DEPENDENCE UNCERTAINTY BOUNDS AND OPTIMIZATION OF AGGREGATE RISK

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Dependence Uncertainty Bounds
and Optimization of Aggregate Risk

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Abstract

An essential tool in financial risk management are risk measures, which allow quantifying and comparing the risk inherent in the uncertain future values of financial positions or insurance claims. Risk measures are used by risk managers to control risk exposures within a company, as well as by financial authorities to determine capital requirements under the Basel and Solvency regulatory frameworks for banks and insurance companies, respectively.

This Thesis considers the optimization of risk measures under two mathematical setups. The first one is in the context of model uncertainty, in particular, dependence uncertainty (DU) between several risks. While the losses within individual business lines or risk types can perhaps be modeled and estimated to a satisfactory degree, their interdependence may be difficult or impossible to model adequately. The framework of DU is an idealization of these circumstances, where the individual risks are modeled perfectly, but their dependence is completely unknown. This impedes evaluating the total (aggregate) risk that the company is exposed to. Paper A provides the theoretical basis as well as numerical means for constructing the “safest” distribution of the aggregate risk, represented mathematically as being minimal with respect to the so-called convex order between random variables. This allows determining the lowest possible value (lower bound) of the Expected Shortfall (ES, a risk measure that is coherent and, in particular, respects the convex order), over all aggregate risks that are consistent with the known distributions of individual risks. Also the upper and lower bound of the most commonly used risk measure, Value-at-Risk (VaR), can be obtained, notwithstanding that VaR does not obey the convex order. Paper B provides numerical examples that complement recent methodological development and theoretical results. In this paper, DU bounds that are calculated using various methods, including simple approximations, are compared. Paper C studies the DU bounds for the expectile, a risk measure that has recently gained attention due to its favorable theoretical properties. Since the expectile is a convex risk measure, the earlier results on convex order apply.

DU bounds are often characterized by extreme negative dependence, and recently a related concept called Σ-countermonotonicity has been introduced. Paper D shows that this concept is relevant for optimizing a subclass of so-called arrangement increasing functions, and illustrates with examples from scheduling and systems assembly. Negative dependence structures can be
attained using an efficient heuristic, called the Rearrangement Algorithm (RA), and a modification of it, the Block RA. Paper E analyzes the performance of the Block RA and a further adaptation that uses Variance Equalization (BRAVE), and demonstrates a superior performance of the latter when applied to the partitioning problem, a classical NP-hard optimization problem.

The second setup of risk optimization is that of finding optimal portfolio weights under full knowledge of the joint asset returns distribution. Rockafellar and Uryasev (2000) formulated the portfolio optimization problem with ES as the objective, and noted that in the case of a discrete asset returns distribution, it is a linear program (LP). This is of practical importance, since any distribution can be approximated by a discrete one, e.g. by Monte Carlo (MC) sampling, and LPs can be solved efficiently. Paper F investigates the quality of the approximate solutions corresponding to such MC samples, and compares the computation times using three different LP formulations. Paper G develops analogous LP formulations for the portfolio optimization problem with the expectile as the objective, and analyzes their performance.
Kurzfassung

Wichtige Instrumente im finanziellen Risikomanagement sind Risikomasse, welche es erlauben, das Risiko inhärent in den ungewissen zukünftigen Werten von finanziellen Anlagen oder Versicherungsansprüchen zu quantifizieren oder zu vergleichen.


DU Schranken sind oft charakterisiert durch starke negative Abhängigkeit, und vor kurzem wurde ein verwandtes Konzept, genannt Σ-countermonotonicity, eingeführt. Paper D zeigt, dass dieses Konzept relevant für die Optimierung einer Unterklassen von sogenannten arrangement increasing Funktionen ist, und es illustriert dies mit Beispielen betreffend Zuordnungsplanung und Systemaufbau. Strukturen negativer Abhängigkeit können erreicht werden unter Benützung eines effizienten Algorithmus, genannt Rearrangement Algorithm (RA), und einer Modifikation von diesem, den Block RA. Paper E untersucht die Leistungen des Block RA und
einer weiteren Adaption, welche die Varianzen-Gleichstellung (Variance Equalization) benützt, und demonstriert eine ausgezeichnete Leistung des Letzteren, wenn dieses angewandt wird auf das Partitionsproblem, ein klassisches NP-schweres Optimierungsproblem.

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Accompanying papers

A  Edgars Jakobsons, Xiaoying Han, Ruodu Wang.
   General convex order on risk aggregation.

B  Paul Embrechts, Edgars Jakobsons.
   Dependence uncertainty for aggregate risk: examples and simple bounds.
   In Podolskij, M., Stelzer, R., Thorbjørnsen, S., and Veraart, A., editors, The Fascination
   of Probability, Statistics and their Applications: In Honour of Ole E. Barndorff-Nielsen,

C  Edgars Jakobsons, Steven Vanduffel.
   Dependence uncertainty bounds for the expectile of a portfolio.

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   Block rearranging elements within matrix columns to minimize the variability of the
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1 Introduction

Risk measures are a central tool for financial risk management in banks and insurance companies, and have also been applied in other industries, for example, to facilitate revenue management or hydropower generation planning. Risk measures summarize random future losses into a number, which provides a way of comparing the risk associated to different decisions at the present time. In the financial industry, risk measures are used mainly for two purposes. From a regulatory point of view, to calculate the regulatory capital that a bank or insurance company is required to hold in order to cover the risks that it is running. From a risk management point of view, as a basis for decision making and limiting risk exposure to acceptable levels, according to the available economic capital.

This thesis considers optimization of risk measures in two contexts: dependence uncertainty bounds, and portfolio optimization. The first issue is more relevant from the regulatory point of view, for example, for the policymakers to choose a risk measure that has suitable properties, or for supervisory authorities to assess model sensitivity on a case-by-case basis. Portfolio optimization is more relevant in asset management, for selecting portfolio positions that minimize market risk.

The summary of this Thesis is structured as follows. Chapter 2 discusses dependence uncertainty (DU), outlining the results on DU bounds for different risk measures in Papers A, B and C. Chapter 3 considers matrix arrangement problems in operations research; we first comment on the connections between extremal negative dependence from probability theory and similar structures in scheduling and systems assembly problems, studied in Paper D. Then, we present and analyze a heuristic algorithm for obtaining approximate solutions to matrix arrangement problems (Paper E). Chapter 4 considers portfolio optimization, analyzing two problems with different risk measures as the objectives. First, the minimization of the portfolio Expected Shortfall and the performance of its linear programming (LP) approximations is discussed, with a focus on the suboptimality of the resulting portfolios, summarizing the findings of Paper F. Then, we outline the ideas and results in Paper G, where analogous LP formulations are developed and investigated for the expectile risk measure.
In the following summary, we shall omit technical details and proofs, with the aim of providing a general overview of the results in this Thesis, as well as explaining the underlying ideas and interpretation. However, full details can be found in the accompanying papers. Two excellent references providing the reader with an introduction to most of the topics considered in this Thesis are McNeil et al. (2015) and Rüschendorf (2013).
2 Dependence uncertainty

It was clearly recognized after the financial crisis of 2007-2009 that model uncertainty had played a crucial role, since the risk exposure inherent in the financial positions was not fully appreciated and thus insufficient capital was set aside for the potential losses (which, as we know, materialized). The capital charges were highly sensitive to the underlying model assumptions, which were not always justified. For example, a trader at JPMorgan Chase was able to accumulate an outsized position in credit default swaps (CDS) because of “inadequate” models. The later investigation revealed that a “$7 billion, or more than 50% of the total $13 billion RWA\(^1\) reduction, could be achieved by modifying risk related models” (US Senate, 2013, p. 171); see also Jacque (2015) for an analysis of this and other examples of flawed financial engineering.

A particular challenge in risk management is dependence modeling, which is an essential step in risk aggregation. A discussion paper by the Basel Committee on Banking Supervision (BCBS, 2010) stresses the importance of modeling dependence beyond linear correlation, as highlighted earlier in Embrechts et al. (2002). BCBS (2010) defines risk aggregation as “the process of combining less-comprehensive measures of the risks within a firm to obtain more comprehensive measures”. In this section, we will use this term in a narrower sense: to mean the process of calculating the aggregate risk measure \(\rho(X_1 + \cdots + X_n)\), where the \(X_i, i = 1, \ldots, n\), are random variables (rvs) corresponding, for example, to losses within different business lines or risk types, and \(\rho\) is a risk measure, such as the popular Value-at-Risk (VaR) or Expected Shortfall (ES). Modeling the individual risks \(X_i\) may be easier than modeling the joint distribution of the random vector \((X_1, \ldots, X_n)\), because data on joint losses may be unavailable or insufficient, and the curse of dimensionality hinders statistical estimation of (multiparameter) multivariate models. Therefore, the whole range of possible values that the aggregate risk measure can attain by assuming various dependence structures, in good faith or otherwise, is of interest. For the policymaker, it allows selecting a risk measure for which the corresponding capital requirements are less susceptible to misspecification in dependence modeling. For the risk manager, this shows where the estimated risk lies within the overall possible range, and serves as a safeguard against overly optimistic models.

As a simplifying idealization, we will assume that the univariate distributions of the risks

\(^1\)RWA is a dollar measure of a bank’s assets, adjusted according to the assets’ risk.
are known, while their interdependence is completely unknown. This type of model uncertainty is referred to as dependency uncertainty (DU). In this context, an aggregate risk $S$ is called an admissible risk for given marginal distributions $F_1, \ldots, F_n$ if it can be expressed as $S = X_1 + \cdots + X_n$ for some $X_i \sim F_i$, $i = 1, \ldots, n$. The admissible risk class is defined by the set of admissible risks for the given marginal distributions:

\[ \Xi_n(F_1, \ldots, F_n) = \{X_1 + \cdots + X_n : X_i \in L^0(\Omega, \mathcal{A}, P), X_i \sim F_i, i = 1, \ldots, n \}. \tag{2.1} \]

We will be interested in finding the range of values a risk measure $\rho$ can take for different aggregate losses $S \in \Xi_n$. In particular, define the best-possible lower bound and the best-possible upper bound as

\[ \underline{\rho} = \inf \{\rho(S) : S \in \Xi_n\} \quad \text{and} \quad \bar{\rho} = \sup \{\rho(S) : S \in \Xi_n\}, \tag{2.2} \]

where the risk measure $\rho$ will be either VaR, ES or (in Section 2.3) the expectile. It turns out that a useful tool for studying the DU bounds in (2.2) is the convex order.

**Definition 1.** Let $X$ and $Y$ be two random variables with finite means. Then $X$ is smaller than $Y$ in convex order, denoted by $X \preceq_{\text{cx}} Y$, if for all convex functions $f$, $E[f(X)] \leq E[f(Y)]$, provided that the two expectations exist.

The usefulness of this ordering stems from the fact that convex risk measures that are determined by the distribution of the loss (also called law-invariant) are consistent with the convex order,\(^2\) in the sense that $X \preceq_{\text{cx}} Y$ implies $\rho(X) \leq \rho(Y)$. ES and the expectile are such risk measures, hence, to determine the corresponding DU bounds, it is sufficient to find elements in $\Xi_n(F_1, \ldots, F_n)$ that are minimal, respectively, maximal in the sense of convex order. Since the convex order is defined only for integrable rvs, we assume in the following that the distributions $F_i$, $i = 1, \ldots, n$, have finite means. As a further application, also the bounds on VaR can be computed based on minimal elements with respect to the convex order in suitably adjusted admissible classes, despite the fact that VaR is not a convex risk measure; see Bernard et al. (2015a, 2014) for this result. Furthermore, note that ES, VaR and the expectile are law-invariant risk measures, and also the convex ordering is defined in terms of the distribution functions. Thus, it suffices to find the distribution function (df) of a minimal and a maximal element in the admissible class. Therefore, we require that the underlying probability space $(\Omega, \mathcal{A}, P)$ is atomless; this condition is equivalent to the existence of a rv with the uniform distribution on $(0, 1)$, and allows us to construct random variables with arbitrary dfs.

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\(^2\)In particular, combining the results in Bäuerle and Müller (2006) and Jouini et al. (2006), follows that any convex, law-invariant risk measure respects the convex order.
It is shown in Tchen (1980) that the comonotonic dependence structure leads to the maximal element (denoted $S^c$) with respect to the convex order in the admissible class $\mathcal{S}_n$; see also Dhaene et al. (2002).

\[ \forall S \in \mathcal{S}_n (F_1, \ldots, F_n) : S \preceq_{\text{cx}} S^c := \sum_{i=1}^n F_i^{-1}(U), \quad \text{where} \ U \sim \text{UNIF}(0, 1). \quad (2.3) \]

As a consequence, this yields the upper DU bound on the Expected Shortfall. Recall the definition of VaR and ES for a rv $X$ interpreted as loss:

\[
\text{VaR}_\alpha(X) = \inf \{ x \in \mathbb{R} : P(X \leq x) \geq \alpha \}, \quad \text{ES}_\beta(X) = \frac{1}{1-\beta} \int_0^1 \text{VaR}_\eta(X) \, d\eta,
\]

where the latter is defined only when $X$ has a finite mean. This condition is satisfied by the rvs we consider, as all risks in the admissible class have a finite mean, equal to $E[X_1] + \ldots + E[X_n]$. Denote by $\text{VaR}_\alpha^+(S)$ and $\text{ES}_\alpha^+(S)$ the risk measures $\text{VaR}_\alpha(S)$ and $\text{ES}_\alpha(S)$ respectively, when the risks $X_i$ are comonotonic. Then the upper DU bound on ES is

\[
\overline{\text{ES}}_\alpha(S) = \sup \{ \text{ES}_\alpha(X_1 + \ldots + X_n) : X_i \sim F_i, i = 1, \ldots, n \} = \text{ES}_\alpha^+(S) = \sum_{i=1}^n \text{ES}_\alpha(X_i),
\]

where the last equality follows because ES is comonotonic additive (see also p. 251 in McNeil et al. (2005) for this example). For VaR, the upper bound is not given by $\text{VaR}_\alpha^+(S)$ in general, since it is not a convex risk measure; see McNeil et al. (2005), p. 241.

A method for obtaining the minimal element in the admissible class with respect to the convex order would lead to the best lower bound for ES and other coherent,3 law-invariant risk measures, as well as provide both DU bounds for VaR. Construction of convex order minimal elements is the focus of Paper A, which is summarized in Section 2.1. To complement the recent methodological advances on the topic of DU, a numerical case study on DU bounds for VaR and ES is conducted in Paper B, and summarized in Section 2.2. An intriguing contender for VaR and ES is the expectile risk measure, which has recently been shown to have favorable theoretical properties, including coherence; hence, the earlier results on convex order can be applied. Paper C provides an introduction to working with the expectile, uses the existing and develops new results on the convex order that lead to DU bounds on the expectile. The paper also considers bounds when some dependence information is available, for example, on the variance of aggregate risk, or a factor structure for the marginal risks. An overview of Paper C is given in Section 2.3.

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3See the seminal paper Artzner et al. (1999), where the four properties (axioms) of coherence are introduced: monotonicity, translation invariance, positive homogeneity and subadditivity.


2.1 General convex order bounds

When there are only two variables, \( n = 2 \), the minimum sum with respect to the convex order is obtained using the \textit{countermonotonic} dependence structure:

\[
F_1^{-1}(U) + F_2^{-1}(1 - U) \preceq_{\text{cx}} S_2 \quad \text{for any } S_2 \in \mathcal{S}_2(F_1, F_2),
\]

where \( U \sim \text{UNIF}(0,1) \). This result can be found in Tchen (1980) and is closely related to the problem on the bounds for the distribution function of \( S_2 \), posed by Kolmogorov and solved by Makarov (1982). Countermonotonicity, however, cannot be generalized to \( n \geq 3 \) in a way that still guarantees the existence as well as minimality of such a dependence structure in \( \mathcal{S}_n(F_1, \ldots, F_n) \). Nonetheless, in two special cases, the minimal element in the admissible risk class is known explicitly: when the admissible class is compatible with mutual exclusivity, and when the class contains a constant element.

Mutual exclusivity was introduced in Dhaene and Denuit (1999) (earlier mathematical results can be found in Dall’Aglio (1972)). In the special case when \( F_1, \ldots, F_n \) have support \( \mathbb{R}_+ \) and \( \sum_{i=1}^{n} F_i(0) \geq n - 1 \), the convex minimum is obtained by the \textit{mutually exclusive} scenario:

\[
Y_1 + \cdots + Y_n \preceq_{\text{cx}} S \quad \text{for any } S \in \mathcal{S}_n,
\]

where \( Y_i \sim F_i \) and \( P(Y_i > 0, Y_j > 0) = 0, i, j = 1, \ldots, n, i \neq j \), i.e. at most one random variable is positive in each outcome. However, this assumption means that the distributions \( F_1, \ldots, F_n \) have atoms at zero with a very large total mass, and hence it is rather restrictive.

Another special case is when the admissible class contains a constant; it is clearly minimal by Jensen’s inequality. In this case, the distributions \( F_1, \ldots, F_n \) are said to be \textit{jointly mixable} (JM). If the marginal distributions are all equal, \( F_i = F, \, i = 1, \ldots, n \), the admissible class is \( \mathcal{S}_n(F, \ldots, F) \); and if it contains a constant, we say that \( F \) is \textit{n-completely mixable} (n-CM). These notions were introduced in Wang et al. (2013) and Wang and Wang (2011), respectively.

2.1.1 Homogeneous case

In Bernard et al. (2014), the case \( F_i = F \) (called \textit{homogeneous} in the following) is considered and a minimal element is constructed by combining the ideas of mutual exclusivity and mixability. Specifically, they define functions \( H(x), D(a) \) for \( a, x \in [0, 1/n] \), and the number \( c_n \):

\[
H(x) = (n - 1)F^{-1}((n - 1)x) + F^{-1}(1 - x),
\]

\[
D(a) = \frac{n}{1 - na} \int_{(n-1)a}^{1-a} F^{-1}(y) \, dy,
\]

\[
c_n = \min \{ c \in [0, 1/n] : H(c) \preceq D(c) \}.
\]
Intuitively speaking, \( H \) represents the sum in a “nearly” mutually exclusive scenario, where one large loss is coupled with \( n - 1 \) small (but non-zero) losses. The distribution of the minimal element is constructed by gradually combining the outcomes of the \( n \) margins, while making sure that exactly the same probability mass \( nx \) has been used up for each margin at every stage \( x \in [0, 1/n] \). Since only one margin takes a large value \( F^{-1}(1 - x) \) at any given outcome, and the other \( n - 1 \) margins take a small value \( F^{-1}((n - 1)x) \), the probability mass in the lower tails is being used up \( n - 1 \) times faster. The function \( D \) keeps track of the expected sum on the remaining (unused) interval, and represents the sum of the risks under the best-case dependence structure, a joint mix. These two scenarios can then be combined, defining a candidate rv for the lower convex bound:

\[
T_a = H(U/n) \mathbb{1}_{\{U \in (0,a)\}} + D(a) \mathbb{1}_{\{U \in [na,1]\}} , \quad \text{where} \quad U \sim \text{UNIF}(0,1).
\] (2.4)

The threshold \( a \) determines the switching point between the two behaviors. Bernard et al. (2014) show that, in order for \( T_a \) to be a lower bound on \( S_n \), meaning that \( T_a \leq c_n S \) for any \( S \in \mathcal{E}_n \), it is sufficient that

(A) the function \( H(x) \) is non-increasing on \( x \in [0,a] \), \( a \leq c_n \),

and in order for \( T_a \) to belong to \( \mathcal{E}_n \), we need that

(B) the distribution \( F \) restricted to the interval \([F^{-1}((n - 1)a), F^{-1}(1 - a)]\) is n-CM.

The threshold \( c_n \) is the optimal one, in the sense that it is the only value of \( a \) for which both assumptions (A) and (B) can be satisfied, and hence only \( T_{c_n} \) can be a minimal element in \( \mathcal{E}_n \). Finding \( c_n \) corresponds to solving an integral equation; the numerical challenges are discussed in Hofert et al. (2015). Bernard et al. (2014) also prove that if \( F \) has a decreasing density on a lower-bounded support (without loss of generality, on \( \mathbb{R}_+ \)), then both assumptions (A) and (B) are indeed satisfied. The assumption (B) in the case of a monotone density follows from Wang and Wang (2011), and for a concave density from Puccetti et al. (2012). In other cases, one may need to resort to verifying these conditions numerically, for example, using the procedure in Puccetti and Wang (2015a) to detect mixability.

### 2.1.2 Inhomogeneous case

In Paper A, we consider the case of (potentially) different marginal distributions \( F_1, \ldots, F_n \) (the inhomogeneous case). The symmetry arguments that were used in the construction of \( T_a \) no longer apply, therefore the lower convex bound obtained in this paper requires a new technique of dynamically weighting the marginal distributions. To find the optimal coupling of the different margins, we introduce the following functional equations (E1)-(E2) in terms of functions \( y, y_1, \ldots, y_n: (0, 1) \to \mathbb{R}_+ \) (we assume \( F_1, \ldots, F_n \) have support \( \mathbb{R}_+ \)). For all \( x \in (0, 1) \),
(E1) \( \sum_{i=1}^{n} \tilde{F}_i(y_i(x)) = x, \)

(E2) \( F_i(y_i(x) - y(x)) + \tilde{F}_i(y_i(x)) = x, \) for each \( i = 1, \ldots, n. \)

The approach is similar to the construction of \( H: \) here, we again combine a large outcome \( y_i(x) \) in one tail, with small outcomes \( y_j(x) - y(x) \) in the other tails, \( j \neq i. \) By (E2), the total probability mass used up in each margin is \( x, \) and (E1) ensures that only one margin is large in any outcome (mutual exclusivity). The function \( y \) probability mass used up in each margin is \( x, \)

The approach is similar to the construction of \( H: \) here, we again combine a large outcome \( y_i(x) \) in one tail, with small outcomes \( y_j(x) - y(x) \) in the other tails, \( j \neq i. \) By (E2), the total probability mass used up in each margin is \( x, \) and (E1) ensures that only one margin is large in any outcome (mutual exclusivity). The function \( y \) keeps track of the length of the remaining unused interval on the support of each margin, making sure that they are the same for each given \( x, \) and therefore the sum of the marginal outcomes is the same, irrespective of which margin we consider as the large outcome. Based on a solution \( (y, y_1, \ldots, y_n), \) we define functions analogous to \( H \) and \( D, \) but adapted for the inhomogeneous case:

\[
    h(x) = \sum_{i=1}^{n} y_i(x) - (n - 1)y(x), \quad x \in (0, 1),
\]

\[
    d(a) = \frac{1}{1 - a} \sum_{i=1}^{n} E \left[ X_i 1_{y_i(a) - y(a) < X_i < y_i(a)} \right], \quad a \in (0, 1),
\]

\[
    s_n = \inf \left\{ s \in (0, 1) : h(s) \leq d(s) \right\}.
\]

The function \( h \) corresponds to the \( (1 - x) \)-quantile of the sum when the margins are approximately mutually exclusive. The function \( d \) in this case is explicitly written out as the sum of conditional expectations on the unused intervals of each margin; this corresponds to the sum of risks in the jointly mixable scenario. We define a candidate lower bound element as follows:

\[
    R_a = h(U) 1_{\{U \in (0,a)\}} + d(a) 1_{\{U \in [a,1]\}}, \quad \text{where} \quad U \sim \text{UNIF}(0, 1).
\]

To show that \( R_a, a \leq s_n \) is a lower bound in \( \Xi_n, \) we use that

(C) \( h, y, y_1, \ldots, y_n \) satisfy monotonicity properties on \( (0, a), \)

which are specified and analyzed in Paper A. To show \( R_a \in \Xi_n, \) it is necessary that

(D) the marginal distributions \( F_i \) restricted to the interval \( [y_i(a) - y(a), y_i(a)] \) are JM.

Again, it is clear that \( s_n \) is the optimal threshold for \( a. \) We show that assumptions (C)-(D) are satisfied for \( a = s_n \) if each \( F_i, i = 1, \ldots, n, \) has a decreasing density on its support \( [0, \infty), \)

using a result on JM from Wang and Wang (2016). It follows that \( R_n \) is a minimal element in the admissible class. In other cases, the conditions (C) and (D) may need to be verified numerically. For details, see Theorem 6 in Paper A, where the assumptions and resulting properties of \( R_n \) are rigorously stated, and which forms the main result of this paper.

The proof that \( R_a \leq_S S \) for any \( S \in \Xi_n, a \leq s_n, \) is based on the fact that for rvs with the same mean, convex order is equivalent to

\[
    \int_{c}^{1} F_{R_a}^{-1}(t) \, dt \leq \int_{c}^{1} F_{S}^{-1}(t) \, dt, \quad \forall c \in (0, 1).
\]

\[\text{See e.g. Theorem 2.5 in Bäuerle and Müller (2006).}\]
To prove this inequality, events of the form $A(x) = \cup_{i=1}^{n} \{ X_i > y_i(x) \}$ are considered, where $X_1 + \cdots + X_n = S$. For our construction of $R_n$, the analogous event is a union of mutually exclusive events, moreover, the expected sum on this event, $E[\mathbb{1}_{A(x)}]$, $x \leq s_n$, is minimized, because when $X_i > y_i(x)$ for some $i$, all the other margins take the smallest possible values, $X_j < y_j(x) - y(x)$, $j \neq i$.

For marginal distributions with decreasing densities, the problem of convex order bounds in the admissible class is thus completely solved. For insurance applications, the generalized Pareto distributions are a relevant example; for applications in reliability - Weibull distributions with parameter $\beta$ are a relevant example; for applications in reliability - Weibull distributions with parameter $\beta \leq 1$ (in particular, the exponential distribution, $\beta = 1$). However, the importance of this result is not limited to these two families of distributions. It also provides a practical method of numerically computing bounds for various functionals (including best-possible bounds for VaR) in more general cases, as explained in the next section.

### 2.1.3 Example and applications

Another method for obtaining (approximate) minimal elements in the admissible class is the Re-arrangement Algorithm (RA), introduced in Puccetti and R"uschendorf (2012). This algorithm uses a discrete approximation of the marginal distributions, and represents the joint distribution by a matrix, where the columns correspond to the outcomes of each margin, and the rows correspond to joint outcomes (see Paper D for an in-depth discussion on matrix arrangement problems and majorization order, which is the discrete analogue of convex order). The pseudo-code of the RA is provided in Section 3.2.

To illustrate the typical dependence structures that correspond to the minimal elements, we consider an example with $n = 3$ and marginal distributions $F_1 = \text{Pareto}(\theta = 3)$, $F_2 = \text{LogN}(\mu = 1, \sigma = 0.5)$ and $F_3 = \text{Gamma}(\alpha = 3, \beta = 1)$. In Figure 2.1, we plot the support of the joint distribution corresponding to the convex-order-minimal aggregate risk obtained using the RA with 1000 discretization points, as well as three curves $\gamma_1, \gamma_2, \gamma_3$ based on the solution of (E1)-(E2) and parameterized by $x \in (0, s_n) = (0, 0.35)$:

\[
\begin{align*}
\gamma_1(x) &= (y_1(x), y_2(x) - y(x), y_2(x) - y(x)), \\
\gamma_2(x) &= (y_1(x) - y(x), y_2(x), y_2(x) - y(x)), \\
\gamma_3(x) &= (y_1(x) - y(x), y_2(x) - y(x), y_2(x)).
\end{align*}
\]

Curve $\gamma_i$ corresponds to the support of the joint distribution on the event $\{ X_i > y_i(s_n) \}$, $i = 1, 2, 3$ (the nearly mutually exclusive part of the dependence structure). Under the joint mixability assumption (D), the remaining part of the joint distribution’s support belongs to the plane $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = d(s_n) = 6.2\}$. The solution $(y, y_1, \ldots, y_n)$ was obtained by numerically solving a system of differential equations, which are based on Equations (E1)-(E2), after taking derivatives. Note that, since the log-normal and gamma distributions do not
Figure 2.1: The best-case dependence structure given by the RA based on a discretization of margins with 1000 points (gray), superimposed with the curves (2.5) obtained from the functional equations (E1)-(E2) (black, up to \( x = s_n = 0.35 \), the endpoints are marked by “+”). The curves overlap the corresponding RA points exactly.

have decreasing densities, it is not known \textit{a priori} that a solution exists, or that it satisfies the required monotonicity properties (C). It was observed in the numerical case study in Paper A, that a solution satisfying these properties exists in more general cases, and the corresponding random variable \( R_{s_n} \) indeed gives similar or superior bounds to other methods considered.

In particular, Section 4 of Paper A shows that the functions \( h, d \) and the corresponding \( R_{s_n} \) can be used to obtain lower bounds on convex risk measures, on convex expectations (\( \mathbb{E}[f(S)] \), where \( f \) is a convex function), as well as upper and lower bounds on VaR. The connection between the bounds on VaR and minimal convex elements in (suitably adjusted) admissible risk classes was shown in Bernard et al. (2014, 2015a). For the upper bound on VaR to be sharp, a decreasing density in the upper tail above \( F^{-1}_i(\alpha) \) is sufficient; this holds for virtually all continuous parametric distributions used in practice, e.g. normal, log-normal, gamma, t, F, \( \chi^2 \), to name a few.

The importance of these convex order bounds is not limited to the computation of bounds on risk measures and convex expectations. This result also provides an illuminating insight into the corresponding extremal dependence structures. For future research, the problem of lower convex order bounds for general admissible classes is still open. In this task, analyzing the dependence structures generated by the RA (see Figure 2.1) may prove useful. Moreover, a minimal element with respect to convex order may not exist in the admissible class (see Section 3.2 in Bernard et al. (2014) for a counterexample), and determining which admissible classes contain such an element is also an open problem.
2.2 Simple bounds and examples

Over the recent years, a number of contributions have been published on the topic of dependence uncertainty for risk measures; see Rüschendorf (2013) for a textbook treatment and Embrechts et al. (2014) for an overview of the state of the art from a regulatory point of view. In this section, we summarize the observations from a numerical study in Paper B, which complements the mainly methodological research on DU.

As mentioned in the beginning of Chapter 2, the upper bound on the Expected Shortfall, denoted \( \text{ES}_\alpha \), is given by the comonotonic dependence structure and is easy to calculate,

\[
\text{ES}_\alpha = \text{ES}_\alpha^+(S_n) = \sum_{i=1}^n \text{ES}_\alpha(X_i),
\]

where \( S_n = X_1 + \cdots + X_n \) and \( X_i \sim F_i, i = 1, \ldots, n \). The best-possible (sharp) upper and lower bounds on VaR under dependence uncertainty (i.e. over the admissible class, see (2.1)), denoted \( \text{VaR}_\alpha \) and \( \text{VaR}_\alpha^- \), respectively, and the lower bound on ES, denoted \( \text{ES}_\alpha^- \), are not straightforward to obtain. These three quantities are the focus of this section. First, note that the sharp bounds on VaR satisfy the following simple inequalities:

\[
\text{LES}_\alpha^+(S_n) \leq \text{VaR}_\alpha^+(S_n) \leq \text{VaR}_\alpha^-(S_n) \leq \text{ES}_\alpha^+(S_n),
\]

where \( \text{LES} \) is the **Left-tail Expected Shortfall**,

\[
\text{LES}_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\alpha(X) \, dq,
\]

and the superscript “+” denotes the values corresponding to the comonotonic sum, \( S_n = \sum_{i=1}^n F_i^{-1}(U), U \sim \text{UNIF}(0, 1) \). In particular,

\[
\text{LES}_\alpha^+(S_n) = \sum_{i=1}^n \text{LES}_\alpha(X_i) \quad \text{and} \quad \text{VaR}_\alpha^+(S_n) = \sum_{i=1}^n \text{VaR}_\alpha(X_i).
\]

The last inequality in (2.6) follows from the definition of ES, which implies that \( \text{VaR}_\alpha(X) \leq \text{ES}_\alpha(X) \) for any rv \( X \); the first inequality holds by symmetry. The middle two inequalities hold because the comonotonic dependence structure is always in the admissible class. Note that also the independence copula is always admissible, and may be considered a more neutral dependence structure than comonotonicity; however, in general, the corresponding values of risk measures cannot be easily computed in terms of quantities pertaining to margins (in practice, one could use, for instance, Monte Carlo sampling to simulate the independent sums) and hence this case is no longer considered “simple” for the purposes of this section.

It is also clear that

\[
\text{E}[S_n] = \sum_{i=1}^n \text{E}[X_i] \leq \text{ES}_\alpha(S_n) \leq \text{ES}_\alpha^+(S_n).
\]

In Paper B, we analyze the behavior of the sharp bounds for different light- and heavy-tailed distributions, and a varying number of risks \( n \). We also compare the sharp bounds with approximate ones, always relative to the ranges given in (2.6) and (2.7) to facilitate the comparisons.
2.2.1 Upper bound on VaR

In a series of papers (Puccetti and Rüschendorf, 2014; Wang, 2014; Puccetti et al., 2013; Wang and Wang, 2015; Embrechts et al., 2015), the asymptotic equivalence of $\text{VaR}_{\alpha}(S_n)$ and $\text{ES}_{\alpha}^{-}(S_n)$ as $n \to \infty$ was shown with increasingly general assumptions on the sequence $(X_i)_{i \geq 1}$ of rvs:

$$\lim_{n \to \infty} \frac{\text{VaR}_{\alpha}(X_1 + \cdots + X_n)}{\text{ES}_{\alpha}^{-}(X_1 + \cdots + X_n)} = 1. \quad (2.8)$$

To estimate the values of $n$ that are large enough for $\text{ES}_{\alpha}^{-}(S_n)$ to be considered a reasonable approximation of $\text{VaR}_{\alpha}(S_n)$, we computed these quantities for $X_i$ with identical log-normal or Pareto distributions with various shape parameters and for different significance levels $\alpha$. We observed that in all but the most heavy-tailed cases, $n \approx 10$ was sufficient (the approximation error was less than 10%). Hence, when aggregating a large (or even a moderate) number of similar risks, one may simply use the comonotonic ES as a reasonable estimate of the most conservative aggregate VaR.

The convergence rate in (2.8) is also known. In the homogeneous case ($F_i = F$, $i \geq 1$), Theorem 3.3 in Embrechts et al. (2015) states that if $E[r|X_1 - E[X_1]|^k]$ is finite for some $k > 1$ and $\text{ES}_{\alpha}(X_1) > 0$, then, as $n \to \infty$,

$$\frac{\text{VaR}_{\alpha}(S_n)}{\text{ES}_{\alpha}^{-}(S_n)} = 1 - O(n^{-1+1/k}). \quad (2.9)$$

For example, if $F$ is the log-normal distribution, the convergence rate is $O(n^{-1})$, since all moments are finite. If $F = \text{Pareto}(\theta)$, then the convergence is slower, since $k < \theta < \infty$. Based on (2.9), one may consider

$$\text{VaR}_{\alpha}(S_n) \approx (1 - Cn^{-1+1/k}) \text{ES}_{\alpha}^{-}(S_n) \quad (2.10)$$

as an approximation for the worst-case VaR, for a suitable $C \in \mathbb{R}$. In Figure 2.2, the differences $1 - \text{VaR}_{\alpha}(S_n)/\text{ES}_{\alpha}^{-}(S_n)$ as $n$ increases are plotted on a logarithmic scale for different Pareto and log-normal distributions. We observe that the rate of convergence seems faster than the theoretical one. Moreover, for small $n$ the rates are not very different between Pareto and log-normal. We therefore conclude that the highest finite moment is not a good predictor of $\text{VaR}_{\alpha}(S_n)$ when the considered dimension $n$ is small.

In order to analyze $\text{VaR}_{\alpha}(S_n)$ at a fixed level, say, $\alpha = 0.99$, and for small $n$, we instead define a different measure of tail-heaviness: the mean-median ratio, defined for $X \sim F$ as

$$M_{\alpha}(F) := \frac{\text{ES}_{\alpha}(X) - \text{VaR}_{\alpha}(X)}{\text{MS}_{\alpha}(X) - \text{VaR}_{\alpha}(X)},$$

where MS is the median shortfall, $\text{MS}_{1-p}(X) := \text{VaR}_{1-p/2}(X)$; see Kou et al. (2013). We then fit similar models to (2.10), but instead of using the highest existing moment as the predictor, we
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Figure 2.2: Relative differences $1 - \frac{\text{VaR}_\alpha(S_n)}{\text{ES}_\alpha(S_n)}$ for $\alpha = 0.99$ on the vertical axis versus $n$ on the horizontal axis, on a log-log scale. Below the dotted line the relative difference is smaller than 10%.

The left panel contains Pareto($\theta$) distributions, $\theta = 1.5, 2, 3, 5, 10$ from top down. The bold line shows the theoretical convergence rate $O(n^{-1/3})$ for Pareto(1.5) according to Equation (2.9). In the right panel the log-normal LogN($0, \sigma^2$) case is plotted, $\sigma = 2.23, 1.66, 1.20, 0.89, 0.70$ from top down, chosen to match the ratio $\text{VaR}_\alpha(X_1)/\text{ES}_\alpha(X_1)$ with that of Pareto. The bold line shows the theoretical convergence rate $O(n^{-1})$ for distributions with all moments finite.

use a function of $M_{\alpha}(F)$. The fitted convergence rates are in the range $O(n^{-2/3})$ (for Pareto(1.5)) to $O(n^{-3.5})$ (for lighter tails), so in many cases faster than the best case $O(n^{-1})$ possible in (2.9). The fitted rates are similar for Pareto and log-normal, when their parameters are such that the corresponding values of $M_{\alpha}(F)$ are similar. For inhomogeneous portfolios ($F_i$ possibly different), we introduce a suitably adapted version of the mean-median ratio $\tilde{M}_\alpha$ and also another predictor called the effective dimension $\hat{n}$, and fit a similar model; see Paper B for further details.

2.2.2 Lower bound on VaR

If the variables $X_i$ are not bounded from below, then the discussion on $\text{VaR}_\alpha$ is similar to that on $\overline{\text{VaR}}_\alpha$ by symmetry arguments. Here, we consider the case when the risks are non-negative; a typical assumption for the claim size distributions in insurance. In this case, a simple lower bound is given by

$$\text{VaR}_\alpha(S_n) \geq \max\{F_i^{-1}(\alpha), i = 1, \ldots, n\} \lor \text{LES}_\alpha(S_n),$$

where $a \lor b := \max\{a, b\}$. For homogeneous risks, the right-hand side is equal to $F_i^{-1}(\alpha)$ for small $n$, and equal to $n \text{LES}_\alpha(X_1)$ for $n \geq F_i^{-1}(\alpha)/\text{LES}_\alpha(X_1)$; when $\alpha = 0.99$, this threshold value is typically below $n \approx 10$, apart from very heavy-tailed distributions. In numerical experiments, we observed that the right-hand side of (2.11) was almost always within 5% of $\text{VaR}_\alpha(S_n)$ (relative to the possible range, see (2.6)). The only exceptions were the cases when the margins have little mass close to 0, so (2.11) should be used with caution in these cases.
A relevant result in the homogeneous case can be found in Puccetti et al. (2013): if df $G$ has a density $g(x) \geq 3/na$ on $[0, a]$, then $G$ is n-CM. Hence, if the conditional density of $X_1$ on $[0, F_1^{-1}(a)]$ is bounded away from zero, then $\text{VaR}_\alpha(S_n) = n \text{LES}_n(X_1)$ for $n$ large enough.

### 2.2.3 Lower bound on ES

Finally, we consider the lower DU bound on the aggregate ES. In the homogeneous case, Cheung and Lo (2013) give a simple bound for non-negative risks:

$$\text{ES}_{1-p}(S_n) \geq \text{ES}_{1-p/n}(X_1).$$

Furthermore, if $F(0) > (n - 1)/n$, then (2.12) holds with equality. Namely, for this bound to be sharp, a sufficient mass at 0 (in fact, at least $p(n - 1)/n$) is required for the marginal distributions. This bound is based on *mutual exclusivity* (see Section 2.1); but recall that the results in Bernard et al. (2014) combine mutual exclusivity and complete mixability. When the df $F$ is compatible with mutual exclusivity in $n$ dimensions, $\text{ES}_{1-p}(T_{c_n})$ also gives a sharp bound (see (2.4) for the definition of $T_{c_n}$); in other cases it gives a better bound than (2.12). A third approach for obtaining an approximation of $\text{ES}_\alpha(S_n)$ is by applying the Rearrangement Algorithm as described in Puccetti (2013).

In Figure 2.3, the convergence of the three bounds is plotted for different Pareto and log-normaldfs. First, we notice that the convergence is slower than for the VaR bounds. Second, we see that the bound given by the RA is not sharp for heavy-tailed margins; this is because it uses a discretization of the margins and therefore underestimates the integral of the infinite tail. This issue is addressed further in Paper C, and a remedy is found in using an alternative way of discretizing the margins. Finally, we observe that (2.12) gives a nearly sharp bound for the Pareto df, because it has a high probability mass near the origin, as well as a heavy tail, which drives the aggregate risk. In contrast, for the log-normal df the bound is noticeably lower (not sharp), in particular, for the light-tailed case, when there is little mass close to the origin.

For a lower bound on ES in the inhomogeneous case, we provide the following result, similar to Theorem 4.1 in Cheung and Lo (2013).

**Theorem 1.** For rvs $X_i \geq 0$, $i = 1, \ldots, n$, let $\zeta = \min\{x \geq 0 : \sum_{i=1}^{n} F_i(x) \geq n - (1 - \alpha)\}$. Then

$$\text{ES}_\alpha(S_n) \geq \frac{1}{1 - \alpha} \sum_{i=1}^{n} \mathbb{E}[X_i \mathbb{1}_{\{X_i > \zeta\}}].$$

Furthermore, if $\sum_{i=1}^{n} F_i(0) \geq n - 1$ and $F_i$ is continuous at $\zeta$, $i = 1, \ldots, n$, then (2.13) holds with equality.

In contrast to Cheung and Lo (2013), we impose the continuity at $\zeta$ for sharpness, but this replaces the $n$-dimensional optimization problem in their paper with a simple root search for a
2.3. BOUNDS ON THE EXPECTILE

In this section, we summarize the results from Paper C, which studies the DU bounds for the expectile, a functional that is well known in regression analysis, but recently has also been considered as a risk measure.

The expectile is introduced in Newey and Powell (1987) as the minimizer of the expectation of an asymmetric quadratic scoring function,

\[ e_{\tau}(X) = \arg\min_{e \in \mathbb{R}} \mathbb{E}\left[ (\tau \mathbb{I}_{(X>e)} + (1-\tau) \mathbb{I}_{(X<e)}) (X-e)^2 \right] . \] (2.14)

Efron (1991) provides an iterative procedure for fitting the coefficients for a linear regression with asymmetric least squares minimization, i.e. for estimating \( \beta \in \mathbb{R}^p \) in the model

\[ y_i = x_i^T \beta + \epsilon_i, \quad y_i \in \mathbb{R}, \quad x_i \in \mathbb{R}^p, \quad \epsilon_i \overset{iid}{\sim} G, \quad e_{\tau}(\epsilon_i) = 0, \quad i = 1, \ldots, N. \]
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Setting $p = 1, x_i = 1, i = 1, \ldots, N$, yields a procedure for computing the expectile of the sample \( \{y_1, \ldots, y_N\} \). Starting from a trial value $b_0 \in \mathbb{R}$, we update iteratively for $k = 0, 1, 2, \ldots$:

$$ b_{k+1} = \sum_{i=1}^{N} w_i^{(k)} y_i / \sum_{i=1}^{N} w_i^{(k)}, \quad \text{where} \quad w_i^{(k)} = |\tau - \mathbb{I}_{\{y_i > b_k\}}| / N. \quad (2.15) $$

This is the method of \textit{iteratively reweighted least squares} (IRLS). The sequence converges to the corresponding expectile, \( \lim_{k \to \infty} b_k = e_\tau(Y) \), where \( Y \sim \text{UNIF}\{y_1, \ldots, y_N\} \).

From (2.14), it follows that $e_\tau(X)$ is the unique solution of the equation implied by the first order conditions

$$ (1 - \tau) \mathbb{E}[(e_\tau - X) \mathbb{1}_{\{X < e_\tau\}}] = \tau \mathbb{E}[(X - e_\tau) \mathbb{1}_{\{X > e_\tau\}}]. \quad (2.16) $$

In fact, (2.15) is Newton’s method for root search applied to (2.16) when $X$ has a discrete distribution. In Paper C, the analogous procedure is provided for $X$ with a general integrable distribution.

The recent paper by Gneiting (2011) brought the issue of \textit{elicitability} to the attention of the risk management community. A risk measure is said to be elicitable if it is a minimizer of the expectation of some scoring function, which depends on the point forecast and the true loss to be observed. Gneiting (2011) showed that VaR is elicitable (if the corresponding quantile is unique), but ES is not. Some authors conclude that this makes back-testing ES problematic (see e.g. Carver (2013, 2014); Hull and White (2014)). Furthermore, in Ziegel (2014); Bellini and Bignozzi (2015); Delbaen et al. (2015), it is shown that the only risk measure that is both elicitable and coherent is the expectile (for $\tau \in [1/2, 1]$). These important properties make the expectile an interesting functional to consider further from a risk management perspective.

While the statistical properties of the expectile in regression analysis are well understood (see e.g. Yao and Tong (1996)), its properties under risk aggregation are not. This section provides a first look at this topic. The main tool in our analysis will be convex order, since coherence implies, in particular, convexity. We will use earlier results on convex order bounds, as well as provide new ones, in order to compute DU bounds on the expectile. Furthermore, in the case of VaR and ES, it has been observed that the DU bounds are in general too wide to be relevant in practice, and in order to obtain narrower bounds, the idea of adding (partial) dependence information has been explored in a series of papers; see Bernard et al. (2015a); Bignozzi et al. (2015); Bernard et al. (2015b); Puccetti et al. (2015); Bernard and Vanduffel (2015b); Bernard et al. (2015c); Bernard and Vanduffel (2015a). With a similar aim, three different setups for the expectile bounds will be considered: only marginal distributions are known, portfolio mean and variance known, and bounds with factor information.
2.3.1 Bounds when only the marginal distributions are known

Since the expectile is consistent with the convex order, the upper bound on \( e_\tau(S) \) for \( S \in \mathcal{Z}_n(F_1, \ldots, F_n) \) is attained by the comonotonic dependence, \( \bar{e}_\tau(S) = e_\tau(S^c) \). In the case of identical margins \( F_i = F_1, i = 2, \ldots, n \), using positive homogeneity, this simplifies to
\[
\bar{e}_\tau(S) = e_\tau(nX_1) = ne_\tau(X_1) = \sum_{i=1}^n e_\tau(X_i).
\]

In general, however, the expectile is not comonotonic additive (as opposed to VaR an ES), and hence, the upper bound \( \bar{e}_\tau \) often needs to be computed numerically. Moreover, since the df of \( S^c \) is typically not available in analytical form, the continuous version of the IRLS method is computationally expensive to apply. However, as \( S^c \) is defined in terms of its quantiles, \( F_{S^c}^{-1}(p) = \sum_{i=1}^n F_i^{-1}(p) \), it is easier to determine the quantile level \( p \) corresponding to the expectile. In Paper C, we provide an iterative procedure (in terms of \( p \)) for computing \( e_\tau(S^c) \) when the marginal distributions admit a density. In general, by subadditivity (recall that we use \( \tau \in [1/2, 1) \)),
\[
e_\tau(S) \leq \sum_{i=1}^n e_\tau(X_i) =: e_\tau^+ \quad \forall S \in \mathcal{Z}_n(F_1, \ldots, F_n),
\]
so \( e_\tau^+ \) is a valid upper bound, but it is typically not the best possible. However, in a numerical example with skew-t margins, we observed that \( e_\tau^+ \) and \( \bar{e}_\tau \) are very close in all cases. Furthermore, for margins from a location-scale family, \( F_i(\cdot) = F((\cdot - \mu_i)/\sigma_i), i = 1, \ldots, n \), for some df \( F \), the comonotonic sum is
\[
S^c = \sum_{i=1}^n F_i^{-1}(U) = \mu + \sigma F^{-1}(U), \quad \text{where} \quad \mu = \sum_{i=1}^n \mu_i, \quad \sigma = \sum_{i=1}^n \sigma_i, \quad U \sim UNIF(0, 1).
\]

Thus, when the margins have the same shape, the upper bound based on subadditivity (2.17) is the best possible, since \( \bar{e}_\tau = \mu + \sigma e_\tau(F^{-1}(U)) = e_\tau^+ \). By extension, we expect that for margins with a similar shape, the bound \( e_\tau^+ \) is nearly optimal.

We also specify the distribution of the convex-order-minimal element in \( \mathcal{Z}_n(F_1, \ldots, F_n) \) when the margins \( F_i \) belong to the location-scale family of a symmetric df \( F \). If the scale of one margin dominates, without loss of generality, \( \sigma_1 \geq \sum_{i=2}^n \sigma_i \), then a minimal element in \( \mathcal{Z}_n \) is
\[
S^\ell = F_1^{-1}(U) + \sum_{i=2}^n F_i^{-1}(1 - U) = \sum_{i=1}^n \mu_i + (\sigma_1 - \sum_{i=2}^n \sigma_i) F^{-1}(U), \quad U \sim UNIF(0, 1).
\]
That is, all other margins are countermonotonic to the dominating one. Otherwise, we use a result on joint mixability from Wang and Wang (2016) to show that if \( F \) furthermore admits a unimodal density, then the minimal element in the admissible class is the constant \( \mu = \sum_{i=1}^n \mu_i \). These convex-order-minimal elements can be used to obtain the best-possible lower bound for the expectile, as well as for other convex risk measures and convex expectations.

The lower bound \( e_\tau \) in more general cases can be computed, for example, using the convex-order-minimal elements constructed in Bernard et al. (2014) for the homogeneous case, and in
CHAPTER 2. DEPENDENCE UNCERTAINTY

Paper A for the inhomogeneous case. A general, but approximate method is the Rearrangement Algorithm (RA), which we adapt to the expectile objective by using a suitable discretization and stopping condition (see Embrechts et al. (2013); Puccetti (2013); Puccetti and Rüschendorf (2015) for the RA formulated with other objectives).

2.3.2 Bounds with variance information

Similarly to the setup in Cheung and Vanduffel (2013), Bernard et al. (2015a) and Puccetti et al. (2015) in the case of VaR and ES bounds, we consider bounds on the expectile, when besides the marginal distributions also the variance of the aggregate risk is known. First, we note that it is not so clear how to combine these two types of information. For the purpose of tractability, we first relax the constraints on the marginal distributions and consider a larger admissible class, the moment space, defined as

\[ \mathcal{M}_2(m, s^2) = \{ S \in L^2(\Omega, \mathcal{A}, P) : E[S] = m, \text{Var}(S) = s^2 \}. \]

It is well known that two-point distributions play a key role in this class, for example, with respect to bounds on stop-loss premiums; see De Vylder and Goovaerts (1982). We say that a rv \( Y \) is diatomic if \( P(Y = a) = p \) and \( P(Y = b) = 1 - p \) for some \( a < b \) and \( p \in (0, 1) \). Diatomic random variables in \( \mathcal{M}_2(m, s^2) \) can be parameterized by \( p \in (0, 1) \); writing \( Y_p \) for the variable with support points \( a_p, b_p \) given as

\[
\begin{align*}
a_p &= m - s \sqrt{1 - p}, \\
b_p &= m + s \sqrt{\frac{p}{1 - p}}.
\end{align*}
\]

We show that \( p = \tau \) maximizes \( e_\tau(Y_p) \) over \( p \in (0, 1) \); see also Figure 2.4. Furthermore, it also yields an upper bound in the moment space, \( e_\tau(Y_p) \geq e_\tau(S) \) for any \( S \in \mathcal{M}_2(m, s^2) \). When the variance constraint \( s^2 \) is small enough, this moment space upper bound is considerably better than the bound based on the marginal distributions. To determine whether the moment space upper bound can be attained also using the given marginal distributions, we apply the Extended Rearrangement Algorithm (ERA), introduced in Bernard et al. (2015a). This algorithm constructs a dependence structure for the margins, such that the sum is approximately diatomic, with the specified locations of the atoms. Similarly to the observations in Bernard et al. (2015a), we note that the moment space bound is nearly sharp; indeed it can be approximated closely for various considered marginal distributions.

The lower bound over the moment space is the trivial \( e_\tau(S) \geq m = E[S] \) for \( S \in \mathcal{M}_2(m, s^2) \) and \( \tau \geq 1/2 \). However, in this case the diatomic rvs \( Y_p \) only approach it in the limit as \( p \to 0 \) or \( p \to 1 \). Therefore, it is not clear how to apply the ERA (i.e. what distribution to set as the target) in order to determine whether this lower bound can be approximated with the given marginal
distributions. In numerical experiments, we observed that setting $Y_p$ with $p$ close to 0 as the target distribution for ERA provided better results than using $p$ close to 1. This is because $e_\tau(Y_p)$ is less sensitive to $p$ close to 0; see Figure 2.4. To conclude, we note that combining variance information with the given marginal distributions in a meaningful way, so as to obtain a non-trivial lower bound on the expectile, is still an open question; in particular, the best-case aggregate distribution may not be approximately diatomic.

Figure 2.4: Expectile $e_\tau(Y_p)$ for a diatomic rv $Y_p$ (standardized to $m = 0, s = 1$), as a function of $p \in (0, 1)$. The maximum is attained at $p = \tau$; the minimum is approached as $p \to 0$ or $p \to 1$.

### 2.3.3 Bounds for factor models

Another type of dependence information was used in Bernard et al. (2015b). The considered setup assumes that there is a risk factor rv $W$, and the dependence between the marginal risk $X_i$ and the factor $W$ is given by a known bivariate distribution $(X_i, W) \sim H_i, i = 1, \ldots, n$; sums of risks that satisfy this structure belong to the factor-constrained admissible class. We focus on location-scale mixtures, which are commonly used factor models with a broad range of applications, going back to Barndorff-Nielsen (1977, 1978). In these models, conditional on a non-negative factor $W$ with distribution $G$, the rvs $X_i$ belong to the location-scale family of distribution $F_0$, as specified by parameters $\mu_i, \gamma_i, \sigma_i \in \mathbb{R}$:

$$X_i = \mu_i + \gamma_i W + \sigma_i \sqrt{W}Z_i, \quad Z_i \sim F_0, \quad Z_i \| W, \quad i = 1, \ldots, n, \quad \text{and} \quad W \sim G.$$

In Bernard et al. (2015b), it is shown that the convex-order-maximal element in the factor-constrained admissible class is attained by conditional comonotonicity, given the factor $W$. We complement this result by characterizing the convex-order-minimal element. For a numerical
example, we consider the hyperbolic skewed Student-$t$ distribution, which is a special case of location-scale mixtures, where $F_0 = \Phi$ (the normal distribution), and the mixing distribution $G$ is the inverse-gamma df; see Aas and Hobæk Haff (2006). We compute the DU bounds on the aggregate expectile in the three setups discussed: only marginal distributions given, variance is known, and the case of factor models; the results are given in Paper C. We remark here that the information on the factor structure increases the lower bound, especially for high $\tau$, because the factor $W$ induces tail dependence for scale mixture models. Specifically, large outcomes of $W$ lead to joint extremes in $X_i, i = 1, \ldots, n$, which, in turn, lead to a higher aggregate risk.

To summarize, we note that there are many similarities between the expectile, VaR and ES, when considering the DU bounds, and this with regard to the methods used, as well as the derived results. As one example, the DU bounds that only consider information on the marginal distributions tend to be very wide, and the expectile seems to be particularly vulnerable when the marginal distributions have two unbounded tails. This is because the expectile takes both tails into account, whereas ES and VaR only focus on the right tail of the loss distribution. Another issue is the non-obvious interpretation of the expectile, although Bellini and Di Bernardino (2015) provide some intuition by relating its acceptance set with the Omega ratio of Keating and Shadwick (2002); a connection also pointed out in Rémillard (2013), p. 128.
3 Matrix arrangement problems

The early works of Rüschendorf (1980, 1982, 1983) used the concept of rearrangements, introduced in Hardy et al. (1952), to study optimization problems over random variables with given margins, but unknown dependence. For measurable functions $f, g : [0, 1] \to \mathbb{R}$, we say that $g$ is a rearrangement of $f$ if $\lambda\{g \geq c\} = \lambda\{f \geq c\}$, $\forall c \in \mathbb{R}$, where $\lambda$ is the Lebesgue measure restricted to $[0, 1]$. It is well known that a random variable $Y$ with distribution function $F$ in an atomless probability space can be expressed as $Y = F^{-1}(U)$ for some $U \sim \text{UNIF}[0, 1]$. Furthermore, rearrangements of the identity function $\text{Id} : [0, 1] \to [0, 1]$ can be used to represent any dependence structure between random variables $X_j \sim F_j$, $j = 1, \ldots, d$ (note that in the following, we denote the number of risks by $d$, for consistency with the accompanying papers). In particular, any random vector $(X_1, \ldots, X_d)$ can be expressed as

$$(X_1, \ldots, X_d) = (F_1^{-1} \circ f_1(U), \ldots, F_d^{-1} \circ f_d(U))$$

for some rearrangements $f_1, \ldots, f_d$ of $\text{Id}$ and $U \sim \text{UNIF}[0, 1]$; see Puccetti and Wang (2015b) for further details. For example, the comonotonic dependence structure is given by $f_j = \text{Id}$, $j = 1, \ldots, d$; this corresponds to perfect positive dependence. Some optimization problems, such as minimizing the variance of the sum, require extremal negative dependence. In two dimensions, this dependence structure is given by countermonotonicity, $f_1 = \text{Id}$ and $f_2 = 1 - \text{Id}$. Generalizing this notion to higher dimensions is the aim of Puccetti and Wang (2015b), where the property of $\Sigma$-countermonotonicity is defined. A random vector $(X_1, \ldots, X_d)$ is said to be $\Sigma$-countermonotonic if $\sum_{j \in J} X_j$ and $\sum_{j \notin J} X_j$ are countermonotonic for all nonempty $J \subseteq \{1, \ldots, d\}$. This property is clearly necessary for the sum $X_1 + \cdots + X_J$ to be minimal with respect to the convex order in the admissible class $\Xi_d(F_1, \ldots, F_d)$ (see Tchen (1980)), but not sufficient. However, a joint distribution with this property always exists, whereas a convex-order-minimal element in $\Xi_d(F_1, \ldots, F_d)$ may not; see Puccetti and Wang (2015b).

The Rearrangement Algorithm (RA) of Puccetti and Rüschendorf (2012), which was mentioned previously in Section 2.1.3, approximates the marginal distributions with uniform distributions on $n$ points, and finds a dependence structure that satisfies a similar negative dependence property to $\Sigma$-countermonotonicity (but weaker, as countermonotonicity is only attained for singleton sets $J = \{j\}$, $j = 1, \ldots, d$). The joint distribution is represented by a matrix
$X = (x_{ij})_{n \times d}$, where the columns correspond to the margins, and the rows correspond to joint outcomes. To represent different dependence structures in the discrete case, we no longer use rearrangements of $I_d$. Instead, permutations of the elements within each column of the matrix are used; see the pseudo-code of the RA provided in Section 3.2. The original application of the RA was to determine DU bounds on the df of the aggregate risk. To this end, a compensating dependence structure in the upper, respectively, lower tail of the marginal dfs is required, so that the aggregate risk has the highest possible mass above, respectively, below the considered threshold.

Recently, an interesting connection with the assembly line crew scheduling (ALCS) problem from Coffman and Yannakakis (1984) and Hsu (1984) was noticed. This problem considers the simultaneous production of $n$ items on $n$ assembly lines. To each line we must assign $d$ workers, specialized in different jobs which are executed in sequence. In total, there are $nd$ workers ($n$ from each of the $d$ specializations), and each has a different completion time of their job. The objective is to minimize the maximal completion time over all items by assigning the workers to the assembly lines optimally. The matrix $X$ in this example represents the completion times for the $nd$ individual workers. Also for this problem, the optimal rearrangement should satisfy a negative dependence property, in the sense that faster workers should compensate the slower workers within each team, to make the completion times of the items as similar as possible. Finding the optimal arrangement, however, is an NP-complete problem for $d \geq 3$ (see e.g. Haus (2015)), therefore efficient heuristics are of interest.

In both of these problems (DU bounds and ALCS), the aim is to optimize an objective function which depends on the elements (and their location) in the matrix, by rearranging the elements within each column. We refer to this type of problems as matrix arrangement problems. In problems where the arrangements are interpreted as dependence structures (in particular, the problem of DU bounds on aggregate risk; see Section 2), the RA has proven to be of great help. In Section 3.1, we comment on the underlying ideas of Paper D, where we abstract away from this probabilistic interpretation and investigate in more generality, which optimization problems require that the optimal solution satisfies the $\Sigma$-countermonotonicity property, and thus may benefit from methods that attain such solutions. Section 3.2 summarizes Paper E, where a modification of the RA is introduced for obtaining $\Sigma$-countermonotonic matrices, and its performance is analyzed.

### 3.1 Negative dependence in optimization problems

The RA was introduced in Puccetti and Rüschendorf (2012) with the objective to minimize the maximum row sum of a matrix over intra-column permutations, i.e. to construct a dependence for the risks in the context of DU, such that their sum is bounded above by the smallest-possible
constant. In Puccetti (2013), the RA was applied to obtain dependence structures such that the aggregate risk (i.e. row sum vector) is close-to-minimal with respect to the convex order; and in Puccetti and Rüschendorf (2015), it was extended to the objective of minimizing the expectation of a supermodular function applied to the random vector (i.e. to the rows of the matrix). In this section, we investigate the underlying structure in these optimization problems, with the aim of characterizing the properties that enable the use of the RA.

The basic iteration of the RA is arranging two vectors in an opposite order. Hence, the task is to characterize the optimization problems that are compatible with such iterations, in the sense that the iterations reduce the objective function value. In particular, we aim to determine which optimization problems require countermonotonicity properties (such as $\Sigma$-countermonotonicity from Puccetti and Wang (2015b)) for the optimizing arrangement. This question is closely related to rearrangement inequalities.

Since the famous book *Inequalities* by Hardy et al. (1952), a series of papers have studied rearrangement inequalities with increasing generality; see Ruderman (1952); Lorentz (1953); London et al. (1970); Minc (1971); Day (1972). As the name suggests, these papers study inequalities between functions under the rearrangement of their arguments, and properties of these functions that yield such inequalities. As in our setup, the entries of the argument vectors are fixed, but the order of the entries is not. An overview of the current state of the art in this field is available in the comprehensive textbook of Marshall et al. (2011).

