

Hence

$$\langle\widehat{\sigma}, \frac{1}{2}d\iota(Z)\widehat{\rho} + \frac{1}{2}d\iota(X)\widehat{\tau} - \frac{1}{2}d\iota(\nabla_Z X + \nabla_X Z)\rho\rangle_\rho = \frac{1}{2}\int_M A + B = \Gamma_{\widehat{\sigma}\widehat{\tau}\widehat{\rho}}.$$

This proves the claim and the theorem. □

The following is an immidiate corollary.

Corollary 0.8 (Geodesic Equation). *The geodesic equation on the space $\mathscr{S}_a$ with respect to the Donaldson metric is*

$$\frac{d^2}{dt^2}\rho_t = d\iota(X_t)d\iota(X_t)\rho_t + d\iota(\nabla_{X_t}X_t)\rho_t, \tag{7}$$

where X_t is the associated vector field of $\partial_t\rho_t$.

The next lemma gives an alternative forumula for the covariant derivative.

Lemma 0.9. *Let $\rho_t : \mathbb{R} \to \mathscr{S}_a$ be a smooth path of symplectic forms with $\rho := \rho_0$ and $\widehat{\rho} := \partial_t|_{t=0}\,\rho_t$. Let X be the associated vector field of $\widehat{\rho}$. Let*

$Y_t : \mathbb{R} \to \mathrm{Vect}(M)$ *be a smooth path of vector fields such that* $*^\rho \iota(Y_t)\rho_t$ *is exact and define*

$$\widehat{\sigma}_t := -d\iota(Y_t)\rho_t, \quad \widehat{\sigma} := \widehat{\sigma}_0, \quad Y := Y_0.$$

Then

$$\nabla^\rho_{\widehat{\rho}}\widehat{\sigma} = -d\iota(\widehat{Y} + \nabla_X Y)\rho, \qquad \widehat{Y} := \left.\frac{d}{dt}\right|_{t=0} Y_t.$$

Proof. We have

$$\begin{aligned}\nabla^\rho_{\widehat{\rho}}\widehat{\sigma} &= -\left.\frac{d}{dt}\right|_{t=0} d\iota(Y_t)\rho_t + \frac{1}{2}d\iota(Y)\widehat{\rho} + \frac{1}{2}d\iota(X)\widehat{\sigma} - \frac{1}{2}d\iota(\nabla_Y X + \nabla_X Y)\rho \\ &= -d\iota(\widehat{Y})\rho - d\iota(Y)\widehat{\rho} + \frac{1}{2}d\iota(Y)\widehat{\rho} + \frac{1}{2}d\iota(X)\widehat{\sigma} - \frac{1}{2}d\iota(\nabla_Y X + \nabla_X Y)\rho.\end{aligned}$$

Using the identity $\mathcal{L}_{[X,Y]} = -[\mathcal{L}_X, \mathcal{L}_Y]$ and Cartan's formula for the Lie derivative we compute

$$\begin{aligned}-2d\iota(Y)\widehat{\rho} + d\iota(Y)\widehat{\rho} + d\iota(X)\widehat{\sigma} - d\iota(\nabla_Y X + \nabla_X Y)\rho \\ &= -d\iota(Y)\widehat{\rho} + d\iota(X)\widehat{\sigma} - d\iota(\nabla_Y X + \nabla_X Y)\rho \\ &= \mathcal{L}_Y\mathcal{L}_X\rho - \mathcal{L}_X\mathcal{L}_Y\rho - d\iota(\nabla_Y X + \nabla_X Y)\rho \\ &= -\mathcal{L}_{[Y,X]}\rho - d\iota(\nabla_Y X + \nabla_X Y)\rho \\ &= -2d\iota(\nabla_X Y)\rho.\end{aligned}$$

Hence,

$$\nabla^\rho_{\widehat{\rho}}\widehat{\sigma} = -d\iota(\widehat{Y} + \nabla_X Y)\rho.$$

This proves the lemma. □

S. Donaldson introduced the following energy functional on the space of symplectic structures in a fixed cohomology class in [1],

$$\mathscr{E} : \mathscr{S}_a \to \mathbb{R}, \qquad \mathscr{E}(\rho) := \int_M \frac{2|\rho^+|^2}{|\rho^+|^2 - |\rho^-|^2}\mathrm{dvol}.$$

