Doctoral Thesis

Systemic Risk: From Generic Models to Food Trade Networks

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Systemic Risk: From Generic Models to Food Trade Networks

A thesis submitted to attain the degree of
DOCTOR OF SCIENCES of ETH ZURICH
(Dr. sc. ETH Zurich)

presented by
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Abstract

This thesis investigates the amplification of failures of dependent components in a network. Such cascade processes are a widely observed phenomenon and are therefore studied in diverse areas such as Physics, Finance, Epidemics, Chemistry and Electrical Engineering. Although the cascade models are as multifaceted as the fields they belong to, they follow similar patterns and share some common overarching principles.

This work is devoted to deepen the understanding of framework models that try to capture these common principles. The focus lies on three generic model classes. These classes are characterized by the load that a node distributes among its functional network neighbors in case of its failure. For each of these model classes we quantify systemic risk by an estimation of the cascade size evolution, i.e. the evolution of the fraction of failed nodes. By this we complement classical risk assessments, as we study the response of the system to a shock rather than the shock itself. Especially, we focus on cascades that start from a small fraction of initial failures, as their devastating nature (intuitively) comes as a surprise. Our modeling approach offers a causal explanation for such extreme events and allows us to study the conditions in which cascades inflict a high damage.

Part I of this thesis is mainly focused on the role of heterogeneous risk diversification strategies. We show that the common intuition that primarily failures of well connected nodes amplify a cascade process is not always correct. Indeed, we identify situations in which the risk of very large cascades can be reduced by the early failure of well connected nodes. However, we note that there is no general rule that applies to all situations. Instead, we highlight that the design and control of connected systems comes with a fundamental system’s design question: Do we prefer systems with generally low risk of large cascades but an increased chance of extremely large breakdowns? Or do we want to reduce the risk of extremely large breakdowns, however coming at the expense of a generally higher risk of large cascades? It is this question that has to be answered prior to deciding on design concepts or control strategies for highly interconnected systems.

We obtain our results with the help of Monte Carlo simulations and analytic derivations of ensemble averages for infinite networks that rely on locally tree-like structures. The latter lead us to fixed point equations that we solve numerically.

In Part II, we study several extensions. These serve also as robustness check with respect to changes in our assumptions in Part I. For instance, we study topologies that can be decomposed into two (or more) networks that are connected in a multiplex fashion. In this case, the cascade outcome becomes very sensitive to the connectivity between the two networks and exhibits sharp regime shifts. This calls for caution of the oversimplification of dependency structures. In a next step, we relax our previous restriction to infinite networks. Precisely, we derive an analytical closed form solution for the final cascade size.
distribution for two special network topologies of finite size. Surprisingly, we find broad, bi-modular, cascade size distributions. This finding questions the validity of the average fraction of failed nodes as suitable systemic risk measure.

While following a generic modeling approach for systemic risk, we successively extend a model in our framework to more realistic application settings. In Part III, we demonstrate how these insights can complement the data analysis of international trade relationships. We examine the international trade of the four major internationally traded staples: maize, rice, soy, and wheat. With the help of a cascade process, we describe the potential reorganization of a trade network in a given year as response to production and demand changes. This way, we provide an economic dependency analysis that goes beyond the identification of important trade partners and can take several risk scenarios into account.
Kurzfassung auf Deutsch


Unsere Resultate gewinnen wir durch Monte Carlo Simulationen und die Herleitung analytischer Ensemblemittelwerte für unendliche Netzwerke, welche auf lokal baumartigen Strukturen beruhen. In den analytischen Herleitungen erhalten wir Fixpunktgleichungen, die wir numerisch lösen.
Im zweiten Teil der Dissertation wenden wir uns verschiedenen Erweiterungen des ersten Teils zu. Diese dienen auch der Robustheitsanalyse unserer Ergebnisse hinsichtlich unserer Annahmen im ersten Teil. Zum Beispiel betrachten wir Netzwerke, die eine Multiplex-Struktur aufweisen, d.h., die sich in zwei (oder mehr) gekoppelte Netzwerke mit verschiedenen Kantenarten zerlegen lassen. Wir beobachten in diesem Fall, dass die mittlere Kaskadengröße sehr sensitiv auf Änderungen der Verbinung zwischen den beiden Netzwerken reagiert und plötzliche unstetige Regimeübergänge aufweist. Dieses Resultat mahnt zur Vorsicht bezüglich zu starker Vereinfachungen topologischer Netzwerkstrukturen. Im nächsten Schritt lockern wir unsere Annahmen in Hinsicht auf die unendliche Größe der bisher betrachteten Netzwerke. Wir leiten eine explizite Gleichung für die Verteilung der finalen Kaskadengröße im Falle zweier spezieller Netzwerkstrukturen beliebiger endlicher Größe her. Überraschenderweise stellen wir fest, dass die Kaskadengrößeverteilungen oftmals breite, bimodale Formen aufweisen. Diese Ergebnisse stellen die mittlere Kaskadengröße als valides systemisches Risikomass in Frage.

Chapter 1

Introduction

Cascade processes are a widely observed phenomenon. In the course of globalization and technological advancement, systems become more interconnected and system components more dependent on the functioning of others (Goldin and Vogel, 2010; Stiglitz, 2002). In particular for socio-economic networks (Schweitzer et al., 2009) and financial networks (Haldane and May, 2011), we observe an increase in coupling strength and complexity at the same time. Examples are global supply chains (Akkermans and Vos, 2003; Vachon and Klassen, 2002), but also technical systems, like power grids (Brummitt et al., 2012a; Carreras et al., 2004; Kinney et al., 2005).

The increasing connectivity in such systems has usually many advantages. For instance, resources can be allocated where they are needed and can thus be used more efficiently. Also, connectivity can provide redundancy and absorb local shocks. Very often, higher connectivity is crucial for the functionality of a system. For instance, it can be necessary to cope with increased requirements in system’s performance. Power grids might need to serve an increased electricity demand and thus have to be expanded. More food for an increasing world population needs to be provided and thus transported to its destination.

However, at the same time, highly connected systems are prone to cascading phenomena and thus carry an eventually high amount of systemic risk. Incidents like the US Northeast power grid blackout of 2003 (Dobson et al., 2007) or the financial crisis in 2007/2008 (Reinhart and Rogoff, 2008) have raised awareness for the downsides of increasing interdependency and complexity.

Particularly in economic systems, geographically local shocks carry the potential to have a global impact. This is of specific relevance for the international trade of staples. For instance, farming conditions are anticipated to face strong regional changes in the course of climate change, which in course could impact the production of staple food (Rosegrant and Cline, 2003; Schmidhuber and Tubiello, 2007). At the same time, the increase of world population is expected to boost demands. This situation clearly opens the way to
cascading effects.

1.1 Cascade processes

Whenever a complex system can be decomposed in basic similar components that depend on or interact with each other, the failure of a few components has the potential to cause further failures of dependent components. In this way, initial failures can get amplified and lead to a system wide break-down. In principle, there are two basic mechanisms that initially can lead to cascading failures. On the one hand, a component’s failure can result from intrinsic component properties. For instance, a technical component’s age or internal conflicts in case of social organizations can cause local failures. On the other hand, the failure can be a consequence of an outer shock, which some components cannot withstand (Tessone et al., 2013). The risk assessment of these (often rare) events is an important and still demanding task in risk management (McNeil et al., 2015). Here, we complement this perspective by focusing on cascading failures as the response of a system to such rare events.

Continuing cascading failures are especially problematic as result of a small number of initial failures, as their devastating outcome comes unexpected. A few component failures are far from rare in most real-world systems. So, every system needs to be able to withstand small shocks.

Consequently, it is a common goal in systemic risk analyses to quantify the risk of large cascades and to analyze under which circumstances these occur. Although the models employed in such analyses are certainly application dependent, they follow similar patterns. With a study of generic cascade models, we contribute to a deeper understanding of these patterns and identify relevant factors in the amplification of cascades.

1.1.1 Generic models

Examples for cascade models include models for such diverse phenomena as fiber bundles that break under stress (Pradhan et al., 2010), epidemic spreading (Pastor-Satorras et al., 2015), financial contagion of bank defaults (Battiston et al., 2012a; Gai and Kapadia, 2010), collective herding behavior in economic and social systems (Bikhchandani et al., 1992; Garcia et al., 2013; Granovetter, 1978; Schweitzer and Mach, 2008; Watts, 2002), and overload failures in power grids or other infrastructure (Motter and Lai, 2002). However, in case of information spreading or marketing (Watts and Dodds, 2007, and references therein), cascades usually have a positive interpretation and the goal is to maximize the outreach of a cascade (Kempe et al., 2003). Here, we focus on negative aspects by studying cascading failures and present several options for the reduction of systemic risk.
We specifically investigate three different cascade model classes which have been identified by Lorenz et al. (2009). The choice of these three classes acknowledges to some extent the fact that the intrinsic dynamics in the above outlined examples are different. At the same time, common overarching principles are discerned that allow for a comparison. Clearly, this requires significant simplifications of real processes. We study models that are reduced only to the main features of cascade processes and follow this way a generic modeling approach.

On the one side, this reduction obviously bears the risk of strong oversimplifications. On the other side, it is impossible to model all details in a complex system. Indeed, very often only basic mechanisms of the complex system are known. Still, in such situations of incomplete information, decisions influencing and shaping such complex systems need to be made. Specifically for such situations, it is important to identify similar patterns in different cascade phenomena to train the intuition needed for decision making under high uncertainty.

Our modeling approach comprises two main sources of simplifying assumptions:

**Complex networks.** Our most important simplification of a real system is introduced by its abstraction to a network. Despite the heterogeneity of system components, we assume that components can be associated with one type of nodes in a network and each possible pairwise interaction or dependence corresponds to a link between two nodes. Despite their similarity, nodes can have several features that capture their heterogeneity and characterize their role in a cascade process.

**Cascade model classes.** We distinguish between three model classes: *Constant Load* (CL), *Load Redistribution* (LRD), and *Overload Redistribution* (OLRD) models. In all classes, network nodes are endowed with exactly two intrinsic variables: robustness and fragility. The fragility corresponds to a load that a node carries, or a loss that it experiences in the course of a cascade. If this load exceeds its robustness, a node fails and either distributes a predefined load (CL), its whole load (LRD) or its overload (OLRD) to its network neighbors.

Alongside the load distribution mechanism, the overall robustness and the network topology are the main factors which determine the evolution of a cascade. The research question how these aspects intertwine and impact systemic risk, is a repeating motif in all our investigations.

### 1.2 Systemic risk reduction

Every research on systemic risk is motivated by the question how we can either design resilient systems or steer existing systems towards a state in which they are less prone to
the amplification of failure cascades.

A trivial answer to this question is to build isolated systems. This is because the existence of links between nodes imply dependencies and hence introduce the possibility for cascade propagation. Without any links, no load can be distributed and no failures amplified. Another trivial answer is to avoid initial failures that trigger cascades. If the robustness of all nodes in a network is high enough, the system is perfectly safe.

Both answers are usually impractical. On the one hand side, as we argued earlier, the existence of most links is either beneficial or even essential for the functioning of a system. Hence, building isolated systems is commonly not an option. On the other side, sufficiently high robustness of system components is associated with high costs or might even be impossible to achieve, such that answer two is not an option as well. Still, these two trivial answers point towards the direction of possible measures to reduce cascade risk: topological adjustments and increasing robustness of the system components. However, we have to be aware that both measures can affect the system performance, as we discuss later.

1.2.1 Diversification from a systemic perspective

Above we introduced topological adjustments as one possible measure to enhance system resilience. Let us note that, from a node’s perspective, topological adjustments very often influence its risk diversification. However, diversification is a two-edged sword in the context of cascade processes. On the one hand side, well diversified nodes, i.e. nodes which depend on many other nodes in the network, are often less exposed to the failure of a single network neighbor. On the other hand side, well diversified nodes are usually more vulnerable when systemic risk is high. According to classical risk management theory, high risk diversification is beneficial for each node, as well as the system that comprises these nodes (Allen and Gale, 2000). The rational is that high risk diversification reduces the failure risk of each node, and thus also of their collection.

However, this view does not consider the interdependencies of such nodes and the impact their failure has on the remaining network. Already Aristotle realized: “... the totality is not, as it were, a mere heap, but the whole is something besides the parts ...” – the whole is something different from merely the sum of the parts (Aristotle). Cascades result less from properties of individual nodes, but emerge from pairwise interactions between nodes on the micro level to systemic risk on the macro level.

Several works (Allen et al., 2011; Battiston et al., 2012a; Gai and Kapadia, 2010; Gleeson and Cahalane, 2007; Roukny et al., 2013) have pointed out that a higher connectivity can give rise to devastating cascades despite the high risk diversification. In these works, the main focus is on the diversification of exposures to other nodes. In this thesis, we
reason that, from a systemic risk perspective, especially the *diversification of damage* a node inflicts to other nodes in case of its failure, is crucial for system safety. This view reflects the recent paradigm shift towards a more systemic perspective, where instead of the failure risk of an individual node the system as a whole is of relevance. Consequently, the safety of particular nodes is not necessarily aligned with the functioning of the system anymore. For instance, a few number of highly connected nodes might face a high failure risk, but their failure is still acceptable if it cannot spread further.

Throughout this thesis, we observe a repeating trade-off concerning system connectivity: highly connected nodes face a significant exposure to cascades, but, at the same time, the diversification of the damage decreases the impact their failure on the system would have. Thus, the prevention of failures and their mitigation, i.e. the prevention of the amplification of further failures, are often alternatives rather than complements, if nodes depend mutually on each other.

### 1.2.2 The interplay between robustness and network topology

As outlined, the network topology as well as the robustness of network nodes are linked to system performance. The question remains, whether systemic risk can be reduced without affecting this performance. In our generic models, this translates to the question, whether we can reduce the risk of large cascades while keeping the overall distribution of robustness among the nodes as well as the interconnectivity fixed.

We test several options to interchange the robustness of nodes in the context of random graph ensembles. Our results pose a system design question that needs to be answered whenever systemic risk is addressed: Which risk do we perceive as problematic? Do we want to reduce the risk of large devastating cascades or do we want to avoid mediocre cascades that occur frequently?

We discuss possible answers and their consequences in several contexts.

### 1.3 Network uncertainty

We are interested in general statements about cascade processes and the identification of indicators for relatively high or low systemic risk. Because of this, we dedicate Part I, and to some extent Part II, of this thesis to the study of random graph ensembles, instead of single network instances. This means, we estimate the average systemic risk in an *ensemble* comprising all networks and robustness configurations, given certain ensemble properties. Precisely, we study an extension of the so called *configuration model*, in which nodes additionally receive a random robustness. This ensemble approach is justified in situations where rarely complete information about the precise interaction patterns between system
components is available, or when the networks change so quickly that the precise structure is unknown in a particular moment in time.

Given this setting, we derive an analytic description of the average cascade size evolution in large systems for specific models in each cascade process class. By this we provide an important methodological extension that enables researcher to deepen their understanding of the main influencing factors leading to the amplification of cascades.

1.4 Further relevant aspects in a systemic risk analysis

In Part II of this thesis, we discuss several extensions of our modeling approach introduced in Part I. At the same time, we provide a critical reflection on the limitations of our previous assumptions. In particular, we address the following three aspects.

**Multiplex.** The network model represents a strong simplification of a real world system. Different types of components or agents that interact in various ways with each other are conceptually joined as nodes and links. We study by an exemplary multiplex network how this simplification can impact our results, as we consider two different link types instead of only one.

From an alternative point of view, we assess systemic risk in two coupled networks, in which cascades can propagate back and forth. We find that excluding this second network layer would lead to a severe underestimation of systemic risk. However, especially in an interconnected world, it is often difficult to determine the boundaries of systems.

**Systemic feedback.** In our cascade models, we assume that load is distributed only locally to the nearest network neighbors. Yet, often some global information about the system state is available and components respond to it. For instance, prices are reported in an economic system, or a warning about a spreading disease is communicated to the population. Furthermore, countermeasures might be taken to prevent the further propagation of a cascade.

Including systemic feedback, we provide with our model a further step to describe systemic risk as system intrinsic. Such a model would not only incorporate the propagation of cascades but also the evolution of conditions that enable them. Furthermore, it reflects the interplay between a cascade process and the change of boundary conditions responsible for their emergence.

**Volatility of cascade sizes.** In Part I, we focus on the study of the average cascade size in infinitely large systems. However, in finite systems, this average is often not a good representation of the actual distribution despite its frequent use. Precisely, we derive closed form solutions for the cascade size distribution on specific network structures of
1.5 International trade of staple food

In Part III of this thesis, we study a real-world system that requires the consideration of all aspects investigated in Part I and Part II.

**Multiplex.** We interpret the yearly international trade of the four major internationally traded staples maize, rice, soy, and wheat as temporal weighted multiplex network. Countries are associated with network nodes. The yearly export volume from one country to another defines the weight of a directed link between these nodes. The resulting network is temporal, as we have information about its evolution from years 1992 to 2013. Its multiplex nature is reflected in the four different types of links, belonging to the trade of a specific staple.

**Systemic feedback.** Furthermore, global information in form of prices and production quantities is usually available to trading agents and influences their decisions. We interpret this as probable source of systemic feedback.

**Volatility of cascade sizes.** The goal of our study is to assess the reorganization of the international food trade network as response to a shock. In our case, this shock can be a decrease in production or an increase in demand. For these shocks, several scenarios are conceivable. On the one hand, local natural hazards, droughts, or pests can threaten the production. On the other hand, population increase or additional alternative food usages, e.g. biofuel production or animal feed, can increase the global demand for crops. As the type of shock scenario and its regional proximity crucially affect the system response, we expect a high variation in cascade outcomes.

**Uncertainty.** As is the case for all complex system models, we have to omit many factors that potentially shape it. On the one hand, this is for the sake of tractability, on the other hand, about many factors we simply lack information. In the considered case of international trade, this includes multilateral and bilateral trade agreements, tariffs, subsidies, and more publicly unavailable information. Clearly, these limitations impede the thorough understanding of all mechanisms that shape the trade network. Yet, for the purpose of systemic risk modeling, we do not need to care about all the details of the trade network formation. Instead, we focus on distortions of already observed structures. In this way, latent factors, unknown to us, are (ceteris paribus) preserved and remain implicitly reflected in the network topology. Still, we have to be aware that there is a high level of uncertainty about the precise reaction of economic actors to shocks.

While we start from a simple model of cascading export restrictions, we successively discuss
model improvements that take the variety of possible decisions into account.

**Sources of systemic risk.** Additionally to cascade risk, an interconnected system is usually exposed to several other risks whose mitigation are a major reason for the interconnectivity in the first place. For instance, the participation in global trade enables the compensation of local production shocks via imports from other parts in the world with plentiful harvest. Also, some regions in the world are not suitable for farming of certain staples and maintaining trade connections with other countries is a measure to compensate for this disadvantage. Thus, a highly interdependent market and the possible emergence of large cascades can also be interpreted as successful mitigation of local shocks. Still, the identification of arising economic interdependencies is a relevant factor in thorough risk analyses.

### 1.6 Thesis outline

The general understanding of commonalities and differences between the cascade processes in our modeling framework is subject of Part I of this thesis. We provide an estimation of the final cascade size under network uncertainty by means of simulations and an analytic branching process approximation valid in large systems.

Part II is devoted to a critical reflection on our assumptions in Part I and asks to which degree our findings are robust with respect to changes in those assumptions. For this purpose, we study several extensions of the models presented in Part I.

In Part III, we exemplify how our insights can inspire systemic risk modeling when more information about a system is provided by data. Precisely, our example refers to the international trade of staple foods. Starting from our generic modeling framework, we show how complexity can successively be added to our cascade model to describe the reorganization of the international trade of staples as response to shocks.
Part I

Modeling systemic risk under network uncertainty

“Il n’est pas certain que tout soit incertain. (Translation: It is not certain that everything is uncertain.)”

Blaise Pascal, Pascal’s Pensees
Chapter 2

Modeling framework

Summary
We introduce our cascade modeling framework and explain its conceptual foundation. We model systems comprised of many similar but heterogeneous agents as network and study the amplification of node failures as response to an initial shock. In extensive simulation studies, we estimate systemic risk as network ensemble average of the evolving cascade size and explore the ramifications of our assumptions. In an overview, we highlight similarities and differences between the studied model classes. In particular, we focus on the interplay between network interconnectivity, different risk diversification strategies of nodes, and the distribution of their robustness. In a next step, we prepare the analytic derivations of the studied ensemble averages to explain our findings.

Based on R. Burkholz, A. Garas, D. Längle, and F. Schweitzer, “Systemic risk from cascading processes in random and scale free networks”. Working Paper. RB contributed to the design of the research questions. RB wrote main parts of the simulation code. RB analyzed and interpreted the simulation results. RB wrote the text in this thesis. RB added the model and framework introduction specifically for this thesis. Some parts are also based on R. Burkholz, A. Garas, and F. Schweitzer, “How damage diversification can reduce systemic risk”. Physical Review E 93, 042313.
Chapter 2. Modeling framework

2.1 Introduction

As outlined in the introduction of this thesis, cascade processes are a considerable source of systemic risk in highly interconnected complex systems. Despite the differences of the observed phenomena, they share some common overarching principles. We study models that are reduced to such basic principles and focus on general cascade properties that are intrinsic to a larger class of models. In this sense, we follow a generic modeling approach. This has advantages and disadvantages.

On the one side, the simplification supports the understanding of cascade dynamics by preventing side tracks originating in complicated model details. Moreover, it fosters computational tractability. The approach is in line with the principle of parsimony, also known as Occam’s razor (Gauch, 2003). This formulates a preference for simpler models, as they are often better falsifiable.

On the other side, most of the time it is reasonable to assume that reality is complicated and that many factors possibly influence the studied system. An oversimplification might lead to wrong conclusions and diminish the value of the modeling approach. Still, we argue that simple models serve as good starting point for successive refinement to match better application requirements. A grounded understanding of simple models can then guide a modeler on the way of adding complex behavior to the models. In Part III of this thesis, we demonstrate this procedure by the example of international trade relationships.

In fact, the models in the studied framework are so flexible designed that many more complicated models could be reduced to them by the right choice of variables. But this reduction is often far from trivial and might not serve the modeling purpose. Still, our theoretical investigations can assist the formulation of hypotheses and the general system design.

The modeling of cascade processes requires certain abstractions from the real world application that comprise essentially three elements: (1) the identification of system components and their dependence structure, (2) the conditions under which a failure occurs, and (3) the interaction between system components once a failure occurs.

We model the dependence structure (1) by a network, a concept that is introduced next. The other two elements characterize the cascade processes and are described in the introduction of the cascade model classes.

2.2 Network model

We associate a system’s components with nodes and their interaction or dependence patterns with links in a network $G = (V, E)$ (Newman, 2010). In the course of the thesis,
we interchangeably use the terms graph instead of network, vertex instead of node, and edge instead of link as in mathematical graph theory. While \( V \) denotes the node set, \( E \) contains links of the form \((i, j)\) with \( i, j \in V \). Such a link connects node \( i \) with node \( j \). Both nodes are called adjacent to each other and adjacent to the link \((i, j)\). Moreover, we name \( i \) and \( j \) (network) neighbors, if they are connected via a link. The number of all neighbors of a node \( i \) is given by its degree \( k_i \). We refer to nodes with a high degree as hubs and nodes with a low degree as leaves. This is not a precise definition, but eases qualitative discussions. The number of nodes in the network is denoted by \( N := |V| \).

If we call \( G \) directed, we interpret a link \((i, j)\) as action from \( i \) towards \( j \), for instance, the failure of node \( i \) would impact \( j \) (but not necessarily vice versa). We refer to the number of links that point to a node \( i \) as its in-degree \( k_i^{\text{in}} \), and to the number of links that start in a node \( i \) as its out-degree \( k_i^{\text{out}} \). The degree \( k_i \) is the sum of its in- and out-degree: \( k_i = k_i^{\text{in}} + k_i^{\text{out}} \).

In case that \((i, j) \in E\) implies already \((j, i) \in E\), we have a mutual dependence of connected nodes and call \( G \) undirected. Sometimes we assume that the links are additionally weighted by weights \( w_{ij} \in E_W \) collected in a weight set \( E_W \). The weight \( w_{ij} \in E_W \) can indicate a direction from node \( i \) to node \( j \) as well as the existence of a link by \( w_{ij} \neq 0 \).

**Figure 2.1:** Example. Graphical representation of an undirected network, an undirected network interpreted as directed network, and a directed network. Nodes are depicted by circles, while undirected links are drawn as lines between nodes. A directed link \((i, j)\) is represented by an arrow that points from the first node \( i \) to the second node \( j \) of the link. The degree \( k_1 \) of node 1 is \( k_1 = 2 \) in all cases and its out-degree is \( k_1^{\text{out}} = 2 \) as well in the right plot.

Modeling a system by a network is possible as long as the system can be disassembled in basic and similar components and the studied interactions between these components can be split in dyadic interactions. Instead of modeling the dynamics of representative agents, we consider many similar (although heterogeneous) system components whose dynamics are determined by their interactions. Also these interactions follow the same or similar laws. This way, we follow a complex system approach.
2.3 Cascade model classes

Having abstracted from system components such as financial institutions, humans in a society, or power stations in an electrical grid to nodes in a network, we need to formalize what their failure means and specify how other components are affected by this.

Especially the latter can vary tremendously from system to system. A generic modeling approach needs to account for these differences to a certain degree. Lorenz et al. (2009) have identified three main classes of cascade processes that we generalize for the application to weighted networks. It is a main goal of this thesis to deepen the understanding of their commonalities and differences.

They all have in common that, on the micro-level, every node \( i \) in a network \( G \) is in one of two states indicated by its state variable \( s_i \): It is functional \((s_i = 0)\) or failed \((s_i = 1)\). A possible switch of \( s_i \) from functional to failed is triggered either initially or in a cascade process that evolves in discrete time steps \( t = 0, \ldots, T \). We assume in the first and second part of this thesis that a node cannot recover. Thus, once it has failed, it stays failed till the end \( T \) of the process. In a finite system (consisting of \( N \) nodes) a cascade ends after \( T = N - 1 \) steps the latest, since in each time step at least one node needs to fail. Otherwise the cascade stops. Nevertheless, it is possible that more than one node fails in each time step.

The state \( s_i \) is determined by two internal node variables: the load \( \lambda_i \) that a node \( i \) carries and the node’s threshold \( \theta_i \) that characterizes its robustness. By Lorenz et al. (2009), the variable \( \lambda_i \) is called fragility and is denoted by \( \phi_i \). Here however, we use the notation \( \lambda_i \) instead to strengthen the association with load or loss that a node experiences. When \( \lambda_i \) exceeds the threshold \( \theta_i \), i.e., \((\lambda_i \geq \theta_i)\), node \( i \) fails. This is equivalent to the net fragility \( z_i := \lambda_i - \theta_i \) exceeding zero. In short, we have \( s_i(t) = H(\lambda_i(t) - \theta_i) = H(z_i(t)) \), where \( H(\cdot) \) denotes the Heaviside function. We assume that only the load \( \lambda_i \) can increase over time, while each threshold stays constant after its initial definition, if not explicitly stated otherwise. For simplicity, each node \( i \) receives the same initial load \( \lambda_i(t = 0) = \lambda_0 \). Thus, all nodes with thresholds smaller than \( \lambda_0 \) fail in the beginning: \( s_i(0) = H(\lambda_0 - \theta_i) \). Each failing node \( i \) distributes a total load \( l_i(t) \) to its remaining functional network neighbors.

It is important to note that this load \( l_i \) can differ from the load \( \lambda_i \) that a node carries and needs to be defined. How the load \( l_i \) is distributed among the neighbors is determined by the considered model. We consider only Markov processes, i.e., the dynamic variables \( s_i(t + 1), \lambda_i(t + 1), l_i(t + 1) \) of all nodes \( i = 1, \ldots, N \) depend only on the variables at time \( t \). The previous history is irrelevant for the determination of \( t + 1 \).

Let’s assume node \( i \) fails at time \( t \) and distributes the load \( l_{ij}(t) \in [0, l_i(t)] \) to its neighbor \( j \). We therefore have \( \sum_{j=1}^{N} l_{ij}(t) = l_i(t) \). \((l_{ij} = 0 \text{ indicates that the link does not exist or that no load is distributed along this link.})\) Thus, the load \( \lambda_j \) that each neighbor carries
increases by \( l_{ij}(t) \). We often call \( l_{ij}(t) \) damage or impact of node \( i \) on node \( j \). Assuming that the loads stay at \( l_{ij}(t_+) = l_{ij}(t) \) for all later times \( t_+ > t \), we can write

\[
\lambda_j(t+1) = \sum_{i=1}^{N} l_{ij}(t)s_i(t) + \lambda_0
\]

as sum over all loads received by failing neighbors. Only nodes that have failed in the previous time steps distribute load to a node \( j \). Nodes that fail at the same time do not distribute load to each other.

If a load increase causes further failures, the failing nodes distribute load again and the cascade process keeps ongoing until no further thresholds are exceeded.

The final fraction of failed nodes, i.e. the final cascade size, serves then as our measure of systemic risk:

\[
\varrho_N(T) = \frac{1}{N} \sum_{i=1}^{N} s_i(T).
\]

If the time dependence is not explicitly stated, we denote the final cascade size by an abbreviation (\( \varrho_N = \varrho_N(T) \)).

We compare this quantity for three main load distribution mechanisms that characterize the load that neighbors distribute \( l_{ij}(t) \): Constant Load (CL), Load Redistribution (LRD) and Overload Redistribution (OLRD). In the course of their introduction, we state small examples provided by Lorenz et al. (2009) in the set-up shown in Fig. 2.2.

**Figure 2.2:** General set-up for the visualization of examples to explain several load distribution mechanisms. The figure is a slight modification of Fig. 1 by Lorenz et al. (2009).

### 2.3.1 Constant Load models

Constant Load (CL) cascade models have the property that the loads are predefined and stay constant in the course of a cascade process (\( l_{ij}(t) = l_{ij}(0) \) for all times \( t \)). Conse-
quently, the loads can also be associated with weights $l_{ij}(t) = w_{ij} \in E_W$ in a weighted network. This simplifies the analysis in comparison to the other load redistribution classes. Even in practice, such cascades are usually easier to control, because it is straightforward to understand what happens next, if all information is available.

Also the time complexity is manageable, since the order of updated failures does not influence the final cascade size. We have defined the time in which a cascade evolves in such a way that several nodes can fail synchronously. But, in this case, it would be sufficient to consider a random order of updated failures for estimating the final cascade size.

Interestingly, many models for cascade estimation problems that are known to be difficult belong to the CL class. The difficulty usually originates from the high uncertainty regarding the precise dynamics and, most importantly, the network structure. The strong simplification is often an attempt to gain a qualitative idea what can happen (in rather short time), while more complicated modeling approaches can hardly improve accuracy because of the high uncertainty.

For example, the SI model in epidemic spreading (Pastor-Satorras et al., 2015) can be mapped to our setting and belongs to the CL class. For instance, travel patterns of people are often unknown and their form of contact unclear. In such a situation, SI-type models prove quite useful (Brockmann and Helbing, 2013). Note that the SI-like models, despite or because of their simplicity have been used to model various contagious processes, like disease spreading (Pastor-Satorras and Vespignani, 2002), innovation spreading (Jackson and Rogers, 2007), crisis spreading (Garas and Argyrakis, 2010), even herding behavior in donations (Schweitzer and Mach, 2008). A form of opinion formation is described by the voter model (Schweitzer and Behera, 2009) that also belongs to the CL class.

Financial contagion of bank defaults (Amini et al., 2010; Battiston et al., 2012b; Gai and Kapadia, 2010) is another example where CL models find application. Here, contracts between financial institutions with a certain maturity predefine (in principle) the amount of loss that needs to be faced in case of a bank default. Still, the determination of the weights and thresholds is far from trivial. Often the data incomplete and like the insolvency proceedings very complicated. Often, $w_{ij}$ would not correspond to a loss, but to a loss relative to a node’s equity in a financial setting. For an in-depth explanation how to link this model to a balance sheet approach, we refer the interested reader to work by Battiston et al. (2012b) and Amini et al. (2010). Here, we just borrow the intuition for cascading losses in a network.

We emphasize that the simplification in our framework has the consequence that we cannot capture all models in their full complexity. For instance, many aspects of the sandpile model (Goh et al., 2003) belong to the CL class, but we cannot describe the creation or deletion of grains. We restrict ourselves to the study of closed systems.
In (Lorenz et al., 2009), two variants of CL models are proposed and called the CL Inward and Outward variant. We study both in more detail on undirected networks, but refer to them as Exposure and Damage Diversification models to facilitate their recognizability and underline the purpose of our study: We explore the consequences of simple diversification strategies on systemic risk.

The first variant, the Exposure Diversification (ED), has first been introduced by Granovetter (1978) to model the formation of riots and then transferred to networks by Watts (2002) to analyze the conditions under which global cascades can emerge, e.g., in the spreading of rumor or opinion formation. Also the k-coreness analysis is based on this model for a special choice of thresholds and serves, for instance, the resilience analysis of social online communities (Garcia et al., 2013).

The load distribution weights and the load that a node carries are defined as:

\[ w_{ji}^{(ED)} = \frac{1}{k_i}; \quad \lambda_{i}^{(ED)}(t + 1) = \frac{1}{k_i} \sum_{j=1}^{N} s_j(t) = \frac{n_i(t)}{k_i}, \]

where \( w_{ji}^{(ED)} = 0 \), if there exists no link between \( i \) and \( j \). \( n_i(t) \) is defined as the number of failed neighbors of \( i \) at time \( t \).

Each node \( i \) is exposed to the possible total loss \( \lambda_i = 1 \), while each single neighbor’s failure inflicts the loss \( 1/k_i \) to node \( i \). Thus, each node’s exposures are diversified and a node experiences as loss which is precisely the fraction of its failed neighbors. The higher the degree \( k_i \) of a node, the better it is diversified. Whether a higher diversification reduces in fact the failure risk of a node is discussed in Chap. 3.

The size of the loss that a node experiences solely depends on properties of the loss receiving node, i.e., its degree, while the total damage that a node inflicts depends on the neighbors’ degrees. In general, the higher the degree of a node, the higher is the damage that its failure is expected to cause, since it affects a higher number of nodes. From a systemic risk perspective, this effect is only problematic, if nodes with a high degree also have a significant failure probability.

In contrast, the Damage Diversification (DD) model mitigates the potential risk amplification by hubs, while preserving the total size of exposures \( \sum_{i=1}^{N} \frac{1}{k_i} \). Instead of diversifying a total exposure of \( \lambda_i = 1 \), the total damage \( l_j \) that a node \( j \) can inflict is limited to \( l_j = 1 \) and distributed equally among the network neighbors. We have:

\[ w_{ji}^{(DD)} = \frac{1}{k_j}; \quad \lambda_{j}^{(DD)}(t + 1) = \sum_{j=1}^{N} \frac{1}{k_j} s_j(t), \]

where again \( w_{ji}^{(DD)} = 0 \), if there exists no link between \( i \) and \( j \). The insight that every link
is counted once by enumerating either all starting nodes or all end nodes lets us verify that both load distribution mechanisms, ED and DD, lead to the same total exposure $E_{tot}$:

$$E_{tot}^{(ED)} = \sum_{i=1}^{N} w_{ji}^{(ED)} = \sum_{i=1}^{N} \frac{1}{k_i} = \sum_{j=1}^{N} w_{ji}^{(DD)} = E_{tot}^{(DD)}.$$ 

In case of DD, failing hubs harm each neighbor only little, even if they affect a high number of nodes in the network. But they usually face also a high failure risk, since every additional neighbor means another potential loss.

Fig. 2.3 illustrates the dynamics of both diversification variants, ED and DD, in a small example taken from Fig. 2 in (Lorenz et al., 2009).

We explore the two models and their impact of the risk diversification strategies on systemic risk in Sec. 2.5.1 via simulations and continue their analysis in Chap. 3-4 as well as in Part II.

### 2.3.2 Load Redistribution models

Load Redistribution (LRD) models add complexity to the dynamics, as the load that a node distributes is now time dependent. For this class of models, the total distributed load $l_i(t)$ coincides with the load $\lambda_i(t)$ that a node carries itself at failure, i.e. $l_i(t) = \lambda_i(t)$.

Now, the order of failures could change the outcome of the cascade evolution. We point out again that several nodes can fail at the same time and load is only distributed to nodes that have not failed yet. Nodes that fail at the same time, do not distribute load to each other.

For instance, models for simplified cascades in power grids (Kinney et al., 2005) fit this framework. Similar load distribution mechanisms can lead to cascades of forest fires (Drossel and Schwabl, 1992) and the same models have been employed to describe earthquakes (Jagla, 2013). One of the most prominent members of the LRD class is the fiber bundle model (Kun et al., 2006, 2000; Pradhan et al., 2010), where a force is applied to a bundle of fibers that is translated into a total load which is shared among the fibers. The threshold of a fiber defines then the amount of load at which it breaks. In case of global load sharing, a failing fiber distributes its load to all intact fibers. This corresponds, in our setting, to a fully connected network topology. We revisit this case in Chap. 9 for finite networks.

The local load sharing rule refers to other network topologies and the load $\lambda_i(t)$ of a failing node (or fiber) is only distributed to its functional network neighbors. If the network structure itself does not change over time it can happen that load is lost. A failing node without functional neighbors cannot distribute its load. This model has been studied by
Figure 2.3: Left Column: Example for a cascade with CL ED load distribution (left) and CL DD load distribution (right) mechanism starting from the set-up in Fig. 2.2. The initial load is set to $\lambda_0 = 0$. The figure is a slight modification of Fig. 2 by Lorenz et al. (2009).

Right Column: Example for a cascade with OLRD LLSC load distribution (left) and OLRD LLSS load distribution (right) mechanism starting from the set-up in Fig. 2.2. The initial load is set to $\lambda_0 = 1$. The figure is a slight modification of Fig. 4 by Lorenz et al. (2009).

Moreno et al. (2002) and is called Local Load Sharing with load Shedding (LLSS) variant by Lorenz et al. (2009).

Formally, it can be mapped to our framework by the choice:

$$l_{ji}^{(LLSS)}(t) = \frac{\lambda_j(t)}{k_j - n_j(t)} \quad \text{for all } t > t_j; \quad \lambda_i^{(LLSS)}(t+1) = \sum_{j=1}^N \frac{\lambda_j(t)}{k_j - n_j(t)} s_j(t),$$

where $l_{ji}^{(LLSS)} = 0$, if there exists no link between $i$ and $j$ or $j$ has not failed yet. $t_j$ denotes the failure time of a node $j$, i.e. $s_j(t_j) = 1$, while $s_j(t_j - 1) = 0$ (if $t_j > 0$). $n_j(t)$ denotes again the number of failed neighbors of node $j$ at time $t$. 
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Apparently, the load does not need to be equally distributed among the surviving nodes. Also general weights of the form \( l_{ji}(t_{+}) = \alpha_{ji}(t)\lambda_{j}(t_{j}) \) with \( \sum_{i} \alpha_{ji}(t) = 1 \) and \( \alpha_{ji}(t) \in [0, 1] \) would fit our framework.

With the picture of fiber bundles in mind, it might be reasonable to assume that no load can be shed, because the applied outer force is not reduced. Instead, at least the closest nodes in the network would need to compensate for a lost fiber. Because of this, the Local Load Sharing with Load Conservation (LLSC) model variant has been investigated (Kim et al., 2005). We relinquish its formal introduction, as we investigate it only in simulations and we feel, the formulas would not improve the process understanding. It can be interpreted in a way that nodes fail, but their links are still intact load is still distributed along them. In an equivalent model formulation, the network structure changes with the evolution of the cascade process. In case of the failure of a node, the remaining functional network neighbors form a fully-connected clique, i.e., each functional network neighbor of the failed node establishes a link to each other functional network neighbor of the failed node. The load distribution follows the same law as in case of LLSS, but on the evolving network.

The model’s complexity makes it at this point impossible to study analytically. Especially for the reason that the model has the interesting feature of increasing interconnectivity that can enhance but also reduce systemic risk.

An example for the two introduced LRD models is provided by Fig. 2.4.

2.3.3 Overload Redistribution models

Overload redistribution models are similar to their LRD counterpart. The only difference is that the overload \( \lambda_{i}(t) - \theta_{i} \) of a failing node \( i \) is distributed instead of the full load \( \lambda_{i}(t) \). Thus, failed nodes are still capable of holding a certain amount of load that equals, for instance, their capacity.

Consequently, OLRD models are often designed to describe traffic redirection that is a result of congestion. Examples include the Internet where the capacity of routers to transmit information (per unit time) is exceeded (Motter, 2004; Motter and Lai, 2002) or power-grids where the performance of power generators degrades because of an overload (Kinney et al., 2005). Another interesting example is the Eisenberg-Noe model (Eisenberg and Noe, 2001) which describes firms connected through a network of liabilities.

In this thesis, we focus on the LLSS and LLSC variants as for LRD models. The LLSS weights can be similarly formulated as:

\[
l_{ji}^{(LLSS)}(t_{+}) = \frac{\lambda_{j}(t_{j}) - \theta_{j}}{k_{j} - n_{j}(t_{j})} \quad \text{for all } t_{+} > t_{j} ; \quad \lambda_{i}^{(LLSS)}(t + 1) = \sum_{j=1}^{N} \frac{\lambda_{j}(t_{j}) - \theta_{j}}{k_{j} - n_{j}(t_{j})} s_{j}(t),
\]
2.4 Ensembles of initial conditions

The introduced cascade processes are completely deterministic for given network $G$, thresholds $\theta_i$ and initial load $\lambda_0$ and also determined by these quantities. Since Part I of this thesis is devoted to the understanding of the commonalities and differences between the presented models, we need to specify the experimental set-up or benchmark of our comparison. As the number of different network, threshold, and initial load configurations is even uncountable, we cannot study every possible case. Another option would be to restrict ourselves to a configuration that is provided by a dataset. But the creation of this dataset might be based on a process that suggests that the application of one of our studied models might fit better than the others. Thus, we might not obtain reasonable

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Figure 2.4: Example for a cascade with LRD LLSC load distribution (left) and LRD LLSS load distribution (right) mechanism starting from the set-up in Fig. 2.2. The initial load is set to $\lambda_0 = 1$. The figure is a slight modification of Fig. 3 by Lorenz et al. (2009).
results for the other models, as we do not test them in their ‘natural environment’, and the comparison would be biased. This is a pitfall that we cannot avoid completely. Neither can we ensure that our findings are generalizable to other network structures and our interpretations might match only the specific configuration at hand. Moreover, we cannot clearly attribute the observed effects to their cause, since too many options would be reasonable candidates.

Because of this, we do not study single entities. Instead, we calculate the average cascade size on an ensemble of different configurations. Evidently, the choice of this ensemble and the probabilistic weight of an instance (i.e. the probability of their occurrence), imply a restriction as well, and so does the focus on an average quantity. We discuss this issue critically in Sec. 2.4.2 and Chap. 9. Also with a focus on ensembles, we cannot guarantee that each realistic network structure for our models is represented in equal proportions or has the same probabilistic weight. At least, we increase the probability that we also consider situations in which an application of the studied models is reasonable.

This approach might also be justified in situations where the underlying network structure is uncertain or changes so quickly (but on a slower time scale then the cascade process) that it is unknown which network determines the systemic risk at a certain point in time.

Before we turn our attention to the introduction of our random graph and threshold model, we insert a small interim section about our understanding and notation of randomness.

### 2.4.1 Notation of randomness

We employ basic concepts of probability theory (Feller, 1968). Thus, a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) underlies most of our investigations, even though we do not mention it explicitly.

It is always assumed that the studied random variables are measurable with respect to this probability space. We denote a random variable \(X : \Omega \to \mathbb{R}\) with capital letters and its realization by a lowercase letter \(x\). The only exception is given by the state variable \(s_i, \ s\) (or \(s_{nb}\)). This is always written as lowercase letter.

To indicate that a probability distribution (pdf) (or its density in case of a continuous random variable) corresponds to a random variable \(X\), we often denote it by \(p_X(x) := \mathbb{P}(X = x)\), if \(X\) is discrete. If \(X\) is a continuous random variable, \(p_X(x)\) stands for its probability density. The same applies to its cumulative distribution function for that we reserve the capital letter \(F\). Thus, we usually define: \(F_X(x) = \mathbb{P}(X \leq x)\). Only in case of the degree \(K\) of a node, we omit the subscript. We also write \(X \sim p_X\) or \(X \sim F_X\) to indicate that \(X\) has pdf or density \(p_X\) or cdf \(F_X\). If two random variables \(X\) and \(Y\) are identically distributed, i.e., their distribution functions are identical, we also write \(X \sim Y\).

The conditional probability distribution \(\mathbb{P}(X = x \mid Y = y)\) or conditional density \(p_{X\mid Y=y}(x)\)
of \( X \) given \( Y \) are defined as usual (Feller, 1968). For \( \mathbb{P}(Y = y) > 0 \) we can write:

\[
\mathbb{P}(X = x \mid Y = y) = \frac{\mathbb{P}(\{X = x\} \cap \{Y = y\})}{\mathbb{P}(Y = y)}.
\]

Intuitively, it expresses the probability to observe the event \( X = x \) with the knowledge that the event \( Y = y \) has occurred already.

For instance, drawing nodes uniformly at random from a network exemplifies the concept. Let \( K \) denote the degree of a randomly picked node and \( \Theta \) its threshold. The probability \( \mathbb{P}(\Theta \leq \theta \mid K = k) \) is the probability that a node with degree \( k \) has a threshold smaller or equal to \( \theta \). At this point, we do not draw nodes from the whole network independently at random. Instead we focus on all nodes with degree \( K = k \). Thus, \( \mathbb{P}(\Theta \leq \theta \mid K = k) \) is the probability to find nodes with a threshold of at least \( \theta \) among all nodes with degree \( k \). We sometimes denote with \( X(y) \) a random variable with distribution \( \mathbb{P}(X = x \mid Y = y) \).