Hollander et al. (1977) considered functions $\phi : \mathbb{R}^n \hat{\times} \mathbb{R}^n \to \mathbb{R}$ which decrease in value, when the two vector arguments are more distant from being comonotonic, and called them *decreasing in transposition*. The distance used was the minimal number of transpositions of the elements in the second vector required to attain comonotonicity. Marshall and Olkin (1979) do not use the terminology of Hollander et al. (1977), because they “prefer not to refer to order-preserving functions as *decreasing*.” Instead, they call such functions *arrangement increasing*. The order that they refer to is the *arrangement preordering*, which is extended to matrices in $\mathbb{R}^{n\times d}$ by Boland and Proschan (1988), using a generalization of transpositions. In particular, an operation called *basic rearrangement* is introduced, which consists of replacing the entries in two rows $k, l$ ($1 \leq k < l \leq n$) of a matrix $X$ by their coordinate-wise minimum $(x_{k1} \wedge x_{l1}, \ldots, x_{kd} \wedge x_{ld})$ and maximum $(x_{k1} \vee x_{l1}, \ldots, x_{kd} \vee x_{ld})$, respectively. If matrix $X$ can be transformed into matrix $Y$ (up to permuting the order of rows) using a sequence of basic rearrangements, then we write $X \preceq_a Y$; this defines the arrangement preordering on $\mathbb{R}^{n\times d}$. Correspondingly, a function $\phi : \mathbb{R}^{n\times d} \to \mathbb{R}$ is called arrangement increasing (AI) if, for any $X, Y \in \mathbb{R}^{n\times d}$,

$$X \preceq_a Y \quad \text{implies} \quad \phi(X) \leq \phi(Y).$$

First, note that by applying a series of basic rearrangements to $X$, we can attain a matrix $X^\dagger$ such that the elements in the columns are sorted in ascending order (comonotonically), and
hence $X \leq_{a} X^\dagger$. In particular, AI functions are maximized by the comonotonic arrangement,

$$\phi(X) \leq \phi(X^\dagger), \quad \text{for all } X \in \mathbb{R}^{n \times d}.$$ 

By analogy, we expect that AI functions are minimized by extreme negative dependence. For example, an AI function $\phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ of two vector arguments (i.e. $d = 2$) is minimized by the countermonotonic arrangement. Writing $x^\dagger (x^{\dagger})$ for the increasing (decreasing) permutation of vector $x$,

$$\phi(x^\dagger, y^{\dagger}) \leq \phi(x, y), \quad \text{for all } x, y \in \mathbb{R}^n.$$ 

This is because the opposite operation of a basic rearrangement is well-defined when $d = 2$. For higher dimensions, however, it is not clear how to attain a minimal element with respect to the arrangement preordering, and even whether it exists. Therefore, we will look for objectives that require negative dependence in the class of AI functions, but expect that not all such functions will be compatible with minimizing through iterations of countermonotonic rearrangements.

Denote the elements of the matrix $X \in \mathbb{R}^{n \times d}$ by $(X)_{ij} = x_{ij}$ and the columns by $x_j$, so that $X = (x_1, \ldots, x_d)$. Three useful classes of AI functions are provided by the following results from Boland and Proschan (1988).

(A) If $\phi$ has the form $\phi(X) = g(x_1 + \ldots + x_d)$, then $\phi$ is AI if and only if $g$ is Schur-convex.

(B) If $\phi$ has the form $\phi(X) = \sum_{i=1}^{n} g(x_{i1}, \ldots, x_{id})$, then $\phi$ is AI if and only if $g$ is supermodular.

(C) If $\phi$ has the form $\phi(X) = \prod_{i=1}^{n} g(x_{i1}, \ldots, x_{id})$, then $\phi$ is AI if and only if $g$ is MTP$_2$.

The definitions and further discussion of these properties can be found in Paper D; in this section we only provide the basic intuition. *Schur-convex* functions are consistent with the majorization order (the discrete analogue of convex order) and include, for example, $g(z) = (1/n) \sum_{i=1}^{n} f(z_i)$, where $z = (z_1, \ldots, z_n)$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ is convex. In the probabilistic interpretation, this example corresponds to an expectation of a convex function. *Supermodular* functions are exactly those, for which the sum increases under basic rearrangements, i.e. $g : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$g(z) + g(w) \leq g(z \wedge w) + g(z \vee w),$$ 

where $\wedge$ and $\vee$ denote the coordinate-wise minimum and maximum, respectively. MTP$_2$ (multivariate totally positive of order 2) functions are obtained by taking the exponential of a supermodular function.

To apply the concept of countermonotonicity (and corresponding rearrangement operations) with the aim of minimizing an AI function, one needs to reduce the matrix argument into two column vectors. Denote by $\mathcal{P}(X)$ the set of matrices that can be obtained from $X \in \mathbb{R}^{n \times d}$ by
applying intra-column permutations. Puccetti and Rüschendorf (2012) consider optimization problems of the form
\[
\min_{X' \in P(X)} \max_{i=1,\ldots,n} g(x'_i, \ldots, x'_d),
\]
and Puccetti and Rüschendorf (2015) consider problems of the form
\[
\min_{X' \in P(X)} \sum_{i=1}^n g(x'_i, \ldots, x'_d),
\]
in both cases imposing a *decomposability* condition on \(g\). Besides further conditions, it is required that there exist functions \(g^{(2)} : \mathbb{R}^2 \to \mathbb{R}\) and \(g^{(d-1)} : \mathbb{R}^{d-1} \to \mathbb{R}\) such that
\[
g(z_1, \ldots, z_d) = g^{(2)}(z_j, g^{(d-1)}(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_d)), \quad j = 1, \ldots, d. \tag{3.1}
\]
This essentially allows reducing multivariate AI functions to AI functions of two vectors, for which countermonotonicity is indeed the minimizing arrangement. Despite the general formulation, the only explicit examples of suitable functions \(g\) provided in the above references are the maximum, minimum, sum, and product operators. The first two examples lead to trivial optimization problems (see Section 4.1 in Puccetti and Rüschendorf (2012) and Section 3.2 in Paper D). The sum of row sums is constant under rearrangements, and the maximal row-sum corresponds to an AI function of type (A). Finally, the product can be reformulated to the summation case by observing that
\[
\prod_{i=1}^d z_i = \exp \left\{ \sum_{i=1}^d \log z_i \right\}
\]
for \(z_i > 0, i = 1, \ldots, d\). It further holds that \(g_1(x) = \sum_{i=1}^n \exp \{x_i\}\) and \(g_2(x) = \max_{i=1,\ldots,n} \exp \{x_i\}\) are Schur-convex functions of \(x \in \mathbb{R}^n\); in particular, after applying the log-transform to matrix entries, we have transformed the objective into an AI function of type (A). In Puccetti and Rüschendorf (2015), supermodular functions \(g\) of rows were considered, with the aim of minimizing AI functions of type (B). A relevant example of supermodular functions are cumulative distribution functions, but these are not decomposable in general. However, the joint df of \(X_i \overset{\text{iid}}{\sim} \text{UNIF}[0,1], i = 1, \ldots, d\), is decomposable and corresponds to the product operator.

In Paper D, we show that if a \(d\)-variate distribution \(F\) admits an Archimedean copula, then it also satisfies the decomposability condition (3.1). Recall that the *Archimedean* copulas have the form
\[
C_{\psi}(u_1, \ldots, u_d) = \psi(\psi^{-1}(u_1) + \ldots + \psi^{-1}(u_d)), \quad (u_1, \ldots, u_d) \in [0,1]^d,
\]
where the generator \(\psi : [0, \infty) \to [0,1]\) is a \(d\)-monotone function, such that \(\psi(0) = 1\) and \(\lim_{x \to \infty} \psi(x) = 0\); see e.g. Joe (2014). By Sklar’s theorem, we also have
\[
F(z_1, \ldots, z_d) = \psi(\psi^{-1} \circ F_1(z_1) + \ldots + \psi^{-1} \circ F_d(z_d)), \quad z \in \mathbb{R}^d, \tag{3.2}
\]
where \( F_j, j = 1, \ldots, d \), are the margins of the \( d \)-variate distribution \( F \). Note that the decomposability ultimately stems from the fact that we aggregate the arguments by summation.

This observation leads us to formulate the following objective function, which is specific and yet flexible enough to include all known AI functions that are non-trivial to minimize, and that require countermonotonicity properties (such as \( \Sigma \)-countermonotonicity). Let

\[
\phi(X) = g \left( \sum_{j=1}^{d} h_j(x_{1j}), \ldots, \sum_{j=1}^{d} h_j(x_{nj}) \right), \quad X \in \mathbb{R}^{n \times d},
\]

where \( g \) is a Schur-convex function and \( h_j : \mathbb{R} \to \mathbb{R}, j = 1, \ldots, d, \) are monotone in the same direction. For example, taking \( g(z) = \sum_{i=1}^{n} \psi(z_i) \) and \( h_j = \psi^{-1} \circ F_j \) for \( F \) as in (3.2), we obtain

\[
\phi(X) = \sum_{i=1}^{n} F(x_{i1}, \ldots, x_{id}),
\]

which is an AI function of type (B). Taking \( g = \max, h_j = \text{Id} \), we recover the maximum row sum. For \( h_j = \text{Id} \), taking \( g(\cdot) = \text{E}[f(\cdot)] \) yields convex expectations of the aggregate risk, and the choice \( g = \text{ES}_\alpha \) yields the aggregate ES objective from Puccetti (2013). In the latter examples, \( h_j = \text{Id} \), because the aggregate loss is assumed to be the sum of marginal risks. If we consider a portfolio of derivatives such as vanilla options, then the corresponding loss would be sum of monotonic transformations of the underlying asset returns. In general, composing an AI function \( \phi \) with monotonic transformations (all increasing or all decreasing) of the columns preserves the AI property, therefore we allow the marginal transformations \( h_j \). This enables considering objectives such as (3.4). However, \( \Sigma \)-countermonotonicity can only be imposed after applying the marginal transformations. In particular, Theorem 2 in Paper D shows that we can reduce the problem of minimizing (3.3) over all \( X' \in \mathcal{P}(X) \) into the problem of minimizing

\[
\tilde{\phi}(H) = g \left( \sum_{j=1}^{d} (H)_{1j}, \ldots, \sum_{j=1}^{d} (H)_{nj} \right)
\]

over \( \Sigma \)-countermonotonic arrangements of matrix \( H \in \mathbb{R}^{n \times d} \), where \( (H)_{ij} = h_j(x_{ij}) \). This is relevant in practice, as there exist efficient algorithms for obtaining arrangements with this property. These algorithms are discussed in the next section.

In Paper D, we also provide detailed examples of problems from operations research that have objective functions of the form (3.3), namely, a stochastic version of ALCS and a reliability problem in systems assembly. The numerical study shows that any \( \Sigma \)-countermonotonic rearrangement of the matrix \( H \) yields an approximate solution of the optimization problem. Therefore, methods for obtaining \( \Sigma \)-countermonotonic arrangements can be considered as useful heuristics for a subclass of matrix arrangement problems, which are notoriously difficult to solve in general.
3.2 Block rearrangement algorithms

The Rearrangement Algorithm (RA) was introduced in Puccetti and Rüschendorf (2012) as a method for obtaining a column-oppositely-ordered (COO) rearrangement of a given matrix $X \in \mathbb{R}^{n \times d}$. The set of COO rearrangements of $X = (x_1, \ldots, x_d)$ is defined by

$$O^c(X) = \left\{ x^* \in \mathcal{P}(X) : x^*_j \perp \sum_{i \neq j} x^*_i, 1 \leq j \leq d \right\},$$

where the relation $u \perp v$ means that the components of vectors $u$ and $v$ are oppositely ordered, i.e. $(u_k - u_\ell)(v_k - v_\ell) \leq 0$ for all $1 \leq k < \ell \leq n$ (the discrete analogue of countermonotonicity).

The RA transforms a given matrix $X$ by sequentially cycling through the columns and arranging them in an opposite order to the sum of other columns; the pseudo-code of the RA is provided in Algorithm 1. The notation $X^\pi$ is defined by $X^\pi_{i,j} = X_{\pi_{i(j)}j}$, where $\pi = (\pi_1, \ldots, \pi_d)$ is a $d$-tuple of permutations on $\{1, \ldots, n\}$. Further, $x^\pi_j$ denotes the $j$th column of $X^\pi = (x^\pi_1, \ldots, x^\pi_d)$.

**Algorithm 1:** Pseudo-code of the Rearrangement Algorithm to obtain a COO arrangement of a given matrix $X \in \mathbb{R}^{n \times d}$.

1. Set $\pi$ to a randomly chosen intra-column arrangement of the elements so that a shuffled matrix $X^\pi$ is obtained;
2. repeat
    3. Set $improved = false$;
    4. for $j = 1, \ldots, d$ do
        5. if $x^\pi_j \nless \sum_{i \neq j} x^\pi_i$ then
            6. Rearrange column $j$ in the matrix $X^\pi$ so that it becomes oppositely ordered to the sum of the other columns, obtaining a new rearrangement $\pi'$;
            7. Set $\pi = \pi'$ and $improved = true$.
        end
    end
    until $improved$ is equal to $false$;

The algorithm terminates within a bounded number of iterations, since there are finitely many matrices in $\mathcal{P}(X)$, and each rearrangement strictly reduces the value of $\sum_{i=1}^n (s^\pi_i)^2$, where $s^\pi = x^\pi_1 + \cdots + x^\pi_d$. However, when the matrix $X$ is large (in terms of dimensions $n$ and $j$ or $d$), the algorithm may take a long time to reach a COO arrangement, therefore a stopping condition (based on a tolerance parameter for the decrease of the maximal row sum) is introduced in Embrechts et al. (2013); this is further discussed in Hofert et al. (2015). In Puccetti (2013), a similar stopping condition is used with the ES objective.

Recall that the aim of the RA is to obtain an arrangement of the input matrix $X$ such that its row sum vector $s^\pi$ is (approximately) minimal with respect to the convex order (more pre-
cisely, its discrete analogue, the majorization order). COO is a necessary condition for such an arrangement, but not a sufficient one. A stronger necessary condition is Σ-countermonotonicity; the set of arrangements of $X$ that satisfy this property is

$$
X^\Sigma(X) = \left\{ X^x \in \mathcal{P}(X) : \sum_{j \in J} x^x_j \perp \sum_{j \in J} x^y_j \text{ for all nonempty } J \subseteq \{1, \ldots, d\} \right\}.
$$

It is shown in Puccetti and Wang (2015b) that this set is nonempty\footnote{Puccetti and Wang (2015b) prove this in more generality, namely, when $X$ denotes a random vector with fixed marginal distributions. The proof uses discrete approximations of $X$ and compactness of the space of copulas.}; and clearly $X^\Sigma(X) \subseteq \mathcal{O}^\Sigma(X)$, so matrices in $X^\Sigma(X)$ provide at least as good, or better, solutions to optimization problems with objectives that are consistent with the convex order; see Section 3.1 and also the numerical case studies in Paper D.

With the aim of obtaining arrangements in $X^\Sigma(X)$, the idea of rearranging several columns (blocks of columns) simultaneously was introduced in Remark 4.1 of Bernard et al. (2015a). The block rearrangement algorithms (Block RA) consider subsets $J \subseteq \{1, \ldots, d\}$ of column indices and apply (the same) permutation to the columns $x_j$, $j \in J$, so that their sum becomes oppositely ordered to the sum of the complementary block. One implementation, the brute force Block RA (see Paper E), repeatedly cycles through all nonempty $J \subseteq \{1, \ldots, d\}$ and applies corresponding rearrangements, until Σ-countermonotonicity is attained. Specifically, there are $(2^d - 2)/2 = 2^{d-1} - 1$ subsets (corresponding to partitions $\{J, J^c\}$) to consider in each cycle, since the empty set and the full set of indices are excluded, and if countermonotonicity for $J$ is verified, then $J^c$ need not be considered. This algorithm is only practical when the number of columns $d$ is less than approximately 15. Note that for $d = 15$, there are 16,383 partitions to check for countermonotonicity, and typically a single cycle of rearrangements through all partitions is not sufficient. In addition, for large $n$, each rearrangement iteration takes longer, since sorting takes $O(n \log n)$ time, and more cycles through all partitions are typically required.

To work with larger matrices, Bernard and McLeish (2015) propose two algorithms that consider a smaller set of partitions, labeled Block RA1 and Block RA2. In Block RA1, a subset of 500 partitions is sampled in each iteration, and the partition is selected that yields the maximal Spearman’s rank correlation between the row sums of the two blocks. This is time-consuming, as computing Spearman’s correlation requires sorting the partial row sums for each considered partition. In numerical experiments, we observed that in a few iterations a maximal rank correlation of less than $-0.999$ is attained, but the variance of the row sums has not decreased significantly. Furthermore, each iteration takes a similar amount of time as 500 standard RA iterations, so this approach offers a poor tradeoff between choosing the best partitions and actually performing enough block rearrangements that decrease the objective. To avoid the time-consuming computation of Spearman’s correlations, the proposed Block RA2 simply chooses a random partition in each iteration.
In Paper E, we propose an alternative block selection rule, which offers a good balance between selecting a partition that significantly improves the objective value, and a fast computation time for each such iteration. This approach is called Block ReArrangement with Variance Equalization (BRAVE). The idea is based on the observation that, in the ideal case, after a rearrangement the row sums become constant. For this to be the case, the distributions of \( \sum_{j \in J} x_j^p \) and \( \sum_{j \in J} x_j^q \) before the rearrangement need to be identical (interpreting vectors in \( \mathbb{R}^n \) as discrete univariate rvs). Thus, our method is based on selecting the partition \( \{J, J^c\} \), for which the distributions of the block sums are as similar as possible. The first central moments are zero by construction, therefore we proceed by matching the second moments, i.e. equalizing the variances. Another way to motivate variance equalization as a block selection rule is as follows. Consider rvs \( U \) and \( V \) with variances \( \sigma_U^2 \) and \( \sigma_V^2 \), respectively, and with Pearson’s correlation \( \rho_{U,V} \). Then,

\[
\text{Var}(U + V) = \sigma_U^2 + \sigma_V^2 + 2\rho_{U,V}\sigma_U\sigma_V \geq \sigma_U^2 + \sigma_V^2 - 2\sigma_U\sigma_V = (\sigma_U - \sigma_V)^2,
\]

since \( \rho_{U,V} \geq -1 \). This shows from another perspective, why similar variances are desirable: even if the row sums of the two blocks after rearrangement have (linear) correlation exactly \( -1 \), the variance of the total row sums is still bounded below by the squared difference of the block standard deviations.

To find the partition of columns that yields two blocks with similar variances, first note that writing \( \beta_j^p = \text{Cov}(x_j^p, s^p) \) and \( s_j^p = \sum_{j \in J} x_j \) for \( J \subseteq \{1, \ldots, d\} \), we can express

\[
|\text{Var}(s_j^p) - \text{Var}(s_j^q)| = |\text{Cov}(s_j^p, s^p) - \text{Cov}(s_j^q, s^p)|, \quad \text{and} \quad \text{Cov}(s_j^p, s^p) = \sum_{j \in J} \beta_j^p.
\]

Thus, finding the partition with the most similar block variances is an example of the \textit{number partitioning} problem, where the set \( \{\beta_1^p, \beta_2^p, \ldots, \beta_d^p\} \) needs to be partitioned into two subsets, such that the subset sums are as close as possible. For this problem, two classical heuristics are available, the greedy algorithm and Karmarkar-Karp (KK) differencing algorithm (Karmarkar and Karp, 1982). The latter typically yields partitions that are closer to optimum, but takes \( O(d^2) \) time, whereas the greedy algorithm takes \( O(d \log d) \) time (due to sorting the numbers initially). These heuristics allow efficiently selecting two blocks that have similar variances, potentially leading to greater improvements in each iteration of BRAVE. In case the objective value does not improve in an iteration, to avoid getting stuck, we either select a random partition in the next iteration, or switch permanently to the standard RA (the latter approach is called BRAVE+RA).

In the numerical experiments in Paper E, we observe that the RA quickly catches up with the considered block rearrangement algorithms, especially when the columns of the matrix are similar. In contrast, when the matrix has \( d \geq 1000 \) columns that are dissimilar and contain
outliers, BRAVE+RA shows the best performance. Hence, BRAVE can be thought of as a useful “pre-solver” for the RA. In these experiments, the results of the variance equalization approach were similar regardless of whether the greedy or KK algorithm was used for block selection, therefore we recommend using the greedy algorithm, as it is faster.

We also show that, besides variance minimization and the assembly line crew scheduling problem, several other discrete optimization problems can be cast as rearrangement problems, in particular, $k$-partitioning, parallel machine scheduling, and the subset sum problem. The performance of block rearrangement algorithms, the standard RA, the greedy algorithm, and Karmarkar-Karp differencing algorithm is then compared in a numerical experiment considering the 2-partitioning problem. In this example, BRAVE was clearly the best performer. Moreover, the choice of the algorithm used for partitioning the covariances in the block selection step mattered in this case. In a curious example of “bootstrapping”, BRAVE using the KK algorithm for block selection significantly improved the solution of the 2-partitioning problem provided by a single application of the original KK algorithm.

Clearly, more numerical case studies are still required in order to determine which version of the Block RA performs best for different kinds of arrangement problems. Another possible direction for future research is developing other efficient block selection rules.
4 Portfolio optimization

Since the seminal paper of Markowitz (1952), portfolio optimization has been the mathematical basis for making decisions in wealth allocation. The classical setup of a single period portfolio optimization problem consists of

- the available wealth (cash) that is to be invested in financial assets (stocks, bonds),
- a selection of assets that are being considered for investment, and the allowed exposures,
- a joint distribution that models the asset values at the end of the investment period,
- a measure of risk or a utility function, which characterizes preferences between different random payouts.

In this section, we focus on the practical issue of solving such problems, once these components have been specified. The original Markowitz formulation uses portfolio variance as the measure of risk, and thus the corresponding optimization problem is a quadratic programming (QP) problem, and can be efficiently solved. This approach is applicable when the utility function of the investor is quadratic, or when the asset returns distribution is elliptical. For other risk measures and asset returns distributions, the portfolio problem often cannot be solved directly (e.g. due to the high computational effort), therefore the true modeled asset returns distribution (typically assumed continuous) is approximated by a discrete one, and the minimization of the risk measure is reformulated as a linear programming (LP) problem. LP formulations are solvable for a large number of assets and portfolio constraints, and have been a crucial tool in portfolio optimization; see Sharpe (1971) for an early example and Mansini et al. (2014) for a recent overview.

Solving a problem based on a discrete approximation of the true asset returns distribution (for example, using a Monte Carlo (MC) sample), yields a suboptimal solution for the original problem. To reduce the suboptimality, a sufficiently large sample should be used, which closely approximates the true distribution. Generating a large sample is typically easy (e.g. for normal mean-variance mixture models, which are commonly used in financial modeling), but solving the corresponding LP problem may become too time-consuming. Therefore, an appropriate sample size should be selected, that offers a good balance between the suboptimality of the
obtained portfolio and the required computation time. What sample size is suitable depends on 
the risk measure, the number of assets, and also on the asset returns distribution, for example, 
how heavy-tailed it is.

To facilitate choosing a suitable MC sample size for the approximate LP problem, we con-
duct a case study using the Student-$t$ distribution, which is an elliptical distribution, so the 
corresponding optimization problem can be solved exactly using the QP formulation. The exact 
solutions are then used as a benchmark for evaluating the suboptimality of the portfolios based 
on LP approximations. In particular, we analyze the suboptimality depending on the sample 
size, number of assets, and the tail-heaviness parameter of the Student-$t$ distribution. In Sec-
tion 4.1, we summarize the case study with the Expected Shortfall as the objective (Paper F), 
and in Section 4.2, we outline Paper G, where three different LP formulations for the portfo-
lio optimization problem with an expectile objective are derived, and a similar case study is 
conducted.

4.1 Conditional value-at-risk optimization

In this section, we consider the portfolio optimization problem with an Expected Shortfall ob-
jective. In the influential papers of Rockafellar and Uryasev (2000, 2002), this risk measure is 
called the Conditional Value-at-Risk (CVaR), therefore we use this name in the following. Let 
$Y$ denote the $d$-dimensional random returns vector of the $d$ considered assets, $x \in \mathbb{R}^d$ denote 
the monetary value invested in these assets, and let $X \subset \mathbb{R}^d$ be the set of permitted exposures 
(portfolio positions). In Rockafellar and Uryasev (2000), a function $F_\alpha : X \times \mathbb{R} \to \mathbb{R}$ is defined 
as

$$F_\alpha(x, \zeta) = \zeta + \frac{1}{1-\alpha} \mathbb{E}\left\{ [-x^\top Y - \zeta]^+] \right\},$$

where $[t]^+ = \max\{0, t\}$. They show that the portfolio CVaR optimization problem can be 
formulated as a simultaneous optimization of the function $F_\alpha$ over the portfolio weights $x$ and 
the variable $\zeta$,

$$\min_{x \in X} \text{CVaR}_\alpha(-x^\top Y) = \min_{(x, \zeta) \in X \times \mathbb{R}} F_\alpha(x, \zeta).$$

Furthermore, it is shown that the obtained optimizer $\zeta^*$ is an $\alpha$-quantile of the loss distribution, 
i.e. $\zeta^* \in \{ q \in \mathbb{R} : \mathbb{P}(-x^\top Y \leq q) = \alpha \}$. If we approximate the asset returns distribution with 
a discrete rv $\tilde{Y}$, such that $\mathbb{P}(\tilde{Y} = y_k) = p_k$, $k = 1, \ldots, n$, then the CVaR optimization problem 
reduces to a linear program

$$\text{minimize: } \zeta + \frac{1}{1-\alpha} \sum_{k=1}^n p_k u_k \quad \text{over: } x \in \mathbb{R}^d, \ z \in \mathbb{R}, \ u \in \mathbb{R}^n \quad \text{(LP)}$$

$$\text{subj. to: } x \in X, \ u_k \geq -x^\top y_k - \zeta, \ u_k \geq 0, \ k = 1, \ldots, n.$$
Rockafellar and Uryasev (2000) conduct a small case study with \( d = 3 \) multivariate normally distributed assets, and conclude that sample size \( n \approx 20,000 \) is sufficient. However, they only compare the objective value of the (LP), which is an estimate of the portfolio’s CVaR, with the true optimal CVaR. This comparison is not accurate, because it aggregates two errors. Namely, if we denote by \( \tilde{Y} \) the random vector following the discrete empirical distribution of the MC sample, by \( x_{\text{LP}} \) the optimal portfolio weights in the LP, and by \( x_{\text{QP}} \) the true optimal weights, then the two errors are

\[
\text{suboptimality} = \text{CVaR}_\alpha(-x_{\text{LP}}^\top \tilde{Y}) - \text{CVaR}_\alpha(-x_{\text{QP}}^\top \tilde{Y}),
\]

\[
\text{bias} = \text{CVaR}_\alpha(-x_{\text{LP}}^\top \tilde{Y}) - \text{CVaR}_\alpha(-x_{\text{QP}}^\top \tilde{Y}),
\]

where \( \text{CVaR}_\alpha(-x_{\text{LP}}^\top \tilde{Y}) \) is the optimal objective of the LP (perceived CVaR). Suboptimality measures how far the true CVaR of the selected portfolio is from the best possible, while bias measures the difference between the perceived CVaR of the selected portfolio and its true CVaR. These two errors typically work in opposite directions, partially canceling each other out, and giving overly optimistic results. Moreover, Rockafellar and Uryasev (2000) do not consider heavy-tailed distributions. These issues are pointed out in Lim et al. (2011), but no experiments with higher sample sizes are performed. In Paper F, we rectify this omission, and find that in some cases, samples of up to a million points are necessary to obtain portfolios for which the risk is within 1% of the optimum.

Solving (LP) (we will call it the Primal formulation) with sample sizes \( n = 10^5 \) or even \( n = 10^6 \) requires long computation times. Ogryczak and Śliwiński (2011) demonstrate that a considerable improvement in the computation time can be achieved by simply using the dual formulation of (LP),

\[
\begin{align*}
\text{maximize:} & \quad \eta \\
\text{over:} & \quad \eta \in \mathbb{R}, \quad \lambda \in [0, 1]^n \\
\text{subj. to:} & \quad \sum_{k=1}^n p_k \lambda_k = 1 - \alpha, \quad \eta + \frac{1}{1 - \alpha} \sum_{k=1}^n p_k \lambda_k y^{(i)}_k \leq 0, \quad i = 1, \ldots, d.
\end{align*}
\]

The optimal portfolio weights are the Lagrange multipliers corresponding to the \( d \) inequality constraints. The coefficients \( \lambda_k \) correspond to the Radon-Nikodym derivatives \( \varphi_k = \lambda_k/(1 - \alpha) \) in the dual (also called robust) representation of CVaR. For a random loss \( L \in L^1(\Omega, \mathcal{A}, P) \), this representation is given in Artzner et al. (1999) (see also Föllmer and Schied (2004)) as

\[
\text{CVaR}_\alpha(L) = \sup_{\Omega \in \mathcal{Q}_\alpha} \mathbb{E}^\Omega[L], \quad \text{where} \quad \mathcal{Q}_\alpha = \{ Q \ll P : dQ/dP \leq 1/(1 - \alpha) \}.
\]

Equivalently, CVaR can be expressed in terms of the Radon-Nikodym derivatives \( \varphi = dQ/dP \),

\[
\text{CVaR}_\alpha(L) = \sup_{\varphi \in \mathcal{M}_\alpha} \mathbb{E}[L\varphi], \quad \text{where} \quad \mathcal{M}_\alpha = \{ \varphi \in L^\infty(\Omega, \mathcal{A}, P) : \mathbb{E}[\varphi] = 1, 0 \leq \varphi \leq 1/(1 - \alpha) \}.
\]
The optimal Radon-Nikodym derivative at each outcome typically takes one of the two values 0 or $1/(1 - \alpha)$ at the endpoints of the interval. Using this observation, another formulation was recently proposed by Espinoza and Moreno (2014), which is based on scenario aggregation. This algorithm reduces the size of (Dual) by aggregating the outcomes (partitioning the scenario set and constraining the values of $\lambda_k$ to be constant on each subset in this partition), iteratively finding an approximate portfolio $\tilde{x}$, and refining the partition $N = \{N_1, \ldots, N_b\}$ of the scenarios $\{1, \ldots, n\}$ by separating the “good” outcomes from the “bad” ones, as indicated by the loss outcomes $-\tilde{x}^\top y_k$ of the approximate portfolio. The probability of an aggregated scenario $N_i \subset \{1, \ldots, n\}$ is $\sum_{k \in N_i} p_k$, and the corresponding asset returns in this aggregated scenario are the expected returns $\sum_{k \in N_i} p_k y_k / \sum_{k \in N_i} p_k$. Using these parameters, (Dual) is then solved, and a new approximate portfolio is obtained. We will refer to this approach as the Aggregation algorithm.

The Dual representation can be solved faster than the Primal, because the $n$ constraints on $\lambda_k, k = 1, \ldots, n$, are constant bounds, whereas in the Primal representation there are $n$ inequality constraints that include several decision variables. However, in the numerical experiments in Paper F, we observe that the Aggregation algorithm performs the best, especially for high $\alpha$ and heavy tails (low parameter $\nu$ for Student-$t$). This is because in these cases there are fewer “bad” scenarios to identify, and it is easier to find the “bad” outcomes. In Figure 4.1, an illustration in the case of $d = 2$ assets is given. The generated outcomes from a light-tailed, respectively, heavy-tailed distribution are plotted, and the cuts that are used by the iterations of the Aggregation algorithm are shown. The Aggregation algorithm solved 7, resp., 6 instances of the Dual formulation, but the maximal size was only 14, resp., 10 aggregated scenarios, instead of the

![Figure 4.1: Cuts corresponding to the iterations (labeled) of the Aggregation algorithm for $d = 2$ assets, based on a sample of $n = 1000$ points from a bivariate distribution. The outcome on the cut line for each iteration is marked with “+”. Left panel: Normal distribution, 7 iterations, 14 aggregated scenarios. Right panel: Student-$t$ with parameter $\nu = 3$, 6 iterations, 10 aggregated scenarios.](image-url)
4.2. EXPECTILE OPTIMIZATION

original $n = 1000$ scenarios. Similar improvements are achieved when there are up to $d = 100$ assets and up to $n = 10^6$ sample points, enabling us to solve very large problem instances in a moderate time (few minutes), especially in the cases when a large sample size is necessary, i.e. for a high probability level $\alpha$ and a heavy-tailed asset returns distribution.

4.2 Expectile optimization

The expectile risk measure was introduced in Section 2.3 as a minimizer of an asymmetric quadratic scoring function. Formulating a portfolio optimization problem using this definition would lead to a difficult nested problem

$$\min_{x \in \mathcal{X}} \arg\min_{e \in \mathbb{R}} \mathbb{E} \left[ (\tau \mathbb{1}_{\{-x^T Y > e\}} + (1 - \tau) \mathbb{1}_{\{-x^T Y < e\}})((-x^T Y - e)^2) \right].$$

As an alternative, one may consider the robust representation of the expectile stated in Delbaen (2013). For a random loss $L \in L^1(\Omega, \mathcal{A}, P)$, the expectile at level $\tau \in [1/2, 1)$ is given by

$$e_\tau(L) = \sup_{\varphi \in \mathcal{M}_\tau} \mathbb{E}[L \varphi], \quad (4.1)$$

where the set $\mathcal{M}_\tau$ of Radon-Nikodym derivatives is defined as

$$\mathcal{M}_\tau = \{ \varphi \in L^\infty(\Omega, \mathcal{A}, P) : \varphi \geq 0, \mathbb{E}[\varphi] = 1, (1 - \tau)m \leq \varphi \leq \tau m \text{ for some } m > 0 \}.$$

However, directly substituting $L = -x^T Y$ into this representation leads to a nested quadratic optimization problem in terms of decision variables $x$ and $\varphi$, with objective $\mathbb{E}[-x^T Y \varphi]$.

Bellini et al. (2015) considered the dual problem to (4.1) and noticed that in this (minimization) problem the portfolio loss $L$ no longer appears in a product with decision variables. Thus, substituting $L = -x^T Y$, one can formulate a joint minimization problem over the portfolio weights $x$ and the dual variables corresponding to the constraints imposed by $\mathcal{M}_\tau$, obtaining a linear program. However, they used the mathematically equivalent expression for $\mathcal{M}_\tau$ from Bellini et al. (2014),

$$\mathcal{M}_\tau = \left\{ \varphi \in L^\infty(\Omega, \mathcal{A}, P) : \varphi > 0, \mathbb{E}[\varphi] = 1, \frac{\text{ess sup}(\varphi)}{\text{ess inf}(\varphi)} \leq \frac{\tau}{1 - \tau} \right\}.$$

Consequently, when formulating the LP corresponding to (4.1) in the case of a finite discrete sample space $\Omega = \{\omega_1, \ldots, \omega_n\}$, the inequality constraints on $\varphi$ were formalized as $n(n - 1)$ constraints

$$(1 - \tau)\varphi_i \leq \tau \varphi_j \quad \text{for all} \quad i \neq j, \quad i, j \in \{1, \ldots, n\}.$$

In the dual problem, which includes the portfolio weights as decision variables (for this reason, we call it the Primal formulation), these constraints correspond to $n(n - 1)$ decision variables.
Therefore, with this formulation it is not possible to solve the portfolio optimization problem for $n > 1000$ scenarios (asset returns outcomes) within a reasonable time.

In Paper G, we formalize the corresponding constraints on the Radon-Nikodym derivative as $2n$ inequalities

$$(1 - \tau)m \leq \varphi_i \leq \tau m \quad \text{for} \quad i \in \{1, \ldots, n\},$$

at the expense of adding one more decision variable $m > 0$. This leads to our Primal LP formulation, which is solvable within half an hour of computation time for up to $n = 10^5$ scenarios (using state-of-the-art optimization software and a 2.2 GHz processor).

With the aim of improving the computation time, we also consider the Dual formulation of the expectile problem (obtained by taking the dual LP of the Primal formulation). Recall that in the case of a CVaR objective, the dual LP formulation from Ogryczak and Śliwiński (2011) lead to considerably improved computation times compared to the original formulation of Rockafellar and Uryasev (2000). The numerical case study in Paper G shows that for the expectile objective such improvements are not attained. While for the CVaR problem in the formulation of Ogryczak and Śliwiński (2011) the constraints on the Radon-Nikodym derivative are simple constant bounds for each scenario (compared to joint inequalities in the formulation of Rockafellar and Uryasev (2000)), for the expectile objective the bounds on $\varphi_i, i = 1, \ldots, n$, all depend on the decision variable $m$. This leads to an inseparable set of $2n$ constraints on $n + 1$ decision variables, and hence the computation times for Primal and Dual formulations of the expectile optimization problem are similar.

Nevertheless, there are similarities between CVaR and the expectile regarding the structure of the set of Radon-Nikodym derivatives. For CVaR$_\alpha, \varphi \in \mathbb{R}^n$ essentially selects the $(1 - \alpha)n$ highest losses and takes the value of its upper bound $1/(1 - \alpha)$ in the corresponding outcomes; for other outcomes it is equal to zero. We notice that the Radon-Nikodym derivative has a similar structure for the expectile problem, with two differences. First, the number of outcomes in each group is now known; it depends on the size of the losses above and below the expectile value. Second, the upper and lower bounds on $\varphi$ are no longer constant. This makes the expectile problem more challenging, but it is still possible to formulate a scenario aggregation approach, because the decision variables $\varphi_i, i = 1, \ldots, n$, take only two values, albeit a priori unknown ones. In particular, from Proposition 8 in Bellini et al. (2014) follows that an optimal solution $(\varphi, m)$ in (4.1) is given by

$$\varphi_i = (1 - \tau)m \mathbb{1}\{\ell_i \leq e_r\} + \tau m \mathbb{1}\{\ell_i > e_r\}, \quad i = 1, \ldots, n,$$

$$m = 1/[(1 - \tau)P(L \leq e_r) + \tau P(L > e_r)],$$

where $e_r = e_r(L)$ is the expectile of the portfolio loss $L \sim \text{UNIF}\{\ell_1, \ldots, \ell_n\}$.

In Paper G, we use this observation to formulate the Aggregation algorithm for portfolio optimization with an expectile objective. This algorithm amounts to solving several instances
4.2. \textit{EXPECTILE OPTIMIZATION}

of the Dual formulation, where the values of the Radon-Nikodym derivative are constrained to be constant on the sets of a partition of the scenarios. This allows representing $\varphi$ as a lower-dimensional decision variable. This constrained (aggregated) problem yields approximate portfolio weights, which are then used to refine the partition in a similar manner as in the CVaR problem, by separating the outcomes for which the loss of this portfolio falls below, respectively, above the expectile level corresponding to this approximate portfolio. After several such refinement iterations (typically, fewer than 25), a portfolio is found that is optimal for the original problem. Since these aggregated problems are of a much smaller size (in terms of the number of decision variables and constraints) compared to the Primal and Dual formulations, the savings in the computation time are considerable. For sample size $n = 10^5$, the solution was obtained within seconds, while for the Primal and Dual formulations it took half an hour or more. For larger sample sizes, the latter formulations become impractical. With the Aggregation algorithm, problems based on up to one million scenarios can be solved in minutes. To conclude, these results demonstrate that, besides having desirable theoretical properties, the expectile is also a practical risk measure for portfolio optimization from a computational perspective.
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**General convex order on aggregate risk.**

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General convex order on risk aggregation

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Abstract

Using a general notion of convex order, we derive general lower bounds for risk measures of aggregated positions under dependence uncertainty, and this in arbitrary dimensions and for heterogeneous models. We also prove sharpness of the bounds obtained when each marginal distribution has a decreasing density. The main result answers a longstanding open question and yields an insight in optimal dependence structures. A numerical algorithm provides bounds for quantities of interest in risk management. Furthermore, our numerical results suggest that the bounds obtained in this paper are generally sharp for a broader class of models.

Keywords: risk aggregation, dependence uncertainty, convex order, risk measures, heterogeneous models, joint mixability.

1 Introduction

In quantitative risk management, under the term \textit{risk aggregation} one discusses the statistical behavior of an aggregate position \( S(\mathbf{X}) \) associated with a risk vector \( \mathbf{X} = (X_1, \ldots, X_n) \), where \( X_1, \ldots, X_n \) are random variables representing one-period individual risks. The most commonly studied aggregate risk position is the sum \( S = X_1 + \cdots + X_n \); it plays an important role in both insurance and finance.

In the quantification of risk aggregation, \textit{model uncertainty} has received much attention recently, especially after the financial crisis of 2008; see discussions in BCBS (2013b). An
insufficient understanding of model uncertainty (or manipulation) and its quantitative consequences may lead to wrong conclusions, undermining the efficiency of risk management (for a case study, see US Senate, 2013, Chapter V).

One of the more challenging types of uncertainty in modern risk management relates to dependence uncertainty. In practice, marginal distributions are easier to analyze with statistical tools, while multivariate dependence is much more difficult to quantify. We refer to Embrechts et al. (2014) for a comprehensive review on the growing literature on this topic and its impact on the recent framework for banking regulation, as for instance discussed in the Basel documents (BCBS, 2012, 2013a). In insurance regulation, discussions on uncertainty in risk management also take place in Solvency 2 and the Swiss Solvency Test; see for instance Sandström (2010) and SCOR (2008).

To address questions of risk aggregation with dependence uncertainty, Bernard et al. (2014) introduced the admissible risk class as the collection of all possible aggregate risks with fixed, known marginal distributions. A practical illustration of this setup is for instance to be found in the Loss Distribution Approach to Operational Risk; see Embrechts et al. (2013). Though, of course, in this case, there is considerable uncertainty at the level of the estimation of the marginal distributions. If one has additional information on the underlying dependence structure, subsets of the admissible risk class can be used to describe the possible aggregate risks; for the case of information on variance, see Cheung and Vanduffel (2013) and Bernard et al. (2015a).

A main tool in our analysis is convex ordering, which in our context is equivalent with second order stochastic dominance or stop-loss ordering. These concepts from the realm of decision making under uncertainty are consistent with risk-averse risk measurement, especially in the by now classical context of coherent risk measures and its utility-based formulation; see for instance Artzner et al. (1999) and Delbaen (2012). In the context of finding upper and lower bounds for quantities related to risk aggregation, convexity arguments play an important role. In the sequel of this paper convex bounds will refer to any bound obtained using arguments based on the concept of convex ordering. It is well-known that the sharp upper convex bound on any admissible risk class is obtained under the comonotonic dependence structure, whereas for the lower convex bound for \( n \geq 3 \) no general solution is known in the literature; see for instance Tchen (1980) and Dhaene et al. (2002). Rüschendorf and Uckelmann (2002) and Section 8.3.1 of Müller and Stoyan (2002) studied special cases of lower convex bounds on risk aggregation for uniform, symmetric and some discrete marginal distributions, and Wang and Wang (2011) studied the case when marginal distributions are identical and have a monotone density. A numerical algorithm (the Rearrangement Algorithm (RA)) for the approximation of the dependence structure leading to a lower convex bound is given in Embrechts et al. (2013). The latter paper contains a general lower convex bound in the homogeneous case, i.e. when all
marginal distributions are identical. Furthermore, under some extra conditions, sharpness of
this bound is proved; see also Bernard et al. (2014).

In this paper, we generalize the results of Bernard et al. (2014) and study a lower convex
bound for non-identical marginal distributions. This generalization is particularly important in
the practice of financial and insurance risk management, where identical marginal distributions
are clearly unrealistic; see Embrechts et al. (2013). It turns out that the problem of finding con-
vex bounds with heterogeneous marginal distributions is considerably more challenging. The
new lower convex bound obtained in this paper is based on a new technique of dynamically
weighting marginal distributions by finding solutions to related functional equations. The de-
pendence structure that leads to the lower convex bound can be interpreted as a combination of
joint mixability, introduced in Wang et al. (2013), and mutual exclusivity, introduced in Dhaene
and Denuit (1999) (earlier mathematical results can be found in Dall’Aglio (1972)). We show
that this new bound is sharp if each of the marginal distributions has a decreasing density on its
support. Numerical results show that the new bounds outperform almost all other results in the
literature, and this in great generality.

We remark that although our results work for a broad class of models, including all models
with decreasing densities, a universal solution for a lower convex bound is still out of reach at
this moment, even for homogeneous models. A full characterization of this lower convex bound
for arbitrary distributions would require further research on joint mixability and other negative
dependence concepts, a rapidly expanding field of research. A recent review on extremal de-
pendence concepts can be found in Puccetti and Wang (2015b).

The rest of the paper is organized as follows. In Section 2 we summarize some preliminaries
on admissible risks, complete and joint mixability, and convex order. Section 3 provides a new
lower convex bound on risk aggregation for heterogeneous marginal distributions. This bound
is shown to be sharp under a monotonicity condition and a condition of joint mixability; in
particular, these conditions are satisfied if the marginal distributions have decreasing densities.
Insurance and financial applications are then discussed in Section 4. Numerical illustrations
and an algorithm are given in Section 5, highlighting the advantages of our results compared to
other numerical methods available in the literature. We conclude in Section 6.

2 Preliminaries

2.1 Admissible risk

In this paper, we assume that all random variables are defined on an atomless probability space
\((\Omega, \mathcal{A}, P)\). Similar to Bernard et al. (2014), we call an aggregate risk the sum
\(S = X_1 + \cdots + X_n\),
where \(X_i\) are non-negative random variables and \(n\) is a positive integer. Note that the non-
negativity is assumed just for notational convenience, and for our results is equivalent to the assumption that $X_1, \ldots, X_n$ are bounded below (since convex order is invariant under translation $X_i \mapsto a_i + X_i, a_i \in \mathbb{R}$).

In this paper we consider the case where for each $i = 1, \ldots, n$ the distribution of $X_i$ is known, while the joint distribution of $X := (X_1, X_2, \ldots, X_n)$ is unknown. We use the notation $X \sim F$ to indicate that $X \in L^0(\Omega, \mathcal{A}, \mathbb{P})$ has distribution function (df) $F$.

**Definition 1.** An aggregate risk $S$ is called an admissible risk for given marginal distributions $F_1, \ldots, F_n$ if it can be written as $S = X_1 + \cdots + X_n$ where $X_i \sim F_i$ for $i = 1, \ldots, n$. The admissible risk class is defined by the set of admissible risks of given marginal distributions:

$$\Xi_n(F_1, \ldots, F_n) = \{X_1 + \cdots + X_n : X_i \sim F_i, \ i = 1, \ldots, n\}.$$

The definition of admissible risks only concerns the distributions of random variables, thus there is a one-to-one relationship between $\Xi_n(F_1, \ldots, F_n)$ and the admissible distribution class defined as

$$\mathcal{D}_n(F_1, \ldots, F_n) = \{\text{distribution of } S : S \in \Xi_n(F_1, \ldots, F_n)\}.$$

Properties of the admissible risk class $\Xi_n(F_1, \ldots, F_n)$ were given in Bernard et al. (2014). A full characterization of $\Xi_n(F_1, \ldots, F_n)$ is challenging and seems far beyond the reach of current methodology. The admissible risk class identifies what risks are possible when the marginal distributions are known. When all risks have the same distribution, i.e. $F_1 = \cdots = F_n$, we say that the risks are homogeneous. When the distributions $F_i$ are allowed to be different, we say that the risks are heterogeneous. For simplicity, we denote $F = (F_1, \ldots, F_n)$.

As mentioned in the introduction, the study of the admissible risk class $\Xi_n(F)$ is of great interest in risk management and this topic has a long history. One of the most important issues is to quantify aggregate risks under extreme dependence structures. Note that all admissible risks of given marginal distributions $(F_1, \ldots, F_n)$ have the same mean if it exists for each $F_i$. It is thus natural to consider variability in the class. In this paper, we measure variability using convex order and focus on extreme aggregate risks in $\Xi_n(F)$ in the sense of convex order.

### 2.2 Complete and joint mixability

Distributions $F_1, \ldots, F_n$ are jointly mixable (JM) (Wang et al., 2013) if there exist $X_i \sim F_i, i = 1, \ldots, n$ such that $X_1 + \cdots + X_n$ is (almost surely) a constant; such $(X_1, \ldots, X_n)$ is called a joint mix. Here, we give an equivalent definition using admissible risks.

**Definition 2.** (i) Univariate distributions $F_1, \ldots, F_n$ are said to be jointly mixable (JM) if the admissible risk class $\Xi_n(F_1, \ldots, F_n)$ contains a constant.
(ii) A univariate distribution $F$ is said to be $n$-completely mixable (n-CM) if the admissible risk class $\Xi_n(F, \ldots, F)$ contains a constant.

We also say that $F$ is n-CM on an interval $I$ if the conditional distribution of $F$ on $I$ is n-CM, and $F_1, \ldots, F_n$ are JM on a hypercube $\prod_{i=1}^n I_i$ if the conditional distributions of $F_1, \ldots, F_n$ on intervals $I_1, \ldots, I_n$, respectively, are JM.

Some examples and recent theoretical results of CM distributions and JM distributions can be found in Wang and Wang (2011, 2016) and Puccetti et al. (2012, 2013). Complete mixability turns out to be crucial for finding lower convex bounds for homogeneous risks; for a detailed discussion, see Bernard et al. (2014). In this paper, joint mixability will be used to obtain sharp lower bounds for heterogeneous risks.

2.3 Convex order and existing results

Convex order describes a preference between two random variables, agreed upon by all risk-avoiding investors. Let us recall the definition.

**Definition 3.** Let $X$ and $Y$ be two random variables with finite means. Then $X$ is smaller than $Y$ in convex order, denoted by $X \preceq_{cx} Y$, if for all convex functions $f$,

$$E[f(X)] \leq E[f(Y)],$$

provided that the two expectations exist.

It is immediate that $X \preceq_{cx} Y$ implies $E[X] = E[Y]$. In the following, we assume that $F_1, \ldots, F_n$ have finite means. Convex order is well-adapted to $\Xi_n(F)$ as all variables in $\Xi_n(F)$ have the same mean. Note that convex order is an order determined by distributions only, hence we do not really need to specify random variables in our discussion. Convex order on aggregate risks has been extensively studied in actuarial science since it is equivalent to stop-loss order (given that the means of two risks are the same), an important concept in insurance premium calculations. More discussions on stochastic orders on aggregate risks can be found in Müller (1997b,a). We say that $T$ is an upper (respectively, lower) convex bound of $\Xi_n(F_1, \ldots, F_n)$ if $T \succeq_{cx} S$ (respectively, $T \preceq_{cx} S$) for all $S \in \Xi_n(F_1, \ldots, F_n)$. From now on, our objective is to find convex bounds of the set $\Xi_n(F_1, \ldots, F_n)$.

We denote by $G^{-1}(t) = \inf\{x : G(x) \geq t\}$ for $t \in (0,1]$ the generalized inverse function for any monotone function $G : \mathbb{R}^+ \rightarrow [0,1]$, and in addition let $G^{-1}(0) = \inf\{x : G(x) > 0\}$ throughout the paper. A well-known result is that the sharp upper convex bound in $\Xi_n(F_1, \ldots, F_n)$ is $F_1^{-1}(U) + \cdots + F_n^{-1}(U)$ where $U$ is uniformly distributed on the interval $(0,1)$, denoted as $U \sim \mathcal{U}(0,1)$. The dependence structure of $X = (F_1^{-1}(U), \ldots, F_n^{-1}(U))$ is called the comonotonic scenario. In particular, one has

$$S_n \preceq_{cx} F_1^{-1}(U) + \cdots + F_n^{-1}(U) \quad \text{for any} \quad S_n \in \Xi_n(F). \quad (1)$$
Note that sharpness immediately follows, since for $i = 1, \ldots, n$, $F_{i}^{-1}(U) \overset{d}{=} X_i$ and hence the above upper bound belongs to $\Xi_n(F)$. We refer to Dhaene et al. (2002) for more details on comonotonicity and Deelstra et al. (2011) for a recent review on the applications of comonotonicity in finance and insurance.

The rest of the paper focuses on the much more complex issue of determining the lower convex bound of $\Xi_n(F)$. When there are only two variables, $n = 2$, the minimum is obtained by the countermonotonic scenario:

$$F_1^{-1}(U) + F_2^{-1}(1 - U) \leq_{\text{cx}} S_2$$

for any $S_2 \in \Xi_2(F_1, F_2)$, where $U \sim U(0, 1)$; and this bound is again sharp. The above two convex order results, in the larger class of supermodular functions, date back to W. Hoeffding in the 40s; see Tchen (1980).

However, the sharp lower convex bound for $n \geq 3$ is missing in the literature due to the fact that countermonotonicity cannot be generalized to $n \geq 3$ without losing its minimality with respect to convex order. In the case when marginal distributions are identical with a monotone density function, the sharp lower bound for general $n$ is obtained in Wang and Wang (2011), together with results on complete mixability. In another special case, when $F_1, \ldots, F_n$ are on $\mathbb{R}^+$ with $\sum_{i=1}^n F_i(0) \geq n - 1$, the convex minimum is obtained by the mutually exclusive scenario:

$$Y_1 + \cdots + Y_n \leq_{\text{cx}} S_n$$

for any $S_n \in \Xi_n(F)$, where $Y_i \sim F_i$ and $P(Y_i > 0, Y_j > 0) = 0$, $i, j = 1, \ldots, n, i \neq j$, i.e. only one random variable can be positive at the same time; see Dhaene and Denuit (1999). However, this assumption means that the distributions $F_1, \ldots, F_n$ have atoms at zero with a very large total mass, and hence it is rather restrictive. Another observation, also restrictive, is that if $F_1, \ldots, F_n$ are JM, then a sharp convex lower bound is based on joint mixability: if $X_1 + \cdots + X_n$ is a constant, where $X_i \sim F_i, i = 1, \ldots, n$, then for $S_n \in \Xi_n(F)$,

$$X_1 + \cdots + X_n = \mathbb{E}[S_n] \leq_{\text{cx}} S_n$$

for any $S_n \in \Xi_n(F)$. However, joint mixability is theoretically difficult to prove, and many distributions are shown to be not JM. Limited results on JM are summarized in Wang et al. (2013) and the more recent Wang and Wang (2016).

One step further, Bernard et al. (2014) studied more general lower convex bounds over the admissible risk class. Roughly speaking, their idea is to combine complete mixability and mutual exclusivity. We summarize their results below. Let $n$ be a positive integer (although only $n \geq 3$ is of interest). Let $F$ be the average of the marginal distributions, i.e.

$$F = \frac{1}{n} \sum_{i=1}^n F_i.$$
The following functions \( H(x) \), \( D(a) \) for \( a, x \in [0, \frac{1}{n}] \) and the number \( c_n \) are defined in Bernard et al. (2014):

\[
H(x) = (n - 1) F^{-1}((n - 1)x) + F^{-1}(1 - x),
\]
\[
D(a) = \frac{n}{1 - na} \int_{(n - 1)a}^{1 - a} F^{-1}(y) \, dy,
\]
\[
c_n = \min \{ c \in [0, \frac{1}{n}] : H(c) \leq D(c) \},
\]
\[
T_a = H(U/n) \mathbb{1}_{U \in [0, na]} + D(a) \mathbb{1}_{U \in (na, 1]}.
\]

Roughly speaking, \( H \) represents the sum in a nearly mutual exclusive scenario, where one large risk is coupled with \( n - 1 \) small risks. \( D \) represents the sum in a scenario of joint mix, where the sum is exactly equal to its mean. The structure of \( T_a \) can be interpreted as a combination of \( H \) and \( D \), and \( c_n \) is a threshold distinguishing the two scenarios.

For some \( a \in [0, \frac{1}{n}] \), Bernard et al. (2014) used the following assumptions (A), (A’) and (B):

(A) \( H(x) \) is non-increasing on \([0, a]\) and \( \lim_{x \to a^-} H(x) \geq D(a) \).

(A’) \( H(x) \) is non-increasing on the interval \([0, c_n]\).

(B) The distribution \( F \) is \( n \)-CM on the interval \( I = [F^{-1}((n - 1)c_n), F^{-1}(1 - c_n)] \).

The assumption (A) is used to obtain a convex lower bound for \( \Xi_n(F_1, \ldots, F_n) \), and the assumptions (A’) and (B) are used to obtain sharpness for the homogeneous model \( \Xi_n(F, \ldots, F) \). Note that (A) always holds trivially for \( a = 0 \), and (A’) is stronger than (A). The main results in Bernard et al. (2014) are summarized in the following theorem.

**Theorem 1** (Bernard et al. (2014)).

(i) Suppose (A) holds for some \( a \in [0, \frac{1}{n}] \), then \( T_a \leq_c S \) for all \( S \in \Xi_n(F_1, \ldots, F_n) \).

(ii) In the homogeneous case \( F_1 = \cdots = F_n = F \), \( T_{c_n} \in \Xi_n(F_1, \ldots, F_n) \) if (A’) and (B) holds.

Theoretically, one only obtains the sharpness of the above bounds for homogeneous risks. As pointed out and illustrated numerically in Bernard et al. (2014), the sharpness of the bound obtained in Theorem 1 generally fails to hold for heterogeneous risks, in particular when the marginal distributions are significantly different. A sharp convex lower bound for heterogeneous risks seems out of reach by their methodology.

In this paper, we give a new lower convex bound for heterogeneous risks which is sharp under a monotonicity condition and a JM condition. Our result is based on a new technique involving a dynamical weighting of the marginal distributions.
3 Convex Bounds on Risk Aggregation

Throughout we suppose $F_1, \ldots, F_n$ are continuous distributions on $\mathbb{R}_+$ with finite means, and $n \geq 3$ is a positive integer. Without loss of generality, we can assume all distributions have left end-point at 0. Since convex order is invariant under shifting by constants, this is equivalent to assuming all of their supports are bounded from below. We denote by $\bar{F}_i$ the survival function of $F_i$, i.e. $\bar{F}_i = 1 - F_i$. In all our discussions, the terms decreasing and increasing are understood in the strict sense.

Our idea to construct an optimal structure is inspired by the arguments of Bernard et al. (2014). In order to have a convexly small element, intuitively one wants the sum $S = X_1 + \cdots + X_n$ to be concentrated around its mean, e.g. a smaller variance is preferred by taking a quadratic $f$ in Definition 3. Typically, for financial risks $P(X_i > E[S]) > 0$ for some $i$ (large losses are possible). When $X_i > E[S]$, $S$ must be greater than its mean, so we would like all the other $X_i$ to be as small as possible, so that the sum $S$ is minimized in convex order. This motivates constructing a dependence structure where, when one of the $X_i$ is large, the others are set to be small (“nearly mutually exclusive”); and when one of $X_i$ is of medium size, all the others are also of medium size and the sum is concentrated around a constant (“nearly jointly mixable”). This idea is very similar to the construction in Bernard et al. (2014) which originates from that in Wang and Wang (2011) and indeed forms the basis for the Rearrangement Algorithm in Embrechts et al. (2013). However, although the idea of complete mixability in Bernard et al. (2014) can be naturally generalized to joint mixability for heterogeneous risks, the construction of the “nearly mutually exclusive scenario” for heterogeneous risks is unclear and cannot be easily generalized. It turns out that to construct such a dependence structure one needs to dynamically assign weights to each margin, while keeping the sum small in convex order. Below we provide a rigorous mathematical formulation for the above idea. At first this weighting may seem rather non-intuitive; further in the paper its importance will hopefully become clear.

3.1 Main results

First, we introduce the following functional equations (E1)-(E2): for $y, y_1, \ldots, y_n : (0, 1) \to \mathbb{R}_+$, such that for $x \in (0, 1)$,

(E1) $\sum_{i=1}^n \bar{F}_i(y_i(x)) = x$,

(E2) $F_i(y_i(x) - y(x)) + \bar{F}_i(y_i(x)) = x$ for each $i = 1, \ldots, n$.

Equations (E1)-(E2) will be key to the rest of the paper. In the following, we shall continue our discussion assuming that (E1)-(E2) have at least one solution. A condition which guarantees the existence and uniqueness of such a solution is:
(F) Each $F_i, i = 1, \ldots, n$, has a decreasing density on its support $[0, \infty)$.

Condition (F) includes, for instance, Pareto and Exponential distributions. The main results in this paper do not require (F), but do require (E1)-(E2) to have a solution; the reader is suggested to keep (F) in mind for a primary example. Our numerical results show that (E1)-(E2) have a solution for a wider class of distributions relevant in practice.

Lemma 2. The system of functional equations (E1)-(E2) has a unique solution if (F) holds.

We put the rather tedious proofs of Lemma 2 and Lemma 3 below at the end of this section.

In the following, we assume $(y, y_1, \ldots, y_n)$ is a solution to (E1)-(E2), and define three functions which play a key role in this paper. For $X_i \sim F_i, i = 1, \ldots, n$,

$$h(x) = \sum_{i=1}^{n} y_i(x) - (n - 1)y(x) \quad \text{for} \quad x \in (0, 1),$$

$$d(a) = \frac{1}{1 - a} \sum_{i=1}^{n} E[X_i \mathbb{1}_{\{y_i(a) - y(a) < x \leq y_i(a)\}}] \quad \text{for} \quad a \in (0, 1), \quad d(0) = \sum_{i=1}^{n} E[X_i],$$

$$s_n = \inf \{ s \in (0, 1) : h(s) \leq d(s) \}.$$

The lemma below lists some useful properties of the solution $(y, y_1, \ldots, y_n)$.

Lemma 3. Suppose $(y, y_1, \ldots, y_n)$ is a solution to (E1)-(E2), and (F) holds. Then on the interval $(0, s_n)$ for each $i = 1, \ldots, n$,

(i) $0 < y < y_i$,

(ii) $y$ and $y_i$ are decreasing,

(iii) $y_i - y$ is increasing,

(iv) $h$ is decreasing.

Remark 1. A numerical procedure (provided in Section 5.2) can be applied to find a solution to (E1)-(E2), when $F_1, \ldots, F_n$ have densities. Moreover, we shall also see that the monotonicity condition (F) is not a necessary one and it is possible to obtain solutions to (E1)-(E2) in more general cases. For instance, in the homogeneous model $F_1 = \cdots = F_n =: F$, one can easily check that $y_1(x) = \cdots = y_n(x) = F^{-1}(x/n)$, and $y(x) = F^{-1}(x/n) - F^{-1}((n - 1)x/n)$ for $x \in (0, 1)$ give a solution to (E1)-(E2) which satisfies (i)-(iii) in Lemma 3 on $(0,1)$. This does not require the assumption (F). Property (iv) is not guaranteed in general, but it is satisfied by a large class of distributions.

