A symplectic homology theory for automorphisms of Liouville domains

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A SYMPLECTIC HOMOLOGY THEORY FOR AUTOMORPHISMS OF LIOUVILLE DOMAINS

a thesis submitted to attain the degree of
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To my adorable niece
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We develop a Floer homology that is tailored for studying automorphisms of Liouville domains (symplectic manifolds with contact-type boundaries). The Floer homology, $\text{HF}_*(W, \phi, a)$, is a graded vector space in $\mathbb{Z}_2$ coefficients which is associated to an exact symplectomorphism $\phi$ of a Liouville domain $W$ and a real number $a$ that is not a period of some Reeb orbit on the boundary $\partial W$. We call such real numbers “admissible slopes.”

We investigate how $\text{HF}_*(W, \phi, a)$ behaves when the data changes continuously. In particular, the Floer homology is invariant (up to isomorphisms) under compactly supported symplectic isotopies of $\phi$ and under varying $a$ through admissible slopes.

Two types of morphisms are significant to the theory, the continuation maps and the transfer morphisms. The former relate the Floer homologies for different slopes. They have the following form

$$\text{HF}_*(W, \phi, a) \rightarrow \text{HF}_*(W, \phi, b),$$

where $a \leq b$, and they make $\{\text{HF}_*(W, \phi, a)\}$ into a directed system of graded vector spaces. A transfer morphism

$$\text{HF}_*(W_2, \phi, a) \rightarrow \text{HF}_*(W_1, \phi, b)$$

is associated to a Liouville domain $W_2$, its codimension-0 Liouville subdomain $W_1$, an exact symplectomorphism
\(\phi : W_1 \rightarrow W_1\), and a pair of positive admissible slopes \((a, b)\) such that \(a \leq b\).

The theory turns out to be efficient for studying the symplectic mapping class group. In particular, it leads to a couple of criteria for detecting symplectomorphisms that are not symplectically isotopic to the identity (relative to the boundary). There are also some applications that go beyond symplectic geometry. Namely, one can find optimal upper bounds for the minimal length of a geodesic on spheres.
RIASSUNTO

Sviluppiamo un’omologia di Floer appositamente creata al fine di studiare automorfismi di domini di Liouville (varietà simplettiche con bordi di tipo contatto). L’omologia di Floer $\text{HF}_*(W, \phi, a)$ è uno spazio vettoriale gradato con coefficienti in $\mathbb{Z}_2$, a cui vengono associati un simplettomorfismo esatto $\phi$ di un dominio di Liouville $W$ ed un numero reale $a$, il quale non risulta essere periodo di qualche orbita di Reeb sul bordo $\partial W$. Chiameremo tali numeri reali “pendenze ammissibili”.

Investighiamo il comportamento di $\text{HF}_*(W, \phi, a)$ al variare con continuità dei dati. Nella fattispecie l’omologia di Floer è invariante (a meno di isomorfismi) rispetto ad isotopie simplettiche a supporto compatto di $\phi$ e rispetto a variazioni di $a$ nelle pendenze ammissibili.

Due tipologie di morfismi sono significative per la teoria, ossia le mappe di continuazione ed i morfismi di trasferimento. I primi collegano le omologie di Floer con pendenze differenti. Assumono la forma seguente

$$\text{HF}_*(W, \phi, a) \rightarrow \text{HF}_*(W, \phi, b),$$

ove $a \leq b$, e rendono \{HF$_*(W, \phi, a)$\} un sistema diretto di spazi vettoriali gradati. Ad un morfismo di trasferimento

$$\text{HF}_*(W_2, \phi, a) \rightarrow \text{HF}_*(W_1, \phi, b)$$

vengono associati un dominio di Liouville $W_2$, il suo subdominio di Liouville $W_1$ di codimensione-o, un simplettomorfismo esatto $\phi : W_1 \rightarrow W_1$ ed una coppia di pendenze ammissibili $(a, b)$ tali per cui $a \leq b$. 

vii
La teoria risulta essere efficiente per lo studio del gruppo delle classi di applicazioni simplettiche. In particolare porta ad un paio di criteri per individuare simplettomorfismi che non sono simpletticamente isotopici all’identità (rispetto al bordo). Esistono altresì alcune applicazioni che si situano oltre la geometria simplettica, ossia è possibile trovare limiti superiori ottimali alla lunghezza minimale di una geodetica su delle sfere.
## CONTENTS

1 INTRODUCTION  
1.1 Transfer morphism  
1.2 Applications  
1.3 Conventions  

2 FLOER HOMOLOGY FOR EXACT SYMPLECTO-MORPHISMS  
2.1 Liouville domains  
2.2 Floer data and admissible data  
2.3 Action functional  
2.4 Grading  
2.5 Taming infinity  
2.6 Chain complex  
2.7 Continuation maps  

3 INVARIANCE UNDER VARYING DATA  
3.1 Dependence on the Liouville form  
3.2 Varyng the slope  
3.3 Naturality  
3.4 Invariance under isotopies  

4 VITERBO’S FUNCTIORIALITY  
4.1 Stair-like Hamiltonians  
4.2 Construction of the transfer morphism  

5 APPLICATIONS  
5.1 The symplectic mapping class group  
5.2 The fibered Dehn twist  
5.3 Detecting nontrivial mapping classes  
5.4 Examples  
5.5 Iterated ratio
5.6 Iterated ratio of the fibered Dehn twist on $D^*S^n$ 75
5.7 An application to closed geodesics 81
A TECHNICAL LEMMAS 87
INTRODUCTION

This thesis is devoted to Floer homology for symplectomorphisms of a Liouville domain. The Floer homology group, \( \text{HF}_*(W, \phi, a) \), is associated to a Liouville domain \((W, \lambda)\), an exact symplectomorphism \( \phi : W \to W \) (see Definition 2.2.1), and a so-called admissible slope \( a \). An admissible slope is any element of \( \mathbb{R} \cup \{\infty\} \) that is not a period of some Reeb orbit on the boundary \( \partial W \). One should think of the groups \( \text{HF}_*(W, \phi, a) \) as an extension to the Liouville domains of the Floer homology groups in [13]. On the other hand, these groups also generalize symplectic homology [8, 9, 14, 15, 29], in that

\[
\text{SH}(W; \mathbb{Z}_2) = \text{HF}(W, \text{id}, \infty).
\]

The construction of our Floer Homology groups relies on a new variant of the standard action functional, which is adapted to the setting of twisted loops associated to exact symplectomorphisms (see Section 2.3).

The Floer homology is invariant (up to isomorphisms) under symplectic isotopies of \( \phi \) that are supported in the interior of \( W \). It is also invariant under varying \( a \) through admissible slopes. However, \( \text{HF}_*(W, \phi, a) \) may change when \( a \) crosses a number that is not an admissible slope. This fact will be extensively used in the applications of the theory. Another useful property of the Floer homology is Viterbo’s functoriality, which we elaborate on in the following section.
1.1 TRANSFER MORPHISM

In his 1999 paper [29], Viterbo constructed a morphism

\[ \text{SH}_*(W_2) \rightarrow \text{SH}_*(W_1) \]  

(1.1.1)

associated to a codimension-0 embedding \( W_1 \rightarrow W_2 \) of a Liouville domain into another Liouville domain that respects the Liouville forms. The map (1.1.1), called the transfer morphism, fits into the commutative diagram

\[
\begin{array}{ccc}
\text{SH}_*(W_2) & \rightarrow & \text{SH}_*(W_1) \\
\uparrow & & \uparrow \\
\text{H}_{*+n}(W_2, \partial W_2) & \rightarrow & \text{H}_{*+n}(W_1, \partial W_1).
\end{array}
\]  

(1.1.2)

Here, \( 2n \) is the dimension of \( W_2 \) and the map

\[ \text{H}_{*+n}(W_2, \partial W_2) \rightarrow \text{H}_{*+n}(W_1, \partial W_1) \]

is the composition of the homomorphism induced by the inclusion \((W_2, \partial W_2) \hookrightarrow (W_2, W_2 \setminus W_1)\) and the excision isomorphism

\[ \text{H}_{*+n}(W_2, W_2 \setminus W_1) \rightarrow \text{H}_{*+n}(W_1, \partial W_1). \]

The transfer morphism can be extended to the framework of the Floer homology for exact symplectomorphism. The setting is as follows. Let \( W_1 \) and \( W_2 \) be as above, and let \( \phi : W_1 \rightarrow W_1 \) be an exact symplectomorphism. Being compactly supported, the symplectomorphism \( \phi \) can be seen as an exact symplectomorphism of \( W_2 \) as well (one extends \( \phi \) by the identity).
Theorem 1.1.1. Let $W_1, W_2, \phi$ be as above, and let $a, b \in \mathbb{R}^+ \cup \{\infty\}$ be positive admissible slopes (with respect to $W_2$ and $W_1$, respectively). Assume $a \leq b$. Then, there exists a linear map

$$\text{HF}_*(W_2, \phi, a) \to \text{HF}_*(W_1, \phi, b),$$

called the transfer morphism, with the following properties. It coincides with the map (1.1.1) for $a = b = \infty$, and $\phi$ equal to the identity. Moreover, the diagram

$$\begin{array}{ccc}
\text{HF}_*(W_2, \phi, a) & \longrightarrow & \text{HF}_*(W_1, \phi, b) \\
\downarrow & & \downarrow \\
\text{HF}_*(W_2, \phi, a') & \longrightarrow & \text{HF}_*(W_1, \phi, b'),
\end{array}$$

consisting of transfer morphisms and continuation maps, commutes for all admissible slopes $a', b' \in \mathbb{R}^+ \cup \{\infty\}$ such that $a \leq a', b \leq b', a' \leq b'$.

It is noteworthy that the diagrams (1.1.2) and (1.1.3) coincide for $\phi = \text{id}$, $a' = b' = \infty$ and $a, b > 0$ small.

1.2 Applications

The theory can be used to detect non-trivial symplectic mapping classes of a Liouville domain $(W, \lambda)$, i.e. compactly supported symplectomorphisms of the Liouville domain up to symplectic isotopies relative to the boundary. The symplectic mapping classes can be seen as elements of $\pi_0 \text{Symp}_c(W, d\lambda)$, where $\text{Symp}_c(W, d\lambda)$ stands for the group of all symplectomorphisms of $W$ that are equal to the identity near the boundary. An important family of such classes is furnished by so-called fibered Dehn twists [6, 24] of Liouville domains with periodic Reeb flow on the boundary (see Definition 5.2.1).
For historical account of symplectic mapping class group in dimensions 2 and 4, we refer to [25]. The papers [24] and [7] contain further results along these lines also in higher dimensions.

**Theorem 5.3.1.** Let \((W,\lambda)\) be a Liouville domain such that the Reeb flow on \(\partial W\) is 1-periodic. Let \(a \in \mathbb{R}\) be a real number that is not a period of any Reeb orbit on \(\partial W\). If the fibered Dehn twist represents a class of order \(\ell \in \mathbb{N}\) in \(\pi_0 \text{Symp}_c(W, d\lambda)\), then

\[
\text{HF}(\text{id}, a) \cong \text{HF}(\text{id}, a + \ell).
\]

(5.3.38)

**Corollary 5.3.3.** Let \((W,\lambda)\) be as in Theorem 5.3.1. If

\[
\dim H(W; \mathbb{Z}_2) < \dim S\mathbb{H}(W; \mathbb{Z}_2),
\]

(5.3.42)

then the fibered Dehn twist represents a class of infinite order in \(\pi_0 \text{Symp}_c(W, d\lambda)\). Here, \(\dim H(W; \mathbb{Z}_2)\) stands for the sum of Betti numbers rather than the dimension of the homology group of a particular degree.

Special cases of Corollary 5.3.3 include the squares of the generalized Dehn twists on \(T^*S^n\) and their extensions to the cotangent bundles of other symmetric spaces (see Corollary 4.5 in [24]).

**Definition 1.2.1.** The homology \(H_*(X; \mathbb{Z}_2)\) of a topological space \(X\) is said to be **symmetric** if there exists \(k \in \mathbb{Z}\) such that

\[
H_{k-j}(X; \mathbb{Z}_2) \cong H_j(X; \mathbb{Z}_2)
\]

for all \(j \in \mathbb{Z}\).
**Corollary 5.3.6.** Let \((W, \lambda)\) be as in Theorem 5.3.1. Assume that the Reeb flow induces a free circle action on \(\partial W\), and that the first Chern class \(c_1(W)\) vanishes. If the homology \(H_\ast(W; \mathbb{Z}_2)\) is not symmetric, then the fibered Dehn twist represents a nontrivial class in \(\pi_0 \text{Symp}_c(W, d\lambda)\).

The corollary applies when \(W\) is a smooth degree \(d \geq 2\) projective hypersurface in \(\mathbb{C}P^m, m > 3\) with a neighbourhood of a smooth hyperplane section removed (see Example 5.4.8).

We define a numerical invariant, the iterated ratio
\[
\kappa(W, \phi) := \limsup_{m \to \infty} \frac{\dim \text{HF}(W, \phi^m, \epsilon)}{m} \in [0, \infty]
\]
of a symplectic mapping class that may be interesting in its own right. It does not depend on the ambient Liouville domain in the following sense.

**Theorem 5.5.3.** Let \(W_1\) and \(W_2\) be Liouville domains as in Theorem 1.1.1, and let \(\phi : W_1 \to W_1\) be an exact symplectomorphism, then \(\kappa(W_1, \phi) = \kappa(W_2, \phi)\).

Consequently, a generalized Dehn twist furnished by an exact Lagrangian sphere in a Liouville domain always induces a class of infinite order (see Corollary 5.6.3).

**Proposition 1.2.2.** Let \((\mathbb{S}^n, g_0)\) be the \(n\)-dimensional sphere with the standard Riemannian metric, and let \(g\) be a Riemannian metric on \(\mathbb{S}^n\) such that \(g \leq g_0\). Then, there exists a non-constant closed geodesic on \((\mathbb{S}^n, g)\) of length less than or equal to \(2\pi\).

In fact, a more general statement is true (see Theorem 5.7.3 and Corollary 5.7.4).
1.3 CONVENTIONS

Let $(W, \omega)$ be a symplectic manifold. A function $H : W \to \mathbb{R}$ and its Hamiltonian vector field $X_H$ are related by

$$dH = \omega(X_H, \cdot).$$

The Hamiltonian isotopy $\psi^H_t : W \to W$ of a time-dependent Hamiltonian $H : \mathbb{R} \times W \to \mathbb{R} : (t, x) \mapsto H_t(x)$ is determined by

$$\partial_t \psi^H_t = X_{H_t} \circ \psi^H_t, \quad \psi^H_0 = id.$$

An almost complex structure $J$ on $W$ is said to be $\omega$-compatible if $\omega(\cdot, J \cdot)$ is a Riemannian metric.
2

2.1 LIOUVILLE DOMAINS

Definition 2.1.1. A Liouville domain is a compact manifold $W$ with a 1-form $\lambda$, called Liouville form, that satisfies the following conditions. The 2-form $d\lambda$ is a symplectic form on $W$, and the Liouville vector field $X_\lambda$, defined by $X_\lambda \cdot d\lambda = \lambda$, points transversally out on the boundary $\partial W$.

If $(W, \lambda)$ is a Liouville domain, then the restriction $\lambda|_{\partial W}$ of $\lambda$ to the boundary $\partial W$ is a contact form. Using the flow of the Liouville vector field $X_\lambda$, a collar neighbourhood of $\partial W$ can be identified with $((0, 1] \times \partial W, r \lambda|_{\partial W})$, where $r$ stands for the real coordinate.

Definition 2.1.2. Let $(W, \lambda)$ be a Liouville domain. There exists a unique embedding

$$\iota : [0, 1] \times \partial W \to W$$

such that $\iota(1, x) = x$ and $\iota^* \lambda = r \lambda|_{\partial W}$. The completion $\widehat{W}$ of $(W, \lambda)$ is the exact symplectic manifold obtained by gluing $W$ and $\mathbb{R}^+ \times \partial W$ via $\iota$.

If $W$ is a Liouville domain and $r \in \mathbb{R}^+$, we denote by $W^r$ the subset of the completion $\widehat{W}$ defined by

$$W^r := \widehat{W} \setminus ((r, \infty) \times \partial W).$$
Here, and in the rest of the thesis, the sets $\mathbb{R}^+ \times \partial W$ and $W$ are identified with the corresponding regions in the completion $\hat{W}$.

2.2 Floer Data and Admissible Data

Definition 2.2.1. A symplectomorphism $\phi : W \rightarrow W$ of a Liouville domain $(W, \lambda)$ is called exact if it is compactly supported in the interior of $W$ and if the 1-form $\phi^*\lambda - \lambda$ is exact.

Given an exact symplectomorphism $\phi$. The definition implies that a primitive function of the 1-form $\phi^*\lambda - \lambda$ is constant on the boundary components. We denote by $F_\phi$ the one for which the sum of its values on the boundary components is equal to 0. In the case of a Liouville domain with connected boundary, this means that $F_\phi$ is compactly supported in the interior of the Liouville domain. The exact symplectomorphisms of a Liouville domain $(W, \lambda)$ form a group, denoted by $\text{Symp}_c(W, \lambda/d)$. The functions associated to composition and inverse are given by

\begin{equation}
F_{\phi_0 \circ \phi_1} = F_{\phi_0} \circ \phi_1 + F_{\phi_1}, \quad (2.2.4)
\end{equation}

\begin{equation}
F_{\phi^{-1}} = -F_{\phi} \circ \phi^{-1}.
\end{equation}

It is convenient to regard exact symplectomorphisms of a Liouville domain as symplectomorphisms of its completion (one can extend $\phi$ by the identity on the complement of the Liouville domain). We will tacitly do so.

Every symplectomorphism generated by a Hamiltonian that is equal to 0 near the boundary is exact. However, an exact symplectomorphism need not be Hamiltonian or even isotopic to the identity.
Definition 2.2.2. A real number $a \in \mathbb{R}$ is called admissible with respect to a Liouville domain $(W, \lambda)$ if it is not a period of any Reeb orbit on the boundary $\partial W$. Infinity $\infty$ is considered admissible with respect to any Liouville domain.

Definition 2.2.3. Let $\phi$ be an exact symplectomorphism of a Liouville domain $(W, \lambda)$. Floer data for $\phi$ is a pair $(H, J)$ of a (time-dependent) Hamiltonian $H_t : \hat{W} \to \mathbb{R}$ and a family $J_t$ of $d\lambda$-compatible almost complex structures on $\hat{W}$ satisfying the following conditions. $H$ and $J$ are twisted by $\phi$, i.e.

\[
H_{t+1} = H_t \circ \phi, \tag{2.2.5} \\
J_{t+1} = \phi^* J_t. \tag{2.2.6}
\]

In addition, there exists $r_0 \in \mathbb{R}^+$ such that

\[
H_t(x, r) = ar, \tag{2.2.7} \\
dr \circ J_t(x, r) = -\lambda, \tag{2.2.8}
\]

on $(r_0, \infty) \times \partial W$, for some admissible $a \in \mathbb{R}$. If we want to specify the slope, we say "$(H, J)$ is Floer data for $(\phi, a)$." 

Remark 2.2.4. Let $(W, \lambda)$ and $J$ be as in Definition 2.2.3. Denote the contact form $\lambda|_{\partial W}$ by $\alpha$ and the induced contact structure on $\partial W$ by $\xi$. Condition (2.2.8) together with $d\lambda$-compatibility of $J_t$ implies that $J_t|_\xi$ is a compatible complex structure on the symplectic vector bundle

$$(\xi, d\alpha) \to \partial W.$$ 

Definition 2.2.5. Let $\phi$ be an exact symplectomorphism. Floer data $(H, J)$ for $\phi$ is said to be regular if $H$ is non-degenerate with respect to $\phi$, i.e.

$$\det \left( d(\phi \circ \psi^H_1)(x) - \text{id} \right) \neq 0,$$
for all fixed points $x$ of $\phi \circ \psi^H$, and if the linearized operator for every solution of the Floer equation (2.3.11) on page 12 is surjective.