Bayes’ Theorem (here formulated for discrete random variables) follows immediately from the concept of conditional probabilities:

\[
\mathbb{P}(X = x) = \sum_y \mathbb{P}(X = x \mid Y = y) \mathbb{P}(Y = y),
\]

where the sum runs over all possible values of \( Y \) with nonzero probability mass, i.e. \( \mathbb{P}(Y = y) > 0 \). Alternatively, for continuous random variables with (Radon-Nikodym) probability densities we have:

\[
p_X(x) = \int p_{X \mid Y = y}(x)p_Y(y) \, dy = \int p_{X,Y}(x,y) \, dy,
\]

where \( p_{X,Y}(x,y) \) is the joint distribution of \( X \) and \( Y \).

Additional to the conditional probability \( \mathbb{P}(X = x \mid Y = y) \), we regard the probability

\[
\mathbb{P}(X = x; Y = y) = \mathbb{P}(\{X = x\} \cap \{Y = y\})
\]

that two events occur together. For instance, in our node drawing example, \( \mathbb{P}(\Theta \leq \theta; K = k) \) is the probability to receive a node with degree \( k \) and threshold at least of the size of \( \theta \) while drawing at random from all nodes in the network.

In Part I and Part II of this thesis, we assume that the network \( G \), the degree \( K_i \) and threshold \( \Theta_i \) of each node \( i \) in this network are random variables that are realized in the beginning of a cascade process and stay constant in the course of the cascade. As these quantities stay constant, we consider \textit{quenched} and not \textit{annealed} structures. Usually, the random variables \( K_1, \cdots, K_n \) are assumed to be independent and identically distributed (iid). This means, they are drawn independently at random and follow the same law (i.e.
they have the same pdf) $p(\cdot)$ as the one of a generic random variable $K$. So, we have $K_i \sim p$ for all nodes $i \in \{1, \ldots, N\}$ as well as $K \sim p$. Also the random variables $\Theta_1, \ldots, \Theta_n$ are not necessarily identically distributed or independent, but at least originate from an order statistics of independent and identically distributed random variables. Usually, we assume that the thresholds are independent, but not necessarily identically distributed.

In this setting, we calculate the fraction of failed nodes as an average with respect to the random graph, the random degrees and the random thresholds.

### 2.4.2 Random graph ensembles: A configuration model approach

It is a paradigm of classical risk management theory that a higher risk diversification reduces the failure risk of the individual, i.e. a node in a network in our case (Allen and Gale, 2000). The question whether this paradigm holds true also with respect to systemic risk where a higher risk diversification also increases the dependence between nodes guides our analysis in Part I and Part II of this thesis.

With our study of a random graph ensemble, we aim at understanding the role of heterogeneous risk diversification strategies of nodes in a network. The diversification strategies in the introduced models are determined by the degrees of a node. The simplest possible assumption, which does not require any further knowledge about the distribution of random graphs, is that nodes with a prescribed diversification strategy, i.e. a prescribed degree, connect randomly with each other without any preferences.

This describes the set-up of the configuration model that defines a random graph ensemble where a graph is drawn uniformly at random from all possible configurations (Molloy and Reed, 1995; Newman, 2010; Newman et al., 2001). It can be understood as entropy maximizing distribution over all graphs with the side constraint of a given degree sequence. The model starts from such a given degree sequence $k_1, \ldots, k_N$ where each degree is assigned to a node in the network that is to be constructed. These degrees can be represented as stubs like in Fig. 2.5. Then, these stubs are paired uniformly at random so that each pair of stubs forms a link in the constructed graph.

This procedure excludes neither the formation of multiple edges (between two nodes) nor self-loops (that connect two stubs of the same node). We would like to avoid that a network has these properties, because they are not the result of a reasonable risk diversification. In finite networks, not accepting such a configuration results in a small bias, since networks are not completely drawn uniformly at random from all possible configurations. This problem can be overcome as proposed by Catanzaro et al. (2005). But the bias also decreases for increasing network size $N$, as the probability for the occurrence of multiple edges and self-loops converges to zero for $N \to \infty$. As our considered networks are usually large enough ($N = 10^5$), we do not employ the more complicated algorithm. (Instead we
From a theoretical perspective, we assume that, at the beginning of a network construction, each degree $k_i$ of a node $i$ is drawn independently at random from a degree distribution $p(k)$, i.e. $K_i \sim p$. This way, we obtain a degree sequence that is the starting point for drawing a network from the configuration model ensemble. This approach introduces some additional noise, because we might compare networks with differing degree sequences. But the difference between this and the original configuration model with prescribed degree distribution is minor for large networks (Newman, 2010). Both converge to the same configuration for increasing network size $N$, if the degree sequence follows the degree distribution. In fact, in our simulations, we even fix the degree sequence as depicted in Fig. 2.6(b).

However, in our analytic numerical approximations of the final cascade size in the next chapters, we usually regard degree distributions as shown in Fig. 2.6(a). We also did calculations where we used degree distributions as input that are defined by the degree sequences from the simulations. The differences of the results are negligible and far smaller than our numerical approximation errors (that are introduced by discretizations and numerical approximations of the Fourier transformation). Consequently, we do not pay attention to this technicality anymore in the rest of this thesis.

We analyze systemic risk for two degree distributions that are of interest for our diversification analysis. The first, $p_P(k)$, defines Poisson random graphs, while the latter, $p_S(k)$,
Chapter 2. Modeling framework

Figure 2.6: (a) The studied degree distributions: Poisson distribution with parameter $\lambda = 2.82$ and cutoff degree $c = 50$ (black), scale free distribution with exponent $\gamma = 3$ and maximal degree $c = 200$ and average degree $z = 3$ (purple) in log-log scale. (b) The corresponding degree sequences of simulated networks with size $N = 10^5$ nodes in the same colors as in (a).

It corresponds to scale free networks:

$$p_P(k) := \frac{1}{S_P} \frac{\lambda^k}{k!}, \quad p_S(k) := \frac{1}{S_S} \frac{1}{k^\gamma}$$

for $k \in \{1, \cdots, c\}$ with normalizing constants

$$S_P := \sum_{k=1}^c \frac{\lambda^k}{k!} \quad \text{and} \quad S_S := \sum_{k=1}^c \frac{1}{k^\gamma},$$

where we adjust $p_S(1)$ (and $S_S$) to set the average degree $z$ of $p_S$ to a specific value. Specifically, we set it to $z = 3$. The Poisson random graphs are of interest as limit of the well studied Erdős-Rényi random graphs (Erdős and Rényi, 1959). They constitute a good starting point for an analysis, as they do not require big samples in simulations, and serve thus as good benchmark for methodological comparisons.

However, many real world systems are expected to be of scale free nature (Barabási, 2009; Boss et al., 2004; Cont et al., 2013). They provide an interesting paradigm for many real networks that have broad degree distributions (Clauset et al., 2009). Thus, a considerable fraction of nodes has a higher degree than others. These hubs are the well diversified nodes which makes them especially interesting for our diversification analysis.

It is known from percolation studies that scale free networks are robust-yet-fragile. This means, they are robust against random shocks as the removal of random nodes as the biggest proportion of the network stays connected. But they are vulnerable against targeted attacks (Cohen and Havlin, 2010). We examine in Part I, if this intuition also applies to our cascade processes.
Additional to the two introduced degree distributions, we discuss regular networks where each node has the same degree $k_i = z$ and a fully connected topology. All our models have already been studied for fully connected networks in the limit of infinite network size ($N \to \infty$), which is also known as thermodynamic limit in Statistical Physics (Lorenz et al., 2009). The CL models have also been investigated on regular networks (Gleeson and Cahalane, 2007; Lorenz et al., 2009). Meanwhile, the LRD models on regular lattices (which are restricted to more structure than regular configuration model random graphs) have caught the interest of several researchers (Pradhan et al., 2010; Tessone et al., 2013), although in combination with different threshold distributions than in our study. Still, we present them to compare heterogeneous risk diversification strategies with homogeneous ones on regular lattices. Especially, the case of full diversification in fully connected networks serves as benchmark in our analysis.

Please note that all degree distributions (except the fully connected one) have average degree $z = 3$. Thus, we complement studies of the effect of the general connectivity on systemic risk by an investigation of the degree heterogeneity.

Limitations of our configuration model approach

We study configuration model random graphs only for large networks. Our analytic approach is correct in the limit $N \to \infty$ and the fraction of failed nodes converges (in probability) to the calculated mean value. This indicates already that we do not consider a real ensemble of different graphs. In fact, the topology of a large configuration model graph is well determined and we can interchangeably talk about a fraction of nodes in the network with a certain property or the probability that a given node has this property. We revisit the issue of finite size effect again in Chap. 9.

The most critical assumption is thus that nodes connect without any preference, which leads to uncorrelated networks. This means the degree $K_i$ of a node and the degree $K_j$ of one of its neighbors are uncorrelated. However, most real world networks show degree-degree correlations.

Moreover, it is an advantage and disadvantage of the configuration model that the clustering coefficient converges to zero in the thermodynamic limit, if the second moment of the degree distribution is finite, i.e. $\sum_k k^2 p(k) < \infty$. This means that the probability that two neighbors of a node are also connected to each other vanishes for $N \to \infty$. The resulting topology is also called locally tree-like.

The analytic approach that we develop in the next chapters is based on this property. Still, most real networks show non-zero clustering. There exist (computationally expensive) approaches to adapt analytic approximations to this observation in clique-based random graphs (Hackett and Gleeson, 2013).
Here, we focus on the effect of simple diversification strategies. More complexity can always be added, if the simpler system has been well understood and higher complexity is needed.

**Extension to weighted networks**

So far, we have specified an ensemble of undirected skeleton networks that defines the links between nodes. On top of that, we can also assume random link weights $W_{ij} \geq 0$, which might introduce a direction of the link. In general, we can have $W_{ji} \neq W_{ij}$. Considering that a weight can also take on the value zero ($w_{ij} = 0$), we can construct quite flexible directed structures. For instance, in a CL model, a node $i$ can send a load $w_{ij}$ to its neighbor $j$ but not vice versa, $w_{ji} = 0$, or the neighbor can send a different amount of load $w_{ji} \neq w_{ij}$ in case it fails before the node.

We assume that all weights $W_{ij}$ in the network are independent random variables. But they do not need to be identically distributed. Instead, we allow the distributions to depend on the degree $k_i$ of the starting node and the degree $k_j$ of the end note. Only weights that have identical link starting node degree and identical link end node degree are identically distributed. They follow the law of a generic random variable $W(k_i, k_j)$, i.e. $W_{ij} \sim W(k_i, k_j)$.

### 2.4.3 Random Thresholds

Before the beginning of a cascade, each node $i$ receives a degree (and network neighbors) and then a threshold $\theta_i$ that is drawn independently at random from the other nodes with the same degree. The threshold cdf can depend on the degree $k_i$ of the node and can be represented by a random variable $\Theta(k_i)$ that takes this dependence into account, i.e. $\Theta_i \sim F_{\Theta(k_i)}$.

Here, we consider three cases for this dependence: (a) random failures (rf), (b) peripheral failures (pf), and central failures (cf).

In case of random failures (a), each threshold is independent from the degree of the corresponding node and drawn independently from the other thresholds. Thus, the threshold of each node $i$ follows the same distribution $\Theta_i \sim F_{\Theta}$. Additionally, we consider two cases in which the threshold and the degree distribution of the nodes are dependent. One would expect in real world situations that the robustness and the connectivity of a node are shaped by similar mechanisms or decisions. We analyze therefore two extreme scenarios: (b) Peripheral failures (pf) describe the case of perfect correlation between degree and threshold distribution. Thus, nodes with a higher degree receive a higher threshold so that nodes with smaller degrees fail initially. The opposite...
Figure 2.7: Visualization of correlation schemes between threshold and degree distribution in a network with $N = 10^5$ nodes and Poisson degree sequence (with $z = 3$). The thresholds follow the order statistics of a normal distribution with mean $\mu = 0.5$ and $\sigma = 0.5$, i.e. $\Theta \sim N(0.5, 0.5^2)$. Nodes with thresholds as indicated by the histogram have degrees defined by the color of the bars: (a) peripheral failures (pf), (b) central failures (cf).

Formally, we construct two order statistics, one for $N$ independently drawn degrees, and one for $N$ independently drawn thresholds. These are sorted separately in increasing order. The degree and threshold at the $i$-th position are then distributed like the random variables $K(i)$ and $\Theta(i)$. We thus have $K(1) \leq K(2) \leq \cdots \leq K(N)$ and $\Theta(1) \leq \Theta(2) \leq \cdots \leq \Theta(N)$. In case of peripheral failures, the degree of a node at position $i$ is then distributed as $K(i)$, and its threshold as $\Theta(i)$. In case of central failures, a node receives a combination of degree and threshold according $K(i)$ and $\Theta(N-i+1)$. An example for a threshold allocation in a network consisting of $N = 10^5$ nodes with introduced Poisson degree sequence is given in Fig. 2.7 for both schemes, pf and cf.

With their analysis, we test two rules of thumb that can serve as guide in the design of complex systems. Moreover, the threshold allocation schemes give a hint which nodes are more critical for the stability of the system: hubs or leaves. Thus, we test some kind of robust-yet-fragile property.

In all our simulations and numerical calculations, we only regard normally distributed thresholds with mean $\mu$ and standard deviation $\sigma$, i.e. $\Theta \sim N(\mu, \sigma^2)$. Still, our theoretical derivations are valid for arbitrary threshold distributions.

The threshold parameters define 2d-phase diagrams that present our systemic risk estimation. We calculate the final fraction of failed nodes for various values of $\mu$ and $\sigma$ and visualize the result color coded as in Fig. 2.8. A dark color corresponds to high systemic
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Figure 2.8: Phase diagram for the initial fraction of failed nodes \( \varrho_0 = \varrho(0) \) in an infinite network with normally distributed thresholds \( (\Theta \sim \mathcal{N}(\mu, \sigma^2)) \). \( \varrho_0 \) is constant along the lines \( \sigma = \mu/\Phi^{-1}(\varrho_0) \).

risk, while the color white indicates negligible average cascade sizes. Exactly a fraction of \( \varrho(0) = F_{\Theta}(\lambda_0) \) nodes fails initially in an infinitely large network, as all nodes with initial net fragility fail in the beginning. Equivalently, a node (with unknown degree) fails with the probability \( F_{\Theta}(\lambda_0) \) also in a finite network. In every presented case in Part I of this thesis, the parameters are chosen so that the initial failure probability is given by \( F_{\Theta}(\lambda_0) = \Phi(-\mu/\sigma) \), where \( \Phi \) denotes the cdf of a standard normal distribution \( \mathcal{N}(0, 1) \). Consequently, \( \varrho_0 := \varrho(0) \) is constant along the lines \( \sigma = \mu/\Phi^{-1}(0) \). This initial fraction of failed nodes in infinite networks is shown in Fig. 2.8.

Similarly, we visualize the final fraction of failed nodes, our measure of systemic risk.

2.5 Model explorations on the macro-level via simulations

In this section, we provide an overview of systemic risk caused by the introduced cascade models. All phase diagrams that color code systemic risk are obtained via Monte Carlo simulations generated with the configuration model for the Poisson and scale free degree sequences as introduced in Sec. 2.4.2. When possible we consider networks consisting of \( N = 10^5 \) nodes and generate 10 independent networks in each simulation. On each network, we run cascades for 50 different independent threshold realizations. Thus, for each parameter combination, we average over 500 final cascade size data points. For LRD and OLRD LLSC models, we have to restrict our exploratory simulations to networks consisting of \( N = 10^3 \) nodes, as the simulations are computationally expensive.

With this study, we transfer the analysis on fully connected networks provided by Lorenz
et al. (2009) to random graph ensembles with heterogeneous degrees. The two examined distributions have both an average degree of $z = 3$. Thus, the differences in their corresponding phase diagrams, originate from the different form of the distributions tails. At this point, we refrain from a detailed comparison and postpone this to later chapters when we deepen the understanding of the models by an analytic approach. Here, we are interested in qualitative shapes of the phase diagrams and prominent differences.

### 2.5.1 Constant Load

The CL ED model has been studied intensively since its introduction by Watts (2002) and has since then inspired several branches of research, such as marketing or opinion formation (Watts and Dodds, 2007) and financial systemic risk (Battiston et al., 2012a; Gai and Kapadia, 2010), or algorithm design (Kempe et al., 2003). Its properties with respect to configuration model type random graphs and variants thereof are well understood (Gleeson and Cahalane, 2007). A thorough simulation study has been conducted by Roukny et al. (2013) that covers also pf and cf as correlations between thresholds and degrees. We review the main contributions to its exploration further in Ch. 3, when we derive analytic ensemble averages for a generalization of the introduced constant load models.

In Fig. 2.9, we present an overview of all studied cases and show simulation studies for the comparison with the CL DD model, which has not been investigated on heterogeneous network topologies before up to our knowledge.

We discuss implications of our findings in detail in Ch. 3 and Ch. 4. Here, we note that the CL DD cascade model seems to expose a system to lower systemic risk in general, while we cannot observe substantial differences between the two studied network topologies. Still, our results show a remarkable effect of different threshold allocations that differ between the two models. Especially, the combination CL ED pf is of interest as it seems to lead to the lowest systemic risk among the studied alternatives.

### 2.5.2 Load Redistribution

The LRD LLSC or fiber bundle model has been studied in great depth and in several variants on various network topologies mainly with the help of simulations (Kun et al., 2006; Pradhan et al., 2010, and references therein). Usually, the thresholds are assumed to follow a uniform or Weibull distribution. As the model’s behavior on heterogeneous but relatively structured networks is quite distinct from the observed one on fully connected networks, both are considered as model variants with different load sharing rules, where the first is termed local while the second is called global. The two variants are assumed to belong to different universality classes.
Figure 2.9: [CL] Overview of simulation results for the Constant Load models. The first row belongs to the CL ED model, while the second row corresponds to CL DD. The left column shows results on scale free networks and the right column on Poisson random graphs. Both consist of $N = 10^5$ nodes and we depict ensemble averages of the final cascade size, where we simulate 50 independent runs on each of 10 independent networks. The thresholds are normally distributed with mean $\mu$ and standard deviation $\sigma$ ($\Theta \sim N(\mu, \sigma^2)$). The big phase diagrams correspond to rf, while the smaller one above belongs to cf and below to pf.

Interestingly, in random configuration type models as we study here, the local load sharing rule leads to similar results as the global load sharing rule (Kim et al., 2005). This can mainly be explained by the fact that the model can be equivalently described with the help of a dynamic network topology where links between neighbors of a failing node are formed. Thus, in the course of a cascade the remaining nodes become more interconnected. As argued by Kim et al. (2005), the small world property of the considered random graphs leads then to almost fully connected network topologies. Because of this, we observe in Fig. 2.10 similar phase diagrams for the LRD LLSC model as on fully connected networks. Here, we study a relatively high initial load $\lambda_0 = 0.5$ and a small average degree $z = 3$. But for smaller loads and degree distributions with higher average degree we would observe qualitatively similar phase diagrams as reported by Lorenz et al. (2009) for fully connected networks. Here, different threshold allocations do not change the results qualitatively.

In contrast, the LRD LLSS cascades expose a system to substantially smaller systemic risk, as shown in Fig. 2.10. This simpler fiber bundle model variant has (up to our knowledge) only been investigated by Moreno et al. (2002) on Barabási Albert networks (Barabási, 2009) in combination with independent random thresholds that follow a Weibull distribution, while Tessone et al. (2013) study the model on regular networks with three
2.5. Model explorations on the macro-level via simulations

Figure 2.10: [LRD] Overview of simulation results for the Load Redistribution models. The first row belongs to the LRD LLSC model, while the second row corresponds to LRD LLSS. The left column shows results on scale free networks and the right column on Poisson random graphs. Initially, each node receives a load $\lambda_0 = 0.5$. The thresholds are normally distributed with mean $\lambda_0 + \mu$ and standard deviation $\sigma$ ($\Theta \sim N(\lambda_0 + \mu, \sigma^2)$). The big phase diagrams correspond to rf, while the smaller one above belongs to cf and below to pf. For the LLSC model, we consider networks with $N = 10^3$ nodes and we depict ensemble averages of the final cascade size, where we simulate 20 independent runs on each of 10 independent networks. For the LLSS model, we consider networks with $N = 10^5$ nodes and we depict ensemble averages of the final cascade size. We simulate 50 independent runs on each of 10 independent networks (rf), or one run on 10 independent networks (cf/pf).

different threshold distributions. Critical behavior is also exhibited in these settings.

Remarkably, we note strong qualitative changes in the phase diagrams considering correlations between threshold and degree distribution. In particular, the cf threshold allocation variant leads to relatively irregular behavior, where a decrease of the mean threshold $\mu$ can first increase suddenly the risk of large cascades, while a further decrease lowers systemic risk again. This is more prominent for scale free networks than for Poisson random graphs.

Thus, in this aspect, we can also see more apparent differences between different degree distributions.

2.5.3 Overload Redistribution

OLRD models are usually investigated in a setting where the load that links or nodes carry depends on the number of shortest paths passing through them (Motter, 2004; Motter and
Figure 2.11: [OLRD] Overview of simulation results for the Overload Redistribution models. The first row belongs to the OLRD LLSC model, while the second row corresponds to OLRD LLSS. The left column shows results on scale free networks and the right column on Poisson random graphs. The thresholds are normally distributed with mean $\mu$ and standard deviation $\sigma$ ($\Theta \sim N(\mu, \sigma^2)$). The big phase diagrams correspond to rf, while the smaller one above belongs to cf and below to pf. We consider networks with $N = 10^3$ nodes and we depict ensemble averages of the final cascade size, where we simulate 20 independent runs on each of 10 independent networks.

Lai, 2002; Zhao et al., 2016), as such work is often motivated by traffic flow dynamics. Thus, the load redistribution is not restricted to the local neighborhood of a failing node, but takes global network information into account.

Instead, we focus on a variation of the LRD models where the overload of a node is distributed instead of the full load that a node carries. Consequently, less load is distributed in total so that we find lower systemic risk than in case of the corresponding LRD models. Remarkably, although known for fully connected networks, the sharp regime shift, as we have observed in all other cases, vanishes. Instead, we only see phase diagrams in Fig. 2.11 that look qualitatively similar to the one representing initial failures as shown in Fig. 2.8.

Interestingly, we observe consistently lower systemic risk for the LLSC model than for the LLSS model. Also the pf threshold allocation has a risk reduction effect in comparison to cf.

For LRD, these findings are permuted. We discuss possible reasons in Ch. 5 and Ch. 6, where we analyze our results in more detail.
2.5.4 Discussion

At first glance, we find only small differences in systemic risk for the two studied degree distributions. Qualitatively, we also observe similar shapes of phase diagrams for the CL ED, LRD LLSC and all OLRD models in comparison to their counterpart on fully connected networks.

A substantial change is introduced by correlations between threshold and degree distribution. Both have the potential to lead to either strong amplifications or to reduce systemic risk substantially in comparison with the uncorrelated case \( rf \). These are of special interest to us, as they provide an indication of the role of hubs in the cascade amplification process.

Remarkably, the relation between the LLSC and LLSS variants seem to be reverted in the LRD and OLRD models. Thus, a growing network connectivity as in case of the LLSC model does not foster cascade amplifications in general.

While highly connected nodes face a higher risk to receive load in both cases and are thus especially vulnerable with respect to LRD or OLRD cascades, they also distribute their own load that they have accumulated to a higher number of nodes in case of their failure. Consequently, their failure is partially mitigated.

Whether the positive and negative effects of a high interconnectivity prevail, seems to be different in case of LRD and OLRD models.

Further aspects as the heterogeneity of diversification strategies among nodes and the time of failure of specific nodes influence the evolution of cascades.

In the following, we deepen the understanding of the contribution of those different effects. For the CL models, and the LLSS model in the LRD and OLRD class, we follow an analytic approach and derive iterative procedures to calculate the network ensemble averages of the evolving cascade size.

These share common principles that we explain next.

2.6 Analytic Framework

It is the main achievement in Part I of this thesis that we are able to derive analytic expressions for the fraction of failed nodes in the limit of infinite network size for model variants in each of the introduced cascade classes. All of these derivations are novel and enable the study of network sizes beyond the limitations of Monte Carlo simulations (which, for instance, cannot sample networks with very right-skewed degree distributions). Especially, the observation that we can correctly describe the time evolution of the fraction of failed nodes and not only its steady state is an interesting finding, which can serve as starting point of further analytic investigations of cascade dynamics.
2.6.1 Local Tree Approximation

Our derivations are based on the fact that the average clustering coefficient in the configuration model converges to zero in the limit $N \to \infty$. This holds only true, if the networks’ degree sequence stems from a degree distribution with finite second moment. Consequently, we can rely on a locally tree like network structure as already mentioned in Sec. 2.4.2. Fig. 2.12 shows an illustration. Because of this, the general approach is called Local Tree Approximation (LTA) (Dodds and Payne, 2009). Sometimes it is also known as Heterogeneous Mean Field Approximation (HMF) or Branching Process Approximation (Amini et al., 2010). Despite including the term approximation in their name, they are correct in the limit of infinite network size under the stated assumptions. The name element approximation comes from their application to more complicated network structures that are approximated by configuration model random graphs who share the same degree sequence with the original network.

The locally tree-like network structure has the advantage that we can treat the failures of a node’s neighbors as independent (since the neighbors are not connected with each other). This does not mean that nodes fail independently of each other. It just acknowledges that in one network realization a node in the network can be connected to an already failed node, while in another it might be connected to a functional one. The probability of being connected to one neighbor is independent from the probability to be connected to another neighbor. Thus, the random realization of connections to nodes, i.e. the graph ensemble approach, introduces the possibility to treat neighbors as independent.

In Part I, all derivations start from the same fundamental building block that we explain next. The specializations towards the specific cascade models under consideration follow in the corresponding chapters.

We have noted already that the average fraction of failed nodes coincides with failure probability of a randomly selected node in the network. After taking the limit $N \to \infty$, the quantity of our interest is therefore:

$$\varrho(t) = \lim_{N \to \infty} \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} s_i^{(N)}(t) \right) = \lim_{N \to \infty} \mathbb{P} \left( s_i^{(N)}(t) = 1 \mid i \in \{1, \cdots, N\} \right) =: \mathbb{P}(s(t) = 1).$$

The symbol $s(t)$ denotes here a random variable for the state of a randomly selected node in an infinite network at time $t$. If we omit the time dependence, we abbreviate the final state, i.e. the state $s(T)$. We call such a randomly selected node also the focal node whose failure probability we want to identify. In Fig. 2.12, it is colored in green. The same principle applies to a neighbor of such a node, which is colored gray in Fig. 2.12. The state of a focal neighbor is denoted by $s_{nb}(t)$. As we see next, we need to determine the
2.6. Analytic Framework

Figure 2.12: Illustration of the local tree approximation. The green node is the focal node. Its conditional failure probability \( P(s = 1 | k = 5) \) can be computed according to Equation (2.1), and depends on the state of its neighbors: Here, the two red ones and the gray one have failed, while the two blue ones are still functional. The neighbors’ conditional failure probabilities \( P(s_{\text{nb}} = 1 | k) \) rely on the failure probabilities of their own neighbors without regarding the green focal point.

Failure probability of a node

We recall that the failure of a node is determined by two (random) variables: its threshold \( \Theta \) and its fragility \( \Lambda(t) \), i.e. the load that it carries at time \( t \). Whenever the carried load exceeds its threshold, a node fails. Thus, we can deduce the relation:

\[
\varrho(t) = P(\Theta \leq \Lambda(t)).
\]

We know that \( \Lambda(t) \) sums up the load that a focal node has received by its failed neighbors in the course of the cascade. Consequently, it depends on the degree \( K \) of a node. From Bayes’ Theorem (Feller, 1968), it follows:

\[
\varrho(t) = \sum_{k=1}^{c} p(k) P(s(t) = 1 | k) = \sum_{k=1}^{c} p(k) P(\Theta \leq \Lambda(t) | k),
\]
where $\mathbb{P} \left( s(t) = 1 \mid k \right)$ denotes the conditional failure probability of a node given that its degree is $K = k$. We also write $\Lambda(k, t)$ and $\Theta(k)$ to indicate that the distributions of $\Lambda(t)$ and $\Theta$ can depend on the degree $k$ of a node.

### Conditional failure probability

The $k$ neighbors of this focal node with degree $k$ have distributed at time $t - 1$ or before a random load $L^{nb}(k, t - 1)$ independently of each other. The distribution is independent because of the locally tree-like network structure. If a neighbor is not failed yet or does not distribute any load, it simply distributes zero load, i.e. $L^{nb}(k, t - 1) = 0$. Thus, we can assign to each neighbor $j = 1, \cdots, k$ a random variable $L^{nb}_{j}(k, t - 1) \sim L^{nb}(k, t - 1)$. These random variables are iid and sum up to $\Lambda(k, t)$:

$$
\Lambda(k, t) = \sum_{j=1}^{k} L^{nb}_{j}(k, t - 1).
$$

Since the random variables are independent, their sum is distributed as the convolution of their distributions.\(^1\) As they are identically distributed, this simplifies to the k-fold convolution: $\Lambda(k, t) \sim p^{*k}_{L^{nb}(k, t - 1)}$.

In summary, we have deduced:

$$
\mathbb{P} \left( s(t) = 1 \mid k \right) = \mathbb{P} \left( \Theta(k) \leq \Lambda(k, t) \right) = \mathbb{P} \left( \Theta(k) \leq \sum_{j=1}^{k} L^{nb}_{j}(k, t - 1) \right) = 
\int_{0}^{\infty} F_{\Theta(k)}(x) P^{*k}_{L^{nb}(k, t - 1)}(x) \, dx.
\quad (2.1)
$$

Consequently, the task is left to derive an analytic expression for the load $L^{nb}(k, t)$ that a single neighbor distributes to a node with degree $k$.

### Failure probability of a neighbor

The first step on this way is guided by the insight that the fragility of a focal neighbor $\Lambda^{nb}(t)$, i.e. the load that it carries, is of similar form as the one of a node. The only difference is that we want to calculate the distribution of $\Lambda^{nb}(t)$ \textit{conditional on the fact that one of its neighbors, i.e., the focal node, has not failed yet}. The failure of the focal node can only be caused by failures of neighbors that have distributed their load before

\(^1\)In case of degree-degree correlations (i.e. correlations between the degree $K$ of a node and its neighbor $K^{nb}$ this would still be true. Only in case of correlations between the degrees of the neighbors, more effort would be needed to calculate the distribution of the sum of dependent variables.)
the failure of the node. This property of a neighbor whose failure does not regard the focal node is called the Without Regarding Property (WOR) by Hurd and Gleeson (2013). With this in mind, we can thus write:

\[
P(s_{nb}(t) = 1 \mid k) = P(\Theta(k) \leq \Lambda^{nb}(k, t)) = P(\Theta(k) \leq \sum_{j=1}^{k-1} L^{nb}_j(k, t - 1))
\]

\[
= \int_0^\infty F_{\Theta(k)}(x) p^*_{L^{nb}(k, t - 1)}(x) \, dx. \tag{2.2}
\]

A second important difference between a node and a neighbor needs to be mentioned. A neighbor’s degree $K^{nb}$ does not follow the pdf $p(k)$ as the one of a node. Instead, we have:

\[
p_{nb}(k) := P(K^{nb} = k) = \frac{p(k)k}{z}, \tag{2.3}
\]

where $z := \sum_k kp(k)$ denotes the normalizing average degree. $p_{nb}(k)$ is proportional to the degree $k$ in the configuration model, because each of a neighbor’s $k$ links could possibly connect the neighbor with the focal node (see, e.g., Newman (2010)). Thus, a node with a higher degree has a higher chance to be connected with the focal node.

Now, Bayes’ Theorem provides an expression for the failure probability $\pi(t) := P(s_{nb}(t) = 1)$ of a neighbor:

\[
\pi(t) = \sum_{k=1}^c p_{nb}(k) P(s_{nb}(t) = 1 \mid k) = \sum_{k=1}^c p_{nb}(k) \int_0^\infty F_{\Theta(k)}(x) p^*_{L^{nb}(k, t - 1)}(x) \, dx. \tag{2.4}
\]

With this definition, we can go even further. The following derivation is useful to understand the relation between our work and the state of the art of LTAs. Let’s denote with $L_{fnb}^{nb}(k, t - 1)$ the random load that a single neighbor distributes to a node with degree $k$ conditional on the event that the neighbor has failed before or at time $t - 1$. Since neighbors of a node fail independently, the number $N_f$ of failed neighbors out of $k$ neighbors is simply binomial distributed. We abbreviate the probability for the event that exactly $N_f = n$ neighbors are failed as

\[
b(n, k, \pi(t - 1)) := \binom{k}{n} \pi(t - 1)^n (1 - \pi(t - 1))^{k-n}.
\]
Then, an alternative to Eq. (2.2) reads as:

\[
P(s_{nb}(t) = 1 \mid k) = \sum_{n=0}^{k-1} P(N_f = n) \mathbb{P}\left( \Theta(k) \leq \sum_{j=1}^{k-1} L_j^{nb}(k, t-1) \mid N_f = n \right) \\
= \sum_{n=0}^{k-1} b(n, k-1, \pi(t-1)) P\left( \Theta(k) \leq \sum_{j=1}^{n} L_j^{nb}(k, t-1) \right) \quad (2.5) \\
= \sum_{n=0}^{k-1} b(n, k-1, \pi(t-1)) \int_0^\infty F_{\Theta(k)}(x) p_{L_{tb}(k, t-1)}^* (x) \, dx.
\]

With these preparations, we are ready to shift our focus to a more detailed exploration of our three model classes.
Chapter 3

Diversification strategies in Constant Load models

Summary

We study the influence of risk diversification on cascading failures originating from Constant Load models in weighted complex networks, where weighted directed links represent exposures between nodes. These weights result from different diversification strategies and their adjustment allows to reduce systemic risk significantly by topological means. As example, we contrast a classical Exposure Diversification (ED) approach with a Damage Diversification (DD) variant. The latter reduces the loss that the failure of high degree nodes generally inflict to their network neighbors and thus hampers the cascade amplification. To quantify the final cascade size and obtain our results, we develop a branching process approximation taking into account that inflicted losses can not only depend on properties of the exposed, but also of the failing node. This analytic extension is a natural consequence of the paradigm shift from individual to system safety. To deepen our understanding of the cascade process, we complement this systemic perspective by a mesoscopic one: an analysis of the failure risk of nodes dependent on their degree. Additionally, we ask for the role of these failures in the cascade amplification.

Based on R. Burkholz, A. Garas, and F. Schweitzer, “How damage diversification can reduce systemic risk”. Physical Review E 93, 042313. RB contributed to the design of the research questions, derived the analytic equations, implemented the code, interpreted the results and wrote the main part of the manuscript. A description of the cascade dynamics has been added specifically for this thesis.


3.1 Introduction

We devote this chapter to a closer analysis of the two Constant Load model variants that we introduced in Sec. 2.3.1. As we have seen already, CL models are strongly linked to the network structure. The loss $l_{ji}$ that a node $i$ experiences in case of the failure of its neighbor $j$ can be interpreted as weight of the link $(j, i)$. Consequently, systemic risk is to a large extend determined by the interconnectivity properties and dependence structure of the studied system.

Increased dependence, under normal conditions, has advantages for the efficient operation of a system. It also leads to an increased risk diversification of its components, since the dependence on single other components is reduced. According to classical risk management theories (Allen and Gale, 2000) a higher connectivity thus decreases the vulnerability of the system as a whole, even with respect to cascading failures. Several authors, e.g. (Allen et al., 2011; Battiston et al., 2012b; Gai and Kapadia, 2010; Gleeson and Cahalane, 2007; Roukny et al., 2013), pointed already out that a higher connectivity can also increase systemic risk, i.e. in our context the expected cascade size.

Although these findings are partially model-dependent, they can be explained intuitively by a basic trade-off between diversification and system connectivity that is reflected in the risk exposures. In fact, most of these works are based on a cascade model by Watts (2002), the ED model, which introduces this trade-off, and can be interpreted as a study of simple diversification strategies of system components or agents that are represented as nodes in a network. Increased diversification of system components reduces their exposure to the failure of single other components. But at the same time, it increases the connectivity of the overall system. Especially, if well connected components, so called hubs, fail they affect a high number of other nodes. Even if the inflicted damage is only little, if it is enough to trigger a few other failures, the cascade might continue and span a large fraction of the entire system.

This implies that hubs, because of their large number of neighbors, considerably affect the network in case of failure. Consequently, policy discussions and risk reduction strategies center around the question of how to prevent the failure of hubs, e.g. by increasing their robustness. Examples include relative capital buffers in finance (Arinaminpathy et al., 2012; Roukny et al., 2013) or immunization in epidemiology (Pastor-Satorras and Vespignani, 2002).

Here, we test in a generic modeling approach a way to complement such regulatory efforts and strategies by the mitigation of the impact of the failures of well connected nodes as it is realized in the DD model. In the DD model, the impact of a failing node in a network is diversified instead of its exposures. This has the potential to reduce systemic risk significantly, since the failure amplification caused by hubs is counterbalanced. For
instance, in a financial network this approach would correspond to a policy where each financial institution is only allowed to get into a limited amount of debt. We are aware that our approach is abstract, and it is based on simplifying assumptions that deviate from real world application scenarios. However, we do hope to motivate researchers in more applied fields and practitioners to consider topological possibilities that can reduce systemic risk.

In this chapter, we present analytic derivations for network ensemble averages as described in Sec. 2.4. The simulation results of Sec. 2.5 serve as benchmark to provide evidence that our approximations are correct. Additionally, we present the dynamics of the cascade size for specific threshold values that we obtained by simulations as well as our analytic approach.

Recall that these derivations are valid in the limit of infinite network size, where two quantities on the system level are given: a) the degree distribution, which defines the number of direct neighbors of a node, and thus limits their respective diversification strategies, and b) the threshold distribution defining the robustness of nodes.

We build on the work of Gleeson and Cahalane (2007) who have derived the LTA for the ED model. However, this approach does not capture processes where the impact of a failing neighbor depends on its specific properties (e.g., its degree or robustness) as it is required for the treatment of the DD variant. Therefore, we extend the branching process approximation to the latter case, and generalize it for the application to weighted random network models. The weight statistics could be deduced from data, taking also into account different properties of neighboring nodes, or from specific model assumptions. This way, we generalize the analytic treatment to match application scenarios better. Moreover, it allows to deepen the understanding of the role of different risk diversification strategies. In fact, it translates a recent paradigm shift to an analytic setting. The focus has been shifted from individual or component-wise risk assessments to systemic risk analysis. Thus, the question of the failure risk of single components is complemented by the question for the consequences of their failure for the rest of the system. A systemic risk analysis is often accompanied by the identification of so-called system-relevant nodes. But this analysis is based on the assumption that various components can have a different impact on the system stability. This is considered in our framework, as failing nodes can cause different damage to their network neighbors.

As a result of the trade-off between system connectivity and risk diversification an increased diversification does not need to reduce the failure risk neither of the system nor its components. Increased diversification does not even need to decrease the failure risk of a component despite the fact that it reduces systemic risk.

Because of this, we accompany our systemic risk measure on the macro-level, the average cascade size, with a comparison of failure probabilities of nodes with different diversifica-
tion strategies on the meso-level. The nodes’ systemic relevance can then be identified on the basis of their cascade amplification role.

3.2 Local Tree Approximation

As expansion of the network and threshold ensembles introduced in Sec. 2.4, we assume that the links between nodes have weights \( w_{ij} \) as introduced in Sec. 2.4.2. These weights define the losses \( l_{ij} = w_{ij} \) that nodes inflict on each other in case of their failure. In general, they can be expressed as independent random variables \( W_{ij} \), which follow distributions that can depend on the degrees of the start and end node of the link: \( W_{ij} \sim p_{W(k_i,k_j)} \). In the beginning of each cascade, the weights are drawn independently at random, after the degrees and thresholds of the nodes have been fixed, and stay constant over the course of the cascade.

This way, a direction of a link is introduced, since in general we can have \( w_{ij} \neq w_{ji} \). Still, we can assume an undirected skeleton network, i.e., we do not need to distinguish between in-degrees and out-degrees. This differs from the usual approach for directed networks (Amini et al., 2010, 2012; Gai and Kapadia, 2010) where the neighbors whose failures impact a node are distinct from the ones who face a loss in case of the node’s failure. In this case, one node is exposed to the other, but not vice versa. In our case, however, once there is a link between two nodes, each can impact the other, but the amount of the loss can be different. Still, this does not limit us to the study of undirected networks, since weights \( w_{ji} \) in one direction can also be set to zero (\( w_{ji} = 0 \)). Instead, it allows us to model situations where some nodes are part of mutual dependencies as well as one way exposures. Thus, our approach is a generalization. In Appendix A.4, we provide an example for an exclusively directed network by defining the weights as net difference between weights in the DD approach: \( w^{(\text{off})}_{ji} = \max (1/k_j - 1/k_i, 0) \).

Some special choices of weights and networks with other weight statistics have been considered already.

3.2.1 The state of the art

For the ED approach, the average fraction of failed nodes at the end of a cascade can be calculated on random networks with given degree distribution \( p(k) \) and threshold distribution \( F_\Theta(\theta) \) (Gleeson and Cahalane, 2007). This LTA were studied in many subsequent works. It was generalized for directed and undirected weighted networks (Amini et al., 2010; Hurd and Gleeson, 2013), it was found to be accurate even for several clustered networks with small mean inter-vertex distance (Melnik et al., 2011), and the influence of degree-degree correlations has been investigated (Dodds and Payne, 2009; Payne et al., 2010).
According to a general framework introduced by Lorenz et al. (2009), the ED and DD approach belong to the constant load class, where the ED is called the inward variant, while the DD is identified as the outward variant. Still, the risk reduction potential of the latter has not been understood so far, since a system’s exposure to systemic risk has been only explored on fully-connected (Lorenz et al., 2009) or regular (Battiston et al., 2012a; Gleeson and Cahalane, 2007; Lorenz et al., 2009) networks, where both model variants coincide.

In order to study the DD approach on more general networks, we generalized and extended the existing approximations, which were proven to be exact for the case of ED (Amini et al., 2010; Hurd and Gleeson, 2013). Now, we can treat more general processes where the directed weights in a directed or undirected network can depend on properties of both nodes, the failing as well as the loss facing one. Here, in contrast to Ref. (Amini et al., 2010), nodes can depend on each other and properties of the failing node influence the amount of loss faced, and in contrast to Ref. (Hurd and Gleeson, 2013) nodes can depend on each other in a non-symmetric way. Moreover, the insight that the iterative steps in the fixed point iteration that lead to the final cascade size correctly describe the cascade dynamics seems to be new.

We show in Section 3.3 that our approach leads to very good agreements with simulations on finite Poisson and scale-free random graphs. Many large systems belong to the latter class (Barabási, 2009; Caldarelli, 2007; Cohen and Havlin, 2010). But often, simulations would require more computational time than our analytic approach or would even be impossible, because nodes with degree far in the right tail of the degree distribution are only realized in large networks that most times cannot be sampled with adequate accuracy. Interestingly, very sparse networks seem to reduce systemic risk for most parameters because of their low connectivity. An example for such a case is provided in the Appendix.

First, we derive the LTA of the cascade size for random graph ensembles with general weight distributions as explained.

### 3.2.2 Analytic approach for general weighted networks

In Sec. 2.6.1, we have derived an expression for the cascade size $\varrho(t)$ that finally depends on the conditional failure probability of a focal neighbor as provided by Eq. (2.2). We recall that it is of the form

$$\mathbb{P}(s_{nb}(t) = 1 \mid k) = \mathbb{P}(\Theta(k) \leq \sum_{j=1}^{k-1} L_{nb}^{ab}(k, t-1)).$$

The task is left to derive the distribution of the load $L_{nb}^{ab}(k, t-1)$ that a focal neighbor has distributed before or at time $t-1$ to a focal node or neighbor with degree $k$. Both a focal
neighbor and a focal node are illustrated in Fig. 2.12 in gray and green respectively. We assume throughout this chapter that the load $L_{nb}^j(k, t - 1)$ is a discrete random variable. All our derivations work analogously for the densities of continuous random variables. We would need to discretize them in numerical calculations anyway. Because of this, we restrict ourselves to the discrete case.

The load distributed by neighbors

Because of the constant load assumption, the load that a failed neighbor distributes in case of its failure is constant over time and is defined by the distributions of the weights $W_{ji}$ that define the load that a failed neighbor distributes. We assume that these distributions are known for given degrees of the nodes that are adjacent to a given weighted link. For instance, if $k_j = d$ and $k_i = k$, we know that $W_{ji} \sim W(d, k)$. This load is distributed only, if a failed neighbor has degree $d$. However, a neighbor can be in one of two states: it is failed with probability $\pi(t - 1)$ and distributes load or it is functional with probability $1 - \pi(t - 1)$. Thus Bayes’ Theorem leads us to

$$P(L_{nb}^j(k, t - 1) = l) = (1 - \pi(t - 1))P(L_{nb}^j(k, t - 1) = l \mid s_{nb}(t - 1) = 0) + \pi(t - 1)P(L_{nb}^j(k, t - 1) = l \mid s_{nb}(t - 1) = 1).$$

for any $l \in \mathbb{R}$. It is certain that a neighbor does not distribute any load, if it has not failed yet. Thus, the probability $P(L_{nb}^j(t - 1) = l \mid s_{nb}(t - 1) = 0)$ to distribute any load $l \neq 0$ in case that the neighbor is not failed is zero and it is 1 otherwise. We can determine $\pi(t - 1)$ with the help of (Eq. 2.4). So, we can focus on the case in which the neighbor is failed and distributes load.