In all the following discussions and results, we throughout assume
(C) $F_1, \ldots, F_n$ are continuous distributions on $\mathbb{R}_+$ and (E1)-(E2) have a solution $(y, y_1, \ldots, y_n)$ which satisfies properties (i)-(iv) in Lemma 3.

From Lemmas 2 and 3 we have seen that (F) is sufficient for (C). Condition (C) can be easily verified numerically for given marginal distributions $F_1, \ldots, F_n$. Indeed, (C) is not restrictive; numerical illustrations suggest that it is satisfied by almost all distributions used in quantitative risk management. See Section 5 for a discussion. However, it is theoretically difficult to show that (C) is satisfied by general choices of distributions; even in the homogeneous case only numerical verification is available, as discussed in Bernard et al. (2014). Note that we do not assume that the distributions $F_1, \ldots, F_n$ have unbounded support, nor do we assume the uniqueness of $(y, y_1, \ldots, y_n)$; we will simply need one solution to (E1)-(E2). Although the uniqueness of $(y, y_1, \ldots, y_n)$ is not guaranteed, we will see that $h$ is unique on $(0, s_n)$ under some extra conditions to be formulated later.

Note that if (C) holds, then $\bar{F}_i(y_i(\cdot))$ is continuous, almost everywhere differentiable on $(0, 1)$, and $0 < d\bar{F}_i(y_i(x))/dx < 1$ for each $i = 1, \ldots, n$; this can be seen from (E1) and the fact that $y_i$ is decreasing. We will use this fact frequently in the subsequent proofs. We first provide some properties of the function $h(x)$.

**Lemma 4.** For $a \in (0, 1)$, we have

$$\int_0^a h(u) \, du = \sum_{i=1}^n \mathbb{E} \left[ X_i \left( 1_{\{ X_i > y_i(a) \}} + 1_{\{ X_i < y_i(a) - y(a) \}} \right) \right].$$

**(2)**

**Proof.** We have that

$$\mathbb{E} \left[ X_i \left( 1_{\{ X_i > y_i(a) \}} + 1_{\{ X_i < y_i(a) - y(a) \}} \right) \right] = \int_0^{\infty} x \, dF_i(x) + \int_0^{y(a) - y(a)} x \, dF_i(x)$$

$$= \int_0^{y_i(t)} dF_i(y_i(t)) + \int_0^a (y_i(t) - y(t)) \, dF_i(y_i(t) - y(t)).$$

**(3)**

By (E2), it follows that

$$\int_0^a (y_i(t) - y(t)) \, dF_i(y_i(t) - y(t)) = \int_0^a (y_i(t) - y(t)) \, dt - \int_0^a F_i(y_i(t))$$

$$= \int_0^a (y_i(t) - y(t)) \, dt + \int_0^a y_i(t) \, dF_i(y_i(t)) - \int_0^a y(t) \, dF_i(y_i(t)).$$

**(4)**
From (3)-(4) and (E1), we have that
\[
\sum_{i=1}^{n} E\left[X_i \left( \mathbb{1}_{\{X_i > y_i(a)\}} + \mathbb{1}_{\{X_i < y_i(a) - y(a)\}} \right) \right] = \sum_{i=1}^{n} \int_{0}^{a} (y_i(t) - y(t)) \, dt - \sum_{i=1}^{n} \int_{0}^{a} y(t) \, dF_i(y_i(t)) \\
= \int_{0}^{a} \left( \sum_{i=1}^{n} y_i(t) - n y(t) \right) \, dt + \int_{0}^{a} y(t) \, dt \\
= \int_{0}^{a} h(t) \, dt.
\]

\[\Box\]

The function \( h(x) \) plays a key role in the construction of a sharp lower convex bound in \( \Xi_n(F_1, \ldots, F_n) \). In order to see this, we first define a candidate for the lower bound. For \( a \in [0, s_n] \), let
\[
R_a = h(U) \mathbb{1}_{\{U \in (0,a]\}} + d(a) \mathbb{1}_{\{U \in [a,1]\}},
\]
where \( U \sim \mathcal{U}(0,1) \). Since convex order depends only on distributions, we are only interested in the distribution of \( R_a \) and do not specify the random variable \( U \). Note that \( R_a \) is a generalization of the random variable \( T_a \) defined in Section 2. When \( U < a \), \( R_a \) is the random variable \( h(U) \); when \( U \geq a \), \( R_a \) is a constant \( d(a) \). The relationship between \( R_a \) and \( \Xi_n(F_1, \ldots, F_n) \) will be discussed later. Intuitively, \( 1 - a \) is the mass of the atom of \( R_a \) at \( d(a) \), so the smaller \( a \) is, the smaller \( R_a \) is in convex order, since it has more mass at a constant.

In the rest of the paper, we will use the following condition (D). It is parallel to (B) in Section 2.

(D) \( F_1, \ldots, F_n \) are JM on the hypercube \( \prod_{i=1}^{n} [y_i(s_n) - y(s_n), y_i(s_n)] =: \prod_{i=1}^{n} I_i \).

The study of joint mixability is a separate research field in probability theory; see Wang et al. (2013) and Wang and Wang (2016). The assumption (F) is sufficient for (D), as was recently shown in Wang and Wang (2016, Theorem 3.2). A numerical procedure to test for joint mixability is provided in Puccetti and Wang (2015a).

We give some properties of the random variable \( R_a \) in the following lemma.

**Lemma 5.** Suppose (C) holds, then

(a) \( E[R_a] = E[S] \) for any \( S \in \Xi_n(F_1, \ldots, F_n) \),

(b) \( R_u \preceq_{\text{cx}} R_v \) for \( 0 \leq u < v \leq 1 \),

(c) \( R_{s_n} \in \Xi_n(F_1, \ldots, F_n) \) if (D) holds.

**Proof.** (a) This follows from the definition of \( d(a) \) and (2).
(b) We have $R_n \preceq_{cx} R$, since $R_n$ is a fusion of $R$, (see Theorem 2.8 of Bäuerle and Müller (2006) and Theorem 3.1 of Bernard et al. (2014)).

(c) If (D) holds, there exist random variables $Y_1, \ldots, Y_n$ s.t. $Y_i$ has the conditional distribution of $F_i$ on $I_i$ and $Y_1 + \cdots + Y_n$ is a constant. Moreover,

$$Y_1 + \cdots + Y_n = \sum_{i=1}^{n} E[X_i \mathbb{1}_{\{X_i \in [y_i(s_n), y(s_n), y(s_n)]\}}] / (1 - s_n) = d(s_n)$$

by the definition of $d$. We now construct $S \in \Xi_n(F_1, \ldots, F_n)$ with the same distribution as $R_{s_n}$ by imposing a particular dependence structure. Let $U \sim U(0, 1)$ be independent of $Y_1, \ldots, Y_n$. We briefly explain the main idea behind the construction before moving forward to the rigorous setting:

- On the set \{ $U \geq s_n$ \}, of probability $1 - s_n$, we let $X_i = Y_i \in I_i$ for each $i$. We call this the body part of the dependence structure.

- On the set \{ $U < s_n$ \}, of probability $s_n$, which we call the tail part of the dependence structure, we let exactly one $X_i$ be in the right tail region, i.e. $X_i > y_i(s_n)$, and all the others be in the left tail region, i.e. $X_j < y_j(s_n) - y(s_n)$, $j \neq i$, being counter-monotonic to $X_i$.

To construct the random variables rigorously, let $K$ be a discrete random variable such that $P(K = i|U) = d\hat{F}_i(y_i(u))/du|_{u = U}$, $i = 1, \ldots, n$. $K$ is properly defined due to the differentiability of $F_i(y_i(u))$ and (E1). We construct for $i = 1, \ldots, n$:

$$X_i = \mathbb{1}_{\{U < s_n\}} \left( y_i(U) - y(U) \mathbb{1}_{\{K \neq i\}} \right) + \mathbb{1}_{\{U \geq s_n\}} Y_i,$$

and check that $X_i \sim F_i$. For $q < y_i(s_n) - y(s_n)$:

$$P(X_i \leq q) = P \left( F_i(y_i(U) - y(U)) \leq F_i(q), \ K \neq i \right)$$

$$= \int_{0}^{s_n} \mathbb{1}_{\{u - F_i(y_i(u)) < F_i(q)\}} \left( 1 - \frac{d\hat{F}_i(y_i(u))}{du} \right) du$$

$$= \int_{F_i(y_i(s_n)) - y(s_n)}^{F_i(y_i(u))} \mathbb{1}_{\{w < F_i(q)\}} dw = F_i(q),$$

using (E2) twice and substitution $w = u - \hat{F}_i(y_i(u))$. Similarly, for $q > y_i(s_n)$:

$$P(X_i > q) = P \left( F_i(y_i(U)) > F_i(q), \ K = i \right)$$

$$= \int_{0}^{s_n} \mathbb{1}_{\{F_i(y_i(u)) > F_i(q)\}} \frac{d\hat{F}_i(y_i(u))}{du} du$$

$$= \int_{F_i(y_i(s_n))}^{F_i(y_i(u))} \mathbb{1}_{\{w < 1 - F_i(q)\}} dw = 1 - F_i(q).$$
using substitution \( w = \tilde{F}_i(y_i(u)) \). Thus, \( X_i \) has the required distribution on \( \mathbb{R}^+ \setminus I_i \), and by construction also on the interval \( I_i \).

We also check that \( S = X_1 + \cdots + X_n \) has the required distribution:

\[
S = \mathbbm{1}_{\{U < s_n\}} \left( \sum_{i=1}^{n} y_i(U) - \sum_{i=1}^{n} y(U) \mathbbm{1}_{\{X_i \neq i\}} \right) + \mathbbm{1}_{\{U > s_n\}} d(s_n) = \mathbbm{1}_{\{U < s_n\}} h(U) + \mathbbm{1}_{\{U > s_n\}} d(s_n) = R_{s_n}.
\]

This completes the proof. \( \square \)

The following theorem is the main result of this paper. We will show that \( R_a \) is a lower convex bound on the set \( \mathcal{Z}_n(F_1, \ldots, F_n) \), and this bound is sharp for \( a = s_n \) under the JM condition (D).

**Theorem 6** (Convex ordering lower bound and sharpness). Suppose (C) holds, then

(i) \( R_a \preccurlyeq S \) for all \( S \in \mathcal{Z}_n(F_1, \ldots, F_n) \) and all \( a \in [0, s_n] \), and

(ii) \( R_{s_n} \) is the sharp lower convex bound in \( \mathcal{Z}_n(F_1, \ldots, F_n) \) if and only if (D) holds.

**Proof.** Recall that we only consider \( n \geq 3 \). The idea of our proof in part (i) is similar to the proof of Theorem 3.1 in Bernard et al. (2014).

(i) Let \( S = X_1 + \cdots + X_n \) with \( X_i \sim F_i \) be any random variable in \( \mathcal{Z}_n(F_1, \ldots, F_n) \) and \( R_a \) be defined in (5). By Lemma 5(a), we have \( \mathbb{E}[R_n] = \mathbb{E}[S] \). Let \( F_S \) and \( F_{R_a} \) be the df of \( S \) and \( R_a \) respectively. Our goal is to show that

\[
\int_0^1 F_{R_a}^{-1}(t) \, dt \leq \int_0^1 F_S^{-1}(t) \, dt, \quad \forall c \in (0, 1).
\]  

(6)

It is well-known that property (6) together with \( \mathbb{E}[R_a] = \mathbb{E}[S] \) is equivalent to \( R_a \preccurlyeq S \) (see for instance Bäuerle and Müller, 2006, Theorem 2.5).

To prove (6), define \( A_S(u) = \bigcup_i \{ X_i > y_i(u) \} \) and let \( W(u) = P(A_S(u)) \). By (E1) and since \( P \) is subadditive, \( W(u) \leq u \) holds; moreover, \( 0 \leq W(u + \epsilon) - W(u) \leq \epsilon \), so \( W \) is continuous and non-decreasing. For \( c \in (0, a] \), let \( u^* = W^{-1}(c) \) (the generalized inverse), so \( W(u^*) = c \) and thus \( u^* \geq c \). Hence \( \{ X_i > y_i(c) \} \subseteq \{ X_i > y_i(u^*) \} \subseteq A_S(u^*) \). Therefore

\[
P(A_S(u^*) \setminus \{ X_i > y_i(c) \}) = c - \tilde{F}_i(y_i(c)) = F_i(y_i(c) - y(c)) = P(X_i < y_i(c) - y(c)).
\]

Since the above two sets have the same measure, we have

\[
\mathbb{E}[\mathbb{1}_{\{X_i < y_i(c) - y(c)\}} X_i] \leq \mathbb{E}[\mathbb{1}_{A_S(u^*) \setminus \{ X_i > y_i(c) \}} X_i].
\]  

(7)
It follows by Lemma 4 that, for \( c \in (0, a) \),

\[
E[1_{\{U < c\}} R_a] = E[1_{\{U < c\}} h(U)] \\
= \sum_{i=1}^{\infty} E\left[\left(1_{\{X_i < y_i(c) - y(c)\}} + 1_{\{X_i > y_i(c)\}}\right) X_i\right] \\
\leq \sum_{i=1}^{\infty} E\left[\left(1_{A_S(u^*)\{X_i > y_i(c)\}} + 1_{\{X_i > y_i(c)\}}\right) X_i\right] \\
= E[1_{A_S(u^*)} S],
\]

where the inequality follows from (7). Thus we have

\[
E[1_{\{U < c\}} R_a] \leq E[1_{A_S(u^*)} S].
\]  

(8)

Note that \( h(x) \) is non-increasing on \( (0, a) \) and \( \lim_{x \to a^-} h(x) \geq d(a) \). Thus for \( c \in (0, a) \),

\[
E[1_{\{U < c\}} R_a] = E[1_{\{U < c\}} h(U)] = \int_{1-c}^{1} F_{R_a}^{-1}(t) \, dt.
\]  

(9)

Also note that, since \( P(A_S(u^*)) = c \),

\[
E[1_{A_S(u^*)} S] \leq \int_{1-c}^{1} F_{S}^{-1}(t) \, dt.
\]  

(10)

It follows from (8), (9) and (10) that for any \( c \in (0, a) \),

\[
\int_{1-c}^{1} F_{R_a}^{-1}(t) \, dt \leq \int_{1-c}^{1} F_{S}^{-1}(t) \, dt.
\]  

(11)

For \( x \in [0, 1 - a] \), let \( G(x) = \int_{1-x}^{1} F_{S}^{-1}(t) \, dt - \int_{1-x}^{1} F_{R_a}^{-1}(t) \, dt \). Note that \( F_{S}^{-1}(t) \) is non-decreasing, and \( F_{R_a}^{-1}(t) = d(a) \) is constant on \( t \in [0, 1 - a] \), hence \( G(x) \) is concave on \( [0, 1 - a] \). Hence, with \( G(0) = E[S] - E[R_a] = 0 \) and \( G(1-a) \geq 0 \) by (11), we have that \( G(x) \geq 0 \) on \([0, 1 - a]\). Thus

\[
\int_{c}^{1} F_{R_a}^{-1}(t) \, dt \leq \int_{c}^{1} F_{S}^{-1}(t) \, dt
\]  

for any \( c \in (0, 1) \), and hence \( R_a \leq_{\text{ex}} S \).

(ii) \( \iff \): This is a direct result of (i) and Lemma 5(c).

\( \Rightarrow \): Suppose \( R_{s_a} \overset{d}{=} S \in \Xi_n(F_1, \ldots, F_n) \), so \( S = X_1 + \cdots + X_n \) for some \( X_i \sim F_i \), \( i = 1, \ldots, n \). Thus (7) is an equality for each \( i \) and \( c \in (0, s_a] \). This implies that \( A_S(u^*)\{X_i > y_i(c)\} = \{X_i < y_i(c) - y(c)\} \), hence \( A_S(u^*) = \{X_i < y_i(c) - y(c)\} \cup \{X_i > y_i(c)\} \) for each \( i \) and \( c \in (0, s_n] \), up to a difference of a P-null set. As a consequence, \( S \) has the same construction as in the proof of Lemma 5 (c) on the set \( A := \{X_i < y_i(s_n) - y(s_n)\} \cup \{X_i > y_i(s_n) - y(s_n)\} \).
Therefore, $S \overset{d}{=} h(U) \mathbb{1}_A$ for some $U \sim \mathcal{U}(0, s_n)$ independent of $X_1, \ldots, X_n$. Since $S \overset{d}{=} R_{s_n}$, we have that $S \mathbb{1}_{A^c} = d(s_n) \mathbb{1}_{A^c}$, that is,

$$n \sum_{i=1}^n X_i \mathbb{1}_{A^c} \overset{a.s.}{=} d(s_n) \mathbb{1}_{A^c},$$

where $A^c = \cap_{i=1}^n \{y_i(s_n) - y(s_n) \leq X_i \leq y_i(s_n)\}$, and thus (D) holds. \qed

Note that when (C) and (D) hold, $s_n$ and $d(s_n)$ is uniquely determined. Moreover, since the smallest element in $S_n \mathbb{1}_{F_1} \cdots \mathbb{1}_{F_n}$ with respect to convex order is unique in law, $h$ is unique on $(0, s_n)$. Even if (D) is not satisfied, Theorem 6 (i) always provides a lower convex bound, which in case of $s_n = 0$, reduces to a constant.

Together with the classical result on comonotonicity (1), this yields the convex bounds for $S \in \mathcal{E}_n(F_1, \ldots, F_n)$ under (C):

$$R_{s_n} \leq_c S \leq_c F^{-1}_n(U) + \cdots + F^{-1}_1(U),$$

where $U \sim \mathcal{U}(0, 1)$. When (D) holds, the upper and lower bounds are both sharp. Note that (F) implies (C) and (D), hence in the case of decreasing densities the problem is fully solved.

**Remark 2.** If $F_1 = \cdots = F_n = F$, one solution to (E1)-(E2) is given by $y_i(x) = F^{-1}(x/n)$ (see Remark 1). In that case, we have that $h_i(x) = H(x/n), d(a) = D(a/n), s_n = nc_n, R_n = T_{na}$ with $H, D, c_n$ and $T$ defined in Section 2. Thus, Theorem 6 implies Theorem 1 which is the main result in Bernard et al. (2014).

**Remark 3.** In this section we only discussed the case for $n \geq 3$. As explained in Section 2.3, the case for $n = 2$ is well-known. Note, however, that our result is also valid for $n = 2$, and it reduces to the counter-monotonic scenario when conditions (C) and (D) hold. This can be seen by the construction using counter-monotonicity in the proof of Lemma 5 (c), and the fact that joint mixability implies counter-monotonicity in the case of $n = 2$.

Similar to the homogeneous risks in Bernard et al. (2014), the optimal dependence structure for heterogeneous risks can be described as follows. The probability space is divided into two subsets:

- For each $i$, if $X_i$ is large, then each $X_j, j \neq i$ is small and $(X_i, X_j)$ are counter-monotonic. This part has probability $s_n$. Since only one of $X_i$ can be large in any outcome, this part represents **mutual exclusivity**.

- For each $i$, if $X_i$ is of medium size, then each $X_j, j \neq i$ is also of medium size, and the sum $X_1 + \cdots + X_n$ is a constant. This part has probability $1 - s_n$ and represents **joint mixability**.

- The optimal dependence structure is a joint mix if $s_n = 0$, and it is a “nearly mutually exclusive structure” if $s_n = 1$. 
Different from the optimal structure in Bernard et al. (2014), the optimal structure in our paper involves functions $y_1, \ldots, y_n$. Essentially, the function $\bar{F}_i(y_i(x))$ represents the weight assigned to each individual risk $X_i$ due to inhomogeneity. In the homogeneous case, it is $x/n$, which means that each individual risk is assigned an equal weight.

3.2 Discussion on assumptions

In this section, we briefly discuss some issues related to the assumptions imposed for the main results. With some more complicated technical details, some of the assumptions can be relaxed slightly.

3.2.1 Continuity of the marginal distributions

The continuity assumed in this section is for ease of notation in (E1)-(E2). In the case when $F_1, \ldots, F_n$ are possibly discontinuous, (E1)-(E2) need to be replaced by the following two equations. For $t_1, \ldots, t_n : (0, 1) \rightarrow (0, 1)$, and $y : (0, 1) \rightarrow \mathbb{R}_+$,

\[(G1) \sum_{i=1}^n t_i(x) = x, \text{ and} \]
\[(G2) F_i^{-1}(1 - t_i(x)) - F_i^{-1}(x - t_i(x)) = y(x) \text{ for each } i = 1, \ldots, n.\]

That is, $t_i(x) = \bar{F}_i(y_i(x))$ in the continuous case. Then a solution to (G1)-(G2) may exist which satisfies the properties in Lemma 3 (substituting $y_i(x) = F_i^{-1}(1 - t_i(x))$). In this case, Theorem 6 is still valid.

For instance, one can add some probability mass to the marginal distributions at zero. A particular example concerns mutual exclusivity. Suppose that $F_1, \ldots, F_n$ are compatible with mutual exclusivity, i.e. $\sum_{i=1}^n F_i(0) \geq n - 1$, and each $F_i$ is continuous on $[0, \infty)$. Let $y(x) = \sup \{t : \sum_{i=1}^n \bar{F}_i(t) > x\}$ and $t_i(x) = \bar{F}_i(y(x))$. We can easily check that $y, t_1, \ldots, t_n$ is a solution to (G1)-(G2) which satisfies (i)-(iv) in Lemma 3 with strict inequalities replaced by non-strict inequalities. Theorem 6 holds for this case, and the optimal structure is given by mutual exclusivity. This is consistent with the construction explained after Theorem 6, which contains a “nearly mutually exclusive” component.

3.2.2 Monotonicity of $h$

One key property to show that $R_s \preceq_{\text{cx}} S$ for all $S \in \mathcal{S}(F_1, \ldots, F_n)$ is that $h$ has to be non-increasing on $(0, s_n)$. Similar to the homogeneous models in Bernard et al. (2014) (see conditions (A) and (A’) in Section 2), if we assume that all properties in Lemma 3 hold on $(0, a)$ for some $a < s_n$ (in particular, $h$ is decreasing on $(0, a)$), we still have the results in Theorem 6 (i): $R_s \preceq_{\text{cx}} S$ for all $S \in \mathcal{S}(F_1, \ldots, F_n)$. This can be used to obtain convex bounds when (C) does not hold, i.e. in case of less regularity of $F_1, \ldots, F_n$. 
However, one only obtains a sharp bound if $a = s_n$, since $R_a \notin \Xi_n(F_1, \ldots, F_n)$ for $a < s_n$. This is implied by a necessary condition for joint mixability (Wang and Wang, 2016, Theorem 2.1). We skip a detailed discussion on cases when (C) does not hold.

### 3.3 Proofs of Lemma 2 and 3

In this section we give proofs of Lemmas 2 and 3.

**Proof of Lemma 2.** Fix any $x \in (0, 1)$ and inductively define the following functions for $z \in (0, x]$:

$$\varphi_i(z) = F_i^{-1}(1 - z_i) - F_i^{-1}(x - z_i), \quad i = 1, \ldots, n,$$

where $z_1(z) = z$, and for $i = 2, \ldots, n$, $z_i(z)$ denotes the unique solution to

$$\varphi_{i-1}(z - z_i) = F_i^{-1}(1 - z_i) - F_i^{-1}(x - z_i).$$

Below we will show that $z_i(z)$, $i = 1, \ldots, n$ are well-defined. The interpretation is that the function $\varphi_k(z)$ returns the value $y(x)$ for the system where (E1) is replaced by $\sum_{i=1}^k F_i(y_i(x)) = z$, and (E2) is relaxed to hold only for $i = 1, \ldots, k$. Similarly, $z_k(z)$ returns the probability mass in the right tail of margin $k$ in this modified system.

Since each $F_i$ is strictly concave, $F_i^{-1}$ is strictly convex, so $F_i^{-1}(1 - \cdot) - F_i^{-1}(x - \cdot)$ is a continuous and decreasing function, with right limit $\infty$ at $0^+$. Clearly, $\varphi_1(\cdot)$ is such a function.

For induction, assume $\varphi_{i-1}$ also is and let

$$\delta_i(z, z_i) = \varphi_{i-1}(z - z_i) - F_i^{-1}(1 - z_i) - F_i^{-1}(x - z_i).$$

Then $\delta_i(z, z_i)$ is continuous in each argument, decreasing in $z$ and increasing in $z_i$. Moreover, $\delta_i(z, 0^+) = -\infty$ and $\delta_i(z, -\infty) = \infty$, so for each $z \in (0, x]$ there is a unique solution $z_i(z) \in (0, z)$ of (14) i.e. of $\delta_i(z, z_i) = 0$, and $z_i(z)$ is continuous and increasing. Hence, by (13) also $\varphi_i$ is a continuous and decreasing function, with right limit $\infty$ at $0^+$, completing the induction. Hence $z_i(z)$ and $\varphi_i(z)$, $i = 1, \ldots, n$ are well-defined functions on $z \in (0, x]$.

Finally, let $z_n^* = z_n(x)$ and $z_i^* = z_i(x - \sum_{j=i+1}^n z_j^*)$ for $i = n - 1, n - 2, \ldots, 1$. Setting $y = \varphi_n(x)$ and $y_i = F_i^{-1}(1 - z_i^*)$, $i = 1, \ldots, n$ yields the values of functions $y, y_1, \ldots, y_n$ at the point $x$. □

In the following, we will use Lemma 5.6 of Wang and Wang (2016), which we state below.

**Lemma 7** (Wang and Wang (2016)). *If a distribution $F$ has a decreasing density $f$ on the support $[0, L]$, $L \in \mathbb{R}^+$, then writing $A = L f(0)$ and $B = L f(L)$, the mean $\mu$ of $F$ satisfies

$$\mu \geq L \frac{AB + 1 - 2B}{2(A - B)}.$$*
The above lemma is obtained using a piecewise linear upper bound on $F$.

**Proof of Lemma 3.**  
(i) Suppose $y_i(x) \leq 0$ for some $x \in (0, 1)$. Then $\bar{F}_i(y_i(x)) = 1 > x$ which violates (E1). Suppose $y_i(x) - y(x) \leq 0$ for some $x \in (0, 1)$. Then $F_i(y_i(x) - y(x)) = 0$, so $\bar{F}_i(y_i(x)) = x$ by (E2), and $\bar{F}_j(y_j(x)) = 0$ for $j \neq i$ by (E1). This contradicts the assumption that densities are decreasing (hence positive) on $[0, \infty)$.

(ii) Define a function $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$f(y_1, \ldots, y_n, y) = \left( F_1(y_1 - y) + 1 - F_1(y_1), \ldots, F_n(y_n - y) + 1 - F_n(y_n), n - \sum_{i=1}^n F_i(y_i) \right)^\top.$$ 

Since $f(y_1(x), \ldots, y_n(x), y(x)) = (x, \ldots, x)^\top$ for $x \in (0, 1)$, the total derivative wrt. $x$ is

$$\frac{\partial f(y_1, \ldots, y_n, y)}{\partial (y_1, \ldots, y_n, y)^\top} \frac{d(y_1, \ldots, y_n, y)^\top}{dx} = (1, \ldots, 1)^\top.$$ 

Writing $a_i = f_i(y_i - y)$ and $b_i = f_i(y_i)$, the Jacobian matrix $\frac{\partial f(y_1, \ldots, y_n, y)}{\partial (y_1, \ldots, y_n, y)^\top}$ takes the form

$$J(y_1, \ldots, y_n, y) = \begin{pmatrix} a_1 - b_1 & 0 & \ldots & 0 & -a_1 \\ 0 & a_2 - b_2 & 0 & \ldots & -a_2 \\ \vdots & 0 & a_3 - b_3 & \ldots & \vdots \\ 0 & \vdots & \vdots & \ddots & -a_n \\ -b_1 & -b_2 & \ldots & -b_n & 0 \end{pmatrix}.$$ 

Let $c = \sum_{i=1}^n \frac{a_i b_i}{a_i - b_i}$ and $\delta_{ij}$ be the Kronecker delta: $\delta_{ij} = \mathbb{1}_{\{i=j\}}$. The inverse $J^{-1}$ is then given by

$$J_{ij}^{-1} = -\frac{1}{c} \frac{a_i}{a_i - b_i} \frac{b_j}{a_j - b_j} + \delta_{ij}, \quad \text{and} \quad J_{n+1,j}^{-1} = -\frac{1}{c} \frac{a_i}{a_i - b_i}.$$ 

$$J_{n+1,n+1}^{-1} = -\frac{1}{c}.$$ 

for $i, j = 1, \ldots, n$. Hence from $d(y_1, \ldots, y_n, y)^\top/dx = J^{-1} \cdot (1, \ldots, 1)^\top$, we obtain

$$y' = -\frac{1}{c} \left( \sum_{j=1}^n \frac{b_j}{a_j - b_j} + 1 \right) < 0 \quad \text{and} \quad y'_i = \frac{1 + a_i y'}{a_i - b_i}.$$ 

To show $y'_i < 0$, or equivalently $a_i y'_i < -1$, it remains to prove the inequality

$$a_i \left( \sum_{j=1}^n \frac{b_j}{a_j - b_j} + 1 \right) > c, \quad i = 1, \ldots, n.$$
We apply Lemma 7 to the conditional distributions of $F_i$ on intervals $[y_i - y, y_i]$, with supports shifted to the origin, i.e. we use $\hat{F}_i(z) = F_i(z + (y_i - y))/(1 - x)$ for $z \in [0, y]$. For each $i$, denote by $\mu_i$ the mean of $\hat{F}_i$ and note that the three quantities $y$, $a_i y/(1 - x)$ and $b_i y/(1 - x)$ correspond to $L$, $A$ and $B$ in Lemma 7 applied to $\hat{F}_i$. Lemma 7 yields

$$\mu_j \geq y \frac{a_j b_j y / (1 - x) + (1 - x) / y - 2b_j}{2(a_j - b_j)}, \quad j = 1, \ldots, n. \quad (15)$$

Define

$$c_0 = \sum_{j=1}^{n} \frac{1}{a_j - b_j} \quad \text{and} \quad c_1 = \sum_{j=1}^{n} \frac{b_j}{a_j - b_j},$$

then (15) sums up to

$$\sum_{j=1}^{n} \mu_j \geq y \left( \frac{cy / (1 - x) + c_0 (1 - x) / y - c_1}{2} \right).$$

Since $x \in (0, s_n)$, we have that $h(x) = \sum_{i=1}^{n} y_i - (n - 1) y > d(x) = \sum_{i=1}^{n} (\mu_i + y_i - y)$ so $\sum_{i=1}^{n} \mu_i < y$. Hence, using the AM-GM (Arithmetic Mean-Geometric Mean) inequality we obtain $1 > \sum_{i=1}^{n} \mu_i / y \geq \sqrt[c_0]{c_0} - c_1$. Rearranging $1 \geq \sqrt[c_0]{c_0} - c_1$, we have

$$\frac{(c_1 + 1)^2}{c_0} > c. \quad (16)$$

Finally, by $a_i > 1/y > b_j, \forall j \neq i$, it follows that

$$c_1 + 1 = \sum_{j \neq i} \frac{b_j}{a_j - b_j} + \frac{b_i}{a_i - b_i} + 1 < \sum_{j \neq i} \frac{a_i}{a_j - b_j} + \frac{a_i}{a_i - b_i} = a_i c_0.$$

As a consequence, we obtain $a_i (c_1 + 1) > c$ from (16), and thus $y' < 0, i = 1, \ldots, n$.

(iii) Since $(y_i - y') = (1 + b_i y')/(a_i - b_i)$, it suffices to show that $b_i y' > -1$, i.e. $b_i (c_1 + 1) < c$.

This can be seen from

$$b_i (c_1 + 1) = b_i \left( \sum_{j \neq i} \frac{b_j}{a_j - b_j} + \frac{b_i}{a_i - b_i} + 1 \right) = \sum_{j \neq i} \frac{b_j b_i}{a_j - b_j} + \frac{a_i b_j}{a_i - b_i} < \sum_{j=1}^{n} \frac{a_j b_j}{a_j - b_j} = c.$$

(iv) A straightforward calculation yields

$$h' = \sum_{i=1}^{n} y'_i - (n - 1) y' = c_0 + y' \left( \sum_{i=1}^{n} \frac{a_i}{a_i - b_i} - n + 1 \right) = c_0 + y' (c_1 + 1) = c_0 - \frac{(c_1 + 1)^2}{c},$$

so $h' < 0$ by inequality (16).
4 Applications to Quantitative Risk Management

4.1 Bounds on convex and coherent risk measures

Convex and coherent risk measures (Artzner et al., 1999; Föllmer and Schied, 2002) are powerful mathematical tools used to calculate capital requirement for a financial institution. The consistency between convex order and convex risk measures can be found in Bäuerle and Müller (2006). As a consequence of Theorem 6, we find the lower bound for convex and coherent risk measures over the admissible risk class. For general definitions of coherent and convex risk measures, we refer to Föllmer and Schied (2004, Chapter 4).

Recall the definition of \( R_{sn} \) in (5) on page 51.

**Corollary 8 (Bounds on convex risk measures).** For any law-invariant and convex risk measure \( \rho \), if (C) holds, then

\[
\inf_{S \in \mathcal{Z}_{n}(F_1, \ldots, F_n)} \rho(S) \geq \rho(R_{sn}),
\]

and the above inequality is an equality if (D) holds.

One of the most commonly used coherent risk measures is the Expected Shortfall (ES), also known as Tail Value-at-Risk (TVaR) or Conditional VaR (CVaR) in the actuarial literature. The ES of \( S \) at level \( p \) is defined as

\[
\text{ES}_p(S) = \frac{1}{1-p} \int_{p}^{1} \text{VaR}_p(S) \, d\alpha, \quad p \in [0, 1),
\]

where \( \text{VaR}_p \) is another popular risk measure, the Value-at-Risk (VaR) at level \( p \):

\[
\text{VaR}_p(X) = \inf \{ x : P(X \leq x) \geq p \}, \quad p \in (0, 1).
\]

Here we give a lower bound on ES over the admissible risk class. For the notation, see page 49.

**Corollary 9 (Bounds on ES).** For \( p \in [0, 1) \), if (C) holds, then

\[
\inf_{S \in \mathcal{Z}_{n}(F_1, \ldots, F_n)} \text{ES}_p(S) \geq \begin{cases} 
\frac{1}{1-p} \left( \text{E}[S] - pd(s_n) \right), & p \leq 1 - s_n, \\
\frac{1}{1-p} \int_{0}^{1-p} h(x) \, dx, & p > 1 - s_n,
\end{cases}
\]

and the above inequality is an equality if (D) holds.

4.2 Bounds on expectations of convex functions

A convex (concave) expectation of a random variable \( X \) is defined as \( \text{E}[f(X)] \) where \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a convex (concave) function. By the definition of convex order, we have a straightforward
corollary about the lower bound on a convex expectation (or upper bound on a concave expectation) over the admissible risk class $\xi(F_1, \ldots, F_n)$,

$$E[f(S)] = E[f(X_1 + X_2 + \cdots + X_n)].$$

(20)

Recall that when $f$ is convex, the upper bound can be computed explicitly with the comonotonic dependence structure.

**Corollary 10** (Bounds on convex expectations). For a convex function $f$, if (C) holds, then

$$\inf_{S \in \xi(F_1, \ldots, F_n)} E[f(S)] \geq \int_0^{s_n} f(h(x)) \, dx + (1 - s_n) f(d(s_n)),
$$

(21)

and the above inequality is an equality if (D) holds.

Note that when $s_n = 0$, (21) degenerates to Jensen’s inequality. However, Lemma 5 (b) implies $R_0 \preceq_{cx} R_n$, so (21) always gives a better bound than Jensen’s inequality when $s_n > 0$.

Another inequality about the lower bound on convex expectation is given in Cheung and Lo (2013), summarized as follows:

**Proposition 11** (Cheung and Lo (2013)). Let $X_1, \ldots, X_n$ be non-negative random variables, $S = X_1 + \cdots + X_n$, and $f$ be a convex function such that $E[f(S)]$ exists.

(i) We have

$$E[f(S)] \geq L_1 := \sum_{i=1}^n E[f(X_i)] - (n - 1)f(0).$$

(ii) If $f$ is strictly convex, then equality holds in (22) if and only if $X_1, \ldots, X_n$ are mutually exclusive random variables.

Inequality (22) provides an easily calculated lower bound on $E[f(S)]$, $S \in \xi(F_1, \ldots, F_n)$. The bound (22) is sharp only if $F_1, \ldots, F_n$ are compatible with mutual exclusivity. When mutual exclusivity is not compatible, it is not clear which bound in (21)-(22) dominates the other. For instance, for bounded distributions with $h(0) \leq d(0)$, (21) becomes Jensen’s inequality which is not strictly comparable to (22). In the numerical illustration in Section 5, we will compare the two bounds.

**Remark 4.** Cheung and Lo (2013) provide two proofs of (22) based on the Breeden-Litzenberger formula and mutual exclusivity. We remark that it can also be obtained from Karamata’s inequality (Karamata, 1932) that

$$f(S) + (n - 1)f(0) \geq \sum_{i=1}^n f(X_i),$$

which is a stronger statement than (22).
4.3 Bounds on Value-at-Risk

The search for bounds on Value-at-Risk under dependence uncertainty is a topic of considerable interest in theory as well as practice; see Embrechts et al. (2014) for a review. Due the fact that VaR does not respect convex order, optimization problems with respect to VaR have always been challenging. A connection between the bounds on VaR and the lower convex bound over an admissible risk class has been given in Bernard et al. (2014, 2015a).

We suppose that $F_1, \ldots, F_n$ are continuous dfs with positive density. For each $i = 1, \ldots, n$, we let $F_{i,p}$ for $p \in (0, 1)$ be the conditional distribution of $F_i$ on $[F_i^{-1}(p), \infty)$, and let $F_i^p$ for $p \in (0, 1)$ be the conditional distribution of $F_i$ on $[0, F_i^{-1}(p))$. Theorem 4.6 of Bernard et al. (2014) gives that

$$\sup_{S \in \mathcal{Z}(F_1, \ldots, F_n)} \text{VaR}_p(S) = \sup \{ \text{ess inf}(S) : S \in \mathcal{E}(F_1, \ldots, F_n) \},$$

and

$$\inf_{S \in \mathcal{Z}(F_1, \ldots, F_n)} \text{VaR}_p(S) = \inf \{ \text{ess sup}(S) : S \in \mathcal{E}(F_1^p, \ldots, F_n^p) \},$$

where ess inf($X$) and ess sup($X$) are the essential infimum and the essential supremum of the support of a random variable $X$. With this connection, the proofs of the following two corollaries are trivial, hence we omit them here.

In the following corollary, we use $(C)_p$ (respectively, $(D)_p$) if $F_1, \ldots, F_n$ satisfy $(C)$ (respectively, $(D)$). The function $h_p$ and the quantity $s_{n,p}$ are the corresponding $h$ and $s_n$ defined for the distributions $F_1, \ldots, F_n$.

**Corollary 12** (Upper bound on VaR). For $p \in (0, 1)$, if $(C)_p$ holds, then

$$\sup_{S \in \mathcal{Z}(F_1, \ldots, F_n)} \text{VaR}_p(S) \leq h_p(s_{n,p}),$$

and the above inequality is an equality if $(D)_p$ holds.

Note that (F) implies (C) and (D). For $(C)_p$ and $(D)_p$ to hold, it suffices to require that each $F_1, \ldots, F_n$ has a decreasing density beyond its $p$-quantile. As $p$ is typically close to 1, this assumption is satisfied by all practical examples. Because of its practical relevance, we state it as a separate corollary; the homogeneous version of this result is given in Wang et al. (2013).

**Corollary 13** (Upper bound on VaR for tail-decreasing densities). For $p \in (0, 1)$, suppose that each $F_1, \ldots, F_n$ has a decreasing density beyond its $p$-quantile, then

$$\sup_{S \in \mathcal{Z}(F_1, \ldots, F_n)} \text{VaR}_p(S) = h_p(s_{n,p}).$$

Similar result holds for the lower bound on VaR. In the following corollary, we use $(C)^p$ (respectively, $(D)^p$) if $F_1^p, \ldots, F_n^p$ satisfy $(C)$ (respectively, $(D)$). Let $\mu_i^p$ be the mean of $F_i^p$, $i = 1, \ldots, n$. 

Corollary 14 (Lower bound on VaR). For \( p \in (0, 1) \), if (C)\(^p\) holds, then

\[
\inf_{S \in \Xi_n (F_1, \ldots, F_n)} \text{VaR}_p (S) \geq \max \left\{ \max_{i=1, \ldots, n} \left\{ F_i^{-1} (p) + \sum_{j \neq i} F_j^{-1} (0) \right\}, \sum_{i=1}^n \mu_i^p \right\},
\]

(25)

and the above inequality is an equality if (D)\(^p\) holds.

The inequality (25) is straightforward, whereas the sharpness under (C)\(^p\) and (D)\(^p\) is not.

5 Numerical Illustration

5.1 Results and comparison

In this section we explain how to calculate the bounds obtained in the previous sections, and present the results from a numerical case study considering 10 different sets of marginal distributions. We compare the results from the method based on Corollaries 9 and 10 with their alternatives: the Rearrangement Algorithm (RA) (Puccetti and Rüschendorf, 2012; Embrechts et al., 2013), the bound given in Bernard et al. (2014) based on an approximation with homogeneous risks, and the bound (22) given in Cheung and Lo (2013). Note that the sharpness of the latter two bounds is either not justified or fails to hold for most of the heterogeneous models of relevance. The RA values can be used as a good approximation to the exact values of the bounds considered.

In Table 1, the bounds for the sets of margins considered in Bernard et al. (2014) are listed, and in Table 2, five further cases are presented. Three quantities are calculated: the variance \( \text{Var}(S) \); the European call option price (or excess of loss) \( \text{E}[f(S)] = \text{E}[(S - K)^+] \); and ES of \( S \) at level 95%. It was mentioned in Cheung and Lo (2013) that their bound on ES involves a complicated optimization procedure for heterogeneous models which is unavailable in their paper, hence for ES we only compare Corollary 9 with the RA and the approximation in Bernard et al. (2014).

We observe in Tables 1 and 2, that the variance bounds from Cheung and Lo (2013) are quite poor (in case a negative bound was obtained, it was replaced by the trivial bound \( \text{Var}(S) \geq 0 \)). The bounds on option price are closer, but still far from sharp. In contrast, the bounds from Corollaries 9 and 10 and the RA are in close agreement, which suggests that the JM condition (D) holds for the considered cases, and the bounds we obtain are sharp. In Table 2, more dissimilar distributions (e.g. from different families) are considered, for which the homogeneous approximation bounds in Bernard et al. (2014) are significantly worse than the ones obtained by the method presented in this paper. The greatest deviation can be observed in the case Pareto-LogN-Gamma (the last column of Table 2). For this case, in the left panel of Figure 1 the plots for the functions \( h(x) \), \( d(x) \) (used for sharp bounds in Theorem 6) are given, along with
\(H(x/n)\) and \(D(x/n)\) in Bernard et al. (2014), which are used for the homogeneous approximation bounds. Notice that the distributions of the constructed lower convex bounds using the two methods differ significantly, and this leads to the discrepancies in Table 2. In the right panel the weights \(\bar{F}_i(y_i(x))\) from the solution to the functional equations (E1)-(E2) are plotted. For comparison, the implied values from the homogeneous approximation are shown in gray.

**Figure 1:** Left panel: plots of functions \(h(x)\), \(d(x)\) (used for sharp bounds) and functions \(H(x/n)\), \(D(x/n)\) from Bernard et al. (2014). Right panel: \(\bar{F}_i(y_i(x))\) from the solution to the functional equations (E1)-(E2), in gray - implied values from the homogeneous approximation.

In the following section, we provide details of the implementation.

### 5.2 Solving the functional equations

In order to solve the functional equations (E1)-(E2), we first obtain a system of ordinary differential equations (ODEs) and an initial condition (as in the Implicit Function Theorem). Differentiating (E1)-(E2) wrt. \(x\) yields a system of ODEs

\[
(E1') \quad - \sum_{i=1}^{n} f_i(y_i(x)) y_i'(x) = 1,
\]

\[
(E2') \quad \left( f_i(y_i(x) - y(x)) - f_i(y_i(x)) \right) y_i'(x) - f_i(y_i(x) - y(x)) y'(x) = 1 \quad \text{for each } i = 1, \ldots, n,
\]

where \(f_i\) is the density corresponding to \(F_i\). For given \(y_i(x)\) and \(y(x)\), this is a system of \(n + 1\) linear equations of the form \(A \cdot (y_1', \ldots, y_n', y')^\top = b\) (easy to solve, see the proof of Lemma 3). Thus, if an initial condition is available, we can solve the ODEs (E1')-(E2') using an Euler-type scheme.

We find an initial condition \(y_i(\epsilon), y(\epsilon)\) at \(x = \epsilon > 0\) (small), using an approximate method:

1. Solve \(\sum_{i=1}^{n} \bar{F}_i(y_i^{(M)}) = \epsilon\) for \(y^{(M)} \in \mathbb{R}\),
2. Let \(y_i^{(m)} = F_i^{-1}(\epsilon - \bar{F}_i(y_i^{(M)}))\) (so that \(F_i(y_i^{(m)}) + \bar{F}_i(y_i^{(M)}) = \epsilon\),
3. Let \( y(\epsilon) = y^{(M)} - \min_{1 \leq i \leq n} y_i^{(m)} \) and \( y_i(\epsilon) = y_i^{(m)} + y(\epsilon) \).

Note that we effectively start by satisfying \((E1)\) with identical \( y_i \equiv y^{(M)} \) and then shifting the \( y_i \) to the right, in order to satisfy \((E2)\). Since the density in the right tail is typically smaller than in the left tail, this shift does not significantly reduce the remaining mass in the right tail. Moreover, for margins with the largest \( \bar{F}_i(y^{(M)}) \), the \( y_i \) are likely to be shifted the least. This method provides very accurate initial conditions in the considered cases (error in \((E1)\) less than \( \epsilon \cdot 10^{-4} \) for \( \epsilon = \Delta x/2 = 10^{-6}/2 \)).

Using the initial condition at \( x = \epsilon \) obtained in this manner, we solve the ODEs for \( x \in [\epsilon, 1] \). This yields the solution \( y_i(x), y(x) \) on a non-uniform grid of \( x \) values (we use the function \texttt{ode113} in \textsc{Matlab}, which is a variable step-size algorithm). Finally, we use linear interpolation to compute \( y_i(x) \) and \( y(x) \) for the desired grid-points \( x \in \{ (k - \frac{1}{2})\Delta x : k = 1, \ldots, 1/\Delta x \} \). We chose the step size \( \Delta x = 10^{-6} \), as is typical in the literature. In all considered cases the interpolated results satisfy the equations (E1) and (E2) with an absolute error of order \( 10^{-8} \) or less (for \( x \in (0, s_n) \)).

In Figure 2, the solutions (quantiles) \( y_i(x) - y(x) \) and \( y_i(x) \) for \( x \) up to \( s_n = 0.61 \) are plotted over the densities of three different Log-Normal margins (the second case in Table 2). We notice that more probability in the right tail of \( \text{LogN}(1, 1) \) is combined with the left tails of the other two margins, i.e. \( \bar{F}_2(y_2(x)) \) is the largest term on the left-hand side of \((E1)\). The remaining support intervals (shown in white) are of length \( y(s_n) \), and \( h(s_n) = d(s_n) \) holds.

![Figure 2: Quantiles \( y_i(x) - y(x) \) and \( y_i(x) \) for \( x = 1\%, 2.5\%, 5\%, 10\%, 25\%, 61\% \).](image)

Finally, to check that the discretization is fine enough, we compared the numerical means of the constructed convexly minimal elements \( R_{s_n} \) with the theoretical means. In all cases the relative error wrt. the theoretical means was of the order \( 10^{-6} \).
5.3 Computation times

The computations were performed on a Lenovo X1 laptop with Intel Core i7 2GHz × 4 processor and 8GB RAM. Computation times are summarized in Table 3. The range of computation times between the ten considered cases are as follows. RA: 28 – 81s, homogeneous approximation bounds from Bernard et al. (2014): 373 – 667s, sharp bounds from Corollaries 10 and 9: 5 – 10s. The considerably longer computation times for the homogeneous approximation bounds are due to the fact that we need to compute the inverse of $F = \frac{1}{n} \sum F_i$, which is computationally expensive, and efficient algorithms are available only for special cases, see e.g. Castellacci (2012). To test the scalability of the algorithms, we also computed the bounds for higher values of $n$ for margins ($n/2$ different Pareto and $n/2$ different LogN). For $n = 20$ the computation time for RA was 1 hour (note that the time is random due to a random initial rearrangement), for homogeneous approximation 13 minutes and for the sharp bounds 13s. Furthermore, it was possible to compute the sharp bounds for up to $n = 100$ different margins in less than 1 minute. At this high number of margins, $s_n$ is very small and the optimal sum $R_{s_n}$ is almost identical to its mean $R_0$.

Table 1: RA results vs theoretical bounds

<table>
<thead>
<tr>
<th></th>
<th>Pareto$(1, \alpha_i), , n = 3$</th>
<th>$X_i \sim \text{Log-Normal}(\frac{3}{10}, 1)$</th>
<th>$n$</th>
<th>Variances</th>
<th>Option price with strike $K = \sum_{i=1}^{n} E[X_i]$</th>
<th>ES at level 0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(\alpha_1, \alpha_2, \alpha_3)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3, 4, 5)</td>
<td>(3.5, 4, 4.5)</td>
<td>3</td>
<td>5</td>
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</tr>
<tr>
<td></td>
<td>RA</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.6057</td>
<td>0.3352</td>
<td>9.0990</td>
<td>10.3695</td>
<td>15.4077</td>
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<tr>
<td>Bernard et al. (2014)</td>
<td>0.6063</td>
<td>0.3360</td>
<td>9.1075</td>
<td>10.3803</td>
<td>15.3885</td>
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<tr>
<td>Cheung and Lo (2013)</td>
<td>0.3114</td>
<td>0.0537</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Corollary 10</td>
<td>0.6063</td>
<td>0.3359</td>
<td>9.1085</td>
<td>10.3938</td>
<td>15.5071</td>
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<td>Option price with strike $K = \sum_{i=1}^{n} E[X_i]$</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>RA</td>
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<td></td>
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</tr>
<tr>
<td></td>
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<td>0.1413</td>
<td>0.7701</td>
<td>0.6419</td>
<td>0.4668</td>
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<td>0.1413</td>
<td>0.7700</td>
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<td>0.4047</td>
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<tr>
<td>Corollary 10</td>
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<td>0.1413</td>
<td>0.7701</td>
<td>0.6420</td>
<td>0.4670</td>
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<td></td>
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<tr>
<td></td>
<td>6.4259</td>
<td>5.8758</td>
<td>16.0777</td>
<td>21.7849</td>
<td>39.0253</td>
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<td>5.8760</td>
<td>16.0778</td>
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<td>38.8893</td>
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Table 2: RA results vs theoretical bounds

<table>
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<tr>
<th></th>
<th>$X_i \sim \text{Pareto}(1, \alpha_i)$</th>
<th>$X_1 \sim \text{LogN}(0, \frac{1}{2})$</th>
<th>$X_1 \sim \text{LogN}(0, 1)$</th>
<th>$X_1 \sim \text{Gamma}(2, \frac{1}{2})$</th>
<th>$X_1 \sim \text{Pareto}(1, 3)$</th>
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<td>$n = 10$</td>
<td>$X_2 \sim \text{LogN}(1, 1)$</td>
<td>$X_2 \sim \text{LogN}(1, 1)$</td>
<td>$X_2 \sim \text{Gamma}(3, \frac{1}{2})$</td>
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</tr>
<tr>
<td></td>
<td>$\alpha = 3, \ldots, 12$</td>
<td>$X_3 \sim \text{LogN}(-1, \frac{3}{2})$</td>
<td>$X_3 \sim \text{Pareto}(1, 3)$</td>
<td>$X_3 \sim \text{Gamma}(4, \frac{1}{2})$</td>
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<td>Variance</td>
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<td>0.3686</td>
<td>33.9848</td>
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<td></td>
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<tr>
<td>RA</td>
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<td>1.5156</td>
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<tr>
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<td>1.5156</td>
<td>1.3957</td>
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<td>ES at level 0.95</td>
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Table 3: Computation times, in seconds

<table>
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<th>Case</th>
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<th>Bernard et al. (2014)</th>
<th>Corollaries 9 and 10</th>
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<td>5</td>
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<td>Table 1, case 2</td>
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<td>Table 1, case 5</td>
<td>81</td>
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<td>Table 2, case 5</td>
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</table>

6 Conclusion

We give a general lower bound on the aggregate risk with respect to convex order for heterogeneous marginal distributions. The bound is shown to be sharp when the marginal distributions all have decreasing densities. The new result partially answers an open question that has existed in the theory of dependence modeling for a long time. Although the proposed lower convex bound is generally implicit and involves solving a non-trivial functional equation, it helps to understand the safest dependence structure with respect to convex order. As opposed to comonotonicity, which is often treated as the most dangerous dependence structure, the safest dependence structure can be interpreted as a combination of joint mixability and mutual exclusivity. This is indeed not surprising if one realizes that joint mixability and mutual exclusivity both give the safest dependence structure when they are compatible with the marginal distributions. Our results directly lead to bounds on quantities including convex and coherent risk measures, the expectation of convex functions and the Value-at-Risk of the aggregate risk. A numerical procedure is provided to identify the distribution representing the lower convex bound and to compute corresponding quantities of interest.

We remark that there are still quite a few open questions on this new method, concerning conditions (C) and (D). Numerical evidence seems to suggest that the main theorem holds for much more general classes of distributions than those with decreasing densities. A theoretical proof of such statements is still beyond our knowledge, and is expected to be very challenging. The new results obtained in this paper are closely associated with the development of theory of joint mixability, on which many open questions are left unanswered at this moment.
Acknowledgments

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Paul Embrechts, Edgars Jakobsons.

**Dependence uncertainty for aggregate risk: examples and simple bounds.**

Dependence uncertainty for aggregate risk: examples and simple bounds

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Abstract

Over the recent years, numerous results have been derived in order to assess the properties of regulatory risk measures (in particular, VaR and ES) under dependence uncertainty. In this paper, we complement this mainly methodological research by providing several numerical examples for both homogeneous as well as inhomogeneous portfolios. In particular, we investigate under which circumstances the so-called worst-case VaR can be well approximated by the worst-case (i.e. comonotonic) ES. We also study best-case values and simple lower bounds.

Keywords: Value-at-Risk, Expected Shortfall, risk aggregation, dependence uncertainty, inhomogeneous portfolio, Rearrangement Algorithm.

1 Introduction

Recent regulatory discussions in the realm of banking and insurance have brought the following quantitative aspects very much to the forefront:

i) The choice of risk measure for the calculation of regulatory capital; examples include Value-at-Risk (VaR) and Expected Shortfall (ES).

ii) The properties of statistical estimators of such risk measures, especially when based on extreme tail observations.

iii) The issue of Model Uncertainty (MU). The latter can be interpreted in many ways: statistical, numerical, functional, . . .
In this paper we will mainly concentrate on a combination of i) and iii), and discuss the (functional) model uncertainty of risk measures VaR and ES for linear portfolios in both the homogeneous as well as inhomogeneous case. The interpretation of MU concerns portfolios with known marginal distributions but unknown interdependence. The latter is often referred to as Dependence Uncertainty (DU). It plays a crucial role in various examples throughout the banking and insurance literature. Going forward, several relevant examples will be given. One general purpose reference is Embrechts et al. (2014).

The considered setting is a static one-period model, however, the results are also applicable to the dynamic case, using marginal distributions conditioned on the available information, e.g. conditional mean and volatility. While conditioning on the volatility may lead to light-tailed marginal distributions, it is not always the case, as shown in McNeil and Frey (2000), where methods from extreme value theory are applied to GARCH-filtered (and still heavy-tailed) one-dimensional time series in order to obtain point estimates of VaR and ES. Moreover, Christoffersen and Diebold (2000) argue that the volatility of financial time series is forecastable only up to 10 days ahead (relevant for market risk) but not for longer time scales (such as 1 year, relevant for insurance). The effect of tail-heaviness of the marginal distributions on the portfolio risk will therefore be analyzed. Furthermore, the conditional dependence between the margins may change over time, as illustrated by Dias and Embrechts (2009), where a parametric copula approach is used to model and detect changes in the dependence structure. This demonstrates some of the difficulties in estimating the dependence. The mainly numerical results presented in our paper aim at understanding better the estimation of the best and worst possible risk measure values under DU. The paper only offers a first step on the way to this goal, and numerous alternative approaches as well as results are available or can be obtained. We very much hope that this paper incites other researchers to look at these and related problems of Quantitative Risk Management.

2 Homogeneous portfolios

We consider random variables $X_i \sim F_i$, $i = 1, \ldots, d$ and $S_d = X_1 + \cdots + X_d$. For the purpose of this paper we assume the marginal distribution functions (dfs) $F_i$ to be known. If $F_i = F$, $i = 1, \ldots, d$, we refer to the homogeneous case, otherwise we refer to the inhomogeneous case. For most of the paper we will concentrate on the homogeneous case, though for many of the results information on the inhomogeneous case can be obtained; see Section 3 for a brief discussion. The joint distribution of $(X_1, \ldots, X_d)$ is not specified, and thus the aggregate value-at-risk $\text{VaR}_\alpha(S_d)$ and expected shortfall $\text{ES}_\alpha(S_d)$ are not uniquely determined. In the following we consider the range of possible values these aggregate risk measures can take, under fixed marginal distributions, but unspecified joint model. This framework in risk management is
referred to as dependence uncertainty; see Embrechts et al. (2015). In this section we focus on the homogeneous case and assume throughout that the support of $F$ is bounded below. Define the generalized inverse

$$ F^{-1}(p) = \inf\{x : F(x) \geq p\}, \quad p \in (0, 1], $$

and $F^{-1}(0) = \inf\{x : F(x) > 0\}$ (the left endpoint of the support); for properties of generalized inverses, see Embrechts and Hofert (2013). Since ES and VaR are translation equivariant, for simplicity also assume $F^{-1}(0) = 0$ (via translation). To give the main results in the literature on DU bounds on risk measures in the following sections, a definition from Wang and Wang (2011) will be useful.

**Definition 1.** For $d \geq 1$, a distribution function $F$ is called $d$-completely mixable ($d$-CM) if there exist rvs $X_1, \ldots, X_d \sim F$ and a constant $k$ such that $X_1 + \cdots + X_d = dk$ a.s.

Examples of distributions that are completely mixable for $d \geq 2$ include uniform, Gaussian, Cauchy and other unimodal symmetric continuous distributions; for higher values of $d$ also distributions with monotone or concave densities on a bounded support; see Puccetti et al. (2012) for further examples.

Some notation from Bernard et al. (2014) will also be needed. With respect to a df $G$ (to be specified), we introduce functions $H(c)$ and $D(c)$, $c \in [0, 1]$ and a constant $c_d$:

$$ H(c) = (d - 1)G^{-1}((d - 1)c/d) + G^{-1}(1 - c/d), $$

$$ D(c) = \frac{1}{1 - c} \int_c^1 H(t)dt = \frac{d}{1 - c} \int_{(d-1)c/d}^{1-c/d} G^{-1}(t)dt, $$

$$ c_d = \inf\{c \in [0, 1] : H(c) \leq D(c)\}, $$

where for $c = 1$ we set $D(1) := D(1-) = H(1) = dG^{-1}(1 - 1/d)$. Also, introduce two conditions

(A) $H$ is non-increasing on $[0, c_d]$, 

(B) The conditional distribution of $G$ on $[(d - 1)c_d/d, 1 - c_d/d]$ is $d$-completely mixable.

These conditions will imply validity and sharpness of some of the bounds stated in the following sections. A special case in which these conditions are satisfied is given in the following lemma.

**Lemma 1** (Bernard et al. (2014)). If the df $G$ admits a non-increasing density on its support, then conditions (A) and (B) hold.

Furthermore, Bernard et al. (2014) motivate numerically that these conditions are satisfied also in some other cases, using examples with Lognormal and Gamma dfs.
2.1 Upper bound on VaR

For a random variable $X \sim F_X$ representing a loss, VaR at confidence level $\alpha \in (0, 1)$ is defined as the $\alpha$-quantile,

$$\text{VaR}_\alpha(X) = F_X^{-1}(\alpha).$$

In turn, the upper bound on $\text{VaR}_\alpha(S_d)$ over all joint models is defined as

$$\overline{\text{VaR}}_\alpha(S_d) := \sup \{ \text{VaR}_\alpha(X_1 + \cdots + X_d) : X_i \sim F_i, i = 1, \ldots, d \}.$$

In the homogeneous case $F_i = F$, $i = 1, \ldots, d$, the upper bound $\overline{\text{VaR}}_\alpha(S_d)$ can be obtained by solving the integral inequality for $c_d$ in (1) with respect to the conditional distribution in the tail (defined below).

**Proposition 2** (Embrechts and Puccetti (2006); Puccetti and Rüschendorf (2013); Wang et al. (2013)). Define $H$, $D$ and $c_d$ in (1) with respect to $F_\alpha$, the conditional distribution of $F$ on $[F^{-1}(\alpha), \infty)$. If $F_\alpha$ admits a non-increasing density on its support, then

$$\overline{\text{VaR}}_\alpha(S_d) = D(c_d).$$

**Remark 1.** A non-increasing density above the $\alpha$-quantile is a natural assumption for high values of $\alpha$, which holds for essentially all distributions used in practice. The proof of Proposition 2 uses Lemma 1 to verify conditions (A) and (B), which are sufficient for the bound in Proposition 2 to be sharp.