The functional and the corresponding negative gradient flow with respect to the Donaldson metric are further studied in [2] and [3]. It is shown in [2] that the gradient of $\mathscr{E}$ with respect to the Donaldson metric is the operator

$$\begin{aligned}\mathrm{grad}\mathscr{E} : \mathscr{S}_a &\to T_\rho\mathscr{S}_a \\ \rho &\mapsto -d *^\rho d\Theta^\rho,\end{aligned}$$

where

$$\Theta^\rho := *\frac{\rho}{u} - \frac{1}{2}\left|\frac{\rho}{u}\right|^2 \rho.$$

We compute its associated vector field.

Lemma 0.10. *The associated vector field* $X_{\mathrm{grad}\mathscr{E}}$ *of* $\mathrm{grad}\mathscr{E}(\rho)$ *is given by the two equivalent equations*

$$*^\rho d\Theta^\rho = \iota(X_{\mathrm{grad}\mathscr{E}})\rho \iff d\Theta^\rho = \rho \wedge \iota\left(X_{\mathrm{grad}\mathscr{E}}\right) g. \tag{8}$$

In the hyperKähler case,

$$X_{\mathrm{grad}\mathscr{E}} = -\sum_{i=1}^{3} J_i X_{K_i},$$

where $K_i := \frac{\omega_i \wedge \rho}{\mathrm{dvol}_\rho}$ *and* X_{K_i} *is the Hamiltonian vector field of* K_i *with respect to the symplectic structure* ρ*.*

Proof. It is immediate that a vector field $X_{\mathrm{grad}\mathscr{E}}$ defined by the first equation of (8) satisfies the two conditions (5) for $\widehat{\rho} = \mathrm{grad}\mathscr{E}(\rho) = -d{*^\rho} d\Theta^\rho$. That the second equation is equivalent to the first follows from the identity $*^\rho \iota(X)\rho = -\rho \wedge \iota(X)g$ proved in [2]. In the hyperKähler case it is shown in [2] that $d\Theta^\rho = *^\rho \sum_i \rho(J_i X_{K_i}, \cdot)$. Hence it follows from the first equation in (8) that $\rho(X_{\mathrm{grad}\mathscr{E}}, \cdot) = -\rho(\sum_i J_i X_{K_i}, \cdot)$. This proves the lemma. □

The Hessian operator of the energy functional $\mathscr{E}$ is the operator $\mathcal{H} : T_\rho\mathscr{S}_a \to T_\rho\mathscr{S}_a$ defined by

$$\mathcal{H}_\rho\widehat{\rho} := \nabla^\rho_{\widehat{\rho}}\mathrm{grad}\mathscr{E}(\rho).$$

Associated to this operator is the Hessian quadratic form $\mathscr{H}_\rho : T_\rho\mathscr{S}_a \to \mathbb{R}$ given by

$$\mathscr{H}_\rho(\widehat{\rho}) := \langle \mathcal{H}_\rho\widehat{\rho}, \widehat{\rho}\rangle_\rho.$$

Since ∇^ρ is the Levi-Civita connection of the Donaldson metric, the Hessian quadratic form equals $\frac{d^2}{dt^2}\Big|_{t=0} \mathscr{E}(\rho_t)$ for a curve $\mathbb{R} \to \mathscr{S}_a : t \to \rho_t$ satisfying $\rho_0 = \rho$, $\frac{d}{dt}\big|_{t=0} = \widehat{\rho}$ and $\frac{d^2}{dt^2}\Big|_{t=0} \rho_t = 0$.

Theorem 0.11 (Covariant Hessian). *Let $\rho \in \mathscr{S}_a$. Then the following holds.*

(i) *The Hessian operator of the energy functional $\mathscr{E} : \mathscr{S}_a \to \mathbb{R}$ is the linear operator*

$$\mathcal{H}_\rho \widehat{\rho} = -d *^\rho d\widehat{\Theta} + d *^\rho (\widehat{\rho} \wedge \iota(X_{\mathrm{grad}\mathscr{E}})g) - d\iota\left(\nabla_X X_{\mathrm{grad}\mathscr{E}}\right)\rho, \tag{9}$$

where $\widehat{\Theta} := \frac{\widehat{\rho} + *^\rho \widehat{\rho}}{u} - \left|\frac{\rho^+}{u}\right|^2 \widehat{\rho}$, ∇ denotes the Levi-Civita connection of the metric g and $X, X_{\mathrm{grad}\mathscr{E}}$ are the associated vector fields to $\widehat{\rho}$ respectively $\mathrm{grad}\mathscr{E}(\rho)$.