### 2.3 Action Functional

**Definition 2.3.1.** [20]. Let $\phi$ be an exact symplectomorphism of a Liouville domain $(W, \lambda)$, and let $H_t : \hat{W} \to \mathbb{R}$ be a Hamiltonian satisfying (2.2.5). The **action functional** $A_{\phi,H}$ is a function defined on the **twisted loop space**

$$\Omega_{\phi} := \{ \gamma : \mathbb{R} \to \hat{W} \mid \phi(\gamma(t+1)) = \gamma(t) \}$$

by

$$A_{\phi,H}(\gamma) := -\int_0^1 (\gamma^*\lambda + H_t(\gamma(t))dt) - F_{\phi}(\gamma(1)).$$

**Proposition 2.3.2.** Let $\phi$ and $H_t$ be as in Definition 2.3.1. The critical points of the action functional $A_{\phi,H}$ are Hamiltonian twisted loops, i.e. the elements of the set

$$P_{\phi,H} := \{ \gamma \in \Omega_{\phi} \mid \dot{\gamma} = X_{H_t} \circ \gamma \}.$$ 

**Proof.** Let $\gamma \in \Omega_{\phi}$ and let $\zeta \in T_\gamma \Omega_{\phi}$. This means $\zeta$ is a section of the vector bundle $\gamma^*TW$ such that

$$d\phi(\zeta(t+1)) = \zeta(t). \quad (2.3.9)$$

The derivative of $A_{\phi,H}$ at the point $\gamma$ in the direction $\zeta$ is given by

$$dA_{\phi,H}(\gamma)\zeta = \left. \frac{d}{ds} \right|_{s=0} A_{\phi,H}(\gamma_s),$$
where $\gamma_s$ is a smooth family of twisted loops satisfying
\[
\left. \frac{d}{ds} \gamma_s(t) \right|_{s=0} = \zeta(t).
\]
Since
\[
\left. \frac{d}{ds} \right|_{s=0} (\gamma_s^* \lambda) = d\lambda(\zeta(t), \dot{\gamma}(t)) \, dt + \frac{d}{dt}(\lambda(\zeta(t))) \, dt,
\]
\[
\left. \frac{d}{ds} \right|_{s=0} H_t(\gamma_s(t)) = dH_t(\zeta(t)) = -d\lambda(\zeta(t), X_{H_t}(\gamma(t))),
\]
and
\[
\left. \frac{d}{ds} \right|_{s=0} F_{\phi}(\gamma_s(1)) = dF_{\phi}(\zeta(1)) = (\phi^* \lambda - \lambda)(\zeta(1)) = \lambda(d\phi(\zeta(1))) - \lambda(\zeta(1)) \equiv \lambda(\zeta(0)) - \lambda(\zeta(1)),
\]
we get
\[
dA_{\phi, H}(\gamma) \zeta = \int_0^1 \omega(\dot{\gamma}(t) - X_{H_t}(\gamma(t)), \zeta(t)) \, dt. \quad (2.3.10)
\]
Here, $\otimes$ used (2.3.9). Since (2.3.10) holds for all $\zeta \in T_\gamma \Omega_{\phi}$, we have $dA_{\phi, H}(\gamma) = 0$ if, and only if, $\dot{\gamma}(t) = X_{H_t}(\gamma(t))$ for all $t$. \qed

Let $(W, \lambda)$ be a Liouville domain. A family $J_t$ of $d\lambda$-compatible almost complex structures on $\widetilde{W}$ which satisfies (2.2.6) gives rise to a Riemannian metric on $\Omega_{\phi}$. The Riemannian metric is given by
\[
\langle \xi, \zeta \rangle := \int_0^1 \omega(\xi(t), J_t \circ \gamma(t) \zeta(t)) \, dt,
\]
where $\gamma \in \Omega_{\phi}$ is a twisted loop, and $\xi, \zeta \in T_{\gamma} \Omega_{\phi}$. The negative gradient flow lines of $\mathcal{A}_{\phi, H}$ with respect to this inner product are solutions $u : \mathbb{R}^2 \to \hat{W}$ of the Floer equation

$$\partial_s u + J_t(u) \left( \partial_t u - X_{H_t}(u) \right) = 0.$$ (2.3.11)

that satisfy the periodicity condition

$$\phi \circ u(s, t + 1) = u(s, t).$$ (2.3.12)

### 2.4 Grading

Let $\phi : \hat{W} \to \hat{W}$ be a symplectomorphism of a symplectic manifold $(\hat{W}, \omega)$. Let

$$H^i_t : \hat{W} \to \mathbb{R}, \quad i = 0, 1$$

be two Hamiltonians such that

$$H^i_{t+1} = H^i_t \circ \phi$$

and such that

$$\det \left( d(\phi \circ \psi^H_i)(x) - \text{id} \right) \neq 0,$$

for all fixed points $x$ of $\phi \circ \psi^H_i$. And, let

$$\gamma_i \in \mathcal{P}(\phi, H^i), \quad i = 0, 1$$

be twisted loops representing the same class in $\pi_0 \Omega_{\phi}$. Consider a homotopy

$$[0, 1] \times \mathbb{R} \to \hat{W}$$

between them through $\Omega_{\phi}$. Write it as a map

$$u : \mathbb{R} \times \mathbb{R} \to \hat{W}$$
such that
\[ u(s, t) = \phi \circ u(s, t + 1), \]  
\hspace{1cm} (2.4.13)
and
\[ u(s, t) = \begin{cases} 
\gamma_0(t) & \text{for } s \leq 0 \\
\gamma_1(t) & \text{for } s \geq 1.
\end{cases} \]  
\hspace{1cm} (2.4.14)

**Definition 2.4.1.** Let \( u \) be as above. The **relative Maslov index** of \( u \) is the number
\[ \mu(u) = \mu_{CZ}(\Psi^1) - \mu_{CZ}(\Psi^0) \in \mathbb{Z}, \]
where
\[ \Psi^i : [0, 1] \rightarrow \text{Sp}(2n), \quad i \in \{0, 1\}, \]
is the path
\[ t \mapsto T_{i, t}^{-1} \circ d\psi_t^H(\gamma_i(0))T_{i, 0} \]
of symplectic matrices, and
\[ T_{s, t} : \mathbb{R}^{2n} \rightarrow T_{u(s, t)}\tilde{W} \]
is a symplectic trivialization satisfying
\[ d\phi \circ T_{s, t+1} = T_{s, t}. \]
The number \( \mu(u) \) does not depend on the choice of trivialization.

We need the notion of a mapping torus in order to define the relative Maslov index for the pair \((\gamma_0, \gamma_1)\) (independent of the choice of homotopy \( u \)).
Definition 2.4.2. Let $\phi$ and $(\hat{\mathcal{W}}, \omega)$ be as above. The mapping torus of $\phi$ is the fibration $p_\phi : \hat{\mathcal{W}}_\phi \to \mathbb{S}^1$ defined by

$$
\hat{\mathcal{W}}_\phi := \frac{\hat{\mathcal{W}} \times \mathbb{R}}{(\phi(x), t) \sim (x, t + 1)}, \quad p_\phi([x, t]) := [t] \in \mathbb{R}/\mathbb{Z}.
$$

Denote the vertical tangent bundle of $\hat{\mathcal{W}}_\phi$ by

$$
T^V\hat{\mathcal{W}} := \ker dp_\phi \to \hat{\mathcal{W}}_\phi.
$$

The symplectic form $\omega$ on $\hat{\mathcal{W}}$ makes $T^V\hat{\mathcal{W}}$ into a symplectic vector bundle. A loop in $\varOmega_\phi$, seen as a map

$$
u : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \to \hat{\mathcal{W}}
$$

satisfying $\phi \circ u(s, t + 1) = u(s, t)$, can be identified with the map

$$
\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \hat{\mathcal{W}}_\phi : (s, t) \mapsto [(u(s, t), t)].
$$

Let $\theta \in \pi_0\varOmega_\phi$. Associated to it is the minimal Chern number $N_\theta$. It is defined as the minimal positive generator of the group

$$
N_\theta \mathbb{Z} := \left\langle c_1(T^V\hat{\mathcal{W}}), \pi_1(\varOmega_\phi, \theta) \right\rangle \subset \mathbb{Z}.
$$

We define the minimal positive generator of trivial group to be $\infty$, and $\mathbb{Z}_\infty := \mathbb{Z}$.

Definition 2.4.3. Let $\hat{\mathcal{W}}, \phi, H^i$ be as above. The relative Maslov index for a pair

$$
(\gamma_0, \gamma_1) \in \mathcal{P}(\phi, H^0) \times \mathcal{P}(\phi, H^1)
$$

of twisted loops representing the same class $\theta \in \pi_0\varOmega_\phi$ is the number

$$
\mu(\gamma_0, \gamma_1) := \mu(u) \pmod{2N_\theta} \in \mathbb{Z}_{2N_\theta},
$$
where 
\[ u : \mathbb{R} \times \mathbb{R} \to \hat{W} \]
is such that (2.4.13) and (2.4.14) hold.

2.5 TAMING INFINITY

**Definition 2.5.1.** Let \( \phi \) be an exact symplectomorphism of a Liouville domain \((W, \lambda)\), and let \((H^-, J^-)\) and \((H^+, J^+)\) be regular Floer data for \( \phi \). **Continuation data** from \((H^-, J^-)\) to \((H^+, J^+)\) consists of Hamiltonians \( H_{s,t}, G_{s,t} : \hat{W} \to \mathbb{R} \) and a family of \( d\lambda \)-compatible almost complex structures \( J_{s,t} \) on \( \hat{W} \) such that the following conditions are satisfied. For each \( s \in \mathbb{R} \), \( H_s, G_s \) and \( J_s \) are twisted by \( \phi \), i.e.

\[
H_{s,t} \circ \phi = H_{s,t+1},
\]

\[
G_{s,t} \circ \phi = G_{s,t+1},
\]

and

\[
\phi^* J_{s,t} = J_{s,t+1}.
\]

In addition, there exist \( r_0 \in (0, \infty) \) and a function

\[
a : \mathbb{R}^2 \to \mathbb{R}
\]

that is increasing with respect to the first coordinate such that

\[
G_{s,t}(x, r) = 0,
\]

\[
H_{s,t}(x, r) = a(s, t)
\]

and the condition (2.2.8) holds for \( J_s(x, r) \), for all \( s, t \in \mathbb{R} \), \( r \geq r_0 \), and \( x \in \partial W \). Finally,

\[
(H_{s,t}, G_{s,t}, J_{s,t}) = (H_{t}^{\pm}, 0, J_{t}^{\pm}), \quad \text{for } \pm s \gg 0.
\]
**Remark 2.5.2.** We can find continuation data from \((H^-, J^-)\) to \((H^+, J^+)\) whenever the slope of \(H^-\) is less than or equal to the slope of \(H^+\).

**Definition 2.5.3.** Let \(\phi, H_{s,t}, G_{s,t}\) and \(J_{s,t}\) be as in Definition 2.5.1. The **energy** of a solution \(u : \mathbb{R}^2 \to \hat{W}\) of the \(s\)-dependent (generalized) Floer equation

\[
\partial_s u - X_{G_{s,t}}(u) + J_{s,t}(u) \left( \partial_t u - X_{H_{s,t}}(u) \right) = 0 \tag{2.5.15}
\]
satisfying periodicity condition (2.3.12) is defined to be

\[
E(u) := \int_{-\infty}^{\infty} \int_0^1 |\partial_s u - X_{G_{s,t}}(u)|^2 \, dt \, ds.
\]

**Lemma 2.5.4.** Let \((W, \lambda), \phi, H^\pm, J^\pm, H_s, G_s, J_s\) be as in Definition 2.5.1, let \(\gamma \in \mathcal{P}(\phi, H^\pm)\) and let \(u : \mathbb{R}^2 \to \hat{W}\) be a solution of (2.5.15), (2.3.12) such that \(\lim_{s \to \pm\infty} u(s, t) = \gamma^\pm(t)\). Then,

\[
E(u) = A_{\phi, H^-}(\gamma^-) - A_{\phi, H^+}(\gamma^+) - \int_{-\infty}^{\infty} \int_0^1 B_{s,t}(u) \, dt \, ds \tag{2.5.16}
\]

where

\[
B_{s,t} := \partial_s H_{s,t} - \partial_t G_{s,t} - \{G_{s,t}, H_{s,t}\},
\]

and \(\{G_{s,t}, H_{s,t}\} := d\lambda(X_{G_{s,t}}, X_{H_{s,t}})\) stands for the Poisson bracket.

**Proof.** Denote by \(\omega\) the symplectic form \(d\lambda\). Using (2.5.15), we get

\[
|\partial_s u - X_{G_{s,t}}(u)|^2 = \nonumber
= \omega (\partial_s u - X_{G_{s,t}}(u), J_{s,t}(u)(\partial_s u - X_{G_{s,t}}))
= \omega (\partial_s u - X_{G_{s,t}}(u), \partial_t u - X_{H_{s,t}}(u)) \tag{2.5.17}
= u^* \omega + dH_{s,t}(\partial_s u) - dG_{s,t}(\partial_t u) + \{G_{s,t}, H_{s,t}\}(u).
\]
By the chain rule,
\[ dH_{s,t}(\partial_s u) = \partial_s (H_{s,t}(u)) - (\partial_s H_{s,t})(u), \] (2.5.18)
and similarly
\[ dG_{s,t}(\partial_t u) = \partial_t (G_{s,t}(u)) - \partial_t G_{s,t}(u). \]

The Stokes theorem implies
\[
\int_{-\infty}^{+\infty} \int_{0}^{1} u^* \omega dt ds = \\
\quad = \int_{0}^{1} (\gamma^+)^* \lambda - \int_{-\infty}^{+\infty} \lambda(\partial_s u(s, 1)) ds - \\
\quad - \int_{0}^{1} (\gamma^-)^* \lambda + \int_{-\infty}^{+\infty} \lambda(\partial_s u(s, 0)) ds \\
\quad = \int_{0}^{1} (\gamma^+)^* \lambda - \int_{0}^{1} (\gamma^-)^* \lambda + \\
\quad + \int_{-\infty}^{+\infty} (\phi^* \lambda - \lambda)(\partial_s u(s, 1)) ds \\
\quad = \int_{0}^{1} (\gamma^+)^* \lambda - \int_{0}^{1} (\gamma^-)^* \lambda + \\
\quad + F_\phi \circ \gamma^+(1) - F_\phi \circ \gamma^-(1). \] (2.5.19)

By Newton-Leibniz formula and (2.5.18), we get
\[
\int_{-\infty}^{+\infty} \int_{0}^{1} dH_{s,t}(\partial_s u) dt ds = \\
\quad = \int_{-\infty}^{+\infty} \int_{0}^{1} \partial_s (H_{s,t}(u)) - (\partial_s H_{s,t})(u) dt ds \\
\quad = \int_{0}^{1} H_{s,t} \circ \gamma^+(t) dt - \int_{0}^{1} H_{s,t} \circ \gamma^-(t) dt - \\
\quad - \int_{-\infty}^{+\infty} \int_{0}^{1} (\partial_s H_{s,t})(u) dt ds. \] (2.5.20)
Similarly,
\[
\int_{-\infty}^{+\infty} \int_{0}^{1} dG_{s,t}(\partial_s u) dt ds = \\
= \int_{-\infty}^{+\infty} \int_{0}^{1} \partial_t(G_{s,t}(u)) - \partial_t G_{s,t}(u) dt ds \\
= \int_{-\infty}^{+\infty} G_{s,1}(u(s, 1)) - G_{s,0}(u(s, 0)) ds - \int_{-\infty}^{+\infty} \int_{0}^{1} \partial_t G_{s,t}(u) dt ds \\
= -\int_{-\infty}^{+\infty} \int_{0}^{1} \partial_t G_{s,t}(u) dt ds.
\]

The lemma follows from (2.5.17), (2.5.19), (2.5.20), and (2.5.21).

Lemma 2.5.5. Let \( W, H_{s,t}, G_{s,t}, J_{s,t} \) and \( r_0 \) be as in Definition 2.5.1, and let \( u : U \to (r_0, \infty) \times \partial W \) be a solution of (2.5.15), where \( U \subset \mathbb{R}^2 \) is open and connected. Then \( r \circ u \) has no local maxima unless it is constant.

Proof. The lemma follows from the standard argument, which can be found, for example, in [30].

2.6 Chain Complex

Theorem 2.6.1. Let \( \phi \) be an exact symplectomorphism of a Liouville domain \( (W, \lambda) \), let \( (H, J) \) be regular Floer data for \( \phi \), and let \( \gamma^-, \gamma^+ \in \mathcal{P}(\phi, H) \). Then, the set

\[ M(\phi, H, J, \gamma^-, \gamma^+) \]

of maps \( u : \mathbb{R}^2 \to \hat{W} \) such that (2.3.11) and (2.3.12) hold, and such that

\[
\lim_{s \to \pm \infty} u(s, t) = \gamma^\pm(t)
\]
is a manifold (possibly empty or with connected components of varying dimensions). The connected component containing $u_0$ has dimension equal to $\mu(u_0)$. The quotient of the $k$–dimensional part $M_k(\cdots)$ of $M(\cdots)$ by the natural $\mathbb{R}$–action is denoted by $\hat{M}_k(\cdots)$.

**Proof.** See page 21.

**Theorem 2.6.2.** In the situation of Theorem 2.6.1, the set $\hat{M}^1(\phi, H, J, \gamma^-, \gamma^+)$ is finite. Its cardinality modulo 2 is denoted by $n(\phi, H, J, \gamma^-, \gamma^+)$. 

**Proof.** See page 21.

**Theorem 2.6.3.** Let $\phi$ be an exact symplectomorphism of a Liouville domain $(W, \lambda)$, let $(H, J)$ be regular Floer data for $\phi$, let $\theta \in \pi_0\Omega_\phi$, and let 

$$\gamma_0 \in \mathcal{P}^\theta(\phi, H) := \mathcal{P}(\phi, H) \cap \theta.$$ 

Then, 

$$(\text{CF}_*(\phi, H, J, \theta), \partial_{H,J})$$ 

is a $\mathbb{Z}_2N_\theta$–graded chain complex (see page 14 for the definition of $N_\theta$), where 

$$\text{CF}_k(\phi, H, J, \theta) := \bigoplus_{\gamma \in \mathcal{P}^\theta(\phi, H), \mu(\gamma, \gamma_0) = k} \mathbb{Z}_2 \langle \gamma \rangle,$$

and $\partial = \partial_{H,J}$ is defined by 

$$\partial \langle \gamma \rangle := \sum_{\gamma' \in \mathcal{P}^\theta(\phi, H), \mu(\gamma, \gamma') = 1} n(\phi, H, J, \gamma, \gamma') \langle \gamma' \rangle.$$
We denote the homology of this complex by

\[ HF_\ast(\phi, H, J, \theta). \]

Note that the grading depends on the choice of \( \gamma_0 \).

**Proof.** See page 21. \( \square \)

### 2.7 Continuation Maps

**Theorem 2.7.1.** Let \( \phi \) be an exact symplectomorphism of a Liouville domain \( (W, \lambda) \), let \((H^\alpha, J^\alpha)\) and \((H^\beta, J^\beta)\) be regular Floer data for \( \phi \), and let

\[ \gamma^\alpha \in \mathcal{P}(\phi, H^\alpha), \quad \gamma^\beta \in \mathcal{P}(\phi, H^\beta). \]

Then, for generic continuation data \((H_s, G_s, J_s)\) from \((H^\alpha, J^\alpha)\) to \((H^\beta, J^\beta)\), the set

\[ \mathcal{M}(\phi, \{H_s, G_s, J_s\}, \gamma^\alpha, \gamma^\beta) \]

of maps \( u : \mathbb{R}^2 \to \hat{W} \) such that (2.5.15) and (2.3.12) hold, and such that

\[ \lim_{s \to -\infty} u(s, t) = \gamma^\alpha(t), \]

\[ \lim_{s \to +\infty} u(s, t) = \gamma^\beta(t) \]

is a manifold (cut out transversely by the Floer equation). The connected component containing \( u_0 \) has dimension \( \mu(u_0) \). The \( k \)-dimensional part of \( \mathcal{M}(\cdots) \) is denoted by \( \mathcal{M}^k(\cdots) \).