Before giving the next proposition, we recall the definition of expected shortfall,

$$\text{ES}_\alpha(X) = \frac{1}{1 - \alpha} \int_\alpha^1 \text{VaR}_q(X) \, dq.$$

Let $\text{VaR}_\alpha^+(S_d)$ and $\text{ES}_\alpha^+(S_d)$ denote $\text{VaR}_\alpha(S_d)$ and $\text{ES}_\alpha(S_d)$ respectively, when the $X_i$ are comonotonic. It is well-known that the *comonotonic* dependence structure gives the worst-case expected shortfall; see Dhaene et al. (2002) and p. 251 in McNeil et al. (2005).

$$\overline{\text{ES}}_\alpha(S_d) := \sup \{ \text{ES}_\alpha(X_1 + \cdots + X_d) : X_i \sim F_i, i = 1, \ldots, d \} = \sum_{i=1}^d \text{ES}_\alpha(X_i) =: \text{ES}_\alpha^+(S_d).$$

Moreover, as $d$ increases, $\overline{\text{VaR}}_\alpha(S_d)$ is asymptotically equivalent to $\text{ES}_\alpha^+(S_d)$. A first result in this direction is:

**Proposition 3** (Puccetti and Rüschendorf (2014)). Suppose $(X_d)_{d \geq 1}$ is a sequence of rvs identically distributed as $F$, where $F$ is integrable and has a decreasing density on $[F^{-1}(\alpha), \infty)$. Then

$$\lim_{d \to \infty} \frac{\overline{\text{VaR}}_\alpha(X_1 + \cdots + X_d)}{\text{ES}_\alpha^+(X_1 + \cdots + X_d)} = 1.$$
Remark 2. Proposition 3 holds under much more general conditions, also in the heterogeneous case; see Embrechts et al. (2015). Results of this type are relevant for regulatory practice for both the banking world (the so-called Basel framework) as well as insurance (Solvency 2); see Embrechts et al. (2015).

In Table 1, the smallest number of margins \( d \) is given for which \( \text{VaR}_\alpha(S_d) \) is within 10\% of \( \text{ES}_\alpha^+(S_d) \), i.e. after which the asymptotically equivalent sequence gives a reasonable approximation. Notice that in all but the most heavy-tailed cases, \( d \approx 10 \) was sufficient. Hence, if the number of margins is greater than that, we may use the easily calculated \( \text{ES}_\alpha^+(S_d) \) as a reasonable estimate of the conservative \( \text{VaR}_\alpha(S_d) \).

Table 1: Thresholds for the number of margins \( d \) for which \( \text{VaR}_\alpha(S_d) \) is within 10\% of \( \text{ES}_\alpha^+(S_d) \). The levels \( \alpha \in \{0.95, 0.99, 0.999\} \) are listed for comparison. The parameter \( \sigma \) for LogN(0, \( \sigma^2 \)) is chosen to match the ratio \( \text{VaR}_\alpha(X_1)/\text{ES}_\alpha(X_1) \) with that of Pareto(\( a \)), i.e. \( \Pr(X_1 > x) = (x + 1)^{-a} \), \( x \geq 0 \).

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( a )</th>
<th>10.0</th>
<th>5.0</th>
<th>3.0</th>
<th>2.5</th>
<th>2.0</th>
<th>1.7</th>
<th>1.6</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pareto(( a ))</td>
<td>( \text{VaR}_{0.95} )</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>12</td>
<td>17</td>
<td>29</td>
</tr>
<tr>
<td>LogN(0, ( \sigma^2 ))</td>
<td>( \text{VaR}_{0.95} )</td>
<td>0.75</td>
<td>0.90</td>
<td>1.13</td>
<td>1.26</td>
<td>1.49</td>
<td>1.71</td>
<td>1.81</td>
<td>1.94</td>
</tr>
<tr>
<td>Pareto(( a ))</td>
<td>( \text{VaR}_{0.99} )</td>
<td>10.0</td>
<td>5.0</td>
<td>3.0</td>
<td>2.5</td>
<td>2.0</td>
<td>1.7</td>
<td>1.6</td>
<td>1.5</td>
</tr>
<tr>
<td>LogN(0, ( \sigma^2 ))</td>
<td>( \text{VaR}_{0.99} )</td>
<td>0.70</td>
<td>0.89</td>
<td>1.20</td>
<td>1.37</td>
<td>1.66</td>
<td>1.95</td>
<td>2.08</td>
<td>2.23</td>
</tr>
<tr>
<td>Pareto(( a ))</td>
<td>( \text{VaR}_{0.999} )</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>11</td>
<td>16</td>
<td>27</td>
<td></td>
</tr>
<tr>
<td>LogN(0, ( \sigma^2 ))</td>
<td>( \text{VaR}_{0.999} )</td>
<td>0.67</td>
<td>0.92</td>
<td>1.33</td>
<td>1.56</td>
<td>1.93</td>
<td>2.28</td>
<td>2.44</td>
<td>2.63</td>
</tr>
</tbody>
</table>

The convergence rate in Proposition 3 is also known.

**Proposition 4** (Embrechts et al. (2015)). If \( \mathbb{E}[|X_1 - \mathbb{E}[X_1]|^k] \) is finite for some \( k > 1 \) and \( \text{ES}_\alpha(X_1) > 0 \), then, as \( d \to \infty \),

\[
\frac{\text{VaR}_\alpha(S_d)}{\text{ES}_\alpha^+(S_d)} = 1 - O(d^{-1+1/k}).
\]

So in particular, if all moments are finite, then the convergence rate is \( O(d^{-1}) \), and if the distribution has a regularly varying tail (see Section 7.3 in McNeil et al. (2005)) with index \(-\rho\), for example \( X_1 \sim \text{Pareto}(\rho) \), then the convergence is slower, since \( k < \rho < \infty \). In Figure 1 the differences \( 1 - \frac{\text{VaR}_\alpha(S_d)}{\text{ES}_\alpha^+(S_d)} \) as \( d \) increases are plotted on a logarithmic scale for different
Figure 1: Relative differences $1 - \frac{\text{VaR}_\alpha(S_d)}{\text{ES}_\alpha^+(S_d)}$ for $\alpha = 0.99$ on the vertical axis versus $d$ on the horizontal, on a log-log scale. Below the dotted line the relative difference is smaller than 10%. The left panel contains Pareto($\alpha$) distributions, $\alpha = 1.5, 2, 3, 5, 10$ from top down. The bold line shows the theoretical convergence rate $O(d^{-1/3})$ for Pareto(1.5) according to Proposition 4. In the right panel the Lognormal LogN(0,$\sigma^2$) case is plotted, $\sigma = 2.23, 1.66, 1.20, 0.89, 0.70$ from top down, chosen to match the ratio $\text{VaR}_\alpha(X_1)/\text{ES}_\alpha(X_1)$ with that of Pareto. The bold line shows the theoretical convergence rate $O(d^{-1})$ for distributions with all moments finite.

Pareto and Lognormal distributions. We observe that, although initially the rate of convergence is slower, for large $d$ it seems faster than the theoretical one. Moreover, for small $d$ the rates are not very different between Pareto and Lognormal. Tail-heaviness is often measured using the notion of regular variation, and also Proposition 4 determines the convergence rate according to the highest existing moment. However, regular variation concerns quantiles asymptotically approaching 1, and Proposition 4 holds for $d$ tending to infinity. In order to analyze $\overline{\text{VaR}}_\alpha(S_d)$ at a fixed level, say, $\alpha = 0.99$, and for small $d$, we need a measure of tail-heaviness that captures the behavior for moderate $\alpha$ and $d$. We refrain from using skewness, kurtosis and higher moments, as they may not exist for power tails. Instead, we will define a different measure of tail-heaviness in the following section.

2.1.1 The normalized mean-median ratio

For $X \sim F$ define the normalized mean-median ratio as

$$M_\alpha(F) := \frac{\text{ES}_\alpha(X) - \text{VaR}_\alpha(X)}{\text{MS}_\alpha(X) - \text{VaR}_\alpha(X)},$$

where MS is the median shortfall, $\text{MS}_{1-p}(X) := \text{VaR}_{1-p/2}(X)$; see Kou et al. (2013). $M_\alpha(F)$ is a measure of tail-heaviness defined for regularly varying, sub-exponential, as well as exponential tails, and only requires the first moment to exist. $M_\alpha(F)$ is invariant under scaling and translation, since the risk measures used in the definition are positively homogeneous and translation equivariant. In Figure 2 $M_\alpha$ is plotted for Gamma, Lognormal and Pareto distributions,
as a function of the shape parameter. For all three families $M_\alpha$ diverges to infinity as the tail becomes heavier, and converges to a finite value for light tails.

For the Gamma distribution $\Gamma(k,1)$ the values are computed numerically; to reach values in the typical range for Lognormal and Pareto, a very small parameter $k$ is required (e.g. $k = 0.0027$ for $M_\alpha(\Gamma) = 3$). For $k = 1$ we recover the exponential distribution $E(1)$, for which

$$M_\alpha(E) = -\log(1-\alpha) + 1 + \log(1-\alpha)\frac{-\log((1-\alpha)/2) + \log(1-\alpha)}{-\log((1-\alpha)/2) + \log(1-\alpha)} = \frac{1}{\log(2)} \approx 1.4427,$$

(2)

independent of $\alpha$ (and scale). And as $k \to \infty$, $\Gamma(k,1)$ converges (under scaling and translation) to the standard Normal distribution $\Phi$, so

$$\lim_{k \to \infty} M_\alpha(\Gamma(k,1)) = \frac{\varphi(\Phi^{-1}(\alpha))/(1-\alpha) - \Phi^{-1}(\alpha)}{\Phi^{-1}(1-(1-\alpha)/2) - \Phi^{-1}(\alpha)} \quad (\approx 1.3583 \text{ for } \alpha = 0.99),$$

(3)

using the analytic formula for ES of the Normal distribution; see p. 45 in McNeil et al. (2005).

For the Lognormal distribution $\text{LogN}(\mu, \sigma^2)$,

$$M_\alpha(\text{LogN}(\mu, \sigma^2)) = \frac{e^{\sqrt{2}(1 - \Phi(\Phi^{-1}(\alpha) - \sigma))}/(1-\alpha) - e^{\sqrt{2}\Phi^{-1}(\alpha)}}{e^{\sqrt{2}\Phi^{-1}(1-(1-\alpha)/2)} - e^{\sqrt{2}\Phi^{-1}(\alpha)}},$$

independent of the scale $e^\mu$. Since for $Z \sim \Phi$ we have $e^{\sqrt{2}Z} \sim \text{LogN}(0, \sigma^2)$, as $\sigma \to 0$, using the Taylor series of $e^x$ about $x = 0$ we recover the same limit 1.3583 as in (3).

For the Pareto distribution $P(a)$, $a > 1$,

$$M_\alpha(P(a)) = \frac{\alpha}{\alpha - 1}(1 - \alpha)^{-1/a} - (1 - \alpha)^{-1/a} = \frac{1}{(a - 1)(2^{1/a} - 1)}$$

independent of the level $\alpha$. As $a \to \infty$, we obtain $1/\log(2)$ as the limit, the same value as in (2). This is due the fact that if $X \sim E(1)$, then $e^{X/a} - 1 \sim P(a)$, and we can again use the Taylor series of $e^x$ about $x = 0$.

We shall use $M_\alpha$ as a measure of tail-heaviness in order to compare the tails of different families of distributions, and from that deduce how close $\text{VaR}_\alpha(S_\delta)$ is to $\text{ES}_\alpha^+(S_\delta)$ for small
values of $d$. Notice that $M_\alpha(F)$ exactly determines the worst-case VaR when $d = 2$ and the density is decreasing beyond $F^{-1}(\alpha) = \text{VaR}_\alpha(X_1)$. This is because for $d = 2$ the worst-case dependence structure for $\text{VaR}_\alpha$ is countermonotonicity above the $\alpha$-quantile of $X_1$ and $X_2$, viz. $X_2 = F^{-1}(\alpha + (1 - F(X_1)))$ when $F(X_1) \geq \alpha$; see Makarov (1982) and Embrechts et al. (2005). Furthermore, when the density is decreasing, the minimal sum $X_1 + X_2$ in the tail is attained when $X_1 = X_2 = \text{MS}_\alpha(X_1)$. Hence $\overline{\text{VaR}}(S_2) = 2 \text{MS}_\alpha(X_1)$. See also (5) and Figure 3 in the next section.

2.1.2 The normalized VaR bound

Since VaR is a positively homogeneous and translation equivariant risk measure, it is easy to re-calculate VaR under these operations. The upper bound on VaR depends only on the conditional distribution $F$ on the interval $[F^{-1}(\alpha), \infty)$. Moreover, it is easy to see that

$$\overline{\text{VaR}}_\alpha(S_d) \in [\text{VaR}_\alpha^+(S_d), \text{ES}_\alpha^+(S_d)].$$

So, in order to focus on the issue of specifying where in this interval $\overline{\text{VaR}}_\alpha$ lies, we define the normalized VaR bound

$$\delta_\alpha(S_d) := \frac{\overline{\text{VaR}}_\alpha(S_d) - \text{VaR}_\alpha^+(S_d)}{\text{ES}_\alpha^+(S_d) - \text{VaR}_\alpha^+(S_d)} \in [0, 1]. \quad (4)$$

Note that $\delta_\alpha(S_d)$ does not depend on the scale and location of $F$. In Figure 3 the values of $\delta_\alpha(S_d)$ are plotted in dependence on $d$. The parameters for Pareto and Lognormal distributions are chosen so that the $M_\alpha$ values match, thus, as explained in the previous section,

$$\delta_\alpha(S_2) = \frac{2 \text{MS}_\alpha(X_1) - 2 \text{VaR}_\alpha(X_1)}{2 \text{ES}_\alpha(X_1) - 2 \text{VaR}_\alpha(X_1)} = 1/M_\alpha(F), \quad (5)$$

and the upper VaR bounds agree for $d = 2$. For $d > 2$ the bounds for the matched Lognormal and Pareto dfs converge at a similar rate, especially for lighter tails, when convergence to 1 is rather fast. The difference $1 - \delta_\alpha$ seems to decrease as a power of $d$, so in Figure 4 we plot this difference versus the dimension $d$ on a logarithmic scale. We observe that, apart from the “kink” at $d = 2$, the dependence is approximately linear on the log-log scale. Based on this numerical evidence, and fixing the value at $d = 2$ according to (5), we consider an approximate model

$$1 - \delta_\alpha \approx (1 - 1/M_\alpha F)^{\lambda(M_\alpha)} \left(\frac{d}{2}\right)^{\lambda(M_\alpha)}.$$

(6)

Taking logs from both sides we obtain a linear dependence, and using linear regression for $d \in \{2 \ldots 15\}$ we fit the slope parameter $\lambda$ (the intercept at $d = 2$ is fixed) for each considered $M_\alpha$. The obtained values of $\lambda$ are plotted on the right panel of Figure 4. They show that the convergence rate is indeed determined by the tail-heaviness $M_\alpha$, and the values of $\lambda(M_\alpha)$ for the
Figure 3: $\delta_\alpha(S_d)$ plotted in dependence on $d$. Pareto$(a)$ with $a = 10, 5, 3, 2, 1.5$ from top down, with tail-heaviness $M_\alpha = 1.55, 1.68, 1.92, 2.41, 3.40$, respectively, and LogN$(\mu, \sigma^2)$ with the parameters chosen to match the $M_\alpha$, $\alpha = 0.99$.

Figure 4: Left panel: normalized VaR bound $\delta_\alpha, \alpha = 0.99$, depending on the dimension $d$. Solid lines correspond to Pareto$(a)$ distributions with $a = 1.5, 2, 3, 5, 10$ and $M_\alpha = 3.40, 2.41, 1.92, 1.68, 1.55$ from top down. Dashed lines correspond to Lognormal distributions with parameters selected to match the $M_\alpha$ values. Right panel: the fitted values of $\lambda(M_\alpha)$ obtained by linear regression for different tail-weight levels $M_\alpha$.

Pareto and Lognormal families are similar. Note also that e.g. for Pareto$(1.5)$ the convergence rate according to Proposition 4 is $O(d^{-1/3})$ as $d \to \infty$, whereas from Figure 4 we read that for small $d$ the difference $1 - \delta_\alpha$ decreases approximately as $d^{-2/3}$ for the corresponding $M_\alpha = 3.4$. Finally, the fitted curves for $\delta_\alpha$ using $\lambda$ values from the linear regression are plotted in Figure 5. The close match shows that the model (6) works well for small dimensions.

### 2.2 Lower bound on VaR

The lower bound on VaR$_\alpha(S_d)$ over all joint models is defined as

$$\underline{\text{VaR}}_\alpha(S_d) := \inf \{ \text{VaR}_\alpha(X_1 + \cdots + X_d) : X_i \sim F_i, i = 1, \ldots, d \}.$$
Figure 5: Normalized VaR bound $\delta_\alpha$, $\alpha = 0.99$ (solid lines) compared to the fitted values (dashed) according to the model (6), depending on the dimension $d$. Left panel: Pareto($\alpha$) distributions with $\alpha = 1.5, 2, 3, 5, 10$ and $M_\alpha = 3.40, 2.41, 1.92, 1.68, 1.55$ from bottom up. Right panel: Lognormal distributions with parameters selected to match the $M_\alpha$ values.

Before providing the basic result for the lower bound on VaR in the homogeneous case, we define the left-tail expected shortfall

$$\text{LES}_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\alpha(X) \, dp,$$

and the comonotonic $\text{LES}_\alpha^+(S_d) = \sum_{i=1}^d \text{LES}_\alpha(X_i) = d \text{LES}_\alpha(X_1)$ (since $X_i \sim F$, $\forall i$).

Proposition 5 (Bernard et al. (2014)). If the support of $F$ is bounded below, then

$$\text{VaR}_\alpha(X_1 + \cdots + X_d) \geq \max\{(d - 1)F^{-1}(0) + F^{-1}(\alpha), d \text{LES}_\alpha(X_1)\}. \quad (7)$$

Furthermore, define $H$ and $D$ as in (1) with respect to $F^\alpha$, the conditional distribution of $F$ on $[F^{-1}(0), F^{-1}(\alpha)]$. If conditions (A) and (B) with respect to $F^\alpha$ are satisfied, then (7) holds with equality, i.e.

$$\text{VaR}_\alpha(S_d) = \max\{H(0), D(0)\}.$$  

Since we assume that the essential support of $F$ is the positive half-axis, (7) reduces to $\text{VaR}_\alpha(X_1 + \cdots + X_d) \geq \max\{F^{-1}(\alpha), d \text{LES}_\alpha(X_1)\}$. In contrast to Proposition 2, in Proposition 5 we include the conditions (A) and (B) explicitly. Recall that previously these were implied by the non-increasing density assumption. While it is reasonable to assume this for the upper tail, in the lower tail this assumption holds for some distributions (e.g. Pareto) but fails for other (e.g. Lognormal). In such cases, one may resort to testing (A) numerically and testing for complete mixability in (B) using the method in Puccetti and Wang (2015a). Note, however, that if $F^{-1}(\alpha) > d \text{LES}_\alpha(X_1)$ then the bound can be sharp also without the $d$-CM condition. Typically, however, letting $d^* = F^{-1}(\alpha)/\text{LES}_\alpha(X_1)$, we may use as a good approximation

$$\text{VaR}_\alpha(S_d) \approx F^{-1}(\alpha) \quad \text{for } d < d^*,$$

$$\text{VaR}_\alpha(S_d) \approx d \text{LES}_\alpha(X_1) \quad \text{for } d \geq d^*.$$  

(8)
An alternative numerical approach for computing an approximation of the sharp bounds on VaR under DU is given by the Rearrangement Algorithm (RA); see Embrechts et al. (2013) or Puccetti and Rüschendorf (2012, 2013) for earlier formulations.\footnote{A website set up by Giovanni Puccetti with the title “The Rearrangement Algorithm project” provides full details and recent developments on the RA; see https://sites.google.com/site/RearrangementAlgorithm/} In Table 2, the values for Pareto, Lognormal and Gamma distributed variables are listed for which the ratio $\text{VaR}_a(S_d)/\text{LES}_a(S_d)$ has reached the limit 1. Due to a non-increasing density, in the Pareto case (7) is always sharp and the limit is reached at $d = \lceil d^* \rceil$. As a side note, for the Pareto distribution at level $p = \alpha$, as $a \to \infty$,

$$d^* = \frac{(1 - p)^{-1/a} - 1}{(1 - (1 - p)^{1-1/a})/((1 - 1/a)/p) - 1} \to \frac{-\log(1 - p)}{1 + ((1 - p)/p) \log(1 - p)},$$

so for $p = 0.99$ we have $d^* > 4.8298$, i.e. no parameter $a$ gives a lower value of $d^*$, hence they are not listed in Table 2. For the Lognormal and Gamma dfs, the values were obtained by the RA, using as discretization parameter $N = 10^5$ and the stopping condition $\varepsilon = 10^{-4}$. In Figure 6 the normalized lower VaR bounds are plotted. The approximate bounds from (8) are the same for the three families. These are sharp for the Pareto, but differ from the sharp bounds (here, computed using the RA) for the Lognormal and Gamma families in the light-tailed cases, since the densities are not decreasing and the conditions in Proposition 5 are not satisfied. In particular, for small $d$ it is the condition (A) that matters, and is not satisfied e.g. for the LogN$\sigma = 0.59$ df with $d = 3$; see Figure 7. With parameter $k \leq 1$, however, $\Gamma(k, 1)$ has a decreasing density, so by Lemma 1 satisfies the conditions and the approximate bound (7) is sharp. Overall, (7) gives an easily calculated lower bound on aggregate VaR, which is close to the sharp bound.

Table 2: Thresholds for the number of margins $d$ for which $\text{VaR}_a(S_d)/\text{LES}_a(S_d)$ has converged to 1. Parameters for the distributions are chosen to match the ratio $d^* = \text{VaR}_a(X_1)/\text{LES}_a(X_1)$.

<table>
<thead>
<tr>
<th>$\alpha = 0.99$, $d^*$</th>
<th>2.36</th>
<th>3.42</th>
<th>5.59</th>
<th>6.55</th>
<th>8.19</th>
<th>11.00</th>
<th>14.91</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pareto($a$), $a = 10$</td>
<td>-</td>
<td>10</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>6</td>
<td>9</td>
<td>11</td>
<td>15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>LogN($0, \sigma^2$), $\sigma = 0.40$</td>
<td>0.40</td>
<td>0.59</td>
<td>0.88</td>
<td>0.98</td>
<td>1.13</td>
<td>1.35</td>
<td>1.60</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td>11</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Gamma($k, 1$), $k = 5$</td>
<td>5.00</td>
<td>2.00</td>
<td>0.77</td>
<td>0.58</td>
<td>0.40</td>
<td>0.25</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td>11</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>
Figure 6: \((\text{VaR}_p(S_d) - \text{LES}_p^+(S_d))/({\text{VaR}_p(S_d) - \text{LES}_p^+(S_d)})\) plotted in dependence on \(d\) for Pareto\(p\), LogN\((0, \sigma^2)\) and \(\Gamma(k, 1)\) distributions with the parameters chosen to match the \(d^*\) (see Table 2). For Pareto, the approximate bounds (8) are sharp, whereas for LogN and Gamma there are small deviations for the lighter-tailed cases in comparison to the sharp bounds given by the RA.

2.3 Lower bound on ES

The lower bound on \(\text{ES}_p(S_d)\) under dependence uncertainty is defined as

\[
\text{ES}_p(S_d) := \inf \{ \text{ES}_p(X_1 + \cdots + X_d) : X_i \sim F_i, i = 1, \ldots, d \}.
\]

In the literature, few results on \(\text{ES}_p\) are available. The two main references are Puccetti (2013) (using the RA) and Bernard et al. (2014). The latter provides the following result in the homogeneous case.

**Proposition 6** (Bernard et al. (2014)). Define \(H\), \(D\) and \(c_d\) as in (1) with respect to \(F\). If condition (A) is satisfied, then

\[
\text{ES}_p(S_d) \geq \begin{cases} 
\frac{1}{1-\alpha}(\text{E}[S_d] - \alpha D(c_d)) & \text{if } \alpha \leq 1 - c_d, \\
\frac{1}{1-\alpha} \int_0^{1-\alpha} H(t)dt & \text{if } \alpha > 1 - c_d.
\end{cases}
\] (9)

Furthermore, if condition (B) is satisfied, then (9) holds with equality.

**Remark** 3. Due to Lemma 1, Proposition 6 is most useful in the case of a decreasing density, e.g. for Pareto or Exponential df. In other cases the conditions (A) and (B) would need to be verified numerically. Alternatively, one may apply the RA. The latter, however, uses a discretization of the marginal distributions, which may underestimate the integral over the infinite tail of the aggregate risk.

A simple lower bound on the aggregate ES can be calculated as follows.

**Proposition 7** (Cheung and Lo (2013)). If \(F^{-1}(0) \geq 0\), then

\[
\text{ES}_{1-p}(S_d) \geq \text{ES}_{1-p/d}(X_1).
\] (10)

Furthermore, if \(F(0) \geq (d - 1)/d\), then (10) holds with equality.
Figure 7: Functions $H$ and $D$ with respect to $F^\alpha$, the conditional distribution of $F = \text{LogN}(0, 0.59^2)$ on $[0, F^{-1}(\alpha)]$, $d = 3$, $\alpha = 0.99$. The vertical scale is normalized to show the interval $[\text{LES}_\alpha^+(S_d), \text{VaR}_\alpha^+(S_d)]$ as $[0, 1]$. **Left panel:** full range $c \in [0, 1]$. **Right panel:** close-up of the range $[0, 0.01]$ shows that the function $H$ is increasing initially. $H(0) = 0.058$, while the sharp lower bound (from the RA) is 0.091.

For this bound to be sharp, a sufficient mass at 0 (at least $p(d - 1)/d$) is required for the marginal distributions, in which case Proposition 6 gives the same bound. In Figure 8 all three bounds are plotted for different Pareto and Lognormal dfs. First, notice that the convergence to the limit ($\text{E}[S_d]$ in this case) is much slower than for the VaR bounds. For the Pareto df, bounds from Proposition 6 are sharp, while the error in the RA bound due to discretization is clearly visible in the heavy-tailed cases (discretization parameter $N = 10^5$ and stopping condition $\varepsilon = 10^{-4}$ were used). The non-sharp bound from Proposition 7, however, is very close, because the Pareto distribution has a large mass near 0. For the Lognormal dfs, the density is small near 0, hence there is a visible approximation error for the lighter tails. The bound from the RA again underestimates the tail-expectation in the heavy-tailed cases due to discretization. Overall, Proposition 6 seems to give the best bound (due to infinite tails, condition (A) is likely to hold).

3 Inhomogeneous portfolios

In this section suppose $X_i \sim F_i$, $i = 1, \ldots, d$, where the $F_i$’s are not necessarily the same, and hence $S_d = X_1 + \cdots + X_d$ is the aggregate loss of a possibly inhomogeneous portfolio. In order to investigate the key determinants of the DU bounds for VaR and ES of an inhomogeneous portfolio, we sample at random different families and parameters for the marginal distributions. In particular, we choose $F_i$ independently for $i = 1, \ldots, d$ from each of the following families with probability 1/3:

- Gamma, $\Gamma(k, \mu/k)$ with $k \sim 1 + \text{Poisson}(1), \mu \sim \mathcal{E}(1/5)$,
Figure 8: \( (\text{ES}_\alpha(S_d) - E[S_d])/(\text{ES}_\alpha^+(S_d) - E[S_d]) \) plotted in dependence on \( d \). Solid lines are the bound from Proposition 6 (Bernard et al., 2014), dashed lines correspond to the RA (Puccetti, 2013) and dotted lines to Proposition 7 (Cheung and Lo, 2013). **Left panel:** Pareto(\( a \)) distributions with \( a = 1.5, 2, 3, 5, 10 \) from top down. **Right panel:** LogN(\( \mu, \sigma^2 \)) distributions with the parameters chosen to match E\( X \) and ES\( \alpha(X) \) with those of Pareto.

- Lognormal, LogN(\( \mu, \sigma^2 \)) with \( \mu \sim \mathcal{N}(0.2, 0.4^2) \), \( \sigma \sim \Gamma(8, 0.1) \),
- Generalized Pareto, GPD(\( k, \nu, 0 \)) with \( k \sim 1/(1.5 + \mathcal{E}(1/5)) \), \( \nu \sim \Gamma(4, 0.3) \).

For the latter Pareto-type df from Extreme Value Theory, see Embrechts et al. (1997). In this manner we sample the marginal distributions of 100 portfolios for each \( d = 2, \ldots, 15 \); in total 1400 portfolios. Then, using only basic properties of these randomly selected marginal distributions and their respective risk quantities, such as VaR\( \alpha \), MS\( \alpha \) and ES\( \alpha \), we will approximate the dependence uncertainty bounds for the aggregated portfolio risk measures.

### 3.1 Upper bound on VaR

In this section we focus on approximating the upper bound for VaR\( \alpha(S_d) \). As the two key drivers, similarly to the homogeneous case, we look at the dimension and the mean-median ratio, properly adapted. Since the marginal distributions can be different, the scale and shape of each margin affects the aggregate VaR. In Figure 9, the left panel, we observe that the nominal portfolio dimension \( d \) is a poor predictor of \( \delta_\alpha(S_d) \). The reason is that the different scales of the marginals have the numerical effect of a dimension reduction for the underlying portfolio; for example, if one marginal has a much larger scale then the others, then the VaR of this marginal gives a good estimate of the aggregate VaR. In this case the dependence uncertainty is small and \( \delta_\alpha \) is close to 0 (similar to \( d = 1 \)). If the marginals are of a similar scale, then the dependence uncertainty is greater (similar to the homogeneous case \( d \)) and \( \delta_\alpha \) is closer to 1.

Let VaR\( ^+(S_d) \), MS\( ^+(S_d) \) and ES\( ^+(S_d) \) denote VaR\( (S_d) \), MS\( (S_d) \) and ES\( (S_d) \) respectively,
when the $X_i$ are comonotonic. Define the effective dimension as
\[ \tilde{d}(S_d) := \frac{\text{ES}^+_\alpha(S_d) - \text{VaR}^+_\alpha(S_d)}{\max_i \{\text{ES}_\alpha(X_i) - \text{VaR}_\alpha(X_i)\}}. \]

On the right panel of Figure 9 we observe that the effective dimension $\tilde{d}$ determines the normalized VaR-bound $\delta_\alpha$ better. In order to take into account also the tail-heaviness of the marginal distributions, define the average mean-median ratio as
\[ \tilde{M}_\alpha(S_d) := \frac{\text{ES}^+_\alpha(S_d) - \text{VaR}^+_\alpha(S_d)}{\text{MS}^+_\alpha(S_d) - \text{VaR}^+_\alpha(S_d)}. \]

We speculate that, similar to the homogeneous case, the rate of convergence depends on the tail-heaviness $\tilde{M}_\alpha(S_d)$, and consider an approximate model (recall the definition of $\delta_\alpha$ in (4))
\[ \delta_\alpha \approx 1 - \tilde{d}^{\lambda(\tilde{M}_\alpha)}. \]

Rearranging and taking log, we obtain a linear dependence $\log(1 - \delta_\alpha) \approx \lambda \log(\tilde{d})$ for which we estimate $\lambda$ using linear regression. The estimated parameters are plotted in Figure 10. As expected, for light tails the convergence is fast, $O(\tilde{d}^{-3})$, and slower for heavy tails. The fitted lines for $\delta_\alpha$ for three levels of the average tail-weight $\tilde{M}_\alpha$, as well as the true VaR bound for portfolios with similar tail-weight, are plotted in Figure 11. We observe that the two key determinants $\tilde{d}$ and $\tilde{M}_\alpha$ provide a reasonable approximation of $\delta_\alpha(S_d)$ and hence of $\text{VaR}_\alpha(S_d)$.

Of course, these conclusions are to be understood in the specific context of the example. An out-of-sample test would be required to determine whether this is a good fit in general.

3.2 Lower bound on VaR

Similar to the homogeneous case (7), an easily calculated but non-sharp bound is
\[ \text{VaR}_\alpha(S_d) \geq \max \{\text{VaR}_\alpha(X_i), i = 1, \ldots, d\} \vee \text{LES}^+_\alpha(S_d). \]
Figure 10: The fitted values of $\lambda(\tilde{M}_a)$ obtained by linear regression for different tail-weight levels $\tilde{M}_a$.

Figure 11: Fitted curves $\delta_{99} = 1 - \tilde{\alpha}^4$ for tail-heaviness levels $\tilde{M}_a = 1.4, 1.8, 2.4$, superimposed on scatterplots of the true $\delta_{99}$ of those sampled portfolios with tail-heaviness within 10% of these levels.

In Figure 12 the error of this approximation is shown, relative to the length of the possible interval $\text{VaR}_a(S_d) \in [\text{LES}_{a}^{+}(S_d), \text{VaR}_{a}^{+}(S_d)]$. We see that of the 1400 portfolios, only 4 have a relative error above 5%. The marginal densities for one of these portfolios is plotted in the right panel of Figure 12. Notice the light left tails of the Lognormal margins (especially for $\sigma < 0.5$); these do not have enough mass to compensate for the heaviest right tail, and hence the approximate bound (11) is not sharp. This is in agreement with the results in the homogeneous case; see Figure 6. Keeping this observation in mind, we can in most cases use (11) as a good approximation for the most favorable aggregate VaR.

### 3.3 Lower bound on ES

In the inhomogeneous case a lower bound on the ES is given in Jakobsons et al. (2015), based on a similar construction to that of (1). In this case, however, due to the lack of symmetry, one cannot combine the tails of the margins at the same rate; a dynamical weighting needs to be found by solving a system of functional equations. Also, to check the validity and sharpness of the obtained bounds, conditions similar to (A) and (B) would need to be verified. Instead, based
on the good performance of the approximate bounds in Section 2.3, we consider the following generalization, based on Theorem 4.1 in Cheung and Lo (2013).

**Theorem 8.** For rvs $X_i \geq 0$, let $\zeta = \min \{ x : \sum_{i=1}^{d} F_i(x) \geq d - (1 - \alpha) \}$. Then

$$ES_{\alpha}(S_d) \geq \frac{1}{1 - \alpha} \sum_{i=1}^{d} E[X_i I_{\{X_i > \zeta\}}].$$

(12)

Furthermore, if $\sum_{i=1}^{d} F_i(0) \geq d - 1$ and $F_i$ are continuous at $\zeta$, then (12) holds with equality.

**Proof.** Let $S'_d = \sum_{i=1}^{d} X_i I_{\{X_i > \zeta\}}$. Clearly $S_d \geq S'_d$, so by monotonicity of ES,

$$ES_{\alpha}(S_d) \geq ES_{\alpha}(S'_d) \geq ES_{\alpha}(S'_M) = \frac{1}{1 - \alpha} \sum_{i=1}^{d} E[X_i I_{\{X_i > \zeta\}}],$$

where $S'_M = \sum_{i=1}^{d} Y_i$ for $Y_i = X_i I_{\{X_i > \zeta\}}$ such that $Y_1, \ldots, Y_d$ are mutually exclusive and hence $S'_M \leq S'_d$ (stop-loss order); see Dhaene and Denuit (1999). If $\sum_{i=1}^{d} F_i(0) \geq d - 1$, then there exist mutually exclusive $X_i \sim F_i$, $i = 1, \ldots, d$ and (12) is an equality. □ □

Also in the inhomogeneous case the RA (Puccetti, 2013) can be applied, with the caveat that the obtained bound may underestimate the expectation in the infinite tail. We computed both bounds at the level $\alpha = 0.99$ for the sampled portfolios, the results are plotted in Figure 13. First, notice that the bound for higher-dimensional portfolios is closer to $E[S_d]$. However, for a fixed $d$, the RA underestimates the bound (due to discretization) when $ES_{\alpha}(S_d)$ is closer to $ES_{\alpha}^+(S_d)$, since this is often due to one or more heavy-tailed margins (e.g. the GPD with parameter $k > 0.5$, which has infinite variance). On the other hand, the RA gives a sharper
bound than (12) in the presence of marginals with light left tails (e.g. LogN(μ, σ²) with σ < 0.5). Marginal densities for example portfolios in each of these two cases are given in Figure 14. Out of the 1400 portfolios, in only 7 cases the RA was worse than (12) by more than 5% points, and in only 6 cases the RA was better by more than 5% points (relative to the possible interval \( \text{ES}_\alpha(S_d) \in [E[S_d], \text{ES}_\alpha^+(S_d)] \)). Hence, depending on the portfolio, either of the two bounds may give a value closer to the sharp lower bound on ES, but in most cases the difference is small (and both of the bounds are close to sharpness).

4 Conclusions and further research

Model uncertainty in general and dependence uncertainty in particular have taken center stage of the regulatory discourse of financial and insurance regulation. In the present paper we have mainly looked at some recent results from the latter, i.e. properties of extremal regulatory risk measures, where we assume full knowledge of the marginal risk distributions, but we do not have, or want, to make assumptions on their interdependence. Whereas numerous analytic results have recently appeared on this issue, there is an increasing need for a better understanding of the underlying numerical problems. This paper provides a first (small) step in this direction. Comparing simple approximate DU bounds for VaR and ES with more advanced ones, we observe that the former often give reasonable estimates of the sharp bounds. Moreover, we identify the cases when the approximate bounds are far from the sharp ones. We also note that the convergence rate of \( \overline{\text{VaR}} \) as the portfolio dimension increases is different for small dimensions
Figure 14: Top panel: marginal densities of portfolio P1 from Figure 13. Notice the light left tails of the Lognormal margins. RA yields $ES_p(S_4) \geq 8.98$ while (12) $ES_p(S_4) \geq 7.73$. The possible range is $[E[S_4], ES^+_{\alpha}(S_4)] = [5.81, 19.20]$, hence the difference is $-9\%$ of this range. Bottom panel: marginal densities of portfolio P2. Notice the heavy right tails of the GPD margins. RA yields $ES_p(S_5) \geq 64.63$ while (12) $ES_p(S_5) \geq 69.90$. The possible range is $[6.73, 103.19]$, hence the relative difference is $5\%$.

compared to the theoretical asymptotic rate. The above examples should serve as illustrations for the recent analytic results in the literature, build intuition and motivate further research in the area of dependence uncertainty.

An important direction of research is investigating what partial information on the dependence structure helps to obtain narrower bounds, which under complete dependence uncertainty tend to be very wide. An early approach considers the case when some multivariate marginals are known, and it is found that it can lead to strongly improved bounds; see Rüschendorf (1991), Embrechts and Puccetti (2010), and Embrechts et al. (2013). Another approach is taken in Bernard and Vanduffel (2015a), where the full-dimensional joint density of the assets in portfolio is assumed to be known, but only on a “trusted region”, which is a subset of the support. They find that the bounds quickly deteriorate as the trusted region deviates from the entire support. Bernard et al. (2015a) argue that variance of the sum of marginals (portfolio variance) is often available, and that this dependence information often leads to substantially narrower VaR bounds. Recently, dynamic factor multivariate GARCH models have been proposed for forecasting financial time series; see Alessi et al. (2009) and Santos and Moura (2014). Consequently, a new and relevant approach would be using factor models in order to reduce the DU to a smaller dimension, namely, the number of factors. Recall the observations from Section 3, which show that for small-dimensional portfolios the bounds are closer to the comonotonic case (narrower). Gordy (2003) even considers the portfolio VaR in a one-factor model, eliminating DU entirely. Of course, any additional assumptions on the structure should be justified, so it is equally important to understand what additional information on the dependence is typically available in practice. An illuminating case study in the applied setting of economic capital computation for a bank is Aas and Puccetti (2014), where the DU bounds are calculated, compared with the value corresponding to a $t$-copula, and other practical issues discussed.
Acknowledgments

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Appendix

Figure 15: The fitted values of $\lambda(\tilde{M}_\alpha), \alpha = 0.99$ obtained by regression for different tail-weight levels $\tilde{M}_\alpha$, for three collections of portfolios: either only Gamma, Lognormal, or Generalized Pareto marginal distributions.

Figure 16: Fitted curve $\delta_\alpha = 1 - \tilde{d}^\lambda, \alpha = 0.99$ for tail-heaviness level $\tilde{M}_\alpha = 1.4$, superimposed on scatterplots of the true $\delta_\alpha(S_d)$ of the sampled portfolios. Portfolios with only Gamma marginal distributions.
Figure 17: Fitted curves $\hat{\alpha} = 1 - \hat{d}^4$, $\alpha = 0.99$ for tail-heaviness levels $\tilde{M}_\alpha = 1.6, 2.0, 2.2$, superimposed on scatterplots of the true $\delta_\alpha(S_d)$ of those sampled portfolios with tail-heaviness within 10% of these levels. Portfolios with only Lognormal marginal distributions.

Figure 18: Fitted curves $\hat{\alpha} = 1 - \hat{d}^4$, $\alpha = 0.99$ for tail-heaviness levels $\tilde{M}_\alpha = 1.6, 2.0, 2.2$, superimposed on scatterplots of the true $\delta_\alpha(S_d)$ of those sampled portfolios with tail-heaviness within 10% of these levels. Portfolios with only Generalized Pareto marginal distributions.
Figure 19: On the horizontal axis: $\text{ES}_\alpha(S_d)$ of the sampled portfolios as approximated by the RA (with discretization parameter $N = 10^5$ and stopping condition $\epsilon = 10^{-4}$). Vertical axis: the difference between the non-sharp bound (12) and the RA bound. For each portfolio the values in $[E(S_d), \text{ES}_\alpha^+(S_d)]$ are normalized to lie within $[0, 1]$. The color corresponds to the dimension from $d = 2$ (dark) to $d = 15$ (light). The marginal dfs are sampled from the families given above each plot. For LogN and Gamma distributions, light lower tails are possible, so (12) is away from sharpness and gives a worse bound than the RA. For Pareto, (12) always gives a better bound due to heavy upper tails. For LogN and Pareto mixed, either of the two bounds may be sharper.
Edgars Jakobsons, Steven Vanduffel.

**Dependence uncertainty bounds for the expectile of a portfolio.**

Dependence uncertainty bounds
for the expectile of a portfolio

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Abstract
We study upper and lower bounds on the expectile risk measure of risky portfolios when
the joint distribution of the risky components is not fully specified. First, we summarize
methods for obtaining bounds when only the marginal distributions of the components are
known, but not their interdependence (unconstrained bounds). In particular, we provide
the best-possible upper bound and the best-possible lower bound (under some condi-
tions), as well as numerical procedures to compute them. We also derive simple analytic
bounds that appear adequate in various situations of interest. Second, we study bounds
when some information on interdependence is available (constrained bounds). When the
variance of the portfolio is known, a simple-to-compute upper bound is provided, and we
illustrate that it may significantly improve the unconstrained upper bound. We also show
that the unconstrained lower bound cannot be readily improved using variance informa-
tion. Next, we derive improved bounds when the bivariate distributions of each of the
risky components and a risk factor are known. When the factor induces a positive depen-
dence among the components, it is typically possible to improve the unconstrained lower
bound. Finally, the unconstrained dependence uncertainty spreads of expected shortfall,
value-at-risk and the expectile are compared.

Keywords: expectiles, convex order, elicitation, coherence, dependence.

1 Introduction and preliminaries

This paper aims to contribute to the broader academic discussion on the properties of risk mea-
sures relevant to risk management and regulation in the banking and insurance industry; see
Embrechts et al. (2014) and Emmer et al. (2015) for an overview. The two most well-known risk measures are the value-at-risk (VaR) and expected shortfall (ES),

$$\text{VaR}_\alpha(X) = \inf \{ x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq \alpha \}, \quad \text{ES}_\beta(X) = \frac{1}{1-\beta} \int_\beta^1 \text{VaR}_q(X) \, dq,$$

where the latter is only defined for random variables (rvs) $X$ with a finite expectation. While VaR is dominantly used in industry, it lacks the property of subadditivity and is thus not coherent in the sense of Artzner et al. (1999). By contrast, ES, which is merely the average of all upper VaRs, is coherent. In fact, it is the smallest coherent risk measure that is more conservative than VaR (see Artzner et al. (1999)). Recently, Gneiting (2011) brought the issue of elicitability to the forefront. A risk measure is said to be elicitable if it is a minimizer of the expectation of some scoring function, which depends on the point forecast and the true observed loss. The work in Gneiting (2011) showed that VaR is elicitable (if the corresponding quantile is unique), but ES is not. While some authors interpret this to mean that ES cannot be back-tested (e.g., Kou and Peng (2014); Chen (2014)), Acerbi and Szekely (2014) argue that elicitability is relevant for relative comparisons between estimators, but not for absolute significance testing. Moreover, Fissler and Ziegel (2016) show that the pair (VaR, ES) is jointly elicitable. Nevertheless, the question arises whether there are non-trivial coherent risk measures that are elicitable. In Ziegel (2014); Bellini and Bignozzi (2015); Delbaen et al. (2015), it is shown that the only risk measure that is both elicitable and coherent is the expectile. The expectile is introduced in Newey and Powell (1987) as the minimizer of the expectation of an asymmetric quadratic scoring function,

$$e_\tau(X) = \arg\min_{e \in \mathbb{R}} \mathbb{E}[(\tau \mathbb{I}_{X > e} + (1-\tau) \mathbb{I}_{X < e})(X - e)^2].$$

It follows that $e_\tau(X)$ is the unique solution of the equation implied by the first order conditions (however, Delbaen (2013) points out that no differentiability or continuity of the distribution function is required):

$$(1-\tau) \mathbb{E}[(e_\tau - X)\mathbb{I}_{X < e_\tau}] = \tau \mathbb{E}[(X - e_\tau)\mathbb{I}_{X > e_\tau}]. \tag{1}$$

Expectiles are well known in regression analysis (Efron, 1991; Yao and Tong, 1996; De Rossi and Harvey, 2009); they are used for forecasting financial time series (Granger and Sin, 2000) and estimating VaR and ES (Taylor, 2008). A penalized least squares approach in portfolio optimization was suggested by Manganelli (2007); the expectile is a special case when a quadratic downside penalty is used. The expectile is also closely related to the Omega performance measure of Keating and Shadwick (2002); see Rémillard (2013), p. 128. The expectile was first explicitly considered as a risk measure in Kuan et al. (2009), and the authors coined the acronym EVaR. This name was later adopted in other articles (De Rossi, 2009; Bellini and Di Bernardino, 2015). However, even the original authors admit that this acronym was already
used for economic-VaR in Aıt-Sahalia and Lo (2000) and recently also for entropic-VaR in Ahmadi-Javid (2012). To avoid confusion, we shall use the notation $e_r$, as in Emmer et al. (2015); Bellini et al. (2014) and McNeil et al. (2015), p. 290.

Throughout, we assume that the random variables represent losses. The expectile-based risk measure $e_r(X)$ is subadditive and thus coherent for $\tau \in [1/2, 1)$; for this property, as well as other features and representations, see Bellini et al. (2014); Delbaen (2013). A discussion on risk management with expectiles can be found in Bellini and Di Bernardino (2015). In the present paper, we further contribute to this discussion by examining their properties when aggregating risks.

Note that by rearranging Equation (1),

$$e_r = E[X] + \theta E[(X - e_r)1_{\{X > e_r\}}], \quad \text{where} \quad \theta := \frac{2\tau - 1}{1 - \tau} \geq 0 \quad \text{for} \quad \tau \in [1/2, 1). \quad (2)$$

Thus, the expectile can also be interpreted as the insurance premium using the Dutch premium principle (see Van Heerwaarden and Kaas (1992)), where the insurer buys an excess-of-loss reinsurance contract for any claim above the premium $e$, with loading factor $\theta$ (and applies zero loading for the retained part). From Equation (2), we observe that expectiles are only implicitly defined, and their computation appears cumbersome. However, if the loss distribution $X \sim F$ is known, the following approach can be used to compute the corresponding expectile. First, define the tail integral, a function which will be useful for shorter notation,

$$\text{TI}_X(x) := \int_x^{\infty} u dF(u) = E[X1_{\{X > x\}}].$$

Analytic expressions for $\text{TI}_X$ are available for many commonly-used distributions, such as Pareto, log-normal, normal, Student $t$, exponential, gamma, and other. Next, applying Newton’s method to Equation (1) yields a practical iterative procedure for computing $e_r(X) = \lim_{k \to \infty} x_k$, given by

$$x_{k+1} = \frac{(1 - \tau) E[X] + (2\tau - 1) \text{TI}_X(x_k)}{(1 - \tau)p_k + \tau(1 - p_k)}, \quad (3)$$

where $p_k = F(x_k)$. An analogue for an empirical distribution using iterative reweighting is mentioned in Newey and Powell (1987) and stated explicitly in Efron (1991). The convergence of this procedure is very fast. It is shown in Bellini and Di Bernardino (2015) that for most common distributions, the expectiles are smaller than quantiles at level $\tau$ (for $\tau$ high enough), while for heavy-tailed (infinite variance) distributions, the opposite holds true. They coincide exactly (for all $\tau$) for a Student $t$ distribution with $\nu = 2$ and asymptotically (as $\tau \to 1$) for the Pareto(2) distribution. Therefore, initializing the procedure at $x_0 = \text{VaR}_r(X)$ appears reasonable.

While the expectile is a familiar object in regression analysis, its properties relevant to risk management are less studied. The focus of this paper is on risk aggregation and measurement

\footnote{Note that for a continuous rv $X$, $\text{ES}_{\beta}(X) = \frac{1}{1-\beta} \text{TI}_X(\text{VaR}_{\beta}(X))$.}
under model uncertainty. Often the total (aggregate) loss that a company faces can be expressed as a sum $S = X_1 + \ldots + X_d$, where the $X_i$ represent, e.g., the losses of different business lines or risk types. The risks $X_i$ are typically modeled separately, and little might be known about their interdependence. We will be interested in finding the range of values a risk measure $\rho(S)$ can take for different aggregate losses $S \in \mathcal{O}$, where $\mathcal{O}$ is the so-called admissible class containing all of the aggregate loss distributions that are consistent with the available marginal and dependence information. In particular, define the best-possible upper bound and the best-possible lower bound as

$$\underline{\rho} = \inf \{ \rho(S) : S \in \mathcal{O} \} \quad \text{and} \quad \overline{\rho} = \sup \{ \rho(S) : S \in \mathcal{O} \},$$

where the risk measure $\rho$ will be either VaR, ES or the expectile. The idea to assess the impact of (partial) dependence information on risk bounds has been explored in a series of recent papers; see Bernard et al. (2015a); Bignozzi et al. (2015); Bernard et al. (2015b); Puccetti et al. (2015); Bernard and Vanduffel (2015b); Bernard et al. (2015c); Bernard and Vanduffel (2015a). In these papers, the risk measure used was the VaR. In this paper, we will mainly focus on the expectile as a challenger for VaR.

The paper is structured as follows: Section 2 considers the case when only marginal distributions are known. We provide the best-possible upper bound, as well as the best-possible lower bound (under some conditions) and provide numerical procedures to practically compute these bounds. We also provide weaker bounds and show they are close to the best-possible ones in various situations of interest. We study the location-scale family and provide analytical expressions for the best possible bounds in this context. Section 3 gives bounds when the mean and variance of the aggregate loss are known. In Section 4, we consider the availability of dependence information through factor models. We provide various bounds in this context, and the results of this and previous sections are applied in an example using the skew-$t$ distribution. In Section 5, the width of the dependence uncertainty interval for the expectile is compared to that of VaR and ES. Finally, Section 6 summarizes the observations.

### 2 Bounds when only the marginal distributions are known

Due to the curse of dimensionality, it is typically easier to statistically fit a one-dimensional distribution function (df) to each $X_i$ than to fit a multivariate distribution to $\mathbf{X} = (X_1, \ldots, X_d)$. Under an idealized version of dependence uncertainty (DU), only the marginal distributions $X_i \sim F_i, i = 1, \ldots, d$ are known, while the dependence structure (copula) is completely unknown. Hence, the aggregate loss $S$ can be any of the elements in the (Fréchet) admissible class $\mathcal{O}$,

$$\mathcal{O}(F_1, \ldots, F_d) = \{ X_1 + \ldots + X_d : X_i \sim F_i, i = 1, \ldots, d \}.$$
The (best-possible) bounds on the expectile are denoted by \( \overline{e}_\tau \) and \( \underline{e}_\tau \). To determine the bounds, it turns out to be sufficient to find elements in \( \mathcal{C}(F_1, \ldots, F_d) \) that are maximal, respectively, minimal in the sense of convex order.\(^2\) We first recall the definition of this ordering concept and then connect it with \( \overline{e}_\tau \) and \( \underline{e}_\tau \).

**Definition 1.** Let \( X \) and \( Y \) be random variables, such that

\[
E[\phi(X)] \leq E[\phi(Y)] \quad \text{for all convex functions } \phi : \mathbb{R} \to \mathbb{R},
\]

provided the expectations exist. Then, \( X \) is said to be smaller than \( Y \) in the convex order; written \( X \preceq_{\text{cx}} Y \).

Consider the convex functions \( \phi_e(x) = (x - e)1_{\{x > e\}} \) indexed by \( e \in \mathbb{R} \). We find that\(^3\)

\[
X \preceq_{\text{cx}} Y \Rightarrow E[(X - e)1_{\{X > e\}}] \leq E[(Y - e)1_{\{Y > e\}}] \quad \forall e \in \mathbb{R} \Rightarrow e_r(X) \leq e_r(Y). \quad (4)
\]

In particular, this shows that upper bound \( \overline{e}_\tau \), resp., lower bound \( \underline{e}_\tau \), is obtained if one can find the maximum, resp., minimum element, in the convex order sense in the admissible class \( \mathcal{C} \). The last implication in Equation (4) comes from the following lemma by taking \( G(e) = E[(Y - e)1_{\{Y > e\}}] \) and noting that \( E[X] = E[Y] \). Specifically, the following lemma connects bounds on the stop-loss premium \( E[(S - e)1_{\{S > e\}}] \) with bounds on \( e_r(S) \).

**Lemma 2.** Suppose \( G : \mathbb{R} \to \mathbb{R} \) is a non-increasing function, such that

\[
E[(S - e)1_{\{S > e\}}] \leq G(e) \quad \forall e \in \mathbb{R}. \quad (5)
\]

Then, \( e_r(S) \leq e^\ast \), where

\[
e^\ast = \inf \left\{ e \in \mathbb{R} : e \geq E[S] + \frac{2\tau - 1}{1 - \tau} G(e) \right\}.
\]

Analogously, a lower bound on the stop-loss premium yields a lower bound on \( e_r(S) \).

**Proof.** From Equation (5), it immediately follows that

\[
e \geq E[S] + \frac{2\tau - 1}{1 - \tau} G(e) \quad \Rightarrow \quad e \geq E[S] + \frac{2\tau - 1}{1 - \tau} E[(S - e)1_{\{S > e\}}]. \quad (6)
\]

Since for both inequalities, the right-hand side is non-increasing in \( e \), the solution of Equation (2) (i.e., \( e_r(S) \)) must be less than or equal to \( e^\ast \). \( \square \)

\(^2\)Likewise, the study of VaR bounds is connected to identifying (in an appropriate admissible class) the elements that are minimum in the sense of convex order, a feature that points to a similarity between the study of bounds on the expectile and the study of bounds on VaR; see Section 2.3 in Bernard et al. (2015a) for these results.

\(^3\)Note that Equation (4) also follows from the more general results in Bäuerle and Müller (2006); Jouini et al. (2006). Indeed, Bäuerle and Müller (2006) has shown that any convex risk measure \( \rho \) with the Fatou property is consistent with the convex order, meaning that \( X \preceq_{\text{cx}} Y \) implies \( \rho(X) \leq \rho(Y) \). Furthermore, Jouini et al. (2006) shows that law-invariant risk measures have the Fatou property. Since the expectile is convex and law invariant, it is consistent with the convex order. See Bellini (2012) for further results on the properties of the expectile and other generalized quantiles with respect to various stochastic orders.
2.1 Upper bound with marginal information

It is shown in Tchen (1980) that the comonotonic dependence structure leads to the maximal element (denoted $S^c$) with respect to the convex order in the admissible class $\mathcal{S}$.

$$\forall S \in \mathcal{S} : \quad S \preceq \max S^c := \sum_{i=1}^{d} F_i^{-1}(U), \quad \text{where} \quad U \sim \mathcal{U}(0, 1) \quad (7)$$

Hence, we find that $\overline{e}_r = e_r(S^c)$. In the case of identical margins $F_i = F$, $i = 2, \ldots, d$, using positive homogeneity, this simplifies to

$$\overline{e}_r = e_r(dX_1) = de_r(X_1) = \sum_{i=1}^{d} e_r(X_i).$$

In general, however, the expectile is not comonotone additive, and hence, the upper bound $\overline{e}_r$ often needs to be computed numerically. Unfortunately, the df of $S^c$ is typically not available in an analytical form, so the iterative procedure Equation (3) is more difficult to apply, since it would involve a nested root search. In particular, to compute $\text{TI}_i(S^c(x_i))$ at each step, one would first need to find a $p_k$, such that $F_i^{-1}(p_k) = x_k$, and then sum up the tail integrals $\text{TI}_i(F_i^{-1}(p_k))$ for the margins.

Since $S^c$ is defined in Equation (7) in terms of its quantile function $F_i^{-1}$, it is easier to work in terms of the probability level $p$ corresponding to the expectile. For continuous marginal distributions with densities $f_i$, we can again apply Newton’s method by differentiating Equation (1) with respect to $p$ using the chain rule. This yields an iterative procedure for computing $e_r(S^c) = F_i^{-1}(p)$ in terms of $p = \lim_{k \to \infty} p_k$, given by

$$p_{k+1} = p_k - e_k / B(p_k) + \frac{\sum_{i=1}^{d} ((1 - \tau) \text{E}[X_i] + (2\tau - 1) \text{TI}_i(F_i^{-1}(p_k)))}{B(p_k)(p_k(1 - \tau) + (1 - p_k)\tau)}, \quad (8)$$

where $e_k = F_i^{-1}(p_k)$ and

$$B(u) = (F_i^{-1})'(u) = \sum_{i=1}^{d} (F_i^{-1})'(u) = \sum_{i=1}^{d} f_i(F_i^{-1}(u)), \quad u \in (0, 1).$$

Again, since analytic expressions for the mean, the tail integral and inverse df of parametric marginal distributions are often available, this is a very fast and accurate method. It is possible that Equation (8) yields $p_{k+1} \geq 1$, in which case we can take $p_{k+1} = (p_k + 1)/2$ instead; similarly, if $p_{k+1} \leq 0$, set $p_{k+1} = p_k/2$. Analogously to Equation (3), it is reasonable to initiate the procedure at $p_0 = \tau$.

In general, by subadditivity (recall that we use $\tau \in [1/2, 1]$),

$$e_r(S) \leq \sum_{i=1}^{d} e_r(X_i) =: e^+_r \quad (9)$$

so $e^+_r$ is a valid upper bound, too, but it is typically not the best possible. In Section 4.3, $e^+_r$ is computed, as well as the best-possible upper bound in an example with skew-$t$ distributions; we can observe that they are very close in all cases.
2.2 Lower bound with marginal information

The analysis of the lower bound is more involved. We first observe that

\[ e_\tau \geq \mathbb{E}[S] = \sum_{i=1}^{d} \mathbb{E}[X_i]. \quad (10) \]

This can be seen either by applying Jensen’s inequality to the degenerate random variable \( m = \mathbb{E}[S] \) to show \( m \leq_{\text{cx}} S \), \( \forall S \in \mathcal{E} \), or by noting that \( e_{1/2}(S) = \mathbb{E}[S] \) and that \( e_{\tau}(S) \) is increasing in \( \tau \) (see, e.g., Bellini and Di Bernardino (2015) for these and other properties). If the admissible class \( \mathcal{E} \) contains the constant \( \mathbb{E}[S] \), then this is the smallest element in convex order, and the lower bound Equation (10) is attained (sharp). This situation is achieved when the components \( X_i \) are “compensating” for each other and corresponds to the notion of joint mixability, which was formally introduced in Wang and Wang (2016) and extends the concept of complete mixability (Wang and Wang, 2011) to the inhomogeneous case; see also Puccetti and Wang (2015b) for an overview of these and related concepts. Precisely, a distribution \( F \) is called \( d \)-completely mixable if there exist rvs \( X_i \sim F_i \) such that \( \sum_{i=1}^{d} X_i = dc \) a.s., \( c \in \mathbb{R} \). Analogously, a \( d \)-tuple of dfs \( (F_1, \ldots, F_d) \) is called jointly mixable if there exist rvs \( X_i \sim F_i \) such that \( \sum_{i=1}^{d} X_i = k \) a.s., \( k \in \mathbb{R} \). Another concept that leads to an explicit smallest element in the Fréchet admissible class is that of mutual exclusivity, which requires that the margins have a large probability mass at an endpoint of the support; see Dhaene and Denuit (1999).

In general, however, the dependence structure that leads to the smallest element in the convex order is known only for \( d = 2 \) (countermonotonicity). If \( d > 2 \), for distributions that are bounded below (and satisfy some further conditions), in the case of identical margins, one can use the method from Bernard et al. (2014) (involves solving an integral equation), or for different margins, the method from Jakobsons et al. (2015) (requires solving a functional equation). A general, but approximate method is the Rearrangement Algorithm (RA) (see Embrechts et al. (2013); Puccetti (2013); Puccetti and Rüschendorf (2015)), which is based on a discretization of the margins. In the following, we describe a simple, yet necessary modification of the RA that provides improved approximations for the best-possible lower bounds in the case of expectation-based risk measures, such as ES and the expectile.

2.2.1 Rearrangement Algorithm

The most general method currently available for computing lower bounds for the common risk measures is the RA; see Embrechts et al. (2013); Puccetti (2013); Puccetti and Rüschendorf (2015). While the quantile-based RA performs well when computing bounds on the (quantile-based) VaR, Embrechts and Jakobsons (2016) indicates that for heavy-tailed margins, the RA
lower bound for ES (Puccetti, 2013) is not sharp, because the tail expectation is underestimated due to discretization. Since also expectiles are defined in terms of the tail expectation (see Equation (2)), it is important to address this issue of the RA. In the following, we first recall the standard discretization of the RA for each margin $i = 1, \ldots, d$ and next provide two modifications that we further investigate.