(ii) *The Hessian of $\mathscr{E}$ is the quadratic form*

$$\mathscr{H}_\rho(\widehat{\rho}) := \int_M \widehat{\Theta} \wedge \widehat{\rho} + \int_M (\iota(X)\widehat{\rho} - \iota(\nabla_X X)\rho) \wedge *^\rho \iota(X_{\mathrm{grad}\mathscr{E}})\rho. \tag{10}$$

(iii) *In the hyperKähler case the Hessian of $\mathscr{E}$ is given by*

$$\mathscr{H}_\rho(\widehat{\rho}) = \int_M \sum_i \left(\widehat{H}_i^2 \mathrm{dvol}_\rho + \omega_i(X, \nabla_{X_{K_i}} X)\right) \mathrm{dvol}_\rho, \tag{11}$$

where $\widehat{H}_i := \frac{(d\iota(X)\omega_i)\wedge\rho}{\mathrm{dvol}_\rho}$ and $K_i := \frac{\omega_i \wedge \rho}{\mathrm{dvol}_\rho}$.

Proof. We prove (i). Let X and $X_{\mathrm{grad}\mathscr{E}}$ be the associated vectorfields of $\widehat{\rho}$ and $\mathrm{grad}\mathscr{E}$. Let $\rho_t : \mathbb{R} \to \mathscr{S}_a$ be a path of symplectic forms such that $\frac{d}{dt}\big|_{t=0} \rho_t = \widehat{\rho}$. By Lemma 0.9

$$\nabla_{\widehat{\rho}} \mathrm{grad}\mathscr{E}(\rho) = -\iota(\widehat{X}_{\mathrm{grad}\mathscr{E}} + \nabla_X X_{\mathrm{grad}\mathscr{E}})\rho$$

where $\widehat{X}_{\mathrm{grad}\mathscr{E}} = \frac{d}{dt}\big|_{t=0} X_{\mathrm{grad}\mathscr{E}}$. By Lemma 0.10 we have $d\Theta^\rho = \rho \wedge \iota\left(X_{\mathrm{grad}\mathscr{E}}\right) g$ and hence

$$d\widehat{\Theta} = \widehat{\rho} \wedge \iota(X_{\mathrm{grad}\mathscr{E}})g + \rho \wedge \iota\left(\widehat{X}_{\mathrm{grad}\mathscr{E}}\right) g,$$

where $\widehat{\Theta} = \frac{d}{dt}\big|_{t=0} \Theta^{\rho_t}$. It follows that

$$\iota(\widehat{X}_{\mathrm{grad}\mathscr{E}})\rho = *^\rho\left(\rho \wedge \iota\left(\widehat{X}_{\mathrm{grad}\mathscr{E}}\right) g\right) = *^\rho d\widehat{\Theta} - *^\rho\left(\widehat{\rho} \wedge \iota\left(X_{\mathrm{grad}\mathscr{E}}\right) g\right).$$

Hence,

$$\nabla_{\widehat{\rho}} \mathrm{grad}\mathscr{E}\,(\rho) = -d *^\rho d\widehat{\Theta} + d *^\rho (\widehat{\rho} \wedge \iota(X_{\mathrm{grad}\mathscr{E}})g) - d\iota\left(\nabla_X X_{\mathrm{grad}\mathscr{E}}\right)\rho.$$

That $\widehat{\Theta} = \frac{\widehat{\rho}+*^\rho\widehat{\rho}}{u} - \left|\frac{\rho^+}{u}\right|\widehat{\rho}$ is proved in [2]. This proves (i).