**Proof.** See page 21. \( \square \)

**Theorem 2.7.2.** In the situation of Theorem 2.7.1, the set \( \mathcal{M}^0(\phi, \{H_s, G_s, J_s\}, \gamma^\alpha, \gamma^\beta) \) is finite. Its cardinality modulo 2 is denoted by \( n(\phi, \{H_s, G_s, J_s\}, \gamma^\alpha, \gamma^\beta) \).
Proof. See below.

\[ \] \[ \] \[ \]

**Theorem 2.7.3.** Let \( \phi, H^\alpha, H^\beta, H_s, G_s, J^\alpha, J^\beta, J_s \) be as in Theorem 2.7.1. Then the linear map

\[
\text{CF}_* (\phi, H^\alpha, J^\alpha, \theta) \rightarrow \text{CF}_* (\phi, H^\beta, J^\beta, \theta)
\]

defined on generators by

\[
\langle \gamma^\alpha \rangle \mapsto \sum_{\gamma^\beta \in P^\theta (\phi, H^\beta)} n(\phi, \{H_s, G_s, J_s\}, \gamma^\alpha, \gamma^\beta) \langle \gamma^\beta \rangle
\]

\[
\mu(\gamma^\beta, \gamma^\alpha) = 0
\]

induces a homomorphism, called *continuation map*, on homology level. The continuation map

\[
\Phi^{\beta \alpha} : \text{HF}_* (\phi, H^\alpha, J^\alpha, \theta) \rightarrow \text{HF}_* (\phi, H^\beta, J^\beta, \theta)
\]

do\ not depend on the choice of continuation data. Moreover, it satisfies the composition formula

\[
\Phi^{\gamma^\beta} \circ \Phi^{\beta \alpha} = \Phi^{\gamma^\alpha}, \quad \Phi^{\alpha \alpha} = \text{id}.
\]

**Proof.** See below.

**Corollary 2.7.4.** Let \( \phi \) be an exact symplectomorphism of a Liouville domain \( (W, \lambda) \), and let \( \theta \in \pi_0 \Omega^\phi \). If \( (H^\alpha, J^\alpha) \) and \( (H^\beta, J^\beta) \) are regular Floer data for \( \phi \), and \( H^\alpha, H^\beta \) have the same slope, then

\[
\text{HF}_* (\phi, H^\alpha, J^\alpha, \theta) \cong \text{HF}_* (\phi, H^\beta, J^\beta, \theta).
\]

**Proof of Theorems 2.6.1, 2.6.2, 2.6.3, 2.7.1, 2.7.2 and 2.7.3.** We prove that there exist compact subsets \( K(\phi, H, J) \) and \( K(\phi, \{H_s, G_s, J_s\}) \) of \( \widehat{W} \) such that all elements of the Moduli
spaces $\mathcal{M}(\phi, H, J, \gamma^-, \gamma^+) \text{ and } \mathcal{M}(\phi, \{H_s, G_s, J_s\}, \gamma^\alpha, \gamma^\beta)$ map $\mathbb{R}^2$ into $K(\phi, H, J)$ and $K(\phi, \{H_s, G_s, J_s\})$, respectively. Since $M(\phi, H, J, \gamma^-, \gamma^+)$ is a special case of $M(\phi, \{H_s, G_s, J_s\}, \gamma^\alpha, \gamma^\beta)$ (for $H_s, J_s$ s-independent, $G_s = 0, \gamma^- = \gamma^\alpha$, and $\gamma^+ = \gamma^\beta$), we consider only the case of $M(\phi, \{H_s, G_s, J_s\}, \gamma^\alpha, \gamma^\beta)$. The boundary $\partial W$ is denoted shortly by $M$.

Let $r_0 \in (1, \infty)$ be such that

$$G_{s,t}(x, r) = 0, \quad H_{s,t}(x, r) = a(s, t),$$

and condition (2.2.8) holds for $J_s(x, r)$, for all $s, t \in \mathbb{R}$, and $(r, x) \in (r_0, \infty) \times M$. The existence of such $r_0$ is guaranteed by the definition of continuation data (Definition 2.5.1). Since the slopes of $H^\alpha$ and $H^\beta$ are admissible, the elements of the sets $\mathcal{P}_{\phi, H^\alpha}$ and $\mathcal{P}_{\phi, H^\beta}$ are contained in the compact set $\widetilde{W} \setminus ((r_0, \infty) \times M)$. Assume there exist $\gamma^\alpha \in \mathcal{P}_{\phi, H^\alpha}, \gamma^\beta \in \mathcal{P}_{\phi, H^\beta}$, and $u \in \mathcal{M}(\phi, \{H_s, G_s, J_s\}, \gamma^\alpha, \gamma^\beta)$ such that

$$u(\mathbb{R}^2) \cap (r_0 + 1, \infty) \times M \neq \emptyset.$$ 

The symplectomorphism $\phi$ is equal to the identity on $(r_0, \infty) \times M$. Therefore

$$u(s, t) = u(s, t + 1)$$

whenever $u(s, t) \in (r_0, \infty) \times M$. This together with

$$\lim_{s \to -\infty} u(s, t) = \gamma^\alpha(t),$$

$$\lim_{s \to +\infty} u(s, t) = \gamma^\beta(t)$$

implies that the sets

$$u([-C, C] \times [0, 1]) \cap ((r_0, \infty) \times M)$$

Thus, we have

$22$
and
\[ u(\mathbb{R}^2) \cap ((r_0, \infty) \times M) \]
coincide for \( C \in (0, \infty) \) large enough. Since
\[ [-C, C] \times [0, 1] \]
is a compact set, the function \( r \circ u \) attains a maximum on \( u^{-1}((r_0, \infty) \times M) \). Let \( U \subset u^{-1}((r_0, \infty) \times M) \) be the connected component of a point at which the maximum is achieved. Connected components of a locally path-connected space are open. Hence \( U \) is an open subset of \( u^{-1}((r_0, \infty) \times M) \). Since \( u^{-1}((r_0, \infty) \times M) \) is open in \( \mathbb{R}^2 \), \( U \) is an open subset of \( \mathbb{R}^2 \) as well.

Lemma 2.5.5 implies that \( r \circ u|_U \) is constant. Denote by \( c \in (0, \infty) \) the value of \( r \circ u \) on \( U \). The set \( U \) is a connected component of \( u^{-1}(\{c\} \times M) \). It follows that \( U \) is a closed subset of \( u^{-1}(\{c\} \times M) \). Since \( u^{-1}(\{c\} \times M) \) is closed in \( \mathbb{R}^2 \), so is \( U \). This implies that \( U \) is a non-empty open and closed subset of \( \mathbb{R}^2 \), i.e. \( U = \mathbb{R}^2 \). However, this is impossible because
\[
\lim_{s \to +\infty} u(s, t) = \gamma^\beta(t).
\]
Since everything happens in a compact subset of \( \widehat{W} \), the arguments for the case of a closed symplectic manifold (instead of \( \widehat{W} \)) apply here as well \([12, 13]\).

We end the proof by two remarks. Bubbling-off cannot occur because the symplectic form on \( \widehat{W} \) is exact. Lemma 2.5.4 implies that elements of \( \mathcal{M}(\phi, \{H_s, G_s, J_s\}, \gamma^\alpha, \gamma^\beta) \) have uniformly bounded energy. □
**Definition 2.7.5.** Let $\phi$ be an exact symplectomorphism of a Liouville domain $(W, \lambda)$, and let $\theta \in \Omega_\phi$. For a fixed admissible slope, the groups $HF_*({\phi, H, J, \theta})$ are canonically isomorphic to each other via the continuation maps. Hence the group

$$HF_*(\phi, a, \theta)$$

is well defined. We define

$$HF_*(\phi, a) := \bigoplus_{\theta \in \pi_0 \Omega_\phi} HF_*(\phi, a, \theta).$$

**Definition 2.7.6.** Let $\phi$ and $\theta$ be as in Definition 2.7.5. The continuation maps give rise to a homomorphism (which we call the same)

$$HF_*(\phi, a_1, \theta) \rightarrow HF_*(\phi, a_2, \theta) \quad (2.7.22)$$

whenever $a_1 \leq a_2$. The groups $HF_*(\phi, a, \theta)$, where $a \in \mathbb{R}$ goes through all admissible slopes, together with homomorphisms (2.7.22) form a directed system of groups. We define

$$HF_*(\phi, \infty, \theta) := \lim_{\substack{\rightarrow a}} HF_*(\phi, a, \theta),$$

$$HF_*(\phi, \infty) := \bigoplus_{\theta \in \pi_0 \Omega_\phi} HF_*(\phi, \infty, \theta).$$

**Example 2.7.7.** In the case of $\phi = \text{id}$, we can identify the groups $HF(\phi, a)$, for certain slopes $a$, with known groups. By construction, $HF(\text{id}, \infty)$ is equal to the symplectic homology of $W$ with $\mathbb{Z}_2$ coefficients

$$HF(\text{id}, \infty) = SH(W; \mathbb{Z}_2).$$

Since the Floer homology for $C^2$-small Hamiltonian reduces to Morse homology, we get

$$HF_*(\text{id}, \epsilon) \cong H_{*+n}(W, M; \mathbb{Z}_2),$$
and

\[ \text{HF}_* (\text{id}, -\varepsilon) \cong \text{H}_{*+n} (W; \mathbb{Z}_2) \]

for \( \varepsilon > 0 \) small enough (cf. [17, Theorem 3.9]). Also

\[ \text{HF}_* (\text{id}, \varepsilon, \theta) = 0 \]

for \( \theta \in \pi_0 \Omega_\phi \) non-trivial and \( |\varepsilon| \) small.
INVARIA NCE UNDER VARYING DATA

3.1 DEPENDENCE ON THE LIOUVILLE FORM

In the next two lemmas, we prove that the Floer homology $HF_\ast(W, \phi, a)$ remains the same if we change the Liouville form by adding an exact 1-form that is compactly supported in the interior of $W$. When dealing with different Liouville forms on the same manifold, we will include the Liouville form as part of the data for Floer homology. I.e. the notation $HF_\ast(W, \lambda, \phi, a)$ will be used instead of $HF_\ast(W, \phi, a)$.

Lemma 3.1.1. Let $(W, \lambda)$ be a Liouville domain, and let $f : W \to \mathbb{R}$ be a function. Then, a map $\phi : W \to W$ is an exact symplectomorphism of $(W, \lambda)$ if, and only if, it is an exact symplectomorphism of $(W, \lambda + df)$.

Proof. Denote the 1-form $\lambda + df$ by $\tilde{\lambda}$. Assume $\phi$ is an exact symplectomorphism of $(W, \lambda)$. Then, there exists a function $F : W \to \mathbb{R}$ such that

$$\phi^*\lambda - \lambda = dF,$$

and $\phi$ is compactly supported. Hence

$$\phi^*\tilde{\lambda} - \tilde{\lambda} = \phi^*(\lambda + df) - \lambda - df$$

$$= \phi^*\lambda - \lambda + d(f \circ \phi - f)$$

$$= d(F + f \circ \phi - f),$$

i.e. $\phi$ is an exact symplectomorphism of $(W, \tilde{\lambda})$. The opposite direction is proven analogously. □
Proposition 3.1.2. Let $\phi : W \to W$ be an exact symplectomorphism of a Liouville domain $(W, \lambda)$, and let $f : W \to \mathbb{R}$ be a function equal to 0 near the boundary. Then,

$$HF_*(W, \lambda, \phi, a) \cong HF_*(W, \lambda + df, \phi, a),$$

for all admissible slopes $a$.

Proof. Denote by $\tilde{\lambda}$ the 1-form $\lambda + df$. Since $\lambda$ and $\tilde{\lambda}$ agree near the boundary (where $f$ is equal to 0), the slope of a Hamiltonian with respect to $(W, \lambda)$ is the same as the one with respect to $(W, \tilde{\lambda})$. Moreover, the completions of $(W, \lambda)$ and $(W, \tilde{\lambda})$ are symplectomorphic. Let $H$ be a Hamiltonian with the slope equal to $a$. Denote by $\tilde{F}_\phi$ the compactly supported function $\hat{W} \to \mathbb{R}$ such that

$$\phi^*\tilde{\lambda} - \tilde{\lambda} = d\tilde{F}_\phi.$$

The functions $\tilde{F}_\phi$ and $F_\phi$ are related by

$$\tilde{F}_\phi = F_\phi + f \circ \phi - f.$$

The lemma follows from the following sequence of equalities

$$\tilde{A}_{\phi, H}(\gamma) := -\int_0^1 \left( \gamma^*\tilde{\lambda} + H_t(\gamma(t))dt \right) - \tilde{F}_\phi(\gamma(1))$$

$$= A_{\phi, H}(\gamma) - \int_0^1 \frac{d}{dt}(f \circ \gamma)dt - f \circ \phi(\gamma(1)) + f(\gamma(1))$$

$$= A_{\phi, H}(\gamma) - f(\gamma(1)) + f(\gamma(0)) - f \circ \phi(\gamma(1)) + f(\gamma(1))$$

$$= A_{\phi, H}(\gamma),$$

for $\gamma \in \Omega_\phi$. \hfill \Box
Proposition 3.2.1. Let $\phi$ be an exact symplectomorphism of a Liouville domain $(W, \lambda)$, and let $a_1$ and $a_2$ be real numbers such that elements of the interval $[a_1, a_2]$ are all admissible. Then

$$HF_*(\phi, a_1, \theta) \cong HF_*(\phi, a_2, \theta)$$

for all $\theta \in \pi_0 \Omega_\phi$. Moreover, the isomorphism is realized by the continuation map.

Proof. Let $(H, J)$ be regular Floer data for $(\phi, a_1)$. Assume that conditions (2.2.7) and (2.2.8) hold on $(r_0, \infty) \times \partial W$. Let $\chi : (r_0, \infty) \to [0, 1]$ be a smooth function equal to 0 near $r_0$ and 1 near $\infty$, and such that $\chi' \in [0, 1]$. Consider the Hamiltonian $\tilde{H} : \mathbb{R} \times \mathring{W} \to \mathbb{R}$ defined by

$$\tilde{H}_t(x, r) := a_1 + (a_2 - a_1)\chi(r),$$

for $(r, x) \in (r_0, \infty) \times \partial W$, and $\tilde{H} = H$ on $W^{r_0}$. The function $r \circ u$ satisfies the maximum principle, where $u$ is a solution of the Floer equation corresponding to the pair $(\tilde{H}, J)$ (see Proposition 4.1 in [30]). Moreover, there is no element of $\mathcal{P}_{\phi, \tilde{H}}$ intersecting $(r_0, \infty) \times \partial W$. This implies that ingredients for the definition of $HF_*(\phi, \tilde{H}, J, \theta)$ are all contained in the compact set $W^{r_0}$, where $H$ and $\tilde{H}$ coincide. Hence, the chain complexes $CF_*(\phi, H, J, \theta)$ and $CF_*(\phi, \tilde{H}, J, \theta)$ are identical. Moreover, for generic $(H, J)$, the map

$$CF_*(\phi, H, J, \theta) \ni \nu \mapsto \nu \in CF_*(\phi, \tilde{H}, J, \theta)$$

can be seen as the continuation map induced by the continuation data that is constant with respect to $s$-coordinate in $W^{r_0}$. This finishes the proof. $\square$
3.3 Naturality

The naturality is a way to construct isomorphisms between Floer homologies for certain pairs of Floer data. The isomorphisms are furnished by the change of variables using Hamiltonian isotopies, and they do not agree with the continuation maps in general. The naturality plays a crucial role in the next section, where we prove that the Floer homology is invariant under isotopies of exact symplectomorphisms. It is also an important ingredient in all of our applications.

Let \( \{\psi_t\}_{t \in \mathbb{R}} \) be an isotopy of a manifold \( \widehat{W} \), let

\[
H : \mathbb{R} \times \widehat{W} \to \mathbb{R} : (t, x) \mapsto H_t(x)
\]

be a smooth function, and let \( \{J_t\}_{t \in \mathbb{R}} \) be a family of almost complex structures on \( \widehat{W} \). We define \( \psi^*H \) to be the smooth function

\[
\psi^*H : \mathbb{R} \times \widehat{W} \to \mathbb{R} : (t, x) \mapsto H(t, \psi_t(x)),
\]

and \( \psi^*J \) to be the family \( \{\psi^*J_t\}_{t \in \mathbb{R}} \). Let \( \phi : \widehat{W} \to \widehat{W} \) be a diffeomorphism. Assume that the isotopy \( \psi \) satisfies

\[
\psi_t \circ \phi \circ \psi_1 = \phi \circ \psi_{t+1}.
\]

Then the map

\[
\Omega_{\phi} \to \Omega_{\phi \circ \psi_1} : \gamma(\cdot) \mapsto \psi_{-1} \circ \gamma(\cdot) =: \psi^*\gamma
\]

is well defined, and induces the bijection

\[
\pi_0 \Omega_{\phi} \to \pi_0 \Omega_{\phi \circ \psi_1} : \theta \mapsto \psi^*\theta.
\]

**Lemma 3.3.1.** Let \( \phi \) be an exact symplectomorphism of a Liouville domain \( (W, \lambda) \), let \( K_t : \widehat{W} \to \mathbb{R} \) be a Hamiltonian that
is linear near infinity, satisfies condition (2.2.5), and such that
\( \psi^K_1 : W \to W \) is an exact symplectomorphism. Let \( \theta \in \pi_0 \Omega_\phi \),
and let \( (H, J) \) be regular Floer data for \( \phi \). Then, the Hamiltonian
\( (\psi^K)^* (H - K) \) and the almost complex structure \( (\psi^K)^* J \) are
regular Floer data for \( \phi \circ \psi^K_1 \). Moreover, the homomorphism
\[ N(K) : CF(\phi, H, J, \theta) \to CF \left( \phi \circ \psi^K_1, \tilde{H}, \tilde{J}, \tilde{\theta} \right) \]
defined on generators by
\[ \gamma \mapsto (\psi^K)^* \gamma \] (3.3.23)
is an isomorphism of chain complexes. Here,
\[ \tilde{H} := (\psi^K)^* (H - K), \]
\[ \tilde{J} := (\psi^K)^* J, \]
\[ \tilde{\theta} := (\psi^K)^* \theta. \]

Note that this isomorphism does not preserve the grading. How-
ever, the relative index is preserved.

**Proof.** Since \( K \) is twisted by \( \phi \), the Hamiltonian isotopy \( \psi^K_1 \)
satisfies \( \psi^K_t \circ \phi \circ \psi^K_1 = \phi \circ \psi^K_{t+1} \). This further implies that
\( \tilde{H} \) and \( \tilde{J} \) are twisted by \( \phi \circ \psi^K_1 \). Near infinity, \( \psi^K_t \) is equal
to \( (r, x) \mapsto (r, \sigma_{at}(x)) \), where \( a \) is some real number, and
\( \sigma_t \) stands for the Reeb flow on the boundary. Hence, \( \psi^K_t \)
preserves \( r \) and \( \lambda \). Therefore, the conditions on \( \tilde{H} \) and \( \tilde{J} \)
early infinity are met. It is easy to check that \( \gamma \) is an element
of \( \mathcal{P}_{\phi, H} \) if, and only if, \( (\psi^K)^* \gamma \) is an element of \( \mathcal{P}_{\phi \circ \psi^K_1, \tilde{H}} \).
Moreover, \( u \) is a solution of the Floer equation (2.3.11) if, and
only if, \( (s, t) \mapsto (\psi^K_t)^{-1} \circ u(s, t) \) is a solution of the
Floer equation
\[ \partial u + \tilde{J}_t(v) \left( \partial_t v - X_{\tilde{H}}(v) \right) = 0. \]
Now, we show that both complexes \( \text{CF}(\phi, H, J, \theta) \) and \( \text{CF}(\phi \circ \psi^H_1, \tilde{H}, \tilde{J}, \tilde{\theta}) \) are graded by the same group, i.e. we show \( N_{\theta} = N_{\tilde{\theta}} \). The map

\[
\widehat{W}_{\phi \circ \psi^H_1} \to \widehat{W}_\phi : [(x, t)] \mapsto [(\psi^K_t(x), t)]
\]

induces a symplectic vector bundle isomorphism \( T^V \widehat{W}_{\phi \circ \psi^H_1} \to T^V \widehat{W}_\phi \). It also induces a bijection between \( \pi_1(\Omega_{\phi \circ \psi^H_1}, \tilde{\theta}) \) and \( \pi_1(\Omega_{\phi}, \theta) \). Hence, the groups

\[
\left\langle c_1(T^V \widehat{W}_{\phi \circ \psi^H_1}), \pi_1(\Omega_{\phi \circ \psi^H_1}, \tilde{\theta}) \right\rangle = \mathbb{Z} N_{\tilde{\theta}}
\]

and

\[
\left\langle c_1(T^V \widehat{W}_\phi), \pi_1(\Omega_{\phi}, \theta) \right\rangle = \mathbb{Z} N_{\theta}
\]

are identical.

Finally, let \( \gamma_0, \gamma_1 \in \mathcal{P}^\theta_{\phi, H} \) and let \( u : \mathbb{R} \times \mathbb{R} \to \widehat{W} \) be a homotopy between them as in Definition 2.4.1. Then,

\[
(\psi^K)^* u : (s, t) \mapsto (\psi^K_t)^{-1} \circ u(s, t)
\]

is a homotopy between \( (\psi^K)^* \gamma_0 \) and \( (\psi^H)^* \gamma_1 \in \mathcal{P}^{\tilde{\theta}}_{\phi, H} \), and

\[
\mu(u) = \mu \left( (\psi^K)^* u \right).
\]

Therefore \( \mu(\gamma_0, \gamma_1) = \mu \left( (\psi^K)^* \gamma_0, (\psi^H)^* \gamma_1 \right) \), and the proof is finished.