The question that we need to answer is: What is the probability $p_{fnb}(d)$ that a failed neighbor $j$ has degree $d$? The probability does not exactly follow the neighbor’s degree distribution $p_{nb}(d)$, since some nodes with a given degree might have a higher failure probability than others and would be thus underrepresented, if we would randomly select a neighbor in the network. In fact, we restrict ourselves to sample from the failed neighbors and ask for the probability that its degree is $d$. Thus, failed neighbors with degree $d$ are represented in this sample proportionally to their failure probability $P(s_{nb}(t - 1) = 1 \mid d)$. The correct distribution of the degree $K_{fnb}$ of a failed neighbor at time $t - 1$ follows from Bayes’ Theorem:

$$p_{fnb}(d) = P(K_{fnb} = d) = \frac{P(s_{nb}(t - 1) = 1 \mid d)p(d)d}{z\pi(t - 1)}.$$

Again employing Bayes’ Theorem lets us deduce the distribution of the loss that a neighbor
inflicts in case of its failure:

\[
P(L_{nb}(k,t-1) = l \mid s_{nb}(t-1) = 1) = \sum_{d=1}^{c} p_{fb}(d) P(L_{nb}(k,t-1) = l \mid s_{nb}(t-1) = 1, K^{fnb} = d)
\]

for any \(l \in \mathbb{R}\).

In summary, we define the pdf \(p_{L_{nb}(k,t-1)}(l) := P(L_{nb}(k,t-1) = l)\) as:

\[
P(L_{nb}(k,t-1) = 0) = (1 - \pi(t-1)) + \sum_{d=1}^{c} \frac{P(s_{nb}(t-1) = 1 \mid d)p(d)d}{z} P(W(d,k) = 0) \tag{3.1}
\]

\[
P(L_{nb}(k,t-1) = l) = \sum_{d=1}^{c} \frac{P(s_{nb}(t-1) = 1 \mid d)p(d)d}{z} P(W(d,k) = l).
\]

for \(l \neq 0\).

The first factor simply corresponds to the probability that a neighbor has degree \(d\) and is failed, while the second factor describes the probability that such a node distributes the load \(l\).

Then, the sum of \(k-1\) independent copies of \(L_{nb}^{t-1}\) is distributed by the \((k-1)\)-fold convolution of this variable. Therefore, we have deduced the missing building blocks to define the dynamics of the fraction of failed nodes \(\varrho(t)\) by

\[
P(s_{nb}(t) = 1 \mid k) = P(\Theta(k) \leq \sum_{j=1}^{k-1} L_{nb}^{j}(k,t-1)) = \sum_{x} F_{\Theta(k)}(x) p^{(k-1)}_{L_{nb}(k,t-1)}(x), \tag{3.2}
\]

\[
P(s(t) = 1 \mid k) = P(\Theta(k) \leq \sum_{j=1}^{k} L_{nb}^{j}(k,t-1)) = \sum_{x} F_{\Theta(k)}(x) p^{(k)}_{L_{nb}(k,t-1)}(x). \tag{3.3}
\]

so that we obtain:

\[
\varrho(t) = \sum_{k=1}^{c} p(k) P(s(t) = 1 \mid k).
\]

The right hand side in the definition of the conditional failure probability of a neighbor \(P(s_{nb}(t) = 1 \mid k)\) in Eq. (3.2) can also be interpreted as the \(k\)-th component of a vector valued function \(G : [0,1]^c \rightarrow [0,1]^c\) that maps the conditional failure probabilities at time \(t-1\) to the corresponding values at time \(t\). This is possible, since the distribution of
\( L^\text{nb}(k, t - 1) \) is solely defined by the knowledge of the vector

\[
\mathbb{P}(s_{\text{nb}} = 1 \mid k) := \left( \mathbb{P}(s_{\text{nb}} = 1 \mid k) \right)_{k \in \{1, \ldots, c\}} \in [0, 1]^c.
\]

For any \( q \in [0, 1]^c \), we define the \( k \)-th component \( G_k(q) \) of \( G(q) \) as:

\[
G_k(q) := \sum_x F_{\Theta(k)}(x) p^{*(k-1)}(x),
\]

or for continuously distributed \( L(q) \) and weights as:

\[
G_k(q) := \int_0^\infty F_{\Theta(k)}(x) p^{*(k-1)}(x) \, dx,
\]

where \( p_L(q) \) is defined analogously to Eq. (3.1) as:

\[
p_L(q)(0) = 1 - \sum_{d=1}^c \frac{q_d \, p(d)}{z} + \sum_{d=1}^c \frac{q_d \, p(d)}{z} p_W(d,k)(0),
\]

\[
p_L(q)(l) = \sum_{d=1}^c \frac{q_d \, p(d)}{z} p_W(d,k)(l) \quad \text{for } l \neq 0.
\]

Although we do not state the \( k \)-dependence explicitly, we note that for each \( k \) we can have a different \( L(q) \).

With this definition, we can formulate Eq. (3.2) equivalently as:

\[
\mathbb{P}(s_{\text{nb}}(t) = 1 \mid k) = G \left( \mathbb{P}(s_{\text{nb}}(t - 1) = 1 \mid k) \right).
\]

Consequently, the cascade process converges to a steady state for \( t \to \infty \), if the function \( G : [0, 1]^c \to [0, 1]^c \) attains a fixed point.

Such a fixed point \( \mathbb{P}(s_{\text{nb}} = 1 \mid k) \) exists according to the Knaster-Tarski Theorem (Tarski, 1955), since the function \( G \) is monotone with respect to a partial ordering and maps the complete lattice \([0, 1]^c\) onto itself.\(^1\) A proof is given in Appendix A.1.

In the models considered, numerical convergence is achieved the latest after \( t = 50 \) fixed point iterations that are equivalent to time steps in the cascade process.

After these steps, we estimate our measure of systemic risk

\[
e = \lim_{t \to \infty} \rho(t),
\]

the final fraction of failed nodes.

\(^1\)Definition of the partial ordering: Two vectors \( x, y \in [0, 1]^c \) are ordered as \( x \leq y \), if and only if \( x_i \leq y_i \) holds for all their components \( i \in I \).
Simplification for homogeneous failure probability of neighbors

In case the impact of a failing neighbor does not depend on its degree \( d \), i.e. \( W(d, k) \sim W(k) \), the vector valued fixed point equation Eq. (3.4) simplifies to a scalar fixed point equation.

To see this, we first observe that Eq. (3.1) reads for \( l \neq 0 \) in the steady state \( t \to \infty \) as:

\[
P(L_{nb}(k) = l) = \frac{c \cdot P(s_{nb} = 1 \mid d) p(d) d}{z} = P(W(k) = l) \pi
\]

and for \( l = 0 \) we have \( P(L_{nb}(k) = 0) = 1 - \pi + \pi P(W(k) = 0) \). Thus, the load distribution does not depend on all the different failure probabilities of neighbors with a given degree \( d \) anymore. Instead, the failure probability \( \pi \) of a randomly selected neighbor defines already the load distribution by a neighbor. Thus, we only need to determine \( \pi \). To give better prominence to the dependence of \( P(s_{nb} = 1 \mid k) \) on \( \pi \), we write it in the form of Eq. (2.5).

Thus, we consider distributions of the load \( L^{fnb}(k) \) that a neighbor distributes to a focal node or neighbor with degree conditional on the fact that it is failed. We simply obtain:

\[
P(L^{fnb}(k) = l) = P(L_{nb}(k) = l \mid s_{nb} = 1) = P(W(k) = l).
\]

Thus, \( L^{fnb}(k) \) and \( W(k) \) are identical distributed. As consequence, \( L^{fnb}(k) \) does not even depend on \( \pi \) anymore and its distribution stays constant for the whole cascade. According to Eq. (2.5), we can thus factorize the conditional failure probability of a neighbor in one part that captures the \( \pi \) dependence and another part that stays constant in each iteration step.

\[
P(s_{nb}(t) = 1 \mid k) = \sum_{n=0}^{k-1} b(n, k-1, \pi) P(\Theta(k) \leq \sum_{j=1}^{n} W_j(k)).
\]

With Eq. (2.4), we arrive at the promised scalar valued fixed point equation:

\[
\pi = \sum_{k=1}^{c} p_{nb}(k) P(s_{nb}(t) = 1 \mid k) = \sum_{k=1}^{c} p_{nb}(k) \sum_{n=0}^{k-1} b(n, k-1, \pi) P(\Theta(k) \leq \sum_{j=1}^{n} W_j(k)),
\]

whose solution is enough to calculate the final fraction of failed nodes as

\[
\varrho = \sum_{k} p(k) \sum_{n=0}^{k} b(n, k, \pi) P(\Theta(k) \leq \sum_{j=1}^{n} W_j(k))
\]
where we have used:

$$P(s = 1 | k) = \sum_{n=0}^{k} b(n, k, \pi) P(\Theta(k) \leq \sum_{j=1}^{n} W_j(k)),$$

(3.7)

For general weight distributions, this equation is not known from the literature. But Amini et al. (2010) have derived the final cascade size distribution for weighted networks on directed skeleton networks, while Dodds and Payne (2009); Gleeson and Cahalane (2007) came to this equation for a special choice of weights, i.e. the ED model.

Despite its generality, this simpler approach is not able to capture the cascade dynamics of the damage diversification model.

### 3.2.3 Specialization to ED and DD models

The only piece missing to calculate the final cascade size for the ED and DD models is the specification of the weight distributions $p_{W(d,k)}$.

In both cases, the weights are completely deterministic. We have:

$$P(W_{(ED)}(d,k) = l) = \delta_{1/k}(l), \quad P(W_{(DD)}(d,k) = l) = \delta_{1/d}(l),$$

where $\delta_y(x)$, defined as $\delta_y(x) := 1$ if $x = y$ and $\delta_y(x) := 0$ otherwise, denotes the discrete delta distribution.

We note that $W_{(ED)}(d,k) = W_{(ED)}(k)$ is independent of the degree $d$ of a neighbor that can potentially inflict the loss $1/k$ to a focal node or neighbor with degree $k$. Consequently, the calculation of the average final fraction of failed nodes can be simplified as outlined in the previous section. Specifically, because of $W_{(ED)}(d,k) = W_{(ED)}(k) \sim \delta_{1/k}$, we have

$$P\left( \Theta(k) \leq \sum_{j=1}^{n} W_{j(ED)}(d,k) \right) = P\left( \Theta(k) \leq \sum_{j=1}^{n} \frac{1}{k} \right) = F_{\Theta(k)}\left( \frac{n}{k} \right).$$

(3.8)

Thus, the fixed point equation Eq. (3.5) specializes to

$$\pi_{(ED)} = \sum_{k=1}^{c} p_{nb}(k) \sum_{n=0}^{k-1} b(n, k - 1, \pi_{(ED)}) F_{\Theta(k)}\left( \frac{n}{k} \right).$$

(3.9)

This coincides with the derivations by Gleeson and Cahalane (2007), if the threshold $\Theta(k)$ is independent of the degree $k$ of the focal neighbor.

For the DD case, we note that the weights are independent of the degree $k$ of the potentially load receiving focal node or neighbor, i.e. $W(d,k) = W(d)$. Also this aspect leads to a
3.2. Local Tree Approximation

**Figure 3.1:** Comparison of numerical calculations and simulations, where lines represent the former and symbols in the same color correspond to the latter. The thresholds $\Theta$ are normally distributed with mean $\mu$ and standard deviation $\sigma$ ($\Theta \sim \mathcal{N}(\mu, \sigma^2)$). The initial load $\lambda_0 = 0$ is set to zero. (a) Cascade size evolution for ED: Black circles belong to Poisson random graphs and $(\mu, \sigma) = (0.3, 0.2)$, dark blue plus signs $+$ to scale free networks with $(\mu, \sigma) = (0.3, 0.2)$. Light blue triangles depict Poisson random graphs with $(\mu, \sigma) = (0.5, 0.5)$, while red and the symbol $x$ belong to scale free networks with the same parameters. (b) As in (a), but for DD with cHMF. (c) Final cascade size for the DD case. Results for scale free networks are depicted in black circles for $\sigma = 0.2$ and purple triangles for $\sigma = 0.5$. Poisson random graphs correspond to light purple plus signs $+$ for $\sigma = 0.2$ and to magenta $x$ for $\sigma = 0.5$.

simplification of the fixed point iterations. As apparent from Eq. (3.1), the load that a neighbor distributes to a focal node or neighbor $L_{nb}(k, t - 1)$ is independent of the degree $k$, i.e. $L_{nb}(k, t - 1) = L_{nb}(t - 1)$. Thus, it is not necessary to calculate for each $k$ a new distribution function and convolutions of completely different distributions.

From Eq. (3.1) follows that the impact $L_{nDD}(k, t - 1)$ that a neighbor has on a focal node or neighbor with degree $k$ is distributed as:

$$
\mathbb{P}\left(L_{nDD}^{\text{nb}}(k, t - 1) = 0\right) = (1 - \pi(t - 1)),
$$

$$
\mathbb{P}\left(L_{nDD}^{\text{nb}}(k, t - 1) = \frac{1}{d}\right) = \frac{\mathbb{P}(s_{nb}(t - 1) = 1 \mid d)p(d) d}{z}
$$

for $d \in \{1, \cdots, c\}$ and $\mathbb{P}\left(L_{nDD}^{\text{nb}}(k, t - 1) = l\right) = 0$ otherwise.

**DD case: Correct Heterogeneous Mean Field Approximation (cHMF)**

Since the convolutions are computationally demanding (in terms of time and especially memory), we approximate $p_{L_n^{\text{nb}}(t-1)}^{s(k-1)}$ by first binning $p_{L_n^{\text{nb}}(t-1)}$ to an equidistant grid and then using Fast Fourier Transformations (FFT) in order to take advantage of the fact that convolutions correspond to simple multiplications in Fourier space (Frigo and Johnson, 2005; Ruckdeschel and Kohl, 2014). This reduces the computation of the fraction of
Figure 3.2: (a) Fraction of failed nodes obtained by (simpHMF) for the DD case on Poisson random networks with $\lambda = 2.82, z = 3,$ and $c = 50$. (b) Difference between the correct version (chMF) and a). (c) Fraction of failed nodes obtained by (simpHMF) for the DD case on scale free networks with $\gamma = 3, z = 3,$ and $c = 200$. (d) Difference between the corresponding correct version (chMF) version and c). The thresholds $\Theta$ are normally distributed with mean $\mu$ and standard deviation $\sigma$ ($\Theta \sim \mathcal{N}(\mu, \sigma^2)$).

Failed nodes for fixed threshold and degree distribution parameters in our set-up to a few minutes. Fig. 3.1(c) shows that our numerical results for the final cascade size coincide with simulations. Even in time, our numerical approach fits very well the cascade size evolution as presented by Fig. 3.1(a) for ED and Fig. 3.1(b) for DD.

Neglecting the neighbors’ degrees in the failure probability (simpHMF)

Considering the computational complexity of the chMF approach as described above, it is worth asking whether we can approximate it with a simpler version as, e.g., the one described in Section 3.2.2, and still obtain reasonably good results.

This would require the weights $W(d, k)$ to be independent of the neighbors’ degree $d$. Consequently, we would assume that every failed neighbor inflicts the loss $1/l$ with probability

$$
P \left( W^{(DDsimp)}(d, k) = \frac{1}{l} \right) = \frac{lp(l)}{z}
$$

for $l \in \{1, \cdots, c\}$.

Thus, $W^{(DDsimp)}(d, k)$ is independent of both $k$ and $d$, i.e. $W^{(DDsimp)}(d, k) = W^{(DDsimp)}$. This approximation allows us to calculate the probabilities

$$
P \left( \Theta(k) \leq \sum_{j=1}^{n} L_j^{fnsimp}(k, t - 1) \right) = P \left( \Theta(k) \leq \sum_{j=1}^{n} W_j^{(DDsimp)} \right)
$$

initially without the need to update it in each fixed point iteration. Then, we can solve again the scalar fixed point equation Eq. (3.5).

Although this approach is more convenient, it is inadequate for the damage diversification variant, especially in combination with skewed degree distributions (as, e.g., in case of scale free networks). If we follow this simplified calculation, we would lose the risk reducing
effect by hubs connected to more nodes. This would lead us to draw opposite conclusions about systemic risk. So, as shown in Fig. 3.2, it is crucial to use the correct HMF to explain our simulation results. The simplified method simpHMF leads to an overestimation of the final fraction of failed nodes.

3.3 Numerical Results

Hereafter, we show how the DD variant can reduce systemic risk in comparison to the ED variant by calculating the fraction of failed nodes for both variants and several degree distributions. Alongside, we study the influence of the presence of hubs and the degree variance. This allows us to discuss up to which extend the overall diversification reduces or increases systemic risk. Later on, we also compare how a higher or lower diversification influences the failure risk of a node on the mesoscopic level.

Our findings do not only lead to different conclusions for the two model variants, but also depend on the threshold distribution parameters under consideration.

Similar to Refs. (Watts, 2002) and (Gleeson and Cahalane, 2007) we study normally distributed thresholds $\Theta \sim \mathcal{N}(\mu, \sigma^2)$ with mean $\mu$ and standard deviation $\sigma$, but we explore the role of the thresholds’ heterogeneity as well as their mean size more extensively by providing a 2d-phase diagram as in Ref. (Lorenz et al., 2009). Although our analytic framework also applies to more general cases, here we assume the thresholds to be independent from the degree $k$ of a node.

Recall that the initial fraction of failed nodes is determined by the nodes with negative thresholds ($\Theta \leq 0$) and is thus given by $F_{\Theta}(0) = \Phi\left(\frac{-\mu}{\sigma}\right)$, where $\Phi$ denotes the cumulative distribution function of the standard normal distribution.

3.3.1 Systemic perspective

In principle, all phase diagrams are of similar shape as the one for fully connected networks of infinite size that has been calculated in Ref. (Lorenz et al., 2009) and is depicted in Fig. 3.3 (a). It shows the effect of perfect diversification. Both variants, ED and DD, coincide for this case as well as for regular networks where every node has the same degree $z$.

We observe that increasing the standard deviation can also reduce systemic risk - even though this increases the fraction of initial failures. Of special prominence is the sharp regime shift from a region of small systemic risk to an almost complete system breakdown. The existence of such a regime shift has been first identified for Poisson random graphs and the ED variant by Gleeson and Cahalane (2007). It is present for most other
Chapter 3. Diversification strategies in Constant Load models

Figure 3.3: Phase diagram for the fraction of failed nodes $\varrho$ with normally distributed thresholds ($\Theta \sim N(\mu, \sigma^2)$) for (a) fully connected networks, and (c) regular networks with degree $z = 3$. The darker the color the higher is the systemic risk. The middle panel (b) shows their difference $\varrho^{(a)} - \varrho^{(c)}$.

topologies as well - for both variants ED and DD. The regime shift separates the white region in Fig. 3.3 (a), where perfect diversification is (nearly) optimal, from a region where other topologies could expose the system to a lower systemic risk. For instance, regular networks with degree $z$ reduce the risk for larger threshold parameter $\sigma$, but also increase the risk for small $\mu$ and $\sigma$. Fig. 3.3 (c) shows the respective phase diagram for $z = 3$. Consequently, we cannot expect that other topologies different from the fully connected one lead to smaller systemic risk for all parameters. But the DD variant can expose the system to a lower systemic risk in comparison with ED on the same topology. And both variants can show lower systemic risk for such a topology than fully connected networks for certain threshold parameters.

We test two degree distributions where the ED and DD variants differ. Both, a Poisson and scale free degree distribution, have been introduced in Sec. 2.4.2. As mentioned, the scale free degree distribution is especially interesting for our diversification analysis because of the existence of hubs. These hubs are the well diversified nodes.

In Fig. 3.4 we compare the two variants ED and DD for the two degree distributions with the same average degree $z = 3$ so that we can study the influence of the presence of hubs rather than the overall connectivity indicated by $z$. We observe that the outcome for the ED variant differs for the two degree distributions only in a narrow threshold parameter range. While the DD variant exposes the system to a smaller systemic risk than the ED variant for all parameters, it especially proves risk reducing for higher degree variation. Scale free random graphs expose the system to lower systemic risk than Poisson random graphs, and Poisson random graphs have lower systemic risk than regular random graphs. The presence of hubs, whose failure causes only small damage in their environment, seems to limit the cascade risk. But in general, a majority of hubs does not reduce systemic risk. Cascades in fully connected networks, where every node is a hub, are more amplified for already identified threshold parameters than in other studied topologies. The DD variant together with a high diversity of degrees, so that a small fraction of high degree nodes is combined with a majority of small degree nodes whose failure does not affect a
3.3. Numerical Results

Figure 3.4: Phase diagrams for the final fraction of failed nodes $\varrho$ calculated numerically for our two different degree distributions with average degree $\bar{z} = 3$, diversification variants ED and DD, and their differences. We have always calculated 200 fixed point iterations. The thresholds $\Theta$ are normally distributed with mean $\mu$ and standard deviation $\sigma$ ($\Theta \sim N(\mu, \sigma^2)$). The fraction of initially failed nodes is given by $F_\Theta(0)$. The darker the color the higher is the systemic risk.

First row: Poisson distribution with parameter $\lambda = 2.82$, and cutoff degree $c = 50$ for the ED (left) and DD (right). The middle panel shows their difference $\varrho^{(ED)} - \varrho^{(DD)}$.

Second row: The difference between the diagrams with Poisson and Scale free degree distributions for the ED variant (left). Similarly for the DD variant (right). In the middle panel the initial fraction of failed nodes $\varrho(0)$ is illustrated. $\varrho_o := \varrho(0)$ is constant along the lines $\sigma = \mu/\Phi^{-1}(\varrho_0)$.

Third row: Scale free distribution with exponent $\gamma = 3$, and maximal degree $c = 200$ for the ED (left) and DD (right). The middle panel again shows their difference $\varrho^{(ED)} - \varrho^{(DD)}$.

A high proportion of other nodes, lowers systemic risk considerably in threshold parameter regions where the system is vulnerable to failure cascades.

We present further examples in the Appendix. In fact, a small set of threshold parameters can be found where the ED variant exposes the system to lower systemic risk than DD. This can also be observed for a degree distribution measured from a snapshot of the Italian interbank market in 1st October, 2002 as published in Ref. (De Masi et al., 2006). But
Figure 3.5: (a) Conditional failure probability for ED and normally distributed thresholds with parameters \((\mu, \sigma) = (0.3, 0.2)\) with \(\pi = 0.2\) (black triangles), \((\mu, \sigma) = (0.15, 0.01)\) with \(\pi = 0.1\) (red \(x\)), and \((\mu, \sigma) = (0.7, 0.2)\) with \(\pi = 0.3\) (light blue +). Here, \(\pi\) is chosen independent of a cascade process. (b) Conditional failure probability for ED and normally distributed thresholds with parameters \((\mu, \sigma) = (0.27, 0.09)\) (black circles), \((\mu, \sigma) = (0.3, 0.3)\) (dark blue +), \((\mu, \sigma) = (0.34, 0.19)\) (light blue triangles), \((\mu, \sigma) = (0.7, 0.6)\) (red \(x\)), and \(\pi\) defined by cascade equilibrium. Lines indicate a scale free distribution, while symbols correspond to Poisson random graphs. (c) As in (b), but for DD and red \(x\) refers to parameters \((\mu, \sigma) = (0.5, 0.5)\). Nodes with degree \(k > 10\) have a similar failure probability as a node with degree \(k = 10\).

The DD variant shows lower systemic risk than ED for most parameters also in this case. Additionally, we provide an example where the sudden regime shift in the phase diagram vanishes. Scale free random graphs with very low average degree \(z\) show in general only small systemic risk (that is even smaller for DD than ED).

The presented results demonstrate that not only the overall connectivity \(z\) and robustness (represented by the threshold parameters) determine the final cascade size. Other topological system properties as the specific distributions of degrees and link weights \(w_{ij}\) have risk reduction potential as well.

The interplay of hubs and leaves seems to have an important effect. To understand their different roles in the cascade amplification, we study next the failure risk of nodes conditional on their degree.

### 3.3.2 Mesoscopic perspective

Two quantities reveal the role of the nodes with a given degree \(k\) in a cascade process: (a) their failure probability \(P(s = 1 \mid k)\) and (b) the cascade amplification that their failure triggers. We measure the second quantity (b) as the increase of the fraction of failed nodes \(\varrho\) as response to an increase of their failure probability \(P(s_{nb} = 1 \mid k)\) as neighbor. Thus,
we call the partial derivative
\[ \frac{\partial \rho}{\partial \mathbb{P}(s_{nb} = 1 \mid k)} \]
cascade amplification by a node with degree \( k \).

For the ED variant the conditional failure probability (a) does not depend explicitly on the degree distribution and thus the diversification strategies of the other nodes. It is determined only by the failure probability of a neighbor \( \pi \), which indicates the state of the system in the cascade, and the threshold distribution. Equation (3.7) and (3.8) give:

\[ \mathbb{P}(s = 1 \mid k) = \sum_{n=0}^{k} b(n, k, \pi) F_{\Theta} \left( \frac{n}{k} \right). \]

We would expect that large degrees, and thus, high diversification decreases the failure risk, since the failure of a high number of neighbors is less probable than the failure of fewer neighbors. As shown in Fig. 3.5(a), this intuition applies only to cases of small risk where the failure probability \( \pi \) of a neighbor is small. Here, \( \pi \) is chosen independently of a cascade evolution. For a few parameters with small threshold standard deviation \( \sigma \), also more irregular shapes of the conditional failure probability are possible (see, e.g., the red circles in Fig. 3.5(a)). Still, in a real cascade process, the cascade keeps on-going in those cases until most nodes are failed. Results of real ED cascade processes are depicted in Fig. 3.5(b), where \( \pi \) is calculated in a fixed point iteration corresponding to the studied threshold distribution. In this case, hubs have a higher risk to fail in a cascade than leaves for almost all threshold parameters. Exceptions occur, e.g., \( (\mu, \sigma) = (0.7, 0.6) \), but then the difference in failure risk between hubs and leaves is negligible. Initially, hubs often have a smaller failure risk than leaves. But if cascades get amplified and \( \pi \) increases, hubs face a larger failure risk than leaves. High diversification becomes unfavorable. Intuitively, nodes with a higher degree are more exposed to an ongoing cascade, as they have a higher chance to be hit by it. But the diversification effect most often saturates for degrees \( k > 10 \). Because of this we restrict Fig. 3.5(a) to \( k \leq 10 \). The failure probability of a node with degree \( k = 10 \) differs only marginally from the one of a node with higher degree.

This explains why we observe similar phase diagrams for Poisson and scale free random graphs for the ED variant. The ED variant responds barely to changes in the tail of a degree distribution, as nodes with large degrees have almost identical failure risk.

In contrast, for the DD variant, the outcome \( \rho \) depends crucially on the degree distribution. Still, hubs have a higher failure risk than nodes with a smaller degree in general, see Fig. 3.5(c). Each additional neighbor introduces the possibility of a loss, if it fails. Thus
the shape of the conditional failure probability

\[
P(s = 1 \mid k) = P\left(\Theta \leq \sum_{j=0}^{k} (L^{\text{nb}}_{\text{DD}})_j\right)
\]

always looks similar as the ones presented in Fig. 3.5(c). Consequently, too many nodes with high degrees could increase the vulnerability of the system.

However, the presence of a few hubs also decreases the overall failure risk by decreasing some of the possible losses. We gain this insight by studying the second indicator (b) that describes the role of nodes in the cascade amplification. For both variants, ED and DD, the partial derivative of \( \varrho \) is of the form:

\[
\frac{\partial \varrho}{\partial P(s_{\text{nb}} = 1 \mid k)} = p(k)k \left[ C_{\text{ED/DD}} - \pi \right].
\] (3.10)

The constant \( C_{\text{ED/DD}} \) can be interpreted as failure probability of a neighbor where one of its neighbors is failed and has degree \( k \). Thus, the infinitesimal change in \( \varrho \) is in fact proportional to the increase of the failure probability of a neighbor.

In case of the ED variant, the constant \( C_{\text{ED}} \) does not depend on the degree \( k \) of a node whose failure probability has increased:

\[
C_{\text{ED}} = \sum_{d} \frac{p(d)d}{z} \sum_{n=0}^{d-1} b(n, d - 1, \pi) F_\Theta \left( \frac{n + 1}{d} \right).
\]

Consequently, the failure of hubs is especially problematic, as the cascade amplification is proportional to the degree \( k \) of a failed node. The only way to reduce systemic risk (while preserving the average degree \( z \)) in comparison to regular random graphs is to introduce nodes to the system with smaller degree than \( z \). Then, only a small fraction of hubs needs to exist in order to preserve the average degree \( z \). But the risk reduction is obtained by the high number of small degree nodes.

In case of the DD variant, risk reduction is also obtained by the existence of hubs. The constant \( C_{\text{DD}} \) decreases with degree \( k \):

\[
C_{\text{DD}} = \sum_{d} \frac{p(d)d}{z} \sum_{j=0}^{d-1} (L^{\text{nb}}_{\text{DD}})_j \left( \Theta \leq \frac{1}{k} + \sum_{j=0}^{d-1} (L^{\text{nb}}_{\text{DD}})_j \right).
\]

The cascade amplification (Equation (3.10)) role of a node is decided by the trade-off between the increasing factor \( p(k)k \) and the decreasing factor \( C_{\text{DD}} - \pi \). Especially less diversified nodes (which have a chance to survive also large failure cascades) can benefit from the diversification of others.


3.4 Discussion

Although risk diversification is generally considered to lower the risk of an individual node, on the system level it can even lead to the amplification of failures. With our work, we have deepened the understanding of cascading failure processes in several ways.

At first, we generalize the method to calculate the systemic risk measure, i.e. the average cascade size, to include directed and weighted interactions. The loss that a node experiences because of the failure of a neighbor can depend now also on properties of the neighbor and not only on properties of the node itself. Moreover, we can correctly calculate the time evolution of the cascade size, not only the final steady state. This allows the analytic analysis of more complicated models. For instance, interventions in time as response to an ongoing cascade could be tested and further questions answered as, e.g., how the cascade can be blocked or further damage reduced.

The final cascade size as macro measure is complemented by a measure on the meso level, by calculating individual failure probabilities of nodes based on their degree (diversification).

This allows us to compare two different diversification mechanisms: ED (exposure diversification) and DD (damage diversification). As we demonstrate, nodes which diversify their exposures well (i.e. hubs), have a lower failure risk only as long as the system as a whole is relatively robust. But above a certain threshold for the failure probability of neighboring nodes, such hubs are at higher risk than other nodes because they are more exposed to cascading failures. This effect tends to saturate for large degrees.

In general, most regulatory efforts follow the too big to fail strategy, and focus on the prevention of the failures of systemic relevant nodes - the hubs. This is mainly achieved by an increase of the thresholds, e.g. capital buffers in a financial context or immunization in case of epidemic spreading. However, in reality this is often very costly.

With our study of another diversification strategy, the damage diversification, we suggest to accompany regulatory efforts by mitigating the failure of hubs. By limiting the loss that every node can impose on others, the damage potential of hubs and, thus, the overall systemic risk is significantly reduced. While this is systemically preferable, the DD strategy is a two-edged sword: Hubs face an increased failure risk, but many small degree nodes benefit from the diversification of their neighbors. This lowers the incentives for diversification as long as no other benefit, e.g., higher gains in times of normal system operation, comes along with a high degree.

As we show, the systemic relevance of a node is not solely defined by its degree, or connectivity. The size of its impact in case of its failure, and thus its ability to cause further failures is crucial. It is a strength of our approach that we can calculate this impact analytically and obtain a more refined and realistic identification of system relevant nodes. A
possible indicator for system relevance is the cascade amplification measure that we have derived.

Additionally, our approach can be transferred to degree-degree correlated networks (Dodds and Payne, 2009; Payne et al., 2009). We would expect that a high degree assortativity in the DD could further reduce systemic risk, since the failure risk of diversified nodes could be reduced by connections to hubs whose failures would impact their neighborhood only little.

With our work we have given one example for systemic risk reduction by topological means. Further possibilities can be explored with the analytical framework that we have provided.
Chapter 4

Correlations in Constant Load models: Potential to reduce systemic risk

Summary

We revisit the Constant Load Exposure and Damage Diversification model variants and study the impact of correlations between threshold and degree distribution on systemic risk with the help of a Local Tree Approximation. This way, we explore an additional dimension to reduce systemic risk without changing system performance or requiring additional resources. Along the way, we gain a deeper understanding of the role of well and less diversified nodes in the evolution of a cascade. We test two extreme scenarios where the threshold and degree distribution are perfectly correlated (pf) or perfectly anti-correlated (cf). Both variants have advantages in different parameter regions where a system could operate. This poses a system design question that is not discussed or decisively answered by systemic risk researchers and policy makers. Is the goal to minimize the risk of large cascades or to avoid as many mediocre cascades as possible? And who has a higher priority in the allocation of limited resources - hubs or leaves? Or is there no preference?

This Chapter has been written specifically for this thesis. RB contributed to the design of the research questions, performed the analytic derivations, implemented the code and interpreted the results.
Chapter 4. Correlations in Constant Load models: Potential to reduce systemic risk

4.1 Introduction

According to Gigerenzer et al. (1999) and Gigerenzer (2015), good decisions in an uncertain world often rely on simple rules that acknowledge the fact that a lot of information about a problem at hand is not available. So called rules of thumb lead to good solutions many times, although they require little time and knowledge.

Especially in complex systems, simplifications are crucial to guide decision and policy makers. In principle, system design decisions can have significant and surprising effects in the presence of sudden regime shifts and phase transitions. These effects can be unexpectedly positive, but also devastating. Because of this, care needs to be taken to regard enough information about the ongoing processes. As we show, one rule might be very successful in one setting, but a bad choice in another.

In this chapter, we test two simple rules of thumb for their effect on systemic risk in case of Constant Load models (see Sec. 2.3.1). Both have been introduced already in Sec. 2.4.3 and concern the allocation of thresholds to nodes. Peripheral failures (pf) refer to a case where nodes with higher degrees also receive higher thresholds, while in the opposite case, central failures (cf), nodes with high degree are equipped with lower thresholds.

For instance, policy discussions about financial systemic risk tend to focus on the former, pf. Primarily, large well connected institutions with big exposures are seen as systemically relevant (Arinaminpathy et al., 2012; Roukny et al., 2013) so that higher capital requirements for these banks suggest a systemic risk reduction (Prasanna et al., 2011). In reality, a situation similar to cf is observed by Arinaminpathy et al. (2012). Thus, large institutions have lower relative capital buffers. Another example corresponds to epidemic spreading, where the immunization of hubs seems to be effective (Pastor-Satorras and Vespignani, 2002), again a pf strategy.

A pf threshold allocation is reasonable in models where primarily the failures of hubs tend to amplify cascade processes. As we have discussed in the previous chapter, pf should therefore reduce systemic for the CL ED model. But if the failure of hubs harms the overall system only little, it is worth considering a cf allocation. Still, we find this reasoning not to hold true for all parameters. A direct comparison between the most promising combinations, ED and pf compared with DD and cf, poses a system design question. While the former leads to negligible systemic risk for most threshold parameters, the latter reduces the risk in regions where relatively big cascades occur. Presented with these two options, assuming that the threshold parameters are out of control, the question that decision makers would need to answer is: Do they prefer a relatively small risk of large cascades that lead to an almost complete system break down or do they prefer to reduce the size of these large cascades while accepting a higher risk of their occurrence at the same time?
4.2 The threshold distribution

With additional uncertainty about the concrete structure of exposures, for instance ED or DD, it is impossible to suggest a rule of thumb that is effective in all cases and cannot have devastating consequences. As we show, other combinations than the proposed ones often have cascade amplification effects.

Still, the presented simple threshold allocation schemes are worth a consideration as they have the potential to reduce systemic risk without changing the system performance (under normal operation). A straightforward way to improve system safety with respect to cascade risk is to increase its robustness, i.e. increasing its average threshold $\mu$. Most of the time, this increase in robustness is associated with a cost or resources that cannot be used for something else. Because of this, an increase of robustness is often linked to a decrease in overall system performance. As we have discussed in Sec. 2.3.1 and the previous chapter, the CL models ED and DD define the same total exposures and exposure distribution. Thus, system performance is equal also in this aspect. Chap. 3 suggests the DD approach as a way to reduce systemic risk by topological means in comparison to ED (without changing system performance). Here, we go one step further and test systemic risk reduction options by allocating thresholds not randomly or optimally, but according to reasonable rules of thumb that seem to be simple to implement and preserve the overall threshold distribution.

4.2 The threshold distribution

As in the previous chapter, we study CL models that define weight distributions on infinitely large configuration-type random graphs as introduced in Sec. 2.4.2.

The analytic derivations in Chap. 3 are explained for such general models that we only need to specify how the nodes’ thresholds $\Theta(k)$ depend on the degree $k$ and define their distributions. As mentioned in Sec. 2.4.3, the thresholds for $N$ nodes follows an order statistic so that the threshold distributions become dependent. Their distribution can be calculated easily on finite samples, but their limit distributions usually become degenerate for increasing sample size $N$. This is not of our concern, since we do not have to calculate threshold distributions for single nodes in an infinitely large network. Instead, our goal is to determine the threshold distribution of fractions of nodes with a given degree $k$.

This can be achieved with a picture similar to Fig. 4.1 or Fig. 4.2 in mind. All nodes in an infinitely large network are represented by an interval $[0, 1]$ and ordered according to their degree. Their position in the interval is defined by the degree cdf $F$. Thus, all nodes with degree $k$ can be associated with the cyan interval $[F(k-1), F(k)]$ in Fig. 4.1. This way, we do not depict nodes, but fractions of nodes in the network or their probability mass when randomly sampling from all nodes in the network. Analogously, we order nodes according to their thresholds and represent them by an interval corresponding to the normal cdf.
Φ. In this chapter, we use the notation Φ instead of \( F_\Theta \) to ease the distinction between degree cdf \( F \) and threshold cdf. In case of pf, fractions of nodes with a certain property, as their degree or threshold, can be found at the same position in the two intervals. Consequently, nodes with degree \( k \) are equipped with threshold values in the interval \( \Phi^{-1}(F(k)), \Phi^{-1}(F(k-1)) \), where \( \Phi^{-1} \) denotes the quantile function corresponding to \( \Phi \). In case of a normal distribution, \( \Phi^{-1} \) simply denotes the inverse of \( \Phi \).

Figure 4.1: Illustration of the threshold assignment for peripheral failures. The cyan interval \( [F(k-1), F(k)] \) represents (the density of) all nodes with degree \( k \) in the network, while the purple subinterval can be associated with all nodes in the network which have degree \( k \) and a threshold between \( y \) and \( x \).

With this information, we can derive the cdf of the threshold \( \Theta(k) \) of a node conditional on the fact that its degree is \( k \) for peripheral failures as

\[
F_{\Theta^{(pf)}(k)}(x) = \mathbb{P}(\Theta^{(pf)}(k) \leq x) = \frac{\left| \left[F(k-1), F(k)\right] \cap [0, \Phi(x)] \right|}{\left| \left[F(k-1), F(k)\right] \right|} = \min\left\{ F(k), \Phi(x) \right\} - F(k-1) \mathbb{I}_{\left\{ \Phi(x) > F(k-1) \right\}},
\]

where \( \mathbb{I} \) denotes the indicator function that is defined for any set \( M \) as \( \mathbb{I}_M(x) = 1 \) if \( x \in M \) or \( \mathbb{I}_M(x) = 0 \) otherwise, i.e. \( x \notin M \). This formula measures formally the overlap between the intervals \( [F(k-1), F(k)] \) and \( [0, \Phi(x)] \) and divides by the length of the former. Since we condition on the fact that the considered nodes have degree \( k \) we only draw randomly from nodes in the interval \( [F(k-1), F(k)] \) and thus need to re-normalize by the probability to draw such nodes with degree \( k \). The probability that among these nodes, a node has a threshold smaller or equal to a value \( x \) depends then on the representation of nodes with this property in the considered cyan interval. This coincides with the length of the purple interval in Fig. 4.1.

The same idea applies also to the case cf. However, the fraction of nodes need to be arranged in opposite order. As illustrated in Fig.4.2, the position in the interval corresponds to probability mass \( 1 - \Phi(x) \). Thus, nodes with threshold bigger or equal to \( x \) correspond to the interval \( [0, 1 - \Phi(x)] \), while nodes with threshold smaller or equal to \( x \) belong to \( [1 - \Phi(x), 1] \). Thus, to calculate the fraction of failed nodes with a threshold smaller or equal to \( x \) within the fraction of nodes with degree \( k \), we need to estimate the overlap between intervals \( [1 - \Phi(x), 1] \) and \( [F(k-1), F(k)] \). The cf thresholds of nodes with degree
4.3 Cascade results

Figure 4.2: Illustration of the threshold assignment in case of central failures. The cyan interval \([F(k-1), F(k)]\) represents (the density of) all nodes with degree \(k\) in the network, while the purple subinterval can be associated with all nodes in the network which have degree \(k\) and a threshold between \(y\) and \(x\). In contrast to peripheral failures, nodes with a bigger threshold can be found on the left (and not on the right) of the unit interval.

\(k\) follow the distribution:

\[
F_{\Theta(\cdot), (\cdot)}(x) = \mathbb{P}\left(\Theta^{(\cdot)}_{\mu, \sigma}(k) \leq x\right) = \frac{|F(k-1), F(k)| \cap |1 - \Phi(x), 1| |}{|F(k-1), F(k)| |} = \frac{F(k) - \max \{F(k-1), 1 - \Phi(x)\}}{p(k)} \mathbb{1}_{\{F(k) > 1 - \Phi(x)\}}(k).
\]

(4.2)

With this choice of threshold distributions, we can calculate the fraction of failed nodes by an LTA as explained in Chap. 3. Fig. 4.3 shows the perfect agreement of our numerical calculations with simulations.

4.3 Cascade results

We give an overview about all possible combinations of ED and DD model variants with pf and cf threshold allocation schemes in Fig. 4.4. Again, we present results for Poisson and scale free degree distributions with the same average degree \(z = 3\) as introduced in Sec. 2.4.2.

We clearly find our intuition confirmed that pf in combination with ED reduces systemic risk tremendously. This effect is even stronger for scale free networks in comparison to Poisson random graphs. This is surprising to some extent, as the presence of a higher fraction of hubs could mean that not all of them can be protected by high thresholds and their failure could still amplify large cascades. But the high diversification also seems to additionally reduce their failure risk. In Poisson random graphs, nodes with thresholds that would survive cascades in a scale free network because of their large degree, can still fail because enough of their smaller number of neighbors can trigger their failure. We account the irregular shape of the phase diagrams corresponding to ED and DD noticeable. Our
Figure 4.3: Comparison of numerical calculations and simulations, where lines represent the former and symbols in the same color correspond to the latter. We compute $T = 50$ fixed point iterations and compare with simulations on networks of size $N = 10^5$. The thresholds $\Theta$ are normally distributed with mean $\mu$ and standard deviation $\sigma$ ($\Theta \sim N(\mu, \sigma^2)$). (a) Final cascade size for pf: Results for scale free networks and $\sigma = 0.5$ are depicted in light blue triangles for the ED model and in red $x$ for the DD model. Poisson random graphs and $\sigma = 0.2$ correspond to black circles for ED and to dark blue plus signs $+$ for DD. (b) As in (a), but for cf.

Simulations confirm this observation and cannot be attributed to numerical instabilities in our approximations. Higher order effects seem to be important for the determination of the phase transition.

The cf in combination with ED leads as expected to increased systemic risk, also in comparison to a random threshold allocation. The system is quite vulnerable to the failure of hubs. Because of this, we observe that for large robustness $\mu$ scale free networks are exposed to even higher systemic risk than Poisson random graphs in this case. But for smaller $\mu$, Poisson random graphs face higher systemic risk. Both situations should be avoided by system designers.

We can therefore attest scale free - but also Poisson - random graphs a robust-yet-fragile property with respect to ED cascades similar to the one with respect to percolation.

For DD, the picture is less definite. Clearly, cf reduces systemic risk for regions where we expect relatively large cascade sizes, but shows higher risk in threshold parameter regions where we would expect small cascades, even in comparison to DD with pf. We still need to point out that there exists a small parameter region where DD rf (random threshold allocation) exposes the system to lower systemic risk than DD cf.

At first sight, ED in combination with pf seems to expose a system to the lowest systemic risk in comparison to all studied cases. Interestingly, although not directly visible in our selection of difference diagrams, we point out that there exists a small threshold parameter
4.3. Cascade results

Figure 4.4: Phase diagrams for the final fraction of failed nodes \( \varrho \) calculated numerically according to a LTA for our two different degree distributions with average degree \( z = 3 \), diversification variants ED and DD, and threshold allocation schemes pf or cf. SF stands for scale free random networks, while Poisson indicates Poisson random graphs. We have always calculated 50 fixed point iterations. The left column presents results for pf and the right column results for cf. The middle column shows their difference \( \varrho^{(pf)} - \varrho^{(cf)} \).

region in which DD pf outperforms ED pf.

An even better candidate to reduce systemic risk in regions of relatively big cascades is the combination DD cf. Because of this, we present a direct comparison of the best candidates so far: ED pf and DD cf in Fig. 4.5.

Fig. 4.5 poses the mentioned system design question most clearly. No combination outperforms another for all threshold parameters. Our analysis boils down to the question what is preferred - small probabilities of events of devastating nature or a higher probability of cascades of mediocre size, but reduced extreme risk?

In fact, such a preference might not be the only criterion for system design. For large enough \( \mu \), where systems usually operate, ED pf seems to be the most promising variant.
Chapter 4. Correlations in Constant Load models: Potential to reduce systemic risk

Figure 4.5: Phase diagram for the fraction of failed nodes $\varrho$ with thresholds distributed according to the order statistics obtained from a normal distribution $N(\mu, \sigma^2)$ on scale free random graphs with average degree $z = 3$. (a) ED with pf, and (c) DD with cf. The middle panel (b) shows their difference $\varrho^{(a)} - \varrho^{(b)}$.

(a) (b) (c)

Figure 4.6: Fraction of failed nodes $P(s(t) = 1; k)$ with degree $k$ in Poisson random graphs that fail at the time indicated by the color of the bar. Light blue corresponds to fractions of nodes that remain functional. Nodes with higher degrees than 20 are omitted ($k > 20$), as their fraction is small in comparison to the others. (a) Peripheral failures with ED. (b) Central failures with DD. The thresholds are normally distributed with parameters $\mu = 0.2$ and $\sigma = 0.3$.