**Standard RA:**
$$x_{k,i} = F_{i}^{-1} \left( \frac{k-1}{N} \right), \quad k = 1, \ldots, N.$$  

**Midpoint RA:**
$$x_{k,i} = F_{i}^{-1} \left( \frac{k-1/2}{N} \right), \quad k = 1, \ldots, N.$$  

**Expectation RA:**
$$x_{k,i} = F_{i}^{-1} \left( \frac{k-1/2}{N} \right), \quad k = 2, \ldots, N - 1,$$
$$x_{N,i} = \text{ES}_{1-i/N}(X_i), \quad x_{1,i} = \text{LES}_{1/N}(X_i) := \int_{0}^{1/N} F_{i}^{-1}(q) \, dq$$

While the standard discretization may seem conservative, it is nonetheless an approximate lower bound, since the RA may stop at a suboptimal rearrangement (see Puccetti (2013)). Moreover, if the distribution is unbounded below, then $F_{i}^{-1}(0) = -\infty$. For high $N$ (such that $pN > d$), this would still give a finite lower bound for ES; however, it would be undefined for the expectile, since it depends on both the upper and the lower tail of the aggregate distribution. The midpoint RA avoids this problem, but still underestimates tail expectations; the expectation RA should solve both issues. To evaluate and compare the sharpness of the bounds obtained using the different discretizations, we consider the homogeneous case with Pareto marginals. In this case, the exact lower bound on ES can be obtained using the method in Bernard et al. (2014). In Table 1, the resulting underestimation errors are listed. Observe that the midpoint RA improves the results considerably, but the errors are still noticeable. The expectation RA, however, gives results that are within 0.1% of the true lower bound. In light of this, we will use the “expectation” version of RA for computing the unconstrained lower bounds on the expectile (i.e., $e_\tau$) in Section 4.3, since it provides more accurate estimates of the tail expectations. One final adjustment concerns the stopping condition for RA. In Puccetti (2013), the RA stops when an iteration reduces the ES by less than a pre-defined $\varepsilon$. To match the stopping condition with the objective, we stop the RA when the reduction in $e_\tau$ becomes smaller than $\varepsilon$. This means that the expectile of the current rearrangement needs to be computed at each iteration. However, the Equation expectile of the previous iteration of the RA makes a very good initial guess for Equation (3), so this is not time-consuming. To summarize, the expectation RA only changes the way the margins are discretized and the stopping condition; the rest of the RA remains as in Puccetti (2013). For other recent developments on the RA, we refer to https://sites.google.com/site/RearrangementAlgorithm/ and Hofert et al. (2015).
Table 1: Relative underestimation as a percent of the exact ES, for d Pareto(θ) distributions, using the Rearrangement Algorithm (RA) with different discretizations of size N = 10^5 and stopping condition ε = 10^{−4}.

<table>
<thead>
<tr>
<th>Standard RA</th>
<th>p = 0.95</th>
<th>p = 0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>1  2</td>
<td>3   4</td>
</tr>
<tr>
<td>θ = 5</td>
<td>0.1  0.2</td>
<td>0.2  0.2</td>
</tr>
<tr>
<td>θ = 3</td>
<td>0.3  0.4</td>
<td>0.5  0.6</td>
</tr>
<tr>
<td>θ = 2.5</td>
<td>0.5  0.7</td>
<td>0.9  1.0</td>
</tr>
<tr>
<td>θ = 2</td>
<td>1.2  1.6</td>
<td>1.9  2.2</td>
</tr>
<tr>
<td>θ = 1.5</td>
<td>5.3  6.6</td>
<td>7.6  8.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Midpoint RA</th>
<th>p = 0.95</th>
<th>p = 0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>1  2</td>
<td>3   4</td>
</tr>
<tr>
<td>θ = 5</td>
<td>0.0  0.0</td>
<td>0.0  0.0</td>
</tr>
<tr>
<td>θ = 3</td>
<td>0.1  0.1</td>
<td>0.1  0.2</td>
</tr>
<tr>
<td>θ = 2.5</td>
<td>0.2  0.2</td>
<td>0.3  0.3</td>
</tr>
<tr>
<td>θ = 2</td>
<td>0.5  0.7</td>
<td>0.8  0.9</td>
</tr>
<tr>
<td>θ = 1.5</td>
<td>3.0  3.8</td>
<td>4.3  4.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Expectation RA</th>
<th>p = 0.95</th>
<th>p = 0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>1  2</td>
<td>3   4</td>
</tr>
<tr>
<td>θ = 5</td>
<td>0.0  0.0</td>
<td>0.0  0.0</td>
</tr>
<tr>
<td>θ = 3</td>
<td>0.0  0.0</td>
<td>0.0  0.0</td>
</tr>
<tr>
<td>θ = 2.5</td>
<td>0.0  0.0</td>
<td>0.0  0.0</td>
</tr>
<tr>
<td>θ = 2</td>
<td>0.0  0.0</td>
<td>0.0  0.0</td>
</tr>
<tr>
<td>θ = 1.5</td>
<td>0.1  0.1</td>
<td>0.1  0.1</td>
</tr>
</tbody>
</table>

2.3 Example: location-scale family

We assume that the X_i belong to the same location-scale family, i.e. \( F_i(\cdot) = F((\cdot - \mu_i)/\sigma_i) \), \( i = 1, \ldots, d \), for some df F. Denote also \( \mu = \sum_{i=1}^{d} \mu_i \) and \( \sigma = \sum_{i=1}^{d} \sigma_i \).

2.3.1 Upper bound

By Equation (7), the convex order-maximal element in \( \Xi(F_1, \ldots, F_d) \) is given by

\[
S^{\varepsilon} = \sum_{i=1}^{d} F_i^{-1}(U) = \mu + \sigma F^{-1}(U), \text{ where } U \sim U(0, 1).
\]
Hence,

\[ \bar{e}_t = \mu + \sigma e_t(F^{-1}(U)) = e_t^+ \]

which can be computed using the procedure described in Equation (3). This also means that when the margins have the same shape, the bound based on subadditivity Equation (9) is the best possible.

2.3.2 Lower bound

As mentioned before, obtaining an element in the admissible class that is minimum in the sense of convex order is often difficult or not even possible to achieve. However, in case the marginal dfs are from the same location-scale family that is symmetric, we can express the minimal element in the admissible class explicitly and, thus, also obtain the best-possible lower bound on the expectile.

**Theorem 3.** Let \( X_i \sim F_i(\cdot) = F((\cdot - \mu_i)/\sigma_i), \) \( i = 1, \ldots, d \) belong to the location-scale family of a symmetric df \( F. \) Suppose without loss of generality that \( \sigma_1 \geq \sigma_i, \) \( i = 2, \ldots, d. \)

(i) If \( \sigma_1 \geq \sum_{i=2}^d \sigma_i \), then a minimal element in \( \Xi(F_1, \ldots, F_d) \) in convex order is

\[ S^\ell = F_1^{-1}(U) + \sum_{i=2}^d F_i^{-1}(1-U) = \sum_{i=1}^d \mu_i + (\sigma_1 - \sum_{i=2}^d \sigma_i) F^{-1}(U), \]

where \( U \sim \mathcal{U}(0,1). \)

Correspondingly,

\[ \bar{e}_t = \sum_{i=1}^d \mu_i + (\sigma_1 - \sum_{i=2}^d \sigma_i) e_t(F^{-1}(U)). \]

(ii) Otherwise, if \( F \) furthermore admits a unimodal density, then the minimal element in the admissible class is the constant \( \mu = \sum_{i=1}^d \mu_i, \) and thus, \( \bar{e}_t = \mu. \)

**Proof.** (i) Case \( \sigma_1 = \sum_{i=2}^d \sigma_i \) is trivial. If \( \sigma_1 > \sum_{i=2}^d \sigma_i \), we use the well-known fact that the convex order is consistent with the ordering of expected shortfall (note that \( \text{VaR}_x(X) = F_X^{-1}(u) \)). In particular, Theorem 3.A.5 in Shaked and Shanthikumar (2007) states that

\[ X \preceq_{\text{cx}} Y \iff \int_0^1 F_X^{-1}(u) \, du \leq \int_0^1 F_Y^{-1}(u) \, du \quad \forall p \in (0,1) \quad \text{and} \ E[X] = E[Y]. \]

Clearly, \( X^g_i = F_1^{-1}(U) \) and \( X^g_i = F_i^{-1}(1-U), \) \( i = 2, \ldots, d \) have the required dfs, so \( S^g = \sum_{i=1}^d X^g_i \in \Xi. \) If \( F \) is continuous, then for any \( S = \sum_{i=1}^d X_i \in \Xi \) and any \( p \in (0,1), \) we have that

\[ \int_0^1 F_S^{-1}(u) \, du = E[S \mathbb{1}_{X_i > F_1^{-1}(p)}] \leq E[S \mathbb{1}_{X_i < F_i^{-1}(1-p)}] \leq \int_0^1 F_S^{-1}(u) \, du. \quad (11) \]

The first inequality follows from the fact that \( \{X_i > F_1^{-1}(p)\} = \{X_i < F_1^{-1}(1-p)\}, \) \( i = 2, \ldots, d \) and \( A = \{X_i < F_i^{-1}(1-p)\} \) minimizes \( E[X_i \mathbb{1}_A] \) over events \( A \) of probability \( 1 - p. \) Similarly, the second inequality follows because \( E[S \mathbb{1}_A] \) is maximal when \( A = \{S > F_S^{-1}(p)\}. \)
If $F$ is not continuous, then the indicators in Equation (11) need to be augmented by adding sets, such as
\[ \{X_1 = F^{-1}_{i1}(p), V \leq (P(X_1 \leq F^{-1}_{i1}(p)) - p)/P(X_1 = F^{-1}_{i1}(p)) \} , \text{ where } V \sim \mathcal{U}(0, 1) \text{ (indep.)} \]
to the first one (and similarly for the others). Since the dfs are symmetric and belong to a location-scale family, the atoms at $F^{-1}_{i1}(p), F^{-1}_{i1}(p)$ and $F^{-1}_{i1}(1-p), i = 2, \ldots, d$ are of the same size.

In Case (ii), the rvs are jointly mixable by Corollary 3.6 in Wang and Wang (2016), and hence, the result follows.

Note that Theorem 3 is of interest beyond the context of expectiles, as the convex order least element also yields lower bounds on, e.g., variance and ES in this admissible class. An early result in this direction was Rüschendorf and Uckelmann (2002), where identical symmetric unimodal distributions with a differentiable density are considered. It is shown that for such a df $F$, there exist $X_i \sim F, i = 1, \ldots, d$, in the form $X_i = RU_i$, where $U_i \sim \mathcal{U}(-1,1)$ and $R$ is a continuous rv. Then, using that the uniform distribution is completely mixable (an explicit construction is given), it follows that also $F$ is completely mixable. This model can be considered as a scale mixture of uniform distributions, where $R$ is the scale factor, common to all margins.

In Section 4, more general factor models are considered, and Theorem 3 is applied to find the minimal element in the convex order and to compute exact lower bounds in an example.

### 3 Bounds when the mean and variance of the sum are known

We consider the case in which additional to the marginal information, also the variance of $S$ is known, i.e., we consider the admissible class $\Xi(F_1, \ldots, F_d, s^2), \Xi(F_1, \ldots, F_d, s^2) = \{X_1 + \ldots + X_d : X_i \sim F_i, i = 1, \ldots, d \text{ and } \text{Var}(X_1 + \ldots + X_d) = s^2\},$

where $s^2 > 0$ is a compatible variance constraint. In this setting, the bounds on $e_\tau$ will be denoted $e_{\tau}^{s^2}$ and $e_{\tau}^{s^2}$. It is not so clear how to determine these best-possible bounds. Instead, we proceed by considering a larger admissible class that is easier to deal with,\(^4\) but gives weaker bounds. Note that in the case of VaR, in Bernard et al. (2015a), it is shown that the weaker bounds are typically close to the best-possible ones. In the following sections, let $m = \sum^d_1 E[X_i], \text{ and denote}$

\[ M_2(m, s^2) := \{S \in L^2 : E[S] = m, \text{Var}(S) = s^2\}. \hspace{1cm} (12) \]

\(^4\)Note indeed that the admissible class $\Xi(F_1, \ldots, F_d, s^2)$ reflects $d + 1$ constraints rendering optimization difficult. By relaxing the $d$ (infinite dimensional) constraints on the marginal distributions and substituting them by the portfolio mean constraint, we enlarge the class (as there are many marginal distributions that yield the same portfolio mean) and effectively obtain two constraints only, which greatly facilitates the optimization.
3.1 Upper bound with variance constraint

Since \( \mathcal{Z}(F_1, \ldots, F_d, s^2) \) is a subset of \( \mathcal{M}_2(m, s^2) \), we find that \( \epsilon_\tau^s \leq B \), where

\[
B := \sup\{e_\tau(S) : S \in \mathcal{M}_2(m, s^2)\}.
\]

(13)

It will become apparent that variables supported on two points play an important role in the class \( \mathcal{M}_2 \).

**Definition 4.** A random variable is \( X \) called *diatomic* if \( P(X = a) = p \) and \( P(X = b) = 1 - p \) for some \( a < b \) and \( p \in (0, 1) \).

The expectile of such a diatomic random variable has a simple expression,

\[
e_\tau(X) = \frac{(1 - \tau)pa + \tau(1 - p)b}{(1 - \tau)p + \tau(1 - p)}
\]

(14)

**Theorem 5.** The maximum expectile \( B \) defined in Equation (13) is given by

\[
B = m + s \sqrt{\frac{q}{1 - q}},
\]

where \( q = (2\tau - 1)^2 \). It is attained by a diatomic rv with support \( \{a, b\} \) and mass \( \tau \) at \( a \), where

\[
a = m - s \sqrt{\frac{1 - \tau}{\tau}},
b = m + s \sqrt{\frac{\tau}{1 - \tau}}.
\]

**Proof.** Denote by \( \mathcal{A} \subset \mathcal{M}_2(m, s^2) \) the subset of diatomic variables. The proof further consists of two steps. First, we construct a variable \( X_\tau \in \mathcal{A} \) that maximizes \( e_\tau \) on \( \mathcal{A} \). Next, we show that \( X_\tau \) also provides the solution to Equation (13). Any \( X_p \in \mathcal{A} \) has two support points \( a_p, b_p \) with mass \( p \) at \( a_p \),

\[
a_p = m - s \sqrt{\frac{1 - p}{p}},
b_p = m + s \sqrt{\frac{p}{1 - p}},
\]

where \( 0 < p < 1 \). Substituting \( a_p \) and \( b_p \) into Equation (14) yields

\[
e_\tau(X_p) = m + s \frac{(2\tau - 1) \sqrt{(1 - p)p}}{(1 - p)\tau + (1 - \tau)p}.
\]

(15)

Using differentiation with respect to \( p \), we find that \( e_\tau(X_p) \) attains its maximum on \( (0, 1) \) when \( p = \tau \) (see also Figure 1). Hence, the variable that maximizes \( e_\tau \) on \( \mathcal{A} \) is given by

\[
X_\tau = \begin{cases} 
m - s \sqrt{\frac{1 - \tau}{\tau}} & \text{with probability } \tau, 
m + s \sqrt{\frac{\tau}{1 - \tau}} & \text{with probability } 1 - \tau. 
\end{cases}
\]
For this rv, defining \( q = (2\tau - 1)^2 \), Equation (15) simplifies to

\[
e_t(X_p) = m + s \left( \frac{\tau - 1/2}{\sqrt{1 - \tau}} \right) = m + s \sqrt{\frac{q}{1 - q}}.
\]

Now, consider any \( S \in \mathcal{M}_2(m, s^2) \). Without loss of generality, we can express \( S = F^{-1}_S(U) \) for some standard uniformly distributed rv \( U \). Letting \( p = F_S(e_t(S)) \), the variable \( S \) can also be written as

\[
S = F^{-1}_S(U)\mathbb{1}_{\{U < p\}} + F^{-1}_S(U)\mathbb{1}_{\{U > p\}}.
\]

Define a diatomic variable \( Y \) such that \( \mathbb{E}[Y] = m \) and \( e_r(Y) = e_r(S) \) by

\[
Y = \left( (m - (1 - p)\mathbb{E}_p(S))/p \right)\mathbb{1}_{\{U < p\}} + \mathbb{E}_p(S)\mathbb{1}_{\{U > p\}}.
\]

From Jensen’s inequality, it follows that \( \text{Var}(Y) \leq \text{Var}(S) = s^2 \). Since \( Y \) is diatomic and the right-hand side of Equation (15) is increasing in \( s \) (recall that \( \tau \in [1/2, 1] \)),

\[
e_r(S) = e_r(Y) \leq e_r(X_p) \leq e_r(X_r),
\]

which completes the proof. \( \square \)

The bound \( B \) does not make use of the specific information on the marginal distributions. When the variance is “too high”, the unconstrained bound \( \overline{\tau} \) will be stronger. In the opposite case, \( B \) will dominate \( \overline{\tau} \). We formulate the following corollary.

**Corollary 6.**

\[
\overline{\tau}^2 \leq \min \{ B, \overline{\tau} \}.
\]
Remarks

(i) A procedure called the Extended Rearrangement Algorithm (ERA) was introduced in Bernard et al. (2015a) and makes it possible to compute an approximation of $\varepsilon^\tau$ from below, using both the marginal, as well as the variance information. This algorithm will be applied in an example in Section 4.3.

(ii) Denote by $C = \sup \{\text{VaR}_\alpha(S) : S \in M_2(m, s^2)\}$ and by $D = \sup \{\text{ES}_\beta(S) : S \in M_2(m, s^2)\}$. A similar proof as in Theorem 5 shows that $C$ and $D$ are attained by the same diatomic variable $X_\tau$ that attains the bound $B$; see also Bernard et al. (2015a). We find that

$$C = D = m + s \sqrt{\frac{\tau}{1-\tau}}$$

and, thus, that $B < C = D$. On the other hand, the numerical value of these upper bounds would coincide for $e_\tau$, VaR$_\alpha$, and ES$_\beta$, if we set $\alpha = \beta = (2\tau - 1)^2$.

3.2 Lower bound with variance constraint

From the proof of Theorem 5, it follows that

$$A := \inf \{e_\tau(S) : S \in M_2(m, s^2)\}$$

is given by $A = m$ ($= E[S]$). Indeed, $e_\tau(X_p) \to m$ as $p \to 0$, respectively $p \to 1$; see also Figure 1. Moreover, this bound cannot be improved by assuming either an upper bounded support or a lower bounded support. Hence, we conclude that working in the moment space $M_2$ does not make it readily possible to improve on $e_\tau$.

4 Bounds for factor models

A factor model is introduced in Bernard et al. (2015b) as a way to include additional information on the dependence structure and, hence, reduce the DU. This model considers rvs $X_i$ and a factor $W$ for which the bivariate distributions $H_i$ of $(X_i, W)$ are known. The aggregate risk $S = X_1 + \ldots + X_d$ then belongs to the factor-constrained admissible class

$${\mathcal{S}}(H_1, \ldots, H_d) = \{X_1 + \ldots + X_d : (X_i, W) \sim H_i, i = 1, \ldots, d\}.$$

In the following, denote by $F_i$ the marginal distribution of $X_i$ and by $F_{i|W}$ the conditional distribution of $(X_i|W = w)$, $i = 1, \ldots, d$ (if defined). The additional information of this factor

---

5Assuming a compact support would improve the lower bound, but we do not elaborate on this case here and refer to De Vylder and Goovaerts (1982).
structure leads to narrower factor-constrained DU bounds,

\[ e_r^l = \inf \{ e_r(S) : S \in \mathcal{E}(H_1, \ldots, H_d) \}, \]

\[ e_r^\tau = \sup \{ e_r(S) : S \in \mathcal{E}(H_1, \ldots, H_d) \}. \]

In the rest of this section, we consider a model where, conditional on a non-negative factor \( W \) with distribution \( G \), the rvs \( X_i \) belong to the location-scale family of distribution \( F_0 \):

\[ X_i = \mu_i + \gamma_i W + \sigma_i \sqrt{W} Z_i, \quad Z_i \sim F_0, Z_i \perp W, \quad i = 1, \ldots, d, \text{ and } W \sim G. \quad (16) \]

Models of this type are called location-scale mixture models and have a broad range of applications, going back to Barndorff-Nielsen (1977, 1978), where a particular location-scale mixture family (generalized hyperbolic) is introduced. In the area of financial modeling, Barndorff-Nielsen (1997) show that this family allows a good fit of asset returns; Eberlein and Keller (1995); Eberlein et al. (1998) apply it for pricing; and Aas et al. (2005) apply it in the context of Garch models. Specific consideration has been given in the literature to sub-families of this class; see, for example, Madan and Seneta (1990); Madan et al. (1998) for the case of the multivariate variance gamma distribution, as well as Demarta and McNeil (2005); Aas and Hobæk Haff (2006) for the case of multivariate skew-t distributions.

4.1 Upper bound

According to Theorem 4.1 in Bernard et al. (2015b), the largest element in the convex order is achieved in the case when, conditional on the factor, the margins are comonotonic. Thus, computing the upper bound on risk in such a model is as easy as computing \( e_r(X_i) \), since the conditionally comonotonic sum belongs to the same class of location-scale mixtures:

\[ S_W^\tau = \mu + \gamma W + \sigma \sqrt{W} F_0^{-1}(U), \quad \text{where} \quad \mu = \sum_{i=1}^d \mu_i, \gamma = \sum_{i=1}^d \gamma_i, \sigma = \sum_{i=1}^d \sigma_i, \]

and \( W \perp U \sim \mathcal{U}(0, 1) \). In particular, we find that \( e_r^\tau = e_r(S_W^\tau) \).

4.2 Lower bound

The minimal element in the convex order sense is given by the following result (counterpart to Theorem 4.1 in Bernard et al. (2015b)).

**Theorem 7.** If \( S_W^l \) is a convex order-minimal element in \( \mathcal{E}_w = \mathcal{E}(F_{1|w}, \ldots, F_{d|w}) \) for each \( w \), then \( S_W^l \) is a minimal element in \( \mathcal{E}(H_1, \ldots, H_d) \) and \( e_r^l = e_r(S_W^l) \).

**Proof.** Since \( S_W^l \in \mathcal{E}_w \), it can be written as \( \sum_{i=1}^d X_{i,w} \) for some \( X_{i,w} \sim F_{i|w}, i = 1, \ldots, d \). Thus, \( (X_{i,w}, W) \) have the required bivariate distributions, and \( S_W^l \in \mathcal{E}_w \). To show it is minimal, consider
any \( T \in \mathcal{E}(H_1, \ldots, H_d) \), and denote \( T_w = T|\{W = w\} \in \mathcal{E}_w \). By the definition of the convex order, \( \mathbb{E}[\phi(S_w^\ell)] \leq \mathbb{E}[\phi(T_w)] \) for any convex function \( \phi \). Using monotonicity and the tower property, we obtain

\[
\mathbb{E}[\phi(S_w^\ell)] \leq \mathbb{E}[\phi(T)],
\]

which completes the proof. \( \square \)

Let \( \sigma^\ell = \max\{0, 2 \max_{i=1}^d \{\sigma_i\} - \sum_{j=1}^d \sigma_j\} \). By Theorem 3, if the df \( F_0 \) in model Equation (16) is symmetric and unimodal, then a minimal element in \( \mathcal{E}_w \) is

\[
S_w^\ell = \mu + \gamma w + \sigma^\ell \sqrt{w} F_0^{-1}(U), \quad U \sim \mathcal{U}(0, 1).
\]

Thus, by Theorem 7, \( S_w^\ell = \mu + \gamma W + \sigma^\ell \sqrt{W} F_0^{-1}(U) \) is a convex order minimal element in the factor-constrained admissible class. Moreover, since \( U \) is independent of \( W \), \( S_w^\ell \) also belongs to the same mixture family as the margins, so the corresponding lower bound \( c^\ell_r = c_r(S_w^\ell) \) can be computed as easily as \( c_r(X) \). Note that the assumption that \( F_0 \) is symmetric is natural, since the location-mixing term \( \gamma_i W \) can be used to add asymmetry to the df of \( X_i \).

### 4.3 Example: skewed Student t distribution

The results in Sections 4.1 and 4.2 apply for general choices of dfs \( F_0 \) and \( G \) in Equation (16). The most well-known location-scale mixture class is that of normal mean-variance mixtures, i.e., the case when \( F_0 = \Phi \). If, in addition, \( W \) follows the generalized inverse Gaussian (GIG) distribution, then the family of generalized hyperbolic (GH) distributions is obtained; see Section 6.2.3 in McNeil et al. (2015). This is a flexible class of distributions that exhibits skewness and heavy tails and is therefore useful for modeling financial data. Moreover, it can also be extended to the multivariate GH distribution,

\[
X = \mu + \gamma W + \sqrt{W} A Z, \quad Z \sim \mathcal{N}(0, I_d), \tag{17}
\]

where vectors in \( \mathbb{R}^d \) are written in bold, \( I_d \) is the identity matrix and \( \Sigma = AA^T \) is the Cholesky decomposition of the scale matrix. The multivariate GH class is closed under linear operations, so it has the portfolio property, which is useful for applications. In this section, a particular subclass of GH is considered: the skew-\( t \) distributions.

The hyperbolic skewed Student \( t \) distribution is a special case of normal mean-variance mixtures, where the mixing distribution is the inverse-gamma df; see Aas and Hobæk Haff (2006). The inverse-gamma distribution \( \Gamma(\alpha, \beta) \) has density

\[
f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha - 1} \exp \left( -\frac{\beta}{x} \right).
\]
Setting $F_0 = \Phi$ and $G = \Gamma(\nu/2, \nu/2)$ in Equation (16) results in $X_i \sim \text{Skew-t}(\mu, \gamma_i, \sigma_i^2)$. The multivariate skew-t subclass of the GH distribution (17) is also closed under linear transformations. Using Theorem 4.1 in Bernard et al. (2015b), the factor-constrained worst-case dependence structure is achieved using $A$ with $\sigma = (\sigma_1, \ldots, \sigma_d)\top$ as the first column and zeros in the others (conditional comonotonicity), resulting in a degenerate matrix $\Sigma$. The corresponding aggregate risk is

$$S'_{W} \sim \text{Skew-t}(\mu, \gamma, \sigma),$$

where

$$\mu = \sum_{i=1}^{d} \mu_i, \quad \gamma = \sum_{i=1}^{d} \gamma_i, \quad \sigma = \sum_{i=1}^{d} \sigma_i.$$

Applying Theorems 3 and 7, the factor-constrained lower bounds in the “dominated” case $\sigma^\ell = 2 \max_{i=1}^{d} \{ \sigma_i \} - \sum_{j=1}^{d} \sigma_j \geq 0$ (see Case (i) of Theorem 3) can be attained using $A$ with $\sigma$ as the first column, except $-\sigma_i$ in row $i$ corresponding to the largest $\sigma_i$ and zeros in other columns (conditional countermonotonicity with respect to the $i$-th margin). The corresponding aggregate risk is then

$$S'_{W} \sim \text{Skew-t}(\mu, \gamma, \sigma^\ell).$$

The inverse df and tail integral (as well as the df and density) of a skew-t distribution can be computed using the methods in Dokov et al. (2008), which rely on the use of a Bessel function, numerical integration and root search (and are computationally intensive). Hence, we can apply the iterative algorithm (3) (using the df and tail integral) to compute its expectile. Thus, we have a method to obtain the upper, respectively lower, factor-constrained DU bound. The unconstrained upper bound on the expectile can be computed using the iterative procedure (8) (based on the tail integral, inverse df and density) and the unconstrained lower bound using the “expectation” version of RA introduced in Section 2.2.1.

Note that the conditionally jointly mixable case cannot be attained using a multivariate GH dependence structure. In this case, $S'_{W} = \mu + \gamma W$ follows a scaled and translated inverse-gamma distribution. In order to apply the iterative procedure (3), we need the df and tail integral (TI) for inverse-gamma. For a general $W \sim \Gamma(\alpha, \beta)$, we calculate, using the substitution $u = \beta/t$,

$$F_W(x) = \int_0^x f_W(t)dt = \frac{1}{\Gamma(\alpha)} \int_{\beta/x}^{\infty} u^{\alpha-1}e^{-u}du,$$

which is the (normalized) incomplete gamma function. In MATLAB, this can be computed using the function \texttt{gammainc(b./x,a,'upper')}.

Similarly, we have

$$\text{TI}_W(x) = \left( \frac{\beta}{\alpha-1} \right) \frac{1}{\Gamma(\alpha-1)} \int_0^{\beta/x} u^{\alpha-2}e^{-u}du,$$

which is given by $b/(a-1)^{*}\text{gammainc(b./x,a-1,'lower')} in MATLAB.

In Table 2, the expectile bounds for two examples of a skew-t distribution are listed. The parameters were selected to be in the range observed when fitting skew-t to daily stock returns.
(scaled by a factor of 250) of companies in the S&P 100 index. Model A is a conditionally jointly-mixable case, and Model B is a “dominated” case. First, notice that the approximate upper bound $e^\tau_\tau$ is very close to the best-possible bound $\bar{e}_\tau$ in all cases. Next, observe that due to the positive dependence the factor model induces, the value of the factor-constrained upper bound is similar to the unconstrained one, whereas the factor structure noticeably improves the lower bound; this is in agreement with the observations in Bernard et al. (2015b). In Model A, the unconstrained lower bound is close to the mean, so the margins are “almost” jointly mixable.

Table 2: Upper and lower dependence uncertainty (DU) bounds for $e_\tau(S)$ for two skew-$t$ examples with $d = 8$. Column $e_\tau$ lists values for a multivariate skew-$t$ distributed $X$ with a diagonal $\Sigma$ matrix.

<table>
<thead>
<tr>
<th>Model A</th>
<th>$E[S] = 1.24, \nu = 4.5, \mu = (-0.2, -0.15, \ldots, 0.15)$, $\gamma = (-0.25, -0.15, \ldots, 0.45), \sigma = (4.5, 5, \ldots, 8)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$</td>
<td>$e_\tau$</td>
</tr>
<tr>
<td>0.8</td>
<td>1.24</td>
</tr>
<tr>
<td>0.9</td>
<td>1.24</td>
</tr>
<tr>
<td>0.95</td>
<td>1.24</td>
</tr>
<tr>
<td>0.99</td>
<td>1.25</td>
</tr>
<tr>
<td>0.999</td>
<td>1.30</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Model B</th>
<th>$E[S] = 1.13, \nu = 5, \mu = (-0.2, -0.15, \ldots, 0.15)$, $\gamma = (-0.25, -0.15, \ldots, 0.45), \sigma = (3.5, 3.5, \ldots, 3.5, 25.5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$</td>
<td>$e_\tau$</td>
</tr>
<tr>
<td>0.8</td>
<td>1.91</td>
</tr>
<tr>
<td>0.9</td>
<td>2.50</td>
</tr>
<tr>
<td>0.95</td>
<td>3.15</td>
</tr>
<tr>
<td>0.99</td>
<td>5.15</td>
</tr>
<tr>
<td>0.999</td>
<td>10.05</td>
</tr>
</tbody>
</table>

In order to illustrate the influence of variance information on the bounds, we first need to find the feasible range for $\text{Var}(S) = s^2$. The law of total variance yields

$$\text{Var}(S) = E[\text{Var}(S | W)] + \text{Var}(E[S | W])$$

$$= E[W \text{Var}(\sum \sigma_i Z_i)] + \gamma^2 \frac{2\nu^2}{(\nu - 4)(\nu - 2)},$$

as long as $\nu > 4$. The first term lies in the range $[0, \sigma^2 \nu/(\nu - 2)]$, corresponding to conditional joint mixability up to conditional comonotonicity. In Figure 2, we plot for Model B the variance-constrained bound $B$ from Theorem 5 (based on the sole knowledge of the first two moments of
the sum $S$), and we compare it with the unconstrained upper bound $\bar{e}$. We also plot the variance-constrained bound obtained by means of the Extended Rearrangement Algorithm (ERA), which in addition to the first two moments of $S$, also takes into account the marginal distributions of the components $X_i$ (see Bernard et al. (2015a) for a description of this algorithm). Variance constraints $s^2$ are taken in the range corresponding to standard deviation $s \in [1.9, 64.6]$. We observe that variance information yields a considerably reduced upper bound $B$, as long as $s^2$ is small enough. As the parameter $\tau$ increases, the bound $B$ becomes weaker and is relevant on a smaller range of $s^2$. The approximate bound computed using ERA is very close to the bound $B$, indicating that $B$ can nearly be attained by constructing the appropriate dependence among the random variables (with the given marginal distributions). This dependence yields a sum $S$ that has a (nearly) diatomic structure, i.e., $S$ becomes distributed as the random variable in Theorem 5. However, the highest variance that $S$ can possibly attain, under the constraint that it is diatomic and consistent with the marginal distributions, occurs when its upper atom is given by $b = \sum_{i=1}^{d} ES_r(X_i)$. Therefore, when the variance constraint is too high, we cannot expect ERA to return a diatomic distribution for $S$; see also Figure 3, where the distribution function of $S$ obtained using ERA is plotted for different variance constraints.

Figure 2: Moment space upper bound $B$ on the expectile, and an approximation of $\overline{e}_{\tau}^2$ computed using the Extended Rearrangement Algorithm (ERA), as a function of the standard deviation constraint $s$ on the horizontal axis. The unconstrained expectile bounds are also plotted for the sake of comparison. The dotted vertical line is the maximum standard deviation of a diatomic random variable, which is consistent with the marginal upper- and lower-tail expectations.

Remarks

(i) More generally, the GIG and, hence, (non-skew) hyperbolic distributions are infinitely divisible (see Barndorff-Nielsen and Halgreen (1977)), so methods from Kim et al. (2010)
Figure 3: Distribution function of $S$ computed using ERA for Model B, $\tau = 0.8$. In the left panel, a standard deviation constraint $s_{\text{constr}} = 25$ is applied and attained by ERA. In the right panel, a constraint $s_{\text{constr}} = 58$ is attempted, but cannot be attained, resulting in a lower actual standard deviation $s_{\text{ERA}} = 43.2$; moreover, the distribution is not diatomic. The dotted lines are the optimal locations of the atoms from Theorem 5.

can be applied to compute the inverse df and the tail integral. In turn, the iterative procedure (3) can be applied to compute the expectile.

(ii) The most time-consuming quantity to compute was the unconstrained lower bound on the expectile, because the RA requires a discretization of the margins, i.e., calculating the skew-$t$ inverse df $d \cdot N$ times (each margin took about 10 min on an Intel i5 2.5 GHz desktop with $N = 10^4$). A similar calculation with Pareto dfs as in Table 1 was done for this discretization size. The maximum error using the expectation RA was 0.4% for $p = 0.99$ and 1.5% for $p = 0.999$; hence, this discretization size was deemed sufficient for our purposes.

(iii) Due to the mixture form of GH distributions, a faster method for discretizing the margins could be using a Monte Carlo sample. Since GH dfs can have heavy tails, a similar approach to the “expectation” discretization for RA was considered, specifically, rejecting any sample points that lie below $F_i^{-1}(1/N)$ or above $F_i^{-1}(1 - 1/N)$ and adding two points equal to the expectations over the corresponding intervals. However, this method resulted in a large variance over repeated trials, so the obtained bounds were not used.

4.4 Adding variance information

In this section, we consider factor models with additional variance information. We define the admissible class

$$\mathcal{S}(H_1, \ldots, H_d, s_W^2) = \{S \in \mathcal{S}(H_1, \ldots, H_d) : \text{Var}(S | W) = s_W^2\},$$
where the conditional variance is known for each outcome of $W$. Consider the problem:

$$\overline{e_f}^{s^2} = \sup \{ e_r(S) : S \in \mathcal{Z}(H_i, \ldots, H_d, s^2_W) \} .$$

**Theorem 8.** Let $e^*$ be given by

$$e^* = m + (2\tau - 1) E \left[ \frac{s^2_W}{\sqrt{s^2_W + (m_W - e^*)^2}} \right] ,$$

where $m := E[S]$ and $m_W := E[S|W]$. Then

$$\overline{e_f}^{s^2} \leq e^* .$$

**Proof.** Let $S_w \in \mathcal{Z}(H_i, \ldots, H_d, s^2_W)$. By De Vylder and Goovaerts (1982) (Case C$_{13}$), the upper bound$^6$ on the stop-loss premium over $S_w \in M_2(m_w, s^2_w)$ (recall the definition of $M_2$ in Equation (12)) is given by

$$E[(S_w - e)1_{(S_w > e)}] \leq \frac{1}{2} \left( m_w - e + \sqrt{s^2_w + (m_w - e)^2} \right) ,$$

which holds for any $e \in \mathbb{R}$. Using monotonicity and the tower property, we obtain an upper bound for the unconditional stop-loss premium $E[(S_w - e)1_{(S_w > e)}]$. Writing $\theta = (2\tau - 1)/(1-\tau)$ and invoking Lemma 2, we find that $e_r(S_w) \leq e^*$, where $e^*$ satisfies

$$e^* = m + \frac{\theta}{2} E \left[ m_w - e^* + \sqrt{s^2_w + (m_w - e^*)^2} \right]$$

$$= m + \frac{\theta}{2} (m - e^*) + \frac{\theta}{2} E \left[ \frac{s^2_w}{\sqrt{s^2_w + (m_w - e^*)^2}} \right] .$$

The stated equation for $e^*$ follows by rearranging. □

**Remark.** If the conditional variances $s^2_w$ are not known, but the total variance $s^2 := \text{Var}(S)$ is available, then we still have that $\overline{e_f}^{s^2} \leq \min \{ \overline{e_f}^{s^2}, B \} .

5 Dependence uncertainty spread comparison

In this section, the dependence uncertainty (DU) spreads of VaR, ES and the expectile are compared, where the DU spread for a risk measure $\rho$ is defined as

$$\overline{\rho}(S) - \underline{\rho}(S) .$$

---

$^6$This upper bound can also be derived using the reasoning in the proof of Theorem 5. Indeed, one shows that the upper bound is attained by a diatomic variable $X_\rho$ (with mean $m_w$ and variance $s^2_w$). Next, one optimizes over $\rho \in (0, 1)$ to obtain Equation (18).
Here, we focus on the Fréchet admissible class (only marginal dfs known); see Section 2. The
behavior of DU spreads of VaR and ES for large-dimensional portfolios is discussed in Em-
brechts et al. (2015). In order to make the resulting capital requirements similar under the
different risk measures, one could, for example, use the same level $\alpha = \beta = \tau$ for all three, but
multiply by different scaling factors.

The approach taken by the Basel Committee on Banking Supervision (BCBS, 2012), when
moving from VaR$_{0.99}$ as the risk measure for the trading book capital requirements to ES,
consists of adjusting the confidence level, apparently so that the numerical value of $ES_\beta(X)$ for a
normally-distributed rv $X$ matches VaR$_{0.99}(X) = 2.3263$. Doing so yields $\beta \approx 0.97423$, which
gets rounded to $\beta = 0.975$. Similarly, Bellini and Di Bernardino (2015) suggest using a para-
meter $\tau$, such that $e_{\tau}(X) = \text{VaR}_{0.99}(X)$ for $X \sim \Phi$; this yields $\tau \approx 0.99855$. Note that rounding
to 0.999 would give $e_{0.999} = 2.4358$; therefore, five significant digits $\tau = 0.99855$ will be used
in this section when comparing the expectile to the other risk measures. ES is not as sensitive
to the level $\beta$, and $ES_{0.975} = 2.3378$ is close enough to VaR$_{0.99}$.

In Figures 4 and 5, the DU spreads are plotted in the homogeneous case for different Pareto
and Student $t$ distributions, respectively, as functions of the dimension $d$. For the Pareto exam-
ple, VaR, $\overline{\text{VaR}}$, $\overline{\text{ES}}$ and $\overline{\varepsilon}$ are computed from the minimal elements in the convex order, obtained
using the methods from Bernard et al. (2014). $\overline{\text{ES}}$ and $\overline{\varepsilon}$ are obtained using the comonotonic
dependence structure.

Since the Student $t$ distribution is symmetric and unimodal, it is completely mixable (Rüschend-dorf and Uckelmann, 2002), so the lower bounds on ES and the expectile are equal to the mean.
To compute the lower bound on VaR, we apply RA. As the Student $t$ density is decreasing from
the median, Bernard et al. (2014) can again be applied for the upper bound on VaR. While VaR
and ES focus only on the losses, the expectile also takes the gains into account. Student $t$ has
two infinite tails, which leads to a larger DU spread for the expectile, especially in the most
heavy-tailed case. Overall, the results indicate that for the chosen adjusted significance levels,
the DU spread is typically the smallest for ES.

6 Final remarks

In the statistics literature, the expectile functional and its properties related to regressions are
well known. Recently, in the context of risk measurement, the expectile risk measure has also
been shown to have appealing theoretical properties. We contribute to the analysis of this new
risk measure by focusing on its properties under dependence uncertainty. We first summarize
and provide improved methods for computing bounds on the expectile of a portfolio in the
case of no information on dependence (unconstrained bounds) and prove analytic bounds for
a location-scale family. Next, we discuss the influence of dependence information on these
unconstrained bounds.

In this regard, we provide simple-to-compute bounds under an additional constraint on the portfolio variance and show that the upper bound can be considerably improved. By contrast, the unconstrained lower bound cannot be improved by only using the information on the first two moments.

Furthermore, we provide bounds in the factor-constrained case. A family of commonly-used distributions, the normal mean-variance mixtures, is considered as a special case. These models are particularly tractable, and we state the conditional best- and worst-case dependence structures explicitly. We note that due to the restriction on dependence that such a factor model induces, the lower $e_\tau$ bounds were significantly improved (for high values of $\tau$). The upper bounds are only slightly reduced, and the simple (unconstrained) upper bound based on subadditivity remains adequate for practical purposes.

We compare the dependence uncertainty spread of the expectile (i.e., the difference between the maximum and minimum possible value of the risk measure when only marginal information is used) with that of VaR and ES. We observe that the results are not favorable to the expectile.
While the expectile has been proposed as the elicitable counterpart to ES, it is not clear that this property is indeed crucial for back-testing, and evidence exists to the contrary (e.g., Costanzino and Curran (2015)). Hence, gaining elicitability may not justify the increase in the dependence uncertainty spread. However, alternative mathematical approaches exist to “provide a broadly similar level of risk capture” (BCBS, 2013a, p. 18), when moving to another risk measure (or even sticking with VaR), such as scaling. Although this makes the interpretation less clear (it was not clear for the expectile to begin with), it would allow reducing the confidence level, hence making statistical analysis more feasible and also reducing model uncertainty (see Table 2).

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**Negative dependence in matrix arrangement problems.**

*Submitted.*
Negative dependence
in matrix arrangement problems

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Abstract
Minimizing an arrangement increasing (AI) function with a matrix input over intra-column permutations is a difficult optimization problem of a combinatorial nature. Unlike maximization of AI functions (which is achieved by perfect positive dependence, namely, arranging all columns in an increasing order), minimization is a much more challenging problem due to the lack of a universal definition and construction of compensating arrangements in more than two dimensions. We consider AI functions with a special structure, which facilitates finding close-to-optimal solutions by employing the concept of $\Sigma$-countermonotonicity and the (Block) Rearrangement Algorithm. We show that many classical optimization problems, including stochastic crew scheduling and assembly of reliable systems, have objective functions with this structure, and illustrate with a numerical case study. This paves a path to obtaining approximate solutions for problems that have so far been considered intractable.

Keywords: Schur-convexity, negative dependence, scheduling, systems assembly, Archimedean copulas, Rearrangement Algorithm.

1 Introduction

In this paper we address a class of matrix arrangement problems. Let $S_n$ be the set of $n$-permutations and we write $x^\pi = (x_{\pi(1)}, \ldots, x_{\pi(n)})$ for a column vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\pi \in S_n$. For a given matrix $X = (x_1, \ldots, x_d) \in \mathbb{R}^{n \times d}$ and a target function $\phi : \mathbb{R}^{n \times d} \to \mathbb{R}$, the aim is to minimize or maximize

$$\phi(X^\pi) \text{ over } \pi \in (S_n)^d, \text{ where } X^\pi = (x_1^{\pi_1}, \ldots, x_d^{\pi_d}) \text{ for } \pi = (\pi_1, \ldots, \pi_d).$$

(1)
That is, one is allowed to permute the elements within each column of the matrix, but not to exchange elements between columns. In this paper, any minimization and maximization of an $\mathbb{R}^{n \times d} \to \mathbb{R}$ function refers to the problem in (1).

As one example, we consider an assembly line crew scheduling problem, where $n$ items are to be produced on $n$ assembly lines. To each line we must assign $d$ workers, specialized in different jobs. In total, there are $nd$ workers ($n$ from each of the $d$ specializations), and each has a different random completion time of their job. The objective is to maximize the probability of finishing all $n$ items within a given deadline by assigning the workers to the assembly lines optimally. The matrix $X$ in this example represents some known parameters of the distributions of the completion time for the $nd$ individual workers.

Another example is a systems assembly problem, where $n$ systems are to be assembled, each composed of $d$ components of different types, connected in parallel. There are $nd$ components available ($n$ of each type), with different individual probabilities of failure. The objective is to maximize the sum of system reliabilities, i.e. the expected number of systems that function satisfactorily. The matrix $X$ in this example represents the individual probabilities of failure for the $nd$ components.

In these contexts, a compensating arrangement is often desirable, in the sense that the fast workers compensate for the slow workers in each team, or the reliable components compensate for the less reliable ones within each system. The teams or systems, respectively, are represented by the rows of the matrix. The decision variable is a vector of permutations, corresponding to the arrangement of elements within each column of the input matrix.

The above two problems of a stochastic nature will be addressed in Section 4. For the moment, let us consider two very simple objective functions

$$\phi_1(X) = \max_{\pi \in S_n^d} \left\{ \sum_{j=1}^{d} x_{ij} \right\}, \quad X \in \mathbb{R}^{n \times d},$$

and

$$\phi_2(X) = \sum_{i=1}^{n} \left( \sum_{j=1}^{d} x_{ij} \right)^2, \quad X \in \mathbb{R}^{n \times d},$$

where throughout, we write $X = (x_1, \ldots, x_d) = (x_{ij})_{n \times d} \in \mathbb{R}^{n \times d}$.

To maximize $\phi_1(X^\pi)$ and $\phi_2(X^\pi)$ over $\pi \in (S_n)^d$ is a relatively easy task; the solution is simply to arrange $X^\pi$ as a similarly ordered matrix (see Section 2.1). However, the minimization of $\phi_1(X^\pi)$ and $\phi_2(X^\pi)$ over $\pi \in (S_n)^d$ is known to be a highly non-trivial task; for $d \geq 3$, the minimization of (2) was shown to be NP-complete in Hsu (1984); Coffman and Yannakakis (1984), see also Haus (2015) for a further analysis of the complexity of this problem. Note that the set of possible arrangements of a given matrix $X \in \mathbb{R}^{n \times d}$ is enormous (of size $(n!)^d$), corresponding to all possible intra-column permutations.

To handle the above minimization of $\phi_1(X^\pi)$ and $\phi_2(X^\pi)$, we borrow some recent developments from probabilistic dependence modeling, which help to solve a large class of problems
including the optimization of (2) and (3). A matrix in $\mathbb{R}^{n \times d}$ can be interpreted as a discrete random vector that is uniformly distributed over $n$ points in $\mathbb{R}^d$, where each column corresponds to a univariate margin and each row - to a possible outcome. Permuting the elements within the columns would therefore alter the joint distribution, but not the (univariate) marginal distributions. Thus, we can interpret different intra-column arrangements of a given matrix as different dependence structures of the corresponding random variable with fixed marginals. The solution to the maximization of $\phi_1(X^\pi)$ and $\phi_2(X^\pi)$ over $\pi \in (S_n)^d$ is naturally translated into dependence modeling as an extremal positive dependence. Although not properly formulated, one would expect a solution for the minimization of $\phi_1(X^\pi)$ and $\phi_2(X^\pi)$ to be translated as an extremal negative dependence. The extremal positive dependence, known as comonotonicity in dependence modeling, is a well-studied and well-understood concept. A considerable amount of research has applied the concept of positive multivariate dependence to optimization problems; see Shaked and Shanthikumar (1990, 1997); Derman et al. (1972); Colangelo et al. (2005).

In contrast, negative dependence has attracted less attention, largely due to the difficulties associated even with defining it; see Ebrahimi and Ghosh (1981); Block et al. (1982); Joag-Dev and Proschan (1983) for early examples. The recent paper Puccetti and Wang (2015b) contains an overview of extremal positive and negative dependence concepts, where a clear asymmetry between the two sides of the coin can be easily spotted.

The recent developments in dependence modeling offer some insights to the problems of minimizing $\phi_1(X^\pi)$ and $\phi_2(X^\pi)$ over $\pi \in (S_n)^d$. For example, it is clear that to minimize $\phi_1(X^\pi)$ one needs to find $\pi \in (S_n)^d$ such that the row-sums of $X^\pi$ are as similar as possible. The concepts of joint mixability (Wang and Wang, 2016) and $\Sigma$-countermonotonicity (Puccetti and Wang, 2015b) are introduced to characterize such dependence scenarios. In the area of financial risk management, a novel matrix rearrangement method has recently been introduced for obtaining dependence uncertainty bounds for risk measures; see Puccetti and Rüschendorf (2012); Embrechts et al. (2013); Puccetti (2013). These ideas from the risk management context have further applications, beyond computing bounds under dependence uncertainty between random variables with given marginal distributions. In particular, we can apply these new methods to problems in scheduling and reliability engineering as mentioned above, by interpreting the elements of the matrix as parameters of entities for which an optimal arrangement is to be found.

In this article, we focus on a general class of objective functions in (1): ones that have the form

$$
\phi(X) = g \left( \sum_{j=1}^d h_j(x_{1j}), \ldots, \sum_{j=1}^d h_j(x_{nj}) \right), \quad X \in \mathbb{R}^{n \times d},
$$

where $g$ is a Schur-convex function (see Section 2.2 for the definition) and (throughout) $h_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, d$, are monotone in the same direction. Here and in the following, the terms “monotone”, “increasing” and “decreasing” are used in the non-strict sense. Some other possible objective functions are also discussed in this paper, but to obtain a meaningful solution
we stick with (4).

For \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), denote by \( x_{[1]} \geq \cdots \geq x_{[n]} \) the components of \( x \) in a decreasing order; that is, the order statistics of \( x \). The objectives in (4) include the following subclass

\[
\phi(X) = L \left( \sum_{j=1}^{d} h_j(x_{ij}) \right), \quad X \in \mathbb{R}^{n \times d},
\]

where \( L : \mathbb{R}^n \to \mathbb{R} \), \( L(x) = \sum_{i=1}^{n} w_i x_{[i]} \), \( w_1 \geq \cdots \geq w_n \geq 0 \) and (throughout) \( f : \mathbb{R} \to \mathbb{R} \) is convex. In particular, two special cases of (5) are

\[
\phi(X) = \sum_{i=1}^{n} f \left( \sum_{j=1}^{d} h_j(x_{ij}) \right), \quad X \in \mathbb{R}^{n \times d},
\]

and

\[
\phi(X) = \max_{i=1,\ldots,n} f \left( \sum_{j=1}^{d} h_j(x_{ij}) \right), \quad X \in \mathbb{R}^{n \times d}.
\]

More examples and their associated practical problems are discussed later in the paper.

Natural choices of objective functions associated with the matrix arrangement problem considered in this article are discussed in Section 2. In Section 3, with the help from recent developments in dependence modeling and risk management, we show that the potential minimizers for objective functions of the form (4) belong to a smaller set of matrices having a \( \Sigma \)-countermonotonic structure, leading to a practical suboptimal solution via efficient algorithms. In Section 4, two problems in stochastic scheduling and systems assembly are shown to have objective functions of the form (6). We apply a heuristic which provides arrangements belonging to the smaller set of \( \Sigma \)-countermonotonic arrangements and demonstrate, by means of a numerical case study, that often any of these special arrangements provides a close-to-optimal solution.

## 2 Notation and preliminaries

An excellent reference for the concepts defined in this section is Marshall et al. (2011); we shall mostly follow the notation from this textbook. For a vector \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), let \( x^\downarrow = (x_{[1]}, \ldots, x_{[n]}) \) and \( x^\uparrow = (x_{[n]}, \ldots, x_{[1]}) \) be the decreasing and increasing arrangements of \( x \), respectively. Vectors \( x, y \in \mathbb{R}^n \) are called *similarly ordered* if \( (x^\uparrow, y^\downarrow) = (x^\downarrow, y^\uparrow) \) for some \( \pi \in S_n \), and *oppositely ordered* if \( (x^\uparrow, y^\uparrow) = (x^\downarrow, y^\downarrow) \) for some \( \pi \in S_n \). These notions correspond, respectively, to *comonotonicity* and *countermonotonicity* of random variables in probability theory (the matrix representation is relevant in a finite sample space; however, these concepts are also defined for general random variables). In the following subsections, we introduce arrangement increasing functions and the related notions of Schur-convexity, supermodularity, and total positivity.
2.1 Arrangement increasing functions

Boland and Proschan (1988) consider functions \( \phi : (\mathbb{R}^n)^d \to \mathbb{R} \) of \( d \) vector arguments in \( \mathbb{R}^n \) (equivalently, functions \( \phi : \mathbb{R}^{n \times d} \to \mathbb{R} \) with a matrix argument, writing \( X = (\mathbf{x}_1, \ldots, \mathbf{x}_d) \)), which increase in value as the components of the vector arguments become more similarly arranged. To rigorously define “more similarly arranged”, they introduce an equivalence relation \( =_a \) and a preordering \( \preceq_a \) between matrices. If \( X, Y \in \mathbb{R}^{n \times d} \) and \((\mathbf{x}_1^T, \ldots, \mathbf{x}_d^T) = Y \) for some \( \pi \in S_n \), then \( X =_a Y \). An operation called basic rearrangement consists of replacing the entries in two rows \( k, l \) \((1 \leq k < l \leq n)\) of a matrix \( X \) by their coordinate-wise minimum \((x_{kl} \land x_{kl}, \ldots, x_{kl} \land x_{ld})\) and maximum \((x_{kl} \lor x_{kl}, \ldots, x_{kl} \lor x_{ld})\), respectively. If matrix \( X \) can be transformed into matrix \( X' =_a Y \) using a sequence of basic rearrangements, then we write \( X \preceq_a Y \); this defines the arrangement preordering on \( \mathbb{R}^{n \times d} \).

**Definition 1.** A function \( \phi : \mathbb{R}^{n \times d} \to \mathbb{R} \) is called arrangement increasing if, for any \( X, Y \in \mathbb{R}^{n \times d} \),

\[
X \preceq_a Y \quad \text{implies} \quad \phi(X) \leq \phi(Y).
\]

Hence, arrangement increasing (AI) functions preserve the arrangement preordering. Three useful classes of AI functions are provided by the following results from Boland and Proschan (1988); see also (Marshall et al., 2011, Section 6.F).

(A) If \( \phi \) has the form \( \phi(X) = g(\mathbf{x}_1 + \ldots + \mathbf{x}_d) \), then \( \phi \) is AI if and only if \( g \) is Schur-convex.

(B) If \( \phi \) has the form \( \phi(X) = \sum_{i=1}^{n} g(x_{i1}, \ldots, x_{id}) \), then \( \phi \) is AI if and only if \( g \) is supermodular.

(C) If \( \phi \) has the form \( \phi(X) = \prod_{i=1}^{n} g(x_{i1}, \ldots, x_{id}) \), then \( \phi \) is AI if and only if \( g \) is MTP\(_2\).

These examples motivate the relevance of the properties of Schur-convexity, supermodularity and MTP\(_2\); we shall properly define them in Sections 2.2 and 2.3. We will frequently refer to the above three types of AI functions throughout the paper. Type (A) functions are indeed our main objectives in (4), albeit in (4) one allows further monotone transformations \( h_1, \ldots, h_d \). Taking the logarithm of a type (C) function yields a type (B) function, and functions of the form (6) where \( h_j = \text{id} \) belong to the intersection of types (A) and (B). The types (A)-(C) by no means provide a complete classification of all AI functions, but merely identify three large, possibly overlapping classes.

It is straightforward to obtain the arrangement that maximizes an AI function \( \phi \):

\[
\phi(X) \leq \phi((\mathbf{x}_1^1, \ldots, \mathbf{x}_d^1)) = \phi((\mathbf{x}_1^1, \ldots, \mathbf{x}_d^1)) \quad \text{for any} \quad X \in \mathbb{R}^{n \times d}.
\]  

This follows from the fact that any matrix \( X \) can be transformed into a comonotonic one by applying at most \( n(n - 1)/2 \) basic rearrangements. Arrangement increasing functions \( \phi \) of two
vectors, \( \phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) (case \( d = 2 \)), were first considered in Hollander et al. (1977) under the name *decreasing in transposition*. For two vector arguments it is also easy to obtain the minimizing arrangement - it is the countermonotonic i.e. oppositely ordered arrangement:

\[
\phi(x^1, y^1) = \phi(x^1, y^1) \leq \phi(x, y) \quad \text{for any} \quad x, y \in \mathbb{R}^n.
\]

For higher dimensions \( d \), however, the minimizing arrangement is not easily constructed, because an oppositely ordered arrangement is only defined for \( d = 2 \). Therefore, while maximizing AI functions is trivial, reversing the direction of the optimization problem often leads to a much more difficult problem. In the following, we investigate in more depth the classes of AI functions which can indeed be (approximately) minimized by applying recent developments in dependence modeling.

### 2.2 Majorization order and Schur-convex functions

In this section, we define the majorization preorder and Schur-convexity - concepts that are relevant to the type (A) of AI functions. For \( x, y \in \mathbb{R}^n \), we say that \( x \) majorizes \( y \), written \( x \succeq_m y \), if

\[
\sum_{i=1}^{k} x[i] \geq \sum_{i=1}^{k} y[i] \quad \text{for} \quad k = 1, \ldots, n - 1, \quad \text{and} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.
\]

**Definition 2.** A function \( g : D \subset \mathbb{R}^n \to \mathbb{R} \) is called Schur-convex if, for any \( x, y \in D \),

\[
x \succeq_m y \quad \text{implies} \quad g(x) \geq g(y).
\]

A function \( g : D \subset \mathbb{R}^n \to \mathbb{R} \) is called strictly Schur-convex if, for any \( x, y \in D \),

\[
x \succeq_m y \text{ and } y \not\succeq_m x \quad \text{imply} \quad g(x) > g(y).
\]

Reversing the inequality for \( g \) would define a Schur-concave function. Note that Schur-convexity requires that the function is symmetric (arguments are exchangeable). From Definition 2, it follows that a majorization-least\(^1\) element in some set \( D \subset \mathbb{R}^n \) minimizes Shur-convex functions over the set \( D \). For example, if the set \( D \) consists of the row-sum vectors of matrices in \( \{X^* : \pi \in (S_n)^d\} \) (column-rearrangements of \( X \in \mathbb{R}^{n \times d} \)), Day (1972) shows that the minimizing arrangement for \( d = 2 \) is the countermonotonic one (with row-sum vector \( x^* = x^1 + x^2 \)). A comprehensive account of the applications of majorization order in statistics, probability and reliability theory is (Marshall et al., 2011, Chapters 11-13).

Examples of Schur-convex functions include (see Proposition 1 on the opposite page)

\(^1\)A least element \( v \) of a preordered set \((A, \succ)\) satisfies \( u \succ v \) for all \( u \in A \). For some preordered sets, such an element may not exist.
a) $g(x) = \max\{x_1, \ldots, x_n\},$

b) $g(x) = \sum_{i=1}^{n} f(x_i), \text{ where } f \text{ is convex},$

c) $g(x) = \prod_{i=1}^{n} f(x_i), \text{ where } f \text{ is log-convex}.$

d) $g(x) = \sum_{i=1}^{n} w_i x_i, \text{ where } w_1 \geq \ldots \geq w_n \geq 0.$

e) $g(x) = \sum_{i=1}^{n} w_i f(x_i), \text{ where } w_1 \geq \ldots \geq w_n \geq 0 \text{ and } f \text{ is increasing and convex}.$

Observe that substituting b) into a type (A) of AI functions, we obtain a function of the form (6). Obviously, applying an increasing transformation to a Schur-convex function would preserve the Schur-convexity. We leave such transformations out of our discussion, since they do not play a role in optimization problems. By taking a logarithm of c), we arrive at b). Functions of the form d) are a special case of $L$-statistics, i.e. linear combinations of order statistics. The form in e) is the most general, in the sense that it includes a), b) and d) as special cases and c) with a log-transform. The Schur-convexity of functions in e) is characterized in the following proposition.

**Proposition 1.** Let $g : \mathbb{R}^n \to \mathbb{R}$ be of the form

$$g(x) = \sum_{i=1}^{n} w_i f(x_i), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

where $w_1, \ldots, w_n \geq 0$ and $f : \mathbb{R} \to \mathbb{R}$ is increasing. The following are equivalent.

(i) $g$ is Schur-convex.

(ii) $g$ is convex.

(iii) $w_1 \geq \ldots \geq w_n$ and $f$ is convex.

Functions of the form (9) belong to the family of Rank-Dependent Expected Utilities (RDEU); see Quiggin (1993). The majorization order, translated into probability theory, is equivalent to convex order between discrete random variables. From there, the equivalence of (i) (called strong risk aversion for RDEU) and (iii) is given in Chew et al. (1987); see also Schmidt and Zank (2008). A precise formulation of (i)$\iff$(ii)$\iff$(iii) is given in Theorem 5.1 of Mao and Wang (2015).

The maximum operator is a special case of (9), taking $f = \text{id}$ and weights $(1, 0, \ldots, 0)$. In Quantitative Risk Management, the Expected Shortfall at some level $\alpha \in (0, 1)$ of a random loss, whose distribution is represented by a vector $x$ (as explained in the Introduction), is of the form (9) by taking equal weights $1/(1-\alpha)$ for the first $n(1-\alpha)$ entries (for simplicity assume $a_\alpha \in \mathbb{Z}$). Matrix arrangement problems where type (A) objectives with $g = \max$ or $g = -\min(\cdot) =$...
max(− · ) are minimized, have been investigated in Puccetti and Rüschendorf (2012), and the Expected Shortfall case in Puccetti (2013). In the above examples, the matrix would correspond to a random vector with a discrete uniform distribution, the sum of columns - to the aggregate loss, and the maximization/minimization problems would correspond to determining the so-called *dependence uncertainty* bounds (recall that different arrangements of the input matrix represent different dependence structures between the components of the random vector). We do not further pursue these types of objectives in this article.

### 2.3 Supermodularity and total positivity

In this section, we define the properties of submodularity and MTP<sub>2</sub>, which are relevant to arrangement increasing functions of types (B) and (C), respectively.

**Definition 3.** A function \( g : D \subset \mathbb{R}^d \rightarrow \mathbb{R} \) is said to be supermodular if it satisfies

\[
g(z \wedge w) + g(z \vee w) \geq g(z) + g(w)
\]

for all \( z, w \in D \),

where \( \wedge \) and \( \vee \) denote the component-wise minimum and maximum, respectively.