**Remark 3.3.2.** The isomorphism \( \mathcal{N}(K) \) from Lemma 3.3.1 induces an isomorphism on the homology level. That isomorphism will be denoted the same, i.e. by

\[
\mathcal{N}(K) : \text{HF}(\phi, H, J, \theta) \to \text{HF} \left( \phi \circ \psi^K_1, \tilde{H}, \tilde{J}, \tilde{\theta} \right).
\]
**Lemma 3.3.3.** Let $\phi$ be an exact symplectomorphism of a Liouville domain $(W, \lambda)$, let $(H^\alpha, J^\alpha), (H^\beta, J^\beta)$ be regular Floer data for $\phi$ with the slope of $H^\beta$ greater than or equal to the slope of $H^\alpha$, let $\theta \in \pi_0 \Omega_\phi$, and let $K$ be a Hamiltonian satisfying conditions (2.2.5) and (2.2.7) such that $\psi_1^K : W \to W$ is an exact symplectomorphism. Then the following diagram commutes

$$
\begin{array}{ccc}
\text{HF}(\phi, H^\alpha, J^\alpha, \theta) & \xrightarrow{\Phi_{\beta \alpha}^K} & \text{HF}(\phi, H^\beta, J^\beta, \theta) \\
\downarrow N(K) & & \downarrow N(K) \\
\text{HF}(\phi \circ \psi_1^K, \cdots) & \xrightarrow{\Phi_{\beta \alpha}^K} & \text{HF}(\phi \circ \psi_1^K, \cdots).
\end{array}
$$

**Proof.** The proof is analogous to the proof of Lemma 3.3.1. The key ingredient is the following fact. The correspondence $u \leftrightarrow (\psi^K)_* u$ is a bijection between the sets of solutions of the appropriate $s$-dependent Floer equations. Here, $(\psi^K)_* u(s, t) := (\psi^K_t)^{-1} \circ u(s, t).$ \hfill $\square$

**Proposition 3.3.4.** Let $\phi$ be an exact symplectomorphism of a Liouville domain $W$, let $a \in \mathbb{R} \cup \{\infty\}$ and let $K_t : \hat{W} \to \mathbb{R}$ be a Hamiltonian with the slope equal to $k \in \mathbb{R}$ such that $K_t \circ \phi = K_{t+1}$, and such that the symplectomorphism $\psi^K_1 : W \to W$ is exact. Then, $N(K)$ induces an isomorphism (denoted the same)

$$
N(K) : \text{HF}(\phi, a) \to \text{HF}(\phi \circ \psi^K_1, a - k). \tag{3.3.24}
$$

**Proof.** The proposition follows from Lemma 3.3.3. \hfill $\square$

Let $H_t$ and $K_t$ be Hamiltonians $\hat{W} \to \mathbb{R}$ whose Hamiltonian isotopies are well defined (for all times). We denote the Hamiltonians $-H_t \circ \psi^H_t$ and $H_t + K_t \circ (\psi^H_t)^{-1}$ by $\overline{H}$ and $H \# K$, respectively. Note that

$$
\psi^H_t = (\psi^H_t)^{-1} \quad \text{and} \quad \psi^{H \# K}_t = \psi^H_t \circ \psi^K_t.
$$
Remark 3.3.5. Let \( \phi \) be an exact symplectomorphism of a Liouville domain \((W, \lambda)\), let \( H_t \) and \( K_t \) be Hamiltonians linear near infinity. Assume that \((\phi, H)\) and \((\phi \circ \psi^H_t, K)\) satisfy condition (2.2.5) and that the symplectomorphisms \( \psi^H_t, \psi^K_t : W \to W \) are exact. Then, \((\phi \circ \psi^H_t, H)\) and \((\phi, H \# K)\) satisfy condition (2.2.5) as well, \( \overline{H} \) and \( H \# K \) are linear near infinity, and their time-1 maps are exact symplectomorphisms. Moreover,

\[
N(K) \circ N(H) = N(H \# K), \quad N(H)^{-1} = N(\overline{H}). \tag{3.3.25}
\]

3.4 
**Invariance under isotopies**

The goal of this section is to associate an isomorphism

\[
I([\phi_t]) : HF(\phi_0, a) \to HF(\phi_1, a)
\]

to a compactly supported isotopy (through \( \text{Symp}_c(W, \lambda/d) \)) \( \{\phi_t\}_{t \in [0,1]} \) between \( \phi_0 \) and \( \phi_1 \in \text{Symp}_c(W, \lambda/d) \). Without loss of generality, we may assume \( \phi_t = \phi_0 \) for \( t \) near 0 and \( \phi_t = \phi_1 \) for \( t \) near 1. By Lemma 3.4.2 below, \( \phi_t = \psi^K_t \circ \phi_0 \) for some compactly supported Hamiltonian \( K : [0,1] \times \hat{W} \to \mathbb{R} \) (note that \( K_t = 0 \) for \( t \) near 0 or 1).

**Definition 3.4.1.** Let \( \{\phi_t\} \) and \( K \) be as above. Consider the Hamiltonian \( K^0 : \mathbb{R} \times \hat{W} \to \mathbb{R} \) that satisfies \( K^0_t \circ \phi_0 = K^0_{t+1} \) for all \( t \in \mathbb{R} \) and \( K^0_t = K_t \circ \phi_0 \) for \( t \in [0,1] \). This Hamiltonian generates the isotopy \( \phi_0^{-1} \circ \phi_t \). We define

\[
I([\phi_t]) := N(K^0) : HF(\phi_0, a) \to HF(\phi_1, a).
\]

**Lemma 3.4.2.** Let \((W, \lambda)\) be a Liouville domain. Any path \( \phi_t \in \text{Symp}_c(W, \lambda/d) \) is a Hamiltonian isotopy generated by a Hamiltonian compactly supported in \( W - \partial W \).
Proof. Let $X_t$ be a vector field of $\phi_t$. For simplicity, we denote $F_{\phi_t}$ by $F_t$. By applying Cartan’s formula on

$$\phi_t^*\lambda - \lambda = dF_t,$$

we get

$$\phi_t^*(X_t \downarrow d\lambda + d(\lambda(X_t))) = d(\partial_tF_t).$$

Hence,

$$X_t \downarrow \omega = d\left(\partial_tF_t \circ \phi_t^{-1} - \lambda(X_t)\right),$$

and the proof is finished. \qed

**Theorem 3.4.3.** Let $\phi_0, \phi_1$ be exact symplectomorphisms of a Liouville domain $(W, \lambda)$, that are in the same connected component of $\text{Symp}_c(W, \lambda/d)$. Then, for each compactly supported isotopy through $\text{Symp}_c(W, \lambda/d)$ between $\phi_0$ and $\phi_1$, and for every admissible $a \in \mathbb{R} \cup \{\infty\}$, there exists an isomorphism

$$\text{HF}(\phi_0, a) \to \text{HF}(\phi_1, a). \quad (3.4.26)$$

The isomorphism depends only on the homotopy class of the isotopies between $\phi_0$ and $\phi_1$.

**Proof.** The existence of the isomorphism follows from the discussion above (the proof is the same as in [13] (Theorem 3.3)). Lemma 3.4.4 below proves that the isomorphism depends only on the homotopy class of the isotopies between $\phi_0$ and $\phi_1$. \qed

**Lemma 3.4.4.** Let $\phi$ be an exact symplectomorphism of a Liouville domain $(W, \lambda)$, let $\theta \in \pi_0\Omega_{\phi}$, let $(H, J)$ be admissible Floer data for $\phi$, and let $K_t: \hat{W} \to \mathbb{R}$ be a compactly supported Hamiltonian such that $K_t \circ \phi = K_{t+1}$, and such that it generates a
loop of Hamiltonian diffeomorphisms that is contractible in the group of all Hamiltonian diffeomorphisms compactly supported in \( W - \partial W \). Then, the naturality isomorphism

\[
N(K) : \text{HF}(\phi, H, J, \theta) \to \text{HF} \left( \phi, \tilde{H}, \tilde{J}, \tilde{\theta} \right)
\]

coincides with the continuation morphism (which is well defined since \( H \) and \((\psi^K)^*(H - K)\) have the same slope). Here,

\[
\tilde{H} := (\psi^K)^*(H - K),
\]

\[
\tilde{J} := (\psi^K)^* J,
\]

\[
\tilde{\theta} := (\psi^K)^* \theta.
\]

Proof. By the assumptions, there exists a family

\[\psi_{s, t} : \hat{W} \to \hat{W}, \quad s \in \mathbb{R}, \ t \in [0, 1],\]

of Hamiltonian diffeomorphisms compactly supported in \( W - \partial W \) such that \( \psi_{s, 0} = \psi_{s, 1} = \text{id} \), for all \( s \in \mathbb{R} \), and such that

\[
\psi_{s, t} = \begin{cases} 
\psi^K_t & \text{for } s >> 0, \\
\text{id} & \text{for } s \ll 0.
\end{cases}
\]

We extend \( \psi_{s, t} \) to an \( \mathbb{R} \times \mathbb{R} \)-family by requiring that it satisfies the following periodicity condition

\[
\psi_{s, t} \circ \phi = \phi \circ \psi_{s, t+1}.
\]

Note that \( \psi_{s, t} = \psi^K_t \) holds for \( s >> 0 \) after the extension as well. Let \( \gamma^\alpha, \gamma^\beta \in \mathcal{P}^\theta(\phi, H) = \mathcal{P}(\phi, H) \cap \theta \). The map \( u \mapsto \psi^* u \), where \( \psi^* u : (s, t) \mapsto \psi^{-1}_{s, t} \circ u(s, t) \), defines a bijection between the sets

\[
\mathcal{M}^0(\phi, \{H, 0, J\}, \gamma^\alpha, \gamma^\beta)
\]
and
\[ M^0(\phi, D, \gamma^\alpha, (\psi^K)^* \gamma^\beta), \]
where
\[ D = \{(\psi_s,)^*(H - K_s), -(\psi_s,)^*K_s, (\psi_s,)^*J\} \]
(for the definition of $\psi^*_s(\cdots)$ see the beginning of section 3.3). The former is empty unless $\gamma^\alpha = \gamma^\beta$, and in that case, it consists of a single element, namely $(s, t) \mapsto \gamma^\alpha(t)$ (the reason for this is that $(H, 0, J)$ is independent of $s$). Hence
\[
n(\phi, D, \gamma^\alpha, (\psi^K)^* \gamma^\beta) = \begin{cases} 1 & \text{if } \gamma^\alpha = \gamma^\beta, \\ 0 & \text{otherwise}. \end{cases}
\]
In other words, the continuation map associated to the continuation data
\[
\{(\psi_s,)^*(H - K_s), -(\psi_s,)^*K_s, (\psi_s,)^*J\}
\]
sends the generator $\gamma$ of $\text{CF}(\phi, H, J, \theta)$ to the generator $(\psi^K)^* \gamma$ of
\[
\text{CF} \left( \phi, (\psi^K)^*(H - K), (\psi^K)^*J, (\psi^K)^*\theta \right),
\]
i.e. it coincides with the naturality isomorphism. This proves the lemma.  \[ \square \]
Throughout this chapter, \((W_2, \lambda)\) is a 2\(n\)-dimensional Liouville domain, \(W_1 \subset W_2\) is a codimension-0 submanifold of \(W_2\) such that \((W_1, \lambda)\) is a Liouville domain in its own right, and \(\phi : W_1 \to W_1\) is an exact symplectomorphism.

Given a real number \(c \in \mathbb{R}\) and regular Floer data \((H, J)\) for \(\phi\) (seen as an exact symplectomorphism of \(W_2\)), one can consider the filtered chain complex

\[
\text{CF}^{<c}_* (W_2, \phi, H, J).
\]

It is defined as a subcomplex of \(\text{CF}_*(W_2, \phi, H, J)\) generated by those elements of \(\text{Crit} \mathcal{A}_{\phi, H}\) which have action less than \(c\). The corresponding homology is denoted by

\[
\text{HF}^{<c}_* (W_2, \phi, H, J).
\]

The chain complex \(\text{CF}^{<c}_* (W_2, \phi, H, J)\) fits into the short exact sequence of chain complexes

\[
0 \to \text{CF}^{<c}_* (\cdots) \to \text{CF}_* (\cdots) \to \frac{\text{CF}_* (\cdots)}{\text{CF}^{<c}_* (\cdots)} \to 0,
\]

which further induces the long exact sequence in homology. We denote by

\[
\text{HF}^{\geq c}_* (W_2, \phi, H, J)
\]

the homology of the chain complex

\[
\text{CF}^{\geq c}_* (W_2, \phi, H, J) := \frac{\text{CF}_* (W_2, \phi, H, J)}{\text{CF}^{<c}_* (W_2, \phi, H, J)}.
\]
The transfer morphism is essentially the map
\[ \text{HF}_*(W_2, \phi, H, J) \to \text{HF}^\geq_*(W_2, \phi, H, J) \]
induced by the natural projection of chain complexes
\[ \text{CF}_*(W_2, \phi, H, J) \to \text{CF}^\geq_*(W_2, \phi, H, J) \]
for a Hamiltonian \( H \) that is \( C^2 \)-close to a so-called stair-like Hamiltonian. The group \( \text{HF}_*(W_2, \phi, H, J) \) is isomorphic to \( \text{HF}_*(W_2, \phi, a) \), where \( a \) is the slope of \( H \), and \( \text{HF}^\geq_*(W_2, \phi, H, J) \) can be identified with \( \text{HF}_*(W_1, \phi, b) \) for a certain slope \( b \). In the rest of the chapter, we describe the construction in more details and prove Theorem 1.1.1.

4.1 STAIR-LIKE HAMILTONIANS

**Definition 4.1.1.** Let \( 0 < a < b \) be such that \( a \) is admissible with respect to \( W_2 \) and \( b \) is admissible with respect to \( W_1 \), and let \( b_0 \) be the greatest period of some Reeb orbit on \( \partial W_1 \) that is smaller than \( b \). The set \( \mathcal{H}_{\text{stair}}(\phi, W_1, W_2, a, b) \) is defined to be the set of (time-independent) Hamiltonians \( H : \tilde{W}_2 \to \mathbb{R} \) having the following property. There exist positive real numbers \( \delta_1, \delta_2, \delta_3, A, B, C \in \mathbb{R}^+ \) and functions

\[
\begin{align*}
    h_1 : & [\delta_1, 2\delta_1] \to \mathbb{R}, \\
    h_2 : & [1 - \delta_2, 1] \to \mathbb{R}, \\
    h_3 : & [1 + \delta_3, 1 + 2\delta_3] \to \mathbb{R},
\end{align*}
\]

such that

1. \( \phi \) is compactly supported in the interior of \( W^\delta_1 \),
2. \( \sup_{p \in W_1} |F_\phi(p)| < A \),
3. \( h_1 \) and \( h_3 \) are convex and strictly increasing,

4. \( h_2 \) is concave, strictly increasing, and \( h_2(r) > rb_0 \) for all \( r \in [1 - \delta_2, 1] \),

and such that

\[
\begin{array}{l|l}
H(p) & \text{for} \\
\hline
-A & p \in W_1^\delta, \\
h_1(r) & p = (x, r) \in [\delta_1, 2\delta_1) \times \partial W_1, \\
b(r - 2\delta_1) & p = (x, r) \in [2\delta_1, 1 - \delta_2) \times \partial W_1, \\
h_2(r) & p = (x, r) \in [1 - \delta_2, 1] \times \partial W_1, \\
B & p \in W_2^{1+\delta_3} \setminus W_1, \\
h_3(r) & p = (x, r) \in (1 + \delta_3, 1 + 2\delta_3) \times \partial W_2, \\
ar + C & p = (x, r) \in [1 + 2\delta_3, \infty) \times \partial W_2.
\end{array}
\]

A typical element of \( \mathcal{H}_{\text{stair}}(\phi, W_1, W_2, a, b) \) is shown in Figure 1.

**Definition 4.1.2.** Let \( W \) be a Liouville domain, and let \( \phi : \hat{W} \to \hat{W} \) be a diffeomorphism. The **support radius** \( \rho(W, \phi) \) of \( \phi \) is defined by

\[
\rho(W, \phi) := \inf \{ r \in \mathbb{R}^+ : \text{supp } \phi \subset W^r \}.
\]

**Lemma 4.1.3.** Let \( a, b, \) and \( b_0 \) be as in Definition 4.1.1. If

\[
C(W_1, \phi) < \min \{ b - b_0, b - a \},
\]

where

\[
C(W_1, \phi) := 2 \max \left\{ \sup_{p \in W_1} |F_{\phi}(p)|, \rho(\phi, W_1, \lambda) \right\}, \tag{4.1.27}
\]

then the set \( \mathcal{H}_{\text{stair}}(\phi, W_1, W_2, a, b) \) is not empty.
Figure 1: A stair-like Hamiltonian
Proof. The constants $\delta_1, \delta_2, \delta_2, A, B, C$ from Definition 4.1.1 are chosen in the following way and the following order.

- Choose
  \[ A \in \left( \frac{C(W_1, \phi)}{2}, \frac{\min\{b - b_0, b - a\}}{2} \right), \]

- choose $\delta_1 \in \left( \frac{A}{b}, \frac{\min\{b - b_0, b - a\}}{2b} \right),$

- choose $\delta_2 \in \left( 0, 1 - \frac{2b\delta_1}{b - b_0} \right),$

- choose $B \in \left( \max\{a, b_0, b - 2b\delta_1 - b\delta_2\}, b - 2b\delta_1 \right),$

- choose $\delta_3 \in \left( 0, \frac{B - a}{a} \right),$

- choose $C \in \left( \max\{0, B - a(1 + 2\delta_3)\}, B - a(1 + \delta_3) \right).$

It follows that the following inequalities hold

\[ 0 < -A - (2b\delta_1) - b\delta_1 < b\delta_1, \quad (4.1.28) \]
\[ 0 < B - (-2b\delta_1) - b(1 - \delta_2) < b\delta_2, \quad (4.1.29) \]
\[ b(1 - \delta_2) + (-2b\delta_1) > (1 - \delta_2)b_0, \quad (4.1.30) \]
\[ B > b_0, \quad (4.1.31) \]
\[ 0 < B - C - a(1 + \delta_3) < a\delta_3. \quad (4.1.32) \]

Due to Lemma A.0.1 and Lemma A.0.2, (4.1.28) implies that $h_1$ exists, (4.1.29), (4.1.30), (4.1.31) imply that $h_2$ exists, and (4.1.32) implies that $h_3$ exists. This finishes the proof. \( \square \)

Lemma 4.1.4. Let $\phi, a, b, H, \delta_1, \delta_2, \delta_3, A, B, C$ be as in Definition 4.1.1. The ranges of the action functional $A_{\phi, H}$ when evaluated on $\phi$-twisted Hamiltonian orbits contained in different regions in $\hat{W}_2$ are given in the following table.
<table>
<thead>
<tr>
<th>region</th>
<th>( \mathcal{A}_{\phi,H} \in )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_1^{\delta_1} )</td>
<td>([A - |F_\phi|, A + |F_\phi|])</td>
</tr>
<tr>
<td>((\delta_1, 2\delta_1) \times \partial W_1)</td>
<td>((A, 2b\delta_1))</td>
</tr>
<tr>
<td>((1 - \delta_2, 1) \times \partial W_1)</td>
<td>((-B, 0))</td>
</tr>
<tr>
<td>(W_2^{\delta_3} \setminus \text{int } W_1)</td>
<td>{(-B)}</td>
</tr>
<tr>
<td>((1 + \delta_3, 1 + 2\delta_3) \times \partial W_2)</td>
<td>((-B, -C))</td>
</tr>
</tbody>
</table>

Note that each \( \phi \)-twisted orbit of \( H \) is contained in one of these regions.

**Proof.** We will prove the statement only for the region

\((1 - \delta_2, 1) \times \partial W_1\).

For the other regions, the proof is similar and even more direct.

The symplectomorphism \( \phi \) is equal to the identity in \((1 - \delta_2, 1) \times \partial W_1\). Therefore, the twisted Hamiltonian orbits coincide with Hamiltonian loops. Additionally, due to the form of the Hamiltonian \( H \) in this region, they can be explicitly described in terms of periodic Reeb orbits on \( \partial W_1 \).

Each 1-periodic Hamiltonian orbit in \((1 - \delta_2, 1) \times \partial W_1\) is given by

\[
t \mapsto (r_0, \gamma(-h'_2(r_0)t))
\]

where \( \gamma : \mathbb{R} \to \partial W_1 \) is a \( h'_2(r_0) \)-periodic Reeb orbit. The action of (4.1.33) is equal to

\[
r_0h'_2(r_0) - h_2(r_0).
\]

Consider the function

\[
f : [1 - \delta_2, 1] \to \mathbb{R} : r \mapsto rh'_2(r) - h_2(r).
\]
Since $f'(r) = r h''_2(r) \leq 0$, $f$ is decreasing. Hence the action of a 1-periodic Hamiltonian orbit in $(1 - \delta_2, 1) \times \partial W_1$ lies in the interval $[f(1), f(r_1)] = (-B, f(r_1))$, where $r_1 \in (1 - \delta_2, 1)$ is the smallest number such that $h'_2(r_1)$ is a period of some Reeb orbit on $\partial W_1$. Since $h'_2$ is continuous and decreasing ($h_2$ is concave), we get $h'_2(r_1) = b_0$. Therefore,

\[ f(r_1) = r_1 b_0 - h_2(r_1) < 0 \]

(because, by definition, $h_2(r) > rb_0$ for all $r \in [1 - \delta_2, 1]$). This finishes the proof. \[ \square \]

**Proposition 4.1.5.** Let $\phi : W_1 \to W_1$ be an exact symplectomorphism and let $\phi_t : W_1 \to W_1$ be the isotopy of exact symplectomorphisms given by

\[ \phi_t := (\psi^\lambda_t)^{-1} \circ \phi \circ \psi^\lambda_t, \]

where $\psi^\lambda_t : \widetilde{W}_1 \to \widetilde{W}_1$ is the Liouville flow. Then,

\[ F_{\phi_t} = e^{-t} F_\phi \circ \psi^\lambda_t \]

and

\[ \rho(\phi_t, W_1, \lambda) = e^{-t} \rho(\phi, W_1, \lambda). \]

Consequently, $C(W_1, \phi_t) = e^{-t} C(W_1, \phi)$.