Furthermore, with our measure of systemic risk, we equally weight nodes with high and low degree. Often, this does not reflect real preferences. Fig. 4.6 gives prominence to the fact that a choice for ED pf or DD cf also is a clear decision for the degree of surviving nodes. While ED pf protects hubs, DD cf protects leaf nodes. Moreover, we observe another advantage of ED pf by focusing on the cascade dynamic as illustrated in Fig. 4.6. Cascades for ED pf seem to evolve more slowly than for DD cf. This gives more time for interventions that could hinder cascades to grow larger.

4.4 Conclusions

We perform an LTA and derive therefore the threshold distribution of nodes with a given degree. This allows us to study the systemic risk reducing potential of two simple rules
of thumb: One that allocates higher thresholds to well diversified nodes, i.e. nodes with higher degree (pf), and one that allocates thresholds in opposite order, i.e. nodes with higher degree receive a lower threshold (cf).

Both represent a way to reduce systemic risk without affecting system performance (under normal system operation) or requiring additional resources. Neither is the dependence structure nor the exposure distribution changed. Available thresholds are simply reallocated according to the degree of nodes.

Both threshold allocation scenarios have different effects in combination with different diversification strategies of nodes, the already discussed ED and DD CL cascade models. Also both can lead to a cascade amplification in the wrong combination.

Lowest systemic risk can be achieved by ED pf and DD cf. While DD has proven to outperform ED quite consistently under random threshold allocation, this picture has changed for our correlation analysis. The region of high systemic in our phase diagrams is smallest for ED pf. Moreover, cascades usually grow slower in time, which would give more options for effective interventions that block large cascades. Finally, hubs have a higher probability to withstand large cascades. Often hubs are more relevant to keep a system functioning. DD cf clearly focuses on leaves instead of hubs in reducing their failure risk. Even though the parameter region of mediocre cascades is larger, cascades are not of such severe size as for ED pf. Thus, DD cf serves better policy makers or system designers with high risk aversion.

Both combinations, ED pf and DD cf, have advantages and disadvantages that need to be taken into account.
Chapter 5

Load redistribution in infinite random graphs

Summary
We present a branching process approximation for Load Redistribution models with Local Load Sharing and load Shedding (LRD LLSS). Accordingly, we derive an iterative procedure to calculate the cascade size evolution as network ensemble average, which allows us to discuss the role of well and less connected nodes in the course of cascade. The heterogeneity and time dependence of the load that a failing neighbor distributes to neighboring nodes are considered so that our analytic results match perfectly our simulations.

Furthermore, we compare our simulation results for the LLSS and LLSC variants and highlight similarities and differences with respect to other cascade models in our framework. This way, we successively identify common overarching principles.

This chapter has been written specifically for this thesis. RB contributed to the development of the research questions and discussions. RB derived the analytic equations and wrote the code for the LTA. RB analyzed and interpreted the results. Results for the LRD LLSC model are partially based on R. Burkholz, A. Garas, D. Längle, and F. Schweitzer, “Systemic risk from cascading processes on random and scale free networks”. Working Paper.
5.1 Introduction

Fiber bundle models have become one of the most prevalent models to describe fracture in materials (Sinha et al., 2015). Especially, the LRD LLSC model has been studied on several topologies (Pradhan et al., 2010), in particular on random graphs that are similar to our approach by Kim et al. (2005).

As discussed in Ch. 2, the network topology can be interpreted as dynamic in this case and changes according to an evolving cascade. This creates a high path dependence of the network structure on the precise steps of the cascade process and, even more problematic, the clustering increases. This destroys the locally tree-like network structure on which an LTA would rely on. Consequently, we cannot calculate the cascade size corresponding to the LRD LLSC model analytically.

But, we can handle the LRD LLSS model. As we see in the following chapter, this is already a non-trivial methodological extension of the existing (and also the so far presented) analytic description of cascade models. The additional challenge is introduced by the variable load that a failing node distributes, because it also depends on the cascade evolution and thus on the time of failure of the node. Thus, additional to the failure probability of neighbors, we also need to keep track of the distribution of the load that a neighbor distributes at a certain point in time before the failure of a node.

In future, it might be possible to tackle the LRD LLSC model by a combination of the following work and an approach by Hackett and Gleeson (2013) to allow for non-zero clustering in an LTA as well. But this goes beyond the scope of this thesis.

Accordingly, we mainly focus on the LRD LLSS model here. For a more in depth analysis of the LRD LLSC model mainly by the means of simulations, we refer to Ref. (Kim et al., 2005; Pradhan et al., 2010). Our own simulation results in Ch. 2 serve mainly the comparison with the LRD LLSS model. We discuss briefly, why we observe increased systemic risk for the LLSC case.

Next, we present our derivations for the analytic ensemble average of the cascade size for the LLSS model and illustrate similarities with the CL DD model. In particular, our focus lies on the role of hubs and leaves in the course of a cascade.

5.2 Local Tree Approximation for the LRD model: Local Load Sharing with load Shedding

In Sec. 2.6.1, we have provided the basis for the Local Tree Approximation of the LRD LLSS model. To deduce the fraction of failed nodes \( \varphi(t) \), we have to derive the distribution of the load \( L^\text{nb}(k, t - 1) \) that a neighbor distributes at time \( t - 1 \) or before to a focal node.
(or neighbor) with degree $k$.

As our following explanation is complicated enough, we restrict ourselves to the case of the LRD LLSS model. A generalization to weighted networks is still straightforward considering the following argumentation steps.

Recall that a focal neighbor with degree $k$ accumulates a load

$$\Lambda^{nb}(k,t) = \sum_{j=1}^{k-1} L_{nb}^{j}(k, t-1)$$

over time. If it exceeds its threshold ($\Lambda^{nb}(k,t) \geq \Theta(k)$) at time $t$ the focal neighbor fails and distributes exactly the load $\Lambda^{nb}(k,t)$ to its $n_h$ functional network neighbors (including the focal node under consideration). Thus, each functional neighbor receives the load $\Lambda^{nb}(k,t)/n_h$.

How does this relate to the load $L_{nb}^{j}(k, t-1)$ that a neighbor spreads before or at time $t$?

### Load distribution by a neighbor

First, we note that $\Lambda^{nb}(k,t)/n_h$ is independent of the degree of the load receiving focal node or neighbor. The degree $k$ belongs here to the failed neighbor. Consequently, we can skip the degree dependence and write $L_{nb}^{j}(k, t-1) = L_{nb}^{j}(t-1)$. In general, the load that is distributed is influenced by the degree of the failing neighbor. But $L_{nb}^{j}(t-1)$ is the load that is distributed by neighbor unconditional on the degree of this neighbor (thus without the knowledge of its degree). The distribution of $L_{nb}^{j}(t-1)$ incorporates the possibility of a neighbor having different degrees.

Second, in contrast to our previous derivations for constant load models, the distributed load is subject to the time $t$ of a neighbor’s failure. Therefore, we need to ensure that we respect the right order and time of failures. Only load that is distributed at time $t$ can trigger new failures at time $t+1$. Moreover, load is not distributed to neighbors that fail at the same time.

To distinguish between failures before time $t$ and at $t$, we introduce a new random variable, the load $\Delta L^{nb}(t)$ that is distributed by a neighbor that fails exactly at time $t$ and distributes load at time $t$.

Let’s assume that we know the distribution of load $L^{nb}(t-1)$ by neighbors before time $t$. We can then update $L^{nb}(t-1)$ to

$$p_{L^{nb}(t-1)}(l) = p_{L^{nb}(t-1)}(l) + p_{\Delta L^{nb}(t)}(l), \quad \text{for all } l \in \mathbb{R}\{0\}. \quad (5.1)$$

Thus, we have reduced our problem to the derivation of the pdf corresponding to the load.
\( \Delta L^{nb}(t) \) that is distributed at time \( t \).

**Load distribution at time \( t \)**

Considering that we have two time steps in the history, failures at time \( t \) and \( t - 1 \), we derive the updated load distribution at time \( t + 1 \). The distribution of \( \Delta L^{nb}(t + 1) \) is determined by several factors, i.e., the degree \( k \) of the failing focal neighbor, the number \( n_o \) of neighbors of this focal neighbor that have failed at \( t - 1 \) before the failure could be triggered, the number \( n_n \) of newly failed neighbors at time \( t \) that distribute enough load so that the threshold \( \Theta(k) \) of the focal neighbor is exceeded and finally the number \( n_a \) of actual neighboring failures that do not receive load by the failing focal neighbor, since they fail at the same time \( t + 1 \).

Let’s assume the \( n_o \) old failed neighbors have distributed a load \( \sum_{i=1}^{n_o} L^{nb}_i(t-1) = l_o \) before or at time \( t \), where each of the \( n_o \) neighbors has distributed a nonzero \( L^{nb}_i(t-1) > 0 \) load. Additionally, the \( n_n \) nodes distribute at time \( t \) the load \( \sum_{j=1}^{n_n} \Delta L^{nb}_j(t) = l_n \), where again each of the \( n_n \) neighbors has distributed a nonzero \( \Delta L^{nb}_j(t) > 0 \). We know that the threshold \( \Theta(k) \) of the focal neighbor under consideration must be bigger than \( l_o + \lambda_0 \), otherwise, it would have failed already. Moreover, \( l_n + l_o + \lambda_0 \geq \Theta(k) \), since the focal neighbor fails at time \( t \). Under these assumptions, this neighbor distributes the load

\[
\Delta L^{nb}(t + 1) = \frac{\lambda_0 + l_o + l_n}{k - n_o - n_n - n_a} = \frac{\lambda_0 + l_o + l_n}{n_s + 1}
\]

to its remaining \( n_s + 1 \) functional network neighbors. \( n_s = k - 1 - n_o - n_n - n_a \) denotes here the number of surviving neighbors that could also have failed before the focal neighbor.

With this observation, we simply need to sum up or integrate all possible probabilities for the cases that lead to the distribution of a load \( l \). This corresponds to a stepwise application of Bayes’ Theorem that determines the state of the neighborhood of a focal neighbor, i.e. the variables that influence the amount of load that is distributed.

**The state of the neighborhood**

We start with conditioning on the degree \( k \) of the focal neighbor and then proceed with the different numbers of failed neighbors \( n_o, n_n, n_a \) and the loads that they distribute (or not distribute). First, we have for \( l > 0 \)

\[
P(\Delta L^{nb}(t + 1) = l) = \sum_{k=1}^{\epsilon} \frac{p(k)k}{z} P(\Delta L^{nb}(t + 1) = l \mid K^{nb} = k).
\]
Next, we recall that we know already the distributions of $L^{nb}(t-1)$ and $\Delta L^{nb}(t)$. Therefore, we can calculate the pdf of $L^{nb}(t)$ as well according to Eq. (5.1) and derive the probability

$$\pi(t + 1) = \sum_{k=1}^{c} \frac{p(k)}{z} \sum_{n_0 = 0}^{k-1} \sum_{n_n = 0}^{k-1-n_0} \sum_{n_a = 0}^{k-1-n_n} \sum_{n_s = 0}^{k-1-n_a-n_n} \pi_t(k) \leq \sum_{j=1}^{k-1} L^{nb}_j(t)$$

that a neighbor fails at time $t+1$ or before. Consequently, a neighbor of the focal neighbor has failed with probability $\pi_t(t)$ at time $t$ or before, with probability $\pi_t(t+1) - \pi_t(t)$ it fails exactly at time $t+1$, and with probability $1 - \pi_t(t+1)$ it is still functional at time $t+1$. Because of the locally tree like network structure in the limit of infinite network size of the configuration model, neighbors fail independently of each other. Thus, the random variables $(N_o, N_n, N_a, N_s)$ that indicate the number of failed or non-failed neighbors at specified times follow a multinomial distribution

$$(N_o, N_n, N_a, N_s) \sim M(k-1, \pi(t-1), \pi(t), \pi_t(t-1), \pi_t(t+1) - \pi_t(t), 1 - \pi_t(t+1)).$$

With Bayes’ Theorem we can further deduce:

$$P(\Delta L^{nb}(t + 1) = l) = \sum_{k=1}^{c} \frac{p(k)}{z} \sum_{n_0 = 0}^{k-1} \sum_{n_n = 0}^{k-1-n_0} \sum_{n_a = 0}^{k-1-n_n} \sum_{n_s = 0}^{k-1-n_a-n_n} \int_{0}^{\infty} \frac{(k-1)!}{n_o!n_n!n_a!n_s!} \times (1 - \pi_t(t+1))^{n_o} (\pi(t+1) - \pi_t(t))^{n_n} (\pi_t(t) - \pi(t))^{n_a} \pi_t(t-1)$$

$$\times P(\Delta L^{nb}(t + 1) = l \mid K^{nb} = k, N_o = n_o, N_N = n_n, N_a = n_a).$$

Instead of conditioning the load that is distributed on the failure of the corresponding neighbors, we calculate directly the probability that a node fails and distributes a specific nonzero load. Next, we consider thus the probabilities $(\pi_t(t) - \pi(t-1))^{n_n}$ and $\pi^{n_a}(t-1)$ already when counting all possible load distribution options that lead to a total load $l$ that is distributed to each of the $n_s + 1$ remaining functional neighbors:

$$P(\Delta L^{nb}(t + 1) = l) =$$

$$= \sum_{k=1}^{c} \frac{p(k)}{z} \sum_{n_0 = 0}^{k-1} \sum_{n_n = 0}^{k-1-n_0} \sum_{n_a = 0}^{k-1-n_n} \sum_{n_s = 0}^{k-1-n_a-n_n} \frac{(k-1)!}{n_o!n_n!n_a!n_s!} \times (1 - \pi_t(t+1))^{n_o} (\pi(t+1) - \pi(t))^{n_n}$$

$$\times P \left( \sum_{j=1}^{n_n} \Delta L^{nb}_j(t) = x \mid \Delta L^{nb}_j(t) > 0 \right) \left( F_{\Theta(k)}(l(n_s + 1)) - F_{\Theta(k)}(l(n_s + 1) - x) \right)$$

$$\times P \left( \sum_{i=1}^{n_s} L^{nb}_i(t-1) = l(n_s + 1) - \lambda_0 - x \mid L^{nb}_i(t-1) > 0 \right),$$

The sum runs over all discrete values $x$ of the load distributed by failures at time $t$, since for the LRD LLSS model the load distribution is discrete. We note that we have $l_n = x$. 

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such that we obtain \( l_o = l(n_s + 1) - \lambda_0 - x \). Then, the load \( (l_n + l_o + \lambda_0)/(n_s + 1) = l \) is newly distributed to a functional neighbor at time \( t + 1 \). The probability

\[
\mathbb{P}(\lambda_0 + l_o \leq \Theta(k) \leq \lambda_0 + l_o + l_n) = F_{\Theta(k)}(l(n_s + 1)) - F_{\Theta(k)}(l(n_s + 1) - x)
\]

captures the case that the focal neighbor can withstand the load \( \lambda_{nb}(k, t) = l_o + \lambda_0 \), but fails because of the load \( \lambda_{nb}(k, t + 1) = l_n + l_o + \lambda_0 \).

### 5.2.1 Cascade dynamics

In summary, we can describe the full cascade dynamics of the LRD LLSS model for infinitely large configuration type random graphs (where the second moment of the degree distribution is finite).

Initially, we start from the pdf of the load that is distributed by initially failing neighbors:

\[
\mathbb{P}(\Delta L_{nb}(0) = \frac{\lambda_0}{n_s + 1}) = \frac{k p(k)}{z} F_{\Theta(k)}(\lambda_0) b(k - 1 - n_s, k - 1, \pi(0))
\]

for \( n_s = 0, \cdots, c - 1 \) and where \( \pi(0) \) is defined as

\[
\pi(0) = \sum_{k=1}^{c} \frac{p(k)k}{z} F_{\Theta(k)}(\lambda_0).
\]  

(5.3)

\(\mathbb{P}(\Delta L_{nb}(0) = 0) = 1 - \pi(0)\) otherwise. Just formally, we define the pdf of the load that is distributed by nodes before time \( t = 0 \) as zero:

\[
\mathbb{P}(L_{nb}(-1) = 0) = 1.
\]

On this basis, we can construct all following time steps iteratively. At time \( t + 1 \) we first determine the failure probability of a neighbor at time \( t + 1 \) as:

\[
\pi(t + 1) = \sum_{k=1}^{c} \frac{p(k)k}{z} \mathbb{P}(\Theta(k) \leq \sum_{j=1}^{k-1} L_{nb}^j(t))
\]

and compute next the distribution of load by a neighbor exactly at time \( t + 1 \). In the following \( n_o \) denotes the number of previously failed neighbors (before or at time \( t - 1 \)), \( n_n \) the number of neighbors that have failed at time \( t \), \( n_a \) the number of neighbors that fail at time \( t + 1 \) like the focal neighbor and \( n_s \) the remaining functional neighbors of the \( k - 1 \) possibly failing neighbors. \( F_{\Theta(k)} \) is the threshold distribution of a focal node or
neighbor with degree \( k \). We obtain:

\[
P(\Delta L_{nb}^{(t + 1)} = l) = \sum_{k=1}^{c} p(k) k \sum_{n_o=0}^{k-1} \frac{(k-1)!}{n_o!n_1!n_2!} (1 - \pi(t + 1))^{n_o} (\pi(t + 1) - \pi(t))^{n_2} \\
\times \left( \sum_{j=1}^{n_o} \Delta L_{j}^{nb}(t) = x ; \Delta L_{j}^{nb}(t) > 0 \right) \left( F_{\Theta(k)}(l(n_s + 1)) - F_{\Theta(k)}(l(n_s + 1) - x) \right) \\
\times \left( \sum_{i=1}^{n_o} L_{i}^{nb}(t - 1) = l(n_s + 1) - \lambda_0 - x ; L_{i}^{nb}(t - 1) > 0 \right),
\]

for \( l > 0 \) and \( P(\Delta L_{nb}^{(t + 1)} = 0) = 1 - (\pi(t + 1) - \pi(t)) \).

Then we update the pdf of the load \( L_{nb}^{(t - 1)} \) that is distributed by a neighbor before or at time \( t - 1 \) to \( L_{nb}^{(t)} \) by

\[
P(L_{nb}^{(t)} = l) = P(L_{nb}^{(t - 1)} = l) + P(\Delta L_{nb}^{(t)} = l), \quad \text{for all } l \in \mathbb{R}\backslash\{0\}
\]

and \( P(L_{nb}^{(t)} = 0) = 1 - \pi(t + 1) \). Finally, with this definition, we determine the fraction of failed nodes \( \varrho(t + 1) \) as:

\[
\varrho(t + 1) = \sum_{k=1}^{c} p(k) P(\Theta(k) \leq \sum_{j=1}^{k} L_{j}^{nb}(t)) = \sum_{k=1}^{c} p(k) \sum_{x} F_{\Theta(k)}(x)p_{L_{nb}^{(t)}}(x).
\]

Despite the fact that the involved load distributions are discrete, we approximate the distributions on an equidistant grid in our numerical calculations. This allows us to compute the convolutions of load distributions by Fast Fourier Transformation as in Chapt. 3 and Chapt. 4. The general procedure is explained in Appendix A.2.

### 5.3 Results

We could repeat the same analysis as in the previous two chapters for the Constant Load models. Conceptually, we only need to specify the threshold distribution, degree distribution, and the initial load \( \lambda_0 \).

In case of random failures, we assume that the thresholds follow a normal distribution independent of the degree of a node, i.e. \( \Theta \sim \mathcal{N}(\mu + \lambda_0, \sigma^2) \). We add \( \lambda_0 \) to the mean threshold value to ensure that nodes fail with the same initial probability as in all other studied cases. With the same set of parameters, but otherwise as in Chapt. 4, we can define the threshold distributions in case of correlations between threshold and degree distribution.
Chapter 5. Load redistribution in infinite random graphs

Figure 5.1: Comparison of numerical LTA calculations and simulations for the LRD LLSS model and Poisson random graphs (with average degree $z = 3$), where lines represent the LTA and symbols in the same color correspond to simulation results. The thresholds $\Theta$ are normally distributed with mean $\mu$ and standard deviation $\sigma$ ($\Theta \sim \mathcal{N}(\mu, \sigma^2)$). The initial load is $\lambda_0 = 0.5$. (a) We show the cascade size evolution. Black circles belong to $(\mu, \sigma) = (0.3, 0.2)$, dark blue plus signs $+$ to $(\mu, \sigma) = (0.5, 0.5)$. Light blue triangles depict $(\mu, \sigma) = (0.3, 0.7)$, while red and the symbol $x$ belong to $(\mu, \sigma) = (0.7, 0.3)$. (b) Final cascade size after $T = 300$ fixed point iterations. Black circles belong to $\sigma = 0.2$ and light blue triangles to $\sigma = 0.3$. Dark blue plus signs $+$ depict $\sigma = 0.5$, while red and the symbol $x$ belong to $\sigma = 0.7$.

We skip a lengthy analysis to avoid unnecessary repetition and only report on the main findings. An overview about systemic risk in all introduced cases can be found in Sec. 2.5.2. We only add information here that goes beyond this overview.

First, we have to prove that our numerical approximation is accurate. Fig. 5.1 shows perfect agreement between our simulation and our LTA results on Poisson random graphs. While Fig. 5.1(b) compares final cascade sizes for several threshold parameters, Fig. 5.1(a) considers the time evolution.

Fig. 5.2 explores the role of the additional variable $\lambda_0$. This extra degree of freedom in the model clearly influences the size of the region of high systemic risk, but does not lead to a qualitative change of its form.

Interestingly, the network topology has a stronger impact on the shape of the region of high systemic risk. Especially for smaller $\lambda_0$, the LRD LLSS model on Poisson random graphs seems to be a mixture between a LRD model on a fully connected network and Constant Load Damage Diversification model (CL DD) on Poisson random graphs. Fig. 5.3 shows all cases for comparison. For small mean threshold $\mu$ and increasing $\sigma$, the LRD LLSS model does not have a second sharp regime shift towards small systemic risk in case of Poisson random graphs. This makes it similar to the CL DD model and distinguishes the
5.3. Results

Figure 5.2: Final fraction of failed nodes $\varrho$ for LRD LLSS after 50 fixed point iterations for Poisson random graphs (with average degree $z = 3$) obtained by an LTA. The thresholds are independently normally distributed with mean $\mu + \lambda_0$ and standard deviation $\sigma$ ($\Theta \sim \mathcal{N}(\mu + \lambda_0, \sigma^2)$). Thus, the fraction of initial failures is identical for all plots. (a) $\lambda_0 = 0.3$, (b) $\lambda_0 = 0.4$, (c) $\lambda_0 = 0.5$.

Poisson random graphs from a fully connected topology.

Figure 5.3: Final fraction of failed nodes $\varrho$ (after 200 fixed point iterations) with independently normally distributed thresholds ($\Theta \sim \mathcal{N}(\mu + \lambda_0, \sigma^2)$) for infinitely large networks. (a) Poisson random graphs (with average degree $z = 3$) (a) LRD LLSS model on fully connected topology with $\lambda_0 = 0.3$, (b) LRD LLSS model on Poisson random graphs with $\lambda_0 = 0.3$, (c) CL DD model on Poisson random graphs.

Both models, CL DD and LRD LLSS, share the independence of the load $L^{nb}(t)$ from the degree $k$ of the load receiving node or neighbor. Consequently, each additional neighbor only increases the chance of additional load uptake so that nodes with a higher degree face a higher failure probability. Fig. 5.4 (b) illustrates how the failure probability grows faster over time for nodes with a higher degree. Initially, all nodes fail with the same probability, but already at $t = 1$, when the load distribution starts, high degree nodes are stronger impacted by a cascade.

We have presented the CL DD model as an option to mitigate the impact that the failure of hubs have on a system. Also in case of the LRD models, the load that a node with higher degree distributes is shared among a higher number of functional network neighbors. But in contrast to the CL DD model, nodes with a high degree and high threshold can accumulate a higher amount of load over time and distribute this among a potentially smaller number of still functional network neighbors. Because of this, it is not clear a
priori whether the presence of hubs supports system stability. We still find that scale free networks are exposed to lower cascade risk than Poisson random graphs with the same average degree $z = 3$ for most parameters.

This similarity with the CL DD model suggests that an arrangement where nodes with a higher degree receive lower thresholds, which we have termed central failures (cf), should further reduce systemic risk. Furthermore, the such designed early failure of hubs would prevent them accumulating high proportions of load.

Indeed, as for the CL DD model, we find that the LRD LLSS model with cf reduces the size of large cascades. Still, for threshold parameter regions of lower systemic risk, random failures (rf) or even peripheral failures (pf), where nodes with higher degree receive higher thresholds, can be advantageous. In these regions, the failure of hubs can be almost completely avoided by rf or pf, because of overall high system robustness. This hinders the amplification of cascades.

**Barabási Albert networks**

Next, we present a small interlude of our presented framework.

Although not mentioned earlier, we have also performed simulations on Barabási-Albert (BA) scale-free networks (Barabási and Albert, 1999) with $m = 4$. These also have a power law degree distribution with exponent $\gamma = 3$, but are not uncorrelated and might
5.4. Local Load Conservation in comparison

Figure 5.5: Final fraction of failed nodes $\varrho$ for simulations of LRD LLSS cascades as average over 100 independent realizations of BA networks consisting of $N = 1000$ nodes. The thresholds are normally distributed with mean $\mu + \lambda_0$ and standard deviation $\sigma (\Theta \sim N(\mu + \lambda_0, \sigma^2))$, where the initial load is $\lambda_0 = 0.4$. We consider three different threshold allocations: (a) pf, (b) rf, (c) cf.

We present our results for the LLSS model, because they are noticeable and surprising. In fact, this model has already been studied by Moreno et al. (2002) on BA networks. But the thresholds have been independently distributed according to a Weibull distribution. First and foremost, we are interested in the effect of correlations between threshold and degree distribution.

Fig. 5.5 shows the average over 100 independent network realizations of the fraction of failed nodes for normally distributed thresholds. Remarkably, the central failures threshold allocation is extremely effective in the reduction of systemic risk. In no other case have we encountered such a shape of the phase diagram and so small systemic risk for very small mean threshold values $\mu$.

How this can be explained needs to be investigated further, but goes beyond the framework studied in this thesis. We hypothesize that the protection of a highly connected core consisting of nodes with a high degree and thus also high threshold in the cf scheme hampers cascades effectively.

5.4 Local Load Conservation in comparison

In Ch. 2, we have seen that the LRD LLSC model leads to higher cascade sizes than the LRD LLSS model in general and have attributed this observation to the growing interconnectivity of networks in case of LLSC, which introduces a high similarity with cascades on fully connected networks (Kim et al., 2005).

This increasing connectivity creates nodes with large degrees, especially if hubs fail early and all their functional neighbors become connected. Because of this, cf usually increases systemic risk in comparison to rf or pf for LLSC in contrast to LLSS. For the initial load
size $\lambda_0 = 0.5$, we cannot observe a high difference between different thresholds allocations in Fig. 2.10. But for smaller initial loads, the explained difference becomes apparent.

A high connectivity amplifies failure cascades, if too high amounts of load are distributed. If many nodes are already close to failure and they all are formed to a fully connected clique, only a few of them need to be hit by the cascade, distribute their load and carry the others down as well in the next step. Thus, the damage diversification effect by high degree nodes is small in comparison to the increase of failure probability that a high degree brings along.

With the OLRD model, where much smaller amounts of load are distributed, we observe the opposite. There, the damage diversification effect by hubs seems to prevail their increased failure probability, simply because most of the nodes of formed cliques are not close to failure.

### 5.5 Conclusions

We have presented an LTA that correctly captures the time evolution of the LRD LLSS model by analytic means. The most crucial innovation in comparison with the LTA for CL models as presented in Ch. 3 is the iterative calculation of the load distribution by a random neighbor that has failed before a given time $t$. In comparison with simulations, our LTA approximation leads considerably faster to results than simulations and captures accurately the cascade size on large networks, even if simulations become impossible in this setting.

The correctness of time evolution might allow in future research to add process complexity in time, for instance, similar as in the model for creep rupture (Danku et al., 2015; Halász et al., 2012; Kun et al., 2008, 2009) and still obtain an analytic description of the cascade size.

We have noted that the load that a neighbor distributes is independent of the degree of the receiving node in case of both studied models, the LLSS and LLSC variant. This is a property that they share with the earlier discussed CL DD model. Consequently, the larger the degree of the node the higher is its failure risk. Despite the high damage diversification for the failure of hubs, an increasing connectivity of the networks (as in case of LLSC model) tends to increase systemic risk, if enough initial load is distributed.

Still, without increasing connectivity in case of the LLSS model, the cf threshold allocation reduces systemic risk remarkably, especially for BA random graphs. Cf leads to an early failure of hubs and prevents thus a high accumulation of load over time until they fail. Furthermore, it disconnects the system early on which lowers the opportunity for cascades to propagate through the entire network.
5.5. Conclusions

On the contrary, the connectivity even increases in case of LLSC so that systemic risk is usually higher for cf in comparison with rf and, in particular, pf.

We have attributed the difference between the LLSS and LLSC variants to a trade-off between increasing failure risk for nodes with high degree and the systemic risk reducing effect by a better damage diversification that is caused by the failure of nodes with a higher degree.

In the next chapter, we revisit the OLRD model, where this trade-off is decided in the opposite direction. There, LLSC leads to lower systemic risk than LLSS. Furthermore, we extend our LTA to match OLRD type models.
Chapter 6

Overload redistribution in infinite random graphs

Summary

Our main contribution in this chapter is our analytic branching process approximation of the cascade size evolution for the OLRD LLSS model. Furthermore, we compare cascade results of the OLRD and LRD model and explain why we observe higher systemic risk for the LRD LLSC model compared with the LRD LLSS model, but observe the reverse effect for the OLRD counterparts. This observation refers to a trade-off between increased exposure and increased damage diversification of well connected nodes. We hypothesize that this trade-off is of specific relevance in the early evolution of a cascade.

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RB has written this chapter specifically for this thesis. RB contributed to discussions and the development of the research questions. RB derived the analytic equations and wrote the code for the LTA. RB analyzed and interpreted the results. Simulations are partially based on R. Burkholz, A. Garas, D. Längle, and F. Schweitzer, “Systemic risk from cascading processes on random and scale free networks”. Working Paper.
6.1 Introduction

Neither the LLSC nor the LLSS model variants in our OLRD class have been studied on configuration model type random graph ensembles up to our knowledge. Still, the aspect of cascading overload failures is a widely observed phenomenon and it is a reasonable assumption that a system has the capacity to absorb a certain amount of load. Or overloaded components do not fail and keep still functioning, but process only a certain amount of load while the remaining load is distributed to other nodes in the system. Often, this redistribution of load follows more complicated procedures. For instance, it depends on the betweenness of nodes in a network. This would require global information about the network topology, which we cannot consider in a Local Tree Approximation (LTA).

Here, we focus on simplified model versions, where load is distributed locally and which we can compare with their LRD counterparts. For the same reasons as in case of LRD, we cannot describe the cascade size evolution of the LLSC model analytically. The increasing clustering violates our assumption of a locally tree like network structure.

For the OLRD LLSS model, we derive an LTA and compare our results with our findings for the LRD model.

6.2 Local Tree Approximation for the OLRD model: Local Load Sharing with load Shedding

The Overload Redistribution Local Load Sharing with load Shedding model is similar to its Load redistribution variant and so is its Local Tree Approximation. An additional challenge is introduced by the fact that the load $L^{nb}(t)$ that a neighbor distributes before or at time $t$ is not discrete, if the thresholds are continuously distributed. Because of this, we deduce the cumulative distribution function of the load $\Delta L^{nb}(t)$ that a neighbor distributes exactly at time $t$ and $L^{nb}(t)$ instead of its probability distribution function (or density in this case).

Let’s assume a focal neighbor with degree $k$ that fails at time $t$ carries the load $\lambda^{nb}(k, t)$, which exceeds its threshold $\theta(k)$. With exactly $n_s + 1$ functional network neighbors, the focal neighbor distributes to each of them the load

$$\frac{\lambda^{nb}(k, t) - \theta(k)}{n_s + 1} = \sum_{i=1}^{n} l_i^{nb}(t - 1) \frac{k - n}{k - n},$$

if $n$ of its network neighbors have failed before and have distributed a load $l_i^{nb}(t - 1)$ and the initial load $\lambda_0$ is set to zero ($\lambda_0 = 0$). As for the LRD model variant, this load does not depend on properties of the receiving functional network neighbors. Consequently, the
load $\Delta L^{nb}(t)$ that is distributed by any neighbor (without the knowledge of its degree or threshold) only depends on the time $t$ of failure of such a neighbor.

The cdf of $\Delta L^{nb}(t)$ is of similar form as Eq. (5.2.1) for the LRD model. Again, we apply Bayes’ Theorem and consider all possible degrees $k$ of a focal neighbor, and all cases in which $n$ neighbors (of the focal neighbor) are failed, among which $n_o$ have failed before without leading to the failure of the focal neighbor, $n_a$ fail at the same time as the focal neighbor, and $n_s$ survive.

As for its LRD counterpart, the cdf of $\Delta L^{nb}$ can be calculated iteratively. Thus, assuming the knowledge of the distribution of $\Delta L^{nb}(t + 1)$, we can calculate:

$$
\mathbb{P}(\Delta L^{nb}(t + 1) \leq l) =
\sum_{k=1}^{c} \frac{p(k)k^{k-1}n_o!n_s!}{z^n} (1 - \pi(t+1))^{n_s} (\pi(t+1) - \pi(t))^{n_a} 
\times \left[ \mathbb{P} \left( 0 \leq \sum_{j=1}^{n} L_j^{nb}(t) - \Theta(k) \leq l(n_s + 1) ; L_j^{nb}(t) > 0 \right) - \sum_{n_a=0}^{n_o} \left( \begin{array}{c} n \\ n_a \end{array} \right) \mathbb{P} \left( 0 \leq \sum_{i=1}^{n_a} L_i^{nb}(t - 1) \right) 
\right] 
+ \sum_{j=1}^{n-n_o} \Delta L_j^{nb}(t) - \Theta(k) \leq l(n_s + 1) ; L_j^{nb}(t - 1) > 0 ; \Delta L_j^{nb}(t) > 0 ; \sum_{i=1}^{n_o} L_i^{nb}(t - 1) - \Theta \geq 0 
\right],$$

for $l > 0$. The first term in square brackets quantifies the probability that a neighbor is failed if it receives at least the load $l$ by exactly $n$ of its neighbors that distribute a nonzero load before or at time $t$ to the focal neighbor. The second term subtracts the probability that the neighbor has failed already before time $t$, because of the load that $n_o$ of its neighbors have distributed to it. $\pi(t+1)$ is defined by Eq. (5.2) exactly as for the LRD LLSS model (or the CL DD model).

Theoretically, we can construct an iterative procedure from this relation as for the LRD case, where each iteration step corresponds to a time step. But, naturally, our numerical implementation cannot regard the continuity of the distribution of $\Delta L^{nb}(t + 1)$. We have to discretize the distribution. It is more convenient to calculate the pdf instead of the cdf of the discretized version for the Fast Fourier Transformations that we have to perform. Thus, we explain our discretization steps for the pdf and the iterative procedure as we implement it next.

### 6.2.1 Discretization and iteration

First, we define a finite interval $[0, b]$ which defines the range of load size that a node can carry before it fails and also the load size that can be distributed from node to node. Thus, the bound $b$ is mainly defined by the support of the threshold distribution $F_{\Theta(k)}$. For a good approximation, we require that $F_{\Theta(k)}(b) \approx 1$ and $F_{\Theta(k)}(-b) \approx 0$. Although we
later on consider normal distributions, which have theoretically support \( \mathbb{R} \), numerically, the choice \( b = 5 \) satisfies our requirements for all considered parameters.

**Discretization**

In the next step, we discretize this interval equidistantly in \( b/h \) bins of length \( h \), and represent each resulting small interval \([v, u]\) by its upper bound \( u \). Furthermore, we add 0 to the set of representatives \( \mathcal{L} := \{0, h, 2h, \ldots, b\} \), which defines also the set of load that can be distributed.

**Binning**

Next, we bin the threshold distribution accordingly and define the discretized thresholds as random variables \( \hat{\Theta}(k) \) by

\[
P\left( \hat{\Theta}(k) = l \right) := F_{\Theta(k)}(l) - F_{\Theta(k)}(l - h)
\]

for all \( l \in \mathcal{L} \setminus \{0\} \) and \( P\left( \hat{\Theta}(k) = l \right) = 0 \) for other \( l \in ]0, b] \setminus \mathcal{L} \).

Accordingly, we also define \( \hat{\Theta}(k) \) on the negative side \([-b, 0]\) with \( P\left( \hat{\Theta}(k) = -b \right) = F_{\Theta(k)}(-b) \). But this part of the threshold distribution only needs to be considered in the definition of the initial load distribution, because all nodes with negative load fail initially. Further iteration steps only depend on values of \( \hat{\Theta}(k) \) in \( \mathcal{L} \setminus \{0\} \).

**Initial load distribution**

Each node \( i \) with negative threshold \( \theta_i \) fails initially and distributes the load \(-\theta_i\) to its functional network neighbors. Consequently, the binned load that is distributed by a failed neighbor \( \Delta L^{nb}(0) \) after the initial failures is distributed by

\[
P\left( \Delta L^{nb}(0) = l \right) = \sum_{k=1}^{\infty} \frac{p(k)k}{z} \sum_{n=0}^{k-1} b(k - 1 - n, k - 1, \pi(0))
\]

\[
\times ( F_{\Theta(k)}(-l(k - n)) - F_{\Theta(k)}((-l - h)(k - n))) \mathbb{1}_{[-b, 0]}((-l - h)(k - n))
\]

\[
\times F_{\Theta(k)}(-b) \mathbb{1}_{(-\infty, -b]}((-l - h)(k - n))
\]

for \( l \in \mathcal{L} \). This expression simply allocates the probability that a load \(-\theta_i/(k - n)\) that is distributed by a node with negative threshold \( \theta_i \) and degree \( k \) to its \( k - n \) functional

---

1The best choice to minimize the discretization error would be the mean value \((u + v)/2\) of the interval. But for implementation simplicity, we take the upper bound instead and choose a sufficiently small \( h \) for good approximation results.
network neighbors to the right bin in the interval \([0, b]\). The initial failure probability of a neighbor \(\pi(0)\) is defined as usual, see e.g. Eq. (5.3) with \(\lambda_0 = 0\).

Additionally, we formally define the pdf of the load that is distributed by nodes before time \(t = 0\) as zero:

\[
P\left(\hat{L}_{nb}(-1) = 0\right) = 1.
\]

Thus, we have defined the initial quantities of interest and can proceed with the iteration steps.

**Iteration step**

The iteration step follows the same reasoning as the derivation of the theoretical cdf of \(\Delta L_{nb}(t + 1)\). We only write it down for the pdf with respect to discretized variables:

\[
P\left(\Delta L_{nb}(t + 1) = l\right) = \sum_{k=1}^{c} \frac{p(k)k}{z} \sum_{n=0}^{k-1} \sum_{n_a=0}^{k-1-n} \frac{(k-1)!}{n!n_a!n_s!} (1 - \pi(t + 1))^{n_s} (\pi(t + 1) - \pi(t))^{n_a} \\
\times \left[ P\left(\sum_{j=1}^{n} \hat{L}_{nb}^j(t) - \hat{\Theta}(k) = l(n_s + 1) ; \hat{L}_{nb}^j(t) > 0\right) - \sum_{0 < x < l(n_s + 1)} P\left(\sum_{j=1}^{n} \Delta \hat{L}_{nb}^j(t) = x ; \Delta \hat{L}_{nb}^j(t) > 0\right) \right],
\]

for \(l \in \mathcal{L}\{0\}\) and \(P\left(\Delta L_{nb}(t + 1) = 0\right) = 1 - (\pi(t + 1) - \pi(t))\). The sum over \(x\) runs over values \(x \in \mathcal{L}\) that fulfill the given constraint. The distribution of the sums of random variables are calculated with the help of a Fast Fourier Transformation (FFT). \(^2\)

The fraction of failed nodes can then be computed as in Eq. (5.4).

### 6.3 Results

For an overview of the final cascade size in all different studied cases for the OLRD type model, we refer to our simulation results depicted in Fig. 2.11 on page 34.

Here, we only give an example for a phase diagram obtained by our numerical approach in Fig. 6.1 and skip all other cases to avoid unnecessary repetition. In principle, also different threshold allocations could be considered analytically with the threshold distributions derived in Ch. 4. Fig. 6.2 provides evidence that our numerical calculations correctly

\(^2\)We add \(b\) on both sides of equations with the summand \(-\Theta(k)\) and consider the positive variable \(b - \Theta(k)\) instead of the negative \(-\Theta(k)\) in the FFT.
Chapter 6. Overload redistribution in infinite random graphs

Figure 6.1: Final fraction of failed nodes $\varrho$ for the OLRD LLSS model after 50 fixed point iterations for Poisson random graphs with parameters $\lambda = 2.82$ and cut-off degree $c = 50$ so that the average degree is $z = 3$. The thresholds are independently normally distributed with mean $\mu$ and standard deviation $\sigma$ ($\Theta \sim \mathcal{N}(\mu, \sigma^2)$).

capture the cascade size dynamics as well as the (average) final fraction of failed nodes.

The main question that is left to answer from our discussions in Ch. 2 and Ch. 5 is why we observe lower systemic risk for the OLRD LLSC model than for the OLRD LLSS variant, although we see the opposite for the LRD analogues. The effects are marginal, as initial failures are barely amplified to large cascades on average. Still, we would expect similar patterns overall.

At first sight, we also note no strong difference in the relation between the failure probabilities of nodes with a given degree for the LRD LLSS model. Fig. 6.3 and Fig. 5.4 on page 80, which corresponds to LRD with LLSS, show similar representations of failed nodes with a certain degree. A closer look then reveals that among the failed nodes - nodes with a higher degree are represented by a slightly higher fraction for LRD than for the OLRD case.

The main reason for this observation is that cascades evolve over a longer time horizon and lead to higher failure probabilities in case of the LRD LLSS model. Nodes with a higher degree are simply more exposed to such cascades. As we see in Fig. 6.3 (b) and Fig. 5.4 (b), the difference between the failure probabilities of nodes with a higher degree and a smaller degree develops over time until mainly leaves follow the hubs in the last time steps.

This insight underlies also our hypothesis why the LLSC variant leads to smaller systemic risk than the LLSS variant, when a non-critical amount of load is distributed by initial failures in the beginning of a cascade. In the early evolution of a cascade, a high interconnectivity is advantageous from a system perspective, because hubs do not face a considerably higher failure risk, but still diversify the damage to their neighborhood well.
in case of their failure. A cascade stops early, when the distributed load is small enough to trigger further failures. This is usually the case for the OLRD model, because a small amount of load, i.e. only the overload, is distributed in general.

In case of the LRD model, the increasing exposure of well connected nodes outweighs their damage diversification, which is not enough to prevent further failures. Thus, the growing interconnectivity in a LLSC cascade means increased amplification of failures.

But the final answer to the question why we observe slightly lower systemic risk for the OLRD LLSC than the OLRD LLSS model would require further statistics on the micro-dynamics of the studied processes. We omit these here, as we consider this case of less relevance for a systemic risk analysis, since the overall systemic risk is quite small in all cases.

### 6.4 Conclusions

We have presented an analytic branching process approximation of the cascade size evolution for the OLRD LLSS model that allows the estimation of the average cascade size in a configuration type random graph for the OLRD LLSS cascade model, if the studied de-
Figure 6.3: Time evolution of failures for Poisson random graphs (with average degree $z = 3$) obtained by an LTA for the OLRD LLSS model. The thresholds are independently normally distributed with mean $\mu = 0.3$ and standard deviation $\sigma = 0.7$. The color indicates the time of failure of the respective fraction of failed nodes. (a) Probability that a node has degree $k$ and is failed at the time indicated by the color. (b) Probability that a node with degree $k$ fails at the time indicated by the color.

gree distribution has a finite second moment. This approach can be generalized to capture also models of OLRD type where the load is not equally distributed among the functional network neighbors, but instead in heterogeneous proportions.

Furthermore, we have discussed in more depth an explanation for the observation that the OLRD LLSC model leads to lower systemic risk than the OLRD LLSS model, although the opposite holds for their LRD counterparts. For the studied threshold distributions and parameters, the OLRD model leads usually to small amounts of load distributions in the first steps of a cascade. A stronger damage diversification of these small amounts, as in case of the LLSC model, can then prevent further failures and stop a cascade process early on. In the LRD LLSC model, the increased exposure of well connected nodes to neighboring failures outweighs the damage diversification effect in case of their failure, because the distributed load is still enough to trigger additional failures. Interestingly, the trade-off between increased damage diversification and increased exposure to failures by hubs is especially relevant in the early stages of a cascade.
Part II

Extensions and limitations of modeling assumptions

“Everything should be made as simple as possible, but not simpler.”

Albert Einstein
Introduction

In Part II, we study extensions of the cascade models in the framework of Part I. For simplicity, we often focus on the simplest case, the CL ED model. Still, our derivations can be transferred to the other model classes as well.

We are mainly interested in the question how robust are results are in general with respect to changes in our modeling assumptions. We challenge this way our previous findings and test, whether more knowledge is required about systems to make reasonable estimations of inherent systemic risk.

We face the problem that the models under investigation are highly-nonlinear, discontinuous and show critical or chaotic behavior. This means, we cannot argue in a fashion that we study a particular effect, e.g., the diversification behavior of nodes with a given degree distribution and add model complexity, for instance we consider furthermore degree-degree correlations, a second network layer, or systemic feedback. And then, the difference between the new results and old results without the model extension can simply be attributed to the new effect. Instead, the results for both effects, the old and new, do not simply add up to a new observation as for linear models. The effects interfere, amplify or dampen each other.

Because of this, we study next three aspects that we find critical in our previous investigations.