Supermodular functions are sometimes called *L*-superadditive, where \( L \) stands for “lattice”. Reversing the inequality in Definition 3 yields submodularity. In reliability theory, Block et al. (1989) explains the intuition that the supermodular/submodular property for a structure function describes whether the system is more series-like or more parallel-like; see Section 4.2 for further details. Examples of supermodular functions \((z_1, \ldots, z_d) \mapsto g(z_1, \ldots, z_d)\) include

a) \( f(z_1 + \ldots + z_d) \), if and only if \( f \) is convex,

b) for a twice differentiable \( f \), \( f(\prod_{i=1}^{d} z_i) \), if and only if \( f'(z) + z f''(z) \geq 0 \),

c) \( f(\min_{j=1}^{d} z_j) \) and \( f(- \max_{j=1}^{d} z_j) \), where \( f \) is non-decreasing,

d) elementary symmetric polynomials,

e) cumulative distribution functions (cdfs).

Observe that if we substitute a) into type (B) of AI functions, we obtain a function of the form (6). Expressing \( f(\prod_{i=1}^{d} z_i) = f(\exp(\log(z_1) + \ldots + \log(z_d))) \), we effectively reduce example b) to a); in particular, if the conditions for \( f \) in b) hold, then \( f \circ \exp \) is convex, hence we have again recovered an objective of the structure (6). If we substitute e) into type (B) of AI functions, the sum of cdfs can be interpreted as the expected number of events that occur; see the example in Section 4.2.

Lorentz (1953) shows that the comonotonic arrangement maximizes the sum of a supermodular function applied to the rows of a matrix; this can also be seen by applying a sequence
of basic rearrangements (see Section 2). Furthermore, for \( d = 2 \) columns, the minimizing arrangement is the countermonotonic one; see (Marshall et al., 2011, Sections 6.D-E) for these and further properties and examples. For \( d \geq 3 \) columns, finding the minimizing arrangement is more challenging; this is related to the fact that the “opposite” operation of a basic rearrangement is no longer well-defined.

Multivariate total positivity of order 2 (MTP₂) is the log-analogue of supermodularity, namely, a positive-valued function \( g \) is MTP₂ if and only if \( \log g \) is supermodular.

**Definition 4.** A function \( g : D \subset \mathbb{R}^d \to \mathbb{R}_+ \) is said to be **multivariate totally positive (MTP₂)**, if

\[
g(z \wedge w)g(z \vee w) \geq g(z)g(w) \quad \text{for all} \quad z, w \in D.
\]

If \( g \) represents a density function, then the MTP₂ property is often seen as a characteristic of multivariate positive dependence; see Karlin and Rinott (1980a); Hu et al. (2003); Colangelo et al. (2005). Reversing the inequality in Definition 4 defines a **multivariate reverse regular of order 2 (MRR₂)** function. This property is considered as a characteristic of negative dependence; see Karlin and Rinott (1980b); Block et al. (1982); Joag-Dev and Proschan (1983). In statistics, the product of densities represents the likelihood of independent (vector-valued) observations. Alternatively, \( g \) may represent a cumulative distribution function, in which case the product is the joint probability of independent events occurring. A useful result (Joe, 1997, p. 55) states: if \( g \) is an MTP₂ density, then its cdf and survival function are also MTP₂. For applications of MTP₂ in reliability theory, see Shaked and Shanthikumar (1990).

It was observed by John Napier in *Mirifici Logarithmorum Canonis Descriptio* (1614) that working with addition is easier than multiplication, and this can be achieved by applying the logarithm. This is also well-known to statisticians, who typically prefer to work with log-likelihoods. Hence, it may be easier to transform type (C) AI functions into type (B) by taking the logarithm, and then continue working with sums of supermodular functions.

3 Main result

3.1 Sigma-countermonotonicity

As explained earlier, matrix arrangement problems are challenging problems of a combinatorial nature. The set of possible arrangements of a matrix \( X \in \mathbb{R}^{n \times d} \) is of size \( (nt)^d \), so finding the solution by brute force search is not possible in practice. Hence, reducing the set of possible optimizers by identifying a necessary property would be advantageous. Puccetti and Wang (2015b) introduce the property of Σ-countermonotonicity for general multivariate random variables. Here, we give the finite (discrete) formulation.
Definition 5. A matrix $X \in \mathbb{R}^{n \times d}$ is said to be $\Sigma$-countermonotonic, if $\sum_{j \in J} x_j$ and $\sum_{j \notin J} x_j$ are oppositely ordered for all nonempty $J \subset \{1, \ldots, d\}$.

A weaker property than $\Sigma$-countermonotonic is column oppositely ordered (COO), which corresponds to considering only singleton sets $J = \{j\}$, $j = 1, \ldots, d$, in Definition 5. In order to apply these properties in the minimization problem with the objective (4), we first transform the entries of the $j$th column of the input matrix $X$ by the function $h_j$, $j = 1, \ldots, d$, and denote the resulting matrix by $H(X)$ (or simply $H$), where $(H)_{ij} = h_j(x_{ij})$. Then, we can work with arrangements of $H$ instead, defining

$$\tilde{\phi}(H) := g\left(\sum_{j=1}^d (H)_{1j}, \ldots, \sum_{j=1}^d (H)_{nj}\right) \quad (= \phi(X)).$$

The following theorem provides the theoretical basis for restricting the minimization problem with objective $\tilde{\phi}$ to $\Sigma$-countermonotonic arrangements of $H$.

Theorem 2. For a given matrix $X \in \mathbb{R}^{n \times d}$ and a function $\phi$ of the form (4), there exists an arrangement $\pi^* \in (S_n)^d$ that minimizes $\phi(X^{\pi^*})$, such that $H(X^{\pi^*}) = H(X)^{\pi^*}$ is $\Sigma$-countermonotonic. Furthermore, if $g$ in (4) is strictly Schur-convex, then $H(X)^{\pi^*}$ is $\Sigma$-countermonotonic for all minimizers $\pi^*$.

The proof of Theorem 2 is similar to those of Propositions 2.4 and 2.6 of Puccetti and R"uschendorf (2015). One difference is that we use the stronger property of $\Sigma$-countermonotonicity instead of COO. The other difference is permitting general monotone transformations $h_j$ in (5). This allows Theorem 2 to include more general objective functions than the above mentioned Proposition 2.4, while still including all known explicit and non-trivial objectives considered in Proposition 2.6 of Puccetti and R"uschendorf (2015); see Section 3.2 for details. In Section 3.3, we provide a further explicit class of objective functions for which Theorem 2 applies, and that, to our knowledge, has so far not been considered in the context of minimizing AI functions.

Proof of Theorem 2. Since there are finitely many arrangements of the matrix $X$, there exists a minimizing arrangement $X^\ast$. If the corresponding $H(X^\ast) = (h_1, \ldots, h_d)$ is not $\Sigma$-countermonotonic, by definition, there exists a nonempty $J \subset \{1, \ldots, d\}$ such that $a = \sum_{j \in J} h_j$ and $b = \sum_{j \notin J} h_j$ are not oppositely ordered. Define $\tilde{\phi}_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{\phi}_2(x, y) = g(x_1 + y_1, \ldots, x_n + y_n)$$

and note that $\tilde{\phi}(H) = \tilde{\phi}_2(\sum_{j \in J} h_j, \sum_{j \notin J} h_j)$ for any $J$. Let $\pi_a, \pi_b \in S_n$ be such that $a^{\pi_a} = a^\uparrow$ and $b^{\pi_b} = b^\downarrow$. Denote $\tilde{H} = (\tilde{h}_1, \ldots, \tilde{h}_d)$, where $\tilde{h}_j = h_j^{\pi_a}$ for $j \in J$ and $\tilde{h}_j = h_j^{\pi_b}$ for $j \notin J$. Since $g$ is Schur-convex and $a + b \succ_m a^\uparrow + b^\downarrow$ (Day, 1972), we have $\tilde{\phi}_2(a, b) \geq \tilde{\phi}_2(a^\uparrow, b^\downarrow)$ (Definition 2). Thus $\tilde{\phi}(H) \geq \tilde{\phi}(\tilde{H})$. For any arrangement, we can also compute a score function

$$V(H) := \sum_{i=1}^n (h_{i1} + \ldots + h_{id})^2.$$
Since \( \mathbf{a} \) and \( \mathbf{b} \) are not oppositely ordered by assumption, it follows that \( V(H) > V(\tilde{H}) \). As the set of arrangements of \( H \) is finite, after a finite number of iterations from \( H \) to \( \tilde{H} \), we obtain a matrix \( \hat{H} \) that minimizes the score \( V(\hat{H}) \). This matrix \( \hat{H} \) must be \( \Sigma \)-countermonotonic, since \( V(\hat{H}) \) is minimized. Furthermore, if \( g \) in (4) is strictly Schur-convex, the same argument shows that any minimizer \( H \) must be \( \Sigma \)-countermonotonic to begin with. □

In particular, the above result means that instead of minimizing \( \phi(X) \) over all arrangements of \( X \), we can minimize \( \tilde{\phi}(\tilde{H}) \) over \( \Sigma \)-countermonotonic arrangements \( \tilde{H} \) of \( H(X) \). In the following section, we connect the theoretical properties of \( \Sigma \)-countermonotonicity and COO with practical algorithms for obtaining arrangements that satisfy these properties.

### 3.2 The Rearrangement Algorithm

A procedure for obtaining a COO arrangement of a given matrix, called the Rearrangement Algorithm (RA), is described in Puccetti and Rüschendorf (2012); Embrechts et al. (2013); Puccetti (2013). The original application of the RA is computing dependence uncertainty bounds for risk measures of sums of random variables with given univariate distributions. In the risk management context, the algorithm has proven to be easily applicable, and to reliably give close-to-optimal bounds (testing in cases when the analytical solution is known). Improvements on the RA, e.g. regarding the stopping conditions, are given in Hofert et al. (2015); for other recent updates, see [sites.google.com/site/RearrangementAlgorithm].

Recently, a modification called the Block Rearrangement Algorithm has been introduced, which finds a \( \Sigma \)-countermonotonic arrangement of a given matrix; see Remark 4.1 in Bernard et al. (2015a), and further analysis in Bernard and McLeish (2015); Boudt et al. (2016). The basic iteration of this algorithm is indeed as described in the proof of Theorem 2. Thus, for type (A) AI functions and, more generally, objectives in (4), the Block RA can directly be applied to obtain an approximate solution.

As for type (B) AI functions, Puccetti and Rüschendorf (2015) generalize the Rearrangement Algorithm for a special class of type (B) functions, in which the supermodular row-aggregation function \( g : \mathbb{R}^d \rightarrow \mathbb{R} \) is decomposable.

**Definition 6.** A supermodular function \( g : \mathbb{R}^d \rightarrow \mathbb{R} \) is decomposable if it is coordinate-wise monotone, and there exist supermodular functions \( g^{(2)} : \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( g^{(d-1)} : \mathbb{R}^{d-1} \rightarrow \mathbb{R} \) such that

\[
g(z_1, \ldots, z_d) = g^{(2)}(z_j, g^{(d-1)}(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_d)), \quad j = 1, \ldots, d. \tag{10}
\]

For a decomposable supermodular function \( g \), minimizing type (B) AI functions is indeed a two-dimensional problem. Expression (10) may seem considerably more general than the
row-aggregating function used in (6), namely,

$$g(z_1, \ldots, z_d) = f(\sum_{j=1}^{d} h_j(z_j)).$$  \hspace{1cm} (11)$$

However, the only explicit examples given by Puccetti and Rüschendorf (2015) satisfying the decomposition (10) are the sum, the product, the minimum and (minus) the maximum operators.

**Remark 1.** It is perhaps useful to note that minimizing the sum of row-minima has a trivial solution: putting the $n$ smallest elements of the matrix, one in each row. Analogously, for the sum of (minus) row-maxima, we put the $n$ largest elements in different rows; see also Section 4.1 in Puccetti and Rüschendorf (2012). Hence, while these examples cannot be directly expressed in the form (4), it is straightforward to obtain the solution, so we do not further discuss these special cases of submodular functions.

In Section 3.3, we identify another useful class of functions that are submodular and have the structure (10): the class of multivariate distributions with an Archimedean copula. This is a flexible subclass of supermodular functions, includes the product function, and is typically non-trivial to minimize. Interestingly, we notice that this class of supermodular functions also admits the form (11), and substituting these functions into type (B) of AI functions, we recover the target (6).

### 3.3 Archimedean copula

As explained in Section 3.2, in order to apply the RA to minimize the sum of supermodular functions of rows (namely, type (B) AI functions), a specific structure (10) for the supermodular function is required. In this section, we focus on the example e) from Section 2.3: multivariate cumulative distribution functions. All cdfs are supermodular, but those, for which the dependence is modeled by an Archimedean copula, additionally have the desired decomposition structure (10), as well as (11).

In order to introduce Archimedean copulas, for the reader’s convenience, we provide a definition and a proposition from McNeil and Nešlehová (2009).

**Definition 7.** A function $\psi : [0, \infty) \to \mathbb{R}$ is called $d$-monotone, where $d \geq 2$, if it is differentiable up to the order $d - 2$ and the derivatives satisfy

$$(-1)^k \psi^{(k)}(x) \geq 0, \quad k = 0, \ldots, d - 2,$$

for any $x \geq 0$ and further if $(-1)^{d-2} \psi^{(d-2)}$ is decreasing and convex. If $\psi$ has derivatives of all orders and if $(-1)^k \psi^{(k)}(x) \geq 0$ for any $x \geq 0$, then $\psi$ is called completely monotone.

**Proposition 3.** Let $\psi : [0, \infty) \to [0, 1]$ be a $d$-monotone function such that $\psi(0) = 1$ and $\lim_{x \to \infty} \psi(x) = 0$. Then $C_{\psi} : [0, 1]^d \to [0, 1]$ defined by

$$C_{\psi}(u_1, \ldots, u_d) = \psi(\psi^{-1}(u_1) + \ldots + \psi^{-1}(u_d))$$  \hspace{1cm} (12)
is a copula. If $\psi$ is completely monotone, then (12) is a copula for any $d \geq 2$.

Any copula of the form (12) is called Archimedean, and the function $\psi$ is the generator of the copula. Note that in the literature, the notation of $\psi$ and $\psi^{-1}$ is sometimes swapped, e.g. in Nelsen (1999). For further details on Archimedean copulas, and copulas in general, see McNeil and Nešlehová (2009); Joe (2014). Distribution functions with an Archimedean copula have the structure (10) (after a marginal transformation) as well as that of (11), as shown in the following proposition.

**Proposition 4.** Let $F : \mathbb{R}^d \to [0, 1]$ be a distribution function with an Archimedean copula. Then $F$ admits the following form:

$$F(z_1, \ldots, z_d) = g(F_1(z_1), \ldots, F_d(z_d))$$

for some decomposable supermodular function $g$, where $F_j$ denote the margins of $F$, $j = 1, \ldots, d$. Furthermore, $F$ admits the form

$$F(z_1, \ldots, z_d) = f(\sum_{j=1}^d h_j(z_j)),$$

for some convex function $f$ and decreasing functions $h_j$, $j = 1, \ldots, d$.

**Proof.** Sklar’s theorem states that any multivariate cdf $F$ can be expressed in terms of its margins $F_j$, $j = 1, \ldots, d$, and a copula $C : [0, 1]^d \to [0, 1]$, that is,

$$F(z_1, \ldots, z_d) = C(F_1(z_1), \ldots, F_d(z_d)).$$

Let $\psi$ be the generator of the Archimedean copula $C$ and let $C^{(k)}_\psi$ be the $k$-variate Archimedean copula with generator $\psi$. Observe that $\psi^{-1} \circ \psi(z) = z$ for $z \in \{x : \psi(x) > 0\}$, since $\psi$ is strictly decreasing on its support; see Embrechts and Hofert (2013) for properties of inverses. By definition,

$$C_\psi(u_1, \ldots, u_d) = \psi(\psi^{-1}(u_1) + \psi^{-1}(u_d))$$

$$= \psi(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \psi^{-1}(u_3) + \ldots + \psi^{-1}(u_d))$$

$$= C^{(2)}_\psi(u_1, C^{(d-1)}_\psi(u_2, \ldots, u_d)).$$

By symmetry, for each $j = 1, \ldots, d$, we have

$$F(z_1, \ldots, z_d) = C^{(2)}_\psi(F_j(z_j), \ldots, F_d(z_d)),$$

which admits the form (13), since any copula is supermodular. Taking $h_j = \psi^{-1} \circ F_j$, $j = 1, \ldots, d$, and $f = \psi$ in (15), we obtain (14). From Definition 7, $f$ is convex and $h_j$ are decreasing. \qed
From Proposition 4, it follows that in order to minimize
\[ \phi(X) = \sum_{i=1}^{n} F(x_{i1}, \ldots, x_{id}), \quad X \in \mathbb{R}_{+}^{n \times d}, \]  
(16)
where \( F \) has an Archimedean copula, we can apply the Block RA to the matrix \( H(X) \) to obtain a \( \Sigma \)-countermonotonic arrangement; we shall see in Section 4.2 that this often provides an approximate solution. A further useful property which connects Archimedean copulas to the optimization with objectives of type (C) is given in Müller and Scarsini (2005).

**Lemma 5.** If \( \psi \) is completely monotone, then \( C_{\psi} : [0, 1]^d \rightarrow [0, 1] \) is MTP2 for any \( d \geq 2 \).

In fact, this result is proved by showing that \( \log \psi \) is a convex function for completely monotone \( \psi \). This implies that type (C) AI functions with \( g \) \( F \), where \( F \) admits a corresponding Archimedean copula, have the structure (6) after taking the logarithm. Thus, we can also apply Theorem 2 and the Block RA in this case.

Notice that the \( \Sigma \)-countermonotonicity condition for \( F \) with an Archimedean copula is equivalent to \( F_j(x_{ij}) \) and \( F_{j'}(x_{i'j'}) \), \( i = 1, \ldots, n \), being oppositely ordered for all nonempty \( J \subseteq \{1, \ldots, d\} \) (where \( F_j = F_{j_1, \ldots, j_k} \) are multivariate marginal distributions and \( x_{ij} = (x_{ij_1}, \ldots, x_{ij_k}) \) for \( J = \{j_1, \ldots, j_k\} \)). One may wonder whether the necessity of this structure remains true for minimizing (16) with an arbitrary distribution function \( F \) (not necessarily associated with an Archimedean copula). As the following example demonstrates, this is in general not true; restricting to the class of distributions associated with an Archimedean copula is important for our approach to work.

**Table 1:** Probability mass function and cdf of a distribution of \((X_1, X_2, X_3)\) on \([0, 1]^3\).

<table>
<thead>
<tr>
<th>( X_j = x_i ), ( i = 1, 2, 3 )</th>
<th>( P(X_1 = x_1) )</th>
<th>( P(X_1 \leq x_1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_j )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0.3</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Consider a distribution on \([0, 1]^3\) as given in Table 1, and suppose that the input matrix and the corresponding marginal distributions are
\[
X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad F_1(0) = 0.5 \quad F_{2,3}(0, 1) = 0.7 \\
F_1(1) = 1 \quad F_{2,3}(1, 0) = 0.6
\]
The marginal probabilities are oppositely ordered, and the corresponding objective value of type (B) is \( F(0, 0, 1) + F(1, 1, 0) = 0.3 + 0.6 = 0.9 \). Arranging the first column *similarly*, however,
would give a smaller objective value, \( F(1, 0, 1) + F(0, 1, 0) = 0.7 + 0.1 = 0.8 \). In fact, this is the minimizing arrangement, moreover \( F_{1,2} \) and \( F_3 \) are also similarly ordered for this arrangement. This shows that for general dependence structures, the arrangement that minimizes the sum of joint cdfs may not have oppositely ordered marginal cdfs. Note that the arrangement that maximizes the sum of cdfs is the comonotonic one, with \( F(0, 0, 0) + F(1, 1, 1) = 0 + 1 = 1 \). This is true for any choice of the distribution function, since AI functions are always maximized by the comonotonic arrangement; see (8).

4 Scheduling and systems assembly problems

In the following subsections, we provide two classical examples of scheduling and systems assembly problems. With some careful analysis, we show that they belong to the class of minimization problems with an objective function of the form (6). The (Block) RA is applied to obtain approximate solutions as discussed in Section 3, and the results are presented.

4.1 Assembly line crew scheduling problem

The deterministic version of the assembly line crew scheduling (ALCS) problem was introduced in Hsu (1984); in the following, we describe a stochastic version of this problem. ALCS considers the production of \( n \) items on \( n \) parallel assembly lines (one item on each line), where each item requires \( d \) different operations. There are \( n \) workers specialized in each of the \( d \) operations, hence \( nd \) workers in total. The time taken by worker \( i \) of specialty \( j \) to complete the operation is a random variable \( T_{ij} \sim F^\theta_{ij} \), where \( F^\theta_j \) is a distribution function (df) that models the completion time of operation \( j \), parameterized by \( \theta \in \mathbb{R} \). All the rvs \( T_{ij}, i = 1, \ldots, n, j = 1, \ldots, d \), are assumed to be independent. Thus, the input for an instance of this problem is the matrix of parameters \( \Theta \in \mathbb{R}^{n \times d} \). We are allowed to assign the workers within each specialty to the \( n \) assembly lines in any order. If the \( i \)-th row of \( \Theta \) represents the \( i \)-th assembly line, then the assignment of workers corresponds to permuting the entries within each column (specialty). Denote by \( C_i = \sum_{j=1}^d T_{ij} \) the completion time of the \( i \)-th item, and by \( C_{\max} = \max_{i=1}^n C_i \) the makespan of the project. We will consider three different objectives for this stochastic problem.

- Maximize the probability of meeting a deadline \( D \in \mathbb{R} \),

\[
P(C_{\max} \leq D) = \prod_{i=1}^n P(C_i \leq D) = \prod_{i=1}^n \left( \prod_{j=1}^d F^\theta_{ij} (D) \right),
\]

where \( F*G \) denotes the df of the sum of independent variables with dfs \( F \) and \( G \), respectively.
• Minimize the expected makespan

\[ E[C_{\text{max}}] = \int_{t=0}^{\infty} P(C_{\text{max}} > t) \, dt. \]  

(18)

• Maximize the expected number of items finished within the deadline \( D \in \mathbb{R} \),

\[ E \left[ \sum_{i=1}^{n} \mathbb{I}_{\{C_i \leq D\}} \right] = \sum_{i=1}^{n} P(C_i \leq D) = \sum_{i=1}^{n} (\Phi_{\theta_{ij}}^d)(D) \cdot \sigma_{ij}^2. \]  

(19)

Suppose the individual completion times of operation \( j \) follow the Normal distribution with mean \( \theta \) and variance \( \sigma_j^2 \), namely \( F_{\theta_j}^j(\cdot) = \Phi((\cdot - \theta)/\sigma_j) \) for \( j = 1, \ldots, d \), and let \( \sigma_+^2 = \sum_{j=1}^{d} \sigma_j^2 \), \( \theta_+ = \sum_{j=1}^{d} \theta_{ij} \). Then \( C_i \sim \mathcal{N}(\theta_+, \sigma_+^2) \) for \( i = 1, \ldots, n \). Note that the same approach would also be applicable if the workers on the same assembly line had dependent completion times, modeled by a multivariate Normal distribution with covariance matrix \( \Sigma \), by taking \( \sigma_+^2 = \mathbf{1}^\top \Sigma \mathbf{1} \).

Since the logarithm is an increasing function, maximizing (17) is equivalent to maximizing

\[ \log(P(C_{\text{max}} \leq D)) = \sum_{i=1}^{n} \log \Phi((D - \theta_+)/\sigma_+). \]  

(20)

Furthermore, the Normal df is log-concave (see Bagnoli and Bergstrom (2005) for this and other examples), so (20) is a sum of a concave function evaluated at row-sums of \( \Theta \).

Hence, the objective (17) and its log-counterpart (20) are Schur-concave, corresponding to the cases c) and b) (with a negative sign, to switch from convex to concave) in Section 2.2, respectively. Thus, finding an arrangement of \( \Theta \) which makes the row-sums “small” in majorization order, yields a larger objective value (17). Moreover, the objective is then large for all deadlines \( D = t \) simultaneously (i.e. large in stochastic order), hence the objective (18), being the integral of tail probabilities, is “small” for this permutation. Therefore, a row-sum vector that is small in majorization order is desirable with respect to both objectives (17) and (18). The objective (19) does not obey the majorization order, because \( \Phi \) is not convex on its domain. For example, suppose that \( \theta_{ij} = i \) for \( i = 1, \ldots, 3 \) and \( j = 1, 2; \) \( \sigma_+^2 = 1 \) and \( D = 4 \). Then the countermonotonic arrangement is

\[ \Theta = \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix} \Rightarrow \sum_{i=1}^{3} P(C_i \leq 4) = 1.5, \quad \text{while} \quad \Theta = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{pmatrix} \Rightarrow \sum_{i=1}^{3} P(C_i \leq 4) = 1.71. \]

An alternative heuristic for functions that are convex on the left and concave on the right would be to take the \( k \in \{0, 1, \ldots, n\} \) slowest workers in each specialty and arrange them comonotonically, while the fastest \( (n - k) \) workers \( \Sigma \)-countermonotonically (which is optimal in the example above). However, we do not pursue this idea further and leave it for future research.
For a simple numerical example, we choose dimensions $n = 15, d = 4$; note that even for such small dimensions a brute force search of all $(15!)^3 \approx 10^{16}$ arrangements is out of the question. We take $\sigma_i^j = j, j = 1, \ldots, d$, and generate the matrix $\Theta$ of location parameters by sampling $\theta_{ij} \sim \sigma_j(5 + 5 \text{Beta}(2, 5))$; the entries of this matrix are then fixed for the remainder of the case study (the choice of the parameters is such that the completion times have a negligible probability of being negative). Thereafter, we consider 1000 random (intra-column) arrangements of the matrix $\Theta$ and compute the value of objectives (17)-(19) with deadline $D = 45$. A histogram of the objective (17) values is given in Figure 1. Taking each random arrangement as the input, we apply the (Block) RA to obtain a $\Sigma$-countermonotonic or COO matrix, and again compute the corresponding probability of meeting the project deadline. We observe that in all cases the resulting arrangements consistently yield high probabilities of success, supporting the assertion that $\Sigma$-countermonotonic matrices are often close-to-optimal. The COO matrices are also close, but show slightly more variation (see Figure 1, right panel, as well as Table 2), since it is a larger set of arrangements. Table 3 in the Appendix shows a $\Sigma$-countermonotonic arrangement of $\Theta$ with the corresponding probabilities of exceeding the deadline.

Figure 1: Left panel: Histogram of $P(C_{\text{max}} \leq 45)$ corresponding to 1000 random initial arrangements of $\Theta$, as well as the result after applying the (Block) Rearrangement Algorithm in order to obtain $\Sigma$-countermonotonic or column oppositely ordered matrices. Worst case is given by the comonotonic arrangement. Right panel: Close-up of the best cases.

Figure 2 shows the expected number of items finished on time (left panel) and the expected makespan (right panel); see also Table 2 for the range of observed values for these objectives. Again, we notice that the $\Sigma$-countermonotonic and COO arrangements consistently give solutions that are significantly better than the random arrangements, with the $\Sigma$-countermonotonic ones showing less variation.
Table 2: Range (minimum and maximum over 1000 experiments) of the objective functions (17)-(19) for \( \Sigma \)-countermonotonic, COO and random arrangements for the matrix \( \Theta \) for the assembly line crew scheduling example with \( n = 15, d = 4 \) and deadline \( D = 45 \). The last column shows the results for the comonotonic arrangement of \( \Theta \) (worst case).

<table>
<thead>
<tr>
<th>Objective</th>
<th>Block RA</th>
<th>RA</th>
<th>random</th>
<th>comonotonic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \max P(C_{\text{max}} \leq 45) )</td>
<td>0.6691</td>
<td>0.6670</td>
<td>0.0430</td>
<td>0.0101</td>
</tr>
<tr>
<td>( \max \sum_{i=1}^{15} P(C_i \leq 45) )</td>
<td>14.6037</td>
<td>14.6007</td>
<td>13.1289</td>
<td>12.6084</td>
</tr>
<tr>
<td>( \min E[C_{\text{max}}] )</td>
<td>44.3512</td>
<td>44.3513</td>
<td>44.8328</td>
<td>50.3584</td>
</tr>
</tbody>
</table>

Figure 2: *Left panel:* Histogram of \( \sum_{i=1}^{15} P(C_i \leq 45) \), based on 1000 random initial arrangements of \( \Theta \), after applying the (Block) Rearrangement Algorithm in order to obtain \( \Sigma \)-countermonotonic or column oppositely ordered matrices, respectively. *Right panel:* Histogram for the objective \( E[C_{\text{max}}] \).

4.2 Systems assembly problem

In reliability theory, a system that is composed of \( d \) individual components, each of which may be functioning properly or not, is characterized by its structure function \( \rho : \{0,1\}^d \rightarrow \{0,1\} \). Denoting the state of component \( j \) by \( x_j \in \{0,1\} \) (representing failed or functional, respectively), \( \rho(x_1,\ldots,x_d) \in \{0,1\} \) returns the state of the system. For example, a *series system* is described by

\[
\rho(x_1,\ldots,x_d) = \min\{x_1,\ldots,x_d\} = \prod_{j=1}^{d} x_j = \begin{cases} 
0 & \text{if } x_j = 0 \text{ for some } j, \\
1 & \text{if } x_j = 1 \text{ for all } j.
\end{cases}
\]
The state of the components is usually not known in advance and is modeled by a random vector 
\( (X_1, \ldots, X_d) \) taking values in \( \{0, 1\}^d \). Then, the reliability of the system is given by

\[
R = \mathbb{E}[\rho(X_1, \ldots, X_d)] = \mathbb{P}(\rho(X_1, \ldots, X_d) = 1).
\]

First, we describe a series systems assembly problem following the setup of Derman et al. (1972), and then proceed to our example with parallel systems. Suppose that a system consists of \( d \) components, each of a different type, and there are \( n \) components of each type available. This enables the assembly of \( n \) systems. The state of the components \( X_{ij}, i = 1, \ldots, n, j = 1, \ldots, d \). Hence, the reliability of system \( i \) is

\[
R_i = \mathbb{P}(Z_{i1} \leq a_{i1}, \ldots, Z_{id} \leq a_{id}) = F_Z(a_{i1}, \ldots, a_{id}).
\]

We can freely choose the order of components of each type when assembling the systems, which corresponds to rearranging elements within the columns \( a_1, \ldots, a_d \) of the input matrix \( (A)_{ij} = a_{ij} \), so that each row corresponds to an assembled system. The expected number of systems that perform satisfactorily is

\[
\sum_{i=1}^d R_i = \sum_{i=1}^d F_Z(a_{i1}, \ldots, a_{id})
\]

Recall from Section 2.3 that cumulative distribution functions are supermodular, therefore the objective function (21) is arrangement increasing, in particular, of the type (B). This yields the main result in Derman et al. (1972), that the comonotonic arrangement \( A^* = (a_1^*, \ldots, a_d^*) \) of components maximizes the expected number of systems that perform satisfactorily. They also note that under a further assumption that the probabilities that components within a system work are independent (\( F_Z \) has the independence copula), the comonotonic arrangement also maximizes the probability that at least \( k \) systems perform satisfactorily, \( 1 \leq k \leq n \).

We now consider the assembly of parallel systems, where at least one component is required to work for a system to perform satisfactorily. In this case, the structure function is

\[
\rho^\parallel(x_1, \ldots, x_d) = \max\{x_1, \ldots, x_d\} = 1 - \prod_{j=1}^d (1 - x_j) = \begin{cases} 0 & \text{if } x_j = 0 \text{ for all } j, \\ 1 & \text{if } x_j = 1 \text{ for some } j. \end{cases}
\]

Using the same model for component states \( X_{ij} = 1_{Z_{ij} \leq a_{ij}} \) as in the series example, the reliability of system \( i \) in the parallel case is

\[
R_i^\parallel = 1 - \mathbb{P}(Z_{i1} > a_{i1}, \ldots, Z_{id} > a_{id}) = 1 - S_Z(a_{i1}, \ldots, a_{id}),
\]
where $S_Z$ denotes the so-called *survival function* of $(Z_{i1}, \ldots, Z_{id})$. Maximizing the reliability (22) is equivalent to *minimizing* a survival function. Since the survival function is also a distribution function, it is supermodular. Therefore, maximizing the expected number of functioning systems, $\sum_{i=1}^{d} R_i^i$, is equivalent to minimizing the sum of supermodular functions (type (B) of AI functions). Derman et al. (1972) note that in the special case of independence copula and $d = 2$ types of components, the countermonotonic arrangement is optimal. Prasad et al. (1991) consider a related problem, and also only give the solution for the case $d = 2$. In Derman et al. (1974) another variation with independent component failures is considered, and a pairwise component interchange heuristic is proposed.

Using the insights from Section 3, we are able to consider dimensions higher than 2, and relax the assumption of independence. In particular, we suppose that for the survival function $S_Z$ of the external factors, the dependence is modeled using an Archimedean copula (see Section 3.3). For a concrete example, we consider the Clayton copula $C_\theta$ with parameter $\theta > 0$ and generator

$$
\psi(t) = (1 + \theta t)^{-1/\theta}, \quad t \in [0, \infty),
$$

$$
\psi^{-1}(u) = (u^{-\theta} - 1)/\theta, \quad u \in (0, 1].
$$

Note that the “survival” function $S_Z$ in (22) actually gives the probability of system failure, therefore we will work with the marginal and joint probabilities of component failures. For $i = 1, \ldots, n$ and $j = 1, \ldots, d$, define $U_{ij} = 1 - F_j(Z_{ij})$ and $p_{ij} = 1 - F_j(a_{ij})$, where $F_j$ is the $j$th margin of $F_Z$ (assumed continuous for simplicity, so that $U_{ij} \sim \text{UNIF}(0, 1)$). Applying Sklar’s theorem, we can take $(P)_{ij} = p_{ij}$ as the input matrix, and express the probability that the $i$th system fails as

$$
1 - R_i^i = P(U_{i1} \leq p_{i1}, \ldots, U_{id} \leq p_{id}) = C_\theta(p_{i1}, \ldots, p_{id}).
$$

Clayton copula induces lower-tail dependence (with index $\lambda_L = 2^{-1/\theta}$). For the present model this implies that the probability of joint failures for reliable components (with small marginal probability $p_{ij}$ of failure) is higher than in the independent case; see Figure 3 for a bivariate sample of $(U_1, U_2) \sim C_\theta$, as well as the conditional probability $P(U_2 \leq 0.1|U_1 = p)$.

For a numerical example, we generate the input matrix $P \in \mathbb{R}^{20 \times 5}$ by sampling the component failure probabilities $p_{ij} \overset{\text{iid}}{\sim} \text{UNIF}(0, 1)$; we fix the entries of this matrix for the remainder of the experiment. Following the approach from Section 3.3, we transform $P$ by applying $\psi^{-1}$ to its entries, thus obtaining $H(P)$. Thereafter, we consider 1000 random arrangements of the matrix $H(P)$ and compute the expected number of failed systems. A histogram of the results is given in Figure 4. Taking each random arrangement as the input, we apply the (Block) RA to obtain a $\Sigma$-countermonotonic or COO matrix, respectively, and again compute the corresponding expected number of failed systems. We observe that in all cases the resulting arrangements consistently yield highly reliable systems, supporting the hypothesis that $\Sigma$-countermonotonic...
Figure 3: **Left panel:** 1000 sample points from a bivariate Clayton copula with parameter $\theta = 2$. **Right panel:** Conditional failure probability $P(U_2 \leq 0.1 \mid U_1 \leq x) = C_\theta(0.1, p)/p$ as a function of $x \in [0,1]$ on the horizontal axis.

matrices are close-to-optimal. The COO matrices are also close, but show more variation (see Figure 4, right panel), since it is a larger set or arrangements. Table 4 in the Appendix shows a $\Sigma$-countermonotonic arrangement of $H(P)$ with the corresponding failure probabilities.

Figure 4: **Left panel:** Histogram of $\sum_{i=1}^{20}(1 - R_i)$ corresponding to 1000 random initial arrangements of $H(P)$, as well as the result after applying the (Block) Rearrangement Algorithm in order to obtain $\Sigma$-countermonotonic or column oppositely ordered matrices. With the Block RA, the result was always 1.7176 (up to four decimal digits), with the standard RA - in the range [1.7176, 1.7202], for the random arrangements - in the range [1.9465, 3.6781], and equal to 5.8037 for the comonotonic arrangement. **Right panel:** Close-up of the best cases.
5 Conclusions

While maximizing an arrangement increasing (AI) functions over intra-column rearrangements of a given matrix is trivial, the corresponding minimization problem is in general intractable. We have shown that for a subclass of AI functions with a special structure, the minimizing arrangement has to satisfy a specific property of negative dependence. This can be exploited to obtain close-to-optimal arrangements using an efficient heuristic called the (Block) Rearrangement Algorithm; this is possible due to the special structure of the considered AI functions. We also show that this subclass of AI functions includes the objective functions of many classical optimization problems, also stochastic ones, and give explicit examples. The numerical case study seems to support the intuition, that any arrangement with the mentioned countermonotonicity property gives a close-to-optimal objective value.

Acknowledgments

The authors benefited from discussions with Marius Hofert, Steven Vanduffel and Paul Embrechts when preparing this article. E. Jakobsons thanks RiskLab Switzerland and the Swiss Finance Institute for financial support. R. Wang would like to thank FIM at ETH Zurich for supporting his visit in 2015 and he acknowledges support from the Natural Sciences and Engineering Research Council of Canada (NSERC RGPIN-435844-2013).
Table 3: A Σ-countermonotonic arrangement for the matrix Θ (obtained using the Block Rearrangement Algorithm) for the assembly line crew scheduling example (Section 4.1) with $n = 15$, $d = 4$. The last four columns show row-sums $\theta_i$ and $P(C_i > 45) = 1 - \Phi((\theta_i - 45) / \sigma_i)$ for a Σ-countermonotonic and comonotonic arrangement of Θ, respectively.

<table>
<thead>
<tr>
<th>Σ-countermonotonic arrangement of Θ</th>
<th>Σ-countermonotonic Θ</th>
<th>Σ-countermonotonic Θ</th>
<th>comonotonic Θ</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Row-sums</td>
<td>P(C_i &gt; 45)</td>
<td>Row-sums</td>
</tr>
<tr>
<td>5.73 8.39 11.84 12.83</td>
<td>38.79</td>
<td>0.025</td>
<td>31.60</td>
</tr>
<tr>
<td>6.90 8.38 10.41 13.13</td>
<td>38.82</td>
<td>0.025</td>
<td>32.86</td>
</tr>
<tr>
<td>6.21 9.26 13.06 10.30</td>
<td>38.83</td>
<td>0.026</td>
<td>33.69</td>
</tr>
<tr>
<td>5.32 10.27 8.93 14.31</td>
<td>38.83</td>
<td>0.026</td>
<td>35.17</td>
</tr>
<tr>
<td>7.29 8.02 10.14 13.39</td>
<td>38.84</td>
<td>0.026</td>
<td>35.43</td>
</tr>
<tr>
<td>7.71 9.43 9.87 11.84</td>
<td>38.84</td>
<td>0.026</td>
<td>36.07</td>
</tr>
<tr>
<td>7.38 7.66 11.68 12.12</td>
<td>38.84</td>
<td>0.026</td>
<td>36.52</td>
</tr>
<tr>
<td>7.14 7.78 11.76 12.17</td>
<td>38.85</td>
<td>0.026</td>
<td>38.42</td>
</tr>
<tr>
<td>7.70 7.76 11.34 12.05</td>
<td>38.85</td>
<td>0.026</td>
<td>39.64</td>
</tr>
<tr>
<td>6.12 9.46 12.67 10.60</td>
<td>38.85</td>
<td>0.026</td>
<td>40.12</td>
</tr>
<tr>
<td>7.13 7.68 12.19 11.86</td>
<td>38.86</td>
<td>0.026</td>
<td>41.51</td>
</tr>
<tr>
<td>7.28 9.42 9.73 12.44</td>
<td>38.87</td>
<td>0.026</td>
<td>42.29</td>
</tr>
<tr>
<td>5.96 7.62 14.66 10.69</td>
<td>38.93</td>
<td>0.027</td>
<td>43.80</td>
</tr>
<tr>
<td>6.02 7.30 9.65 15.98</td>
<td>38.95</td>
<td>0.028</td>
<td>46.20</td>
</tr>
<tr>
<td>5.16 7.21 9.64 16.96</td>
<td>38.98</td>
<td>0.028</td>
<td>49.60</td>
</tr>
</tbody>
</table>

$P(C_{max} \leq 45) = 0.6721$  $P(C_{max} > 45) = 0.0101$

$\sum_{i=1}^{15} P(C_i \leq 45) = 14.6079$  $\sum_{i=1}^{15} P(C_i > 45) = 12.6084$

$E[C_{max}] = 44.3519$  $E[C_{max}] = 50.3584$
Table 4: A Σ-countermonotonic arrangement for the matrix $H(P)$ (obtained using the Block Rearrange-ment Algorithm) for the systems assembly example (Section 4.2) with $n = 20, d = 5$ and Clayton($\theta = 2$) copula. The last four columns show the row-sums and individual system failure probabilities for a Σ-countermonotonic and comonotonic arrangement of $H(P)$, respectively ($R^i_\parallel$ is the reliability of the $i^{th}$ system).

<table>
<thead>
<tr>
<th>Σ-countermonotonic arrangement of $H(P)$</th>
<th>Σ-countermonotonic</th>
<th>comonotonic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row-sums</td>
<td>$1 - R^i_\parallel$</td>
<td>Row-sums</td>
</tr>
<tr>
<td>6.23</td>
<td>4.48</td>
<td>1.48</td>
</tr>
<tr>
<td>6.04</td>
<td>9.94</td>
<td>1.26</td>
</tr>
<tr>
<td>13.13</td>
<td>3.26</td>
<td>4.10</td>
</tr>
<tr>
<td>3.41</td>
<td>3.18</td>
<td>14.22</td>
</tr>
<tr>
<td>3.15</td>
<td>2.67</td>
<td>1.09</td>
</tr>
<tr>
<td>2.11</td>
<td>2.13</td>
<td>0.91</td>
</tr>
<tr>
<td>1.49</td>
<td>23.69</td>
<td>1.06</td>
</tr>
<tr>
<td>1.48</td>
<td>2.06</td>
<td>0.86</td>
</tr>
<tr>
<td>1.09</td>
<td>1.05</td>
<td>0.81</td>
</tr>
<tr>
<td>0.79</td>
<td>0.82</td>
<td>39.44</td>
</tr>
<tr>
<td>0.57</td>
<td>0.68</td>
<td>0.60</td>
</tr>
<tr>
<td>0.48</td>
<td>87.49</td>
<td>0.30</td>
</tr>
<tr>
<td>0.34</td>
<td>0.51</td>
<td>139.28</td>
</tr>
<tr>
<td>0.32</td>
<td>0.51</td>
<td>0.30</td>
</tr>
<tr>
<td>0.31</td>
<td>0.30</td>
<td>0.10</td>
</tr>
<tr>
<td>0.28</td>
<td>0.28</td>
<td>222.46</td>
</tr>
<tr>
<td>0.15</td>
<td>0.16</td>
<td>265.89</td>
</tr>
<tr>
<td>2637.05</td>
<td>0.10</td>
<td>0.04</td>
</tr>
<tr>
<td>0.14</td>
<td>0.08</td>
<td>10927.86</td>
</tr>
<tr>
<td>0.04</td>
<td>0.07</td>
<td>0.01</td>
</tr>
</tbody>
</table>

$\sum_{i=1}^{20}(1 - R^i_\parallel)$: 1.7176  5.8037
Kris Boudt, Edgars Jakobsons, Steven Vanduffel.

Block rearranging elements within matrix columns to minimize the variability of the row sums.

Submitted.
Block rearranging elements within matrix columns to minimize the variability of the row sums

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Abstract

We study the rearrangement problem of finding the intra-column rearrangement (permutation) of a matrix such that the row sums show minimum variability. The Rearrangement Algorithm (RA) of Puccetti and Rüschendorf (2012) is a heuristic that aims at solving this problem by rearranging (swapping) the elements within the columns of the matrix so that each column becomes oppositely ordered to the sum of all other columns. We show that, under a mild assumption on the variability measure, the RA can be improved by rearranging blocks of columns rather than single columns, and label this approach as the Block Rearrangement Algorithm (Block RA). We propose a block selection heuristic that aims at finding the optimal blocks efficiently. We describe that several problems in operations research, such as the assembly line crew scheduling problem, the k-partitioning problem, the parallel machine scheduling problem and the subset sum problem, can be cast as rearrangement problems. Specifically, we use the Block RA to solve the 2-partitioning problem and show it outperforms the well-known greedy algorithm and the Karmarkar-Karp differencing algorithm (Karmarkar and Karp, 1982).

Keywords: assembly line crew scheduling, matrix rearrangement, k-partitioning, Karmarkar-Karp differencing algorithm.

\footnotesize This work has benefited from useful comments of Ludger Rüschendorf and participants at various seminars and conferences. We thank Kristof Verbeken for his initial research assistance.
1 Introduction

In this paper, we study the rearrangement problem of finding an intra-column rearrangement of an \( n \times d \) matrix, such that the row sums show minimum variability. This problem is observationally equivalent to the problem of optimal assembly line crew scheduling (ALCS); see Hsu (1984). The most obvious way of solving the problem would be by brute force enumeration. However, as there are \( (n!)^{d-1} \) rearranged matrices to consider, doing so is not feasible unless \( n \) and \( d \) are small. In fact, whenever \( d > 2 \), the problem is NP-hard, explaining why it is typically dealt with using heuristic approaches. In this regard, a very recent algorithm applicable to this purpose is the so-called Rearrangement Algorithm (RA) of Puccetti and Rüschendorf (2012). The RA aims at solving the rearrangement problem by rearranging the elements within the columns of the matrix so that each column becomes oppositely ordered to the sum of all other columns. This heuristic has a strong theoretical support by the fact that a necessary condition for row sums to have minimum variance is that the elements in each column are oppositely ordered to the row sums of the other columns. A related, but less performant algorithm can be found in Coffman and Yannakakis (1984). These authors order the elements of the columns in the opposite order to the row sums of all preceding columns, an approach that is also inherent in Graham’s seminal list scheduling heuristic\(^2\); see Graham (1966). The RA was initially motivated by the problem of assessing the impact of model uncertainty on the estimated risk of a portfolio, where model uncertainty is due to a lack of (sufficient) knowledge on the dependence that exists among the risky components. In this application, the columns of the matrix represent the discretized distributions of the risks involved.

The quality of the approximation obtained by the RA can be further improved by rearranging blocks of columns. Indeed, if any block of two or more columns in the output matrix of the RA can be further rearranged, so that their row sums become anti-monotonic with the row sums of the remaining columns, then doing so improves the solution. Furthermore, the RA considers the columns in the matrix one at a time, so it can be expected that, for the same quality of solution, a substantial reduction in the number of iterations can be achieved by rearranging blocks of columns (instead of single columns) of the matrix. These basic insights lead to so-called block rearrangements. The idea of using block rearrangements to improve the RA is not new. It first appeared in Bernard et al. (2015a) (see their Remark 4.1) and was further used in Bernard et al. (2015c) under the name Block RA (Remark 3.3). Compared to the mentioned literature, we make the following contributions.

First, we show that, under a mild assumption on the variability measure, the rearrangement

\(^{1}\text{rearranging refers to the act of permuting (swapping) elements}\)

\(^{2}\text{In Graham (1966), the so-called parallel machine scheduling problem is solved by iteratively assigning subsequent jobs to the machine whose current completion time is minimum.}\)
of (blocks of) columns can only reduce the variability of the row sums, and that the solution of the RA can typically be further improved by block rearrangements.

Second, we propose a carefully motivated heuristic to find an approximately optimal block in each iteration and compare it to the alternative of random block selection. The numerical experiments confirm that for matrices with a large number of dissimilar columns, the proposed heuristic for block selection requires fewer iterations than the random block selection approach.

Third, we describe how several problems in operations research, such as the assembly line crew scheduling problem, the k-partitioning problem, the parallel machine scheduling problem and the subset sum problem, can be cast as rearrangement problems. In particular, we use the Block RA for obtaining approximate solutions to the classic number partitioning problem and show that it outperforms the well-known greedy algorithm and the Karmarkar-Karp differencing algorithm (Karmarkar and Karp, 1982).

The remainder of the paper is organized as follows. In the remainder of this section, we describe the rearrangement problem and discuss variability measures (objective functions) that are compatible with the idea of block rearrangements. In Section 2, we characterize optimal solutions of the rearrangement problem. The proposed algorithms are given in Section 3, and a simulation analysis is conducted to analyze their performance. In Section 4, we link the rearrangement problem to a series of classical problems in operations research. Concluding remarks are given in Section 5.

1.1 The problem setup

The input for the rearrangement problem consists of \(d\) \(n\)-dimensional column vectors \(X_1, \ldots, X_d \in \mathbb{R}^{n \times 1}\) that are combined into a matrix \(X := (X_1, X_2, \ldots, X_d) \in \mathbb{R}^{n \times d}\). In the rearrangement problem, we aim at creating a new \(n \times d\) matrix such that (i) the \(j\)th column-vector is a permutation of \(X_j\) \((j = 1, 2, \ldots, d)\) and (ii) the row sums have minimal variability. The precise description of what is exactly meant with “variability” may depend on the underlying optimization problem at hand. Some examples include taking the variance of the row sums, their maximum or their minimum, or the difference between these (the range). Denote by \(X^\pi := (X^\pi_1, X^\pi_2, \ldots, X^\pi_d) \in \mathbb{R}^{n \times d}\) the resulting matrix after rearranging the \(n\) elements of the \(j\)th column \((j = 1, 2, \ldots, d)\) by a permutation \(\pi_j : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}\), i.e., \((X^\pi)_{ij} = (X)_{\pi_j(i)j}\), where \(\pi := (\pi_1, \pi_2, \ldots, \pi_d)\) denotes the vector of the \(d\) permutations. Throughout, we will call the vector of permutations \(\pi\) a rearrangement and \(X^\pi\) a rearranged matrix. Denote the vector of the corresponding row sums by \(S^\pi := \sum_{j=1}^d X^\pi_j\).

This paper focuses on a restricted version of rearrangements, called block rearrangements. A block rearrangement consists of rearranging the entire row of a block of columns. More formally, let \(\delta \in \{0, 1\}^d\) indicate by \(\delta_j = 1\) the columns \(j\) to be rearranged, \(j = 1, \ldots, d\). A rearrangement of \(X^\pi\) into \(X^{\delta^\pi}\) is a block rearrangement if there exists a \(\delta \in \{0, 1\}^d\) and a
permutation \( \pi^\delta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \) such that \( \pi'_j = \pi_j \circ \pi^\delta \) for \( \delta_j = 1 \) and \( \pi'_j = \pi_j \) for \( \delta_j = 0 \).

We formally state the rearrangement problem as

\[
\min_{\pi} V(S^\pi),
\]

where \( V(\cdot) \) is a scalar-valued function that measures the variability of the row-sum vector (or the outcomes of a random variable) in its argument.\(^3\) The problem of rearranging elements within the columns of a matrix, in order to obtain nearly-constant row sums, is exactly the optimization problem in assembly line crew scheduling (ALCS); see e.g. Coffman and Yannakakis (1984) and Haus (2015). Assume that the production of some item requires the serial execution of \( d \) jobs and that there are \( n \) identical assembly lines available. Assume further that for the \( j^{th} \) job \((j = 1, 2, \ldots, d)\), there is a pool of \( n \) specialized workers available, and the \( i^{th} \) worker \((i = 1, 2, \ldots, n)\) requires labor time \((X)_{ij} = x_{ij}\) to complete his job. The objective of the ALCS is then to find a rearrangement \( \pi^* \) for which the maximal time spent by any assembly line (the bottleneck of the schedule) is minimized:

\[
\min \max_{\pi} S^\pi_i.
\]

In order to further illustrate ALCS, in Tables 1 and 2 a simple example with \( d = 8 \) jobs and \( n = 8 \) assembly lines is given. Table 1 shows a randomly chosen arrangement of the workers within each specialty (each worker spends between 1 and 10 units of time to complete their job). One clearly observes large discrepancies between the total times each assembly line requires to produce an item, with completion times ranging from 27 to 45. Using the objective (2) and applying the brute force Block RA (see in Section 3) yields the well-balanced crew schedule in Table 2, which is a global optimum of the minimization problem.

\(^3\)Note that if \( \pi^* \) is a solution of (1), then swapping two rows of \( X^{\pi^*} \) leads to a solution that is as good. For clarity in presentation, it is therefore recommended to rearrange (swap) the rows of the rearranged matrix so that its row sums appear in an increasing order.
Table 1: Example of a randomly arranged crew schedule. The \((i, j)\)-th element corresponds to the time spent by worker \(i\) of specialty \(j\).

<table>
<thead>
<tr>
<th>(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)</th>
<th>(S^\pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>([3, 1, 3, 6, 2, 1, 6, 5])</td>
<td>27</td>
</tr>
<tr>
<td>([4, 7, 5, 3, 2, 4, 4, 2])</td>
<td>31</td>
</tr>
<tr>
<td>([4, 7, 5, 2, 2, 3, 7, 2])</td>
<td>32</td>
</tr>
<tr>
<td>([2, 6, 2, 3, 5, 6, 5, 5])</td>
<td>34</td>
</tr>
<tr>
<td>([8, 6, 2, 8, 2, 1, 4, 7])</td>
<td>38</td>
</tr>
<tr>
<td>([6, 10, 2, 2, 5, 7, 8, 1])</td>
<td>41</td>
</tr>
<tr>
<td>([8, 10, 3, 3, 5, 5, 2, 6])</td>
<td>44</td>
</tr>
<tr>
<td>([9, 7, 9, 3, 1, 10, 2, 4])</td>
<td>45</td>
</tr>
</tbody>
</table>

Table 2: Crew schedule in Table 1 rearranged to minimize the differences in the production times.

<table>
<thead>
<tr>
<th>(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)</th>
<th>(S^\pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>([4, 10, 3, 3, 2, 10, 2, 2])</td>
<td>36</td>
</tr>
<tr>
<td>([2, 6, 9, 2, 5, 1, 4, 7])</td>
<td>36</td>
</tr>
<tr>
<td>([9, 10, 5, 2, 2, 4, 2, 2])</td>
<td>36</td>
</tr>
<tr>
<td>([6, 7, 2, 6, 5, 1, 8, 1])</td>
<td>36</td>
</tr>
<tr>
<td>([4, 7, 2, 3, 1, 7, 7, 5])</td>
<td>36</td>
</tr>
<tr>
<td>([8, 6, 2, 3, 2, 6, 5, 4])</td>
<td>36</td>
</tr>
<tr>
<td>([3, 1, 3, 8, 5, 5, 6, 6])</td>
<td>37</td>
</tr>
<tr>
<td>([8, 7, 5, 3, 2, 3, 4, 5])</td>
<td>37</td>
</tr>
</tbody>
</table>

1.2 Condition on the variability measure

The algorithms that we propose to deal with the rearrangement problem (1) are well motivated under a certain assumption on the variability measure \(V(\cdot)\). This assumption is weak in the sense that most common variability measures (such as the variance of the row sums, their maximum or (minus) minimum) satisfy these conditions. To formulate this condition, it is convenient to interpret the matrix \(X = (X_1, X_2, \ldots, X_d) \in \mathbb{R}^{n \times d}\) as a random vector; each column is a variable and each row is a possible outcome. Furthermore, all outcomes occur with probability \(1/d\). A rearrangement \(\pi\) thus yields a rearranged matrix \(X^\pi\) in which the variables \(X^\pi_j\) assume the same values (outcomes) as \(X_j\), but these values can be seen as being obtained under other scenarios. In other words, rearranging the values within the columns of the matrix \(X\) to obtain the matrix \(X^\pi\) yields random variables \(X^\pi_j\) that have the same (marginal) distribution function as the \(X_j\) \((j = 1, 2, \ldots, d)\), but the interdependence among the variables has changed, i.e., the joint distributions of \((X_1, X_2, \ldots, X_d)\) and \((X_1^\pi, X_2^\pi, \ldots, X_d^\pi)\) are different. In this regard, two types of dependence are of particular interest. We say that the \(d\) random variables \(X_1^\pi, X_2^\pi, \ldots, X_d^\pi\) are perfectly positively related (co-monotonic dependence), if all the elements in the column \(X_i^\pi\) have the same order (rank) as the corresponding elements in \(X_j\), for any \(i \neq j\). By contrast, when the elements in \(X_i^\pi\) appear in the opposite order of the ones in \(X_j\), then we have a perfect negative relation (anti-monotonic dependence) between these two columns. Without further ado, we use the notation \(X\) to either describe a matrix or a random vector with given marginal
distributions $F_j(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{x_i \leq x\}}, j = 1, 2, \ldots, d$.

The essential requirement that we impose on $V$ is consistency with the so-called convex order, which is a probabilistic device used to describe variability. We say that a random variable $W$ is smaller than $Z$ in convex order, written $W \preceq_{cx} Z$, if $E[f(W)] \preceq E[f(Z)]$ for all convex functions $f(\cdot)$ such that the expectations exist.\footnote{Note that $E[Z] \preceq_{cx} Z$ (Jensen’s inequality). Furthermore, $W \preceq_{cx} Z$ implies that $Z$ has the same mean as $W$, but a greater or equal variance. For details of this ordering, see Müller and Stoyan (2002).}

We formulate the following assumption.

Assumption 1. For any random variables $W$ and $Z$, the function $V(\cdot)$ is such that

$$W \preceq_{cx} Z \implies V(W) \preceq V(Z).$$

Note that variability measures of the form $V(S\pi) = \sum_{i=1}^{n} f(S_{\pi}^i)$ in which $f(\cdot)$ is a convex function clearly satisfy the stated assumption; in particular, it is satisfied by the variance (rearrangements do not change the mean value of the row sums). Another suitable class is given by L-statistics (linear combinations of order statistics) with decreasing coefficients, thus also the maximum and (minus) the minimum of row sums satisfies this condition.\footnote{See also Section 2.3 in Bernard et al. (2015a), where the connection between the convex order and the minimum value of a random sum is established. The notion corresponding to the convex order in the discrete case (when $S$ is a vector in $\mathbb{R}^n$), is the majorization order. Functions $V : \mathbb{R}^n \to \mathbb{R}$ that respect the majorization order are called Schur-convex; see Boland and Proschan (1988) and Jakobsons and Wang (2016) for further details on the compatibility of rearrangement methods with Schur-convex objective functions.}

## 2 Characterization of optimal rearrangements

Let $X := (X_1, X_2, \ldots, X_d) \in \mathbb{R}^{n \times d}$ be a given matrix for which the vector $S$ of the $n$ row sums has variability $V(S)$. Under Assumption 1, it holds that, if there exists a dependence (or a rearrangement $\pi$) between the $X_j$’s such that $S^\pi$ becomes constant and thus equal to the mean, then it is the solution to the rearrangement problem (1). This specific situation corresponds to mixability of random variables, a probabilistic notion that has recently been studied in Wang and Wang (2011, 2016). In general, one cannot expect to obtain a constant $S^\pi$, in which case finding the optimal distribution of $S^\pi$ and, moreover, the optimal rearrangement $\pi$ remains a challenging task.

However, it is clear that a candidate solution (i.e., a rearrangement $\pi$) can be improved, if one can find another rearrangement $\pi'$ that yields a sum $S^\pi'$ that is smaller in convex order than the sum $S^\pi$. At this point, we recall the well-known fact that a sum $X + Y$ with given marginal distributions for $X$ and $Y$ is minimal with respect to the convex order if $X$ and $Y$ are oppositely ordered (Day, 1972). Hence, if the row sums of a block of columns are not oppositely ordered with the row sums of the remaining columns (complementary block), rearranging the block of columns so that, after rearrangment, they become oppositely ordered, will reduce the
variability of the row sums of the total matrix; see also Remark 3 in Bernard et al. (2015a). The following proposition is thus proven.

**Proposition 1 (Improvement of a solution by block rearrangement).** Let $V$ be a variability measure satisfying Assumption 1. Suppose that for $X^\pi \in \mathbb{R}^{n \times d}$, there exists $\delta \in \{0, 1\}^d$ such that $\sum_{j=1}^d \delta_j X_{j}^\pi$ and $\sum_{j=1}^d (1 - \delta_j) X_{j}^\pi$ are not oppositely ordered. Let $\pi' = (\pi'_1, \ldots, \pi'_d)$ be an intra-column rearrangement such that $\sum_{j=1}^d \delta_j X_{j}^{\pi'}$ and $\sum_{j=1}^d (1 - \delta_j) X_{j}^{\pi'}$ are oppositely ordered. Then,

$$V(\sum_{j=1}^d \delta_j X_{j}^{\pi'}) + \sum_{j=1}^d (1 - \delta_j) X_{j}^{\pi'}) \leq V(\sum_{j=1}^d \delta_j X_{j}^\pi) + \sum_{j=1}^d (1 - \delta_j) X_{j}^\pi.$$ 

The main result is thus that, each time a block is rearranged so that its row sums become oppositely ordered with the row sums of the complementary block, the variability of the row sums decreases. It further follows that, if the objective $V$ is of the form $V(S) = \sum_{i=1}^n f(s_i)$, where $f$ is a strictly convex function, then the “$\leq$” in Proposition 1 can be replaced by a “$<$”. A minimum can then only be attained when all partitions into two blocks have the property that the corresponding partial row sums corresponding to these blocks are oppositely ordered. Such a matrix is called $\Sigma$-countermonotonic; see also Section 3.4 in Puccetti and Wang (2015b) for a probabilistic study of the concept of $\Sigma$-countermonotonicity of a random vector. The block rearrangement algorithms that we propose as heuristics for the rearrangement problem are in fact designed to find $\Sigma$-countermonotonic matrices. These algorithms are discussed further in the next section.

### 3 (Block) rearrangement algorithms

Let $X$ be a given matrix with a corresponding row-sum vector $S$. In order to find the optimal rearranged matrix $X^\pi$, we first consider the RA of Puccetti and Rüschendorf (2012), which rearranges single columns one-by-one, and then we discuss block rearrangements. Similarly to Embrechts et al. (2013), all algorithms will start by applying a random permutation of the elements within each column. Since the RA rearranges the columns sequentially, starting with the first column, the order of the columns matters for the performance of this algorithm. Throughout the paper, we assume the convention that the columns of $X$ are sorted in terms of decreasing variability, i.e., $V(X_i) \geq V(X_j)$, for all column numbers $1 \leq i \leq j \leq d$. This section concludes with a numerical study on how the choice of the algorithm affects the trajectory of the objective function, as a function of the number of iterations (rearrangements).
3.1 The Rearrangement Algorithm

The RA is a heuristic that aims at solving the rearrangement problem by iteratively rearranging the columns of the matrix one-by-one, so that each column becomes oppositely ordered to the sum of the other columns. In other words, the RA is not strictly designed to achieve the condition of optimality, i.e., a $\Sigma$-countermonotonic matrix (see also Proposition 1 and its discussion), but only approximately achieves this situation by restricting to vectors $\delta \in \{0, 1\}^d$ for which $\sum_{j=1}^{d} \delta_j = 1$. In the initial publication (Puccetti and Rüschendorf, 2012), the proposed algorithm does not specify convergence conditions in terms of the objective function $V$, instead the algorithm continues until no changes are made during a full cycle through the columns. Algorithm 1 states the pseudo-code of the RA adapted to solving the general variability minimization problem in (1).