**Proof.** Denote $\psi^\lambda_t$ by $\psi_t$ for simplicity. The Cartan formula implies $\psi_t^* \lambda = e^t \lambda$ and $(\psi_t^{-1})^* \lambda = e^{-t} \lambda$. Therefore

\[ dF_{\phi_t} = \phi_t^* \lambda - \lambda \]
\[ = (\psi_t^{-1} \circ \phi \circ \psi_t)^* \lambda - \lambda \]
\[ = \psi_t^* \phi^* (\psi_t^{-1})^* \lambda - \lambda \]
\[ = e^{-t} \psi_t^* (\phi^* \lambda - \lambda) \]
\[ = e^{-t} d(F_\phi \circ \psi_t). \]
Hence $F_{\phi_t} = e^{-t}F_{\phi} \circ \psi_t$. Since
\[ \text{supp } \phi_t \subset W^r \iff \text{supp } \phi \subset W^{e^t r}, \]
the equality $\rho(W_1, \phi_t) = e^{-t} \rho(W_1, \phi)$ holds.

\section{Construction of the Transfer Morphism}

\textbf{Definition 4.2.1.} Let $a$ and $b$ be as in Definition 4.1.1.\textbf{Transfer data} for $(\phi, W_1, W_2, a, b)$ is Floer data $(H, J)$ for $(\phi : W_2 \to W_2, b)$ satisfying the following. There exists a Hamiltonian $\overline{H} \in \mathcal{H}_{\text{stair}}(\phi, W_1, W_2, a, b)$ such that $H = \overline{H}$ in $[2\delta_1, 1 - \delta_2] \times \partial W_1$ and $[1 + 2\delta_3, \infty) \times \partial W_2$, and such that the action functional is positive at twisted Hamiltonian orbits in $W_1^{2\delta_1}$ and negative at the other twisted orbits (this holds for $H$ that is $C^2$-close to $\overline{H}$). Additionally,
\[ dr \circ J_t = -\lambda \]
in $[2\delta_1, 1 - \delta_2] \times \partial W_1$.

\textbf{Lemma 4.2.2.} Let $(H, J)$ be a regular transfer data for $(\phi, W_1, W_2, a, b)$. By definition, there exist $\delta_1, \delta_2 \in (0, 1)$ such that $2\delta_1 + \delta_2 < 1$ and $H_t(x, r) = b(r - 2\delta_1)$ for all $(r, x) \in [2\delta_1, 1 - \delta_2] \times \partial W_1$. Let $G_t : \hat{W}_1 \to \mathbb{R}$ be a Hamiltonian defined by
\[ G_t(p) := \begin{cases} H_t(p) & \text{for } p \in W_1^{1-\delta_2} \\ b(r - 2\delta_1) & \text{for } p \in [2\delta_1, \infty) \times \partial W_1. \end{cases} \]

Then,
\[ \text{HF}_*(W_1, \phi, G, J) = \text{HF}^{\geq 0}_*(W_2, \phi, H, J). \]
Proof. The generators of the chain complexes $\text{CF}_*(W_1, \phi, G, J)$ and $\text{CF}^{\geq 0}_*(W_2, \phi, H, J)$ coincide. Therefore it is enough to prove the following. The solutions

$$u : \mathbb{R}^2 \to \widehat{W}_2, \quad \phi(u(s, t + 1)) = u(s, t)$$

of the Floer equation which connect two Hamiltonian twisted orbits in $W_1$ satisfy $u(\mathbb{R}^2) \subset W_1$. This follows from the part a) of Theorem 4.5 in [16]. See also [16, Proposition 4.4].

We are going to define the transfer morphism in couple of steps. Definition 4.2.3 defines it for positive slopes with additional technical conditions. Definition 4.2.4 and Definition 4.2.7 eliminate the technical conditions. And, finally, Proposition 4.2.6 extends the transfer morphism to the case of infinite slopes.

**Definition 4.2.3.** Let $a, b \in \mathbb{R}^+$ be positive real numbers such that $a$ and $b$ are admissible with respect to $W_2$ and $W_1$, respectively, and such that $a < b$. Let $b_0$ be the greatest period of some Reeb orbit on $\partial W_1$ that is smaller than $b$. Assume

$$C(W_1, \phi) < \min\{b - b_0, b - a\}.$$

The **transfer morphism** is defined to be the map

$$\text{HF}_*(W_2, \phi, a) \to \text{HF}_*(W_1, \phi, b),$$

induced by the natural map of chain complexes

$$\text{CF}_*(W_2, \phi, H, J) \to \text{CF}^{\geq 0}_*(W_2, \phi, H, J),$$

where $(H, J)$ is regular Transfer data for $(\phi, W_1, W_2, a, b)$. 

47
**Definition 4.2.4.** Let $a \in \mathbb{R}^+$ be admissible with respect to $W_2$, and let $b \in \mathbb{R}^+$ be admissible with respect to $W_1$. Assume $a < b$. The transfer morphism is defined as the composition

$$HF_*(W_2, \phi_1, a) \longrightarrow HF_*(W_1, \phi_1, b)$$

Here,

$$\phi_t := (\psi_{ct}^\lambda)^{-1} \circ \phi \circ \psi_{ct}^\lambda, \quad t \in [0, 1],$$

for $c \in \mathbb{R}^+$ such that

$$C(W_1, \phi_1) < \min\{b - b_0, b - a\},$$

and $T$ is the transfer morphism of Definition 4.2.3.

**Lemma 4.2.5.** In the situation of Definition 4.2.4, the transfer morphism does not depend on the choice of $c$.

**Proof.** Let $c_1$ and $c_2$ be positive real numbers such that

$$C(W_1, \phi_i) < \min\{b - b_0, b - a\}, \quad i \in \{1, 2\},$$

where

$$\phi_i := (\psi_{c_i}^\lambda)^{-1} \circ \phi \circ \psi_{c_i}^\lambda, \quad i \in \{1, 2\}.$$

It is enough to prove that the diagram

$$\begin{array}{ccc}
HF_*(W_2, \phi_1, a) & \longrightarrow & HF_*(W_1, \phi_1, b) \\
\downarrow & & \downarrow \\
HF_*(W_2, \phi, a) & \longrightarrow & HF_*(W_1, \phi, b) \\
\downarrow & & \uparrow \\
HF_*(W_2, \phi_2, a) & \longrightarrow & HF_*(W_1, \phi_2, b)
\end{array}$$
commutes, where the vertical arrows stand for the isomorphisms induced by symplectic isotopies. These isomorphisms, as it follows from their definition in Section 3.4, coincide with naturality isomorphisms with respect to some Hamiltonians that are compactly supported in the interior of \( W_1 \). Hence, it is enough to prove that the diagram

\[
\begin{array}{ccc}
\text{HF}(W_2, \phi_1, a) & \longrightarrow & \text{HF}(W_1, \phi_1, b) \\
\mathcal{N}(H) \downarrow & & \downarrow \mathcal{N}(K) \\
\text{HF}(W_2, \phi_2, a) & \longrightarrow & \text{HF}(W_1, \phi_2, b)
\end{array}
\]

commutes, where \( H, K : \mathcal{W}_2 \rightarrow \mathbb{R} \) are compactly supported in the interior of \( W_1 \). This follows from the following fact. The naturality with respect to a Hamiltonian \( \mathcal{W}_2 \rightarrow \mathbb{R} \) that is compactly supported in the interior of \( W_1 \) converts transfer data for \((\phi_1, W_1, W_2, a, b)\) into transfer data for \((\phi_2, W_1, W_2, a, b)\).

\[\square\]

**Proposition 4.2.6.** The diagram

\[
\begin{array}{ccc}
\text{HF}_*(W_2, \phi, a) & \longrightarrow & \text{HF}_*(W_1, \phi, b) \\
\downarrow & & \downarrow \\
\text{HF}_*(W_2, \phi, a') & \longrightarrow & \text{HF}_*(W_1, \phi, b')
\end{array}
\]

consisting of transfer morphisms and continuation maps, commutes whenever \( a, b, a', b' \in \mathbb{R}^+ \) are such that all the maps in the diagram are well defined.

**Proof of Theorem 1.1.1.** The proof follows from the proof of Proposition 4.7 in [16]. \[\square\]

**Definition 4.2.7.** Let \( a \in \mathbb{R}^+ \) be admissible with respect to both \( W_1 \) and \( W_2 \). Let \( \varepsilon > 0 \) be a positive number such that
the elements of \([a, a + \varepsilon]\) are all admissible with respect to \(W_1\). The transfer morphism

\[
HF_*(W_2, \phi, a) \to HF_*(W_1, \phi, a)
\]

is defined to be the composition of the transfer morphism

\[
HF_*(W_2, \phi, a) \to HF_*(W_1, \phi, a + \varepsilon)
\]

of Definition 4.2.4 and the inverse of the continuation map

\[
\Phi^{-1} : HF_*(W_1, \phi, a + \varepsilon) \to HF_*(W_1, \phi, a).
\]

**Lemma 4.2.8.** The transfer morphism of Definition 4.2.7 is well defined and does not depend on the choice of \(\varepsilon\).

**Proof.** Lemma A.0.3 implies that \(\varepsilon\) from Definition 4.2.7 exists. Due to Proposition 3.2.1, the continuation map

\[
\Phi : HF_*(W_1, \phi, a) \to HF_*(W_1, \phi, a + \varepsilon)
\]

is an isomorphism. Hence, its inverse \(\Phi^{-1}\) is well defined. Independence of the choice of \(\varepsilon\) follows from Proposition 4.2.6.

Proposition 4.2.6 enables us to extend the transfer morphism to the case of infinite slopes. The proposition is still true after the extensions of the transfer morphism.
5.1 THE SYMPLECTIC MAPPING CLASS GROUP

Let \((W, \lambda)\) be a Liouville domain. The group of all symplectomorphisms \(W \to W\) that are compactly supported in the interior of \(W\) is denoted by \(\text{Symp}_c(W, d\lambda)\).

**Definition 5.1.1.** The *symplectic mapping class group* of a Liouville domain \((W, \lambda)\) is defined to be the group \(\pi_0 \text{Symp}_c(W, d\lambda)\). In other words, it is the group of symplectomorphisms up to symplectic isotopies. (Both the symplectomorphisms and the isotopies are assumed to be compactly supported in the interior of \(W\).) The elements of the symplectic mapping class group are called *symplectic mapping classes*.

The Floer homology is not defined for an arbitrary symplectomorphism of a Liouville domain. Nevertheless, the symplectic mapping class group is investigable by the Floer homology for exact symplectomorphisms. This is due to the following lemma.

**Lemma 5.1.2** (Giroux). *For any Liouville domain \((W, \lambda)\), the inclusion*

\[ j : \text{Symp}_c(W, \lambda/d) \to \text{Symp}_c(W, d\lambda) \]

*is a homotopy equivalence.*
Proof (following [6]). Denote the symplectic form $d\lambda$ by $\omega$. We construct a homotopy inverse of $j$ as follows. Let $Y^\phi$, $\phi \in \text{Symp}_c(W, \omega)$ be a vector field on $W$ defined by

$$\lambda - \phi^*\lambda = \omega (Y^\phi, \cdot) .$$

Since $Y^\phi$ is compactly supported in $W - \partial W$, its flow $\psi^\phi_t$ is well defined. We denote by $T$ the map

$$\text{Symp}_c(W, \omega) \rightarrow \text{Symp}_c(W, \lambda/d) : \phi \mapsto \phi \circ \psi^\phi_1 .$$

First, we prove that $T$ is well defined, i.e. that

$$T(\phi) \in \text{Symp}_c(W, \lambda/d) .$$

Note that $\psi^\phi_t =: \psi_t$ is a symplectomorphism for all real $t$, and that it preserves the form $\lambda - \phi^*\lambda$. Using Cartan’s formula, we get

$$\frac{d}{dt} (\psi^*_t \lambda) = \psi^*_t (Y^\phi \cdot d\lambda + dY^\phi \cdot \lambda)$$

$$= \psi^*_t (\lambda - \phi^*\lambda + d (\lambda (Y^\phi)))$$

$$= \lambda - \phi^*\lambda + d (\lambda (Y^\phi) \circ \psi_t) .$$

Hence

$$\psi^*_t \lambda - \lambda = (\lambda - \phi^*\lambda) t + d \left( \int_0^t \lambda (Y^\phi) \circ \psi_s ds \right) . \quad (5.1.34)$$

This implies that

$$(\phi \circ \psi_1)^*\lambda - \lambda = \psi^*_1 \phi^*\lambda - \psi^*_1 \lambda + \psi^*_1 \lambda - \lambda$$

$$= \psi^*_1 (\phi^*\lambda - \lambda) + \psi^*_1 \lambda - \lambda \quad \quad (5.1.35)$$

is an exact form.
Now, we prove that $j \circ T$ and $T \circ j$ are homotopic to the identity. The homotopies are given by

$$[0, 1] \times \text{Symp}_c(W, \omega) \to \text{Symp}_c(W, \omega)$$

$$(t, \phi) \mapsto \phi \circ \psi_t^\phi,$$

and

$$[0, 1] \times \text{Symp}_c(W, \lambda/d) \to \text{Symp}_c(W, \lambda/d)$$

$$(t, \phi) \mapsto \phi \circ \psi_t^\phi.$$ 

In what follows, we check that the latter is well defined. Similarly as in (5.1.35), we get

$$(\phi \circ \psi_t^\phi)^* \lambda - \lambda = \left( (\psi_t^\phi)^* \lambda - \lambda \right) + (\phi^* \lambda - \lambda).$$

The both forms in the brackets (on the right-hand side of the equation above) are exact: the latter because of $\phi \in \text{Symp}_c(W, \lambda/d)$, and the former because of (5.1.34) in addition. \qed

### 5.2 The fibered Dehn twist

This section is devoted to a construction of an interesting class of exact symplectomorphisms. These symplectomorphisms are defined on Liouville domains having periodic Reeb flow on their boundaries.

**Definition 5.2.1.** [6, 24]. Let $(W, \lambda)$ be a Liouville domain with 1-periodic Reeb flow on the boundary. Let $v : \mathbb{R} \to \mathbb{R}$ be a smooth function which is equal to 0 on $(-\infty, 0)$ and to $-1$ on $(0.95, +\infty)$. The (right-handed) **fibred Dehn twist** is an element $\tau$ of $\text{Symp}_c(W, \lambda/d)$ defined by

$$\tau(p) = \begin{cases} (r, \sigma_{v(r)}(x)) & \text{for } p = (r, x) \in \mathbb{R}^+ \times \partial W, \\ p & \text{otherwise,} \end{cases}$$

53
where $\sigma_t : \partial W \to \partial W$ stands for the Reeb flow. The isotopy class of $\tau$ does not depend on the choice of $v$.

**Remark 5.2.2.** The fibered Dehn twist of Definition 5.2.1, is a time-1 map of the Hamiltonian $K : \widehat{W} \to \mathbb{R}$

$$K(p) := \begin{cases} -V(r) & \text{for } p = (r, x) \in \mathbb{R}^+ \times \partial W, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $V : \mathbb{R} \to \mathbb{R}$ is the unique primitive of $v$ equal to 0 near 0. Note that

$$K(x, r) = r - \int_0^1 v(t) \, dt - 1$$

near infinity.

The disk-cotangent bundle $D^*S^n$ of the sphere $S^n$ with the standard Riemannian metric (rescaled by $2\pi$) is an example of a Liouville domain that satisfies the conditions from Definition 5.2.1. The fibered Dehn twist for $D^*S^n$ represents the same symplectic mapping class as the square of the so-called generalized Dehn twist $[4, 22]$, which was considered by Seidel in his thesis.

Given a Lagrangian sphere $L$ in a symplectic manifold $W$, one can use the fibered Dehn twist (or the generalized Dehn twist) on $D^*S^n$ to construct a symplectomorphism of $W$ that is supported in a neighbourhood of $L$. This is due to the Weinstein theorem, which guarantees the existence of a neighbourhood of $L$ in $W$ that is symplectomorphic to $D^*S^n$. Note, however, that the symplectomorphism between $D^*S^n$ and the neighbourhood is not unique. Sometimes, different identifications of the neighbourhood with $D^*S^n$ lead to symplectomorphisms of $W$ that are not symplectically isotopic to each other $[11$, **Theorem A**].
We describe below a long exact sequence due to Giroux [6] which provides further examples of symplectic mapping classes. The fibered Dehn twist can be seen as a special case of this construction (see Proposition 5.2.6).

Let \((W, \lambda)\) be a Liouville domain. We denote by \(\text{Symp}(\tilde{W}, \lambda/d)\) the group of symplectomorphisms \(\phi : \tilde{W} \to \tilde{W}\) such that the 1-form \(\phi^*\lambda - \lambda\) is exact and compactly supported. The elements of the group \(\text{Symp}(\tilde{W}, \lambda/d)\) are not necessarily compactly supported.

**Lemma 5.2.3.** Let \((W, \lambda)\) be a Liouville domain, and let \(\phi \in \text{Symp}(\tilde{W}, \lambda/d)\). Then, there is a contactomorphism \(\phi_\partial : \partial W \to \partial W\) and a function \(f : \partial W \to \mathbb{R}^+\) such that

\[
\phi_\partial^* \beta = f \beta
\]

and

\[
\phi(r, x) = \left( \frac{r}{f(x)}, \phi_\partial(x) \right)
\]

for all \(x \in \partial W\) and all \(r \in \mathbb{R}^+\) large enough.

**Proof.** There exists \(r_0 \in \mathbb{R}^+\) such that the restriction of \(\phi\) to \((r_0, \infty) \times \partial W\) is a \(\lambda\)-preserving embedding

\[
(r_0, \infty) \times \partial W \to \mathbb{R}^+ \times \partial W.
\]

Let

\[
\phi(r, x) = (g(r, x), \sigma(r, x)) \in \mathbb{R}^+ \times \partial W
\]

for \((r, x) \in (r_0, \infty) \times \partial W\). Since \(\phi\) preserves \(\lambda\) and \(\omega\), it preserves the vector field \(X_\lambda = r \partial_r\) (because \(X_\lambda\) is characterized by the relation \(X_\lambda \cdot \omega = \lambda\)). Thus, \(\phi\) commutes with the flow of \(r \partial_r\), i.e.

\[
(g(e^t r, x), \sigma(e^t r, x)) = (e^t g(r, x), \sigma(r, x)).
\]
This implies that \( \sigma(r, x) \) does not depend on \( r \). The map \( \sigma(r, \cdot) \), denoted by \( \phi_\partial \), is a contactomorphism, because

\[
 r\beta = \phi^*(r\beta) = g(r, x)\phi_\partial^*\beta
\]

implies

\[
 \phi_\partial^*\beta = \frac{r}{g(r, x)}\beta
\]

and \( \frac{r}{g(r, x)} \) is a positive function. Another consequence is that \( \frac{r}{g(r, x)} =: f(x) \) does not depend on \( r \). Hence, the proof is finished.

\[\square\]

**Definition 5.2.4.** In view of Lemma 5.2.3, one can associate a contactomorphism \( \phi_\partial : \partial W \to \partial W \) to every symplectomorphism \( \phi \in \text{Symp}(\hat{W}, \lambda/d) \). This gives rise to the homomorphism

\[
 \Theta : \text{Symp}(\hat{W}, \lambda/d) \to \text{Cont}(\partial W) : \phi \mapsto \phi_\partial,
\]

which we call the **ideal restriction map**.

Let \( \text{Symp}(W, \lambda/d) \) be the group of symplectomorphisms

\[
 \varphi : \hat{W} \to \hat{W}
\]

such that \( \varphi^*\lambda - \lambda = dF \) for a function \( F : \hat{W} \to \mathbb{R} \) compactly supported in \( W - \partial W \).