First, we ask for the consequences of simplifying network structures. As we have mentioned in Ch. 2, a network is a model for interaction or dependency patterns between similar components in a system. Usually, these interactions do not follow the exact same rules. Because of this, we study in Ch. 7, what happens if we introduce a second network layer and another type of link. This introduces a feedback mechanism between the two network layers.

In Ch. 8, we further ask for the consequences if a similar feedback does not result from a different network layer. Instead, the thresholds of nodes are coupled to the macro state of the system, thus respond to global information on the cascade size evolution. This could be also interpreted as global load distribution that shows similarities with LRD models on fully connected networks. Thus, we investigate the variability of our cascade model
Chapter 6. Introduction

classes.

Ch. 9 we devote to the weakest point of our branching process approximation, which assumes infinitely large networks. These serve as estimate of the average cascade size. But we show by several examples that a focus on infinite networks crucially underestimates the variability of cascade outcomes in finite systems. The size of a system clearly needs to be considered in the design or control of its interdependencies.
Chapter 7

Systemic risk in multiplex networks with asymmetric coupling and threshold feedback

Summary

We study cascades on a two-layer multiplex network, with asymmetric feedback that depends on the coupling strength between the layers. In this extension, we focus on the Constant Load Exposure Diversification model on each layer. Based on an analytical branching process approximation, we calculate the systemic risk measured by the final fraction of failed nodes on a reference layer. The results are compared with the case of a single layer network that is an aggregated representation of the two layers. We find that systemic risk in the two-layer network is smaller than in the aggregated one only if the coupling strength between the two layers is small. Above a critical coupling strength, systemic risk is increased because of the mutual amplification of cascades in the two layers. We even observe sharp phase transitions in the cascade size that are less pronounced on the aggregated layer. Our insights relate to a scenario where firms decide whether they want to split their business into a less risky core business and a more risky subsidiary business. In most cases, this may lead to a drastic increase of systemic risk, which is underestimated in an aggregated approach.

Based on R. Burkholz, M. V. Leduc, A. Garas, and F. Schweitzer, “Systemic risk in multiplex networks with asymmetric coupling and threshold feedback”. Physica D: 323-324, 64 - 72. RB contributed to the design of the research questions. RB derived the analytic formulas, implemented parts of the fixed point iterations, contributed to the interpretation of the results and the writing.
7.1 Introduction

In Part I of this thesis, we have considered simplifying dependency structures that were solely defined by the degree distribution of a network. Also the type of dependency has been the same. Here, we get an impression how limiting this assumption can be by introducing a second network layer and interactions between those layers for one of the models that we have introduced, the Constant Load Exposure Diversification model. Although we follow a generic modeling approach, we motivate our model choice by firms with core and subsidiary businesses each represented in one of the two layers. Although this application is rather academic considering our simplifying network and process assumptions, it identifies a possible application area where we would expect similar effects as we can observe in our model. Moreover, our multiplex approach underlines that a systemic risk analysis can be substantially misleading, if the nature of different network layers is neglected (Burkholz et al., 2016b).

In many situations, cascading failures can be influenced by the combination of different types of interactions between the individual components of the system. For instance, this is the case in interbank systems, where banks are exposed to each other via different types of financial obligations (loans, derivative contracts, etc.) (e.g. Ref. (Arinaminpathy et al., 2012) and (Montagna and Kok, 2013)). The bankruptcy of a bank can thus cascade to other banks in non-standard ways. Another example are firms diversifying their activities across different business units, each of which is exposed to cascade risk in its own field of activity.

An important question we wish to investigate in this chapter is how diversification across different types of interactions can affect the risk of cascading failures. For that purpose, we study the case of a firm that diversifies its activities across a core-business unit and a subsidiary-business unit. Each business unit is exposed to other firms’ business units in the same sector of business activity (either core or subsidiary). This means that a business unit can fail (i.e. go bankrupt) as a result of a cascade of failures (i.e. bankruptcies) in the same sector of business activity.

The question of the structuring of a firm into sub-units has been studied from a different angle in the financial economics literature (e.g. (Lewellen, 1971) and (Maksimovic and Phillips, 2002)) and often focuses on the efficiency of the allocation of its resources across different industries. Another question that has received some attention is that of whether a firm can diversify the risk of its income streams by operating in different business areas. For example, Levy and Sarnat (1970), Smith and Schreiner (1969) and Amihud and Lev (1981) studied how conglomerates can diversify the risk associated with their revenue streams from the perspective of portfolio theory.

Here, we use a complex networks approach and we view the system of firm activities as an
interconnected multi-layered network (see (D’Agostino and Scala, 2014; Gao et al., 2012; Garas, 2015; Kivelä et al., 2014)). The distinct layers of this network contain individual networks defined by a particular type of interaction according to a given business activity, while the inter-connections between layers allow for cross-layer interactions. In this setting we develop a model where failures (i.e. bankruptcies) on two different network layers affect firms asymmetrically: The first layer represents exposures between the firms in the core business while the second layer represents exposures between firms in the subsidiary business. Failure (i.e. bankruptcy) of a firm’s core business unit implies failure of its subsidiary business unit, whereas failure of a firm’s subsidiary business unit only causes a shock to the firm’s resistance threshold in its core business unit (see Fig. 7.1 for an illustration). We find that when the coupling strength from the core to the subsidiary layer is varied only slightly, there is a sharp transition between a safe regime, where there is no cascade of failures, and a catastrophic regime, where there is a full cascade of failures. Moreover, when comparing the two-layer network to the single-layer network formed by aggregating the two layers, we find that cascades can be larger on the two-layer network than on the aggregated one. On the other hand, by varying the strength of the feedback between the two layers, we identify the existence of a regime where the two-layer network is safer than the aggregated one and another regime where the reverse holds. This points to the critical importance of the coupling of the layers when structuring a firm into different business units. Also, dealing with aggregated network data that ignores the fine structure of the coupling between different layers can lead to significant underestimation or overestimation of cascade risk.

This Chapter is structured as follows. In Sec. 7.2, we describe the two-layer cascade model. In Sec. 7.3, we derive a branching process approximation as an approximation for large networks and use it to analyze the aforementioned phenomena. These phenomena are presented in Sec. 7.4 where we compare our analytical results with simulations and analyze further the observed phase transitions. In Sec. 7.5, we conclude and interpret the consequences of our theoretical investigations for the application to networks of firms that might decide about merging their core and their subsidiary business.

7.2 Model

We first consider a finite model with \(N\) firms. Each firm can be represented by a node \(i\) present on each of two different layers: layer 0 (the core-business layer) and layer 1 (the subsidiary-business layer). Each layer \(l \in \{0, 1\}\) has a topology represented by an adjacency matrix \(G_l\). On each layer \(l\), node \(i\) can be in one of two states \(s_i^l \in \{0, 1\}\), healthy \((s_i^l = 0)\) or failed \((s_i^l = 1)\). \(s_i^0 = 1\) represents the bankruptcy of firm \(i\)’s core-business unit, whereas \(s_i^1 = 1\) represents the bankruptcy of its subsidiary-business unit. This state is determined by two other variables as in the previous chapters: a node’s fragility or loss on
Chapter 7. Systemic risk in multiplex networks with asymmetric coupling and threshold feedback

Figure 7.1: Illustration of a system with asymmetrically coupled layers. A failure (or bankruptcy) on the Core Business layer implies a failure on the Subsidiary Business layer. This coupling is illustrated by an inter-layer dependency link (red arrow). On the other hand, a failure on the Subsidiary Business layer only decreases a node’s failure threshold on the Core Business layer. This coupling is illustrated using dashed black arrows. The intra-layer links represent business relations or other forms of interactions due to normal business.

A given layer $\lambda_i^l$, which accumulates the load a node carries, and its threshold $\theta_i^l$ on that layer, which determines the amount of load it can carry without failing. Whenever the fragility exceeds the threshold $\lambda_i^l \geq \theta_i^l$, the node fails on that layer and cannot recover at a later point in time.

On each layer, we assume that a cascade of failures spreads according to the threshold failure mechanism of the ED model as introduced in Sec. 2.3.1. Thus a node fails if a sufficient fraction of its neighbors have failed. The fragility of a node $i$ of degree $k_i^l$ on layer $l$ (i.e. a node with $k_i^l$ neighbors on layer $l$) can be expressed as

$$\lambda_i^l(k_i^l) = \frac{1}{k_i^l} \sum_{j \in \text{nb}_l(i)} s_j^l = \frac{n_i^l}{k_i^l}$$

(7.1)

where $\text{nb}_l(i)$ is the set of nodes in $i$’s neighborhood on layer $l$ and $n_i^l$ is the number of failed neighbors on layer $l$. This failure mechanism is useful to model a firm diversifying its exposure to failure risk across neighbors: the more neighbors a node has, the less it is exposed to the failure of a single neighbor. A cascade of failures thus starts with an initial fraction of failed nodes. These failures can then spread to their neighbors in discrete time steps. The load $\lambda_i^l$ of a node $i$ is thus updated at each time step $t$. We have studied this model extensively on single-layer networks in Chapt. 3 and Chapt. 4.

In a financial or economic setting, where the nodes are assumed to represent firms that possess simplified versions of balance sheets, the fragility and threshold can be expressed in
terms of the assets and capital of a firm. In this case, the fragility \( \lambda_i^l \) of a node \( i \) represents the loss that a firm encounters divided by its total assets \( A_i^l \) in layer \( l \). According to the ED model, \( i \) has the same financial obligation \( w_i^l = A_i^l / k_i^l \) to each of its neighbors in layer \( l \), we therefore have \( \sum_{j \in \text{nb}(i)} w_i^l = A_i^l \) and

\[
\lambda_i^l(k_i^l) = \frac{\sum_{j \in \text{nb}(i)} s_j^l w_i^l}{A_i^l} = \frac{n_i^l}{k_i^l}.
\] (7.2)

The threshold \( \theta_i^l \) of a node in layer \( l \) signifies similarly the ratio between a node’s capital buffer \( C_i^l \) and its total assets \( A_i^l \):

\[
\theta_i^l = \frac{C_i^l}{A_i^l}.
\] (7.3)

Consequently, when a node \( i \) fails and its fragility exceeds its threshold \( (\lambda_i^l(k_i^l) \geq \theta_i^l) \), equivalently its loss \( n_i^l w_i^l \) exceeds its capital buffer \( C_i^l \). A more detailed explanation of the ED model in the context of balance sheets can be found, for instance, in Battiston et al. (2012b); Brummitt and Kobayashi (2015). This financial interpretation (among other reasons) inspired several authors to extend the ED model to a multiplex setting (Brummitt and Kobayashi, 2015; Brummitt et al., 2012b; Lee et al., 2014; Yağan and Gligor, 2012). In Brummitt and Kobayashi (2015); Brummitt et al. (2012b); Yağan and Gligor (2012), the failure of a node is determined by considering or aggregating weighted variables of all network layers, while Lee et al. (2014) combines the approach of Brummitt et al. (2012b) with dependency links (Buldyrev et al., 2010). Yağan and Gligor (2012) assumes a hierarchy of layers that is interpreted as debts of different seniority in financial networks where a failure in one layer can only occur after the failure of the node in all preceding layers.

Here we investigate an asymmetry between layers, through which both layers can affect each other via a non-symmetric dependency. More precisely, the two network layers are related by *partial dependency links* (see Fig. 7.1). These are directed links connecting a node on layer 0 to its alter ego on layer 1. These links are characterized by weights \( r_{01}, r_{10} \in [0,1] \) affecting the size of a shock to a node’s threshold on a given layer following a failure on the other layer. Precisely, the failure of node \( i \) on layer 1 reduces its threshold \( \theta_i^0 \) on layer 0 in the following way

\[
\theta_i^{0,r} = (1 - r_{10})\theta_i^0,
\] (7.4)

while the failure of node \( i \) on layer 0 reduces its threshold \( \theta_i^1 \) on layer 1 in the following way

\[1\text{ Please note that }A_i^l \text{ does not denote a random variable for once.}\]
\[ \theta^{i,r}_1 = (1 - r_{01})\theta^i_1. \]  

(7.5)

In the remainder of this chapter, we will set \( r_{01} = 1 \). Thus, the failure of node \( i \) on layer \( 0 \) (the core-business layer) automatically leads to its failure on layer \( 1 \) (the subsidiary-business layer), since its shocked threshold becomes \( \theta^{i,r}_1 = 0 \) and the failure condition \( \lambda^i_1 \geq \theta^{i,r}_1 \) is trivially satisfied even in the absence of any failed neighbors on layer \( 1 \). On the other hand, we allow \( r_{10} \) to take values in \([0, 1]\).

For \( r_{10} = 1 \), both layers are fully inter-dependent, meaning that the failure of a node on one layer implies its failure on the other. In this special case, we have normal dependency links between the two layers as introduced in Buldyrev et al. (2010). For \( r_{10} = 0 \), the failure of a node on layer \( 1 \) does not affect its failure on layer \( 0 \). For \( r_{10} \in (0, 1) \), we have an asymmetric inter-dependency between layers. In the remainder, we call \( r_{10} \) the coupling strength between the two layers.

This models our case of interest, where layer \( 0 \) is interpreted as a firm’s activities in a core business, whereas layer \( 1 \) can be interpreted as a firm’s activities in a subsidiary business. The failure of the firm in the subsidiary business does not necessarily imply its failure in the core business, but the loss incurred reduces its ability to withstand the failure of neighbors in the core-business. The size of this loss is given by a fraction of a firm’s threshold, i.e. \( r_{10} \cdot \theta^0_i \). In the following, we test for the consequences of varying values of \( r_{10} \). In the extreme case of \( r_{10} = 0 \), the subsidiaries are bankruptcy remote, i.e. their failure (interpreted as bankruptcy) has no negative effect on their core-business. At least in the financial crisis of 2008, this has been rarely the case and subsidiaries that were designed to be bankruptcy remote (as for example structured investment vehicles or special purpose entities) inflicted a loss to their parents, as for example it was the case of Citi (p.60, Stoeckel (2013)). Therefore, in reality usually it is \( r_{10} > 0 \).

On the other hand, the failure of the firm in the core business however implies its failure in the subsidiary business. It is thus logical to choose the fraction of failed nodes on layer \( 0 \) when the cascade has reached steady state as the appropriate measure of systemic risk, i.e.

\[ \varrho^{(N)}_0 = \lim_{t \to \infty} \frac{1}{N} \sum_{i=1}^{N} s^i_0(t). \]  

(7.6)

This quantity can also be interpreted as the failure or default probability of a firm in the core business layer \( 0 \).
7.3 Local tree approximation

7.2.1 Aggregation

We compare our measure of systemic risk in a multiplex setting \( \varrho_0^{(N)} \) with the corresponding measure in a network where we aggregate the two layers to a single one. This aggregation could be interpreted as decision of all (or most) of the firms to merge their businesses of the core and subsidiary layer. Accordingly, the assets as well as the capital buffers of both layers are added to form the assets and capital buffer of an aggregated firm:

\[
A_{agg}^i = A_0^i + A_1^i, \quad C_{agg}^i = C_0^i + C_1^i.
\]

(7.7)

For simplicity, we also assume a simple ED model on the aggregated network. Consequently, merged businesses are equally exposed to all of their neighbors. Thus, a node \( i \) that had degree \( k_0^i \) in layer 0 and degree \( k_1^i \) in layer 1 has now degree \( k_{agg}^i = k_0^i + k_1^i \) and is exposed equally to each neighbor by \( w_{agg}^i = A_{agg}^i / k_{agg}^i \).

Since merging the businesses should not change the size of their exposures, we assume \( w_{agg}^i = w_0^i = w_1^i \) for consistency with the ED model. It follows from

\[
\frac{A_0^i + A_1^i}{k_0^i + k_1^i} = \frac{A_0^i}{k_0^i} = \frac{A_1^i}{k_1^i}
\]

(7.8)

and \( C_l^i = \theta_l^i \cdot A_l^i \) that the thresholds in this case are given by:

\[
\theta_{agg}^i = \frac{C_{agg}^i}{A_{agg}^i} = \frac{C_0^i}{A_0^i} = \frac{k_0^i \theta_0^i + k_1^i \theta_1^i}{k_{agg}^i}.
\]

(7.9)

7.3 Local tree approximation

In order to make the model analytically tractable, in this section we restrict ourselves to a special class of configuration type multiplex networks. For these, we assume that each node is characterized by two degrees, \( k_0, k_1 \), and two thresholds, \( \theta_0, \theta_1 \), which can be different on layers \( l \in \{0, 1\} \). These values are drawn independently from the degree distributions \( p_l(k_l) \) and the cumulative threshold distributions \( F_l(\theta_l) \).

As on single layer networks, in the limit of infinite network size \( N \to \infty \), the clustering coefficient that quantifies the chance that any two neighbors of a given node are also neighbors converges to zero. It means that the network layers are locally tree-like, i.e., it does not contain short cycles. This then allows us to develop a branching process approximation for the final fraction of failed nodes \( \varrho_l := \lim_{N \to \infty} \varrho_l^{(N)} \) on each layer \( l \) in

\[\text{The probability that a node } i \text{ has common neighbors in layer 0 and layer 1 vanishes in the limit of infinite network size } N \to \infty. \text{ Therefore, joining the set of neighbors in layer 0 and layer 1 leads to a set of new neighbors with size } k_{agg}^i = k_0^i + k_1^i.\]
the limit of infinite network size \((N \to \infty)\). This approximation is valid for arbitrary degree distributions with finite second moments (Molloy and Reed, 1995; Newman et al., 2001). For our model, the risk measure is \(\varrho = \varrho_0\), i.e., the fraction of failed nodes in layer 0 only. However, in order to calculate \(\varrho\), we need to calculate both \(\varrho_l\), as failures on the two layers are mutually dependent.

A node fails initially if it has a negative threshold. Consequently, \(F_0(\theta_0 = 0)\) and \(F_1(\theta_1 = 0)\) define, for each layer, the initial fraction of failed nodes. These failures can lead to cascades that evolve until the steady states \(\varrho_l\) on the two layers are reached. We can express the \(\varrho_l\) as averages with respect to the degree distribution \(p_l(k_l)\) which we take as input to our model:

\[
\varrho_l = \sum_{k_l} p_l(k_l) \mathbb{P}(s_l = 1|k_l). \tag{7.10}
\]

\(\mathbb{P}(s_l = 1|k_l)\) is the conditional probability that a node with a given degree \(k_l\) on layer \(l\) fails in layer \(l\). In order to calculate this probability, the local tree approximation is essential as it allows us to treat failures of neighbors of a given node as independent events that happen with a failure probability \(\pi_l\). Consequently, the number \(n_l\) of failed neighbors of a node of degree \(k_l\) in layer \(l\) follows a binomial distribution so that \(n_l\) neighbors are failed with probability

\[
B(n_l, k_l, \pi_l) := \binom{k_l}{n_l} (1 - \pi_l)^{k_l-n_l} (\pi_l)^{n_l}. \tag{7.11}
\]

This allows us to express \(\mathbb{P}(s_l = 1|k_l)\) as:

\[
\mathbb{P}(s_l = 1|k_l) = \sum_{n_l=0}^{k_l} B(n_l, k_l, \pi_l) \mathbb{P}(s_l = 1|k_l, n_l, \varrho_{s,1-l}), \tag{7.12}
\]

where \(\mathbb{P}(s_l = 1|k_l, n_l, \varrho_{s,1-l})\) is the probability that a node fails in layer \(l\) given that exactly \(n_l\) of its \(k_l\) neighbors failed. This failure results from the fact that the fraction of failed nodes \(n_l/k_l\) exceeded the threshold. This was the original threshold \(\theta_l\) with a probability \((1 - \varrho_{s,1-l})\) because the node did not fail in layer \((1 - l)\). Or it was the reduced threshold \(\theta_l(1 - r_{(1-l)|l})\) with a probability \(\varrho_{s,1-l}\) because the node failed in layer \((1 - l)\) before. Consequently, we have:

\[
\mathbb{P}(s_l = 1|k_l, n_l, \varrho_{s,1-l}) = (1 - \varrho_{s,1-l}) F_l \left( \frac{n_l}{k_l} \right) + \varrho_{s,1-l} F_l \left( c_l \frac{n_l}{k_l} \right). \tag{7.13}
\]

\(F_l(n_l/k_l)\) is the probability that the original threshold is exceeded, whereas \(F_l(c_l n_l/k_l)\) is the probability that the reduced threshold is exceeded, where \(c_0 := 1/(1 - r_{10})\) and \(c_1 := \infty\).

Note that \(\varrho_{s,1-l}\) differs from \(\varrho_{l-1}^*\) used in Eqn. (7.10). It gives us only the conditional probability that the node has failed in layer \((1 - l)\), given that it has not failed in layer \(l\).
It depends on a neighbor’s failure probability on the other layer, \( \pi_{1-l}^* \), via

\[
\varrho_{s,1-l} = \sum_{k_{1-l}} p_{1-l}(k_{1-l}) \sum_{n_l=0}^{k_{1-l}} B(n_l, k_{1-l}, \pi_{1-l}^*) F_{1-l} \left( \frac{n_{1-l}}{k_{1-l}} \right).
\]  

(7.14)

Eqn. (7.14) has the same structure as the branching process approximation for the fraction of failed nodes on single layers (Gleeson and Cahalane, 2007). It also has a structure similar to Eqs. (7.10)-(7.12), with the only difference of a simpler response function, i.e.,

\[
P(s_l = 1|k_l, n_l, \varrho_{s,1-l}) = F_{1-l}(n_{1-l}/k_{1-l}),
\]

which is the probability that the fraction of failed neighbors exceeds the original threshold in layer \((1 - l)\).

In order to compute Eqn. (7.12), we need to know the failure probability \( \pi_l \) of a neighbor in layer \( l \). To achieve this, we iteratively solve a system of coupled fixed-point equations for the probabilities \( \pi_l \), defined as

\[
\pi_l = L_l(\pi_l) := \frac{1}{z_l} \sum_{k_l} p_l(k_l) \sum_{n_l=0}^{k_l-1} B(n_l, k_l - 1, \pi_l) P(s_l = 1|k_l, n_l, \varrho_{s,1-l})
\]

(7.15)

where \( z_l = \sum_{k_l} p_l(k_l) k_l \) and \( P(s_l = 1|k_l, n_l, \varrho_{s,1-l}) \) is defined by Eqn. (7.13)

We note again similarities between Eqn. (7.15) that describes the failure probability of a neighbor and Eqs. (7.10), (7.13) that describe the failure probability of a node, but also two main differences:

The first difference is that in Eqn. (7.15) the degree distribution of a neighbor follows \( p_l(k_l) k_l/z_l \) instead of \( p_l(k_l) \) for a node (Newman, 2010), analogously to single network layers. It is proportional to the degree \( k_l \), since every link of a neighbor increases the probability of the neighbor to be connected to the node under consideration, i.e. a randomly selected node that fails with probability \( \varrho_l \) in layer \( l \).

The second difference is that the binomial distribution in Eqn. (7.15) depends on \( k_l - 1 \) instead of \( k_l \) because we have to take into account the neighbors of the neighbors of the node under consideration in order to determine the failure probability of the node’s direct neighbors. The set of all neighbors of a node is also called its first order neighborhood, while the set of neighbors of the neighbors the node’s second order neighborhood. We take the failure probability of a neighbor as input to calculate the failure probability of the node under consideration as in Eqs. (7.10), (7.13). Therefore, \( \pi_l \) is conditioned on the event that the node under consideration has not failed before the neighbor. Only \( k_l - 1 \) neighbors of the neighbor with degree \( k_l \) can have possibly failed before, because the node under consideration is a neighbor of this neighbor.
7.3.1 Aggregation

The fraction of failed nodes on a single aggregated layer $\varrho_{\text{agg}}$ can be computed with less effort and with a similar approach as in Gleeson and Cahalane (2007), although the thresholds depend on the nodes’ degrees as in Burkholz et al. (2016a).

For given degrees $k_0$ in layer 0 and $k_1$ in layer 1 we denote the cumulative distribution function of $\theta_{\text{agg}}$ by $F_{\text{agg}}^{(k_0,k_1)}$ to signify its degree dependence. Then we iteratively solve the scalar fixed point equation

$$
\pi_{\text{agg}} = \sum_{k_0,k_1} p_0(k_0)p_1(k_1) \frac{z_0 + z_1}{k_0 + k_1} \times \sum_{n_{\text{agg}}=0}^{k_0+k_1-1} B(n_{\text{agg}}, k_0 + k_1 - 1, \pi_{\text{agg}}) F_{\text{agg}}^{(k_0,k_1)} \left( \frac{n_{\text{agg}}}{k_0 + k_1} \right) \tag{7.16}
$$

for the failure probability of a node’s neighbor $\pi_{\text{agg}}$. Then, its solution $\pi_{\text{agg}}^*$ serves as input for the calculation of the average fraction of failed nodes on the aggregated network using:

$$
\varrho_{\text{agg}} = \sum_{k_0,k_1} p_0(k_0)p_1(k_1) \sum_{n_{\text{agg}}=0}^{k_0+k_1-1} B(n_{\text{agg}}, k_0 + k_1 - 1, \pi_{\text{agg}}^*) F_{\text{agg}}^{(k_0,k_1)} \left( \frac{n_{\text{agg}}}{k_0 + k_1} \right). \tag{7.17}
$$

7.4 Results

7.4.1 Comparison with computer simulations

We now compare our numerical solution of the fixed point equations (7.15) for the multiplex model with computer simulations, using the illustrative case of an uncorrelated, two-layer Erdős-Rényi network (Erdős and Rényi, 1959). For the computer simulations, we implement the time-dependent model as described in Sec. 7.2, i.e. we simulate the evolution of cascades until they reach the steady state. The network size is $N = 10,000$ for each layer. Further, we sample over 100 network realizations for every initial condition.

The degree distributions $p_l(k_l)$ on each layer are approximately Poisson distributions

$$
p_l(k_l) = \frac{z_l^{k_l}}{k_l!} e^{-z_l} \tag{7.18}
$$

with identical mean degrees $z_0 = z_1$. The thresholds are normally distributed, i.e. $\theta_0 \sim \mathcal{N}(\mu_0, \sigma_0^2)$ and $\theta_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ with different parameters $\mu_l$, $\sigma_l$. In our case, we fix the threshold distribution on layer 0 to parameters $\mu_0 = 0.3$ and $\sigma_0 = 0.1$, which ensures that the failure probability for the nodes in an uncoupled network would be very small, to be...
7.4. Results

Figure 7.2: Sharp Regime Change in Threshold-Feedback Model: \( \varrho_0 \) for different combinations of threshold distributions, as the coupling strength \( r_{10} \) is varied. The threshold distribution on the core-business layer is set to \( \mathcal{N}(0.3, 0.1^2) \) while the threshold distribution on the subsidiary-business layer is \( \mathcal{N}(0.6, 0.1^2) \) (black curve), \( \mathcal{N}(0.5, 0.5^2) \) (green curve) and \( \mathcal{N}(0.2, 0.5^2) \) (blue curve). The two layers are independent Erdős-Rényi networks with mean degrees \( z_0 = z_1 = 5 \). The dotted lines are the curves predicted by our analytics, while with the open symbols we show simulation results on Erdős-Rényi networks with 10000 nodes where each point is averaged over 100 realizations. The error bars indicate the size of the standard error.

precise: 0.0045. We vary the parameters \( \mu_1 \) and \( \sigma_1 \) of layer 1, to study the emergence of large cascades on layer 0 because of the coupling with strength \( r_{10} \).

Figure 7.2 demonstrates that a small variation in the coupling strength \( r_{10} \) may result in a rapid shift from a regime with almost no failures (\( \varrho_0 \approx 0 \)) to a regime of complete system failure (\( \varrho_0 \approx 1 \)). The coupling strength at the onset of this regime shift, \( r_{10}^c \), depends on the parameters of the threshold distribution on layer 1 (i.e. \( \mu_1 \) and \( \sigma_1 \)). To be precise, \( r_{10}^c \) is increasing in \( \mu_1 \). For certain parameter constellations (e.g. \( \mu_1 = 0.6 \) and \( \sigma_2 = 0.1 \)), we also find no failure cascades at all, which will be further discussed in Fig. 7.4. Finally, we note the excellent match between the numerical and the simulation results. The differences in the slope result from the fact that we have simulated finite networks, whereas the fixed point equations holds for infinite networks.

7.4.2 Impact of the coupling strength

To gain a broader understanding of how the coupling between the two layers can cause such a rapid transition from a low-risk regime (i.e. \( \varrho_0 \approx 0 \)) to a catastrophic one (i.e. \( \varrho_0 \approx 1 \)), we calculate the \((\mu_1, \sigma_1)\) phase diagram for various coupling strengths. The results are shown in Fig. 7.3, where dark areas indicate parameter constellations with a very high fraction of failed nodes. The left column shows the measure of systemic risk \( \varrho_0 \), which is equivalent to the fraction of failed nodes in layer 0, whereas the middle column shows the corresponding fraction of failed nodes in layer 1.
Chapter 7. Systemic risk in multiplex networks with asymmetric coupling and threshold feedback

As a reference case, the right column shows a single-layer network, which we constructed in order to put the results for the two-layer network into perspective. Nodes in this combined single-layer network have a degree $k_{agg} = k_0 + k_1$ and a threshold $\theta_{agg} = (k_0 \theta_0 + k_1 \theta_1)/(k_0 + k_1)$, as we discussed already in Sec. 7.2.1, by deriving a node’s new balance sheet that results from merging the core business and the subsidiary.

In Sec. 7.3.1 we have explained how to compute our systemic risk measure from the given degree distributions. Since in our example the thresholds $\theta_0$ and $\theta_1$ follow independent normal distributions, the aggregated thresholds are normally distributed as well for given degrees $k_0$ and $k_1$. Their distribution can be found by taking the convolution of the probability density functions of $\theta_0 k_0/(k_0 + k_1)$ and $\theta_1 k_1/(k_0 + k_1)$. This yields $\theta_{agg} \sim \mathcal{N}(\mu_{agg}, \sigma^2_{agg})$, where $\mu_{agg} = (\mu_0 k_0 + \mu_1 k_1)/(k_0 + k_1)$ and $\sigma_{agg} = \sqrt{(\sigma_0 k_0)^2 + (\sigma_1 k_1)^2}/(k_0 + k_1)$.

Our reference case is motivated by two considerations: (a) We want to estimate the error made if a multiplex network is approximated by a single layer network, i.e. the properties of the different layers are simply aggregated in one layer. (b) For the application scenario at hand, namely the management decision of firms to merge their core and subsidiary business units into one business, we want to understand the impact on the resulting risk exposure.

To calculate the phase diagram, we assume that all firms make the same decision, which for example could be motivated by herding behavior.

Since we vary only $(\mu_1, \sigma_1)$, we plot all phase diagrams with respect to these two parameters. Because the combined layer network no longer contains the coupling strength $r_{10}$, the respective phase plots do not change by varying $r_{10}$. They are merely repeated for the purpose of comparison with the other columns. We hypothesize that nodes in the combined layer network have a smaller failure probability compared to the two-layer network because they share their capital buffers. Their degree is also larger, which additionally implies that the risk is better diversified among the neighbors. On the other hand, because of the larger degrees, there is a higher connectivity in the combined layer network. This has the potential to amplify small cascades more than in the less connected separate layers. We will investigate, by means of numerical solutions of the fixed point equations, which of these antagonistic effects may dominate in a given parameter region.

By comparing the first and third columns in Fig. 7.3, we can identify a different risk profile for small and for large coupling strengths. For small values of $r_{10}$ the cascades on layer 1 cannot propagate to layer 0, therefore we do not observe any systemic risk. This is different for the combined layer network, where large cascades can occur for a small range of parameters.

The picture is inverted for larger values of $r_{10}$. Here we find, by increasing $r_{10}$, an increasing region of high systemic risk that is driven by the mutual amplification of cascades between the two layers. This leads to a very sharp phase transition, i.e. a clear separation of regions
with complete breakdown and regions with no breakdown. We note that this differs from the observation for the combined layer network, where the phase transition can also be observed. However, there are extended regions where the systemic risk is at intermediate levels, as indicated by the gray areas.

We also wish to identify how the onset of systemic risk on layer 0 depends on the coupling
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\[ \rho_0 \] \[ \rho_1 \] \[ \rho_{\text{agg}}^{*} \]

\[ \sigma_0 = 0.1 \text{ and } \sigma_1 = 0.3 \text{ as } \mu_1 \text{ and } r_{10} \text{ are varied. The two layers are independent Erdős-Rényi networks with mean degrees } z_0 = z_1 = 5 \text{ and the thresholds on each layer are independently distributed, i.e. } \theta_0 \sim \mathcal{N}(0.3, 0.1^2) \text{ and } \theta_1 \sim \mathcal{N}(\mu_1, 0.3^2), \text{ where } \mu_1 \in [0, 1]. \] \[ \rho_{\text{agg}}^{*} \text{ is the fraction of failures on the aggregated network, i.e. a network where } k_{\text{agg}} = k_0 + k_1 \text{ and } \theta_{\text{agg}} = (k_0 \theta_0 + k_1 \theta_1)/(k_0 + k_1). \]

\begin{align*}
\text{strength } r_{10}. \text{ Thus, in addition to fixing } \mu_0 \text{ and } \sigma_0, \text{ we set } \sigma_1 \text{ to a small value and plot the phase diagram with respect to } (\mu_1, r_{10}). \text{ Fig. 7.4, for the same columns as in Fig. 7.3, shows that there is indeed a critical value } r_{10}^c \text{ which is independent of } \mu_1. \text{ Below } r_{10}^c \text{, we do not observe any systemic risk in layer 0, whereas in the combined layer network, we find a considerable systemic risk for small values of } \mu_1. \text{ Above } r_{10}^c, \text{ we see in the two-layer network a sharp phase transition between full and no systemic risk that depends on a critical value } \mu_1^c. \text{ Consequently, we study the transition line } r_{10}(\mu_1) \text{ in the following subsection.}
\end{align*}

\subsection*{7.4.3 Scaling behavior}

In Fig. 7.3 we observe that the sharp phase transition scales almost linearly for large enough coupling strength \( r_{10} \), i.e.

\[ \sigma_1 = m_1 \mu_1 + m_0 \text{ for } \mu_1 \geq 0.4, r_{10} \geq r_{10}^c \] (7.19)

In order to determine the scaling parameter \( m_1 \), we take the phase transition line from the phase diagrams of Fig. 7.3, which were also calculated for \( r_{10} = 0.3, 0.5, 0.6, 0.7, 0.9 \). These lines are approximated by a linear regression, and the slopes obtained this way are plotted in Fig. 7.5 (left, red dots) against the coupling strength \( r_{10} \). We observe a non-monotonous dependence of the slope, with a saturation effect for large \( r_{10} \). Thus, we want to determine \( r_{10}^c \) and \( m_1(r_{10}) \) for \( r_{10} \geq r_{10}^c \).

To better understand this dependence, we perform a linear approximation of \( L_0 \) in Eqn. (7.15) as

\[ L_0(\pi_0) \approx a + b \pi_0 \] (7.20)

which is valid for small values of the failure probability \( \pi_0 \). Instead of solving the full
7.4. Results

Figure 7.5: (Left): Red Circles: Slope $m_1$ of the linear phase transition obtained by linear regression of $\varpi_0^*$ that we calculated by solving the fixed point equations (7.15) numerically for different values of $r_{10}$. The blue curve shows the approximate values for the slopes provided by Eqn. 7.26. (Middle): Phase diagram for the final cascade size on a single layer without threshold feedback for Poisson random graphs with average degree $z = 5$ and normally distributed thresholds $F_\theta \sim \mathcal{N}(\mu, \sigma^2)$. (Right): Final cascade size on layer 0, i.e. $\varpi_{0}^*$, for $r_{10} = 0.3$. The red line marks the phase transition defined by the criterion $\varpi_{\text{single}}^* \geq \varpi_{c1}^*$. The green line corresponds to $\varpi_{\text{single}}^* \geq 0.45$.

system of fixed point equations Eqn. (7.15), we only solve the linearized system in order to deduce criteria for the growth behavior of $\pi_0$ in the fixed point iterations. The fixed point of the linear equation (7.20) exceeds one if

$$\frac{a}{1 - b} \geq 1. \quad (7.21)$$

Under this condition, initial failures in both layers result in large cascades in layer 0.

Equation (7.21) leads to conditions for the parameters $\mu_{c1}^*$, $\sigma_{c1}^*$ and $r_{c10}^*$ if we know the expressions for $a$ and $b$. These are obtained from the linearization of Eqn. (7.15):

$$a = \mathbb{P} (s_0 = 1|k_0, n_0 = 0, \varpi_{s,1}^*),$$

$$b = \sum_{k_0=1}^{\infty} \frac{k_0(k_0 - 1)}{z_0} p_0(k_0) \times$$

$$\times \left[ \mathbb{P} (s_0 = 1|k_0, n_0 = 1, \varpi_{s,1}^*) - \mathbb{P} (s_0 = 1|k_0, n_0 = 0, \varpi_{s,1}^*) \right]. \quad (7.22)$$

These parameters depend on the failure probability $\varpi_{s,1}^*$ of a node in the other layer, i.e. layer 1.

For the following discussion, we first concentrate on the worst case scenario $\varpi_{s,1}^* = 1$, i.e. all nodes failed on layer 1. We are interested in identifying the regime where even in the worst case, cascades in layer 1 do not propagate to layer 0. In other words, we want to approximate the critical coupling $r_{c10}^*$ above which we can observe regions of high systemic
risk in the left column of Fig. 7.4. With \( s_{*1} = 1 \) the parameters \( a \) and \( b \) simplify to

\[
\begin{align*}
a &= F_0(0), \\
b &= \sum_{k_0=1}^{\infty} \frac{k_0(k_0 - 1)}{z_0} p_0(k_0) \left[ F_0 \left( \frac{1}{k_0 (1 - r_{10})} \right) - F_0(0) \right].
\end{align*}
\]

(7.23)

We note that these are independent of \( \mu_1, \sigma_1 \). For the set of parameters used in Fig. 7.4, we obtain with the help of Eqn. (7.22) and Eqn. (7.21) the value of the critical coupling \( r_{10} = 0.204 \). In comparison with the numerical calculation \( r_{10} = 0.18 \), which takes the full set of fixed point equations into account, this is a good approximation. That means large cascades can propagate from layer 1 to layer 0 above a critical coupling strength \( r_{10}^c \simeq 0.2 \). This holds independently of \( \mu_1 \) and \( \sigma_1 \).

Next we explain the dependence \( m_1(r_{10}) \) as shown in Fig. 7.3. By this we estimate the slope of the phase transition line (which is of the form \( \sigma_1 \propto m_1 \mu_1 \)). Again, we deduce this relation from Eqs. (7.21), (7.22), but this time we cannot use \( s_{*1} = 1 \). For \( s_{*1} \neq 1 \), we automatically obtain a dependence on the threshold parameters \( \mu_1 \) and \( \sigma_1 \) of layer 1. Combining Eqn. (7.22) and Eqn. (7.21), we can define a critical value \( \varrho_{s,1}^c \) such that Eqn. (7.21) is satisfied for \( s_{*1} \geq \varrho_{s,1}^c \):

\[
\varrho_{s,1}^c := 1 + F_0(0)(z_0 - 1) - c_0(0) \
\]

(7.24)

where

\[
c_0(r) := \sum_{k_0=1}^{\infty} \frac{k_0(k_0 - 1)}{z_0} p_0(k_0) F_0 \left( \frac{1}{k_0 (1 - r_{10})} \right). \]

(7.25)

This is a reformulation of Eqn. (7.21), which gives us an estimation of when to expect large cascades in layer 0. Still, we do not know \( s_{*1} \) without solving the full system of fixed point equations (7.15). In the following we test two approximations for \( s_{*1} \) which lead us to a linear dependence between \( \mu_1 \) and \( \sigma_1 \).

**Case (1)** The initial value \( s_{1}(t = 0) = F_1(0) = \Phi(-\mu_1/\sigma_1) \) is already enough to determine the growth of \( \pi_0 \) in the early stages of the fixed point iterations. Especially for larger values of \( r_{10} \), a large cascade in layer 0 might be already triggered just by the initial failures in layer 1. Therefore, we set \( \varrho_{s,1}^c \simeq F_1(0) \), which results in

\[
\sigma_1 \propto -\mu_1/\Phi^{-1} \left( \varrho_{s,1}^c \right), \]

(7.26)

i.e.,

\[
m_1 = -1/\Phi^{-1} \left( \varrho_{s,1}^c \right) \]

(7.27)
where $\Phi^{-1}$ denotes the cumulative distribution function of the standard normal distribution. We plot this slope using a blue line in Fig. 7.5(left), to demonstrate the good agreement of our approximation with the numerical slopes $m_1$ for $r_{10} \geq 0.4$.

**Case (2)** For smaller coupling strengths $r_{10}$ we test a second proxy for $\varrho^*_{s,1}$. A lower bound for $\varrho^*_{s,1}$ is given by the final cascade size on a single layer $\varrho^*_{\text{single}}$, where no feedback mechanism with another layer exists. This would coincide with $\varrho^*_{s,1}$, if layer 0 would not exist. That means, the layer 0 cannot further amplify the failures. Cascades in single layers have been studied in Burkholz et al. (2016a); Gleeson and Cahalane (2007). We use their approach to plot a phase diagram for the final cascade size for Poisson random graphs with average degree $z = 5$, shown in the middle panel of Fig. 7.5.

When we compare the plots of Fig. 7.3 (left) with Fig. 7.5 (middle), we observe that in the parameter regions in Fig. 7.3 (left) where no cascades occur, also no cascades occur in Fig. 7.5 (middle). This holds only for strong couplings $r_{10}$. Consequently, in this limit, our second approximation for $\varrho^*_{s,1}$ gives similar results as our first approximation.

However, the second case for $\varrho^*_{s,1}$, is the more general one as we can also correctly cover the transition line for values of $0.2 \leq \mu_1 \leq 0.4$. In this range, the transition line is largely independent of the value $r_{10}$. It basically coincides with the transition line for $\varrho^*_{\text{single}}$, the fraction of failed nodes for single layer networks.

For values of $\mu_1$ larger than 0.4 and for weak coupling, e.g. for $r_{10} = 0.3$, we observe in Fig. 7.5 (right, red line) that the transition line is no longer accurately described. The condition $\varrho^*_{\text{single}} \geq \varrho^*_{s,1} = 0.55$ gives only an upper bound for the transition line, but not a lower one. We observe that already cascade sizes of $\varrho^*_{\text{single}} \geq 0.45$ lead to large cascade amplification on layer 0, due to nonlinear effects. I.e. the green dotted line in Fig. 7.5 (right) gives an almost perfect match with the numerically calculated transition line.

That means, we can even generalize our statement that the transition line for $\varrho^*_{\text{single}}$ determines the phase transition. This holds not only for small $\mu_1$, but also for larger $\mu_1$ and a broad range of the coupling strength $r_{10}$.

## 7.5 Conclusions

Our work essentially addresses the problem of whether systemic risk is increased or decreased if, instead of a single-layer network, a two-layer representation is used. In the latter, the nodes of the network appear on two layers 0 and 1 with different properties. Specifically, they have different degree distributions $p_0(k_0), p_1(k_1)$ and different threshold distributions $F_0(\theta_0), F_1(\theta_1)$. There is an asymmetric coupling between the two layers such that nodes that failed on layer 0 also fail on layer 1. However, nodes that did not fail on layer 0, may still fail on layer 1 because of a cascade dynamics on layer 1. In this
case, their failing threshold on layer 0 is reduced by a fraction $r_{10}$, where $r_{10}$ denotes the coupling strength between the two layers.

The mutual feedback between the two layers can then result in the amplification of failure cascades, which we study analytically. We can calculate a variable $\varrho_l^*$ which is the final fraction of failed nodes on each layer $l \in \{0, 1\}$. Our measure of systemic risk is $\varrho^* \equiv \varrho_0^*$, i.e. we only consider whether nodes have failed on layer 0. Obviously, if $r_{10}$ is small, no failure cascade in layer 1 can propagate to layer 0. In this case, whether or not we observe failure cascades only depends on the conditions in layer 0. These conditions are expressed by the parameters of the threshold distribution, $\mu_0$ and $\sigma_0$, which are chosen so that no failure cascade occurs. By varying $r_{10}$, we then study the impact of failures on layer 1 on failures on layer 0. We derive an analytical approach to calculate $\varrho_l^*$, which leads to a system of coupled fixed point equations solved numerically.

Our results are visualized through phase diagrams showing, for various parameter constellations, the value of the main risk measure $\varrho^* = \varrho_0^*$. The most prominent feature is the existence of a very sharp phase transition between a regime of no systemic risk and a regime of full collapse. This means that small changes in the parameters $\mu_1$ and $\sigma_1$, describing the threshold distribution on layer 1, can lead to an abrupt regime shift. Our task was then to approximate this line of transition in terms of the coupling strength $r_{10}$ and the parameters of the threshold distribution. Subsequently, we use these insights to compare the systemic risk in single-layer and two-layer networks.

The derivation of mathematical approximations for the phase transitions that are compared, and confirmed, by numerical solutions of the full problem has a value on its own. Here, we focus on the conclusions that can be made based on these calculations. First of all, we understand that systemic risk is reduced in the two-layer network only if the coupling strength between the two layers is rather small. Above a critical value $r_{10}^c = 0.2$, which is also estimated analytically, failures on layer 1 are amplified on layer 0 and thus lead to an increase of the systemic risk. If we compare this with the reference case of a single layer network, we find that the systemic risk is smaller there, for most ranges of the parameters. Hence, above a critical coupling, it is not beneficial to split the network into two layers, if systemic risk shall be mitigated.