We prove (ii). By part (i)

$$\begin{aligned}
\mathscr{H}_\rho(\widehat{\rho}) &= \langle \mathcal{H}_\rho(\widehat{\rho}), \widehat{\rho}\rangle_\rho \\
&= \langle -d *^\rho d \left(\frac{\widehat{\rho}+*^\rho\widehat{\rho}}{u} - \left|\frac{\rho^+}{u}\right|^2 \widehat{\rho}\right), \widehat{\rho}\rangle_\rho \\
&\qquad + \langle d *^\rho \left(\widehat{\rho} \wedge \iota(X_{\mathrm{grad}\mathscr{E}})g\right), \widehat{\rho}\rangle_\rho + \langle -d\iota\left(\nabla_X X_{\mathrm{grad}\mathscr{E}}\right)\rho, \widehat{\rho}\rangle_\rho \\
&=: A + B + C.
\end{aligned}$$

By the definition of the Donaldson metric

$$\begin{aligned}
A &= \int_M \left(*^\rho d\left(\frac{\widehat{\rho}+*^\rho\widehat{\rho}}{u} - \left|\frac{\rho^+}{u}\right|^2 \widehat{\rho}\right)\right) \wedge *^\rho \iota(X)\rho \\
&= \int_M \left(\frac{\widehat{\rho}+*^\rho\widehat{\rho}}{u} - \left|\frac{\rho^+}{u}\right|^2 \widehat{\rho}\right) \wedge (-d\iota(X)\rho) \\
&= \int_M \left(\frac{\widehat{\rho}+*^\rho\widehat{\rho}}{u} - \left|\frac{\rho^+}{u}\right|^2 \widehat{\rho}\right) \wedge \widehat{\rho}.
\end{aligned}$$

Likewise,

$$\begin{aligned}
B &= -\int_M \widehat{\rho} \wedge (\iota(X_{\mathrm{grad}\mathscr{E}})g) \wedge \iota(X)\rho \\
&= -\int_M (\iota(X)\widehat{\rho}) \wedge \left(\iota\left(X_{\mathrm{grad}\mathscr{E}}\right)g\right) \wedge \rho - \int_M g(X_{\mathrm{grad}\mathscr{E}}, X)\,\widehat{\rho} \wedge \rho \\
&= \int_M (\iota(X)\widehat{\rho}) \wedge *^\rho\left(\iota\left(X_{\mathrm{grad}\mathscr{E}}\right)\rho\right) - \int_M g(X_{\mathrm{grad}\mathscr{E}}, X)\,\widehat{\rho} \wedge \rho.
\end{aligned}$$

Since $\langle -d\iota(X)\rho, -d\iota(Y)\rho\rangle_\rho = \int_M(\iota(X)\rho) \wedge *^\rho\iota(Y)\rho = \int_M g(X,Y)\mathrm{dvol}_\rho$ for X associated to $\widehat{\rho}$ and Y an arbitrary vector field we have

$$\begin{aligned}
C &= \int_M g(\nabla_X X_{\mathrm{grad}\mathscr{E}}, X)\mathrm{dvol}_\rho \\
&= \int_M \left(\iota(X)dg(X_{\mathrm{grad}\mathscr{E}}, X) - g(X_{\mathrm{grad}\mathscr{E}}, \nabla_X X)\right)\mathrm{dvol}_\rho \\
&= \int_M dg(X_{\mathrm{grad}\mathscr{E}}, X) \wedge (\iota(X)\rho) \wedge \rho - \int_M g(X_{\mathrm{grad}\mathscr{E}}, \nabla_X X)\mathrm{dvol}_\rho \\
&= \int_M g(X_{\mathrm{grad}\mathscr{E}}, X)\,\widehat{\rho} \wedge \rho - \int_M \iota(\nabla_X X)\rho \wedge *^\rho \iota(X_{\mathrm{grad}\mathscr{E}})\rho.
\end{aligned}$$

Hence,

$$A+B+C=\int_M\left(\frac{\widehat{\rho}+*^\rho\widehat{\rho}}{u}-\left|\frac{\rho^+}{u}\right|^2\widehat{\rho}\right)\wedge\widehat{\rho}$$
$$+\int_M\left(\iota(X)\widehat{\rho}-\iota(\nabla_X X)\rho\right)\wedge *^\rho\iota(X_{\mathrm{grad}\mathscr{E}})\rho.$$

This proves (iii).