**Proposition 5.2.5.** The ideal restriction map of Definition 5.2.4

\[
 \Theta : \text{Symp}(W, \lambda/d) \to \text{Cont}(\partial W)
\]

is a Serre fibration with fibre over the identity equal to the group \( \text{Symp}_c(W, \lambda/d) \) of all exact symplectomorphisms of \( W \).
Proof (following [6]). By definition, a Serre fibration is a continuous map that satisfies the homotopy lifting property for cubes, $I^m$. To prove the proposition, we should show that for any homotopy

$$f : I \times I^m \to \text{Cont}(\partial W)$$

$$(t, y) \mapsto f_t(y)$$

and for any lift $\widetilde{f}_0$ of $f_0$ (i.e. $\Theta \circ \widetilde{f}_0 = f_0$), there exists a homotopy

$$\widetilde{f} : I \times I^m \to \text{Symp}(W, \lambda/d)$$

such that $\Theta \circ \widetilde{f} = f$. Let $K_t(y) : \mathbb{R}^+ \times \partial W$ be the contact Hamiltonian generating the path

$$t \mapsto f_t(y) \in \text{Cont}(\partial W)$$

of contactomorphisms, i.e. $K_t(y)$ is an $\mathbb{R}^+$-equivariant Hamiltonian on the symplectization $\mathbb{R}^+ \times \partial W$ such that the following diagram commutes

$$\begin{array}{ccc}
\mathbb{R}^+ \times \partial W & \xrightarrow{\psi_t^K(y) \circ \widetilde{f}_0} & \mathbb{R}^+ \times \partial W \\
pr_2 \downarrow & & \downarrow pr_2 \\
\partial W & \xrightarrow{f_t(y)} & \partial W.
\end{array}$$

Let $\chi : W \to [0, 1]$ be a cut-off function that is equal to 0 on $W^1$ and to 1 outside $W^2$. Denote by $G_t(y) : W \to \mathbb{R}$ the Hamiltonian defined by

$$G_t(y)(p) := \chi(p)K_t(y)(p), \quad p \in W.$$

Now, we can take

$$\tilde{f}(t, y) := \psi_t^G(p) \circ \widetilde{f}_0.$$
This proves that $\Theta$ is a Serre fibration. The fibre is equal to
\[ \Theta^{-1}([\text{id}]) = \text{Symp}_c(W, \lambda/d). \]
\[ \square \]

By Proposition 5.2.5, there is a homotopy long exact sequence

\[
\begin{array}{cccccc}
\vdots & & & & & \\
\downarrow & & & & & \\
\pi_k \text{Symp}_c(W, \lambda/d) & \xrightarrow{\Delta} & \pi_{k-1} \text{Symp}_c(W, \lambda/d) & \downarrow \\
\downarrow & & & & & \\
\pi_k \text{Symp}(W, \lambda/d) & \\
\downarrow & & & & & \\
\pi_k \text{Cont}(\partial W) & \\
\end{array}
\]

(5.2.36)

In particular, there is a connecting homomorphism
\[ \Delta : \pi_1 \text{Cont}(\partial W) \to \pi_0 \text{Symp}_c(W, \lambda/d). \]  
(5.2.37)

In view of Lemma 5.1.2, the inclusion
\[ \text{Symp}_c(W, \lambda/d) \to \text{Symp}_c(W, d\lambda) \]
induces isomorphism on homotopy groups, so that the groups $\pi_i \text{Symp}_c(W, \lambda/d)$ in (5.2.36) and (5.2.37) can be replaced by $\pi_i \text{Symp}_c(W, d\lambda)$.

**Proposition 5.2.6.** Let $(W, \lambda)$ be a Liouville domain with 1-periodic Reeb flow $\sigma_t : \partial W \to \partial W$ on the boundary. Then, the image $\Delta([\sigma])$ of the class $[\sigma] \in \pi_1 \text{Cont}(\partial W)$ under the connecting homomorphism $\Delta$ is represented by the fibered Dehn twist.
Proof. The proposition follows from Lemma 5.2.7 (applied in the case of $Y_t$ being the Reeb vector field), Lemma 5.2.8 and Remark 5.2.2.

Lemma 5.2.7. Let $(W,\lambda)$ be a Liouville domain. We denote by $\alpha$ the induced contact form on the boundary. Let $\sigma_t : \partial W \to \partial W$ be a 1-periodic family of contactomorphisms generated by a vector field $Y_t$. Then the family $\{\sigma_t\}$ determines an element of $\pi_1 \text{Cont}(\partial W)$ whose image under the connecting homomorphism $\Delta$ can be represented by the time-1 map of a (non-compactly supported) Hamiltonian $\widehat{W} \to \mathbb{R}$ that is equal to $-r\alpha(Y_t)$ near infinity.

Proof. The element in $\pi_1 \text{Cont}(\partial W)$ determined by $\{\sigma_t\}$ is denoted (by a slight abuse of notation) by $[\sigma]$. By definition, $\Delta([\sigma])$ is equal to $[\psi_1] \in \pi_0 \text{Symp}_c(W, d\lambda)$, where $\psi_t$ is a path in $\text{Symp}(W, \lambda/d)$ such that $\Theta(\psi_t) = \sigma_t$ (see Definition 5.2.4). Let $\chi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function equal to 0 near 0 and 1 on $[1, \infty)$. We can choose $\psi_t$ from above to be $\psi_t^H$ where $H : \widehat{W} \to \mathbb{R}$ is the Hamiltonian that is equal to

$$(r, x) \mapsto -\chi(r) r\alpha(Y_t)$$

on $\mathbb{R}^+ \times \partial W$ and 0 elsewhere. This finishes the proof.

Lemma 5.2.8. Let $(W, \lambda)$ be a Liouville domain, and let $H_t$ and $K_t$ be two Hamiltonians on $\widehat{W}$ that are equal near infinity and whose flows are well-defined for all times. If the maps $\psi_t^H$ and $\psi_t^K$ are compactly supported in $W - \partial W$, then they represent the same class in $\pi_0 \text{Symp}_c(W, d\lambda)$.

Proof. The Hamiltonian $H \# \overline{K}$ (see page 33) is compactly supported and generates the Hamiltonian isotopy $\psi_t^H \circ (\psi_t^K)^{-1}$ (compactly supported as well). This isotopy, however, may not be supported in the interior of $W$ for all times. Let $\phi_t^\lambda$
be the family of diffeomorphisms of \( \hat{W} \) that is generated by the Liouville vector field \( X_\lambda \). The diffeomorphism \( \phi_\lambda^t \) is not a symplectomorphism (except for \( t = 0 \)), however if we conjugate a symplectomorphism by it we get a symplectomorphism. Since the isotopy \( \psi_t^H \circ (\psi_t^K)^{-1} \) is compactly supported, there exists \( s \in \mathbb{R}^+ \) such that

\[
(\phi_s^\lambda)^{-1} \circ \psi_t^H \circ (\psi_s^K)^{-1} \circ \phi_s^\lambda
\]

is compactly supported in the interior of \( W \). In particular, the maps \( (\phi_s^\lambda)^{-1} \circ \psi_1^H \circ \phi_s^\lambda \) and \( (\phi_s^\lambda)^{-1} \circ \psi_1^K \circ \phi_s^\lambda \) represent the same class in \( \pi_0 \text{Symp}_c(W, d\lambda) \). Note that

\[
(\phi_t^\lambda)^{-1} \circ \psi_1^H \circ \phi_t^\lambda \in \text{Symp}_c(W, d\lambda)
\]

for all \( t \in \mathbb{R}^+ \). Hence \( (\phi_s^\lambda)^{-1} \circ \psi_1^H \circ \phi_s^\lambda \) and \( \psi_1^H \) represent the same class in \( \pi_0 \text{Symp}_c(W, d\lambda) \). Similarly, the same holds for \( (\phi_s^\lambda)^{-1} \circ \psi_1^K \circ \phi_s^\lambda \) and \( \psi_1^K \), and the proof is finished. \( \square \)

### 5.3 Detecting Nontrivial Mapping Classes

**Theorem 5.3.1.** Let \( (W, \lambda) \) be a Liouville domain such that the Reeb flow on the boundary \( \partial W \) is 1-periodic, and let \( a \in \mathbb{R} \) be an admissible slope. If the fibered Dehn twist represents a class of order \( \ell \in \mathbb{N} \) in \( \pi_0 \text{Symp}_c(W, d\lambda) \), then

\[ \text{HF}(W, \text{id}, a) \cong \text{HF}(W, \text{id}, a + \ell). \quad (5.3.38) \]

**Proof.** Remark 5.2.2 implies that the \( \ell \)-th power of the fibered Dehn twist \( \tau^\ell \) is the time-1 map of a Hamiltonian \( K : \hat{W} \to \mathbb{R} \) that is equal to \( \ell r \) near infinity. The naturality provides us with the isomorphism (see Proposition 3.3.4)

\[ \mathcal{N}(K) : \text{HF}(W, \text{id}, a + \ell) \xrightarrow{\sim} \text{HF}(W, \tau^\ell, a) \quad (5.3.39) \]

60
for all admissible $a \in \mathbb{R}$. On the other hand, if we assume that $\tau^\ell$ represents trivial class in $\pi_0 \text{Symp}_c(W, d\lambda)$, by Lemma 5.1.2, it represents the trivial class in $\text{Symp}_c(W, \lambda/d)$ as well. Theorem 3.4.3 implies

$$\text{HF}(W, \text{id}, a) \cong \text{HF}(W, \tau^\ell, a). \quad (5.3.40)$$

From (5.3.39) and (5.3.40), we get

$$\text{HF}(W, \text{id}, a) \cong \text{HF}(W, \text{id}, a + \ell). \quad (5.3.41)$$

(In general, the isomorphism (5.3.41) preserves neither the grading nor the class in $\pi_0 \Omega_\phi$.)

\[\square\]

**Remark 5.3.2.** Strictly speaking, the construction of the groups $\text{HF}_*(W, \phi, a)$ for $\phi \neq \text{id}$ was not necessary for the proof of Theorem 5.3.1. In this remark, we rewrite the proof of Theorem 5.3.1 without mentioning $\text{HF}_*(W, \phi, a)$ for $\phi \neq \text{id}. \text{ Let } K: \widehat{W} \to \mathbb{R} \text{ be a Hamiltonian equal to } \ell r \text{ near infinity such that } \psi^K_1 = \tau^\ell, \text{ and let } G_t: \widehat{W} \to \mathbb{R} \text{ be a compactly supported Hamiltonian such that } G_{t+1} = G_t \text{ and } \psi^G_t = \tau^\ell. \text{ The Hamiltonian } G \text{ exists because } [\tau^\ell] = 1 \in \pi_0 \text{Symp}_c(W, \lambda/d) \text{ and because of Lemma 3.4.2. Consider the Hamiltonian } G\#\overline{K}. \text{ It has the slope equal to } -\ell \text{ and generates the isotopy } \psi^G_t \circ (\psi^K_1)^{-1}. \text{ In particular, its time-1 map is equal to the identity. Now, the naturality with respect to this Hamiltonian provides the isomorphism (5.3.41).}

**Corollary 5.3.3.** Let $(W, \lambda)$ be as in Theorem 5.3.1. If

$$\dim H(W; \mathbb{Z}_2) < \dim SH(W; \mathbb{Z}_2), \quad (5.3.42)$$

then the fibered Dehn twist represents a class of infinite order in $\pi_0 \text{Symp}_c(W, d\lambda)$. Here, $\dim H(W; \mathbb{Z}_2)$ stands for the sum of Betti numbers rather than the dimension of the homology group of a particular degree.
**Proof.** Assume, by contradiction, that the fibered Dehn twist represents a class of order $\ell \in \mathbb{N}$ in $\pi_0 \text{Symp}_c(W, d\lambda)$ (and, equivalently, in $\pi_0 \text{Symp}_c(W, \lambda/d)$). Then Theorem 5.3.1 implies

$$\text{HF}(W, \text{id}, a) \cong \text{HF}(W, \text{id}, a + \ell)$$

for all admissible $a \in \mathbb{R}$. In particular,

$$\text{HF}(W, \text{id}, k\ell - \varepsilon) \cong \text{HF}(W, \text{id}, -\varepsilon) \cong H(W; \mathbb{Z}_2),$$

for $k \in \mathbb{Z}$ and $\varepsilon > 0$ small enough (see also Example 2.7.7). It follows that

$$\dim \text{HF}(W, \text{id}, k\ell - \varepsilon) = \dim H(W; \mathbb{Z}_2). \quad (5.3.43)$$

Since

$$\text{SH}(W; \mathbb{Z}_2) = \text{HF}(W, \text{id}, \infty) = \lim_{\longrightarrow a} \text{HF}(W, \text{id}, a)$$

has dimension greater than or equal to

$$d := \dim H(W, \mathbb{Z}_2) + 1,$$

there exist elements $\alpha_j \in \text{HF}(W, \text{id}, a_j), j \in \{1, \ldots, d\}$ such that

$$\iota(\alpha_1), \ldots, \iota(\alpha_d) \in \text{SH}(W; \mathbb{Z}_2)$$

are linearly independent. Here, $\iota$ stands for the homomorphisms

$$\iota : \text{HF}(W, \text{id}, a) \to \lim_{\longrightarrow a} \text{HF}(W, \text{id}, a).$$

Let $\varepsilon > 0$ be small enough, and let $k \in \mathbb{N}$ be such that

$$k\ell - \varepsilon > \max\{a_1, \ldots, a_d\}.$$
Then the images of $\alpha_1, \ldots, \alpha_d$ under the continuation maps

$$HF(W, \text{id}, a_j) \to HF(W, \text{id}, k\ell - \varepsilon), \quad j \in \{1, \ldots, d\}$$

are linearly independent. However, this leads to contradiction, because of (5.3.43) and $d > \dim H(W; \mathbb{Z}_2)$.

**Definition 5.3.4.** Let $m \in \mathbb{N}$ be a natural number. A $\mathbb{Z}_m$-graded vector space $V_*$ is called **symmetric** if there exists $k \in \mathbb{Z}_m$ such that

$$V_{k-j} \cong V_j$$

for all $j \in \mathbb{Z}_m$.

**Corollary 5.3.5.** Let $(W, \lambda)$ be a Liouville domain, let $o \in \pi_0 \Omega_{\text{id}}$ be the class of contractible loops, and let $N$ be the minimal Chern number $N_o$ (see page 14). Assume the Reeb flow induces a free circle action on the boundary $\partial W$. If the homology $H_*(W; \mathbb{Z}_2)$ rolled up modulo $2N$ is not symmetric, then the fibered Dehn twist represents a nontrivial class in $\pi_0 \text{Symp}_c(W, d\lambda)$.

**Proof.** Assume the contrary, i.e. that the fibered Dehn twist represents the trivial class in $\pi_0 \text{Symp}_c(W, d\lambda)$. Let $1 > \varepsilon > 0$. Theorem 5.3.1 implies

$$HF(W, \text{id}, -\varepsilon) \cong HF(W, \text{id}, 1 - \varepsilon).$$

This isomorphism preserves neither the grading nor the class in $\pi_0 \Omega_{\text{id}}$. However, it preserves the relative grading. Hence there exist a class $o' \in \pi_0 \Omega_{\text{id}}$ and $c \in \mathbb{Z}_{2N}$ such that

$$HF_*(W, \text{id}, -\varepsilon, o) \cong HF_{*+c}(W, \text{id}, 1 - \varepsilon, o'). \quad (5.3.44)$$
Since \( \epsilon \in (-1, 0) \), \( 1 - \epsilon \in (0, 1) \) and numbers in \((-1, 0) \cup (0, 1)\) are all admissible, we get (by Proposition \( 3.2.1 \) and Example \( 2.7.7 \))

\[
\begin{align*}
\text{HF}_{*}(W, \text{id}, -\epsilon, o) & \cong \text{H}_{*+n}(W; \mathbb{Z}_2), \\
\text{HF}_{*}(W, \text{id}, 1 - \epsilon, o') & \cong \begin{cases} 
\text{H}_{*+n}(W, \partial W; \mathbb{Z}_2) & \text{if } o = o' \\
0 & \text{if } o \neq o'.
\end{cases}
\end{align*}
\]

The group \( \text{H}(W; \mathbb{Z}_2) \) is never 0. Hence (because of (5.3.44)), \( o = o' \) and

\[
\text{H}_{*}(W; \mathbb{Z}_2) \cong \text{H}_{*+c}(W, \partial W; \mathbb{Z}_2).
\]

By Poincaré duality

\[
\text{H}_{*}(W, \partial W; \mathbb{Z}_2) \cong \text{H}^{2n-*}(W; \mathbb{Z}_2).
\]

Since we are working with field coefficients, we have

\[
\text{H}^{2n-*}(W, \mathbb{Z}_2) \cong \text{Hom}(\text{H}_{2n-*}(W; \mathbb{Z}_2), \mathbb{Z}_2) \cong \text{H}_{2n-*}(W; \mathbb{Z}_2).
\]

Therefore

\[
\text{H}_{2n-*}(W; \mathbb{Z}_2) \cong \text{H}_{*}(W; \mathbb{Z}_2).
\]

This contradicts the fact that \( \text{H}_{*}(W; \mathbb{Z}_2) \) is not symmetric.

\[\square\]

**Corollary 5.3.6.** Let \((W, \lambda)\) be as in Theorem 5.3.1. Assume that the Reeb flow induces a free circle action on \( \partial W \), and that the first Chern class \( c_1(W) \) vanishes. If the homology \( \text{H}_{*}(W; \mathbb{Z}_2) \) is not symmetric, then the fibered Dehn twist represents a nontrivial class in \( \pi_0 \text{Symp}_c(W, d\lambda) \).

**Proof.** The corollary is a special case of Corollary 5.3.5 (the case in which \( N = \infty \)).

\[\square\]
Example 5.4.1. Let \((Q, g)\) be a closed Riemannian manifold. We denote by \(D^*Q\) and \(S^*Q\) the disk cotangent bundle and the unit cotangent bundle of \(Q\), respectively. The standard Liouville form \(\lambda_{\text{can}}\) on \(T^*Q\) equips \(D^*Q\) with the structure of a Liouville domain. The Reeb flow on \(S^*Q\) coincides with the geodesic flow of \(Q\) under the obvious identification of tangent and cotangent bundles of \(Q\). It is periodic if, and only if, the geodesics of \((Q, g)\) are all closed. Examples of Riemannian manifolds with all geodesics periodic are spheres \(S^m\), complex and quaternionic projective spaces \(\mathbb{C}P^m\) and \(\mathbb{H}P^m\), and the Cayley plane (octonionic projective plane) \(\mathbb{C}aP^2\) with the standard metrics.

Assume \((Q, g)\) is one of these manifolds. We can rescale \(g\) so that the Reeb flow on \(S^*Q\) is 1-periodic. By a theorem of Viterbo (see Theorem 5.4.2 below), the symplectic homology \(\text{SH}_*(D^*Q; \mathbb{Z}_2)\) is isomorphic to the singular homology \(H_*(\Lambda Q; \mathbb{Z}_2)\) of the free loop space of \(Q\). The homology of \(\Lambda Q\) is explicitly computed in [31], and it turns out to be infinite dimensional in \(\mathbb{Z}_2\) coefficients. Hence Corollary 5.3.3 implies that the corresponding fibered Dehn twist is of infinite order in \(\pi_0\text{Symp}^c_c(D^*Q, d\lambda_{\text{can}})\).

Theorem 5.4.2 (Viterbo). Let \((Q, g)\) be a closed Riemannian manifold, and let \(\Lambda Q\) be its free loop space. Then

\[
\text{SH}_*(D^*Q; \mathbb{Z}_2) \cong H_*(\Lambda Q; \mathbb{Z}_2).
\]

Proof. Proofs can be found in [28], [1], [19], and [2].

Remark 5.4.3. Corollary 5.3.5 does not imply nontriviality of the fibered Dehn twists for the Liouville domains considered in Example 5.4.1. The reason is the following. Let
(Q, g) be as in Example 5.4.1. The disk cotangent bundle $D^*Q$ is homotopic to $Q$, which is a closed manifold, and therefore (by Poincaré duality), the homology $H_*(D^*Q; \mathbb{Z}_2)$ is symmetric.

**Remark 5.4.4.** The fibered Dehn twists on $D^*S^2$ and $D^*S^6$ are smoothly isotopic to the identity relative to the boundary (see Lemma 6.3 in [23] and also Lemma 10.2 in [5]). The same is true for $D^*\mathbb{C}P^m$, $m \in \mathbb{N}$ [24, Proposition 4.6]. On the other hand, the fibered Dehn twist on $D^*S^m$ for $m \not\in \{2, 6\}$ is not even smoothly isotopic to the identity relative to the boundary (see, for example, Lemma 10.1 in [5]). Interestingly, the only manifolds among $S^m$ and $\mathbb{C}P^m$, $m \in \mathbb{N}$, admitting almost complex structures are $S^2, S^6$ and $\mathbb{C}P^m$, $m \in \mathbb{N}$. This is not a coincidence (see Proposition 5.4.5 below).

**Proposition 5.4.5.** Let $(Q, g)$ be a Riemannian manifold whose geodesics are all 1-periodic. Assume there exists an almost complex structure $J$ on $Q$ that preserves the norm of vectors in $TQ$. Then, the square of the fibered Dehn twist on $D^*Q$ is smoothly isotopic to the identity relative to the boundary.

It is an interesting question whether in the situation of Proposition 5.4.5 the fibered Dehn twist itself is smoothly isotopic to the identity relative to the boundary.