Our findings are motivated by a scenario where firms have to decide whether to split their business into a less risky core business, essential for their survival, and a subsidiary business, which can be more risky. Such a split leads to the representation of the (same) firm on two different levels 0 and 1, where the latter has a larger risk of failure cascades. This decision only leads to less failure risk for the firm if the coupling between the core business layer and the subsidiary business layer is weak enough. Close to the transition line between the no-risk and the high-risk regime, slight changes in the firms’ failure thresholds or in the coupling between the different businesses may potentially cause the system to collapse completely.
7.5. Conclusions

There is another important conclusion to be drawn for firms that already have their activities split between a less risky core business and a more risky subsidiary business. If such firms estimate systemic risk based on an aggregated network representation with only one layer instead of the two-layer representation, they may systematically underestimate the real risk, in particular if the coupling is still strong. As we have shown, under such conditions the systemic risk in the aggregated network is lower than in the two-layer network. Hence, there can be drastic consequences from drawing conclusions based on an inappropriate aggregated picture.

Closing, we would like to point out that other network properties could influence systemic risk calculations. For example assortativity is known to affect the dynamics of a network (D’Agostino et al., 2012; Zhou et al., 2012), and can influence the evolution of cascades especially during crisis periods where it may vary due to network re-organization. However, studying such influence is out of the scope of our current work, and is left for future study.
Chapter 8

Systemic feedback transforming constant load cascades to effective load redistribution

Summary
In this small interim chapter, we extend the Constant Load Exposure Diversification model by a simple systemic feedback mechanism. Interestingly, we observe similar phase diagrams as for the Load Redistribution model (without systemic feedback). Although these similarities can be explained analytically, the mechanisms underlying both models differ. Interventions would require appropriately different countermeasures and therefore a good understanding of cascades on the microlevel. Comparisons of macro final cascade sizes are not enough to identify micro-dynamics correctly.

RB wrote this chapter specifically for this thesis. RB contributed to the design of the research questions, derived the formulas, wrote the code and performed the analysis.
8.1 Introduction

In Part I of this thesis, systemic risk originates solely from local interactions and dependencies of system components. Especially in social systems, a successive downward trend of a cascade might be observable by members of the system and prompt reactions. These reactions can be interventions, but also a trend reinforcement, i.e. amplifications of the cascade. As long as all system components are affected by these, we term the phenomenon systemic feedback.

Examples for negative trend reinforcement include fire sales by distressed financial institutions (Cifuentes et al., 2005) that reduce the assets’ value by all other institutions that hold these assets. At the same time, increasing haircuts, i.e., the amount of collateral required for a transaction, can worsen the situation (Gorton and Metrick, 2012). The common trust in the bail-out of big institutions can have both a negative as well as a positive effect. It might lead to excessive risk taking or prevent panics that could amplify cascades. Another example are opinion formation cascades of binary decisions, where TV broadcasts inform frequently about the dissemination of the opinion, for instance the percentage of voters for a party in a poll.

In the previous chapter, the second subsidiary layer has introduced a feedback on the core layer. From a theoretical perspective, this feedback applied to all nodes in the core layer, as their probability increased to be connected to a failed node in the subsidiary layer. There, we encountered already a way to deal with such a feedback analytically.

This chapter we devote to the analysis of a simple feedback mechanism that is inspired by the previous chapter, although the feedback originates from the same layer. We show that this extension of the CL ED model leads to similar macroscopic results as the LRD (LLSC) model. This similarity can be explained by a comparison of the LTA calculations and the approach on fully connected networks. Still, an observation of the macroscopic phase diagrams would not allow us to distinguish between the two models despite the different underlying mechanisms.

8.2 Local Tree Approximation incorporating systemic feedback

Additional to the local interactions between network nodes in the ED model, systemic feedback can affect the nodes’ thresholds (as in the previous chapter). The values of these thresholds are coupled to the state of the overall system and reflect thus a degrading of robustness in the course of a cascade.

We assume that the threshold $\theta_i(t)$ of each node $i$ is proportional to the size of the func-
Chapter 8. Systemic feedback transforming constant load cascades to effective load redistribution

Figure 8.1: (a) Final fraction of failed nodes $\varrho$ after 40 fixed point iterations for infinite Poisson random graphs with parameters $\lambda = 2.82$ and cut-off degree $c = 50$ so that the average degree is $z = 3$. The thresholds are independently normally distributed with mean $\mu$ and standard deviation $\sigma$ ($\Theta \sim \mathcal{N}(\mu, \sigma^2)$). Systemic feedback is considered in each time step. (b) Final fraction of failed nodes $\varrho$ after 40 fixed point iterations of the LRD (LLSS or LLSC) model on an infinitely large fully connected network with initial load $\lambda_0 = 0.8$ and independently normally distributed thresholds ($\Theta \sim \mathcal{N}(\mu + \lambda_0, \sigma^2)$).

Thus, the distribution of the time dependent threshold $\Theta(t)$ of a focal node or neighbor can be expressed with respect to the distribution of the initial distribution of $\Theta := \Theta(0)$ by:

$$P (\Theta(t) \leq \theta) = P ((1 - \varrho(t - 1)) \Theta \leq \theta) = F_{\Theta} \left( \frac{\theta}{1 - \varrho(t - 1)} \right).$$

With this threshold distribution, the cascade dynamics can then be described analogously to Chapt. 3 as

$$\pi(t) = \sum_{k=1}^{c} \frac{p(k)}{z} \sum_{n=0}^{k-1} \binom{k-1}{n} (1 - \pi(t - 1))^{k-1-n} \pi(t - 1)^n F_{\Theta} \left( \frac{n}{k \pi(t - 1)} \right),$$

$$\varrho(t) = \sum_{k=1}^{c} p(k) \sum_{n=0}^{k} \binom{k}{n} (1 - \pi(t - 1))^{k-n} \pi(t - 1)^n F_{\Theta} \left( \frac{n}{k (1 - \varrho(t - 1))} \right). \quad (8.1)$$

Fig. 8.1(a) shows the results of our calculations for infinitely large Poisson random graphs.
8.3 Similarity with LRD on fully connected networks

(with average degree \( z = 3 \)) as introduced in Sec. 2.4.2. Results for scale free networks with the same average degree are almost identical. Next to it, in Fig. 8.1(b), we plot a corresponding phase diagram for an LRD LLSC/LLSS model with initial load \( \lambda_0 = 0.8 \) on a fully connected topology. Qualitatively, the shapes look extremely similar.

We explain this finding by an approximation of Eq. (8.1) that translates into an iterative procedure to calculate the fraction of failed nodes for LRD models on fully connected networks as provided by Lorenz et al. (2009).

8.3 Similarity with LRD on fully connected networks

We approximate the number of failed neighbors \( N_f \) of a node with degree \( k \), which is binomial distributed, in Eq. (8.1) by its average \( N_f \approx k \pi (t-1) \) and receive

\[
\varrho(t) = \sum_{k=1}^{c} p(k) F_{\bar{\Theta}} \left( \frac{k \pi (t-1)}{k (1 - \varrho(t-1))} \right) = F_{\bar{\Theta}} \left( \frac{\pi (t-1)}{1 - \varrho(t-1)} \right).
\]

Assuming that the difference between the failure probability of a neighbor \( \pi(t-1) \) is and the failure probability of a node is negligible, we can substitute \( \pi(t-1) \) by \( \varrho(t-1) \) and arrive almost at the iterative formula

\[
\varrho(t) = F_{\bar{\Theta}} \left( \lambda_0 + \frac{\varrho(t-1)\lambda_0}{1 - \varrho(t-1)} \right),
\]

which Lorenz et al. (2009) have derived for the (average) fraction of failed nodes in case of LRD LLSS/LLSC cascades in an infinitely large fully connected network. If we choose the threshold distribution for \( \tilde{\Theta} \) according to \( \Theta + \lambda_0 \) we obtain matching results.

8.4 Discussion

We have presented an example, where the introduction of systemic feedback to a CL model leads to a similar cascade outcome as for an LRD model without systemic feedback (on a different network topology).

This has two implications for our perspective on the studied modeling framework.

(1) The boundaries between the different model classes are not sharp. Additional model complexity, as for instance systemic feedback, might convert one in the other. Also several states between a global load distribution as in case of the LRD model on fully connected networks and a local one are possible alternatives. Similarly, several fiber bundle models have been constructed to transfer models belong to one universality class to models in another class (Kun et al., 2006).
2) Obviously, in a real world modeling approach it is not sufficient that our models match observations of a macro state. We need further information about micro dynamics to deduce the nature of a (cascade) process.
Chapter 9

Cascade size distribution for finite networks

Summary

In the previous chapters, we have approximated the average cascade size in large random graphs by the cascade size in infinitely large systems. But in finite systems, large deviations from this average are very probable. We present explicit closed-form solutions for the full final cascade size distribution on finite fully connected and finite star-shaped networks consisting of \( N \) nodes for a general class of cascading processes and arbitrary threshold distributions. With three familiar examples from our model framework, i.e. the CL ED, the CL DD and the LRD LLSS models, we demonstrate the relevance of considering the full probability distribution of the final cascade size, as we obtain distributions of broad shape. Despite the high symmetry of the networks and response function studied, we find surprisingly asymmetric cascade size distributions. This property persists for large networks, as the convergence of the cascade size distribution to a delta distribution is relatively slow. This finding questions the average as suitable risk measure.

Based on R. Burkholz, F. Schweitzer, and H. Herrmann, “Cascade size distribution for finite networks”. Working Paper. RB contributed to the design of the research questions and to discussions. RB derived the formulas. RB analyzed and interpreted the results. RB wrote the first draft of the manuscript.
Chapter 9. Cascade size distribution for finite networks

![Example networks: (a) fully connected network with $N = 10$ nodes, (b) star-shaped network with $N = 10$ nodes.](image)

**Figure 9.1:** Example networks: (a) fully connected network with $N = 10$ nodes, (b) star-shaped network with $N = 10$ nodes.

### 9.1 Introduction

Until now, we have considered infinitely large random networks and have measured the vulnerability of systems arising from their interconnectivity. In most applications, the main focus lies on the estimation of systemic risk in *finite* systems. Still, it is common to approximate the average final cascade size in a given finite system by a theoretical estimate in an infinite system. We have given plenty of examples in the previous chapters for such (Heterogeneous) Mean Field Approaches (HMF) and followed an established line of research (Burkholz *et al.*, 2016a,b; Dorogovtsev and Mendes, 2003; Gleeson and Cahalane, 2007; Jackson and López-Pintado, 2013; Lorenz *et al.*, 2009). We could also verify that these estimates approximate the average cascade size in large but finite networks very well. Still, this average can be misleading as risk measure. If the studied systems are finite, cascade sizes different from the average often occur with high probability. We quantify this probability by presenting closed form solutions for the full probability distribution of the final cascade size for a general cascade process class for two network topologies of finite arbitrary size $N$. Our results for fully connected networks allow the assessment of finite size effects in comparison with mean field theories and might serve as well as proxy for dense networks. Despite the elegance of closed form solutions, they often enable explicit optimization and control strategies for cascade size reduction (Liu *et al.*, 2014). For instance, this has been achieved for a specific cascade model for power grid failure on fully connected networks and uniformly distributed load sizes (Dobson *et al.*, 2005). As we non-trivially generalize this case to a whole class of different cascade models and to arbitrary response functions, we hope to foster further developments. Moreover, closed form solutions ease the calculation of risk measures different from the average. For instance, Value at Risk or Expected Shortfall as discussed by Embrechts *et al.* (2014) could be such candidates, although we emphasize that these risk measures would present a distorted picture of a
broad bi-modular probability distribution as well.

Our formula summarizes a high number of combinatorial terms that increase in network size. So, it might provide a possibility to save computational time in the calculation of the average cascade size in clique-based random graphs (Hackett and Gleeson, 2013).

Additionally to fully connected networks, we derive a formula for the cascade size distribution in star-shaped networks to study the role of hubs, which are potential main spreaders in cascade processes.

9.2 The cascade processes

In contrast to our previous analysis, we assume a fixed and finite undirected network $G = (V, E)$ consisting of $N = |V|$ nodes instead of random graphs. Both considered topologies are illustrated in Fig. 9.1. All of our derivations rely on their high symmetry. Our results are general with respect to other aspects: They apply to a large class of different cascade models, arbitrary network size $N$, and any threshold distribution.

Since we fix the network topology, the only random variables in our cascades are the nodes’ thresholds $\Theta_1, \ldots, \Theta_N$. On fully connected networks, they are identical distributed according to a cumulative distribution function (cdf) $F$. We denote it with $F$ instead of $F_\Theta$ to ease the notation. As the thresholds are the only available random input variables, $F$ can be uniquely attributed to them.

As before, once the thresholds are drawn independently at random in the beginning of a cascade, the evolution of failures proceeds deterministically. A node $i$ fails whenever the load $\lambda_i(t)$ that it carries exceeds its threshold $\theta_i$, as explained in Sec. 2.3.

How many nodes have failed in the end depends on the joint distribution of thresholds $\Theta_1, \ldots, \Theta_N$ so that the final fraction of failed nodes $\varrho$ is a random variable as well. For simplicity, we omit the $N$ dependence and use the notation $\varrho = 1/N \sum_{i=1}^{N} s_i(T)$\(^1\). For fully connected networks, we denote it with $\varrho_{fc}$, while for star-shaped networks we write $\varrho_s$. For both cases, we present a closed-form solution for the probability distribution of $\varrho$ on finite networks with $N$ nodes for general cascading processes where the load $\lambda[n]$ that a node with exactly $n$ failed neighbors carries depends only on the number of failed neighbors $n$ and the networks size $N$. The load might additionally depend on the type of a node, as its degree or failure history of some neighbors as we will explain later. $\varrho$ takes discrete values in the set $\{0, 1/N, 2/N, \ldots, 1\}$. Thus, we have determined the distribution if we can calculate the probability for each event $\mathbb{P}(\varrho = n/N)$, where $n$ denotes the number of failed nodes at the end of the cascading process.

\(^1\)Although $\varrho$ is a random variable, we denote it with a small letter, because its capital version can be confused with a $P$.  

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9.3 Cascade size distribution on fully connected networks

We offer two different representations of the final cascade size distribution for fully connected networks: one that enumerates all possible combinations of failures that could have led to a cascade outcome, and one which can be deduced by the inclusion-exclusion principle and whose implementation requires less computational resources. Both representations rely on combinatorial arguments, which we explain next. First, we note that each non-failed node in the network is exactly in the same state. They all have the same degree, the same neighbors and carry the same load \( \lambda[n] \) if \( n \) other nodes have failed already.

If \( \varrho_{fc} = n/N \) is the final cascade outcome we know that all surviving \( N-n \) nodes must have thresholds larger than this load \( \lambda[n] \). The probability for this event is

\[
P(\Theta_i > \lambda[n]) = (1 - F(\lambda[n]))^{N-n},
\]

since we assume that the thresholds are independently distributed. There are \( \binom{N}{n} \) different combinations of \( n \) surviving nodes out of the total \( N \) nodes. The remaining \( n \) nodes fail altogether with a probability \( p_n \) that needs to be determined. In summary, we can write the cascade distribution as:

\[
P(\varrho_{fc} = \frac{n}{N}) = \binom{N}{n} (1 - F(\lambda[n]))^{N-n} p_n. \tag{9.1}
\]

The probability \( p_n \) that all \( n \) of the remaining nodes fail is equivalent to the failure of all nodes in a network with \( n \) nodes, since non-failed nodes do not influence the amount of load that any of the \( n \) nodes carries. The load required to cause a node to fail is determined by the node’s threshold, while the entire configuration of all thresholds defines the order of the nodes’ failure. But not all threshold configurations lead to \( n \) failures. For instance, it is not enough that all \( n \) nodes have a threshold smaller than \( \lambda[k] \). If each node has a threshold \( \Theta > \lambda_0 \), no node fails initially and the cascade cannot start. In each time step, enough failures need to occur to trigger new failures. This can be translated into a criterion that identifies the threshold configurations that lead to exactly \( k \) failures. Whenever a node’s threshold lies in the interval \( (\lambda[j-1], \lambda[j]) \), it fails after \( j \) nodes have failed before (with \( j = 1, \ldots, N-1 \)). We denote the number of nodes with this property with \( m_j \). All \( m_0 \) nodes with threshold in \( (-\infty, \lambda[0]) \) fail initially, while all \( m_N \) nodes with thresholds in \( (\lambda[N-1], \infty) \) never fail. This definition implies that all nodes with a threshold corresponding to \( m_j \) fail before or at the same time as the nodes corresponding to \( m_l \) if \( l > j \), since \( \lambda[l] > \lambda[j] \). Each possible threshold configuration translates into a vector \( \mathbf{m}_N = (m_0, \ldots, m_N) \). In the first cascade time step, all \( m_0 \) nodes fail. We require
9.3. Cascade size distribution on fully connected networks

$m_0 > 0$, otherwise nothing happens. In the second time step, all nodes fail whose failure premises at least $m_0$ failed neighbors. This includes all nodes that correspond to $m_1$. In order to cause the failures of at least all nodes that belong to $m_2$ (or any $j > 2$), the total number of failures before (or at the same time) $m_0 + m_1$ needs to be at least 2. Continuing this argument, for each $j = 1, \ldots, n - 1$ we require $m_j \geq j + 1 - \sum_{l=0}^{j-1} m_l$ and in total $\sum_{l=0}^{n-1} m_l = n$. We define the set of all such causal configurations for a cascade of size $n/N$ as

$$I_n := \left\{ m_{n-1} \in \{0, \ldots, n\}^n \mid \sum_{l=0}^{j} m_l \geq j + 1 \quad (\forall j \in 0, \ldots, n - 2), \sum_{l=0}^{n-1} m_l = n \right\}$$

and sum the probabilities of all those configurations in order to calculate $p_n$. Since each node’s threshold lies in the interval $(\lambda[j - 1], \lambda[j])$ with probability $F(\lambda[j]) - F(\lambda[j - 1])$, we have

$$p_n = \sum_{m \in I_n} \frac{n!}{\prod_{j=0}^{n-1} (m_j!)} \left( F(\lambda[0])^{m_0} \prod_{j=1}^{n-1} (F(\lambda[j]) - F(\lambda[j - 1]))^{m_j} \right).$$

The enumeration of all configurations in $I_n$ is computationally very expensive, and the summation of mostly small summands numerically inaccurate. But we can derive an iterative formula for $p_n$ with the help of the probabilistic inclusion-exclusion principle (Szpankowski, 2001):

$$p_n = \sum_{l=0}^{n-1} (-1)^{n+l+1} \binom{n}{l} F(\lambda[l])^{n-l} p_l,$$

9.3.1 Model examples on fully connected networks

Constant load models

As first example, we calculate the cascade size distributions for the ED model as introduced in Sec. 2.3.1, which has been well studied on infinite configuration type random graphs (Burkholz et al., 2016a; Dodds and Payne, 2009; Gleeson and Cahalane, 2007; Hurd and Gleeson, 2013). We recall that it coincides with the DD model on fully connected networks. Each node carries the fraction of its failed neighbors as load. In a fully connected network with $N$ nodes, every node has $N - 1$ neighbors and consequently carries the load $\lambda[n] := n/(N - 1)$, if $n$ neighbors have failed. The average final cascade size on infinite fully connected networks has already been calculated by Lorenz et al. (2009). However, Fig. 9.2 (a) reveals that the average is not an appropriate risk measure on finite networks,
Figure 9.2: Distribution of the fraction of failed nodes $\varrho_{fc}$ on a fully connected network with normally distributed thresholds with mean $\mu + \lambda_0$ and standard deviation $\sigma$ (i.e. $\Theta \sim N(0.5 + \lambda_0, \sigma^2)$) for the (a) ED model with $N = 50$ nodes and initial load $\lambda_0 = 0$, and (b) fiber bundle model with $N = 30$ nodes and initial load $\lambda_0 = 1$.

since the distribution can be broad and even of bi-modular shape. Thus, it would be misleading to expect a cascade size outcome that is similar to the average. Fig. 9.2 (a) shows the cascade size distribution for $N = 50$ nodes and normally distributed thresholds with mean $\mu = 0.5$ and different standard deviations $\sigma$. The network and the threshold distribution are highly symmetric so that the nodes are very similar. Thus, it is surprising that the cascade size distribution is of such asymmetric and broadly distributed shape (see parameters $\sigma = 0.4$ and $\sigma = 0.5$ in Fig. 9.2 (a)). For decreasing $\sigma$, the peak at a value close to 0 increases, while the smaller peak close to 1 decreases until the latter one completely vanishes. For $\sigma \leq 0.1$, no node fails ($\varrho_{fc} = 0$) with high probability. For increasing $\sigma$, the two peaks of the cascade size distribution move closer together, overlap and merge to a single peak at $\varrho_{fc} = 0.5$ for $\sigma \geq 0.7$. This leads to an axisymmetric distribution around $\varrho_{fc} = 0.5$, as expected. The peak becomes more pronounced for increasing $\sigma$ as the standard deviation of the cascade size distribution decreases. Still, for most parameter values it is important to take the whole cascade size distribution into account rather than focusing on its average to assess the cascade risk correctly.

For increasing network size $N$, the distribution converges to a point measure, whose value has been calculated by Lorenz et al. (2009). The speed of this convergence decides if the presented phenomena are just an artifact of rather small networks or can be observed in for realistic network sizes relevant for applications.
Figure 9.3: (a) Final cascade size distribution for fully connected networks of several sizes \( N \) obtained via \( 10^4 \) independent simulations of the CL ED model. The thresholds are independently normally distributed with mean \( \mu = 0.5 \) and standard deviation \( \sigma = 0.4 \) \((\Theta \sim N(0.5, 0.4^2))\). (b) Position of the two local maxima of the distributions in (a) with respect to the network size \( N \).

Convergence speed

The presented formulas cannot be solved numerically accurately for large network sizes. Equ. (9.2) requires the multiplication of large numbers \( \binom{n}{l} F(\lambda[l])^{n-l} \) with rather small numbers \( p_l \). Consequently, we simulate cascades on large networks instead to obtain a impression for which network sizes we can expect the observed bi-modularity to vanish. Fig. 9.3 presents our simulations results for the ED model with threshold distribution parameters \( \mu = 0.5 \) and \( \sigma = 0.4 \). Strikingly, we find that the bi-modularity of the final cascade size distribution is still present for networks of size \( N = 10^7 \). Only for \( N = 10^8 \), we observe an unimodular sharply peaked distribution.

This finding provides evidence that a systemic risk analysis cannot solely rely on the estimation of the average final cascade size. The whole distribution needs to be considered.

LRD models

As second example, we consider the LRD model. On fully connected networks the LLSC and LLSS variants, which have been introduced in Sec. 2.3.2, coincide. On a fully connected topology, they correspond to the fiber bundle model with global load sharing (Daniels, 1945). Each node initially carries a load \( \lambda[0] = \lambda_0 \) and can accumulate additional load over time. Each failing node distributes all of its accumulated load equally among its non-failed neighbors. Thus, in case of \( n \) failed nodes, a total load of \( n \ast \lambda_0 \) is shared among the remaining \( N - n \) neighbors (additionally to the initial load). It follows that each node with \( n \) failed neighbors carries \( \lambda[n] = \lambda_0 + n \lambda_0 / (N - n) \).

Usually, this model is studied under the assumption that thresholds are uniformly dis-
tributed and a single node fails initially. This node distributes a load $L_0$ to the remaining $N$ functional network neighbor at time $t = 1$. (We consider a network of size $N + 1$ for simplicity.) By defining the initial load as $\lambda_0 = L_0/N$ and uniformly distributed thresholds $\Theta \sim \text{Unif}[0, 1]$, our approach at time $t = 0$ matches the described setting at $t = 1$. Consequently, the final cascade size distribution coincide in the end. For the special (and simplifying) case of uniformly distributed thresholds, a formula for the final cascade size distribution has been derived by (Dobson et al., 2005).

In our framework analysis, we focus on normally distributed thresholds instead. The final cascade size distribution for this case is shown in Fig. 9.2 (b). Again we find broad distributions. Noticeably, a total break down of the network ($\varrho_{fc} = 1$) often occurs with a much higher probability than slightly smaller final cascade sizes. There seems to be a tendency towards complete failure in the system if a certain cascade size is reached and high load is accumulated.

OLRD models

In case of OLRD models, we face the additional obstacle that the load $\lambda[n]$ that a node with $n$ failed neighbors carries depends on the thresholds of the nodes that have failed before. Consequently, $\lambda[n]$ is a random variable $\Lambda[n]$ that is defined by

$$\Lambda[n] = \sum_{i=1}^{n} \Theta_i \frac{n}{N}$$

conditional on the fact that $\Theta_i \leq 0$. So, it depends on the whole history of the cascade process. Unfortunately, the inclusion-exclusion does not hold in such a case and our derivations do not apply to OLRD models.

At this point, we can formulate the cascade size distribution only as multi-dimensional integral over all possible threshold configurations. The result is not practically computable.

### 9.4 Final cascade size distribution on star-shaped networks

Next, we shift our focus to a finite star-shaped network consisting of $N$ nodes as illustrated in Fig. 9.1 (b). This is the simplest possible topology where a hub and leaves are present. In fact, we only have two categories of nodes, the single hub in the center with high degree $d = N - 1$, and $N - 1$ leaf nodes, i.e., the nodes with degree 1. Under the constraint that all nodes in a network are part of a single connected component, a star is the topology that minimizes the overall connectivity (and thus the total number of links). Consequently, we would expect relatively small systemic risk, since the low connectivity hinders cascades to
9.4. Final cascade size distribution on star-shaped networks

We proceed further in time. We discuss later, if this intuition really applies.

A cascade ends after maximally three time steps. The central node has a prominent role, as it is the only node in the network that can further distribute accumulated load. Initial failures might happen at \( t = 0 \) before the central node fails (among other nodes) at \( t = 1 \), and this failure possibly causes neighboring nodes to fail at \( t = 2 \). If the center does not fail, initial failures cannot become further amplified and the cascade stops after \( t = 0 \). If the center fails initially, it can cause further failures at \( t = 1 \), but then the cascade stops.

Because of the limited number of possibilities, we can take the history of the center into account when we formulate a closed-form solution for the distribution of the final cascade size \( \varrho_* \) for general cascading processes.

We allow the central node’s cdf \( F_c \) and load \( \lambda_c \) to be different from a leaf node’s cdf \( F \) and its load \( \lambda_l \). As before, we assume that the load \( \lambda_c[n] \) that the center carries only depends on the number of its failed neighbors \( n \) (and the network size \( N \)). A leaf node can fail initially with probability \( F(\lambda_l[0]) \) (with 0 neighboring failures) or, if the center has failed before, it can carry the load \( \lambda_l[1,j,l] \), which depends on two additional variables that are determined by the history of the center. \( j \) denotes the number of failed leaf nodes that have distributed load to the center, before the center has failed. \( l \) indicates the number of nodes among which the accumulated load of the center is shared (when the center fails).

We simply add the probabilities for all different cases and obtain:

\[
P \left( \varrho_* = \frac{n}{N} \right) = (1 - F_c(\lambda_c[0])) \left( \frac{N-1}{n} \right) F(\lambda_l[0])^n (1 - F(\lambda_l[0]))^{N-1-n} + F_c(\lambda_l[0]) \left( \frac{N-1}{n-1} \right) \sum_{j=0}^{n-1} \left( \frac{n-1}{j} \right) F(\lambda_l[0])^j \times (F(\lambda_l[1,0,N-1-j]) - F(\lambda_l[0]))^{n-1-j} (1 - F(\lambda_l[1,0,N-1-j]))^{N-n} + \left( \frac{N-1}{n-1} \right) \sum_{j=1}^{n-1} \left( \frac{n-1}{j} \right) F(\lambda_l[0])^j (F_c(\lambda_c[j]) - F_c(\lambda_l[0])) \times (F(\lambda_l[1,j,N-1-j]) - F(\lambda_l[0]))^{n-1-j} (1 - F(\lambda_l[1,j,N-1-j]))^{N-n}.
\]  

The first summand considers the case when the center does not fail, while the second term adds the probability for the case when the center fails initially. Then, each of the other \( n-1 \) failures of leaves can either occur initially or because of a load distribution by the center. The size of this load might depend on the number of nodes \( l \) that fail initially together with the center, since these nodes cannot receive load after the failure of the center. The index \( j \) in the third term finally takes the events into account when \( j \) leaves have failed before the center.

Even though each cascade is over after maximally three time steps, the final cascade size
can still be large if enough load is distributed by the center. Thus, low interconnectivity is not necessarily an advantage. Here, load can be distributed almost globally to network neighbors, a familiar problem in case of the failure of hubs. The small number of times steps in the cascade evolution additionally make interventions more difficult, because these need to be implemented fast.

Still, we expect that systemic risk is low in some of our cascade model examples that reduce the impact of the failure of hubs.

9.4.1 Model examples on star-shaped networks

ED models

In Equ. (9.3), the loads are defined as $\lambda_l[0] = \lambda_c[0] = 0$, $\lambda_l[1, j, l] = 1$ and $\lambda_c[j] = j/(N - 1)$ for all values of $j$ and $l$ in case of an ED model. We would expect a bi-modal cascade size distribution because of the prominent role of the central node. If the center can withstand the failure of some of its neighbors, the cascade stops early without causing high damage with high probability. But if the center fails, the further failure of leaves might be amplified, because they need to carry a higher load. Each event, the survival and the failure of the center, account for one peak in the cascade size distribution.

Fig. 9.4 (a) confirms this intuition for most cases. The plot shows the final cascade size distribution for normally distributed thresholds with parameters $\mu = 0.5$ and different standard deviations $\sigma$. For all $\sigma \leq 3$, we observe a bi-modal distribution whose left peak is higher than the right. These peaks move closer together towards 0.5 and merge to a single peak for $\sigma = 4$. Although intuitively convincing, the bi-modal shape of the distribution relies crucially on the size of the load distribution mechanism of the ED model. We compare these results with the DD model where damage is diversified.

DD models

In DD models the loss, the failure of hubs inflict, is reduced. It has been introduced in Lorenz et al. (2009) and contrasted with the ED model on configuration model random graphs in Burkholz et al. (2016a). Each failing node distributes a total load of 1 equally among its network neighbors. While the model coincides with the ED model on fully connected networks, for star-shape networks we have $\lambda_l[1, j, l] = 1/(N - 1)$ and $\lambda_c[j] = j$ again for all values of $j$ and $l$. Here, the center fails with high probability, but this failure does not cause significant further damage. Because of this, we only observe unimodular cascade size distributions in Fig. 9.4 (b). Generally, the cascade risk is reduced in comparison with the ED model.
Systemic risk reduction in CL models

In both Constant Load models, systemic risk could be further reduced by providing the center with high thresholds. Similar to our findings in Chapt. 4, it costs less resources to increase the threshold of the center in case of ED. We would only need to ensure that the maximal load $\lambda_c = 1$ cannot exceed the threshold $\Theta_c$. In contrast, the maximal load that the central node can carry is $N - 1$ in a DD model. Instead, the $(N - 1)$ leaf nodes can be cost effectively protected, as each only needs to withstand a maximal load of $1/(N - 1)$.

Thus, ED and DD models on star-shaped topologies could be sheltered effectively from systemic risk.

Still, we note a difference between the two models that we have not paid attention to yet. The connectivity of the system at the end of a cascade is not relevant for our notion of systemic risk, although it might be essential for system performance. From this perspective, the DD model can be quite problematic, as the center fails with high probability and the network becomes disconnected. If this corresponds to a short shut-down in a crisis and connections can be easily established afterwards, the DD model weights might still represent a good system design option.

LRD models

Our third example is the LRD LLSS model, i.e. the fiber bundle model (now with a form of local load sharing). We need to take the two additional parameters $j$ and $l$ into account, as they define the amount of load that the center accumulates before it distributes...
Chapter 9. Cascade size distribution for finite networks

Figure 9.5: Distribution of the fraction of failed nodes $\varphi_*$ on a star-shaped network with $N = 50$ nodes for the LRD LLSS model with $\lambda_0 = 1$. The thresholds are normally distributed as $\Theta \sim \mathcal{N}(\mu + \lambda_0, \sigma^2)$.

It among its non-failed neighbors. Initially, failed leaves carry only the load $\lambda_l[0] = \lambda_0$ and distribute all of it to the center (if the center has not failed initially). Thus, if $j$ leaves fail initially before the center, the central hub accumulates $\lambda_c[j] = \lambda_0(j + 1)$. The formula also holds for the case $j = 0$, when the center fails initially. The load $\lambda_c[j]$ is distributed to the remaining leaves $l$ that might fail after the center, i.e. $\lambda[1, j, l] = \lambda_0(j + 1)/l$.

Fig. 9.5 shows the resulting cascade size distribution for normally distributed thresholds for several parameters and a star-shaped network consisting of $N = 50$ nodes. We observe broad and partly irregular distributions. Some have two local maxima, while others show only one. We do not see the same abrupt tendency towards full system break down as in the case of fully connected networks. (Still, there are parameter configurations where $\varphi_* = 1$ almost certainly.) Interestingly, several cascade size outcomes are similarly probable, although only two events determine the cascade: The central node fails or does not fail. Still, diverse load sizes are distributed if the center fails, because the load size depends mainly on the number of initial leaf failures, for which several possibilities exist.

**OLRD models**

The additional obstacle that the load $\lambda_i$, a node carries, depends on the thresholds of the nodes that have failed before, is not as restricting as in fully connected networks, because we only need to consider three cascade time steps. The final cascade size distribution can be calculated analogously to Equ. 9.3. We only need to specify the distributions of the random variables $\Lambda_c[j]$ and $\Lambda[1, j, l]$ for the OLRD LLSS model.

Initially, nodes carry a load $\lambda_l[0] = \lambda_0 = 0$. $j$ initially failing nodes with thresholds $\theta_i \leq 0$ distribute the load $-\sum_{i=1}^j \theta_i$ to the center. Consequently, the load carried by the center
9.4. Final cascade size distribution on star-shaped networks

is distributed as

\[ \Lambda_c[j] = - \sum_{i=1}^{j} \Theta_i^f \]

where \( \Theta_i^f \) denotes the random threshold of a node conditional on its initial failure, i.e. \( \Theta_i \leq 0 \). It simplifies our calculations, if we consider instead of the pdf of \( \Lambda_c[j] \) a distribution that is not conditional on the failure of the nodes \( i \). This distribution is not a probability distribution. Because of this, we denote it by \( d_{\Lambda_c[j]} \). It sums up the probability mass of all threshold configurations, where \( j \) nodes have thresholds smaller than zero.

Let’s assume that each threshold distribution has a density \( p_\Theta \) or \( p_c \) for the central node. \( d_{\Lambda_c[j]} \) can then be written as convolution

\[ d_{\Lambda_c[j]} = d_\Theta \ast_j \Theta_c(z) \]

where \( d_\Theta \) is defined by

\[ d_\Theta(x) := p_\Theta(-x), \text{ if } x \geq 0, \text{ and } d_\Theta(x) := 0, \text{ if } x < 0. \]

Similarly, the load \( \Lambda[1,j,l] \) that leave nodes receive at time \( t = 2 \) after \( j \) nodes have failed initially and triggered the failure of the center, is distributed as

\[ \Lambda[1,j,l] = -\Theta_c - \sum_{i=1}^{j} \Theta_i^f \]

where \( \Theta_i^f \) follow the distribution of \( \Theta_i \) conditional on \( \Theta_i \leq 0 \) and \( \Theta_c^f \) follows the distribution of \( \Theta_c \) conditional on \( 0 \leq \Theta_c \leq -\sum_{i=1}^{j} \Theta_i^f \). Again, we define a density \( d_{\Lambda[1,j,l]} \) as

\[ d_{\Lambda[1,j,l]}(z) := \int_{\Theta_c}^{\infty} p_c(x - zl) d_{\Lambda_c[j]}(x) \, dx \]

for \( z \geq 0 \). This is the probability that a threshold configuration of \( j \) initially failing nodes and a central node leads leads to the distribution of the load \( z \) to the remaining \( l \) nodes (unconditional on all mentioned failures).

With these definitions we can adapt Equ. 9.3 for the OLRD LLSS model to

\[
P(\varphi = \frac{n}{N}) = \binom{N - 1}{n} (1 - F(0))^{N-1-n} \int_0^\infty (1 - F_c(x)) d_{\Lambda_c[n]}(x) \, dx \\
+ \binom{N - 1}{n - 1} \sum_{j=0}^{n-1} \binom{n - 1}{j} \int_{-\infty}^0 (F(-y) - F(0))^{n-1-j} (1 - F(-y))^{N-n} p_c(y) \, dy \\
+ \binom{N - 1}{n - 1} \sum_{j=1}^{n-1} \binom{n - 1}{j} \int_0^\infty (F(z) - F(0))^{n-1-j} \\
\times (1 - F(z))^{N-n} d_{\Lambda[1,j,N-1-j]}(z) \, dz.
\]

As the numerical calculation of the given formula is more involved (although possible), we
omit the discussion of examples.

9.5 Conclusions

In summary, we have presented closed-form solutions for the cascade size probability distribution corresponding to a general class of cascade models on fully connected and star-shaped networks with respect to an arbitrary threshold distribution. Fully connected networks are especially of interest as a proxy for dense networks, while stars are suitable for investigating the role of hubs.

With our derivations, we have gained further insights into the microscopic evolution in time of a cascading process. In finite networks, neighbors of a node do not fail independently. The failure of each node depends on the specific threshold configuration, and is thus determined regarding all other threshold values.

We have seen in Chapt. 4 that these threshold configurations and the studied cascade model influence the time until a cascade reaches its steady state. Moreover, the example of a star-shaped network has led us to the insight that the presence of hubs can not only amplify, but also accelerate or block cascades, when they represent channels connecting otherwise disconnected network components. Such fast cascade evolutions can be problematic for the implementation of interventions.

Our results allow further studies of final cascade size distributions with respect to various shapes of the threshold distribution, but also to the network size $N$. Furthermore, explicit formulas can enable the optimization of several parameters for the design of robust systems and invite the treatment of model expansions like restorations as in Liu et al. (2014).

Most importantly, we have shown for the CL ED, CL DD and LRD LLSS models that a good risk assessment in finite systems of finite size $N$ demands the consideration of full cascade size distributions. Bi-modular distributions can occur even in large networks, for instance, in the order of magnitude of $N = 10^7$ nodes.

With our examples, we hope to encourage the further study of full cascade size distributions on finite networks.
Part III

Modeling systemic risk in a data driven approach

“The goal is to turn data into information, and information into insight.”

 Carly Fiorina
Introduction: Systemic risk and international trade of staple food

In Part I of this thesis, we have investigated generic models for cascade processes under network uncertainty. Then, in Part II, we have critically reflected on our assumptions and successively introduced further complexity to our models. Two sources of feedback, one originating from another network layer, another originating from the system response to the cascade evolution, have been discussed. Moreover, we have emphasized the high variation of the final cascade size that finite systems can exhibit. All these three aspects, the feedback between multiplex layers, systemic feedback by variables on the macro level, and system finiteness are of high relevance also in Part III.

However, in Part III, we successively shift our perspective from generic models to an Overload Redistribution cascade model that we motivate by observed data and that we adjust to our modeling purpose. This allows us to transfer some of our insights to an application context.

In general, the high amount of variables in our models, which need to be inferred from data, pose a practical challenge. For each network node, a failure threshold needs to be determined, and, for each link, a load that is transferred in case of the failure of one node needs to be defined. Our choice is simplified by an economic motivation of these variables. In fact, they can be defined by measurable quantities.

In Part III, we study the international trade of the four major internationally traded staples: maize (also known as corn), rice, soy, and wheat. While we interpret the world’s countries as nodes in an international trade network, weighted links represent aggregated trades of a specific crop in a given year between two countries. We specify in Chapter 10 how we obtain such networks from a dataset provided by the Food and Agriculture Organisation of the United Nations (FAO) for the public.\(^2\)

These networks contain information about the dependence of countries on the international market and about economic relationships between countries with respect to staple foods. The trade relationships result from economic mechanisms matching supply and demand.

\(^2\)http://faostat3.fao.org/home/E
in a given year. Thus, they determine the availability of food on the country level and are therefore an important factor for efforts to ensure food security. The FAO defines food security as a “situation that exists when all people, at all times, have physical, social, and economic access to sufficient, safe, and nutritious food that meets their dietary needs and food preferences for an active and healthy life” (Food Agriculture Organization, 2001). We focus on the aspect of stability in the following chapters and study the fragility of the trade network with respect to shocks.

There is a wide range of risk factors for agricultural production. Especially in the course of climate change, weather conditions are expected to become more extreme and can pose a challenge to agricultural production (Schmidhuber and Tubiello, 2007). Crops are sensitive to extreme temperatures, droughts or floods put yields at risk (Puma et al., 2015, and references therein), and water famine can hamper irrigation (Misra, 2014). The decline of global pollinators additionally challenges agricultural production (Potts et al., 2010). Also crop pests and pathogens can reduce yields (Garrett, 2013), while local political instabilities and wars might hinder farming. The list of possible shocks on the production side can be continued (Puma et al., 2015; Rosegrant and Cline, 2003).

Additional to that, shocks on the demand side can lead to a reorganization of the trade network. New crop usage possibilities, as for instance, biofuel production (Mitchel, 2008; Zilberman et al., 2012), or increases for the demand of feed for meat production might compete with human consumption of crops and increase the total demand (Spiertz and Ewert, 2009, and references therein). Also the increasing demand due to population growth is of concern and might influence the export behavior of food producing countries (Suweis et al., 2015). Although such demand increases might not occur immediately, also more gradual changes lead to trade reorganizations.

Price adjustments are the results of such changes in production and demand and influence the future evolution of the food system. Moreover, rapid price changes might refer to additional speculation effects (Lagi et al., 2015, and references therein). Food prices and their variability are an important indicator for food security, as they determine the food consumption options of the poorest and impact the profitability of agricultural production. Food prices provide an orientation for market participants and are thus not only a result of economic interactions, but are also a source of systemic feedback.

We return to the special role of prices in Chap. 12, when we introduce a more refined cascade model that incorporates price dynamics.

First and foremost, we are interested in the reorganization of the international trade network in response to shocks on the production or demand side. While the causes of such shocks can be manifold and require careful modeling, we make simple assumptions about the size of the shocks and study several scenarios that have the purpose to reveal the dependency structure between countries. Our focus lies on the response of the system.
to shocks and cascading readjustment of trade links.

Such cascading phenomena have been termed multiplier effect (Giordani et al., 2014) among others. Although its relevance has been acknowledged (Giordani et al., 2014; Puma et al., 2015), to our knowledge, it has been neglected in resilience analyses of international food trade networks (Puma et al., 2015; Suweis et al., 2015; Wu and Guclu, 2013).

In Chap. 11, we first study the effect of cascading export reductions as response of shocked countries that propagate through a trade network according to a simple variant of an overload redistribution cascade model. Ref. (Giordani et al., 2014) provides evidence that national export bans and protective market behavior (including subsidies and other trade policies) can be observed and impact food prices. How exactly these export bans influence international trade is subject to different risk scenarios. In Ref. (Puma et al., 2015), shocked countries discard completely their exports and reallocate oversupplies according to a GDP ranking of possible trade partners. This is an extreme shock scenario and does not encounter several cascade steps.

In contrast, we study distortions of the system as response to relatively small initial shocks and still preserve basic trade relationships. Our approach has the advantage that many of the mechanisms that have formed the trade networks under study are still reflected in our model. Moreover, in Chap. 12, we extend our generic cascade model to incorporate further observed trade characteristics and include systemic feedback in form of evolving food prices.

Furthermore, we introduce a multiplex perspective, as we consider maize, rice, soy and wheat trade. All of these networks are coupled, as they can substitute each other to a certain degree. For instance, rising maize prices make soy more attractive as feed in comparison to maize and vice versa. Also people with a preference for rice as staple crop might consider to eat wheat instead, if rice prices grow exorbitantly. Because of this, crop prices and thus also the corresponding trade dynamics are strongly coupled. Before we acknowledge this coupling, we start with an exploration of the studied data sets and treat them as separate in our simple OLRD cascade model.

As preparation, we first give an overview of the studied datasets and present a descriptive data analysis in the next chapter.
Chapter 10

International trade of staple food: A descriptive data analysis

Summary
We interpret the annual international trade of maize, rice, soy, and wheat as four different temporal weighted networks and compare their basic properties. In this regard, we identify the main economic actors and discuss the evolution of major trade relationships. This supports our understanding of the system and forms our intuition for later improvement of our cascade models. For this purpose, our main observations concern the overall connectivity of the networks and the heterogeneity of the dependencies. We find that the number of participants in trade as well as the interconnectivity of the networks grow over time for all considered staples. While this can be interpreted as increasing market integration, the high dissimilarity of trade volumes suggests still big differences in the influence and dependence of countries on the market. Moreover, the stability of such major trade relationships is quite different from crop to crop. Especially for wheat and partially also for rice, big trade volumes can change rapidly over time. Surprisingly, in particular for maize, we observe a structural shift from 2012 to 2013 in which established trade relationships decline substantially while others gain more in volume.

RB wrote this chapter specifically for this thesis. RB designed the research questions, contributed to discussions, wrote the code, and performed the analysis.
10.1 Introduction

We explain in this chapter our network models for the international trade of maize, rice, soy and wheat, aggregated on a yearly basis. We restrict our analyses to the years 1992-2013, although data is already available from 1986 onwards, because the system undergoes a dynamical transition with the dissolution of the Soviet Union. After this event, we can observe a period of economic growth and globalization, in which not many countries change their borders or names.

As the trade networks reveal economic food trade relationships between countries, which are relevant for food security, their structure and evolution has been subject of several studies. Wu and Guclu (2013) address specifically the structure of the maize trade network and identify main importers and exporters in the years 2000-2009. Several cluster analyses classify countries on the basis of node variables including their Maize import, export and production quantities to identify similar trade patterns (Diaz-Bonilla et al., 2000). Also for the years 1992-2009, Puma et al. (2015) describe the structure and evolution of the wheat and rice trade networks. We are not aware of similar studies for soy trade. To some extent, soy is considered as part of a high number of different types of food by MacDonald et al. (2015) who explore the international trade flows of calories and money between countries. Furthermore, for aggregates of different food types, Ercsey-Ravasz et al. (2012) develop a dynamic food flux model to measure vulnerability of countries to contamination of food which has been produced in another country. Their analysis includes reports of the betweenness centrality of nodes in their network.