We prove (iii). Assume the hyperKähler case. The following identities are proved in [2],

$$\mathrm{grad}\mathscr{E}(\rho)=d\sum_i^3 dK_i\circ J_i^\rho$$
$$\widehat{\Theta}=\sum_i^3\widehat{K}_i^2\omega_i^\rho-\frac{1}{2}\sum_i^3 K_i^2\widehat{\rho}$$
$$\int_M\widehat{\Theta}\wedge\widehat{\rho}=\int_M\sum_i\left(\widehat{K}_i^2\mathrm{dvol}_\rho-\frac{1}{2}K_i^2\widehat{\rho}\wedge\widehat{\rho}\right),$$

where $\widehat{K}_i=\frac{\omega_i^\rho\wedge\widehat{\rho}}{\mathrm{dvol}_\rho}$, $\rho(J_i^\rho\cdot,\cdot):=\rho(\cdot,J_i\cdot)$ and $\omega_i^\rho=\omega_i-K_i\rho$. From (i) we have

$$\mathscr{H}_\rho(\widehat{\rho})=\int_M\widehat{\Theta}\wedge\widehat{\rho}-\int_M\widehat{\rho}\wedge\iota\left(X_{\mathrm{grad}\mathscr{E}}\right)g\wedge\iota(X)\rho$$
$$+\int_M g\left(\nabla_X X_{\mathrm{grad}\mathscr{E}},X\right)\mathrm{dvol}_\rho$$
$$=:\int_M\widehat{\Theta}\wedge\widehat{\rho}+D+E.$$

By Lemma 0.10 we have $X_{\mathrm{grad}\mathscr{E}}=-\sum_i J_iX_{K_i}$. Therefore

$$D=-\int_M\widehat{\rho}\wedge\iota\left(X_{\mathrm{grad}\mathscr{E}}\right)g\wedge\iota(X)\rho=\int_M\sum_i\iota\left(X_{K_i}\right)\omega_i\wedge\iota\left(X\right)\rho\wedge\widehat{\rho}$$

and

$$E=\int_M g\left(\nabla_X X_{\mathrm{grad}\mathscr{E}},X\right)\mathrm{dvol}_\rho=\int_M\sum_i\omega_i\left(X,\nabla_X X_{K_i}\right)\mathrm{dvol}_\rho.$$

It now follows from Lemma 4.3 in [2] that

$$\begin{aligned}\mathscr{H}_\rho(\widehat{\rho}) &= \int_M \sum_i \left(\widehat{K}_i^2 \mathrm{dvol}_\rho - \frac{1}{2} K_i^2 \widehat{\rho} \wedge \widehat{\rho} \right) + D + E \\ &= \int_M \sum_i \left(\widehat{H}_i^2 \mathrm{dvol}_\rho + \omega_i \left(X, \nabla_{X_{K_i}} X \right) \right) \mathrm{dvol}_\rho.\end{aligned}$$

This proves (iii) and the theorem. □

Remark 0.12. *The Hessian operator* $\mathcal{H} : T_\rho \mathscr{S}_a \to T_\rho \mathscr{S}_a$ *given by* (9) *is a non-local differential operator of degree two. It is non-local because of the last term* $d\iota\left(\nabla_X X_{\mathrm{grad}\mathscr{E}}\right)\rho$*, which involves solving the equation*

$$-d\iota(X)\rho = \widehat{\rho}, \qquad *^\rho \iota(X)\rho \in \mathrm{im} d$$

for the associated vector field X *of* $\widehat{\rho}$*. Its leading term*

$$\begin{aligned}-d *^\rho d\widehat{\Theta} &= -d *^\rho d \left(\frac{\widehat{\rho} + *^\rho \widehat{\rho}}{u} - \left| \frac{\rho^+}{u} \right|^2 \widehat{\rho} \right) \\ &= -2d\frac{*^\rho}{u} d\widehat{\rho}^{+\rho} + d *^\rho \left(\frac{du}{u^2} \wedge (\widehat{\rho} + *^\rho \widehat{\rho}) \right) + d *^\rho \left(d \left| \frac{\rho^+}{u} \right|^2 \wedge \widehat{\rho} \right) \\ &= (d^{*^\rho} \frac{1}{u} d + d \frac{1}{u} d^{*\rho}) \widehat{\rho} + d *^\rho \left(\frac{du}{u^2} \wedge (\widehat{\rho} + *^\rho \widehat{\rho}) \right) \\ &\qquad + d *^\rho \left(d \left| \frac{\rho^+}{u} \right|^2 \wedge \widehat{\rho} \right)\end{aligned}$$

is an elliptic differential operator.

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