**Lemma 5.4.6.** Let $(Q, g)$ be as in Proposition 5.4.5. Denote by $SQ \subset TQ$ the unit tangent bundle. Then the geodesic flow induces a loop of diffeomorphisms $SQ \to SQ$ whose square is contractible.

**Proof.** Let $J$ be as in Proposition 5.4.5, and let $\gamma_{\xi}$ be the unique geodesic on $Q$ such that $\dot{\gamma}_{\xi}(0) = \xi \in TQ$. The (restriction of the) geodesic flow is given by

$$
\Psi_t : SQ \to SQ : \xi \mapsto \gamma_{\xi}(t).
$$
The antipodal map
\[ SQ \rightarrow SQ : \xi \mapsto -\xi \]
is isotopic to the identity via the isotopy
\[ [0, \pi] \times SQ \rightarrow SQ : (s, \xi) \mapsto \cos s\xi + \sin sJ\xi. \]
Hence the loop
\[ \mathbb{R}/\mathbb{Z} \ni t \mapsto \Psi_t \in \text{Diff}(SQ) \]
is homotopic to the loop
\[ t \mapsto -\Psi_t(-\cdot) \]
(\(\Psi_t\) is pre- and postcomposed by the antipodal map). Since
\[ -\Psi_t(-\xi) = \Psi_t^{-1}(\xi), \quad \xi \in SQ, \]
the loop \( t \mapsto \Psi_t \) is homotopic to its inverse \( t \mapsto \Psi_t^{-1} \). Therefore, the loop \( t \mapsto \Psi_t^2 \) is contractible. \(\square\)

**Proof of Proposition 5.4.5.** We take \((W, \lambda) = (D^*Q, \lambda_{\text{can}})\). The square of the fibered Dehn twist is given by
\[ p \mapsto \begin{cases} (r, \sigma^2_{v(r)}(x)) & \text{for } p = (r, x) \in \mathbb{R}^+ \times \partial W, \\ p & \text{otherwise,} \end{cases} \tag{5.4.45} \]
where \(\sigma_t\) is the Reeb flow on \(\partial W\), and \(v : \mathbb{R} \rightarrow \mathbb{R}\) is a smooth function that is equal to 0 on \((-\infty, 0)\) and \(-1\) on \((0.95, +\infty)\). By Lemma 5.4.6, \(\sigma^2\) seen as a loop
\[ \mathbb{R}/\mathbb{Z} \rightarrow \text{Diff}(\partial W) \]
is homotopic to the constant loop \( t \mapsto \text{id} \). Let
\[ \Phi^s : \mathbb{R}/\mathbb{Z} \rightarrow \text{Diff}(\partial W), \quad s \in [0, 1] \]
be a homotopy between the loops \( t \mapsto \text{id} \) and \( t \mapsto \sigma_t^2 \). An isotopy between (5.4.45) and the identity is given by

\[
[0, 1] \times \hat{W} \to \hat{W} \\
(s, p) \mapsto \begin{cases} 
(r, \Phi^s_{v(r)}(x)) & \text{for } p = (r, x) \in \mathbb{R}^+ \times \partial W, \\
\quad p & \text{otherwise}.
\end{cases}
\]

Albers and McLean found in [3] a large family of contact manifolds such that every strong filling of them has infinite dimensional symplectic homology. It is an interesting question, in the view of Corollary 5.3.3, whether any of their exact fillable examples has a periodic Reeb flow.

**Example 5.4.7.** Assume \( W \) is a higher-genus closed surface with an open disk removed. Since \( H_\ast(W; \mathbb{Z}_2) \) is not symmetric, Corollary 5.3.5 implies that the fibered Dehn twist represents a nontrivial class in \( \pi_0 \text{Symp}_c(W) \). The Fibered Dehn twist in this case is the Dehn twist along (a translate of) the boundary circle.

**Example 5.4.8.** Let \( V \) be a smooth degree \( d \geq 2 \) projective hypersurface in \( \mathbb{CP}^m \), \( m > 3 \), and let \( H \) be a hyperplane transverse to \( V \). By removing a neighbourhood of \( V \cap H \) in \( V \), we get a Liouville domain \( W \) such that the Reeb flow induces a free circle action on its boundary \( \partial W \). By [27], the manifold \( W \) has the homotopy type of a bouquet of \( (m-1) \)-spheres. Moreover, the number of the spheres is equal to \((d-1)^m \) (see Theorem 1 and consequence (iii) on page 487 in [10]). Hence \( c_1(W) = 0 \) and the homology \( H_\ast(W; \mathbb{Z}_2) \) is not symmetric. Corollary 5.3.6 implies that the fibered Dehn twist of \( W \) is not symplectically isotopic to the identity relative to the boundary. (See also Example 7.14 in [7].)
5.5 Iterated Ratio

**Definition 5.5.1.** Let $\phi : W \to W$ be an exact symplectomorphism of a Liouville domain $W$. The *iterated ratio* $\kappa(W, \phi)$ is defined to be the number

$$\kappa(W, \phi) := \limsup_{m \to \infty} \frac{\dim HF(W, \phi^m, \varepsilon)}{m}.$$ 

Here $\varepsilon > 0$ is small enough (smaller than any positive period of some Reeb orbit on $\partial W$).

It follows directly from the definition that $\kappa(W, \text{id}) = 0$. If the limit

$$\lim_{m \to \infty} \frac{\dim HF(W, \phi^m, \varepsilon)}{m}$$

exists, then

$$\kappa(W, \phi^\ell) = \ell \kappa(W, \phi).$$

**Remark 5.5.2.** The iterated ratio is invariant under compactly supported symplectic isotopies. It measures the linear growth rate of $\dim HF(W, \phi^m, \varepsilon)$. There is yet another similar invariant, the Floer theoretic entropy,

$$h_{\text{Floer}}(W, \phi) := \limsup_{m \to \infty} \frac{\log (\dim HF(W, \phi^m, \varepsilon))}{m},$$

which measures the exponential growth rate of $\dim HF(W, \phi^m, \varepsilon)$. If $W$ is a surface and $\phi$ an area-preserving diffeomorphism that has a pseudo-Anosov component, then $h_{\text{Floer}}(W, \phi) > 0$ [26, page 167]. Consequently, $\kappa(W, \phi)$ is equal to infinity.

**Theorem 5.5.3.** Let $(W_2, \lambda)$ be a Liouville domain, let $W_1 \subset W_2$ be a codimension-0 submanifold such that $(W_1, \lambda)$ is a Liouville domain in its own right, and let $\phi : W_1 \to W_1$ be an exact symplectomorphism. Then, $\kappa(W_1, \phi) = \kappa(W_2, \phi)$. 

69
Proof of Theorem 5.5.3. Let $\varepsilon > 0$ be small. It is enough to prove that the number

$$|\dim \text{HF}(W_1, \phi^m, \varepsilon) - \dim \text{HF}(W_2, \phi^m, \varepsilon)|$$

is bounded (by a constant not depending on $m$). Let $\psi_m : W_1 \rightarrow W_1$ be an exact symplectomorphism that is isotopic to $\phi^m$ through exact symplectomorphisms, and such that $C(W_1, \psi_m) < \frac{\varepsilon}{2}$ (see (4.1.27) on page 41 for the definition of $C(W_1, \psi_m)$). Such a symplectomorphism exists for each $m \in \mathbb{N}$ due to Proposition 4.1.5. Choose transfer data $(H^m, J^m)$ for $(\psi_m, W_1, W_2, \frac{\varepsilon}{2}, \varepsilon)$, such that $H^m = H^1$ and $J^m = J^1$ on $W_2 \setminus W_1$. The short exact sequence

\[
\begin{array}{c}
0 \\
\downarrow \\
\text{CF}^<_{\leq 0}(W_2, \psi_m, H^m, J^m) \\
\downarrow \\
\text{CF}_{\leq 0}(W_2, \psi_m, H^m, J^m) \\
\downarrow \\
\text{CF}^>_{\geq 0}(W_2, \psi_m, H^m, J^m) \\
\downarrow \\
0
\end{array}
\]
of chain complexes induces the long exact sequence in homology

\[
\begin{align*}
\vdots \\
\downarrow \\
\HF^<^0_*^\left(W_2, \psi_m, H^m, J^m\right) \\
\downarrow \\
\HF_*^\left(W_2, \psi_m, H^m, J^m\right) \\
\downarrow \\
\HF^\geq^0_*^\left(W_2, \psi_m, H^m, J^m\right) \\
\vdots
\end{align*}
\]

By applying identifications as in Section 4.2, the long exact sequence is transformed into

\[
\begin{align*}
\vdots \\
\downarrow \\
\HF^<^0_*^\left(W_2, \psi_m, H^m, J^m\right) \\
\downarrow \\
\HF_*^\left(W_2, \psi_m, \frac{\epsilon}{2}\right) \\
\downarrow \\
\HF_*^\left(W_1, \psi_m, \epsilon\right) \\
\vdots
\end{align*}
\] (5.5.46)
The chain complexes \( \text{CF}_{*}^{<0}(W_2, \psi_m, H^m, J^m), m \in \mathbb{N} \) are generated by twisted orbits contained in \( W_2 \setminus W_1 \). In this region \( \psi_m = \text{id} \) and \( (H^m, J^m) = (H^1, J^1) \). Hence

\[
\text{CF}_{*}^{<0}(W_2, \psi_m, H^m, J^m), \quad m \in \mathbb{N}
\]

are generated by the same set of generators. This does not imply that

\[
\text{HF}_{*}^{<0}(W_2, \psi_m, H^m, J^m), \quad m \in \mathbb{N}
\]

are isomorphic, because the differentials may differ. However,

\[
\dim \text{HF}_{*}^{<0}(W_2, \psi_m, H^m, J^m) \leq \dim \text{CF}_{*}^{<0}(W_2, \psi_m, H^m, J^m) = \dim \text{CF}_{*}^{<0}(W_2, \psi_1, H^1, J^1),
\]

and \( \dim \text{CF}_{*}^{<0}(W_2, \psi_1, H^1, J^1) \) does not depend on \( m \). The long exact sequence (5.5.46) implies

\[
\left| \dim \text{HF}(W_2, \psi_m, \frac{\varepsilon}{2}) - \dim \text{HF}(W_1, \psi_m, \varepsilon) \right| < C,
\]

where

\[
C := 2 \dim \text{CF}_{*}^{<0}(W_2, \psi_1, H^1, J^1).
\]

Finally, note that

\[
\text{HF}(W_2, \psi_m, \frac{\varepsilon}{2}) \cong \text{HF}(W_2, \phi^m, \frac{\varepsilon}{2}) \cong \text{HF}(W_2, \phi^m, \varepsilon),
\]

and

\[
\text{HF}(W_1, \psi_m, \varepsilon) \cong \text{HF}(W_1, \phi^m, \varepsilon).
\]

This finishes the proof.
The following proposition shows that symplectomorphisms which are supported in the cylindrical part of a Liouville domain have finite iterated ratio.

**Proposition 5.5.4.** Let \((W, \lambda)\) be a Liouville domain, and let \(\phi : W \to W\) be an exact symplectomorphism that is compactly supported in \((0, 1) \times \partial W\). Then

\[ \kappa(W, \phi) < \infty. \]

**Proof.** The idea of the proof is the following. We construct an exact symplectomorphism \(\psi_m : W \to W\) that is symplectically isotopic to \(\phi^m\) relative to the boundary, and such that it consists of \(m\) “copies” of \(\phi\) with disjoint supports. By choosing a suitable Hamiltonian, one can ensure that each “copy” contributes the same number of generators to the Floer chain complex. Since the number of “copies” grows linearly with respect to \(m\), the iterated ratio has to be smaller than the number of generators furnished by a single “copy.”

Now, we implement the idea rigorously. By the assumption, there exists \(\delta \in (0, \frac{1}{2})\) such that \(\phi\) is compactly supported in \((2\delta, 1) \times \partial W\). Let \(\varepsilon > 0\) be small, and let \(H_t : \widehat{W} \to \mathbb{R}\) be a Hamiltonian equal to

\[ (r, x) \mapsto \varepsilon r \]

on \((\delta, 2\delta) \times \partial W \cup (1, \infty) \times \partial W\) such that

\[ H_{t+1} = H_t \circ \phi, \]

and such that

\[ \det \left( \text{d} (\phi \circ \psi^H_1)(x) - \text{id} \right) \neq 0 \]

for all fixed points \(x\) of \(\phi \circ \psi^H_1\). The non-degeneracy condition implies that there are finitely many Hamiltonian
twisted orbits. We denote by \( A \) and \( B \) the numbers of such orbits in \( W^\delta \) and \((2\delta, 1) \times \partial W\), respectively.

Fix \( m \in \mathbb{N} \). Denote by \( \varphi_0 : W \to W \) the symplectomorphism

\[
\varphi_0 := \psi_t^\lambda \circ \phi \circ (\psi_t^\lambda)^{-1},
\]
where

\[
T := (m - 1) \ln \delta - m \ln(1 + \delta),
\]
and \( \psi_t^\lambda \) is the Liouville flow. Let \( \varphi_j, j \in \{1, \ldots, m - 1\} \) be the symplectomorphism given by

\[
\varphi_j := \psi_{jc}^\lambda \circ \varphi_0 \circ (\psi_{jc}^\lambda)^{-1},
\]
where \( c := \ln(1 + \delta) - \ln \delta \). Similarly, denote

\[
H_0^t := H_t \circ (\psi_t^\lambda)^{-1},
\]
\[
H_j^t := H_0^t \circ (\psi_{jc}^\lambda)^{-1}, \quad j \in \{1, \ldots, m - 1\}.
\]

The symplectomorphisms \( \varphi_0, \varphi_1, \ldots, \varphi_{m-1} \) are all symplectically isotopic to \( \phi \) relative to the boundary, and they are compactly supported in

\[
\begin{align*}
(r_0, r_0e^c) \times \partial W,
(r_0e^c, e_0e^{2c}) \times \partial W,
\ldots,
(r_0e^{(m-1)c}, r_0e^{mc}) \times \partial W,
\end{align*}
\]
respectively. Here,

\[
r_0 := \frac{\delta^m}{(1 + \delta)^m}.
\]
and $c$ is as above. (It follows that $r_0 e^{mc} = 1$ and consequently the sets (5.5.47) are all subsets of $W$. Hence, $\varphi_j$ is compactly supported in the interior of $W$ for all $j \in \{0, \ldots, m-1\}$.) Let $\psi_m : W \to W$ be the exact symplectomorphism

$$
\psi_m := \varphi_0 \circ \cdots \circ \varphi_{m-1},
$$

and let $G_t : \widehat{W} \to \mathbb{R}$ be the Hamiltonian defined by

$$
G_t(p) := \begin{cases} 
H_t^0(p) & \text{for } p \in W^{r_0}, \\
H_t^j(p) & \text{for } p \in S_j, j \in \{1, \ldots, m-1\} \\
\varepsilon r & \text{otherwise},
\end{cases}
$$

where

$$
S_j := \left( r_0 e^{(j-1)c}, r_0 e^{jc} \right) \times \partial W.
$$

By construction, $G_t$ has exactly $A + mB$ Hamiltonian $\psi_m$-twisted loops, and they are all non-degenerate. This implies

$$
\dim \text{HF}(W, \phi^m, \varepsilon) = \dim \text{HF}(W, \psi_m, \varepsilon) \leq A + mB.
$$

The first equality follows from $\phi^m$ being symplectically isotopic to $\psi_m$ relative to the boundary. Hence,

$$
\kappa(W, \phi) \leq \lim_{m \to \infty} \frac{A + mB}{m} = B < \infty,
$$

and the proof is finished. \qed

5.6 Iterated Ratio of the Fibered Dehn Twist on $D^*S^n$

In this section, we compute the iterated ratio for the fibered Dehn twist $\tau : D^*S^n \to D^*S^n$ [6, 24] of
the disk cotangent bundle of the sphere $S^n$ endowed with the standard Riemannian metric. The Floer homology $HF(D^*S^n, \tau^m, \varepsilon)$, $m \in \mathbb{N}$, is isomorphic (as a $\mathbb{Z}_2$-vector space) to $HF(D^*S^n, id, 2\pi m + \varepsilon)$. This follows from Proposition 3.3.4 and Remark 5.2.2. Hence

$$\kappa(D^*S^n, \tau) = \lim_{m \to \infty} \frac{\dim HF(D^*S^n, id, 2\pi m + \varepsilon)}{m}.$$ 

**Proposition 5.6.1.** Let $m$ and $n$ be positive integers, let $\varepsilon$ be an element of $(0, 2\pi)$, and let $D^*S^n$ be the disk cotangent bundle of $S^n$ with respect to the standard Riemannian metric. Then, the vector space

$$HF_k(D^*S^n, 2\pi m + \varepsilon)$$

has the following isomorphism type.

- If $n = 2$,

  $$\otimes \cong \begin{cases} 
  \mathbb{Z}_2 & \text{for } k \in \{0, 1, 2m + 1, 2m + 2\}, \\
  \mathbb{Z}_2^2 & \text{for } k \in \mathbb{N} \text{ and } 2 \leq k \leq 2m, \\
  0 & \text{otherwise}.
  \end{cases}$$

- If $n > 2$,

  $$\otimes \cong \begin{cases} 
  \mathbb{Z}_2 & \text{for } k \in A_m, \\
  0 & \text{otherwise},
  \end{cases}$$

where

$$A_m := \{\ell(n - 1), \ell(n - 1) + n : \ell \in \mathbb{Z} \& 0 \leq \ell \leq 2m\}.$$ 

In particular,

$$\dim HF_k(D^*S^n, 2\pi m + \varepsilon) = \dim HF_k(D^*S^2, 2\pi m + \varepsilon) = 4m + 2.$$
Proof. We will consider only the case of $S^2$. The proof for other cases is analogous and even simpler.

The Reeb flow on $S^*S^2 := \partial D^*S^2$ is periodic with minimal period equal to $2\pi$. Hence, there exists a homology long exact sequence

\[ \cdots \rightarrow \HF_k(D^*S^2, 2\pi \ell + \varepsilon) \xrightarrow{\Delta} \HF_{k-1}(D^*S^2, 2\pi \ell + \varepsilon) \rightarrow \HF_k(D^*S^2, 2\pi (\ell + 1) + \varepsilon) \rightarrow \cdots \]

\[ \Delta \rightarrow \HF_{k+\Delta\ell+1}(S^*S^2) \]

where $\ell \in \mathbb{N} \cup \{0\}$, and $\Delta$ is a shift in grading [21, Lemma 3.6]. The shift $\Delta$ is equal to $-(2\ell - 1)$ [18, Proposition 5.12]. The long exact sequence (5.6.48) implies

\[ \HF_k(D^*S^2, 2\pi \ell + \varepsilon) \xrightarrow{\cong} \HF_k(D^*S^2, 2\pi (\ell + 1) + \varepsilon), \]

for $k < 2\ell$. Therefore

\[ \HF_k(D^*S^2, 2\pi \ell + \varepsilon) \cong \SH_k(D^*S^2), \quad \text{for } k < 2\ell. \quad (5.6.49) \]
We prove the proposition by induction. Using (5.6.48) with \( \ell = 0 \), we get

\[
\dim \text{HF}_2(D^*S^2, 2\pi + \varepsilon) = \dim \text{HF}_3(D^*S^2, 2\pi + \varepsilon) + 1,
\]
\[
\text{HF}_4(D^*S^2, 2\pi + \varepsilon) \cong \mathbb{Z}_2.
\]
\[
\text{HF}_k(D^*S^2, 2\pi + \varepsilon) = 0, \quad \text{for } k \notin \{0, 1, 2, 3, 4\},
\]
\[
\dim \text{HF}_3(D^*S^2, 2\pi + \varepsilon) \leq 1.
\]

Consequently,

\[
\dim \text{HF}_2(D^*S^2, 2\pi + \varepsilon) \leq 2. \quad (5.6.50)
\]

The equation (5.6.49) implies

\[
\begin{align*}
\text{HF}_0(D^*S^2, 2\pi + \varepsilon) &\cong \text{SH}_0(D^*S^2) \cong \mathbb{Z}_2, \\
\text{HF}_1(D^*S^2, 2\pi + \varepsilon) &\cong \text{SH}_1(D^*S^2) \cong \mathbb{Z}_2, \\
\text{HF}_2(D^*S^2, 4\pi + \varepsilon) &\cong \text{SH}_2(D^*S^2) \cong \mathbb{Z}_2^2.
\end{align*}
\]

(The computation of the symplectic homology for \( D^*S^2 \) can be done by using Theorem 5.4.2 and [31, page 21].) Consider the segment

\[
\cdots \to \text{HF}_2(D^*S^2, 2\pi + \varepsilon) \to \text{HF}_2(D^*S^2, 4\pi + \varepsilon) \to 0
\]

of the long exact sequence (5.6.48) with \( \ell = 1 \). It implies

\[
\dim \text{HF}_2(D^*S^2, 2\pi + \varepsilon) \geq \dim \text{HF}_2(D^*S^2, 4\pi + \varepsilon) = 2.
\]

Therefore (see (5.6.50))

\[
\text{HF}_2(D^*S^2, 2\pi + \varepsilon) \cong \mathbb{Z}_2^2
\]

and

\[
\text{HF}_3(D^*S^2, 2\pi + \varepsilon) \cong \mathbb{Z}_2
\]
This proves the basis of the induction. Assume now the claim holds for 1, \ldots, m. The groups
\[ HF_k(D^*S^2, 2\pi(m + 1) + \varepsilon) \]
for \( k \in \{0, \ldots, 2m - 1\} \cup \{2m + 4\} \) can be computed directly from (5.6.48) with \( \ell = m \). The isomorphism (5.6.49) computes the groups \( HF_{2m}(\cdots) \) and \( HF_{2m+1}(\cdots) \). Finally, the groups \( HF_{2m+2}(\cdots) \) and \( HF_{2m+3}(\cdots) \) are computed in the same way as \( HF_{2}(D^*S^2, 2\pi + \varepsilon) \) and \( HF_{3}(D^*S^2, 2\pi + \varepsilon) \) in the basis of the induction. (We are not going to repeat the argument here.) This finishes the proof. \( \square \)

**Corollary 5.6.2.** Let \( n > 1 \), and let \( \phi : D^*S^n \to D^*S^n \) be the fibered Dehn twist. Then,
\[ \kappa(D^*S^n, \phi) = 4. \]

**Proof.** The proof follows directly from Proposition 5.6.1. \( \square \)

**Corollary 5.6.3.** Let \( L \) be a Lagrangian sphere in \( 2n \geq 4 \) dimensional Liouville domain \( W \), and let \( \tau_L \) be a generalized Dehn twist furnished by the sphere. Then,
\[ \kappa(W, \tau_L^2) = 4. \]

Consequently, the symplectomorphism \( \tau_L : W \to W \) and all of its iterates are not symplectically isotopic to the identity relative to the boundary.