In general, the network analysis of international food trade networks could comprise similar approaches as the ones that have been applied to the world trade web (Fagiolo et al., 2009b), which covers additional products from different economic sectors. For instance, the vector centrality of countries in the network could be interpreted as dollar experiment where one dollar is tracked to different destinations via a random walk model (Serrano et al., 2007). Or, the weighted network structures could be further described. It could be tested whether they exhibit a core-periphery structure (Fagiolo et al., 2009a). Similarly, studies on virtual water networks answer research questions that can be relevant for food trade webs as well. The connection between them has been emphasized by Sartori and Schiavo (2015) who study also topological features of aggregated food trade flows from 1986 to 2010.

Here, we only summarize the main features of the maize, rice, soy and wheat networks separately and compare our main findings. Of special interest are the years 2010-2013 in our descriptive data analysis, as we can observe some structural change in the importance of specific trade links.

We restrain from testing a number of different network metrics and centrality measures,
10.2 The network model

Our analysis is based on a data set about global food production and trade which is provided by the Food and Agriculture Organization of the United Nations (FAO) for the public.\footnote{http://faostat3.fao.org/home/E}

We interpret aggregated trades between countries in a given year as weights in an international trade network for a specific crop, where the set of nodes $V$ consists of 176 countries. A country list is provided in Table C.1 in the Appendix. We mainly use ISO 3166-1 alpha-3 codes as country names. Their translation to the official name can also be found in this table. A geographic visualization is provided by Fig. 10.1, which depicts the considered countries in a color that we associate with them throughout this thesis and that refers to their regional proximity.

In each year $y$, the countries are connected via directed weighted links $w_{ij}(y)$ describing the export of a quantity $w_{ij}(y) \in W(y)$ (in tonnes) of a certain crop from country $i$ to country $j$. Trades are reported by importers as well as exporters. In case their declarations deviate,
we refer to the minimum of both values and thus follow a conservative estimate of the trade interconnectivity. This way, we only consider trade between FAO reporting countries. As our cascade models require anyway further information about production amounts and other economic variables, e.g., their gross domestic product (GDP) or population size, we have to restrict ourselves to a set of nodes for which this data is available. Thus, we consider more than 85% of all international trade for maize, soy and wheat according to the total reported FAO trades. Only for rice, we refer to ca. 65%, as a high proportion of trade is issued between Asian countries which do not report to FAO. In future work, the remaining variables could be partially inferred by trade reports of other countries that trade with the missing ones. But here, we focus only on reported information.

This way, we define for each of the four staples a temporal network

\[ G^{(c)}(y) = (V, W^{(c)}(y)) \] for \( y = 1992 \) to \( 2013 \)

that consists of 22 time slices \( G^{(c)}(y) \), one for each year \( y \). All of these slices consist of the same nodes \( V \), but vary in their weights \( W^{(c)}(y) \). The superscript \( (c) \) indicates the considered crop: maize (m), rice (r), soy (s), or wheat (w).

### 10.3 Further available information

Additional to trade volumes, we have information about financial flows along links. Consequently, we can calculate the average price per quantity along a link. In this thesis, we utilize this information only for the calculation of the average world price per quantity in a given year. But in future, we plan to develop a model for price flows as well, since they describe an important aspect of food security and determine who can afford food when it becomes a scarce resource.

Alongside information about links, we consider several variables that are associated with countries. From the trade flows, we can deduce the exports \( \exp_i(y) \) and imports \( \imp_i(y) \) by each country \( i \) in a given year \( y \).

Adding data on production quantities \( \text{prod}_i(y) \), we can compute each country’s demand as \( \text{dem}_i(y) = \text{prod}_i(y) + \imp_i(y) - \exp_i(y) \). Furthermore, we have information about the area harvested \( \text{area}_i(y) \), the yield \( \text{yield}_i(y) \) and amount of produced seeds \( \text{seed}_i(y) \). These can be important predictors of future production quantities or reflect expectations of farmers.

Another important factor for future investments are prices in USD/tonne that farmers receive at farm gate for their products \( \text{prodPrice}_i(y) \). These also depend on the usage

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of the products. According to the Commodity Balances by FAO\(^4\), we can differ between feed for livestock \((\text{feed}_i(y))\), seeds, waste \((\text{waste}_i(y))\), further processing \((\text{processed}_i(y))\), human consumption \((\text{food}_i(y))\), and other uses \((\text{other}_i(y))\) including, for instance, biofuels. These variables are especially interesting for the analysis of causes for changing demands. They might enable a more fine grained cascade analysis in future work. They might help to relate certain imports and exports to different usage and thus help distinguish between different types of crops. Furthermore, these insights could enable better estimations of the consequences of the loss of certain trades or production capacities.

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<tr>
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<tbody>
<tr>
<td>(\text{prodPrice}/100\text{US}$)</td>
<td>0.89</td>
<td>2.19</td>
<td>1342.11</td>
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<tr>
<td>(\text{cprod}/10^4\text{tonnes})</td>
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<td>0.05</td>
<td>0.97</td>
</tr>
<tr>
<td>(\text{prod}/10^7\text{tonnes})</td>
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<td>0.32</td>
<td>25.53</td>
</tr>
<tr>
<td>(\text{area}/10^6\text{Ha})</td>
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<td>0.76</td>
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<tr>
<td>(\text{yield}/10^5\text{Hg/ha})</td>
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<td>0.26</td>
<td>2.25</td>
</tr>
<tr>
<td>(\text{seed}/10^5\text{Hg/ha})</td>
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<td>0.31</td>
<td>18.20</td>
</tr>
<tr>
<td>(\text{stockVar}/10^6\text{tonnes})</td>
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<td>-0.03</td>
<td>38.87</td>
</tr>
<tr>
<td>(\text{supply}/10^6\text{tonnes})</td>
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<td>192.04</td>
</tr>
<tr>
<td>(\text{feed}/10^6\text{tonnes})</td>
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<td>0.14</td>
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</tr>
<tr>
<td>(\text{processed}/10^6\text{tonnes})</td>
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<td>0.18</td>
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<tr>
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<td>12.49</td>
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<tr>
<td>(\text{other}/10^6\text{tonnes})</td>
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<td>0.19</td>
<td>21.71</td>
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<tr>
<td>(\text{dem}/10^6\text{tonnes})</td>
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<td>(\text{exp}/10^6\text{tonnes})</td>
<td>0.00</td>
<td>0.31</td>
<td>51.19</td>
</tr>
<tr>
<td>(\text{imp}/10^6\text{tonnes})</td>
<td>0.00</td>
<td>0.31</td>
<td>16.70</td>
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Table 10.1: Summary of country variables relating to maize. Values are rounded to two digits. The minimum values of stockVar are \(-37.57\) (1992-1999), \(-26.86\) (2000-2007), and \(-8.54\) (2008-2011).

The stock variation \((\text{stockVar}_i(y))\) provides further insights into the perception of countries of the economic situation. An increase of stocks can be seen as preparation for increasing prices, while a decrease of stocks can be interpreted as exploitation of a good opportunity or attempt to mitigate a tense situation. We do not capture speculation explicitly in our models. It is only considered implicitly by expectations that shape the introduced variables. Unfortunately, the Commodity Balance variables are only available for the years 1992-2011. If necessary, we assume the values of 2011 for the missing years 2012-2013. For 2013, this assumption might be erroneous for maize trade, as we observe a substantial shift in trade patterns. Because of this, we avoid to employ the usage variables, as far as possible. A data summary for country variables relating to maize can be found in Table 10.1. This table also serves as list of considered variables. For summary statistics relating to rice, soy or wheat we refer to Appendix C.2.

Economic variables that are not associated with a certain crop\(^5\) are a country’s gross domestic product \(gdp_i(y)\) (in US dollar), and its value in 2005 prices \(gdpVal05_i(y)\), the gross domestic product per capita \(cgdp_i(y)\) (in US dollar), and its value in 2005 prices \(cgdpVal05_i(y)\). In principle, we could also consider the value added by agriculture, forestry and fishing \(gdpAgr_i(y)\), its value in 2005 prices \(gdpAgrVal2005_i(y)\), and their share of gdp \(AgrShareGDP_i(y)\), or in 2005 prices \(AgrShareGDPVal2005_i(y)\). But these variables did not turn out to be noticeable in our models yet. One of the noticeable variables is a country’s population size \(pop_i(y)\). A summary of this data can be found in Table 10.2.

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<tr>
<td></td>
<td>median</td>
<td>mean</td>
<td>max</td>
</tr>
<tr>
<td>(pop/10^6)</td>
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<td>31.22</td>
<td>1262.71</td>
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<tr>
<td>(gdp/10^1\text{US$})</td>
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<td>16.13</td>
<td>966.06</td>
</tr>
<tr>
<td>(cgdp/10^3\text{US$})</td>
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<td>6.87</td>
<td>81.69</td>
</tr>
<tr>
<td>(gdpVal05/10^3\text{US$})</td>
<td>2.73</td>
<td>8.98</td>
<td>101.45</td>
</tr>
<tr>
<td>(gdpAgr/10^3\text{US$})</td>
<td>0.98</td>
<td>5.57</td>
<td>130.40</td>
</tr>
<tr>
<td>(gdpAgrVal05/10^3\text{US$})</td>
<td>1.02</td>
<td>5.50</td>
<td>128.64</td>
</tr>
<tr>
<td>(AgrShareGDP)</td>
<td>0.10</td>
<td>0.15</td>
<td>0.73</td>
</tr>
<tr>
<td>(AgrShareGDPVal05)</td>
<td>0.08</td>
<td>0.13</td>
<td>0.71</td>
</tr>
</tbody>
</table>

Table 10.2: Summary of economic country variables. The minimum is zero in all cases because of missing values.

Additional to variables on the country level, we define global variables that measure the macroeconomic state of the system. The total production \(prod_{tot}(y)\), the population of all considered countries \(pop_{tot}(y)\), and their total \(gdp_{tot}(y)\), or \(gdp\) per capita \(cgdp_{tot}(y)\) might influence the average price per tonne of a crop \(p_{avg}(y)\).

Furthermore, we might express country variables as share of such global variables as measure of their importance for the global picture.

Additionally to the described data, further information about consumer price indices and deflators is available by FAO and would support the estimation of evolving food preferences and price elasticities. These might be relevant, when we further consider to add complexity to our multiplex view of the trade of crops. But at the stage of this thesis, we disregard them. Furthermore, data on inputs, investments, and land use, etc. might describe other relevant aspects of the system that are relevant on the long-term. At this stage, we focus on short time horizons and the exposure of actual food trade networks to cascading effects. Because of this, we omit this information in the course of this thesis.

Instead, we devote the following chapter to a further descriptive analysis of the introduced trade networks and their evolution in time.

\(^5\)http://faostat3.fao.org/download/M/MK/E
10.4 General trends

As result of the green revolution, total agricultural production of the four major internationally traded crops is increasing over time. Although the total world population grows as well, Fig. 10.2 (a) demonstrates that the production per head still increases for maize and for soybeans most of the time. In contrast, the production per head of rice grows only slowly since 2002, while the one of wheat shows relatively high fluctuations without a clear growth trend. Still, the total world wide production increases for all crops and with it, the total amount that is traded internationally. As we see in the next section, the interconnectivity of the trade networks and the number of countries that engage in international trade increase over the years.

Fig. 10.2 (b) answers further the question whether also the proportion of the total trade volume increases relative to the growing production. Interestingly, the highest fraction of the total production that is traded belongs to soy and this fraction increases. In contrast, rice is mainly produced for national consumption. Still, our cascade analysis concerning its international trade in the next chapter is of special interest, because export bans are a common instrument to protect the national rice supply. For maize, the proportions do not change much over time. So, traded goods and national consumption grow similarly with increasing production. For wheat, the proportion of trades seems to increase slowly, although the production per capita does not.

The question left to answer is who is engaged in production and in international trades. Is the increase in production mainly a result of intensified farming in a few regions in the
The number of producing countries itself does not change noticeably in the observed time horizon for wheat or rice, but for maize and soy a few new countries start producing. In all cases, the production amount increases for most of the countries and, in general, we observe a trend towards better diversification of production. Still, the differences between larger and smaller producers stay enormous.

Increasing market integration is revealed by the fact that the number of importing nations increases for soy, and for the other crops also till 2002 (but seems to saturate afterwards). Similarly, the number of exporters increases, especially for soy, while the market participants in case of the remaining crops are (more or less) settled till 2004. Despite our observation that barely new countries engage in production, the number of trade links increases for all crops and the trade networks become more dense. But the size of the trade volumes along a link are very different. In general, the weight distributions are highly right skewed and this does not change over time. Fig. 10.3 (a) shows the evolution of the Gini coefficient (Gini, 1912, 1997) for the positive weight distributions, which serves as measure for the dissimilarity between positive trade volumes. We find that the traded volumes are quite dissimilar. Their distribution is also shown in Fig. 10.4 for the years 1992 and 2013 in direct comparison. The positive trade volumes of wheat are most similar according to the Gini index, but they seem to be also subject to stronger fluctuations. This is a sign of strong changes especially the larger trade volumes. These exist in higher proportions for wheat than for the other crops, as can be deduced from the Fig. 10.4. For maize and for rice, trade volume fluctuations seem to be smaller, and no clear trend is visible. The dissimilarity of Maize trade volumes seems to decrease slightly in the last
years. This can be mainly attributed to a lowering biggest trade volume. Interestingly, we find that the dissimilarity increases for soy, where the number of international market participants is growing the most. As we see Fig. 10.4 (c) and later again, this is especially caused by a faster growth of the biggest trade volumes in comparison to the other soy trades. Additionally, many new trade links of smaller size are established over time.

![Histograms of the logarithm of the positive trade volumes](image)

**Figure 10.4:** Histogram of the logarithm of the positive trade volumes in the years 1992 and 2013 for (a) maize, (b) rice, (c) soy, and (d) wheat.

Which countries engage most in trade is different from crop to crop and is analyzed in detail in the next sections. In general, the USA and ARG have always a considerable share in exports as big producers. Except for wheat, the same holds for BRA and, except for rice, also for UKR. We turn our attention to these hubs especially when we study the propagation of cascades, since they are potential main spreaders.

Interestingly, the USA, CAN or FRA are countries with a relatively high gross domestic product per capita. Also BRA, ARG and CHN have a substantial gdp as developing countries with large populations. According to the economic theory of structural-change, we would expect that countries with a low income (and thus low cgdp) mainly produce for
Figure 10.5: We plot the sum of all productions (i.e., of maize, rice, soy, and wheat) per capita of each country versus a country’s gross domestic product per capita in 2013. The size of the bubbles is proportional to the amount of imports. All variables are reported as share of the maximum observed value.

subsistence, while agricultural production is less relevant for countries with high incomes. Fig. 10.5 shows the relation between per head production and per head gdp in 2013. We observe that least developed countries, or countries with a low gdp per capita are in fact quite import dependent. As we also see later on, international traded goods are produced to a large extend in developed or developing countries where big agricultural businesses achieve substantial production amounts.

The economic decisions of such big producers and of big consumers might even have strong implications for the prices at which crops are traded. Although there is no sign of decreasing per capita production, the average price per quantity of each traded crop increases since 2005 (with a small intermediate decline in 2009 and 2010) illustrated by Fig. 10.3 (b). The years 2007/2008 are often referred to as food crisis and prices continue to grow further since 2011. In fact, in terms of purchasing power of the major importer and exporter nations (with growing GDPs), real prices do not seem to have increased. Still, the populations of importing countries with slower growing GDPs and quite unequal wage distributions are still impacted by these price trends.

Several possible causes are discussed, e.g., see Ref. (Lagi et al., 2015, and references
As main factors, an increased demand for bio fuels (mainly obtained from maize or soy), price speculation, an increasing demand for meat in developing countries, whose production requires a higher amount of feed, and bad weather are investigated. All of the different causes affect either the production or demand side. Our cascade models serve the purpose to study the implications of such factors on the trade network topology.

We note high correlations between the price evolution of the different crops. This is not surprising, as the staples can substitute each other partially. As explained in the introduction of Part III, when one crop becomes more expensive the preferences of consumers might shift towards a cheaper crop. Thus, the demand for the latter increases, which leads to a price increase of the at first cheaper crop. The coupling of the prices suggests that a cascade model for food trade should consider a multiplex perspective, where the different networks are coupled by prices, for instance.

Before we discuss further details of a good cascade model, we focus on directly observed trade relationships between countries in the next sections.

10.5 International trade of maize

As the nodes in our trade networks refer to countries, they incorporate geographic information. Usually we do not illustrate networks on maps, because the high number of trade links challenges the identification of trade partners. Nevertheless, we encode regional information by similar colors that we associate with countries and that are depicted in Fig. 10.1. A trade link is then associated with the color of the exporter nation. Fig. 10.6 displays four snapshots of the temporal network of maize trades in 1992, 2000, 2008, and 2013.

As mentioned earlier, we observe a clear increase in the number of economic actors and in coupling strength of the network. In 1992, the USA dominated the international market. Considering that it produces almost 46% of the world wide maize production, this is not astonishing. ARG, CAN and AUS are additional bigger exporters that serve other continents, while CHN mainly serves the Asian market, and FRA and HUN export primarily within Europe. Over the years, further countries cultivate maize so that the share of the production by the USA declines to 35% in 2013. Accordingly, additional exporters and importers enter the market.

Fig. 10.7 displays the production of 20 largest producers relative to total production in 2000 and 2012. Additionally, a darker color indicates how much of this production covers their demand. The USA clearly is the top producer (and stays the biggest exporter till 2012). Also the production of CHN grows noticeably till 2012 but its demand increases even faster so that CHN becomes a net importer. In fact, a number of main producers have a higher demand than they can serve by their own production: Besides CHN, also
Figure 10.6: [Maize] Evolution of the international maize trade network. Snapshots of the years: (a) 1992, (b) 2000, (c) 2008 and (d) 2013. The color of a link \((i, j)\) corresponds to the exporting nation \(i\), while the link thickness is proportional to a logarithmic transformation of the export quantity: \(\log (1 + w_{ij})\). Links with larger weights are plotted on top of the smaller ones. Square node shapes indicate that the respective country is a net importer, while circles refer to net exporters. The node size is proportional to a log transformation of their net imports or net exports. The twenty biggest nodes have their country code assigned. Node colors follow the scheme introduced in Fig. 10.1, which encodes regional proximity.

MEX, IDN, EGY, and TUR belong to this category.

On the other side, ZAF develops from a net importer in 1992 to a net exporter in 2013. Still, also for net exporters, an increase in production is often associated with an increase
10.5. International trade of maize

**Figure 10.7:** [Maize] Share of national production in total (world wide) production for the 20 biggest producers. The segment in darker color refers to the demand of the depicted country. If it is negative the demand encompasses the sum of both segments, the lighter and the darker one, and the lighter segment refers to the countries’ production. (a) 1992. (b) 2013.

**Figure 10.8:** [Maize] Share of the national variable with respect to the total world wide economic variable corresponding to maize. The most inner circle belongs to a country’s demand, the second inner circle corresponds to a country’s export quantity, the second outer circle characterizes the share of a country’s imports in comparison to the sum of all imports, and the most outer circle belongs to a country’s share in production. (a) 1992. (b) 2013.

of their demand relative to their own production. This can also be seen in Fig. 10.8, which illustrates the share of imports, exports, demand, and production of each country in a given year. Most surprisingly, in 2013, BRA passes ARG and BRA and takes over the position as top exporter, while the USA reduce their exports. At the same time, the top
Chapter 10. International trade of staple food: A descriptive data analysis

10.6 International trade of rice

As for maize, we observe a general trend of increasing coupling strength in the international trade of rice over the years. Fig. 10.10 illustrates the evolution of the temporal rice trade network. Still, about 4% of the total production is traded in 2013. This is less than half of the share (in total production) that is traded of maize (almost 10%).

In Asia, the population has usually a strong preference for rice as staple and regard it
10.6. International trade of rice

Figure 10.10: Evolution of the international rice trade network. Snapshots of the years: (a) 1992, (b) 2000, (c) 2008 and (d) 2013. The color of a link \((i, j)\) corresponds to the exporting nation \(i\), while the link thickness is proportional to a logarithmic transformation of the export quantity: \(\log(1 + w_{ij})\). Links with larger weights are plotted on top of the smaller ones. Square node shapes indicate that the respective country is a net importer, while circles refer to net exporters. The node size is proportional to a log transformation of their net imports or net exports.

as important for their culture. Consequently, national production is sometimes subject to export (or even import) restrictions or other protective policies. Rice is often first produced for national consumption, while overproduction is traded.

Consequently, the biggest five producers are Asian countries. Fig. 10.11 depicts the share of the 20 biggest producers in production and their demands. CHN leads clearly with
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Figure 10.11: [Rice] Share of national production in total (world wide) production for the 20 biggest producers. The segment in darker color refers to the demand of the depicted country. If it is negative the demand encompasses the sum of both segments, the lighter and the darker one, and the lighter segment refers to the countries’ production. (a) 1992. (b) 2013.

almost 40% production share in 1992, which decreases in the course of generally growing production to a bit more than 30% in 2013. Interestingly, it develops from a net exporter to a net importer. Also IDN and BRA, both big producers, are in fact net importers.

IND, as second largest producer, is also the biggest exporter. The next biggest exporter is THA and on third position is already the USA. This is apparent from Fig. 10.12, which illustrates the evolution of production, import, export, and demand distributions among the countries.

Interestingly, main importers after SAU come from Africa in 2013: BEN, SEN, and ZAF. Thus, we would expect that they are also involved in the biggest trade volumes.

Fig. 10.13 confirms this hypothesis and shows the evolution of the biggest five network weights. In contrast to Maize, the difference between the largest volumes is not that enormous. Also the biggest volume, the export from IND to SAU, stays the biggest in 2010 and 2013, but it fluctuates in between substantially. In 2008, it even vanishes. The export from the USA to MEX increases meanwhile almost consistently. Otherwise THA exports the highest volumes in 2010, but in the next three years exports from IND to the main African importers in 2013 take over quickly. Thus, we observe again a quick rise of trade relationships that have been almost negligible before in the years 2011-2013.

10.7 International trade of soybeans

Soy is traded internationally to a large extend. 34% of the total production is traded in 2013 as results of a relative steep increase of this fraction (see Fig. 10.2 (b)).
10.7. International trade of soybeans

Figure 10.12: [Rice] Share of the national variable with respect to the total world wide economic variable corresponding to rice. The most inner circle belongs to a country’s demand, the second inner circle corresponds to a country’s export quantity, the second outer circle characterizes the share of a country’s imports in comparison to the sum of all imports, and the most outer circle belongs to a country’s share in production. (a) 1992. (b) 2013.

Still, the temporal network of international trades seems to be less dense than the others, although we observe growing interconnectivity. With an increase of market participants, also a diversification of production and trade links is associated.

While in 1992 the USA produces 52% of the total production, this share decreases to 33% in 2013 and BRA produces a similar amount of soy. Nevertheless, the order of the first five main producers does not change, as can be seen in Fig. 10.15. Their demand does. Whereas CHN is a net exporter in 1992, it imports far more than it produces in 2013. Fig. 10.16 confirms this insight and shows that CHN is by far the biggest importer in 2013. More than 65% of all imports go to CHN. BRA has overtaken the USA in as main exporter in 2013. Together, they export 79% of all traded soybeans. Together with ARG, the third largest exporter, it is almost 87%.

Consequently, we expect that the biggest trade volumes belong to exports from one of the three biggest exporters to CHN. As illustrated in Fig. 10.17, these three links are, in fact, the biggest ones and this has not changed in the last three years of our observation period. Only in 2013, the export from BRA to CHN exceeds the one from the USA to CHN substantially. Both trade volumes grow much faster than the others since 1999/2000 and are of similar order of magnitude. Thus, the structural change from 2012 to 2013 is not necessarily noticeable (in contrast to our observations for maize and rice).
Figure 10.13: [Rice] Evolution of the five links with the biggest weights (i.e., trade volume) in comparison to the other link weights in the same year. A link is illustrated in the color of the corresponding exporting nation. The year of reference is (a) 2010. (b) 2013.

Still, in comparison to maize, we find high similarities in the main economic actors and in the fact that BRA seems to overtake the USA as biggest exporter and biggest trade volumes to the main importer. Additionally, maize and soy serve as substitutes of each other for feed and both grow under similar farming conditions. Consequently, the coupling between both markets is substantial.

10.8 International trade of wheat

Also in case of wheat, a substantial share of the total production is traded internationally, i.e. 16% in 2013. As we see in Fig. 10.18, again the USA, CAN, and ARG, as well as several European countries, for instance, FRA, DEU, and HUN, together with AUS are the main exporting nations. Especially since 2000, UKR and RUS add as big producers and exporters several links to the network and support the increase of interconnectivity. Fig. 10.4(d) shows that a higher fraction of links correspond to larger network weights than for the other crops. Thus, we observe a tendency towards big trade volumes.

The main producer of wheat, CHN, is not prominent in the network, because it mainly produces for its own consumption. It even is a net importer, as can be seen in Fig. 10.19. Also the second largest producer in 2013, IND, is not noticeably engaged in trade. Fig. 10.20 illustrates that most exports belong to the USA, FRA, CAN, AUS, and RUS. In general, imports are more equally shared between a higher number of countries than exports. This
tendency becomes stronger, as especially CHN and JPN reduce their share in total imports over the years. Still, they stay among the biggest importers after to BRA, IDN, and DZA. Interestingly, the largest trade volume in 2010 belongs to the export of wheat from ARG
Figure 10.15: Share of national production in total (world wide) production for the 20 biggest producers. The segment in darker color refers to the demand of the depicted country. If it is negative the demand encompasses the sum of both segments, the lighter and the darker one, and the lighter segment refers to the countries’ production. (a) 1992. (b) 2013.

Figure 10.16: Share of the national variable with respect to the total world wide economic variable corresponding to rice. The most inner circle belongs to a country’s demand, the second inner circle corresponds to a country’s export quantity, the second outer circle characterizes the share of a country’s imports in comparison to the sum of all imports, and the most outer circle belongs to a country’s share in production. (a) 1992. (b) 2013.
10.9 Discussion

We have studied the international trade relationships between 176 countries from 1992 to 2013 with respect to their yearly trade of the four major internationally traded staples: maize, rice, soy, and wheat. By associating such countries with nodes in temporal weighted networks, where each snapshot corresponds to one year, we reveal the basic dependency patterns. Our observation of increasing number of market participants and increasing coupling strength suggests a growing market integration.

With respect to systemic risk, this increasing interconnectivity might be problematic, as it can support the amplification of cascading losses that can reach a substantial fraction of the world. Thus, local crises have the potential to become global. Nevertheless, the increasing connectivity can also reduce the risk of food shortages in countries that have limited farming capacities and are prone to local shocks, for instance, extreme weather or natural

Figure 10.17: [Soy] Evolution of the five links with the biggest weights (i.e., trade volume) in comparison to the other link weights in the same year. A link is illustrated in the color of the corresponding exporting nation. The year of reference is (a) 2010. (b) 2013.

or BRA. Also the second largest trade volume shows a consistent growth trend: Still, the exports from AUS to IDN, which are of similar magnitude as the largest trade volume in 2013, are subject to strong fluctuations as well.

Accordingly, the dissimilarity of positive trade volumes changes as discussed in context of the evolution of the Gini coefficient shown in Fig. 10.3 (a).

Further empirical visualizations of degree and weight distributions of
Figure 10.18: [Wheat] Evolution of the international wheat trade network. Snapshots of the years: (a) 1992, (b) 2000, (c) 2008 and (d) 2013. The color of a link \((i,j)\) corresponds to the exporting nation \(i\), while the link thickness is proportional to a logarithmic transformation of the export quantity: \(\log(1 + w_{ij})\). Links with larger weights are plotted on top of the smaller ones. Square node shapes indicate that the respective country is a net importer, while circles refer to net exporters. The node size is proportional to a log transformation of their net imports or net exports. The twenty biggest nodes have their country code assigned. Node colors follow the scheme introduced in Fig. 10.1, which encodes regional proximity.

Disasters. For every country, already the possibility of further imports is an additional mitigation option as response to shocks. In this sense, increasing market integration also means better risk diversification.
Figure 10.19: [Wheat] Share of national production in total (world wide) production for the 20 biggest producers. The segment in darker color refers to the demand of the depicted country. If it is negative the demand encompasses the sum of both segments, the lighter and the darker one, and the lighter segment refers to the countries’ production. (a) 1992. (b) 2013.

Figure 10.20: [Wheat] Share of the national variable with respect to the total world wide economic variable corresponding to rice. The most inner circle belongs to a country’s demand, the second inner circle corresponds to a country’s export quantity, the second outer circle characterizes the share of a country’s imports in comparison to the sum of all imports, and the most outer circle belongs to a country’s share in production. (a) 1992. (b) 2013.

Still, there are substantial differences between countries in their abilities to engage in trade and to benefit from it. We observe a high heterogeneity in production, imports, and exports, among which the imports are usually best diversified between a higher number of countries. Still, also the distributions of trade volumes are highly right skewed. This
suggests a high dependence of several countries on the market and a vulnerability to the loss of a few important trade relationships.

High fluctuations in these trade relationships, as for instance in case of wheat trade, can mean a high flexibility of the market, and still pose a challenge to the market participants because of planning insecurity. This can also introduce higher uncertainty to models for this trade or cascading processes that distort such trade relationships.

It is apparent from our analysis that special care needs to be taken of adequate models for the trade of the USA, BRA, ARG, FRA, JPN, CHN, JPN, RUS, and UKR, as they account for noticeable fractions of all trades.

Consequently, local shocks concerning these big network hubs can have strong implications for the overall system. As we observe a substantial structural change from 2012 to 2013, and also changing distributions of production shares over the whole studied time horizon, we can assume that our data comprises already several responses to shocks.

Still, this data allows us not necessarily to gain insights about the causes of these responses, i.e. the type of shocks. For instance, we have observed a completely unexpected decline of maize exports from the USA to JPN in 2013. Is this decline now initiated by the USA or JPN or results from a preference of both? The total exports by the USA as well as the total imports by JPN, decrease. This is a sign that neither of them mitigates the loss substantially, although JPN increases its imports from BRA. A possible explanation would be a stressed Japanese economy and an increased demand of the USA, especially for bio
fuel production. A hint supporting this hypothesis is an increase in prices that American farmers receive for their products - also in relation to producers in other countries. Thus, the value of Maize seems to increase for the USA and might be less affordable by JPN.

By this example, we see that detailed analysis of several hypotheses and their coherence with the data is required to infer causal responses to shocks. This is only possible for a limited number of cases and cannot incorporate scenarios that do not follow the trend observed in the data before. We discuss this problem further in Chap. 12, where we take several steps towards a more realistic cascade model.

Considering the uncertainty about the decisions of all economic actors and possible shock scenarios, it is reasonable to start first with a simple cascade model that can reveal already some basic trends and economic food dependencies that go beyond the analysis of network neighbors and incorporate more topological features of the network patterns.

Such a model needs to make strong assumptions on the causal propagation of shocks. We motivate those assumptions by economic arguments and show how they lead to an OLRD type cascade model in the next chapter.
Chapter 11

Overload redistribution: A dependency analysis for international trade of staple food

Summary

We introduce an OLRD type cascade model that can be interpreted as propagation of export bans in a weighted and directed economic network. Its implications for the international trade of maize, rice, soy, and wheat are compared with respect to two different shock scenario types.

In our analysis, we investigate economic trade dependencies that go beyond the identification of direct trade relationships. In contrast to our previous studies, we encounter a temporal dimension with the yearly evolution of systemic risk. For the identification of basic trends, we complement the average fraction of failed nodes by several indicators that measure additional aspects of the cascade process. Most information is carried by the average fraction of trade that is withdrawn in response to shocks. It serves especially well in the identification of trade intermediary countries and gives thus rise to a cascade betweenness centrality.

Our simple model is limited in its ability to estimate systemic risk, when countries have the option to increase imports instead of solely reducing their exports. However, it gives valuable information about the role of countries in international trade of staple food.

Furthermore, it is our first step towards a more refined cascade model.

RB wrote this chapter specifically for this thesis. RB designed the research questions, contributed to discussions, derived the formulas, wrote the code and performed the analysis.
11.1 Introduction

In the previous chapter, we have explored several topological features of four different temporal networks, where each belongs to the yearly evolution of international trade relationships between 176 world countries with respect to one of the four major internationally traded crops: maize, rice, soy, and wheat.

There, we have restricted ourselves to the analysis of direct trade relationships by the comparison of link weight distributions and the identification of major exporters and importers.

Usually, further information is not taken into account in an economic analysis of trade dependencies. But further topological features of a network can become relevant, as the interconnectivity and market dependency grow, because the system can become more prone to cascades. Thus, local changes in another part of the world can impact a country, although there is no direct trade link.

For a detailed and realistic risk scenario analysis, we would need to understand the forces and mechanisms that lead to the formation of trades. This would require more information than is accessible or possible to include in a small computable model. For instance, multilateral and bilateral trade agreements in relation to tariffs, all kinds of direct and indirect subsidies or other economic dependencies affect the attractiveness of a trade partner. Moreover, the time of harvests and thus the relation of supply and demand in a year shape the aggregated links that we consider and might lead to surprises on the aggregated level. But also factors like cultural similarity, geographic proximity, or simply a historic path dependence and the experience of past successful trades can influence economic decisions. Those human decisions do not need to be rational or just consider less information or more information than we have available. In summary, our subject of interest is a complex system that we cannot model in detail.

Still, we capture many of the processes that have shaped it indirectly. The idea is that these processes are reflected in the network topology indirectly and small distortions in some (exogenous) variables cause a reorganization of the network that starts from the originally observed one. This reorganization we describe by a cascade process.

For instance, a country whose production has decreased or demand has increased might seek for a compensation of its deficit by increasing its imports or decreasing its exports. Thus, every otherwise established trade link in a network is questioned, and might be partially redirected to the shocked country. This redirection causes a decline of imports for other countries and creates thus another shock that might further propagate through the network and successively change the network topology of international trades. In principle, the competition with other trade links might not only have local implications, but can also impact the global market, e.g., lead to price increases, and thus introduce
11.1. Introduction

systemic feedback to a cascade model. With and without systemic feedback, we implicitly assume a memory of old trade relationships and high stability in the rules that lead to the formation of trade links. Furthermore, our modeling approach is only reasonable, if the variables that are changed in a shock scenario only cause negligible changes in the processes that we do not observe but that shape the network topology.

Consequently, smaller shocks should be better represented by our model than large shocks, as extreme scarcity of food supply probably changes substantially the behavior of all affected economic agents. Especially extreme scenarios are difficult to study, as data is hardly available for these cases. Puma et al. (2015) assume, for instance, that shocked countries discard completely their exports and reallocate oversupplies according to a GDP ranking of possible trade partners. Thus, the network topology, traditional trade relationships, or other political considerations, are almost irrelevant in their model. Meanwhile, Sartori and Schiavo (2015) discuss the weighted and unweighted degree and second order distribution of the virtual water trade network and link it to food security and cascading effects without studying actual cascades. Nevertheless, systemic risk and the result of cascade processes depend crucially on details of the microscopic topology. By incorporating such features, we thus advance our systemic risk analysis further.

In this chapter, we focus on a simple OLRD-type cascade model that describes the evolution of export restrictions of shocked countries. Export bans are not an unusual instrument to ensure the food supply of the population, especially considering the trade of rice (Giordani et al., 2014; Puma et al., 2015). Although the adjustment of imports might be an additional option, we restrict our analysis at this point to the more extreme case, where imports cannot be increased.

Our general approach to study the response of a network to small distortions as cascade could have several use cases. First, it provides a way to test the implications of hypothetical shocks on the production or demand side in several risk scenarios. Second, it can be interpreted as part of descriptive topological data analysis to judge the trade coreness or centrality of nodes in the trade network. Third, in principle the model could be employed for the prediction of future trade networks on the basis of past observed trade relationships and expected future production amounts of countries. But this would require a more complicated and detailed cascade modeling ansatz than we can provide in this thesis. Still, this might be a goal of future research.

Here, we restrict ourselves to a simple dependency analysis between countries that incorporates cascade effects and goes beyond an analysis of direct neighbors, as it incorporates the topology of the complete observed network topology.
11.2 A cascade model for cascading export restrictions

As we shift our attention to the application of international food trade, we need to extend our interpretation of a node’s state $s_i(t)$ in a cascade process and consequently reflect on our earlier measure of systemic risk.

In Part I and II, we have assumed that $s_i$ can only switch from functional ($s_i = 0$) to failed ($s_i = 1$), but not vice versa. Next, we also allow for the (partial) recovery of node by canceling exports. Such a recall of exports we associate with an overload redistribution similarly to our models in Part I of this thesis.

We thus define the threshold $\theta_i$ of a node $i$ as its original demand

$$\theta_i := \text{dem}_i(0) = \text{prod}_i(0) + \text{imp}_i(0) - \text{exp}_i(0)$$

in a non-shocked network at time $t = 0$ (which does not change in this chapter due to price changes taking the price elasticity of a country’s population into consideration). As before, the threshold stays constant the whole cascade over, which evolves according to discrete time steps $t = 0, \cdots, T$. This time scale is different from the yearly evolution of the food trade networks that is measured in $y$. A complete cascade is assumed to develop in each single year and defines in its end state at time $T$ a new trade network as result of a distortion of the originally observed one.

The load $\lambda_i(t)$ that a node carries depends on the cascade time $t$ and is defined by the deficit of food in the country in comparison to its demand. This deficit can either be caused by an initial shock to the production or demand of a country or result from an export reduction by in-neighbors, which could not be compensated fully (yet). We define $\lambda_i(t)$ as amount of staple food present in a country:

$$\lambda_i(t) = \text{prod}_i(t) - \text{exp}_i(t) + \text{imp}_i(t) - \text{shock}_i.$$ (11.1)

Here, shock$_i$ denotes the initial shock size that a country is exposed to. To emphasize that it refers to an additional exogenous variable that is to be defined, it is as separate summand in our definition of $\lambda_i(t)$, although it formally decreases the already considered production of a country or increases the demand (so technically the threshold) of a country.

If $\lambda_i(t) < \theta_i$, the state of node $i$ is defined as active (or former failed) in the next time step: $s_i(t+1) = 1$. Thus, a similar relation

$$s_i(t+1) = H(\theta_i - \lambda_i(t))$$

holds as before, where $H$ denotes the Heaviside function. However, the failure of a node
11.2. A cascade model for cascading export restrictions

does not occur when its threshold is exceeded as in our previous investigations. Instead, a node $i$ is activated (fails) when its load $\lambda_i$ becomes smaller than its threshold ($\lambda_i < \theta_i$). An additional difference to our previous assumptions is that when a node distributes load it reduces also the load $\lambda_i$ that it carries and thus supports its recovery. When a node is active at $t+1$ ($s_i(t+1) = 1$) it diminishes as many of its exports as necessary to compensate for its supply deficit, if this is feasible. Thus, it tries to call back in total $l_i(t+1) = \theta_i - \lambda_i(t+1)$. If all of its exports are not enough to recover fully from a shock, it tries to minimize its deficit and simply calls back everything that is possible. We thus have $l_i(t+1) = \min(\theta_i - \lambda_i(t+1) , \exp_i(t))$. The exports that are left are therefore:

$$\exp_i(t+1) = \exp_i(t) - l_i(t+1) = \exp_i(t) - \min(\theta_i - \lambda_i(t+1) , \exp_i(t)) .$$

In the next time step, the load that a node carries $\lambda_i(t+1)$ is updated according to Equ. (11.1). If the compensation has been successful (i.e. $\lambda_i(t+2) = \theta_i$), the considered node recovers fully and its state switches back to fully supplied (or functional) ($s_i(t+2) = 0$).

We still need to specify how the neighbors of a node are affected by the withdrawal of exports. At this point, we assume that a country has no preference for trading partners and wants to displease each of them as little as possible. Thus, the proportions of exports to each country in comparison to the others are kept constant. Let $w_{ji}(t)$ denote the quantity of exports of a given crop ($c$) from country $j$ to country $i$ in given year $y$ at time $t$ in a cascade process. Thus, we have $w_{ji}(0) \in W^{(c)}(y)$ as defined in Ch. 10. From our assumption follows that we update in each time step the exports from an active country by

$$w_{ji}(t+1) = w_{ji}(t) - \frac{w_{ji}(0)}{\exp_j(0)} l_j(t+1) s_j(t+1),$$

where we always have $\sum_{j=1}^{N} \frac{w_{ji}(0)}{\exp_j(0)} = 1$. As consequence, the imports by each country might be reduced because of export reductions by in-neighbors in the network:

$$imp_i(t+1) = imp_i(t) - \sum_{j=1}^{N} \frac{w_{ji}(0)}{\exp_j(0)} l_j(t+1) s_j(t+1).$$

Additional to the nodes’ state variables, the changing weights, exports, and imports carry interesting information about the extent of the countries’ exposure to the studied cascade processes. Because of this, we accompany our previous measure - the final cascade size,

\[ Remark: \text{We could still fit the definition of } \theta_i \text{ and } \lambda_i \text{ in our previous framework by setting } \theta_i = 0 \text{ for all nodes and } \lambda_i(t) = dem_i - pro_i(t) - imp_i(t) + exp_i(t), \text{i.e., the loss of food with respect to their demand. As this would hide the heterogeneity of demands and possible losses, we favor the definition above.} \]
which corresponds here to the fraction of countries with a supply deficit - by several other measures that are introduced next.

### 11.3 Measures of systemic risk

By turning our attention to an application, we have to broaden our understanding of systemic risk and need to be able to consider several aspects in our analysis.

The fraction $\varrho(c)$ of countries with a supply deficit of a specific crop $c$ gives information about the outreach of a cascade, but does not allow to judge the deepness of deficits that the affected countries face. For instance, a high number could pinpoint to a good diversification of a shock among many countries so that each of them faces only little deficits. On the contrary, it could also be interpreted as problematic, since a considerable number of countries is affected that cannot compensate for the loss by other means, e.g., substituting food types.

We need to incorporate further information about the affected countries to judge about the deepness of a food deficit. Because of this, we complement $\varrho(c)$ by several additional indicators.

The total unfulfilled demand is not indicative, because it coincides with the shock size that we define and it does not relate it to a country’s perception of its deficit. Because of this, we report additionally to $\varrho(c)$ the percentage at which a country’s demand is not fulfilled and weight it by their amount of population. Finally, we calculate the percentage of world population whose original demand is not met and term it $\delta$ (for deficit):

$$
\delta(c)(y) := \frac{1}{\text{popTot}(y)} \sum_{i=1}^{N} \left( \frac{\varrho_i(c) - \lambda_i(c)(T)}{\text{dem}_i(c)} \right) \text{pop}_i(y).
$$

This measure can only be a proxy for a change in actual food deficit in comparison to an observed distribution. It does neither question the observed distribution nor does it consider that the demand is not equally distributed between the inhabitants of a country or that a significant percentage of the demand might not be used for human consumption at all. In general, the change in demand of a country with relatively high demand despite a small population is weighted less than the change in demand of a country with high population and relatively low demand. More advanced measures might be worth considering in future that incorporate information about the food insecurity of countries.

Furthermore, even if a country does not face a deficit at the end of a cascade, a combined reduction of imports and exports might still have negative consequences. For instance, the value added by the country to a product may be lost. Or, the benefit from the time gap between the own export and the import may be lost. This benefit could result from profits
11.4 Shock scenarios

The modeling of realistic shock scenarios is a challenge on its own that we cannot discuss in depth in this thesis. Shocks can originate from natural hazards, weather anomalies, pests, increasing wastes, etc on the production side or from increasing demands because of additional crop usage possibilities, speculation, or simply increasing population, etc. Of course, also the transportation of goods could face shocks, but this is not considered in this thesis.

Here, we focus on cascade effects resulting from trade interdependencies between countries. Consequently, we are mainly interested in the hypothetical question how a country $i$ is affected in case of a shock originating in another country $j$. For this purpose, we study $N$ different shock scenarios in each of which a single country $j$ is shocked.

**Shock 1.** In the first scenario, each country $i$ is shocked by the same amount $s_a$. We simply choose $s_a(y) = 0.25 \cdot \text{prodTot}(c)(y)/N$, i.e., 25% of the average production by a country, because at least 5% of all shocks in our data are this severe. If the demand and the production of a country are smaller than $s_a(y)$ we assume that the shock is only of the size of the maximum of both. This ensures that we do not assume that a country’s
Figure 11.1: Visualization of aggregated data in 2013 (i.e., trades and variables for maize, rice, soy, and wheat are added). (a) Production versus demand. The size of the bubbles is proportional to the inverse average weighted out-degree of a country relative to its exports. All variables are reported as share of the maximum observed value. (b) Weighted in-degree versus weighted out-degree. The bubble size corresponds to a country’s demand.

demand can more than double in size due to a shock or more than its production is lost if we consider a shock on the production side.

In future work, we could analyze our cascade size measures in dependence of varying shock size and thus identify critical shocks at which cascades reach a global scale similarly to Ref. (Tessone et al., 2013).

In the explained shock scenarios, right skewed production and demand distributions are not problematic from a systemic perspective, since the assumed shock size does not scale accordingly. Since big producers and consumers usually have high out- or in-degrees, their shock often affects a high number of countries in the system. Thus, the total amount of food deficit that is introduced to the system by a shock is shared between many countries so that each might be affected only little.

Fig. 11.1 (a) supports this claim. The bigger the bubble corresponding to a country, the lower is the shock proportion that it inflicts on average to each of its out-neighbors. In this sense, a high amount of well connected big producers or consumers can mean lower systemic risk similarly as in the case of the CL damage diversification model that we have mainly discussed in Ch. 2-4.

However, for shocks with a spatial dimension, especially the global production needs to be geographically well diversified from a systemic perspective. A small number of big producers mean considerable systemic risk. For instance, a pest or a natural hazard in
11.4. Shock scenarios

Figure 11.2: [Average shock] (a) Average shock size over time, where shock1 stands for a scenario where each country is shocked by the same amount sa\(^{(c)}(y)\), while shock2 corresponds to shocks comprising 25% of a country’s the production or demand. (a) Maize, (b) rice, (c) soy, and (d) wheat.