**Algorithm 1:** Pseudo-code of the RA to minimize the variability $V$ of the row sums of a matrix $X$.

1. Set target value $a^*$ and maximum number of iterations $M$;
2. Set $\pi$ to a randomly chosen intra-column arrangement of the elements so that a matrix $X^\pi$ is obtained;
3. Compute $a := V(S^\pi)$ with $S^\pi := \sum_{j=1}^{d} X_j^\pi$, and let $m = 0$ and $j = 1$;
4. while $a > a^*$ and $m < M$ do
   5. Rearrange column $j$ in the matrix $X^\pi$ so that (after rearrangement) it becomes oppositely ordered to the sum of the other columns;
   6. Compute $a := V(S^\pi)$ with $S^\pi := \sum_{j=1}^{d} X_j^\pi$;
   7. Set $j = j + 1$ and $m = m + 1$;
8. if $j > d$ then
end

Note that the objective cannot deteriorate in an iteration. The choice of the target value $a^*$ is somewhat arbitrary. Nevertheless, once a candidate solution to the rearrangement problem is obtained, one can always lower the target value based on the value $a$ of the output matrix, and then use this matrix (and its corresponding rearrangement) in Step 4. An alternative termination condition based on the objective ($V = -\min$ or $V = \max$) is introduced in Embrechts et al. (2013), and further discussed in Hofert et al. (2015).
3.2 (Brute force) Block RA

The RA, as specified in Algorithm 1, attains a weaker necessary condition for optimality than $\Sigma$-countermonotonicity, and the solution can be readily improved whenever there is a partition into two blocks with the property that the row sums of the first block of columns is not oppositely ordered to the row sums of the remaining columns (see Proposition 1).

In order to address this issue, we consider algorithms that rearrange blocks of columns. For the notation, we use a $d$-dimensional binary vector $\delta \in \{0, 1\}^d$ to indicate which columns belong to the first block ($\delta_j = 1$), and which ones to the complementary block ($\delta_j = 0$). Each $\delta$ thus corresponds to a partition of the matrix $X^\pi$ into two submatrices ($X^\pi_\delta$ and $X^\pi_{1-\delta}$). Denote their corresponding row sums by $S^\pi_\delta := \sum_{j=1}^{d} \delta_j X^\pi_j$ and $S^\pi_{1-\delta} := \sum_{j=1}^{d} (1 - \delta_j) X^\pi_j$.

In the so-called brute force Block RA in Algorithm 2, we require that for all possible partitions into two blocks, the corresponding row sums of the two blocks are oppositely ordered (i.e., $X^\pi$ is $\Sigma$-countermonotonic). This implementation ensures that a further improvement of the objective using block rearrangements is not possible (see Proposition 1). That is, we have reached a local optimum.

Algorithm 2: Pseudo-code of the brute force Block RA to minimize the variability $V$ of the row sums of a matrix $X$, with brute force rearrangement of all possible partitions.

1. Set target value $a^*$ and maximum number of iterations $M$;
2. Choose a random enumeration $(\delta^{(k)})_{k=1}^{2^{d-1}-1}$ of all distinct partitions $\delta \in \{0, 1\}^d$;
3. Set $\pi$ to a randomly chosen intra-column arrangement of the elements;
4. Set $a := V(S^\pi)$ with $S^\pi := \sum_{j=1}^{d} X^\pi_j$, and let $m = 0$, improved = true;
5. while $a > a^*$ and $m < M$ and improved is equal to true do
6. Set improved = false;
7. for $k \in K := 1, \ldots, 2^{d-1} - 1$ do
8. Set $\delta = \delta^{(k)}$, construct $X^\pi_\delta$ and $X^\pi_{1-\delta}$ and compute the row sums $S^\pi_\delta$ and $S^\pi_{1-\delta}$;
9. if $S^\pi_\delta$ is not oppositely ordered to $S^\pi_{1-\delta}$ then
10. Rearrange the rows of the submatrix $X^\pi_\delta$ so that the row sums $S^\pi_\delta$ become oppositely ordered to $S^\pi_{1-\delta}$, and compute $a := V(S^\pi_\delta + S^\pi_{1-\delta})$;
11. Set improved = true;
end
12. Set $m = m + 1$.
end

* To verify this condition, one needs to consider $(2^d - 2)/2 = 2^{d-1} - 1$ different partitions of the matrix into two blocks. The division by two is because for each partition $\delta$, the corresponding $1 - \delta$ does not need to be considered. Furthermore, the cases in which $\delta$ is zero (resp. one) for all elements, do not require consideration.
An easy way to enumerate all partitions is to use a bijection from the set \( \{0, 1\}^d \) to integers \( \{0, \ldots, 2^d - 1\} \) via their binary expansion. The reason we use a random enumeration (i.e. a random ordering of the partitions) is to avoid successive partitions that differ by only a few columns; from this perspective, using the Gray code\(^9\) would be the worst choice, as any two successive values would differ in only one bit (i.e. column). Another point to note is that in Bernard and McLeish (2015) (see their Block RA2), to check whether an improvement has occurred, the variances of the row sums are computed before and after each loop through the set of all partitions. This is equivalent, since the variance is a strictly convex functional (so any rearrangement would yield a positive reduction in the variance).

### 3.3 Random Block RA

In the brute force Block RA from the previous section, evaluating all possible vectors \( \delta \) to verify the condition of \( \Sigma \)-countermonotonicity corresponds to considering \( 2^d - 1 \) partitions of the matrix (see Footnote 8). Doing so becomes infeasible for even moderate values of \( d \). One solution, proposed in Bernard and McLeish (2015), is to consider a smaller (randomly obtained) subset \( K \) of the set of all partitions. In Algorithm 3, we implement a similar idea, in each iteration considering one random partition obtained by Bernoulli sampling, in which the probability for a matrix column to belong to either of the blocks is always 50\% (uniform sampling over all partitions). We call this algorithm the (random) Block RA (BRA) in the following.

**Algorithm 3:** Pseudo-code of the procedure random BRA to minimize the variability \( V \) of the row sums of a matrix \( X \). Blocks are chosen randomly.

1. Set target value \( a^* \), and maximum number of iterations \( M \);
2. Set \( \pi \) to a randomly chosen intra-column arrangement of the elements;
3. Set \( a := V(S_\pi) \) with \( S_\pi := \sum_{j} X_{\pi j}^2 \) and let \( m = 0 \);
4. **while** \( a > a^* \) and \( m < M \) **do**
   5. Generate a random vector \( \delta \in \{0, 1\}^d \) by Bernoulli sampling with equal probability for each column to be assigned to a block;
   6. Rearrange the rows of the submatrix \( X_\delta^\pi \) so that the row sums \( S_\delta^\pi \) become oppositely ordered to \( S_{\pi - \delta}^\pi \), and compute \( a := V(S_\delta^\pi + S_{\pi - \delta}^\pi) \);
7. Set \( m = m + 1 \).

A variation of this algorithm would correspond to a non-uniform sampling: using different probabilities for each column to be included in the first block, \( \delta_j \sim \text{Bernoulli}(p_j), \ j = 1, \ldots, d \).

\(^9\)Gray code is an encoding of integers into strings of binary digits, see http://mathworld.wolfram.com/GrayCode.html.
Another variation would be sampling the size $N$ of the first block $P(N = i) = q_i, i = 1, \ldots, d-1$, and then selecting the first block uniformly over all blocks of size $N$ (for example, $q_1 = 1$ and $q_i = 0, i \geq 2$, would lead to RA with a random column selection instead of sequential). However, these alternatives pose the practical problem of having to calibrate the block-selection probabilities \textit{ex ante}; we do not investigate them further.

3.4 Block RA with Variance Equalization (BRAVE)

In the previous sections, we have observed that the number of possible partitions of the matrix columns is prohibitively large even for moderate values of $d$. Therefore, a method for choosing those partitions which yield the greatest improvement is desirable. Bernard and McLeish (2015) (see their Block RA1) propose selecting the partition $\tilde{\delta}$ by

$$\tilde{\delta} = \arg\max_{\delta \in \mathcal{P}} \phi(S^\pi_\delta, S^\pi_{1-\delta}),$$

where $\phi$ is Spearman’s rank correlation and $\mathcal{P} \subset \{0,1\}^d$ is a randomly selected subset of all partitions, consisting of $|\mathcal{P}| = \min(2^{d-1} - 1, 512)$ partitions to keep the above problem tractable. They also note that using Pearson’s correlation as $\phi$ would make it computationally easier, since $\phi(S^\pi_\delta, S^\pi_{1-\delta})$ for any $\delta$ can then be computed from the covariance matrix of $X^\pi$. However, Spearman’s correlation has the advantage that it is equal to $-1$ for oppositely ordered vectors, so the rearranged matrix is $\Sigma$-countermonotonic if and only if Spearman’s correlation is $-1$ for all partitions.

The above approach has two drawbacks. First, since the method needs to compute the correlation for each considered partition, in practice, it cannot consider all possible partitions; and finding $\tilde{\delta}$ is rather time-consuming even with $|\mathcal{P}| = 512$. Second, using correlations ignores the scales of the partial row sums, so this method may choose a partition where the two partial row sums are of very different scales, resulting in little improvement in $V(P)$.

We propose a new block selection rule, which is based on \textit{variance equalization}. In particular, consider an initial matrix $X^\pi$. An optimal choice for the two blocks $X^\pi_\delta$ and $X^\pi_{1-\delta}$ would clearly occur when, after rearranging $S^\pi_\delta$ in an opposite order to $S^\pi_{1-\delta}$, it holds that $S^\pi_{\delta'} + S^\pi_{1-\delta'} = E[S^\pi]$, where $\pi'$ denotes the new rearrangement. A necessary condition to fulfill this condition is that, before rearranging, $S^\pi_\delta - E[S^\pi_\delta]$ and $E[S^\pi_{1-\delta}] - S^\pi_{1-\delta}$ have the same distribution. Since by construction the first moments of these distributions are already matched, this motivates selecting an optimized partitioning vector $\delta^*$ so that the second moments of the distributions are matched as closely as possible. Namely, we choose the partition $\delta^*$ for which the difference in the variances of the two distributions is minimized, i.e.

$$\delta^* = \arg\min_{\delta} |\text{Var}(S^\pi_\delta) - \text{Var}(S^\pi_{1-\delta})|.$$
Equivalently,
\[ \delta^* = \arg\min_{\delta} |\text{Cov}(S^\pi_{\delta}, S^\pi) - \text{Cov}(S^\pi_{1-\delta}, S^\pi)|, \]

since \( \text{Var}(S^\pi_{\delta}) - \text{Var}(S^\pi_{1-\delta}) = \text{Cov}(S^\pi_{\delta}, S^\pi_{1-\delta}) - \text{Cov}(S^\pi_{1-\delta}, S^\pi_{1-\delta}) \). The latter formulation of the block selection problem is relatively easy to solve. In fact, denoting \( \beta^\pi_j = \text{Cov}(X^\pi_j, S^\pi) \),

we can express \( \text{Cov}(S^\pi_{\delta}, S^\pi) = \sum_{j=1}^{d} \delta \beta^\pi_j \). Finding two blocks with row sums that have equal variance is thus an example of the number partitioning problem, where the set \( \{\beta^\pi_1, \beta^\pi_2, \ldots, \beta^\pi_d\} \) needs to be partitioned so that \( \sum_{j=1}^{d} \delta \beta^\pi_j = \sum_{j=1}^{d} (1 - \delta_j) \beta^\pi_j \) for the corresponding \( \delta \).

A simple and intuitive heuristic for this problem is the so-called greedy algorithm, which iterates through the numbers and places the largest remaining number (in absolute value) into the subset that yields the smallest difference between the respective sums.\(^{11} \) Another possibility is to use the Karmarkar-Karp differencing algorithm (Karmarkar and Karp, 1982), which tends to be more effective in finding partitions with equal variance, but is slower.

It is unlikely that the above block selection rule yields an optimal rearrangement after one iteration, and thus, in the following iterations one needs to ensure that the chosen blocks are sufficiently different from the blocks in the previous iteration. To this end, starting from the second iteration, we modify the partitioning heuristics by first partitioning \( \{\beta^\pi_j : \delta_j = 1\} \) (where \( \delta \) is the previous partition), obtaining a difference \( \tilde{\beta} \) in variances. We then partition \( \{\tilde{\beta}\} \cup \{\beta^\pi_j : \delta_j = 0\} \). This then induces a partitioning of all columns, which is typically different from the partition in the previous iteration.

The pseudo-code describing the sequence of steps for the Block RA with blocks selected to have similar variances is shown in Algorithm 4. Similarly to the other algorithms, it requires setting a target value for the objective function \( a^* \) and the maximum number of iterations \( M \). In order to avoid that the algorithm gets stuck in a partition for which the blocks are already oppositely ordered (and the rearrangement does not reduce the variability measure), it is specified that when this happens, a randomly chosen block selection is used instead of the one based on variance equalization. An alternative option, which we explore further in Section 3.5, is to permanently switch to the standard rearrangement algorithm. We call this algorithm BRAVE+RA and provide its pseudo-code in Algorithm 5.

\(^{11}\)The greedy algorithm has a running time of \( O(d \log d) \) and is flexible enough to deal with real numbers. Most partitioning algorithms are designed to work with integer numbers or positive real numbers. See Korf (1998) for a review of alternative solutions.
Algorithm 4: Pseudo-code of the BRAVE algorithm to minimize the variability $V$ of the row sums of a matrix $X$. Blocks are selected to have similar variances. If no improvement is observed, a random block is chosen.

1. Set target value $a^*$ and maximum number of iterations $M$;
2. Set $\pi$ to a randomly chosen intra-column arrangement of the elements;
3. Set $a := V(S^\pi)$ with $S^\pi := \sum_{j=1}^d X_j^\pi$ and let $m = 0$;
4. while $a > a^*$ and $m < M$ do
   5. Block selection: Find $\delta^* = \arg\min_\delta |\text{Cov}(S^\pi_{\delta^*}, S^\pi) - \text{Cov}(S^\pi_{1-\delta^*}, S^\pi)|$;
   6. Rearrange the rows of the submatrix $X^\pi_{\delta^*}$ so that the row sums $S^\pi_{\delta^*}$ become oppositely ordered to $S^\pi_{1-\delta^*}$, and compute $a_{\delta^*} := V(S^\pi_{\delta^*} + S^\pi_{1-\delta^*})$;
   7. if $a_{\delta^*}$ is equal to $a$ then
      8. Set $\delta$ to a randomly chosen partition;
      9. Rearrange the rows of the submatrix $X^\pi_{\delta}$ so that the row sums $S^\pi_{\delta}$ become oppositely ordered to $S^\pi_{1-\delta}$, and compute $a := V(S^\pi_{\delta} + S^\pi_{1-\delta})$;
   else
      10. Set $a = a_{\delta^*}$;
   end
   11. Set $m = m + 1$.
end
Algorithm 5: Pseudo-code of the BRAVE+RA algorithm to minimize the variability $V$ of the row sums of a matrix $X$. Blocks are selected to have similar variances. If no improvement is observed, the algorithm switches permanently to the standard RA.

1. Set target value $a^*$ and maximum number of iterations $M$;
2. Set $\pi$ to a randomly chosen intra-column arrangement of the elements;
3. Set $a_{\delta^*} := V(S^\pi)$ with $S^\pi := \sum_{j=1}^d X^\pi_j$, and let $m = 0$, switch2RA = false;
4. while $a > a^*$ and $m < M$ and switch2RA is equal to false do
   5. Block selection: Find $\delta^* = \arg\min_{\delta} |\text{Cov}(S^\pi_{\delta^*}, S^\pi) - \text{Cov}(S^\pi_{1-\delta^*}, S^\pi)|$;
   6. Rearrange the rows of the submatrix $X^\pi_{\delta^*}$ so that the row sums $S^\pi_{\delta^*}$ become oppositely ordered to $S^\pi_{1-\delta^*}$, and compute $a_{\delta^*} := V(S^\pi_{\delta^*} + S^\pi_{1-\delta^*})$;
   7. if $a_{\delta^*}$ is equal to $a^*$ then
      8. switch2RA = true;
   else
      9. Set $a := a_{\delta^*}$;
   end
   10. Set $m = m + 1$.
end
11. if switch2RA is equal to true then
   12. Set $j = 1$;
   13. while $a > a^*$ and $m < M$ do
      14. Rearrange column $j$ in the matrix $X^\pi$ so that it becomes oppositely ordered to the sum of the other columns, and compute $a := V(S^\pi)$;
      15. Set $j = j + 1$ and $m = m + 1$;
      16. if $j > d$ then
         17. Set $j = 1$.
   end
end
3.5 Numerical illustration

All rearrangement algorithms described in the previous sections are designed to improve the objective function (satisfying Assumption 1) at each iteration. They differ in terms of the selected (block of) column(s) that is rearranged at each iteration and thus in terms of the magnitude of the improvement. In this section, we compare the effectiveness of the RA and various versions of the Block RA for minimizing the variance of the row sums, by rearranging matrices that have \( n = 100 \) or \( n = 1000 \) rows and \( d = 100, 1000, \) or \( 10,000 \) columns. Specifically, we compare the RA, random BRA and BRAVE, as well as the BRAVE+RA procedure, which aims at combining the best of the two worlds by using BRAVE for the initial rearrangements and then switching to RA when no improvement is found.\(^{12}\)

We expect that the first few iterations in the block rearrangement algorithms will have the largest (absolute) impact on the objective functions and that afterwards the improvements will be relatively small. We further expect that these algorithms (in particular, BRAVE) will yield greater initial improvements than when the RA is used and that the gains will tend to be larger when the number of columns \( d \) is large and the matrix columns are dissimilar. We verify this intuition by sampling and rearranging \( N = 100 \) matrices for twelve possible configurations. Besides varying \( n \in \{100, 1000\} \) and \( d \in \{10^2, 10^3, 10^4\} \), we also vary the choice of the distribution used to generate the matrices. The matrix elements are random draws from either the uniform distribution on \([0, 1]\) or from the Pareto distribution with shape parameter equal to 2 and a unit scale parameter. These two distributional choices represent two very different scenarios in the sense that in the case of the uniform distribution, all columns are similar and the matrix is close to being compatible with complete mixability. By contrast, in the Pareto case, even though all columns are generated using draws from the same distribution, the columns can be very dissimilar due to the fat tail of the distribution. We have tested several other distributional choices (such as exponential and beta), but our findings can be summarized using these two cases.

In Figures 1 and 2, we plot the progress of each algorithm in terms of reducing the variance of the row sums. In particular, we sampled \( N = 100 \) different matrices with entries from the uniform or Pareto distribution, respectively, and plotted the average value of the natural

\(^{12}\)The results for the BRAVE algorithms are based on the greedy algorithm for finding the blocks. Using the Karmarkar-Karp algorithm, we obtained similar results, but the computation times were longer. We also tested the Block RA1 algorithm of Bernard and McLeish (2015) with block selection as in (3). These simulations showed that block selection based on maximal Spearman correlation of the row sums in a few iterations leads to a rearranged matrix with maximal Spearman correlation that is close to −1, but the improvement in the objective function \( V \) is relatively slow compared to the RA, BRAVE and BRAVE+RA algorithms. Moreover, the Spearman correlation approach requires sorting the partial row sums for all considered partitions at every iteration, and is time-consuming. For the sake of brevity, we do not report these results.
logarithm of the variance of $S^\pi$ after $i = 1, \ldots, 500$ iterations. Namely, the plots show

$$\frac{1}{N} \sum_{j=1}^{N} \log(\text{Var}(S^\pi_{i,j})), $$

where $(S^\pi)_{i,j}$ vector or row sums after the $i$th iteration of the $j$th experiment run. The solid gray line corresponds to the trajectory of the RA, the black dotted line is the average evolution of the objective when random BRA is used, and the gray and black dashed lines correspond to rearrangement using BRAVE and BRAVE+RA.

Figure 1: The effect of the rearrangement algorithms on the trajectory of the natural logarithm of the variance of the row sums as a function of the number of iterations (rearrangements), when the matrix entries are sampled independently from a standard uniform distribution. The lines show the averages over 100 replications for matrices with the given number of rows $n$ and number of columns $d$.

Consider first the trajectories in Figure 1, which shows the results from rearranging matrices with entries sampled from the standard uniform distribution. As expected, we see that for the
initial rearrangements, in all cases considered, larger improvements in the objective function are obtained by rearranging blocks rather than single columns. However, the RA eventually catches up in each setup. The larger the number of columns, the more iterations the RA requires to catch up with the block RA. Since the columns have similar distributions, the performance of the random BRA and BRAVE is similar. Overall, it seems that in the case of matrices with similar columns, the classical RA catches up quickly with the block RA and there are no significant gains in using the block rearrangement algorithms, provided a moderately high number of iterations is used.

Figure 2: The effect of the rearrangement algorithms on the trajectory of the natural logarithm of the variance of the row sums as a function of the number of iterations (rearrangements), when the matrix entries are sampled independently from a Pareto distribution with shape parameter 2 and unit scale. The lines show the averages over 100 replications for matrices with the given number of rows $n$ and number of columns $d$.

The conclusions are different in the case of high-dimensional matrices for which the columns can be very different. This is shown in Figure 2, where the matrix elements are obtained by
drawing from a fat-tailed Pareto distribution. Because of the heavy tails, some columns contain
outlying values and thus the selection method for the block of columns to be rearranged mat-
ters. We see that in this case, the BRAVE+RA algorithm (black dashed line) outperforms the
RA (solid gray line) in terms of finding a lower objective function within a given number of
iterations, for \( d \geq 1000 \). Importantly, the proposed BRAVE heuristic, where blocks are selected
based on variance equalization, clearly dominates the random block selection.

Overall, the numerical experiments confirm the intuition that the block rearrangement algo-
rithms achieve larger initial improvements in the objective function than the classical RA. When
the columns are similar (homogeneous), random block selection and the BRAVE approach lead
to similar results. In addition, after a moderately large number of iterations, the classical RA
catches up with block rearrangement algorithms and even outperforms them. When the ma-
trix has columns that are highly heterogeneous, then the block selection method matters. The
BRAVE approach outperforms the random block selection and the BRAVE+RA heuristic out-
performs the classical RA. As a consequence, BRAVE can be seen as a useful “pre-solver” for
the RA.

4 Applications of rearrangement algorithms in operations re-
search

The rearrangement algorithms are suitable heuristics for the problem of assembly line crew
scheduling and the assessment of model risk; see e.g. Haus (2015) and Embrechts et al. (2013).\textsuperscript{13}
In this section, we show that several problems in operations research can be cast as rearrange-
ment problems, and thus rearrangement algorithms can in principle be applied. We first describe
the \( k \)-partitioning problem and formulate it as a matrix arrangement problem. We provide re-
results of a numerical experiment showing the superiority of the BRAVE heuristic in solving the
2-partitioning problem, compared to using the RA or two classical algorithms for the number
partitioning problem (greedy and Karmarkar-Karp). We briefly describe that, besides the assem-
bley line crew scheduling problem and the \( k \)-partitioning problem, also the the parallel machine
scheduling problem and the subset sum problem can be cast as a rearrangement problem.

\textsuperscript{13}Other noteworthy references include Puccetti and Rüschendorf (2012), Aas and Puccetti (2014) and Hofert
et al. (2015) for studies on the use of the RA for assessing model uncertainty when only the marginal distributions
of the risky components are known as well as Bernard and Vanduffel (2015a), Bernard et al. (2015a) and Jakobsons
and Vanduffel (2015) for extensions that make it possible to deal with cases when partial dependence information
is available.
4.1 Using (B)RA(VE) to solve the k-partitioning problem

As an application of our results, we show that block rearrangement algorithms can be used as heuristics for the classic k-partition problem (Chopra and Rao, 1993), which is often encountered in fields such as telecommunications (fixed-spectrum frequency assignment problem), physics (spin glass problem), network planning, and sports team scheduling. This problem concerns the task of deciding whether a given multiset \( T \) of numbers with the total sum equal to \( s \) can be partitioned into \( k \) subsets \( T_1, T_2, \ldots, T_k \), such that all subsets have the same sum \( s/k \). In the optimization version of this problem, the objective is to find the smallest difference \( \max(T_i) - \min(T_i) \) (some algorithms also provide the corresponding \( k \) subsets). This problem is NP-hard. The case \( k = 2 \) is often simply called the number partitioning problem.

Hence, we assume that a set of \( d \) real numbers with the total sum \( s \) is given. We add \((k-1)d\) zeros to this set, so that the total cardinality is \( kd \), and we can represent the numbers as a matrix \( X \in \mathbb{R}^{k \times d} \), where the last \( k-1 \) rows contain only zeros. Let \( S \) denote the corresponding vector or row sums. Clearly, partitioning the set into \( k \) subsets with sums that are as similar as possible, amounts to finding a rearranged matrix \( X^{*} \) such that \( S^{*} \) is as constant as possible (and in the ideal situation, only assumes the value \( s/k \)). Using this formulation, we can apply the Block RA to find approximate solutions. In the case \( k = 2 \), a well-known algorithm for the partitioning problem is the Karmarkar-Karp differencing algorithm (KK); see Karmarkar and Karp (1982). However, the matrix corresponding to the solution given by KK may not be \( \Sigma \)-countermonotonic, and hence may be outperformed by the Block RA. We illustrate this with an example.

Example 2 (Comparing Karmarkar-Karp’s algorithm with the Block RA). Consider the set \( T = \{771, 121, 281, 854, 885, 734, 486, 1003, 83, 62\} \) with the total sum \( s = 5280 \). The task is to partition \( T \) into two subsets \( T_1 \) and \( T_2 \) such that their total sums \( t_1 \), respectively, \( t_2 \) are as close as possible. Applying the differencing algorithm of Karmarkar-Karp yields \(|t_1 - t_2| = 26\). However, applying the brute force Block RA reveals that a perfect partition exists, namely \( T_1 = \{771, 281, 854, 734\} \) and \( T_2 = \{121, 885, 486, 1003, 83, 62\} \). In fact, when randomly permuting the elements within the columns of the initial matrix \( X \), we always found the optimal partition in less than 1000 iterations. Note also that applying the greedy algorithm would yield a discrepancy of 32 between the two subset sums; also application of the classic RA does not yield an improvement.\(^{14}\)

As noted by Boettcher and Mertens (2008), a further drawback of the Karmarkar-Karp differencing algorithm is that it gives the value of the discrepancy, but implementing an algo-

\(^{14}\)In fact, from Proposition A.1 in Bernard et al. (2015c) it follows that the greedy algorithm yields an arrangement where each column is oppositely ordered to the sum of the other columns and is thus - in the particular context of the partition problem - as good as the RA.
algorithm that finds the actual partitions is not straightforward. Karmarkar and Karp (1982) illustrate a “bottom-up” analysis to obtain the partitions, while Boettcher and Mertens (2008) recommend using graphs. An alternative is to apply the proposed Block RA to the matrix $X$, augmented with an additional column, which consists of a zero and the discrepancy given by the Karmarkar-Karp difference algorithm. After rearrangement by the Block RA, the first $d$ columns of the rearranged matrix correspond to a partition with the same discrepancy as the solution to Karmarkar-Karp difference algorithm. For the example mentioned above, the partitions obtained are $T_1^{KK} = \{771, 734, 1003, 83, 62\}$ and $T_2^{KK} = \{121, 281, 854, 885, 486\}$.

4.2 Numerical study

In this section, we present a numerical case study comparing performance of the various rearrangement algorithms with that of the classical number partitioning algorithms: greedy and Karmarkar-Karp (KK). To generate instances of the 2-partitioning problem, we take into account the observations in Gent and Walsh (1998) and Mertens (1998); namely, that the size $d$ the set to be partitioned, and the possible range of the numbers, affects the difficulty of the optimization problem in a specific way. In particular, instances with $d$ integers drawn uniformly from $\{1, \ldots, 2^d\}$ are typically difficult to solve, and only one partition, on average, is perfect (with the difference in sums $\Delta := |t_1 - t_2| \leq 1$). We therefore generate $N = 1000$ random instances as described above, for each $d = 10, 11, \ldots, 20$. We choose these relatively low values for $d$, as this allows solving the problem exactly, using a dynamic programming formulation of this problem; see e.g. Gent and Walsh (1998). Note, however, that the heuristic algorithms can also be applied to much larger problem instances. The corresponding average (log-)suboptimality is computed as

$$\frac{1}{N} \sum_{i=1}^{N} \log(1 + \Delta_i^{\text{Heur}} - \Delta_i^{\text{Opt}}),$$

where $\Delta_i^{\text{Heur}}$ and $\Delta_i^{\text{Opt}}$ are the differences given by the heuristic, respectively, by the optimal solution, for the $i$th problem instance. The adjustment by 1 inside the logarithm is to count as $\log(1) = 0$ the cases when a heuristic attains the optimal solution.

In Figure 3, left panel, the average suboptimality is plotted for the various heuristics, depending on $d$. For the rearrangement algorithms, 200 iterations are used. The poorest performer is the RA, followed by the greedy algorithm and the (random) BRA. The best results are given by the two versions of the Block RA that use variance equalization when selecting the blocks; they have a lower suboptimality than KK. By BRAVE(greedy) and BRAVE(KK) we refer to the two versions of the BRAVE algorithm, which use the greedy, respectively, the KK algorithm to partition the column covariances (see Section 3.4). BRAVE(KK) is clearly the best performer. Note that in this case study, the initial arrangement of the input matrix is co-monotonic: the first
Figure 3: Left panel: Suboptimality of the greedy and KK solutions, as well as the solutions of the rearrangement algorithms after 200 iterations. Right panel: The development of average suboptimality of BRAVE(KK) solution, as a function of the number of iterations (rearrangements). The lines correspond to selected values of $d \in \{10, \ldots, 20\}$.

row contains the $d$ numbers to be partitioned (denote them by $a_1, \ldots, a_d$), and the second row - only zeros. Correspondingly, in the first iteration of BRAVE(KK), the covariance of the $j^{th}$ column with the row sum vector $(s, 0)^T$ is equal to $sa_j/2 - s/2 \cdot a_j/2 = sa_j/4$, so the covariances to be partitioned are proportional to the given numbers. Therefore, the partition obtained after one iteration is the same as given by KK applied to the original set of numbers. We see, however, that further iterations bring the solution closer to optimality. To illustrate this convergence, the right panel of Figure 3 shows the suboptimality of BRAVE(KK) for different sizes $d$ of the set to be partitioned, as a function of the applied rearrangements (iterations).

4.3 Parallel machine scheduling problem

Given $d$ jobs with processing times $p_1, \ldots, p_d$ and $k$ parallel identical machines which can execute at most one job at a time, the parallel machine scheduling problem aims at assigning each job to exactly one machine, so as to minimize the latest completion time of all jobs. It is clear that this problem can be cast as a $k$-partitioning problem. Hence, the rearrangement algorithms can be applied to this problem as competitors to other known heuristics proposed in Frangioni et al. (2004) and Alvim and Ribeiro (2004). Exact algorithms are presented in Dell’Amico and Martello (1995) (branch-and-bound algorithm), Mokotoff (2004) (cutting-plane algorithm) and Dell’Amico et al. (2008) (algorithm based on a specialized binary search and a branch-and-price scheme). Of course, it cannot be expected that these algorithms yield the optimal solution in polynomial time.
4.4 The subset sum problem

Another closely related problem that can be approximately solved with rearrangement algorithms is the subset sum problem. For a given set of numbers with the total sum equal to $s$ and a given constant $c \leq s$, the task is to find a subset for which the sum is as close as possible to $c$. The subset problem can be cast as a 2-partitioning problem in a straightforward way (see also Korf (1998)). It is sufficient to add to the set a new number $t$ such that $t + s = 2c$. One can then apply a partitioning algorithm, and the partition that does not contain the number $t$ yields the solution to the subset sum problem.

5 Conclusion

In this paper, we discussed the problem of finding the intra-column rearrangement of a matrix that minimizes the variability of its row sums. Our first contribution is that we characterize the optimal solution to this challenging combinatorial problem and we present methodological innovations relative to the current literature. In particular, as an alternative to rearranging single columns (RA) or randomly chosen blocks of columns (BRA), we propose the BRAVE algorithm, which rearranges blocks that are as similar as possible (in terms of the variance of their row sums) and, if the objective function no longer improves, the algorithm either uses a random block selection or switches permanently to rearranging single columns (BRAVE+RA). Our second contribution is to show that (block) rearrangement algorithms can be used as heuristics for solving other problems such as the $k$-partitioning problem. For the problem of 2-partitioning (number partitioning), block rearrangement algorithms, and in particular the BRAVE algorithm, appear superior to well-known existing algorithms. For large dense matrices, the numerical results show that while block rearrangement algorithms (in particular BRAVE) yield a sharp initial improvement in the objective function, the RA catches up quickly. When the number of iterations is sufficiently large, the RA is competitive with the block rearrangement algorithms.
Edgars Jakobsons.

**Suboptimality in portfolio conditional value-at-risk optimization.**

Suboptimality in portfolio
conditional value-at-risk optimization

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Abstract
In this paper, the portfolio optimization problem with conditional value-at-risk as the objective is considered. We summarize commonly used methods of solution and note that the linear programming approximation is the most generally applicable and easy one to use (the LP uses a Monte Carlo sample from the true asset returns distribution). The suboptimality of the obtained approximate portfolios is then analyzed using a numerical example, with up to 101 assets and Student-t distributed returns, ranging from light to heavy tails. The results can be used as an estimate of the portfolio suboptimality for more general asset returns distributions, based on the number of assets, tail-heaviness, and the fineness of the discretization. Computation times using different techniques available in the literature are also analyzed.

Keywords: portfolio optimization, conditional value-at-risk, discretization error, suboptimality, linear programming, heavy tails.

1 Introduction

The Conditional Value-at-Risk (CVaR) has recently superseded Value-at-Risk (VaR) as the risk managers’ “favorite” risk measure due to its desirable theoretical properties of convexity and coherence (Artzner et al., 1999; Acerbi and Tasche, 2002a,b), as well as their practical implications on portfolio optimization. CVaR has gained further practical relevance by being proposed as the basis for trading book capital requirements by the Basel Committee on Banking Supervision; see Embrechts et al. (2014). Its popularity can be illustrated by the plethora of aliases used in the literature, including tail VaR (TVaR), average VaR (AVaR), expected shortfall (ES), expected tail loss (ETL) and conditional tail expectation (CTE).
One of its main advantages from a practical perspective is that the portfolio CVaR can be optimized efficiently using a linear programming (LP) approximation introduced in Rockafellar and Uryasev (2000). LP formulations are solvable for a large number of assets and constraints, and have been a crucial tool for practical portfolio analysis since the birth of the field (Markowitz, 1952); see Sharpe (1971) for an early example and Mansini et al. (2003, 2007) for a recent overview. The LP approximations are usually based on a discrete approximation of the true asset returns distribution (often assumed continuous). This is also the typical approach for portfolio CVaR optimization, often without regarding the errors that solving an approximate problem produces, both in terms of the resulting suboptimality of the obtained portfolio, as well as an inaccurate estimation of its true CVaR. In Rockafellar and Uryasev (2000) the aggregate of these two errors is considered in the case of a multivariate Gaussian distribution for the returns of three assets and CVaR with confidence levels up to 99%. The distribution is approximated using a Monte Carlo sample of size 20,000 and the obtained objective value is deemed to be close enough to the true optimal CVaR; the quality of the obtained portfolios is not investigated. Thereafter, the same sample size is used in Andersson et al. (2001) for a portfolio of 197 bonds and CVaR at 99.9% level. In Krokhmal et al. (2002), only 500 scenarios are used for the discrete joint distribution of 94 stock returns. Lim et al. (2011) point out the problem of portfolio suboptimality, and illustrate this issue by considering sample sizes up to 400 (typical for historical data) and heavy-tailed portfolios. A line of research has focused on finding portfolios that are robust to estimation errors, e.g. Vahn et al. (2014) develop a performance-based regularization method for improved out-of-sample performance of a portfolio driven by a short history of observations. However, in the idealized case when the underlying distribution of returns is known, an alternative is to generate a larger sample of scenarios in order to achieve a close-to-optimal portfolio. In the case of a bounded underlying distribution, Brown (2007) and Takeda and Kanamori (2009) give probabilistic bounds on the CVaR estimation error, in dependence on the sample size $N$ and the CVaR level $\alpha$. In the case of unbounded returns, however, a thorough analysis is required to see what sample sizes are sufficient so that the quality of the approximate solutions is still satisfactory when a high confidence level or a large number assets is considered.

Our contribution is a broad numerical case study, investigating the quality of the solutions obtained from the LP approximation for a range of up to 101 assets, heavy-tailed (Student-$t$) asset returns distributions and sample sizes up to 1,000,000. The discretization error is split into two parts and analyzed separately, with the main focus on the suboptimality of the obtained portfolio, showing that in typical cases it can reach several percent. This serves as a warning against using small sample sizes for the LP approximation. The results can be used as a guide for choosing a suitable sample size for the intended application, based on the number of assets, tail-heaviness of the returns distribution, and the CVaR confidence level.
This paper is organized as follows. In Section 2 we introduce the portfolio CVaR optimization problem and summarize the most common methods for solving it. In Section 3 the quality and computation time of the linear programming approximation is analyzed using a numerical case study, and Section 4 summarizes the findings.

2 Portfolio CVaR optimization

In this section we provide the formal definitions of the value-at-risk and conditional value-at-risk. Then, we introduce the portfolio CVaR optimization problem and provide an overview of methods for solving it. For a random variable $L \in L^p(\Omega, \mathcal{F}, P; \mathbb{R})$, $p \geq 0$, the VaR at confidence level $\alpha \in (0, 1)$ is defined as

$$\text{VaR}_\alpha(L) = \inf \{ q : P(L \leq q) \geq \alpha \}.$$  

This is the generalized inverse distribution function of $L$, so $\text{VaR}_\alpha(L)$ is the $\alpha$-quantile of $L$. If $p \geq 1$, the corresponding CVaR is then defined as

$$\text{CVaR}_\alpha(L) = \frac{1}{1 - \alpha} \int_0^1 \text{VaR}_u(L) du.$$  

In general, for some risk factors $Y \in L^p(\Omega, \mathcal{F}, P; \mathbb{R}^m)$, we can define the portfolio loss as a function of the decision variable $x \in \mathcal{X} \subset \mathbb{R}^d$ and the value of risk factors, $f : \mathcal{X} \times \mathbb{R}^m \to \mathbb{R}$, namely, $L \overset{\Delta}{=} f(x, Y)$. The optimal decision problem with CVaR objective is then

$$\text{minimize: } \text{CVaR}_\alpha(f(x, Y)) \quad \text{subj. to: } x \in \mathcal{X}.$$  

For our purposes, however, we will throughout assume that the portfolio consists of positions in $d$ assets, and $Y$ is the vector of returns of these assets over the considered time period (hence $m = d$). The decision vector $x \in \mathcal{X}$ is then the monetary value invested in each of the $d$ assets, and the portfolio loss is the bilinear function $f(x, y) = -x^\top y$. The feasible decision set $\mathcal{X}$ is assumed convex, often polygonal, e.g. for the normalized budget constraint $1^\top x = 1$, a minimum expected return $\mu^\top x \geq R$ or diversification constraints $\underline{b} \leq x \leq \overline{b}$, where these bounds are to be interpreted componentwise.

We now present the portfolio CVaR optimization approach from the seminal papers of Rockafellar and Uryasev (2000, 2002). Define the function $F_\alpha : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ as

$$F_\alpha(x, \zeta) = \zeta + \frac{1}{1 - \alpha} \mathbb{E}\left\{ \left[ -x^\top Y - \zeta \right]^+ \right\},$$  

where $[t]^+ = \max\{0, t\}$. If $Y$ admits a density $p(y)$, then this can be expressed using a $d$-dimensional integral

$$F_\alpha(x, \zeta) = \zeta + \frac{1}{1 - \alpha} \int_{y \in \mathbb{R}^d} \left[ -x^\top y - \zeta \right]^+ p(y) dy. \quad (1)$$
Rockafellar and Uryasev (2000) show that instead of using a two-stage optimization, we can optimize simultaneously over portfolio weights $x$ and the dummy variable $\zeta$,

$$
\min_{x \in X} \text{CVaR}_\alpha(-x^T Y) = \min_{(x, \zeta) \in X \times \mathbb{R}} F_\alpha(x, \zeta).
$$

(RU)

Furthermore, it is shown that one obtains as a by-product the optimizer $\zeta^*$, which is an $\alpha$-quantile of the loss distribution, i.e. $\zeta^* \in \{q \in \mathbb{R} : P(-x^* Y \leq q) = \alpha\}$.

2.1 Methods for solving the portfolio CVaR optimization problem

In this section we summarize commonly used methods for solving the optimization problem (RU). We assume that the true multivariate distribution of the asset returns is known so as to avoid the issue of model uncertainty and the resulting suboptimality of the portfolio. In practice, of course, we would need to statistically fit a multivariate distribution using historical data or expert opinion. The methods for portfolio CVaR optimization with a known asset returns distribution are described in the following two subsections.

2.1.1 Continuous distributions

If the asset returns distribution is continuous, as is usually assumed, then the following methods are available for portfolio CVaR optimization, from the general to more specific cases.

i) Numerically compute the integral in the definition (1) for each candidate portfolio considered by an optimization method that does not require derivatives, e.g. Nelder and Mead (1965). This is by no means numerically easy; see e.g. Dokov et al. (2008); Kim et al. (2010), where the issues of computing the one(!)-dimensional integral for skew-t and infinitely divisible distributions respectively are discussed. Hence, it is only practical for low dimensions (note that e.g. MATLAB has built-in numerical integration routines for only up to 3-dimensional domains.) This was the approach taken by Hellmich and Kassberger (2011), where a portfolio of 4 assets with multivariate generalized hyperbolic (mGH) distributed returns was optimized, however the numerical optimization procedure is not explained in detail.

ii) If the distributions of linear combinations of the asset returns belong to a parametric family, we can reduce the multi-dimensional integral in (1) to a one-dimensional integral, with the additional cost of finding the $\alpha$-quantile of the portfolio loss distribution. This may involve a numerical root search, see e.g. Surya and Kurniawan (2014) for the mGH case. In particular, for each candidate portfolio $x$ we would need to:

- compute the parameters of the loss distribution $G_x$ (the distribution function of $-x^T Y$);
- find the $\alpha$-quantile $z_\alpha = G_x^{-1}(\alpha)$ of the portfolio loss distribution, and
- numerically compute the tail expectation

$$
\text{CVaR}_\alpha(-x^T Y) = \frac{1}{1 - \alpha} \int_{z_\alpha}^{\infty} u \, dG_x(u).
$$
This procedure needs to be repeated for each candidate portfolio considered by the selected derivative-free optimization method. As a consequence, this approach is also computationally expensive. Note that in Hellmich and Kassberger (2011), although portfolios of mGH distributed assets belong to the parametric family of (univariate) GH distributions, they opted for the approach i) in order to avoid the root search.

iii) The optimization simplifies even further if the CVaR can be computed analytically for the parametric family of portfolio returns distributions. Andreev et al. (2005) provides formulae for several common univariate distributions, however the expressions are given in terms of the \( \alpha \)-quantile, hence the root-search cannot be avoided in general. Landsman and Valdez (2003) focus on the elliptical family, which is useful for portfolio optimization, since the multivariate elliptical family is closed under linear transformations. Elliptical distributions \( Y \sim E_d(\mu, \Sigma, \psi) \) are defined by the corresponding characteristic function

\[
\varphi_Y(t) = \exp \left( i t^\top \mu \right) \psi \left( \frac{1}{2} t^\top \Sigma t \right), \quad t \in \mathbb{R}^d.
\]

The elliptical family has the property that any linear combination \( x^\top Y \) of the margins (assets) with weights \( x \in \mathbb{R}^d \) belongs to a particular location-scale family (determined by \( \psi \)), with parameters \( \mu^\top x \) and \( \sqrt{x^\top \Sigma x} \) respectively; see also Section 3.3 in McNeil et al. (2005). Amongst the four properties of coherent risk measures (monotonicity, sub-additivity, translation invariance and positive homogeneity), Landsman and Makov (2011) identify the latter two as sufficient for reducing the portfolio risk optimization with multivariate elliptical asset returns to minimizing a combination of a linear functional and a square root of a quadratic functional. For a risk measure \( \rho \) we define

\[
\begin{align*}
\text{translation invariance:} & \quad \text{for } c \in \mathbb{R}, \rho(c + L) = c + \rho(L), \\
\text{positive homogeneity:} & \quad \text{for } s \geq 0, \rho(sL) = s \rho(L).
\end{align*}
\]

Since the CVaR is coherent and hence, in particular, a translation invariant and positive homogeneous risk measure, for elliptically distributed \( Y \) it has a simple analytic form

\[
\text{CVaR}_{\alpha}(x^\top Y) = -\mu^\top x + \lambda_{\alpha} \sqrt{x^\top \Sigma x},
\]  

(\text{LS})

where \( \lambda_{\alpha} \) is a coefficient that is different for each characteristic generator \( \psi \) and confidence level \( \alpha \), but does not depend on the portfolio \( x \). In fact, \( \lambda_{\alpha} = \text{CVaR}_{\alpha}(Z) \), where \( Z \sim E_1(0, 1, \psi) \). Landsman and Valdez (2003) provide explicit analytical formulae for \( \lambda_{\alpha} \) for several commonly used elliptical distributions; Shaw (2011) discusses the multivariate Student-\( t \) distribution in particular, in the context of portfolio CVaR optimization. Landsman (2008b,a) shows that (LS) is convex in \( x \), and under affine equality constraints the corresponding optimization problem has an analytic solution. However, this method does not apply if inequality constraints are present, such as the no-short-sales constraint or diversification constraints. In Rockafellar and Uryasev
(2000) it is noted that in the Gaussian case, if the minimum expected return constraint \( \mathbf{x}^{\top} \mu \geq R \) is present and active for the optimal portfolio, then we can replace it with an equality constraint and the objective function (LS) then becomes

\[
\text{CVaR}_\alpha (-\mathbf{x}^{\top} \mathbf{Y}) = -R + \lambda \mathbf{x}^{\top} \mathbf{\Sigma} \mathbf{x}.
\]

Embrechts et al. (2002) point out that this applies to all elliptical distributions; see also McNeil et al. (2005), Section 6.1.5. Thus, to find the optimal portfolio, it is sufficient to solve the Markowitz (1952) quadratic programming problem

\[
\begin{align*}
\text{minimize: } & \mathbf{x}^{\top} \mathbf{\Sigma} \mathbf{x} \\
\text{subj. to: } & \mathbf{1}^{\top} \mathbf{x} = 1, \quad \mathbf{x} \in \mathcal{X}.
\end{align*}
\]

(QP)

Alternatively, note that if the asset returns are assumed to have \( \mu = 0 \), then there can be no minimum expected returns constraint, but the problem of minimizing (LS) still reduces to (QP). We will use this special case in Section 3 as a benchmark for which the portfolio CVaR optimization problem can be solved exactly.

In the famous paper by Fama (1965) it is argued that the Gaussian distribution does not fit daily log-returns well due to observed heavy tails. Praetz (1972); Blattberg and Gonedes (1974) consider stable and Student-\( t \) distributions as alternatives, and conclude that the latter offers a better fit. More recently distributions such as multivariate generalized hyperbolic (mGH) and other normal mean-variance mixture distributions have been considered, as they can model the asymmetry observed in stock returns; see e.g. Aas and Hobæk Haff (2006); Hu and Kercheval (2007) and Section 3.2 in McNeil et al. (2005). However, Student-\( t \) offers a good trade-off for our case study: it models heavier tails than Gaussian, but it is also an elliptical distribution and hence particularly tractable for portfolio CVaR optimization. This family of distributions will be used in Section 3 to analyze the effects of heavier tails on portfolio CVaR estimation and optimization.

### 2.1.2 Discrete distributions

In the special case when the asset returns follow a finite discrete distribution, \( P(\mathbf{Y} = y_k) = p_k \) for \( k = 1, \ldots, N \), the optimization problem (RU) simplifies to minimizing

\[
F_\alpha (\mathbf{x}, \zeta) = \zeta + \frac{1}{1 - \alpha} \sum_{k=1}^{N} p_k \left[ -\mathbf{x}^{\top} \mathbf{y}_k - \zeta \right]^+
\]

over \( \mathbf{x} \) and \( \zeta \). Note that if we use the empirical distribution of historical data or a Monte Carlo sample, then all outcomes have equal probability \( p_k = 1/N, k = 1, \ldots, N \). Rockafellar and Uryasev (2000) show that in the discrete case the CVaR optimization problem reduces to a
linear program
\[
\begin{align*}
\text{minimize:} & \quad \zeta + \frac{1}{1 - \alpha} \sum_{k=1}^{N} p_k u_k \\
\text{over:} & \quad x \in \mathbb{R}^d, \quad \zeta \in \mathbb{R}, \quad u \in \mathbb{R}^N \quad (\text{LP}) \\
\text{subj. to:} & \quad x \in X, \quad u_k \geq -x^\top y_k - \zeta, \quad u_k \geq 0, \quad k = 1, \ldots, N.
\end{align*}
\]

There are three most common ways of obtaining a finite discrete approximation of the true (assumed continuous) underlying asset distribution:

i) Using the empirical distribution from historical data. This avoids the issue of statistically fitting a distribution to the asset returns. However, the observed outcomes may not include enough “large” losses, in particular since the CVaR is based only on a small fraction \((1 - \alpha)\) of the data. Krokhmal et al. (2002) use this approach based on 500 observations. As we shall see in Section 3, such a small sample size does not yield satisfactory results in terms of portfolio optimality. Moreover, Kondor (2014) warns that small sample sizes relative to the number of assets may lead to unbounded CVaR (so-called false arbitrage).

ii) Gaussian quadrature (sometimes called cubature in the multivariate case) methods are used for finding a discrete approximation that matches the first \(n\) moments of the model distribution, with a minimal number of discretization points; see Miller and Rice (1983); DeVuyst and Preckel (2007) and references therein. Since heavy-tailed distributions such as stable and Student-\(t\) are common in risk management, this method may not be applicable when only a limited number of moments are finite.

iii) Monte Carlo sampling from the fitted distribution. This method uses the generated sample as a discrete approximation; e.g. Rachev et al. (2008) optimize a portfolio of 47 assets with generalized stable returns distribution approximated by 3000 scenarios. The main advantage is that this method is easy to implement for commonly used asset distributions (efficient sampling algorithms are available), and it is possible to generate arbitrarily large samples, improving the quality of the discrete approximation. However, as the sample size increases, so does the time needed for solving the corresponding LP. Thus we need to choose a moderate sample size that offers a good trade-off between quality and speed of the optimization. This is discussed in more detail in the next section.

To summarize, apart from the special case of elliptical distributions (when CVaR optimization reduces to mean – standard deviation optimization), the most practical approach is to solve the LP based on a discrete empirical distribution. The latter is given either by the historical data or by a sample simulated from the fitted multivariate distribution. Since the LP solves an approximate problem, the selected portfolio will be suboptimal. How much higher the CVaR
Paper F

is for this suboptimal portfolio compared to the best possible CVaR depends on the true distribution and on the number of points used to approximate it. Furthermore, since the empirical distribution is itself random, the corresponding portfolio and its suboptimality are also random.

Finally, notice that not only the portfolio is suboptimal, but also the estimate of its CVaR obtained as the objective value of the LP (perceived CVaR) is not accurate. However, we will not focus on this problem, since there are efficient methods available for computing the CVaR, once a candidate portfolio is selected. For instance, any of the methods in Section 2.1.1 can be applied to compute the CVaR for a single portfolio of interest. Alternatively, one can generate a larger sample (than tractable for the LP) from the asset returns distribution, compute the corresponding portfolio returns, and average the worst \( (1 - \alpha)100\% \) losses. This would give a more accurate and unbiased estimate of the selected portfolio’s CVaR. Nevertheless, this does not give any information about how far this suboptimal CVaR is from the best possible. In the next section we investigate this issue by considering a numerical example.

3 Numerical case study

To enable computing the exact optimal CVaR as well as the discrete approximations, we will use elliptical distributions, for which the formula (LS) holds. In particular, we will consider multivariate normal \( \mathcal{N}(\mu, \Sigma) \) and multivariate Student \( t(\mu, \Sigma, \nu) \) distributions for asset returns. The parameters are selected as follows. As the location parameter \( \mu \) the zero vector is used, since for asset returns data it typically is much smaller than the standard deviation and not significantly different from zero. Moreover, this choice simplifies the portfolio CVaR optimization problem to (QP). To obtain a realistic dependence structure, for the scale matrix \( \Sigma \) we use the covariance matrix of financial time series. In particular, in this section we present the results using the covariance matrix of daily returns of up to 101 constituents of the FTSE100 index (see the list of companies in Table 10). This is computed using data from Bloomberg (2014) of daily closing prices on trading days from 2-Jan-2003 to 18-Sep-2014, adjusted for splits and dividends. Complete results are given in the Appendix, along with results obtained using analogous data for S&P100 constituents (listed in Table 11). Note that due to missing data (not all of the stocks were traded on the exchanges for the entire period) the covariance matrix was estimated from pairwise covariances (the obtained matrix was positive semi-definite). Since \( \Sigma \) is only the scale, not the covariance matrix for Student-\( t \), it is not intended that these parameters are a good fit for the original data, only a reasonable choice for a case study. Blattberg and Gonedes (1974) estimate the degrees of freedom parameter for most stock daily returns to be in the range \( \nu \in (4, 5) \), with occasional values above and below. Thus in our case study we consider a representative set of values \( \nu \in \{3, 5, 10, \infty\} \). Note that the limiting case \( \nu = \infty \) is the normal distribution. The optimization was performed over fully invested portfolios \( x \in \mathbb{R}^d \),
\[ \sum_{i=1}^{d} x_i = 1 \text{ of } d \in \{3, 5, 10, 25, 50, 101\} \text{ assets, under the no-short-sales constraint } x \geq 0. \]

As the benchmark, the minimum variance portfolios were computed for each \( d \) using (QP). Since \( \mu = 0 \), according to (LS) this immediately gives us the CVaR optimal portfolio (same for all \( \nu \) and \( \alpha \)) in the case of elliptically distributed asset returns. Furthermore, the corresponding CVaR can be computed from (LS), using values of \( \lambda_\alpha \) given on p. 45 in McNeil et al. (2005) as

\[ \lambda_\alpha = f(z_\alpha)/(1 - \alpha) \quad \text{for the normal distribution,} \]
\[ \lambda_\alpha = \frac{\nu + z_\alpha^2}{\nu - 1} f(z_\alpha)/(1 - \alpha) \quad \text{for Student-}t, \ \nu > 1, \tag{2} \]

where \( f \) in each case is the density function of the corresponding distribution (with location and scale parameters 0 and 1 respectively), and \( z_\alpha = F^{-1}(\alpha) \) is its \( \alpha \)-quantile. Similar expressions are available for other elliptical distributions; see Landsman and Valdez (2003). The values for \( \alpha \in \{0.9, 0.95, 0.99, 0.999\} \) and \( \nu \in \{\infty, 10, 5, 3\} \) are listed in Table 1. In agreement with intuition, taking a higher confidence level \( \alpha \) or heavier tails results in larger coefficients. Note that these coefficients also account for the standard deviation of the Student-\( t \) distribution when \( \nu > 2 \) (it is \( \sqrt{\nu/(\nu - 2)} \) for scale parameter 1). For \( 1 < \nu \leq 1 \) the expectation, and hence CVaR, is not defined. We can also observe that since Student-\( t \) has regularly varying tails, the ratio \( \lambda_\alpha/z_\alpha \) converges to \( \nu/(\nu - 1) \), as \( \alpha \to 1 \); see Section 7.2.3 in McNeil et al. (2005) (shape parameter \( \xi = 1/\nu \)).

Table 1: Quantile \( z_\alpha \) and coefficient \( \lambda_\alpha \) for the normal distribution (\( \nu = \infty \)) and Student-\( t \) with \( \nu \in \{10, 5, 3\} \) degrees of freedom, as given by the formulae (2).

<table>
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<th>( z_\alpha )</th>
<th>( \nu )</th>
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<th>95%</th>
<th>99%</th>
<th>99.9%</th>
<th>\lambda_\alpha</th>
<th>( \nu )</th>
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<th>95%</th>
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3.1 Portfolio suboptimality

In Rockafellar and Uryasev (2000) only the Gaussian case with \( d = 3 \) assets was considered, and just one instance was solved for selected sizes (up to \( 2 \times 10^4 \)) of the simulated random sample. To investigate more thoroughly the quality of the LP solution depending on the sample size, number of assets, underlying asset returns distribution and confidence level, we solved many instances of the LP, changing the parameters and random samples. We used sample sizes \( N \in \{10^3, 10^4, 10^5, 10^6\} \); \( d \in \{3, 5, 10, 25, 50, 101\} \), \( \nu \in \{\infty, 10, 5, 3\} \) and \( \alpha \in \{0.95, 0.99, 0.999\} \).
In each case, we generated 100 samples of the given size, and for each sample computed the corresponding LP-optimal portfolio and its CVaR as given by the objective value of (LP). Then, also the true CVaR for this portfolio was computed using the formula (LS).

![Figure 1: Suboptimality of the LP solution portfolio CVaR (in black) and underestimation of its true CVaR (in gray), both relative to the optimal CVaR. For each sample size \(N \in \{10^3, 10^4, 10^5\}\) the boxplot shows the spread of results of LP over 100 simulations. Parameters \(\alpha = 99\%, \nu = 5\) and \(d = 25\).](image)

In Figure 1, a boxplot of the obtained results is given for the case \(\alpha = 99\%, \nu = 5, d = 25\). For each sample size the spread of portfolio suboptimality over the 100 repetitions is plotted, as well as the underestimation of the obtained portfolio’s true CVaR. Namely, if we denote by \(\tilde{Y}\) the random vector following the discrete empirical distribution of the MC sample, by \(x_{\text{LP}}\) the optimal portfolio weights in the LP and by \(x_{\text{QP}}\) the true optimal weights, then

\[
\text{subopt} = \left[ \text{CVaR}_\alpha\left(-x_{\text{LP}}^\top Y\right) - \text{CVaR}_\alpha\left(-x_{\text{QP}}^\top Y\right) \right]/\text{CVaR}_\alpha\left(-x_{\text{QP}}^\top Y\right), \\
\text{bias} = \left[ \text{CVaR}_\alpha\left(-x_{\text{LP}}^\top \tilde{Y}\right) - \text{CVaR}_\alpha\left(-x_{\text{LP}}^\top \tilde{Y}\right) \right]/\text{CVaR}_\alpha\left(-x_{\text{QP}}^\top Y\right),
\]

where \(\text{CVaR}_\alpha\left(-x_{\text{LP}}^\top \tilde{Y}\right)\) is the optimal objective of the LP (perceived CVaR). As expected, the discretization errors reduce as the sample size increases. Note that we have separated the error in two parts, while in Rockafellar and Uryasev (2000) only the aggregate error was analyzed. The two errors affect the result in opposite directions, and the CVaR underestimation (bias) is typically higher than the suboptimality. This means that the perceived CVaR is often lower than the best possible CVaR, although the obtained portfolio is suboptimal, i.e. the results are overly optimistic. This is not always the case, however: while the obtained portfolio is always suboptimal (outcomes are above the 0% line in the plot), the perceived CVaR may also occasionally be larger than the portfolio’s true CVaR (underestimation is negative). Figure 1 is representative of the general results – the plots obtained for other values of \(\nu, d\) and \(\alpha\) show similar features. Note that given a candidate portfolio, for a general returns distribution we can obtain a better estimate of its true CVaR by using a new, larger sample with, say, \(N = 10^7\) (while optimization
with this sample size would be too costly; see Section 3.2.) As a Monte Carlo estimator, it would be consistent and unbiased. This, however, is not true for the estimated CVaR that we get as the optimal objective value of the LP, as we can observe in Figure 1 (bias boxplots are not centered around zero), because the LP selects those assets for the portfolio, which “happened” to not have large joint losses in the generated sample.

The portfolios obtained from the LP using $N = 10^5$ sample points seem rather close to optimal in Figure 1. To zoom in closer, in Tables 2 and 3 the 50th and 90th percentile of suboptimality among the 100 (random) instances is given, for samples of size $N = 10^4$ and $N = 10^5$ respectively. We see that samples of size $N = 10^4$ give satisfactory results in the case of lighter tails or a small number of assets, while the performance deteriorates in the heavy-tailed, higher-dimensional case. In these cases we would need a sample of size at least $N = 10^5$ to reduce the portfolio suboptimality to under 1%.

Table 2: 50th and 90th percentile of suboptimality, given in % of the optimal CVaR$_{0.99}$. Portfolios were obtained by solving 100 instances of the LP based on Monte Carlo samples of size $N = 10^4$.

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<th>$d$</th>
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<th>10</th>
<th>25</th>
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Table 3: 50th and 90th percentile of suboptimality, given in % of the optimal CVaR$_{0.99}$. Portfolios were obtained by solving 100 instances of the LP based on Monte Carlo samples of size $N = 10^5$.

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In Table 6 (see Appendix), the full results for $N = 10^5$ and $\alpha \in \{0.95, 0.99, 0.999\}$ are listed. With this higher sample size, the results are satisfactory except in the numerically most unfavorable cases - when $\alpha = 99.9\%$, $d \geq 10$, and $\nu \leq 10$. This is when particular care is required, e.g. by taking an even larger sample size, as portfolio suboptimality can reach several percent when $N = 10^5$. In Table 4 the suboptimality for $\alpha = 99.9\%$ and $N = 10^6$ is listed,
showing that this sample size is sufficient. To optimize the portfolio with such a high sample size, efficient algorithms suited for large-scale problems were used; see Section 3.2.