**Proof.** By the Weinstein neighbourhood theorem, there exists a neighbourhood \( V \) of \( L \) in \( W \) that is symplectomorphic to the disk-cotangent bundle \( D_\rho^*S^n \) for small enough radius \( \rho \in \mathbb{R}^+ \). The manifold \( V \) is not, in general, a Liouville domain when considered with the 1-form \( \lambda \). Namely, the Liouville vector field \( X_\lambda \) might not be pointing out on
the boundary $\partial V$. There is, however, a 1-form $\lambda_0$ furnished by the canonical Liouville form on $D^*_\rho S^n$ that makes $V$ into a Liouville domain. The restriction $(\lambda - \lambda_0)|_L = \lambda|_L$ is an exact form because $L$ is an exact Lagrangian submanifold. This, together with the inclusion $L \hookrightarrow V$ being homotopy equivalence, implies that there exists a function $f : V \to \mathbb{R}$ such that $\lambda - \lambda_0 = df$. Let $\chi : V \to [0, 1]$ be a function equal to 0 on $V^{\frac{1}{2}}$ and to 1 on the complement of $V^{\frac{3}{2}}$. Consider the Liouville form

$$\lambda_1 = \lambda_0 + d(\chi f).$$

Note that $\lambda_1$ is equal to $\lambda_0$ on $V^{\frac{1}{2}}$ and can be extended by $\lambda$ on the complement of $V$. Without loss of generality, assume that $\tau_L$ is compactly supported in the interior of $V^{\frac{1}{2}}$. Theorem 5.5.3 implies

$$\kappa(V^{\frac{1}{2}}, \lambda_1, \tau^2_L) = \kappa(W, \lambda_1, \tau^2_L). \quad (5.6.51)$$

The Liouville domain $(V^{\frac{1}{2}}, \lambda_1)$ is isomorphic to the disk-cotangent bundle $D^*_\varepsilon S^n$ for some radius $\varepsilon$ because $\lambda_1$ is equal to $\lambda_0$ on $V^{\frac{1}{2}}$. Besides, the symplectomorphism $\tau^2_L$ is symplectically isotopic to the fibered Dehn twist. Hence, due to Proposition 5.6.1,

$$\kappa(V^{\frac{1}{2}}, \lambda_1, \tau^2_L) = 4. \quad (5.6.52)$$

Finally, the 1-form $\lambda - \lambda_1$ is, by construction, the exterior derivative of a function $W : V \to \mathbb{R}$ that is equal to 0 near the boundary. Hence Proposition 3.1.2, (5.6.51), and (5.6.52) imply

$$\kappa(W, \tau^2_L) = 4.$$

Moreover,

$$\kappa(W, \tau^{2m}_L) = 4m, \quad m \in \mathbb{N}.$$
Since \( \kappa \) is invariant under symplectic isotopies, and since \( \kappa(W, \text{id}) = 0 \), the generalized Dehn twist \( \tau_L \) represent a symplectic mapping class of infinite order. \( \square \)

5.7 AN APPLICATION TO CLOSED GEODESICS

Definition 5.7.1. The visible rank \( \tau(W, a) \) of a Liouville domain \( W \) and an admissible slope \( a \) is defined to be

\[
\tau(W, a) := \dim_{\mathbb{Z}_2} \iota(\text{HF}(W, \text{id}, a)),
\]

where

\[
\iota : \text{HF}_*(W, \text{id}, a) \to \text{HF}_*(W, \text{id}, \infty) = \text{SH}_*(W; \mathbb{Z}_2)
\]

is the natural morphism.

Lemma 5.7.2. Let \( W_1, W_2 \) be as in Chapter 4. Assume there exists \( r \in (0,1) \) such that \( W_2^r \subset W_1 \). Then, the transfer morphism

\[
\text{SH}(W_2) \to \text{SH}(W_1)
\]

is an isomorphism.

Proof. Consider the map

\[
\tilde{W}_1 \to \tilde{W}_2
\]

defined by \( p \mapsto j(p) \) for \( p \in W_1 \), and

\[
p \mapsto (\text{pr}_1(j(p)) \cdot r, \text{pr}_2(j(x)))
\]

for \( p = (r, x) \in \mathbb{R}^+ \times \partial W_1 \). It is a diffeomorphism that respects the Liouville forms. Here \( j : W_1 \hookrightarrow W_2 \) stands for the inclusion, and \( \text{pr}_1, \text{pr}_2 \) are the projections

\[
\text{pr}_1 : \mathbb{R}^+ \times \partial W_2 \to \mathbb{R}^+
\]
and

\[ \text{pr}_2 : \mathbb{R}^+ \times \partial W_2 \to \partial W_2. \]

We will identify \( \widehat{W}_1 \) and \( \widehat{W}_2 \) via this map. By the assumption, there exists \( s \in \mathbb{R}^+ \) such that \( W_2 \subset W_1^s \). The transfer morphisms

\[
\begin{align*}
SH_*(W_1^s) &\to SH_*(W_1), \\
SH_*(W_2) &\to SH_*(W_2^r)
\end{align*}
\]

are isomorphisms due to Lemma 4.16 in [16]. They fit (because of the functoriality of the transfer morphisms [16]) into the commutative diagram

\[
\begin{array}{ccc}
SH_*(W_1^s) & \longrightarrow & SH_*(W_2) \\
& \downarrow & \searrow \\
& \to & SH_*(W_1) \\
& & \longrightarrow \rightarrow \ SH_*(W_2^r)
\end{array}
\]

Hence (5.7.53) is an isomorphism as well. \( \square \)

**Theorem 5.7.3.** Let \( W_1, W_2 \) be as in Chapter 4. Assume there exists \( r \in (0, 1) \) such that \( W_2^r \subset W_1 \). Then,

\[ r(W_2, a) \leq r(W_1, a) \]

for all slopes \( a \in \mathbb{R}^+ \) that are admissible with respect to both \( W_1 \) and \( W_2 \).

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
SH_*(W_2) & \longrightarrow & SH_*(W_1) \\
& \uparrow & \uparrow \\
HF_*(W_2, a) & \longrightarrow & H_*(W_1, a),
\end{array}
\]

82
whose existence follows from Theorem 1.1.1. The upper horizontal arrow of the diagram is an isomorphism due to Lemma 5.7.2. Hence
\[ r(W_2, a) \leq r(W_1, a). \]

\[ \square \]

**Corollary 5.7.4.** Let Q be a closed manifold, and let \( g_0 \) and \( g_1 \) be two Riemannian metrics on Q such that \( g_1 \leq g_0 \), i.e. \( g_1(v, v) \leq g_0(v, v) \) for all \( v \in TQ \). Assume

\[ r(D^*Q, a) > \dim_{Z_2} H(Q, Z_2), \quad (5.7.55) \]

where \( D^*Q \) is the disk-cotangent bundle of Q with respect to \( g_0 \) and \( a \) is a positive real number that is not equal to the length of any closed \( g_0 \)-geodesic on Q. Then there exists a closed \( g_1 \)-geodesic of length less than or equal to \( a \).

**Proof.** Denote \( D^*Q \) by \( W_2 \) and the disk-cotangent bundle of Q with respect to \( g_1 \) by \( W_1 \). The condition \( g_1 \leq g_0 \) implies (see Lemma 5.7.6 below) \( W_1 \subset W_2 \).

We argue by contradiction, i.e. assume there exists no (non-constant) closed \( g_1 \)-geodesic on Q with the length less than or equal to \( a \). Then, the numbers in \( (0, a] \) are all admissible with respect to \( W_1 \). Hence

\[ r(W_1, a) = r(W_1, \varepsilon) \leq \dim HF(W_1, \varepsilon) = \dim H(W_1; Z_2) = \dim H(Q; Z_2), \]

where \( \varepsilon > 0 \) is small enough. This, together with (5.7.55), implies

\[ r(W_1, a) < r(W_2, a), \]

a relation in contradiction with Theorem 5.7.3. \( \square \)
Example 5.7.5. Let \((S^n, g_0)\) be the \(n\)-dimensional sphere with the standard Riemannian metric, and let \(\varepsilon \in (0, 2\pi)\). The visible rank \(r(D^*S^n, 2\pi + \varepsilon)\) is equal to 6 and, therefore, greater than the sum of the Betty numbers of \(S^n\). Hence any Riemannian metric \(g\) on \(S^n\) with \(g \leq g_0\) has a non-constant closed geodesic of length less than or equal to \(2\pi + \varepsilon\). Since \(\varepsilon\) is an arbitrary number in \((0, 2\pi)\), \(g\) has a non-constant closed geodesic of length less than or equal to \(2\pi\).

Lemma 5.7.6. Let \(Q\) be a manifold, and let \(g_0, g_1\) be two Riemannian metrics on \(Q\) such that \(g_1 \leq g_0\). Then,

\[
D^*Q(g_1) \subset D^*Q(g_0),
\]

where \(D^*Q(g_i), i = 0, 1\) stand for the disk cotangent bundle of \(Q\) with respect to \(g_i, i = 0, 1\).

Proof. It is enough to prove that, for every \(\alpha \in T^*Q\), the inequality

\[
|v|_0 \leq |w|_1
\]

holds, where \(|\cdot|_i\) is the norm induced by \(g_i, i = 0, 1\), and \(v, w\) are unique vectors in \(TQ\) such that

\[
g_0(v, \cdot) = \alpha = g_1(w, \cdot).
\]

Using the polarization identity

\[
2g_1(v, w) = |v|_1 + |w|_1 - |v - w|_1
\]

and the inequality \(|v|_0 \geq |v|_1\), we get

\[
|w|_1^2 - |v|_1^2 = |w|_1^2 + |v|_1^2 - |w - v|_1^2 - |v - w|_1^2 - |v|_1^2 + |w - v|_1^2 - |v|_0^2
= 2g_1(w, v) - |v|_1^2 + |w - v|_1^2 - |v|_0^2
= 2\alpha(v) - |v|_1^2 + |w - v|_1^2 - |v|_0^2
= 2|v|_0^2 - |v|_1^2 + |w - v|_1^2 - |v|_0^2
= |v|_0^2 - |v|_1^2 + |w - v|_1^2 \geq 0.
\]
**TECHNICAL LEMMAS**

**Lemma A.0.1.** Let \( r_0, \beta_0, \beta_1 \) be real numbers and let \( \ell, \alpha \) be positive real numbers. Then, the following conditions are equivalent.

- There exists a convex, strictly increasing function
  \[
  h : [r_0, r_0 + \ell] \to \mathbb{R}
  \]
  such that the function
  \[
  \mathbb{R} \to \mathbb{R} : r \mapsto \begin{cases} 
  \beta_0 & r \leq r_0, \\
  h(r) & r \in [r_0, r_0 + \ell], \\
  \alpha r + \beta_1 & r \geq r_0 + \ell
  \end{cases}
  \]
  is \( C^\infty \).

- The following inequalities hold
  \[
  \alpha r_0 < \beta_0 - \beta_1, \\
  \beta_1 - \beta_0 + \alpha r_0 + \alpha \ell > 0. \quad \text{(A.0.56)}
  \]

*Proof.* We first assume there exists such a function and prove (A.0.56). Since \( h \) is increasing, we get

\[
\beta_0 = h(r_0) < h(r_0 + \ell) = \beta_1 + \alpha r_0 + \alpha \ell,
\]

or equivalently

\[
\beta_1 - \beta_0 + \alpha r_0 + \alpha \ell > 0.
\]
The Newton-Leibniz formula, together with $h$ being convex, implies

$$\alpha r_0 + \alpha \ell + \beta_1 - \beta_0 = h(r_0 + \ell) - h(r_0)$$

$$= \int_{r_0}^{r_0 + \ell} h'(r) \, dr$$

$$< \int_{r_0}^{r_0 + \ell} \alpha \, dr$$

$$= \alpha \ell.$$

Hence (A.o.56). (The function $h'$ is continuous and not constant. Therefore the strict inequality holds.)

Now, we construct $h$ under assumption (A.o.56). Consider the famous cut-off function

$$\mathbb{R} \rightarrow \mathbb{R} : r \mapsto \begin{cases} e^{-\frac{1}{1-r^2}} & |r| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $f_2 : \mathbb{R} \rightarrow \mathbb{R}$ the function given by

$$f_2(r) := \frac{\alpha}{c\ell} f_1 \left( \frac{2r - 2r_0 - \ell}{\ell} \right),$$

where

$$c := \int_{\mathbb{R}} f_1(r) \, dr.$$  

It is bigger than 0 on $(r_0, r_0 + \ell)$ and equal to 0 elsewhere. In addition $\int_{\mathbb{R}} f_2(r) \, dr = \alpha$. Hence the function

$$f_3 : \mathbb{R} \rightarrow \mathbb{R} : r \mapsto \int_{r_0}^{r} f_2(t) \, dt$$

is equal to 0 on $(-\infty, r_0]$, strictly increasing on $(r_0, r_0 + \ell)$, and equal to $\alpha$ on $[r_0 + \ell, \infty)$.  

88
If
\[ \beta_1 - \beta_0 + r_0 \alpha + \ell \alpha = \int_{r_0}^{r_0 + \ell} f_3(r) dr =: c_1, \]
then the function
\[ h(r) := \beta_0 + \int_{r_0}^{r} f_3(t) dt \]
satisfies the conditions. Assume \( \beta_1 - \beta_0 + r_0 \alpha + \ell \alpha > c_1 \).
Let \( k_1 \geq 1 \) be a real number such that
\[ \int_{r_0}^{r_0 + \ell} f_4(r) dr = \beta_1 - \beta_0 + r_0 \alpha + \ell \alpha, \]
where
\[ f_4(r) := f_3(k_1 r - k_1 r_0 + r_0). \]
Such a number exists because
\[ g_1 : t \mapsto \int_{r_0}^{r_0 + \ell} f_3(tr - tr_0 + r_0) dr \]
is a continuous function satisfying
\[ g_1(1) = c_1 < \beta_1 - \beta_0 + r_0 \alpha + \ell \alpha, \]
\[ \lim_{t \to \infty} g(t) = \ell \alpha > \beta_1 - \beta_0 + r_0 \alpha + \ell \alpha. \]
The last inequality is due to (A.0.56). In this case, we can take \( h \) to be
\[ h(r) := \beta_0 + \int_{r_0}^{r} f_4(t) dt. \]
Assume now \( \beta_1 - \beta_0 + r_0 \alpha + \ell \alpha < c_1 \). Let \( \varepsilon \in (0, 1) \) and \( k_2 \in [1, \infty) \) be real numbers such that
\[ \int_{r_0}^{r_0 + \ell} f_3(r) f_5(r) dr = \beta_1 - \beta_0 + r_0 \alpha + \ell \alpha, \]
where
\[ f_5(r) := \varepsilon + \frac{1 - \varepsilon}{\alpha} f_3(k_2 r - k_2 r_0 - k_2 \ell + r_0 + \ell). \]

The function
\[ h(r) := \beta_0 + \int_{r_0}^{r_0 + \ell} f_3(r) f_5(r) \, dr \]
satisfies the conditions. In what follows, we show that the numbers \( \varepsilon \) and \( k_2 \) exist. Assume the contrary. Since the function
\[ g_2 : [0, 1] \times [1, \infty) \to \mathbb{R} \]
\[ (s, t) \mapsto \int_{r_0}^{r_0 + \ell} f_3(r) \left( s + \frac{1 - s}{\alpha} f_3(tr - tr_0 - t\ell + r_0 + \ell) \right) \, dr \]
is continuous, \( g_2(1, t) = c_1 > \beta_1 - \beta_0 + r_0 \alpha + \ell \alpha \), and the space \( (0, 1] \times [1, \infty) \) is connected, the inequality
\[ g_2(s, t) > \beta_1 - \beta_0 + r_0 \alpha + \ell \alpha \]
holds for all \( (s, t) \in (0, 1] \times [1, \infty) \). The function
\[ r \mapsto f_3(tr - tr_0 - t\ell + r_0 + \ell) \]
is equal to 0 on \( (-\infty, r_0 + \frac{t-1}{t} \ell) \), hence
\[ g_2(s, t) \leq sc_1 + \frac{\ell}{t} \max f_3 \left( s + \frac{\max f_3}{\alpha} \right). \]

Therefore, for \( s \) small enough and \( t \) large enough, we get
\[ g_2(s, t) < \beta_1 - \beta_0 + r_0 \alpha + \ell \alpha \]
(the number on the right-hand side is positive due to (A.0.56)). This is a contradiction and the proof is finished. \( \square \)
Lemma A.0.2. Let $r_0, \beta_0, \beta_1$ be real numbers and let $\ell, \alpha$ be positive real numbers. Then, the following conditions are equivalent.

- There exists a concave, strictly increasing function
  \[ h : [r_0, r_0 + \ell] \to \mathbb{R} \]
  such that the function
  \[ \mathbb{R} \to \mathbb{R} : r \mapsto \begin{cases} 
  \alpha r + \beta_0 & r \leq r_0, \\
  h(r) & r \in [r_0, r_0 + \ell], \\
  \beta_1 & r \geq r_0 + \ell 
  \end{cases} \]
  is $C^\infty$.

- The following inequalities hold
  \[ 0 < \beta_1 - \beta_0 - \alpha r_0 < \ell \alpha. \] (A.o.57)
  Moreover, given $\alpha_1 \in \mathbb{R}$, the inequality
  \[ h(r) > \alpha_1 r \]
  holds for all $r \in [r_0, r_0 + \ell]$ if, and only if,
  \[ \alpha r_0 + \beta_0 > r_0 \alpha_1 \quad \text{and} \quad \beta_1 > \alpha_1 (r_0 + \ell). \]

Proof. The function $h : [r_0, r_0 + \ell] \to \mathbb{R}$ is concave, strictly increasing if, and only if, the function
  \[ [-r_0 - \ell, -r_0] \to \mathbb{R} : r \mapsto -h(-r) \]
  is convex and strictly increasing. Hence Lemma A.o.1 implies the first part of the lemma. The second part follows from the concavity of the function $h$. \qed
Lemma A.0.3. Let $M$ be a closed contact manifold with contact form $\alpha$. Then, the set $S(M, \alpha)$ of all periods of periodic Reeb orbits on $(M, \alpha)$ is a closed subset of $\mathbb{R}$.

Proof. Let $d : M \times M \to [0, \infty)$ be a metric on $M$. Denote by $\sigma_t : M \to M$ the Reeb flow of $(M, \alpha)$. Consider the function

$$f : M \times \mathbb{R} \to [0, \infty) : (x, t) \mapsto d(x, \sigma_t(x)).$$

The set $f^{-1}([0])$ is a closed subset of $M \times \mathbb{R}$, because $f$ is a continuous map. Since $M$ is compact, the projection

$$\text{pr}_2 : M \times \mathbb{R} \to \mathbb{R}$$

is a closed map. Hence

$$\text{pr}_2\left(f^{-1}([0])\right) = S(M, \alpha)$$

is a closed subset of $\mathbb{R}$. \qed
BIBLIOGRAPHY