Table 4: 50th and 90th percentile of suboptimality, given in % of the optimal CVaR_{0.999}. Portfolios were obtained by solving 100 instances of the LP based on Monte Carlo samples of size $N = 10^6$.

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To check if these conclusions are sensitive to the covariance matrix $\Sigma$ used, in Tables 6 and 7 we list the results using $\Sigma$ corresponding to the FTSE100 and S&P100 dataset respectively. Comparing the two tables, we observe that using the FTSE100 covariance matrix for sampling, the suboptimality is consistently higher, up to twice that obtained using the S&P100 covariance matrix. As a hypothesis, this may be due to the higher correlations between the components of the S&P100 index, see Figure 2. The mean correlation amongst FTSE100 components is 0.21, while amongst those of S&P100 it is 0.29. This leads to smaller diversification opportunities, and thus the portfolio choice has a smaller impact on the risk.

![Figure 2: Histogram of the returns correlations amongst the FTSE100 and S&P100 constituents, resp.](image)

### 3.2 Computation times

In Section 3.1 we observed that large sample sizes $N$ are required to obtain portfolios that are close to optimal. In order to solve the corresponding large-scale LPs in a reasonable time, the choice of solution technique becomes crucial. Specialized methods are available in the literature, for example, using nondifferential optimization (Lim et al., 2010) or cutting-plane methods.
maximize: $\eta$ over: $\eta \in \mathbb{R}, \lambda \in [0, 1]^N$ 
subj. to: $\sum_{k=1}^{N} p_k \lambda_k = 1 - \alpha, \quad \eta + \frac{1}{1 - \alpha} \sum_{k=1}^{N} p_k \lambda_k y_k^{(i)} \leq 0, \quad i = 1, \ldots, d.$

The optimal portfolio weights are the Lagrange multipliers corresponding to the $d$ inequality constraints. Recently, a scenario aggregation procedure was introduced in Espinoza and Moreno (2014). This algorithm reduces the size of the Dual formulation by aggregating the outcomes (partitioning the scenario set), iteratively finding an approximate portfolio and refining the partition by separating the “good” outcomes from the “bad” ones, as indicated by the values of the approximate portfolio. We will refer to this approach as the Aggregation algorithm. Similarly, in the following (LP) will be referred to as the Primal formulation. In addition, since linear programs are typically solved using either the simplex method or the interior point (barrier) method, both methods were tested for comparison.

The computations were performed with IBM ILOG CPLEX 12.5, using MATLAB interface on a 2.2GHz AMD Opteron 6174. To estimate the computation times in dependence on the parameters $\nu$, $d$ and $N$, for each combination 10 instances corresponding to 10 MC samples were solved using the 3 different formulations and 2 solution methods (cases where the CPU time is expected to exceed 30 minutes were not run). Since the true optimal CVaR is of the order $10^{-2}$, to obtain enough significant digits, the tolerance parameter was set to $10^{-6}$ for the Primal and Aggregation formulations; for the Dual formulation tolerance $10^{-3}/N$ was required to get the same optimality gap. The obtained CPU times are listed in Tables 8 and 9. We observe that the Primal formulation is solved faster with the simplex method when $\alpha$ is high, while the barrier method performs better for lower values of $\alpha$; the computation times are similar for $\alpha = 99\%$. This observation is in agreement with Gotoh and Takano (2007), although there the application is a newsvendor problem. For the Dual and Aggregation formulations, the simplex method is faster for all considered $\alpha$ values. Overall, the best computation times are achieved using the Dual formulation.

In Figure 3, the computation times (using the simplex method) are plotted against $d$ and $N$ on a logarithmic scale; we observe an approximately linear dependence on the plotted domain for the Primal and Dual formulations, and on $d \geq 10, N \geq 3 \cdot 10^4$ for the Aggregation algorithm. Hence, we fit a linear model for log-time on the relevant domain (the input variables $d$ and $N$ are on an approximately regular grid in the log-scale). This yields $\tau = Cd^a N^b$ as a model for the computation time $\tau$ in minutes. In Table 5 the fitted coefficients are given for the different formulations and solution methods, along with the fitted time $\bar{\tau}$ for the values $d = 25, N = 10^5$. Using the simplex method, the computation time for the Primal formulation has an approximately quadratic dependence on the sample size $N$, while it is approximately $O(N^{1.5})$ for the
Figure 3: Computation times in minutes for $\alpha = 99\%$, plotted in a logarithmic scale against $d$ and $N$.

Table 5: Fitted parameters to the model $\tau = C d^a N^b$ for computation time $\tau$ in minutes. $\bar{\tau}$ is the fitted time for the case $d = 25, N = 10^5$. Matrix $\Sigma$ computed using FTSE100, respectively, S&P100 dataset.
Dual and Aggregation formulations. With the barrier method for Primal formulation, the rate improves to a similar level, but the reference time $\bar{\tau}$ is still much larger than for the other formulations. As the portfolio dimension $d$ increases, the computation time grows approximately as $O(d^{0.7})$ for Primal and Dual formulations, whereas for the Aggregation algorithm as $O(d^{0.6})$. Combined with a low reference time $\bar{\tau}$ and rate $O(N^{1.2})$ for $\alpha = 99.9\%$, the Aggregation algorithm outperforms the other approaches for large $N$; see Table 9. Furthermore, while the tail-heaviness $\nu$ does not have a significant effect on the CPU times with the Dual formulation, when using the Aggregation algorithm the optimization requires less time for heavy tails; see Figure 4. With $d = 101$ and sample size $N = 10^6$ the computation for $\nu = 3$ takes on average 2.6 minutes, whereas for $\nu = \infty$ it is 5.4 min. This is because in the heavy-tailed case it is easier to identify the “bad” scenarios and partition the scenario set accordingly in fewer iterations. Hence exactly in the cases where a large sample size is required (high $\alpha$ and heavy tails, as identified in Section 3.1), the Aggregation algorithm excels.

![Figure 4: Boxplot of computation times using the Aggregation algorithm for $d = 101$, $\alpha = 99.9\%$, and $\Sigma$ computed from FTSE100 dataset, plotted in a logarithmic scale against $N$ and $\nu$.](image)

### 4 Conclusion

We summarized the commonly used methods of portfolio CVaR optimization. It was argued that in general the most practical method is solving the LP corresponding to a discrete approximation of the true underlying asset returns distribution, while other methods are suitable only in special cases, e.g. low number of assets or joint elliptical distribution of returns. Using a numerical example based on realistic distributions and parameter values, we provide an estimate of how suboptimal the portfolios obtained from the LP approximation are (can be). We observe that for light-tailed asset returns or portfolios of size up to $d = 10$ the suboptimality is under 1% in most instances of the LP with sample size $10^4$, which is consistent with the results of the case study in Rockafellar and Uryasev (2000). However, for heavy-tailed asset returns distributions, or portfolios composed of 50 or more assets, it is likely that the suboptimality
exceeds 5%. Therefore, under these circumstances a sample of size $10^5$ is required for a good balance between the quality of solution and the computation time. This sample size yields a portfolio within 1% of the optimum with 90% confidence, except in the most adverse cases, and the required computation time is less than 1 minute. In the numerically problematic case $\nu \leq 10$ (typical for daily asset returns) and number of assets $d \geq 25$, the portfolio suboptimality may still exceed 1%, especially for high values of $\alpha$. In these cases it is recommended to use an even higher sample size (e.g. $10^6$) and optimization techniques that are suited for large-scale linear programs.

Acknowledgments

The author thanks Paul Embrechts and two anonymous referees for valuable comments and suggestions, and acknowledges financial support from RiskLab Switzerland and the Swiss Finance Institute.
### Appendix

Table 6: 50\(^{th}\) and 90\(^{th}\) percentile of suboptimality, given in % of the optimal CVaR. Portfolios were obtained by solving 100 instances of the LP based on Monte Carlo samples of size \(N\), and the covariance matrix of the FTSE100 companies.

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<th>(\nu_d)</th>
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Table 7: 50th and 90th percentile of suboptimality, given in % of the optimal CVaR. Portfolios were obtained by solving 100 instances of the LP based on Monte Carlo samples of size $N$, and the covariance matrix of the S&P100 companies.

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Table 8: Geometric mean computation times (in minutes) for the Primal formulation, depending on the sample size $N$, number of assets $d$ and significance level $\alpha$. Samples were generated using the covariance matrix of FTSE100 constituents.

| $\alpha = 95\%$ | Simplex method | | | | | | Barrier method | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $d$ | $N$ | 10K | 30K | 100K | 300K | 1M | $d$ | $N$ | 10K | 30K | 100K | 300K | 1M |
| 3 | 0.02 | 0.22 | 3.2 | - | - | 3 | 0.02 | 0.07 | 0.3 | 6.3 | - |
| 5 | 0.03 | 0.36 | 5.0 | - | - | 5 | 0.02 | 0.09 | 1.4 | 6.4 | - |
| 10 | 0.05 | 0.57 | 8.0 | - | - | 10 | 0.03 | 0.11 | 1.7 | 6.4 | - |
| 25 | 0.08 | 0.87 | 11.6 | - | - | 25 | 0.16 | 0.54 | 2.2 | 8.3 | - |
| 50 | 0.11 | 0.97 | 10.4 | - | - | 50 | 0.21 | 0.79 | 3.0 | 11.0 | - |
| 101 | 0.34 | 2.31 | 22.3 | - | - | 101 | 0.31 | 1.47 | 6.2 | 19.2 | - |

| $\alpha = 99\%$ | Simplex method | | | | | | Barrier method | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $d$ | $N$ | 10K | 30K | 100K | 300K | 1M | $d$ | $N$ | 10K | 30K | 100K | 300K | 1M |
| 3 | 0.01 | 0.07 | 1.0 | 10.4 | - | - | 3 | 0.04 | 0.16 | 0.7 | 13.8 | - |
| 5 | 0.01 | 0.14 | 1.9 | 19.4 | - | - | 5 | 0.04 | 0.16 | 3.3 | 13.0 | - |
| 10 | 0.02 | 0.18 | 2.3 | 24.0 | - | - | 10 | 0.04 | 0.19 | 3.6 | 13.4 | - |
| 25 | 0.04 | 0.32 | 4.0 | - | - | 25 | 0.24 | 0.94 | 4.6 | - | - |
| 50 | 0.06 | 0.39 | 4.4 | - | - | 50 | 0.33 | 1.24 | 5.7 | - | - |
| 101 | 0.14 | 1.00 | 10.0 | - | - | 101 | 0.33 | 2.25 | 10.3 | - | - |

| $\alpha = 99.9\%$ | Simplex method | | | | | | Barrier method | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $d$ | $N$ | 10K | 30K | 100K | 300K | 1M | $d$ | $N$ | 10K | 30K | 100K | 300K | 1M |
| 3 | 0.00 | 0.01 | 0.1 | 1.3 | 13.0 | 3 | 0.07 | 0.27 | 1.3 | - | - | - |
| 5 | 0.00 | 0.02 | 0.2 | 2.0 | 18.0 | 5 | 0.06 | 0.25 | 5.9 | - | - | - |
| 10 | 0.00 | 0.03 | 0.3 | 2.4 | 21.3 | 10 | 0.07 | 0.28 | 6.1 | - | - | - |
| 25 | 0.01 | 0.05 | 0.5 | 3.7 | - | - | 25 | 0.32 | 1.35 | 6.8 | - | - | - |
| 50 | 0.02 | 0.10 | 0.7 | 4.8 | - | - | 50 | 0.44 | 1.90 | 9.4 | - | - | - |
| 101 | 0.04 | 0.21 | 1.8 | 12.1 | - | - | 101 | 0.26 | 3.10 | 14.5 | - | - | - |
Table 9: Geometric mean computation times (in minutes) for the Dual and Aggregation formulations using the simplex method, depending on the sample size $N$, number of assets $d$ and significance level $\alpha$. Samples were generated using the covariance matrix of FTSE100 constituents.

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Table 10: Ticker symbols and names of companies in the FTSE100 stock index in September 2014.

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## Table 11: Ticker symbols and names of companies in the S&P100 stock index in September 2014.

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Edgars Jakobsons.

**Scenario aggregation method for portfolio expectile optimization.**

*Submitted.*
Scenario aggregation method for portfolio expectile optimization

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Abstract

The statistical functional expectile has recently attracted the attention of researchers in the area of risk management, because it is the only risk measure that is both coherent and elicitable. In this article, we consider the portfolio optimization problem with an expectile objective. Portfolio optimization problems corresponding to other risk measures are often solved by formulating a linear program (LP) that is based on a sample of asset returns. We derive three different LP formulations for the portfolio expectile optimization problem, which can be considered as counterparts to the LP formulations for the Conditional Value-at-Risk (CVaR) objective in the works of Rockafellar and Uryasev (2000), Ogryczak and Śliwiński (2011) and Espinoza and Moreno (2014). When the LPs are based on a simulated sample of the true (assumed continuous) asset returns distribution, the portfolios obtained from the LPs are only approximately optimal. We conduct a numerical case study estimating the suboptimality of the approximate portfolios depending on the sample size, number of assets, and tail-heaviness of the asset returns distribution. Further, the computation times using the three LP formulations are analyzed, showing that the formulation that is based on a scenario aggregation approach is considerably faster than the two alternatives.

Keywords: expectile, portfolio optimization, linear programming, scenario aggregation.

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This work has benefited from stimulating discussions with Fabio Bellini and Christian Colombo during a visit to the University of Milano-Bicocca in January 2016.
1 Introduction

The expectile is introduced in Newey and Powell (1987) as the minimizer of the expectation of an asymmetric quadratic scoring function with respect to a point forecast \( e \in \mathbb{R} \) of a random variable \( L \),

\[
e_{\tau}(L) = \arg\min_{e \in \mathbb{R}} \mathbb{E}[(\tau \mathbb{1}_{\{L>e\}} + (1-\tau)\mathbb{1}_{\{L\leq e\}})(L - e)^2].
\]  

(1)

Until recently, it was known almost exclusively amongst statisticians and was applied in regression analysis; see e.g. Efron (1991); Yao and Tong (1996); De Rossi and Harvey (2009). Currently, the expectile is making a debut in the area of risk management, due to functional properties that make it suitable as a risk measure for regulatory and portfolio management purposes.

The axioms of coherence, introduced in the seminal paper of Artzner et al. (1999), are often considered as indispensable for any risk measure used in practice, as motivated by their interpretation, as well as the resultant mathematical tractability. However, the most commonly used risk measure, Value-at-Risk (VaR) (Jorion, 1996), is not coherent. In particular, VaR does not satisfy the axiom of subadditivity, which requires that the risk measure captures diversification. Recently, Gneiting (2011) brought another property, elicitation, to the foreground. A risk measure is said to be elicitable if it is a minimizer of the expectation of some scoring function, which depends on the point forecast and the true observed loss. Gneiting (2011) further showed that the well-known coherent risk measure Expected Shortfall (ES, also known as the Conditional Value-at-Risk, CVaR) is not elicitable. Some authors associate elicitation with being able to back-test risk models based on the quality of forecasts that have been made in the past, a crucial requirement from a regulatory perspective (Carver, 2013, 2014; Hull and White, 2014). Other authors argue that elicitation is relevant for relative comparisons between estimators, but not for absolute significance testing (Acerbi and Szekely, 2014).

In Ziegel (2014); Bellini and Bignozzi (2015); Delbaen et al. (2015), it is shown that the only risk measure that is both elicitable and coherent is the expectile (for \( \tau \in [1/2, 1) \), when positive outcomes for the random variable \( L \) represent losses). This makes the expectile an interesting object for research with the aim of assessing its suitability for measuring risk in practice. First such studies include Kuan et al. (2009); Bellini and Di Bernardino (2015); Bellini and Bignozzi (2015); Jakobsons and Vanduffel (2015).

Another property of the expectile can be observed from the first order conditions on the optimization in (1), namely, \( e_{\tau}(L) \) is the unique solution of

\[
(1 - \tau) \mathbb{E}[(e_{\tau} - L)\mathbb{1}_{\{L\leq e_{\tau}\}}] = \tau \mathbb{E}[(L - e_{\tau})\mathbb{1}_{\{L>e_{\tau}\}}].
\]

In particular, while VaR an ES focus only on extreme losses, the expectile balances gains and losses, which is desirable for portfolio management. Because of this, the expectile is
also closely related to the Omega performance measure of Keating and Shadwick (2002); see Rémillard (2013), p. 128. Bellini and Di Bernardino (2015) also point out the expectile has an intuitive interpretation in terms of its acceptance set, namely, a risk is acceptable (for a regulator) if its gains-loss ratio (Bernardo and Ledoit, 2000) is sufficiently high. See Bellini and Di Bernardino (2015) and references therein for further properties of the expectile risk measure.

Before we conclude this section, we provide a simple iterative procedure for computing the expectile of a discrete rv, based on the method in Efron (1991) for fitting a linear regression with asymmetric least squares minimization. A special case of this method yields a procedure for computing the expectile of a discrete rv \( L \) such that \( P(L = \ell_i) = p_i, i \in \{1, \ldots, n\} \). Starting from a trial value \( b_0 \in \mathbb{R} \), we update iteratively for \( k = 0, 1, 2, \ldots \):

\[
b_{k+1} = \frac{\sum_{i=1}^{n} w_i^{(k)} \ell_i}{\sum_{i=1}^{n} w_i^{(k)}}, \quad \text{where} \quad w_i^{(k)} = |\tau - 1_{\ell_i > b_k}| p_i. \tag{IRLS}
\]

This is the method of iteratively reweighted least squares (IRLS).\(^1\) The sequence converges to the corresponding expectile, i.e., \( \lim_{k \to \infty} b_k = e_r(L) \). In practice, this procedure typically converges within a few iterations.

The rest of this article is organized as follows. In Section 2, we introduce the portfolio optimization problem with an expectile objective and derive three different LP formulations for this problem, when the asset returns follow a finite discrete distribution. In Section 3, we analyze the quality of the portfolios obtained from the LPs, when the latter are based on a Monte Carlo sample from the true (assumed continuous) asset returns distribution. We also compare the computation times required for solving each of the three LP formulations. Finally, Section 4 concludes.

### 2 Portfolio optimization

We consider a standard single period portfolio optimization problem, where a given amount \( w > 0 \) of wealth is to be invested in \( d \) available assets. If the current value of an asset is \( S_j(0) > 0 \) and its value at the end of the period (investment horizon) is \( S_j(1) \in L^1(\Omega, \mathcal{A}, \mathbb{P}) \), then the return of this asset is given by \( Y_j = S_j(1)/S_j(0) - 1, j = 1, \ldots, d \). Denoting by \( \mathbf{Y} = (Y_1, \ldots, Y_d)^\top \) the vector of asset returns and by \( \mathbf{x} \in \mathbb{R}^d \) the vector of monetary amounts invested in the corresponding assets at the beginning of the period, the random variable \( L = -\mathbf{x}^\top \mathbf{Y} \) corresponds to the portfolio loss. The possible portfolio positions (values of \( \mathbf{x} \)) have to satisfy a budget constraint \( \mathbf{x}^\top \mathbf{1} \leq w \) and typically further constraints, such as the expected returns constraint \( \mathbf{x}^\top \mathbf{\mu} \geq R \) (where \( \mathbf{\mu} = \mathbb{E}[\mathbf{Y}] \) and \( R \in \mathbb{R} \)), no-short-sales constraint \( \mathbf{x} \geq 0 \)

\(^1\)The analogous procedure for continuous distributions is provided in Jakobsons and Vanduffel (2015); in fact, it is Newton’s method applied to the optimization problem (1).
diversification constraints $\underline{b} \leq \mathbf{x} \leq \overline{b}$ for $\underline{b}, \overline{b} \in \mathbb{R}^d$. We denote the set of feasible portfolios by $X$, and remark that it is often polygonal. The corresponding portfolio optimization problem with the expectile risk measure as the objective is

$$\min : \ e_\tau(-\mathbf{x}^\top \mathbf{Y}) \quad \text{over : } \mathbf{x} \in \mathbb{R}^d$$

subj. to : $\mathbf{x} \in X$.

This is a $d$-dimensional optimization problem, where the expectile objective is only implicitly defined in (1). Furthermore, the definition involves expectations of the portfolio loss over events of the form $\{-\mathbf{x}^\top \mathbf{Y} > e\}$. In general, such an expectation corresponds to a $d$-dimensional integral and is numerically cumbersome to evaluate. However, in special cases, the problem can be simplified, for example, when the asset returns vector follows a multivariate elliptical distribution.

Elliptical distributions $\mathbf{Y} \sim E_d(\mu, \Sigma, \psi)$ are defined by the corresponding characteristic function

$$\varphi_{\mathbf{Y}}(\mathbf{t}) = \exp \left( \mathbf{i}^\top \mu \right) \psi \left( \frac{1}{2} \mathbf{t}^\top \Sigma \mathbf{t} \right), \quad \mathbf{t} \in \mathbb{R}^d.$$

The elliptical family has the property that any linear combination $\mathbf{x}^\top \mathbf{Y}$ of the margins (assets) with weights $\mathbf{x} \in \mathbb{R}^d$ belongs to a particular location-scale family (determined by $\psi$), with parameters $\mu^\top \mathbf{x}$ and $\sqrt{\mathbf{x}^\top \Sigma \mathbf{x}}$ respectively; see also Section 3.3 in McNeil et al. (2005).

$$\mathbf{x}^\top \mathbf{Y} \overset{d}{=} \mu^\top \mathbf{x} + \mathbf{Z} \sqrt{\mathbf{x}^\top \Sigma \mathbf{x}}, \quad \text{where } \mathbf{Z} \sim E_1(0, 1, \psi). \quad (2)$$

Furthermore, since the expectile is a coherent risk measure for $\tau \in [1/2, 1]$ (Ziegel, 2014), it satisfies the axioms of monotonicity, subadditivity, translation invariance and positive homogeneity. The latter two are stated as follows:

- translation invariance: for $c \in \mathbb{R}$, $e_\tau(c + L) = c + e_\tau(L)$,
- positive homogeneity: for $s \geq 0$, $e_\tau(sL) = se_\tau(L)$.

Applying these properties in (2) and using $\mathbf{Z} \overset{d}{=} -\mathbf{Z}$, we obtain

$$e_\tau(-\mathbf{x}^\top \mathbf{Y}) = -\mu^\top \mathbf{x} + \lambda_\tau \sqrt{\mathbf{x}^\top \Sigma \mathbf{x}}, \quad \text{where } \lambda_\tau = e_\tau(\mathbf{Z}). \quad (3)$$

Thus, under an expected returns constraint $\mu^\top \mathbf{x} = R$, the expectile optimization problem is solved by the minimal variance portfolio of Markowitz (1952). See also Proposition 6.13 in McNeil et al. (2005), where this result is stated in full generality, for all law-determined risk measures that are positive homogeneous and translation invariant. This useful property will enable using the multivariate Student-$t$ distribution (which is elliptical) as a benchmark for the numerical case study in Section 3.

---

2The inequalities for vectors are to be understood component-wise. The notation $\mathbf{1} = (1, \ldots, 1)^\top \in \mathbb{R}^d$ is used.
In general, for multivariate distributions that do not have such symmetry properties, the portfolio problem cannot be solved analytically. Examples of particular relevance for modeling skewed asset returns distributions can be found in the class of normal mean-variance mixtures. A prime example is the class of generalized hyperbolic distributions (Barndorff-Nielsen, 1977), as well as subclasses of it, such as the NIG (Barndorff-Nielsen, 1997), VG (Madan and Seneta, 1990) and the skew- \( \text{t} \) distribution (Aas and Hobæk Haff, 2006).

However, it is easy to obtain large Monte Carlo (MC) samples from these multivariate mixture distributions. In the case of a discrete asset returns distribution (e.g., based on the MC sample or the empirical distribution of historical returns), the portfolio optimization problems with risk objectives such as the mean absolute deviation (Konno and Yamazaki, 1991), CVaR (Rockafellar and Uryasev, 2000), and the Omega ratio (Kapsos et al., 2014) can be formulated as Linear Programs (LP). These results are of considerable practical importance, since LP formulations are solvable for a large number of assets and constraints, and have been a crucial tool for portfolio selection since the advent of computers; see Sharpe (1971); Ang (1975) for early examples and Mansini et al. (2003, 2007, 2014, 2015) for a recent overview.

In the following sections, we derive three LP formulations for the portfolio optimization problem with an expectile objective. In Section 2.1, the primal formulation is derived, in Section 2.2, the dual formulation, and in Section 2.3, a scenario aggregation approach is presented.

### 2.1 Primal LP formulation

The definition of the expectile in (1) is useful for fitting regression models in statistical analysis. For the portfolio optimization problem, however, using this definition would lead to a nested optimization problem, which are known to be computationally costly. A so-called dual representation of the expectile that is more suitable for our purpose is provided in Delbaen (2013) in terms of a set of generalized scenarios (probability measures \( Q \ll P \)); see also Delbaen (2012), p. 36. This representation stated in terms of the corresponding Radon-Nikodym derivatives \( \varphi = \frac{dQ}{dP} \) is as follows. For \( L \in L^1(\Omega, \mathcal{A}, P) \) and \( \tau \in [1/2, 1) \),

\[
e_{\tau}(L) = \sup_{\varphi \in \mathcal{M}_\tau} \mathbb{E}[L\varphi],
\]

where

\[
\mathcal{M}_\tau = \{ \varphi \in L^\infty(\Omega, \mathcal{A}, P) : \varphi \geq 0, \ \mathbb{E}[\varphi] = 1, \ (1 - \tau)m \leq \varphi \leq \tau m \ \text{for some} \ m > 0 \}. \quad (5)
\]

In fact, a risk measure is coherent if and only if it admits a representation of this type, called robust representation; see Proposition 4.14 in Föllmer and Schied (2004).

When the probability space is finite and discrete, \( \Omega = \{\omega_1, \ldots, \omega_n\} \), the representation in (4) can be formulated as an LP. In particular, we define the following optimization problem for
computing $e_t(L)$, where $L$ is a discrete rv such that $P(L = \ell_i) = P(\omega_i) = p_i$, $i \in \{1, \ldots, n\}$. Recall that we assume $\tau \in [1/2, 1)$.

$$\max: \sum_{i=1}^{n} \ell_i p_i \varphi_i \quad \text{over: } m \in \mathbb{R}, \varphi \in \mathbb{R}^n$$

(D1)

$$\text{subj. to: } \sum_{i=1}^{n} p_i \varphi_i = 1, \quad (1-\tau)m \leq \varphi_i \leq \tau m, \quad m \geq 0, \quad \varphi_i \geq 0, \quad i = 1, \ldots, n.$$

The decision variables $\varphi_i$ represent the Radon-Nikodym derivative evaluated at outcome $\omega_i$. Taking the dual of the problem (D1) yields

$$\min: \zeta \quad \text{over: } \zeta \in \mathbb{R}, \ u, v \in \mathbb{R}^n$$

(P1)

$$\text{subj. to: } p_i \zeta - u_i + v_i \geq 0, \ u_i \geq 0, \ v_i \geq 0, \quad i = 1, \ldots, n,$$

$$(1-\tau)\sum_{i=1}^{n} u_i - \tau \sum_{i=1}^{n} v_i \geq 0.$$  

Notice that in this formulation, the loss values $\ell_i$ are separated from the decision variables. This allows us to express the losses in terms of the portfolio weights, and formulate a joint optimization problem that is still linear, as follows. In the portfolio problem, the random variable $L$ represents portfolio losses, $L = -x^T Y$, where $x \in \mathbb{R}^d$ is the vector of portfolio weights and $Y$ is a $d$-dimensional random vector of asset returns. Suppose that the set of admissible portfolio weights is given by $X \subset \mathbb{R}^d$, which is a polytope generated by some linear constraints. We also assume that $Y$ follows a discrete distribution, with $P(Y = y_i) = p_i$, $i \in \{1, \ldots, n\}$. Substituting $\ell_i = -x^T y_i$ into the problem (P1) yields the primal LP formulation of the portfolio optimization problem.

$$\min: \zeta \quad \text{over: } x \in \mathbb{R}^d, \ z \in \mathbb{R}, \ u, v \in \mathbb{R}^n$$

(Primal)

$$\text{subj. to: } p_i x^T y_i + p_i \zeta - u_i + v_i \geq 0, \ u_i \geq 0, \ v_i \geq 0, \quad i = 1, \ldots, n,$$

$$x \in X, \quad (1-\tau)\sum_{i=1}^{n} u_i - \tau \sum_{i=1}^{n} v_i \geq 0.$$  

This is an LP with $d + 1 + 2n$ decision variables and $n + 1$ joint inequality constraints (apart from those imposed by $X$). This formulation can be considered as the expectile counterpart of the LP formulation for the CVaR objective in Rockafellar and Uryasev (2000). For the CVaR problem, the corresponding LP has only $d + 1 + n$ decision variables, because the corresponding set of generalized scenarios has a simpler structure; see Theorem 4.47 in Föllmer and Schied (2004).

In Bellini et al. (2015), a similar LP formulation to (Primal) for the expectile problem was derived, using the following set of Radon-Nikodym derivatives from Bellini et al. (2014), which is mathematically equivalent to (5).

$$\mathcal{M}_\tau = \left\{ \varphi \in L^\infty(\Omega, \mathcal{A}, P) : \varphi > 0, \quad \mathbb{E}[\varphi] = 1, \quad \frac{\text{ess sup}(\varphi)}{\text{ess inf}(\varphi)} \leq \frac{\tau}{1-\tau} \right\}.$$
The latter inequality constraints were formalized in the discrete case as \( n(n - 1) \) constraints

\[
(1 - \tau)\varphi_i \leq \tau \varphi_j \quad \text{for all} \quad i \neq j, \quad i, j \in \{1, \ldots, n\}.
\]

With this formulation, solving the corresponding LPs within a reasonable computation time is possible only up to \( n = 1000 \), because this already yields more than 999,000 decision variables in the primal formulation of the portfolio optimization problem. In our (Primal) formulation, the number of decision variables grows linearly in \( n \), because in (D1) there are only \( 2n \) constraints on \( \varphi \), at the expense of only one additional decision variable \( m \). This allows solving LPs that are based on much larger sample sizes; see Section 3.

2.2 Dual LP formulation

It was demonstrated in Ogryczak and Śliwiński (2011), that the portfolio optimization problem with the CVaR objective can be solved considerably faster by using the dual formulation instead of the primal one. Since the expectile portfolio optimization problem has a similar structure, we also formulate the dual LP to (Primal). In Section 3.2, the computation times using the different formulations will be analyzed to determine which formulation is computationally the most efficient.

To state the dual problem explicitly, we specify the set of admissible portfolios as

\[
\mathcal{X} = \{x \in \mathbb{R}^d : \mathbf{1}^\top x = 1, x_k \geq 0, k = 1, \ldots, d\},
\]

which corresponds to a fully invested portfolio with wealth normalized to 1 (without loss of generality, since coherent risk measures are positively homogeneous), and under the no-short-sales constraint. Note that further linear constraints can be added, which would then correspond to additional decision variables in the dual problem.

The dual problem to (Primal) for portfolio expectile optimization is the following.

\[
\begin{align*}
\max & : \eta \\
\text{over} : & \eta, m \in \mathbb{R}, \varphi \in \mathbb{R}^n \\
\text{subj. to} : & \sum_{i=1}^{n} p_i \varphi_i = 1, \quad (1 - \tau)m \leq \varphi_i \leq \tau m, \quad m \geq 0, \quad \varphi_i \geq 0, \quad i = 1, \ldots, n, \\
& \sum_{i=1}^{n} p_i y_i^{(k)} \varphi_i + \eta \leq 0, \quad k = 1, \ldots, d,
\end{align*}
\]

where \( y_i^{(k)} \) denotes the return of asset \( k = 1, \ldots, d \) in outcome \( i = 1, \ldots, n \). The optimized objective value \( \eta^* \) is the expectile \( e_\tau(-x^*\mathbf{y}) \) for the optimal portfolio \( x^* \). The optimal portfolio weights are given by the Lagrange multipliers corresponding to the last \( d \) inequality constraints.
2.3 Aggregation algorithm

Recently, Espinoza and Moreno (2014) proposed an iterative solution procedure for the portfolio CVaR optimization problem, based on (dis-)aggregation of scenarios. In Jakobsons (2016), this method was found to be advantageous in terms of computation time, in particular, for a large sample size $n$ and a high probability level $\alpha$ of the CVaR objective. In this section, we formulate the Aggregation algorithm for the portfolio expectile optimization problem.

The idea is based on the property that, for the optimal solution of (Dual), the Radon-Nikodym derivative takes values in the range $\varphi_i \in [(1 - \tau)m, \tau m]$, $i = 1, \ldots, n$, with most outcomes $\varphi_i$ at the endpoints of this interval. In fact, from Proposition 8 in Bellini et al. (2014) follows that an optimal solution of (D1) is given by

$$\varphi_i = (1 - \tau)m \mathbb{1}\{\ell_i \leq e_r\} + \tau m \mathbb{1}\{\ell_i > e_r\}, \quad i = 1, \ldots, n,$$

$$m = 1/[(1 - \tau) P(L \leq e_r) + \tau P(L > e_r)],$$

where $e_r = e_r(L)$. Further, if a solution has decision variables $\varphi_i \in ((1 - \tau)m, \tau m)$ that lie strictly inside the interval, they correspond to losses $\ell_i = e_r$. This leads to the idea of reducing the number of decision variables by representing the values of $\varphi_i$ over several outcomes with a single variable; this does not affect the optimality, as long as these outcomes correspond to losses that are all either above or below the level $e_r$. This approach is called scenario aggregation.

This method is implemented by partitioning the set of all outcomes and constraining the values $\varphi_i$ to be constant within each set of this partition. Specifically, for a partition $N = \{N_1, \ldots, N_\tilde{n}\}$ of the set of indices $\{1, \ldots, n\}$ into $\tilde{n}$ subsets, we add constraints $\varphi_i = \varphi_j$ whenever $i, j \in N_k$ for some $k = 1, \ldots, \tilde{n}$. This problem can be represented using fewer decision variables, as follows.

$$\text{max : } \eta \quad \text{over : } \eta, m \in \mathbb{R}, \varphi \in \mathbb{R}^{\tilde{n}} \quad \text{(Aggregated)}$$

$$\text{subj. to : } \sum_{i=1}^{\tilde{n}} \tilde{p}_i \varphi_i = 1, \quad (1 - \tau)m \leq \varphi_i \leq \tau m, \quad m \geq 0, \quad \varphi_i \geq 0, \quad i = 1, \ldots, \tilde{n},$$

$$\sum_{i=1}^{\tilde{n}} c_i^{(k)} \varphi_i + \eta \leq 0, \quad k = 1, \ldots, d,$$

where

$$\tilde{p}_i = \sum_{j \in N_i} p_j, \quad \text{and} \quad c_i = \sum_{j \in N_i} p_j y_j, \quad i = 1, \ldots, \tilde{n}.$$ 

Let $\tilde{x}$ be the vector of portfolio weights given by the Lagrange multipliers corresponding to the last $d$ inequality constraints in the problem (Aggregated). This portfolio may not be optimal for the original problem, because we have implicitly imposed additional constraints on (Dual).
The expectile for the loss of this portfolio can be computed, e.g. by solving (P1) or (D1) with \( \ell_i = -\mathbf{x}^\top y_i \). However, we recommend using (IRLS) for this purpose, as it is considerably faster.

Denote the optimal objective value of the constrained problem (Aggregated) by \( e^N_\tau \), the optimal value of the original problem (Dual) by \( e^{OPT}_\tau \), and the expectile (as computed using (IRLS)) of the approximate portfolio \( \tilde{x} \) by \( e^\tilde{x}_\tau \). Then the following inequalities hold:

\[
e^N_\tau \leq e^{OPT}_\tau \leq e^\tilde{x}_\tau.
\]

The first inequality holds because (Aggregated) is equivalent to a more constrained version of the maximization problem (Dual); the second - because the portfolio \( \tilde{x} \) is feasible for the portfolio optimization problem (Primal) when \( X \) is given by (6). Hence, solving the (Aggregated) problem yields an interval for the true optimal portfolio expectile.

The value \( e^\tilde{x}_\tau \) also induces a partitioning \( \{M_1, M_2, M_3\} \) of the scenario indices \( i \in \{1, \ldots, n\} \),

\[
M_1 = \{i : -\mathbf{x}^\top y_i < e^\tilde{x}_\tau\}, \quad M_2 = \{i : -\mathbf{x}^\top y_i = e^\tilde{x}_\tau\}, \quad M_3 = \{i : -\mathbf{x}^\top y_i > e^\tilde{x}_\tau\}.
\]

This partitioning can then be used to refine our previous partitioning \( N \) by splitting the sets that contain both “good” and “bad” scenarios, namely, \( \ell_i \) that are below, respectively, above the expectile level of the current portfolio. These ideas are combined in the following scenario aggregation algorithm for portfolio expectile optimization.

---

**Aggregation algorithm**

**Require:** Stopping gap \( \delta > 0 \)

1. Initiate \( N \leftarrow \{1, \ldots, n\} \)
2. loop
   3. Solve (Aggregated), obtain \( e^N_\tau \) and \( \tilde{x} \).
   4. Compute \( e^\tilde{x}_\tau = e_\tau(-\tilde{x}^\top Y) \) using (IRLS) and construct \( M_1, M_2, M_3 \).
   5. if \( e^\tilde{x}_\tau - e^N_\tau \leq \delta \) then
      6. return Solution(\( \tilde{x} \))
   7. else
      8. Let \( N \leftarrow \{N \cap M_j : N \in N, j \in \{1, 2, 3\}\} \).
end

At the first iteration of the Aggregation algorithm, the (IRLS) procedure in step 4 can be initialized at \( b_0 = E[-\tilde{x}^\top Y] \), and in subsequent iterations the expectile of the previous portfolio can be used as the starting value for (IRLS). Note that the Aggregation algorithm will need to solve several instances of (Aggregated). However, because of the scenario aggregation, these instances will be of a considerably smaller size than (Dual), which may lead to savings in the total computation time. The computation times are analyzed in Section 3.2.
Lemma 1. The Aggregation algorithm with $\delta = 0$ returns a solution after a finite number of iterations.

Proof. The proof is parallel to that of Lemma 1 in Espinoza and Moreno (2014). We only need to show that in each iteration the number of sets in the partition $N$ strictly increases; the number of iterations is then bounded by the number of scenarios $n$. To this end, we define a Lagrangian relaxation of (Aggregated) by penalizing the violation of the last $d$ inequality constraints by $\xi \in \mathbb{R}^d$.

$$L(\xi) = \max : \eta + \xi^T \left( - \sum_{i=1}^{\tilde{n}} \mathbf{c} \tilde{\varphi}_i - \eta \right) \quad \text{over} : \eta, m \in \mathbb{R}, \tilde{\varphi} \in \mathbb{R}^\tilde{n}$$

$$\text{subj. to} : \sum_{i=1}^{\tilde{n}} \tilde{p}_i \tilde{\varphi}_i = 1, \quad (1 - \tau)m \leq \tilde{\varphi}_i \leq \tau m, \quad m \geq 0, \quad \tilde{\varphi}_i \geq 0, \quad i = 1, \ldots, \tilde{n}.$$  

Suppose that in step 8 of the Aggregation algorithm, the number of sets in the partition does not increase, i.e. $N$ is already a refinement of $\{M_1, M_2, M_3\}$. Then, since $e^x_\tau$ is the optimal objective value of (D1) with $\ell_i = -\mathbf{x}^T y_i$, by (7) there exists a solution $(m^*, \varphi^*)$ of (D1) such that $\varphi^*$ is constant on each of the sets $M_1, M_2, M_3$. Hence, this solution is also constant on the sets in partition $N = \{N_1, \ldots, N_{\tilde{n}}\}$, so we can define $\tilde{\varphi}_i^* = \varphi_j^*$ using any $j \in N_i$ for $i = 1, \ldots, \tilde{n}$. Then, $(\eta, m^*, \tilde{\varphi}^*)$ is feasible for $L$ for any $\eta \in \mathbb{R}$. With these feasible values, setting $\xi = \tilde{x}$, the objective of $L$ is equal to $e^x_\tau$, so $e^x_\tau \leq L(\tilde{x})$. Further, since $\tilde{x}$ is the optimal Lagrange multiplier from (Aggregated) corresponding to the relaxed constraints, we obtain $L(\tilde{x}) = e^N_\tau$. Combining with (8), it follows that $e^x_\tau = e^N_\tau$, which completes the proof. □

Remark 1. The formulations (Dual), (Aggregated), and hence the Aggregation algorithm can be adapted to accommodate further portfolio constraints of the form $A \mathbf{x} = \mathbf{b}$ beyond the definition of $\mathcal{X}$ in (6). If $\mathbf{b} \in \mathbb{R}^q$, we replace the decision variable $\eta$ by vector $\eta \in \mathbb{R}^q$, the objective by $\mathbf{b}^T \eta$, and put $(A^T \eta)^{(k)}$ in the constraints instead of $\eta$.

3 Numerical case study

In this section, we analyze the LP problem of expectile portfolio optimization from two perspectives. First, note that typically it is assumed that the true asset returns distribution is continuous, therefore the LP formulations are based on a discrete approximation of the true distribution. As a consequence, the obtained portfolios may be suboptimal. In Section 3.1, we analyze the suboptimality of the portfolios given by LP approximations, based on the sample size and other factors. Second, to determine the best approach for solving the LPs, we also compare the computation times using the formulations (Primal), (Dual), and the Aggregation algorithm; see Section 3.2.
Setting $\mu = 0$ in (2), we note that for elliptical asset returns distributions, the portfolio expectile optimization problem is solved by the minimum variance portfolio, namely,

$$
x^* = \text{argmin} : x^T \Sigma x \quad \text{over} : x \in \mathbb{R}^d, \quad \text{subj. to} : x \in X.
$$

This is a quadratic programming problem with linear constraints and only $d$ decision variables, which can be solved efficiently using standard optimization software. Further, note that this yields the true optimal portfolio for the corresponding elliptical asset returns distribution, without relying on discrete approximations. Therefore, these distributions can be used as benchmark for evaluating the quality of approximate solutions obtained from the LPs. In this case study, we will consider $d$-variate normal $N_d(0, \Sigma)$ and Student $t_d(0, \Sigma, \nu)$ distributions for asset returns. The parameters are selected as follows. The location vector is set to zero, because this enables obtaining the benchmark portfolio using (QP), furthermore, for asset returns data the means are indeed typically close to zero. To obtain a realistic dependence structure, for the scale matrix $\Sigma$ we use the covariance matrix of financial time series. In particular, the covariance matrix is computed using the daily returns of up to $d = 101$ constituents of the FTSE 100 index from 2-Jan-2003 to 18-Sep-2014 (the same dataset as used in Jakobsons (2016)). Since $\Sigma$ is only the scale, not the covariance matrix for Student-$t$, it is not intended that these parameters are a good fit for the original data, only a reasonable choice for a case study. Blattberg and Gonedes (1974) estimate the degrees of freedom parameter for most stock daily returns to be in the range $\nu \in (4, 5)$, with occasional values above and below. Hence, in our case study we consider a representative set of values $\nu \in \{3, 5, 10, \infty\}$. Note that the limiting case $\nu = \infty$ is the normal distribution. The optimization is performed over fully invested portfolios under the no-short-sales constraint, as defined in (6).

From (3) follows that the portfolio loss expectile for any portfolio $x \in X$ in this case study can be computed using the formula

$$
e_{\tau}(x - Y) = \lambda_{\tau} \sqrt{x^T \Sigma x}, \quad \text{where} \quad \lambda_{\tau} = e_{\tau}(Z), \quad Z \sim t_1(0, 1, \nu).
$$

We will consider expectile levels $\tau \in \{0.99, 0.999, 0.9999\}$. In Table 1, the values of $\lambda_{\tau}$ are listed for the considered values of $\tau$ and Student-$t$ parameters $\nu$.

Since the expectile is currently not being used as a regulatory risk measure, it is not clear what values of $\tau$ would be considered reasonable. To give an intuition for how far in the tail of the loss distribution the expectile levels $\tau \in \{0.99, 0.999, 0.9999\}$ are, we compare them to the levels of other well-known risk measures. Table 2 lists the corresponding probability levels $\alpha$ and $\beta$ that would yield the same value for $\text{VaR}_\alpha(L)$ and $\text{ES}_\beta(L)$ as that of $e_{\tau}(L)$. Note that a Student-$t$ distribution with $\nu = \infty$ is the normal distribution; and in this case the corresponding probabilities are e.g. 95.7%, 99.3%, 99.9% for VaR. These are close to the levels typically considered for VaR. The comparison in the case of the normal distribution is relevant, since apparently, a similar approach was taken by the Basel Committee on Banking Supervision (BCBS,
Table 1: Expectile $e_\tau(L)$ at levels $\tau \in \{0.99, 0.999, 0.9999\}$ for a (standard) Student-$t$ distributed rv $L$ with $\nu \in \{\infty, 10, 5, 3\}$ degrees of freedom.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\nu = \infty$</th>
<th>10</th>
<th>5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99</td>
<td>1.72</td>
<td>2.03</td>
<td>2.50</td>
<td>3.63</td>
</tr>
<tr>
<td>0.999</td>
<td>2.44</td>
<td>3.15</td>
<td>4.43</td>
<td>8.12</td>
</tr>
<tr>
<td>0.9999</td>
<td>3.06</td>
<td>4.42</td>
<td>7.31</td>
<td>17.63</td>
</tr>
</tbody>
</table>

2012), when moving from VaR$_{0.99}$ as the risk measure for the trading book capital requirements to ES$_{0.975}$. Adjusting the probability level $\beta$, so that the numerical value of ES$_\beta(L)$ for a normally distributed rv $L$ matches VaR$_{0.99}(L) = 2.3263$, yields $\beta \approx 0.97423$, which is rounded to $\beta = 0.975$ (adjusting $\tau$ so that $e_\tau(L)$ matches VaR$_{0.99}(L)$ would yield $\tau \approx 0.99855$).

Table 2: Probability levels $\alpha$ and $\beta$ such that VaR$_\alpha(L)$, respectively, ES$_\beta(L)$ is equal to the expectile at level $\tau \in \{0.99, 0.999, 0.9999\}$ of a Student-$t$ distributed rv $L$ with $\nu \in \{\infty, 10, 5, 3\}$ degrees of freedom. The corresponding values of $e_\tau(L)$ are listed in Table 1. The values of $\tau, \alpha, \beta$ are listed in percent.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\nu = \infty$</th>
<th>10</th>
<th>5</th>
<th>3</th>
<th>$\alpha$ s.t. VaR$<em>\alpha(L) = e</em>\tau(L)$</th>
<th>$\beta$ s.t. ES$<em>\beta(L) = e</em>\tau(L)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>99</td>
<td>95.71</td>
<td>96.50</td>
<td>97.29</td>
<td>98.19</td>
<td>89.18</td>
<td>90.62</td>
</tr>
<tr>
<td>99.9</td>
<td>99.26</td>
<td>99.49</td>
<td>99.66</td>
<td>99.80</td>
<td>98.08</td>
<td>98.58</td>
</tr>
</tbody>
</table>

3.1 Suboptimality

When a continuous asset returns distribution is approximated by a discrete one, and a portfolio is selected by solving an LP based on the discrete approximation, the selected portfolio may not be optimal for the original asset returns distribution. To estimate the suboptimality of the portfolios that are obtained using such LP approximations, we conducted a large number of experiments, changing the number of assets $d \in \{3, 5, 10, 25, 50, 101\}$ and Student-$t$ parameter $\nu \in \{3, 5, 10, \infty\}$, in each case generating 100 MC samples of size $n, 10^3 \leq n \leq 10^6$, and solving the corresponding instances of LPs for $\tau \in \{0.99, 0.999, 0.9999\}$.

Let $Y \sim t_d(0, \Sigma, \nu)$ be a rv with the true asset returns distribution and $\tilde{Y} \sim \text{UNIF}\{y_1, \ldots, y_n\}$ be the discrete approximation based on an MC sample. Further, denote by $x_{\text{LP}}$ the portfolio obtained by solving the LP approximation, and by $x_{\text{QP}}$ the true optimal portfolio, given by (QP).
We first point out that, to estimate the suboptimality, one cannot use the objective value of the LP, $e_r(-x_{\text{LP}}^TY)$ (the *perceived* expectile), because it is a biased estimate of the true expectile of this portfolio, $e_r(-x_{\text{QP}}^TY)$. The latter can be computed using (9), and so can the true optimal expectile $e_r(-x_{\text{QP}}^TY)$. Hence, we can separate the error into two parts, bias and suboptimality, computed relative to the true optimum as

$$
\text{subopt} = \frac{e_r(-x_{\text{LP}}^TY) - e_r(-x_{\text{QP}}^TY)}{e_r(-x_{\text{QP}}^TY)}, \quad \text{bias} = \frac{e_r(-x_{\text{LP}}^TY) - e_r(-x_{\text{LP}}^\hat{Y})}{e_r(-x_{\text{QP}}^TY)}.
$$

In Figure 1, the suboptimality and bias is plotted for different sample sizes $n$ in the case $\tau = 0.999, \nu = 10,$ and $d = 25$. The boxplot shows the range of the results over 100 runs, each based on an MC sample of $n$ observations of the asset returns vector. We observe that the bias (underestimation) is typically larger than suboptimality, which means that the perceived expectile is often lower than the true optimum, while the actual expectile of the selected portfolio is higher than the optimum. Sometimes, however, the perceived expectile can be higher than the actual expectile of the selected portfolio (negative underestimation), but suboptimality is always positive (above the dotted line). For sample size $n = 1000$, the suboptimality is typically around 10%, but it can reach up to 30%. Similar observations apply also for a lower expectile level, $\tau = 0.99$; see Table 3, where the median and the 90th percentile of the suboptimality is listed for different $\nu$ and $d$. Table 3 also shows that the suboptimality is higher when the number of assets $d$ is large and when the asset returns distribution is more heavy-tailed (lower parameter

![Figure 1: Suboptimality of the LP solution portfolio expectile (black), and underestimation of its true expectile (gray), both relative to the optimal expectile. For each sample size $n \in \{10^3, 10^4, 10^5\}$, the boxplot shows the spread of the results over 100 simulations using parameters $\tau = 0.999, \nu = 10,$ and $d = 25.$](image-url)
Table 3: Portfolio suboptimality in percent of the optimal portfolio expectile at level $\tau = 0.99$ with sample size $n = 1000$, depending on the number of assets $d$ and Student-t parameter $\nu$. The 50th and the 90th percentile of the outcomes in 100 experiment runs are listed.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$d$ = 3</th>
<th>5</th>
<th>10</th>
<th>25</th>
<th>50</th>
<th>101</th>
<th>50th percentile</th>
<th>90th percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>0.4</td>
<td>0.7</td>
<td>1.9</td>
<td>3.4</td>
<td>4.5</td>
<td>5.4</td>
<td>1.1</td>
<td>1.7</td>
</tr>
<tr>
<td>10</td>
<td>0.7</td>
<td>1.2</td>
<td>2.6</td>
<td>4.6</td>
<td>5.7</td>
<td>7.8</td>
<td>1.7</td>
<td>1.9</td>
</tr>
<tr>
<td>5</td>
<td>0.7</td>
<td>2.2</td>
<td>3.3</td>
<td>5.9</td>
<td>8.2</td>
<td>9.4</td>
<td>2.9</td>
<td>5.5</td>
</tr>
<tr>
<td>3</td>
<td>1.6</td>
<td>4.2</td>
<td>6.4</td>
<td>8.6</td>
<td>11.3</td>
<td>13.4</td>
<td>4.9</td>
<td>11.9</td>
</tr>
</tbody>
</table>

In these cases, the suboptimality exceeds 10%. Clearly, higher sample sizes are required to obtain portfolios that are closer to the optimum, as indicated by Figure 1.

In Table 4, the results with sample size $n = 10^5$ for $\tau = 0.9999$ are listed. For such a high expectile level, even an approximation with $n = 10^5$ sample points may lead to portfolio suboptimality in excess of 10%. Table 6 in the Appendix shows the results for the sample sizes that we considered as sufficient, namely, $n = 10^4$ for $\tau = 0.99$, $n = 10^5$ for $\tau = 0.999$, and $n = 10^6$ for $\tau = 0.9999$. With these sample sizes, the suboptimality of the selected portfolios was typically under 5% even with a high number of assets and a heavy-tailed returns distribution. The LPs corresponding to these sample sizes are computationally demanding to solve, because of the large number of decision variables and constraints. Therefore, it is important to use the most efficient (least time-consuming) LP formulation. This question is addressed in the next section.
3.2 Computation times

In this section, we analyze the computation times using the (Primal), (Dual) formulations, and the Aggregation algorithm for different problem instances corresponding number of assets $3 \leq d \leq 101$, Student-$t$ parameter $\nu \in \{3, 5, 10, \infty\}$, and expectile levels $\tau \in \{0.99, 0.999, 0.9999\}$. The computations were performed with IBM ILOG CPLEX 12.5, using MATLAB interface on a 2.2GHz AMD Opteron 6174. To estimate the computation times depending on the parameters $\nu, d, n$ and $\tau$, for each combination, 10 instances corresponding to 10 MC samples of size $n$ were solved using the three different formulations.

The two most common computational methods for solving LPs are the simplex algorithm and barrier (interior point) method. We compared the computation times for the three formulations using the two solution methods, and observed that, especially for the (Primal) and (Dual) formulations, the barrier method was considerably faster. This is consistent with the observations in the review paper by Gondzio (2012), that “interior point methods are competitive when dealing with small problems of dimensions below one million constraints and variables and are beyond competition when applied to large problems of dimensions going into millions of constraints and variables”; see also Bixby (2002).

In Figure 2, the geometric average computation times (using the barrier method) for the (Primal), (Dual) formulations, and the Aggregation algorithm are plotted in a logarithmic scale against the number of assets $d$ and sample size $n$. First, note that only with the Aggregation algorithm it was possible to solve instances with sample size $n > 10^5$ with an average time of less than 45 minutes. We observe an approximately linear dependence (in the logarithmic scale) on $d$ and $n$. To estimate the growth of the computation time in terms of $d$ and $n$ for each formulation and solution method, we fit a linear model for log-time (the input variables $d$ and $n$ are on an approximately regular grid in the log-scale). This yields

$$T(d, n) \approx C d^a n^b$$

as a model for the computation time $T$ in minutes. In Table 5, the fitted coefficients are listed. Note that to fit the parameters for the (Primal) and (Dual) formulations, we used $10^3 \leq n \leq 3 \times 10^4$ for the simplex method and $10^3 \leq n \leq 10^5$ for the barrier method. For the Aggregation algorithm, $3 \times 10^4 \leq n \leq 10^6$ were used for both methods. While the barrier method is clearly faster for the (Primal) and (Dual) formulations, the differences are less significant for the Aggregation algorithm, because the instances of (Aggregated) are of a much smaller size, where the two methods are competitive. Overall, for the considered levels $\tau$ and sample sizes in the thousands, the Aggregation algorithm is clearly the most efficient formulation, with computation times that are by a factor of 50-200 smaller than for the other two formulations. The advantage increases as we consider larger sample sizes and higher expectile levels $\tau$. 
Figure 2: Computation times in minutes for $\tau = 0.999$, plotted in a logarithmic scale against $d$ and $n$.

Table 5: Fitted parameters to the model $T = C d^a n^b$ for computation time $T$ in minutes. $T^*$ is the fitted time for the case $d = 25, n = 10^5$. 

<table>
<thead>
<tr>
<th>Simplex</th>
<th>Barrier</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau = 0.99$</td>
<td>$\tau = 0.99$</td>
</tr>
<tr>
<td></td>
<td>$a$</td>
</tr>
<tr>
<td>Primal</td>
<td>0.37</td>
</tr>
<tr>
<td>Dual</td>
<td>0.59</td>
</tr>
<tr>
<td>Aggregation</td>
<td>1.29</td>
</tr>
<tr>
<td>$\tau = 0.999$</td>
<td>$\tau = 0.9999$</td>
</tr>
<tr>
<td>Primal</td>
<td>0.10</td>
</tr>
<tr>
<td>Dual</td>
<td>0.66</td>
</tr>
</tbody>
</table>
A further observation is that the parameter $\nu$ did not have a significant effect on the computation times using the (Primal) and (Dual) formulations. For the Aggregation algorithm, however, we see in Figure 3 that $\nu$ has a significant effect, in particular, the cases with heavier tails (lower parameter $\nu$) take less time. For example, with sample size $n = 10^6$ and $\tau = 0.9999$, the cases with normal distribution ($\nu = \infty$) take 5 minutes on average, whereas with Student-$t$ distributed returns ($\nu = 3$) only 1.4 minutes. This is because the Aggregation algorithm is based on iteratively refining the partition of the scenarios into subsets that contain either only “good” or only “bad” outcomes. When the asset returns distribution is more heavy-tailed, it is easier to identify the outliers with large joint losses. This also explains why the Aggregation algorithm performs faster for high expectile levels $\tau$: there are fewer “bad” scenarios to identify.

Remark 2. The conclusions regarding the three formulations for the expectile problem are somewhat different from those drawn in Jakobsons (2016) for the CVaR problem. There, the dual formulation by Ogryczak and Śliwiński (2011) was considerably faster, because it replaced $n$ joint constraints in the primal formulation of Rockafellar and Uryasev (2000) with $n$ constant bounds for the Radon-Nikodym derivatives, namely, $\varphi_i \leq 1/(1 - \beta)$, $i = 1, \ldots, n$, according to the robust representation of CVaR$_\beta$. In the case of the expectile, the constraints on $\varphi_i$ are made dependent through $m$. This leads to similar growth rates $O(n^{1.5})$ for the computation time in terms of $n$ for both (Primal) and (Dual) algorithm (using the barrier method). Also, because of this inseparable set of $n$ constraints for $n + 1$ decision variables, scenario aggregation yields even greater relative savings in the computation time for the expectile problem.
4 Conclusions

In this paper, we considered the portfolio optimization problem where the risk is measured using the expectile. The expectile is a statistical functional that has recently gained attention due to its theoretical properties of coherence and elicitability, which make it an appealing candidate for use as a regulatory risk measure, as well as for managing portfolio risk. For elliptical joint asset returns distributions, the portfolio optimization problem with any coherent risk measure (such as the expectile) as the objective is solved by the minimal variance portfolio. In contrast, for general asset returns distributions, the portfolio optimization problem may be very challenging. A practical method for obtaining close-to-optimal portfolios is approximating the true (respectively, modeled) continuous asset returns distribution using a discrete sample, and solving an LP formulation corresponding to the risk measure. In order to enable this approach for the expectile risk measure, we derived three LP formulations, which we refer to as primal, dual and Aggregation algorithm. In the numerical study, we observed that large samples (up to a million sample points) may be necessary to obtain a portfolio that is within 5% of the optimum, especially for a high expectile level $\tau$, a heavy-tailed asset returns distribution, and a large number of assets. The only formulation that was practical to use (in terms of the computation time) for solving problem instances of this size was the Aggregation algorithm. Also for lower sample sizes this algorithm offers a $50 - 200$ times reduction in the computation time, compared to the other two considered formulations. Overall, this paper confirms from another perspective that the expectile can be a useful risk measure, in particular, that it is possible to solve large scale portfolio optimization problems with the expectile objective.

Acknowledgments

The author gratefully acknowledges the financial support from RiskLab Switzerland and the Swiss Finance Institute.
Appendix

Table 6: Portfolio suboptimality in percent of the optimal portfolio expectile, depending on the expectile level $\tau$, sample size $n$, Student-t parameter $\nu$, and number of assets $d$. The 50th and the 90th percentile of the outcomes in 100 experiment runs are listed.

$$\tau = 0.99, n = 10^4$$

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$d$</th>
<th>50th percentile</th>
<th>90th percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>3 5 10 25 50 101</td>
<td>0.0 0.1 0.2 0.4 0.5 0.8</td>
<td>0.1 0.3 0.4 0.6 0.7 1.1</td>
</tr>
<tr>
<td>10</td>
<td>0.1 0.1 0.3 0.7 0.8 1.2</td>
<td>0.2 0.4 0.5 1.0 1.1 1.7</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.1 0.2 0.6 1.0 1.3 2.1</td>
<td>0.3 0.6 1.1 1.6 1.9 2.8</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.3 0.6 1.3 2.1 2.9 3.9</td>
<td>0.8 1.6 2.3 3.4 3.6 5.4</td>
<td></td>
</tr>
</tbody>
</table>

$$\tau = 0.999, n = 10^5$$

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$d$</th>
<th>50th percentile</th>
<th>90th percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>3 5 10 25 50 101</td>
<td>0.0 0.0 0.1 0.2 0.3 0.4</td>
<td>0.0 0.1 0.1 0.2 0.3 0.4</td>
</tr>
<tr>
<td>10</td>
<td>0.0 0.1 0.1 0.3 0.4 0.6</td>
<td>0.1 0.1 0.2 0.4 0.6 0.7</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.0 0.2 0.4 0.7 0.8 1.3</td>
<td>0.2 0.4 0.6 1.1 1.3 1.5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.3 0.5 1.1 2.0 2.4 3.5</td>
<td>1.1 1.2 1.7 3.1 3.8 5.2</td>
<td></td>
</tr>
</tbody>
</table>

$$\tau = 0.9999, n = 10^6$$

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$d$</th>
<th>50th percentile</th>
<th>90th percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>3 5 10 25 50 101</td>
<td>0.0 0.0 0.1 0.2 0.3 0.4</td>
<td>0.0 0.1 0.1 0.3 0.4 0.5</td>
</tr>
<tr>
<td>10</td>
<td>0.0 0.0 0.1 0.2 0.3 0.5</td>
<td>0.1 0.1 0.2 0.4 0.5 0.6</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.1 0.1 0.3 0.6 0.7 1.1</td>
<td>0.2 0.3 0.6 0.7 1.0 1.6</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.1 0.5 1.0 2.0 2.6 3.4</td>
<td>0.6 0.9 2.0 3.0 4.7 4.7</td>
<td></td>
</tr>
</tbody>
</table>
Table 7: Geometric mean computation times (in minutes) of the optimal portfolio, depending on the expectile level \( \tau \), sample size \( n \), and number of assets \( d \). The LPs corresponding to the Primal, Dual, and Aggregation formulation were solved using the barrier method.

### \( \tau = 0.99 \)

<table>
<thead>
<tr>
<th>( d )</th>
<th>( n = )</th>
<th>Primal</th>
<th>Dual</th>
<th>Aggregation</th>
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### \( \tau = 0.999 \)

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### \( \tau = 0.9999 \)

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