The USA would threaten a substantial proportion of the global harvest (of all considered staples). Still, it would not be reasonable to move a significant fraction of the production instead to a place that is more prone to shocks just to diversify global production.

We emphasize that the type of shock that we consider also impacts our assessment of cascade risks. Because of this, we also discuss a second type of shock scenarios where the size distribution of the production and demand is of relevance.

**Shock 2.** Again, we study a cascade that originates in a single country, but this time the size of the shock \(\text{shock}_{1}^{(c)}(y)\) is proportional to the maximum of the country’s production and demand. Similar to our first shock scenario, we assume that \(\text{shock}_{1}^{(c)}(y)\) strikes 25% of a country’s production or demand. Thus, we have \(\text{shock}_{1}^{(c)}(y) = \max(\text{prod}_{1}^{(c)}(y), \text{dem}_{1}^{(c)}(y))\).
For both shock scenario types, we calculate the introduced systemic risk measures for each shock on a country and report then the mean quantity of all single country shocks.

11.5 Cascade results

The interpretation of our cascade analysis results depends crucially on a reader’s perception of international market dependence. Systemic risk studies usually emphasize the drawbacks of interconnectivity and interdependence. Certainly, collateral costs of transportation for the environment are a negative externality. The dependence on volatile markets, unreliable trade partners, or unforeseen shocks in another part of the world are examples for problems that can come along with globalization.

Nevertheless, international trade has also many advantages for the engaging countries, as already discussed earlier. In relation to food, it enables varied diets (for people who can afford it) and allow for specialization, which can lead to more efficient production. Also from a risk perspective, trade offers the opportunity to mitigate harvest losses or compensate for a scarcity of farming resources. Thus, autarky as alternative to imports might not be an option for many countries. Still, the degree of dependence on the market and on other countries can be partially regulated and is an important information for policy makers or trade participants.

Because of this, we provide first steps in the identification of trade interdependencies taking cascade effects into account, which serve as model for the response of the international trade network to shocks.

11.5.1 Relations of interdependencies

Differences in shock scenarios

Fig. 11.3 shows our results for the international trade of maize in 2010: a network of trade interdependencies. Each shock of a single country is represented by the links leaving the corresponding node, which are colored according to the shocked node. The link weight is proportional to the demand deficit at the end of the studied cascade of a node where the link points to. The demand deficit that a shocked node faces on its own is not shown.

Shock 1. In case that each node is shocked initially by the same amount, as illustrated by Fig. 11.3 (a), we observe a relatively good diversification of shocks starting in main exporters. One exception is the high exposure of KOR towards CHN and ESP is hit by cascades in many cases.

Still, in comparison with our second shock scenario where countries are shocked proportional to their production or demand, shocks are well shared among international countries.
11.5. Cascade results

Figure 11.3: [Maize] Cascade dependencies for international trade of maize in 2010. Each link corresponds to a shock scenario where the country that passes its color to the link is shocked initially. Only links with large weights are visible. The link width is proportional to the demand deficit of the country in which the link ends after $T = 5 \cdot 10^4$ cascade steps. Node colors follow the scheme introduced in Fig. 10.1, which encodes regional proximity. The bar size of each country represents its weighted degree, i.e., the sum of all in-coming and out-going edge weights in the cascade dependency network. Shock scenario in case country $i$ is shocked initially: (a) $\text{shock}_i = s_{a(m)}(2010)$, (b) $\text{shock}_i = 0.25 \max (\text{prod}_{i(m)}(2010), \text{dem}_{i(m)}(2010))$.

**Shock 2.** Fig. 11.3 (b) shows the exposures for the second scenario and portrays clearly the risks involved in spatially concentrated production. Local shocks in the USA threaten a significant share of the global maize production.

We still note that similar countries are affected in both shock scenarios. Thus, we can restrict our comparison of cascade interdependence networks for different crops to one scenario without missing general patterns.

**Trade of different staples in comparison**

Fig. 11.4 provides a collocation of our cascade results in 2013 where all countries are shocked equally.

**Wheat.** In this case, the network for wheat seems to be most balanced and diversified, as several countries face demand deficits of similar size. Still, especially African countries have to face heavy losses in case of shocks in Europe, while Asian countries depend more on America and AUS.
Maize. A similar tendency can be observed for maize. Noticeably, MEX and VEN are relatively dependent on the USA despite their own maize productions.

Rice. In case of rice trade, Asia and the USA are clearly crucial for the world supply. In particular African countries rely on Asian exports. Also KOR and JPN as Asian countries are highly dependent, mostly on CHN and the USA.
Soy. Most remarkable is the interdependence network for soy in 2013. CHN is obviously by far the most dependent country on the soy market. Shocks in the USA, CAN, ARG, BRA and URY would all be problematic for CHN. Interestingly, shocks of the other two central exporters PRY and UKR would not affect the Chinese supply.

Soy trade is special. In Ch. 10, we have seen that the international soy trade network is the least connected among all trade networks (although the highest proportion of the production is traded internationally) and in 2013 it also shows the highest heterogeneity in the positive trade volumes. Both observations suggest the natural hypothesis that more homogeneous, well connected trade networks also balance the response of the system to shocks. This means that more countries are affected by shocks and also share its burden.

Limitations of focus on trade dependencies. But these thoughts do not consider the amount of burden that affected countries can bear. Moreover, the presented figures do not show the extend to which countries absorb shocks because they have no further options to reduce their exports. Thus, a low observed connectivity or small weights can in fact imply a low cascade risk, but high systemic risk in that sense that countries are severely exposed to local shocks.

We address these issues partially in our systemic risk measures.

11.5.2 General trends in the evolution of systemic risk

Information beyond trade dependency. The discussed aspects are partially reflected in our one-dimensional indicators, i.e., the risk measures introduced in Sec. 11.3. For instance, also the demand deficit of initially shocked countries are considered in the final cascade size $\varrho$ and in the share of the global population $\delta$ that faces a demand deficit. Moreover, $\delta$ addresses partially the issue to which extend a country can absorb shocks. It measures a demand deficit relative to the original demand of country. Thus, a deficit of a country with a generally high observed demand and relatively small population, as e.g. the USA, is weighted less than a deficit of a country whose supply mainly serves for human consumption.

Information loss by summary. In the following, we calculate the introduced indicators for systemic risk as average with respect to all country shocks. Their evolution is shown in Fig. 11.5, Fig. 11.7, Fig. 11.9, and Fig. 11.11. To estimate the cascade amplification, we also depict the average initial shock sizes in Fig. 11.2.

This enables us to summarize rough trends. Still, we have discussed the limitations of this approach in Ch. 9. In particular for the second shock scenario where countries are shocked proportional to their demand or production amount, Fig. 11.3 (b) implies that we have to expect high differences with respect to whom is shocked and also whose deficit we analyze.

Therefore, we go into more detail for special cases in Sec. 11.5.3 and compare shocks of
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Figure 11.5: [Maize] Systemic risk measures as average over $N(y)$ shocks in each of which a single country $i$ (with positive demand or production) is shocked according to one of two shock scenarios: (a) shock$_i = sa^{(m)}(y)$, (b) shock$_i = 0.25 \max(\text{prod}^{(m)}(y), \text{dem}^{(m)}(y))$. All measures are reported as share of the maximal value in the considered time period 1992-2013. The rounded maximum is given in the legend.

Figure 11.6: [Maize] Share of countries’ contribution to the depicted risk measures in 2010, i.e., the amount by which a cascade affects them on average. Maximally $T = 5 \cdot 10^4$ cascade steps are considered. The inner circle corresponds to $\delta^{(m)}$, the middle circle to $\varepsilon^{(m)}$, and the outer one to $\varrho^{(m)}$. (a) shock1, (b) shock2.

Different countries and how other countries are affected. This way, we also touch on the topic of node centrality with respect to their role in the studied cascade phenomena. For instance in financial networks, this topic is of great interest as well (Battiston et al., 2012c). Interestingly, we find that the most central countries are not necessarily the big hubs or leaves that we have identified in Ch. 10.
Figure 11.7: [Rice] Systemic risk measures as average over $N(y)$ shocks in each of which a single country $i$ (with positive demand or production) is shocked according to one of two shock scenarios: (a) shock$_i = sa_i^r(y)$, (b) shock$_i = 0.25 \max \{\text{prod}_i^r(y), \text{dem}_i^r(y)\}$. All measures are reported as share of the maximal value in the considered time period 1992-2013. The rounded maximum is given in the legend. Maximally $T = 5 \cdot 10^4$ cascade steps are considered.

**Similarity between shock types.** To start with, we focus on a rough summary of trends over time. Interestingly, we observe a high similarity between our one-dimensional indicators for both two considered shock scenario types. This might be a sign that the topological features of the studied networks are of higher relevance for the cascade process than the studied shock scenarios.

**Fraction of failed nodes $\rho(c)$.** We find a clear trend in all trade networks that the fraction of nodes $\rho(c)$ with a demand deficit at the end of the studied cascades grows over the years. Only in the last years, it settles down at ca. 20% for all staples.

The first explanation that comes to mind is that more countries engage in trade over time and could thus possibly be affected by a cascade. But we have considered this possibility and have calculated $\rho(c)$ with respect to a time varying number of nodes, i.e., all actively trade engaged nodes. Still, we find the share of nodes that are affected by a cascade increasing. We mainly attribute this to the rising interconnectivity of the studied networks. It is a sign of an increasing shock diversification, as encountered shocks are distributed among a higher number of countries.

However, the deficit that these countries encounter can still increase over time, as the average shock in our studies grows as well with the increasing production.

How this shock is perceived by countries is indicated by their demand deficit in relation to their total demand.

**Fraction of demand deficient population $\delta(c)$.** The fraction of demand deficient population $\delta(c)$ seems to be relatively low in the studied shock scenarios, i.e. in the scale of...
Figure 11.8: [Rice] Share of countries’ contribution to the depicted risk measures in 2009, i.e., the amount by which a cascade affects them on average. Maximally $T = 5 \times 10^4$ cascade steps are considered. The inner circle corresponds to $\delta(r)$, the middle circle to $\varepsilon(r)$, and the outer one to $\varrho(r)$. (a) shock1, (b) shock2.

The evolution is also quite stable and fluctuates only between a small range of values - despite the increasing shock size, network interconnectivity, population and fraction of failed nodes $\varrho(c)$.

The studied changes are comparatively negligible considering the high worldwide demand for the staples. The main source for systemic risk, as we measure it, concerns the benefits resulting from intermediate trade.

**Export withdrawal $\varepsilon(c)$**. The degree of trade reorganization is primarily measured by $\varepsilon(c)$ that quantifies the average proportion of trade that is canceled in the course of a cascade.

How do we have to interpret this quantity? On the one side, a high adaptability of the network can be seen as sign of good mitigation of shocks. On the other side, a high reduction of intermediate trade also means a loss of the benefits associated with it.

We find that maximally 1% of all trades are at stake, while it can be less (for maize and soy) in case of smaller shocks when each country is shocked by the same amount. Otherwise, this amount is surprisingly similar for all studied staples. Only the precise evolution of $\varepsilon(c)$ varies, as it is partially determined by the evolution of the average shock size in relation to the amount of total trade in a given year. Usually, we find it growing. Remarkably, $\varepsilon(c)$ declines for soy, as the average shock size grows slower than the network interconnectivity.

The remaining question is whether the measured values of $\varepsilon(c)$ are problematic and if they
Figure 11.9: Systemic risk measures as average over $N(y)$ shocks in each of which a single country $i$ (with positive demand or production) is shocked according to one of two shock scenarios: (a) shock$_i = s a^{(s)}(y)$, (b) shock$_i = 0.25 \max(\text{prod}^{(s)}(y), \text{dem}^{(s)}(y))$. All measures are reported as share of the maximal value in the considered time period 1992-2013. The rounded maximum is given in the legend. Maximally $T = 5 \cdot 10^4$ cascade steps are considered.

Figure 11.10: Share of countries’ contribution to the depicted risk measures in 2010, i.e., the amount by which a cascade affects them on average. Maximally $T = 5 \cdot 10^4$ cascade steps are considered. The inner circle corresponds to $\delta^{(s)}$, the middle circle to $\varepsilon^{(s)}$, and the outer one to $\varrho^{(s)}$. (a) shock1, (b) shock2.

The cascade length $T^{(c)}$ and $N_c^{(c)}$. A substantial reorganization of trade is indicated by the high cascade length. We have calculated maximally $T^{(c)} = 5 \cdot 10^4$ cascade steps, but in
several cases the cascade would continue. Only wheat trade is reorganized relatively early, on average after ca. $T^{(w)} = 1.6 \cdot 10^4$ steps.

In case of all staples, the evolution of the two reported quantities is quite volatile. Especially in the first half of the observation period, we encounter big differences over orders of magnitudes between two consecutive observations. Also $T^{(c)}$ and $N_{tr}^{(c)}$ can differ a lot in their trend. Short cascades can still involve a high number of shock distribution events that happen at the same time.

These can only occur, if there is a substantial fraction of intermediary countries that export and import high amounts which can be withdrawn in the course of a cascade.

Thus, we ask next which countries are the main shock channels, along which cascades propagate.

**Cascade centrality.** In the remainder, we present the figures that depict our findings for each of the four studied staples. Alongside, we show the distribution of specific country contributions to our main systemic risk indicators $\varrho^{(c)}$, $\delta^{(c)}$, and $\epsilon^{(c)}$, i.e. how these countries are affected on average.

Our main focus is on $\delta^{(c)}$, as this is the risk measure carrying the most interesting information. We see that it also corresponds to the most heterogeneous distribution among countries. A high value can further be linked to the trade centrality or cascade betweenness of a node.

The other two distributions, relating to $\varrho^{(c)}$ and $\delta^{(c)}$, indicate a good diversification of the cascade outcome.

Additionally, we could also depict the average systemic risk that is caused by the shock of a country. This would identify potential cascade sources. As this question is answered to a large extend by our economic dependency analysis, we focus on the issue how countries are potentially affected by cascades and local shocks.

A more detailed discussion follows in the next section alongside the figures that we have based our trend summary on. Primarily, we highlight the special characteristics concerning each staple. The interested reader is, of course, encouraged to delve further into the results.

### 11.5.3 Staple characteristics

**International trade of maize**

Fig. 11.5 presents the evolution of the average indicators that we have introduced in Sec. 11.3 for international trade of maize in our observation period 1992-2013. Interestingly, we observe a clear growth trend in the fraction $\epsilon^{(m)}$ of trade withdrawals only in case of shock scenario 1, but not shock scenario 2. This is despite the high similarity in
the corresponding average shock evolutions as shown in Fig. 11.2 (a).

However, $\varepsilon^{(m)}$ is smaller in case of shock1, even though it increases. This can be expected, as on average, the system is also shocked less. In both scenarios, the shock size is generally increasing, although not clearly in relation to the total issued trade. For the other three staples, the average shock decreases in relation to the total trade.

A possible indicator why we observe a relatively stable $\varepsilon^{(m)}$ for shock2, where countries are shocked proportional to their production or demand, is provided in Fig. 11.6 (b). A considerable proportion of trades is withdrawn by the USA, the main producer of Maize. As its shock is immense in comparison to the others, this one shock scenario dominates our average measure $\varepsilon^{(m)}$ to a large extend. Furthermore, each shock that reaches the USA is distributed further to a high number of other countries which are also exporter. The dominant role of the USA in scenario 2 is relatively stable over time. Consequently, $\varepsilon^{(m)}$ does not respond strongly to structural network changes affecting our results for scenario 1.

In contrast, exports are withdrawn by the USA only to smaller extend than by several European countries in scenario 1 as shown in Fig. 11.6 (a). This confirms our impression of the central European role in the international trade of Maize (in addition to the USA, BRA, and ARG).

**International trade of rice**

Cascades in the international trade of rice lead to the highest maximal proportion $\varrho^{(r)}$ of nodes with demand deficit as seen in Fig. 11.7. This is a sign of relatively good diversification of shocks.

This is also apparent from Fig. 11.8, which shows that the demand deficit $\delta^{(r)}$ is relatively equal distributed among the countries in a similar amount shocks, as shown by their similar contribution to $\varrho^{(r)}$.

The contribution to the evolution of the export withdrawal $\varepsilon^{(r)}$ depends again on the considered shock scenarios. In comparison to maize, there are more countries with a similar contribution also in case of shock2, where countries are shocked proportional to their production or demand. Interestingly, in scenario 2, the largest share does not correspond to the main producers, but to one of the main exporter nations: THA. In scenario 2 instead, the biggest share does not belong to THA but to the USA.

**International trade of soy**

Soy is special, since we observe a decreasing trend for the share of export withdrawal $\varepsilon^{(s)}$ for both shock scenarios as depicted by Fig. 11.9.
Figure 11.11: [Wheat] Systemic risk measures as average over $N(y)$ shocks in each of which a single country $i$ (with positive demand or production) is shocked according to one of two shock scenarios: (a) shock$_i = sa^{(w)}(y)$, (b) shock$_i = 0.25 \max(\text{prod}^{(w)}(y), \text{dem}^{(w)}(y))$. All measures are reported as share of the maximal value in the considered time period 1992-2013. The rounded maximum is given in the legend. Maximally $T = 5 \cdot 10^4$ cascade steps are considered.

We recall that soy is the staple of which the highest proportion is internationally traded - but only by a relatively small number of countries. Moreover, we have noticed an increasing heterogeneity of link weights.

Both observations are a sign that less intermediaries are present in comparison to the other trade networks. Thus, there is less potential of trade that can be reorganized in the course of a cascade. And this tendency increases over time.

Fig. 11.8 (b) shows clearly that the production and exports can be attributed to three countries: USA, BRA, and the USA. Interestingly, the NLD, PRY, and CAN add to the picture as further exporters, when shocks are not dominated by shocks of big producers (see Fig. 11.8 (a)).

**International trade of wheat**

We have argued before that the international trade of wheat is most balanced among all considered trade networks. Fig. 11.12 supports this impression, as it shows that the highest amount of countries usually face a demand deficit in the end of a cascade.

However, primarily African and Asian people are affected, as shown by the distribution corresponding to $\delta^{(w)}$.

We cannot completely exclude the possibility that cascades for the other crops result in a similar outcome, once they are fully converged. Nevertheless, we doubt this, as our findings reflect how the countries are engaged in trade and to which extend they rely on
11.6 Discussion

Figure 11.12: [Wheat] Share of countries’ contribution to the depicted risk measures in 2012, i.e., the amount by which a cascade affects them on average. Maximally $T = 5 \cdot 10^4$ cascade steps are considered. The inner circle corresponds to $\delta^{(w)}$, the middle circle to $\varepsilon^{(w)}$, and the outer one to $\varrho^{(w)}$. (a) shock1, (b) shock2.

imports. As wheat cascades are the shortest among all, we can ensure their convergence (see Fig. 11.11).

In contrast to soy, wheat is imported and exported again over several steps, especially in Europe, while the total amount of potentially canceled trade $\varepsilon^{(w)}$ increases over time. The latter is shared between a higher number of primarily European intermediaries, as apparent from Fig. 11.12. Remarkably, this picture is not significantly different for the two considered shocks. When the shock is proportional to the production or demand of a shocked country, the USA, AUS, RUS, and CAN simply add to the set of trade adjusters because of their considerable production.

11.6 Discussion

In summary, we have transferred our overload redistribution model with load shedding to directed weighted networks. We have further modified it to describe cascading export restrictions in international trade networks. Specifically, we have analyzed its implications for the trade of maize, rice, soy, and wheat in comparison.

Our modifications are designed in such a way that we can calculate the demand deficit that countries experience because of local shocks, potentially in other parts of the world. To judge the deepness of a deficit, we would need information about a country’s buffer, i.e. its stocks. These would be the natural choice of a node’s threshold, while its demand deficit
would correspond to the load that it carries. In lack of data about stocks, we assume that they are non-existent. However, they could be easily incorporated in our model.

To capture additional aspects of the trade reorganization in response to shocks, we have developed several indicators of systemic risk that complement the average fraction of failed nodes, which we have focused on in the previous chapters.

We find especially the average export withdrawal a relevant measure for the degree of trade reorganization in response to shocks. Furthermore, its definition comprises interesting information about the cascade betweenness centrality of nodes in a network. With respect to international trade, the most central nodes can be associated with trade intermediate countries.

In general, we do not observe a prominent structural change in 2012/2013 in the studied risk measures, although we have found the largest weights reorganizing in the previous chapter. We argue that the studied shocks are not large enough to show a response. They are much smaller than the trades along the biggest links, which are still enormous in comparison to other trades.

However, the increasing network interconnectivity in 1992-2013 leads to an increasing number of nodes that are finally affected by cascading export restrictions. This does not imply that the fraction of people that face a demand deficit grows as well. In contrast, this stays relatively stable over time. Shocks are relatively well diversified, i.e. distributed in similar shares between a considerable number of countries, which is highest for wheat trade.

This shock distribution process develops over a very high number of cascade time steps and results in trade withdrawals of different size relative to the total amount of trade. Only for soy, such trade distortions are decreasing.

This observation can be attributed to the fact that soy trade is dominated by the exports of only three main producers and the share of intermediate trade shrinks. This is especially problematic in case of geographically concentrated shocks that hit the main producers. Similarly, the production of maize is not well diversified.

This limits also the possibilities to redirect imports to other producers in case of a shock. However, this circumstance is not considered by the discussed model, as countries can mitigate shocks only by export reduction. Imports cannot become increased or redirected, as it would usually be the case. In this sense, we study a rather extreme shock scenario, in which countries are not willing to export more of a scarce resource.

The advantage of our model is that the distortions preserve the observed network structure as far as possible. In consequence, many of the hidden factors that have shaped this structure are still considered. In this sense, we do not study such an extreme risk scenario that the mechanisms leading to the setup of trades are completely overruled. As we are
particularlly interested in severe system responses to small shocks, we argue that it is a reasonable assumption that many of the trade formation mechanisms are still in place.

Consequently, the next step towards a more realistic cascade model has to include active decisions of shocked countries concerning their imports, as this is a realistic alternative to export adjustments in case of small shocks. Moreover, we have to add responses on information that is available about the system state on the macro level. The main example for such an information is the average price of a good. Also the total production amount might be available (or estimates thereof). A further reorganization of the network structure might be in order in a next step. Countries value certain trades more than others and their preferences might also be reflected in their decisions which trades they increase or decrease. We thus have to relax our assumption that trade partners are shocked proportional to the observed trade weight. In the next chapter, we give an outlook about possible extensions in line with these considerations.

Although the presented scenarios are not realistic in several aspects, we still find our model interesting because of the information that it reveals about the trade topology. As it respects an observed network structure, it serves very well for the identification of trade intermediary countries.
Chapter 12

Outlook: Towards a cascade model tailored to the international trade of maize

Summary

We address several unrealistic assumptions in our previous model for the trade reorganization of the international trade of staple foods as response to small shocks. Our outline how these could be resolved includes preliminary model investigations for the trade of maize.

The first step in a model improvement has to incorporate import increases as possible strategy for countries to mitigate shocks. This requires careful modeling of the countries’ demands and export decisions under changed economic conditions. Furthermore, their preferences concerning their trade partners have to be determined. Moreover their changing preferences with respect to different staples have to be considered. This would finally introduce a multiplex view.

The presented models lack predictive power yet and pose practical challenges. We discuss possible answers to these challenges and suggest solutions.

RB wrote this chapter specifically for this thesis. RB designed the research questions, contributed to discussions, decided about the modeling, and performed the analysis.
12.1 Introduction

In the previous section, we have studied a simple model for cascading export restrictions and its implications for the international trade of maize, rice, soy, and wheat. This model comprises several assumptions that are unrealistic for our modeling purpose.

**Modeling purpose.** We ask how the international trade network reorganizes in response to rather small initial shocks on the production or demand side, i.e. when countries face bad harvests or increase their demand. This reorganization is modeled by an evolving cascade process and thus results from local distortions of an observed trade structure for which we create an alternative. In future, the goal is to start from networks observed in previous years and predict trades in a given year \( y \) by cascades running on the observed trade network in \( y - 1 \).

As reasoned in the introduction of Part III, our idea relies strongly on the assumption that the network that we start from contains valuable information about the mechanisms that distort it. We have to assume that these mechanisms are relatively stable. However, in an extreme shock scenario, countries would completely reconsider their behaviour and seek for possibilities to attain scarce resources. We could only speculate about their strategies. Consequently, we focus on implications of small shocks on international trade of staples that can lead already to considerable distortions of trade relationships.

**Critical assumptions.** The previously considered export restrictions alone signify already rather scarce resources, if they cannot be accompanied by import increases. As most critical drawbacks of our cascade model, we regard that countries (1) are very limited in their response strategies to a shock and (2) do not consider prices or other information about the system state in their decisions.

(1) comprises that neither imports can be increased nor the countries’ trade preferences in a changed economic environment are considered. We have assumed that exports are withdrawn proportional to the observed export volumes. However, countries might value their trade relationships differently based on their past experiences. For instance, trading inhabitants of countries might not like to stress their long established trade relationships with partners in countries they export to. Instead, they might be willing to completely cancel other links that they consider as less relevant. Alternatively, they might seek for different sources and increase their imports. Even though these might be more expensive, if countries regard a shock as exceptional, they might still be willing to pay extra to preserve valuable trade relationships.

(2) Prices clearly influence the decisions of economic actors. Exporters and importers might ponder best alternative deals. Consumers adjust their demand accordingly. Food is relatively price inelastic in comparison to other goods, because people need it in any case. Yet, consumers have the option to substitute one staple by another. On the long...
run, also producers can decide to farm more profitable crops. Consequently, the studied international trade networks of maize, rice, soy, and wheat are highly coupled. We have neglected this coupling so far, but prepared its consideration by our joined study of the four major internationally traded staples.

**Preliminary analysis.** Because of time constraints, we cannot include the coupling in our models. In the following, we further have to restrict ourselves to present preliminary results for the trade of maize, as our improvements require careful and time-consuming modeling decisions that are specifically adjusted to a given staple.

Also the presented results reflect that our work is in progress and only serve the outline of the next steps that need to be considered in model improvements. All of our presented parameter estimations show signs of over-fitting. More care of the precise model selections needs to be taken.

We still employ the models to study a scenario that is coherent with our trade observations. Moreover, we discuss the drawbacks of our results and further steps to address them. First, we focus on including the possibility of import increases. This incorporates a model for the average world price. There, we employ partially a multiplex perspective by regarding the production amounts of all four previously considered staples. In a next step, we ask additionally for the countries' preferences in their trade relationships and develop a preliminary for the volumes traded by countries in a given year.

### 12.2 Import increases as shock mitigation

Regarding import increases in our model poses a considerable additional challenge. While export restrictions involve the decision of a single country, import increases require the willingness of another country to export more or redirect its exports in parts. Furthermore, a country needs to decide about the amount that it is willing to import rather than reducing its imports.

Thus, we consider a much higher distortion of the considered trade networks than in our previous model. To stay as close as possible to the observed structures, we assume in a first approach that countries cannot redirect existing exports. However, they can adapt their demands to a changing economic situation. For instance, it can be more profitable for producers in a country to export rather than to serve the local market, although their product would be bought in principle, but at a lower price. Consequently, countries can decide to increase or decrease their exports. Or, they relinquish to compensate for a shock. New links cannot be created, only volumes along existing trades can be increased or decreased.

Therefore, we have to develop a model for the countries demands that can change in
Figure 12.1: [Maize] Average world price of one tonne of maize. Black circles and the confidence intervals correspond to our linear model, while red diamonds depict the observed data.

response to production or other countries’ demand shocks. Additionally, their preferences for export in relation to imports need to be determined. Both model approaches require information about the average world price of maize. Thus, we start with our estimation of the price.

12.2.1 World prices

Prices change in the course of a year. These changes can be attributed to seasonal fluctuations in production and demand, but also to different corp qualities that are traded. Each executed trade is furthermore a result of alternative comparisons or bargaining. Consequently, trades are issued at different prices and in different times of a year. We only have aggregated information about the average prices along trade volumes. On this basis, we further aggregate this information to calculate the average price of a tonne of maize in a given year.

This price is a construct and can only be an indicator of an economic situation. So are countries not deciding about imports and exports but trading agents in these countries. Still, these aggregates can summarize trends in the decisions of stakeholders and describe the evolution of a system on the macro level.

We have therefore compared several linear (and multiplicative) models to best represent the evolution of the average world price $p(y)$. Our best result from ordinary least square
regression so far is of the form

\[ p(y) = \sum \delta_q x_q + N, \]

where \( x_q \) are predictors and \( N \) denotes homoscedastic, unbiased noise. Among other variables, we have tested for the explanatory power of several variables including the total production of maize, rice, soy, and wheat and the productions per head, the total population of all regarded countries, the world wide gross domestic product in total and per head, the production by the main producers, e.g. the USA, the number of producing countries the standard deviation of production among countries, the total amount of maize that is used for feed and the amount that is used for other purposes than food, feed, seeds or that are further processed or wasted. Among these variables we have selected the best model employing joined backward and forward selection optimizing Akaike’s information criterion.\(^1\)

Our results are economically reasonable and lead to remarkably good \( R^2 \) and adjusted \( R^2 \) values, as presented in Table 12.1. Fig. 12.1 compares our fit with the observed data. However, we have to acknowledge the small amount of data (22 points) that we have and the relatively high number of parameters that we estimate.

The explanatory power of our model can be questioned. We still regard our model as reasonable input in a risk scenario analysis.

### 12.2.2 Demands and exports

#### Demands

Populations may not only have different consumption preferences, they also adjust their demand according to varying criteria. Obviously, producing and non-producing countries are distinct. For the former, information about their production (e.g. production and seed amounts, also of previous years, their yield, the are harvested, etc.) usually influence their demand considerably. This information is irrelevant for countries that do not produce, but their supply can be highly influenced by the production and demand of the main producers. Importers rely to a large extent on the total global production and might base their import decisions also on their GDP or GDP per capita in relation to other GDP, i.e. on the information how affordable the imports are. Also, the usage type of demanded goods can be reflected in their overall demand. Naturally, the population amount and the share of the total population are relevant indicators, if the maize is consumed to a large extend by humans. Of course, also prices are considered, i.e. in our models the average

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\(^1\)We have used the function \textit{step} for the joined backward and forward selection.
Chapter 12. Outlook: Towards a cascade model tailored to the international trade of maize

**Dependent variable:**
Average world price of maize

<table>
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<th>Predictor</th>
<th>Coefficient (SE)</th>
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<td>prodTotR</td>
<td>−0.521*** (0.115)</td>
<td>prodTotS</td>
<td>−1.510*** (0.169)</td>
<td>popTot</td>
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<tr>
<td>cprodTotM</td>
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<td>prodSd</td>
<td>−1.537*** (0.245)</td>
<td>feedTot</td>
<td>0.387*** (0.056)</td>
<td>prodTotM popTot</td>
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Observations 22
R² 0.996
Adjusted R² 0.992
Residual Std. Error 0.029 (df = 10)
F Statistic 239.307*** (df = 11; 10)

*Note:* *p<0.1; **p<0.05; ***p<0.01

### Table 12.1: Regression results. Linear model for the average world price of one tonne of maize: Summary statistics for the ordinary least squares fit. The data variables have been scaled to the interval [0,1]. Predictor variables are: total production of maize, rice, and soy. (The wheat production amount has no significant predictive power), total population, total GDP, total GDP per capita, production amount of maize per head, standard deviation of maize productions between all considered countries, total amount of maize that is used for feed, two indicated interaction terms, and an intercept.

To acknowledge the differences between countries, we have fitted seven different linear models to their demand and have chosen the best representation. However, these models clearly over-fit, as we have often employed a high amount of parameters. Consequently, a report of our estimation would be too space consuming and not carry adequate information. Anyway, we have to revise the models and look for patterns that are common to a high number of countries to estimate a joined model. At the moment we have fitted our models separately to each countries demand to obtain good fits.

The predictions of these models are not reliable. We still test their implications, but restrain from premature conclusions.

To some extend, it is also questionable how good our model results can become. We can barely identify changing preferences of countries. For instance, the USA has increased its demand significantly in 2013, reduced its exports and, at the same time, increased its imports. When we decrease the production of the USA in this year as part of a risk scenario, we receive as result a demand reduction that is even higher than the considered shock.

This unrealistic result can be explained by the correctly reflected observation that the US American demand has been much lower in times, when it had produced comparable amounts with what we assume in our shock scenario. Still, the initially high demand in 2013 is barely reflected. To address this issue, we could consider estimations procedures...
where we weight present observations and those close in time higher than observations that are more distant in time.

Furthermore, we have to regard joined estimations of the demands, where we respect economic constraints. At the moment, the sum of all demands does not add up the the total production. This is, of course, impossible. We could overcome this problem by the formulation of a linear or quadratic program that minimizes the estimation error, while fulfilling linear constraints. As estimation error we could, for instance, choose the residual sum of squares so that we employ the methodology of quadratic programming.

**Export decisions**

In our estimation of a country’s exports we face the same challenges as in the estimation of their demands.

The only difference is that we can additional use a country’s demand as predictor, while we have excluded present exports from our demand models. We thus assume that countries first decide about how much they want to consume and adjust then their exports and imports accordingly.

Furthermore, we estimate the exports of countries independently. But in a more realistic model, we would have implement the constraint that a country’s exports cannot exceed the sum of its imports and production.

### 12.2.3 Cascade dynamics

On the basis of a model for the world price, the countries’ demands and exports, we have implemented the following cascade dynamics. As outlined in the introduction, we assume minimal distortions of the observed network structure, as this is the most realistic input that we have about trade relationships in a given year. We further assume that countries have a preference for increasing imports rather than decreasing exports to preserve good trade relationships.

We consider, as previously, two different shock scenarios that are explained in detail in Sec. 11.4. Thus we start from a shock of a single country and recalculate thus its production as well as the total world wide production. Next, we update the world price accordingly and estimate afterwards the new demands of countries that can change according to the new situation. Next, their export preferences are determined.

On this basis, a cascade is employed to decide iteratively about the new international trade structure. A shocked country tries to first increase its imports by asking whether the nations they import from have higher export preferences than they have currently realized. Trades are then formed proportional to the trade volumes in the previous time
step to match the joined preferences of the exporting countries as well as importing shocked ones. If shocks cannot be mitigated fully this way, countries try to reduce as much of their exports as necessary, again proportional to the export volumes of the previous cascade time step.

Export and import preferences are met until no further adjustments are possible in agreement of exporter and importer.

12.2.4 Preliminary cascade results

We study the same two shock scenarios as in Ch. 11 for our model of cascading export restrictions. Fig. 12.2 shows the corresponding dependency networks for 2010. This allows us to compare our results with their equivalent in the cascading export restrictions model depicted in Fig. 11.3. We calculate the demand deficit with respect to the observed values and not the ones adapted to the changed economic situation, as we want to quantify the change introduced by a shock.

Remarkably, we find the USA highly exposed to shocks of other countries. This is explained by our unrealistic model of the US American demand, which is very sensitive to changes in the total production of maize.

Interestingly, the difference between the two shock types is smaller than in our previous study. Correspondingly, also the topological change with respect to the original distorted network structure is similar. Fig. 12.3 (a) depicts the evolution of this topological change, which we quantify by summing the absolute values of all trade differences in all shocks scenarios of single countries in a given year. We present the change as fraction of the total observed trades in a given year. As we would expect from our previous studies, a higher distortion is still introduced by shocks that are proportional to the production or demand of a country.

However, these results are barely meaningful, as we observe additional to demand deficits also excesses of demands. This phenomenon is even more pronounced, when we compare the equipment with maize of countries with their adjusted demand that they could reduce in response to a lower world wide production. According to our model, countries facing deficits and surplus cannot trade with each other, if they do not trade according to our observed data. This is, of course, an unrealistic risk scenario.

The inadequacy of our findings could in principle originate from a false assumption of trade preferences. To keep the network structure as close to our observation as possible, we have matched trades according to a network that belongs to a different economic situation with higher overall productions. Yet, trade relationships between countries might very well depend on the experienced shocks.

To test this hypothesis, we complement our analysis by a model of trade preferences of
12.3 Trade volumes reflecting trade preferences

We add to our previous models an adjustment of trade preferences that can respond to changes of the economic situation. For the inference of the cascade evolution rules, it would be sufficient to write the trade volume $w_{ij}$ between two countries $i$ and $j$ as function of their production, their imports, and exports.

The standard model class in economics that answers this purpose comprises gravitational models for international trade (Tinbergen, 1965).

Figure 12.2: Cascade dependencies for the international trade of maize in 2010 corresponding to a model with observed country preferences. Each link corresponds to a shock scenario where the country that passes its color to the link is shocked initially. Only links with large weights are visible. The link width is proportional to the demand deficit of the country in which the link ends after $T = 100$ cascade steps. Node colors follow the scheme introduced in Fig. 10.1, which encodes regional proximity. The bar size of each country represents its weighted degree, i.e. the sum of all in-coming and out-going edge weights in the cascade dependency network. Shock scenario in case country $i$ is shocked initially: (a) shock$_i = sa^{(m)}(2010)$, (b) shock$_i = 0.25 \max \left( \text{prod}^{(m)}(2010), \text{dem}^{(m)}(2010) \right)$.

countries that are responsive to shocks, i.e. we develop a model for the trade volumes $w_{ij}(y)$.
12.3.1 Gravitation models

In a gravitation model, the basic assumption is that nodes have properties that make them attractive trade partners similar to masses in the model for gravitation in physics. This can be the production or export amount of the exporter, the import amount of an importer, their GDP, population sizes etc. Other effects act like the distance in the model for gravitation and reduce their attraction and thus the trade between them.

In such a model, the non-zero weights $w_{ij}$ are a function of possibly time dependent variables $x_q$, node variables $n_i^{(l)}$, $n_j^{(m)}$ and distance variables $d_{ij}^{(k)}$ concerning the link. Their relation is of the functional form:

$$w_{ij} = \prod_q x_q^{\delta_q} \prod_l (n_i^{(l)})^{\alpha_l} \prod_m (n_j^{(m)})^{\beta_m} \prod_k (d_{ij}^{(k)})^{\gamma_k} N_{ij}.$$  

The parameters $\alpha_l, \beta_m, \gamma_k, \delta_q$ need to be determined. This can be achieved by taking the logarithm of the equation above so that it becomes linear in the parameters. The assumption of linear additive noise enables then the fit of generalized linear models. Special care needs to be taken of heteroscedasticity (Silva and Tenreyro, 2006). As possible node variables we test the imports and exports of the countries, their GDPs, population sizes, production amounts, and average export and import prices, and the average world price. As distance variables we have the average prices of the volume and geographical distance. Additionally, we allow the year of a trade as predictor.

But this model does not fit the data well. Also other functional forms did not lead to better results.

This can have several reasons: One is a lack of sufficient data to judge the generalized distance between countries. Also missing data about possibly important predictors like fuel prices, storage volumes (for each country), bio-fuel production, trade agreements, subsidies, value chain information, different types of maize, and order of trade execution etc. might distort the picture.

However, trade of food is also known to follow different rules than the trade of other goods. In consequence, the model approach might not respect the dependencies between different trades adequately. The complexity of the trade as well as the dependencies between trades turns the inference of the trade volumes to a hard problem of many dependent variables with a comparable low number of observations.

Yet for our purpose, it is not necessary to understand the full formation of trades. We only need to model distortions of the system due to changes in production or demand. This means we can use all network information of previous years as predictors.
12.3.2 Modeling distortions

Our second approach fits our observations considerably better. Instead of modeling the volumes \( w_{ij}(t) \) in year \( t \) by a multiplicative model, we obtain good results by

\[
\begin{align*}
    w_{ij}(y) &= \sum_q \delta_q x_q + \sum_l \alpha_l n_{i}^{(l)} + \sum_m \beta_m n_{j}^{(m)} + \sum_k \gamma_k d_{ij}^{(k)} \\
    &\quad + \varepsilon_1 w_{ij}(y-1) + \varepsilon_2 w_{ij}(y-2) + N_{ij}.
\end{align*}
\]

This time, we include also variables from the previous two time steps as predictors. The node variables are, for instance, the GDP, cGDP (GDP per head), production and production per head, population size, demands of countries, and the average world price at \( y \), \( y-1 \), and \( y-2 \). Also variables describing the overall system state are considered: the total maize production, total GDP, their equivalents per head, the total population, average prices, etc.

Most importantly, we further regard indicators of the importance of the link for the exporter or importer. For instance, the importance of a link for an exporter \( i \) with respect to a node variable \( n^{(m)} \) is given by a comparison of \( n^{(m)} \) among the nodes that it exports to. Let \( F_{i,m} \) denote the empirical cumulative distribution function of \( n^{(m)} \) of the nodes that \( i \) exports to. The importance of the export from \( i \) to \( j \) from the perspective of the exporter \( i \) is then given by \( F_{i,m}(n_{j}^{(m)}) \). Most of the importance variables are significant and improve the fit. For links that have not been existent in the previous times, we fit a model of similar form, but omit all variables corresponding to the years \( y-1 \) or \( y \).

We fit the model independently on six different sets of links that we have identified by k-means clustering of on the link weight evolutions. This time, we achieve 0.983 as multiple \( R^2 \) and 0.982 as adjusted \( R^2 \) on 11044 degrees of freedom. When we cannot consider information on the previous years we obtain 0.79 as multiple \( R^2 \) and 0.778 as adjusted \( R^2 \) on 3931 degrees of freedom. Again, we do not present our fit results in more detail because they would be space-consuming and have to be revised again.

The resulting residuals of our models can be of considerable size. This could also be attributed to the stochastic nature of trade volumes that cannot be deterministically determined without considering further effects that we do not have information about.

Our model has to be questioned for other reasons. When exports of the exporting country corresponding to a trade link are set to zero, our model can predict negative trade weights. We have also tried to determine similar models for \( w_{ij}(y)/\exp_i(y) \) to overcome this problem, but our results so far have not been convincing. A more realistic model could be obtained again by the formulation of a linear or quadratic program with constraints that respect possible trade flows. Thus respective link weights need to add up to the total imports and exports of countries. Furthermore, zero exports and imports have to correspond to
non-existent trade volumes. This makes a multiplicative model a more realistic ansatz. We still employ the model that we have introduced to obtain preliminary results. However, we assume that the trade preferences of countries are only proportional to predicted reasonable values. To be precise, we assume

\[
ep_{ij}(t) = \max \left( \hat{w}_{ij}(y), 0 \right) \exp_i(y, t) \quad \text{and} \quad ip_{ij}(t) = \max \left( \hat{w}_{ij}(y), 0 \right) \imp_j(y, t)
\]

as value a link \((i, j)\) for an exporter, denoted by \(ep_{ij}\), of for an importer, denoted by \(ip_{ij}\). The dependence on \(y\) and \(t\) shall indicate that the variables correspond to variables that can change according to our trade reorganization model. \(\hat{w}_{ij}(y)\) corresponds to the weights that we have obtained by our introduced model that acknowledges a changed economic situation because of a shock. \(\exp_i(y, 0)\) and \(\imp_j(y, 0)\) are defined by respective sums of these weights. In later cascade times \(t\), \(\exp_i(y, t)\) and \(\imp_j(y, t)\) correspond only to sums of weights which match partially the trade preferences of the involved trade partners.

Because of the drawbacks that we have discussed, the weights cannot become realized completely. Neither can the exports and demands that we have estimated independently from the weights.

We therefore interpret the inappropriate model as preferred trade outcomes of countries, if these are realizable. A simple matching of these preferences is discussed next.

**Cascade dynamics**

There are several meaningful ways to define a cascade process based on given demand, export, and weight informations. Again, we could start from an observed trade structure and distort it according to preferences of shocked countries and possible trade counterparts. As neither our demand and export preferences nor our trade volume preferences change in the course of a cascade, we have decided to not mix structures and preferences. We therefore employ a simple matching of new preferences similar to our previous model.

For each link weight, trades are established by the maximal amount that both countries agree on according to their export and import preferences. In a next step, they revise their valuation of each trade considering only remaining possible trade matches of the previous time step. This ensures that the order of country consideration in one time step cannot influence our results. Successively, we built a trade network respecting the estimated trade preferences as far as reasonable.

Our model cannot really be interpreted as distortion of observed structures to construct an alternative trade network. This could instead be realized by a different reading of our model fits. Here, we assume that the demand of a country expresses its desire of a good considering its price etc. Furthermore, we could see it as quantity that changes in the
12.3. Trade volumes reflecting trade preferences

Figure 12.3: [Maize] Evolution of the topological change with respect to two shock types: Black circles correspond to shock scenario 1, while red diamonds belong to shock scenario 2. The observed network structure and the structure after a shock are compared. The absolute value of all trade differences for all shocks of single countries is added and shown as fraction of the total observed trade. (a) Model with observed trade preferences, (b) model with adjusted trade preferences.

course of a cascade, if we interpret it as the actual amount of maize a country has access to. Such a cascade model is left for more reliable estimations of the necessary ingredients. Here, we test first the direct implications of our model fits giving all preferences equal weight in the formation of a trade network.

12.3.3 Preliminary results

As for the previous model, we present our cascade results corresponding to the two introduced shock scenario types. Fig. 12.3 (b) shows that our model results have nothing to do anymore with observed network structures. The change in all trades is at least 100 times bigger than all observed trades.

The resulting network of trade dependencies is densely connected, and its structure not informative at all. Because of this, we omit to show it.

Furthermore, our cascades result again in a situation where countries with a demand deficit and a demand excess coexist without the option to trade, because a trade would not match their preferences. We attribute this to the model inadequacy that demands, export preferences, and weights are not modeled jointly. Reasonable models have to ensure that a final matching of trades is possible. In future work, this can be realized by the implementation of adequate linear constraints.
12.4 Summary and outline of possible model improvements

In summary, the goal is to estimate the response of a food trade network to a shock. This shock can originate from a production decrease or demand increase (e.g. caused by population growth) in specific countries of the world. Its size and location is an assumption in a stress test scenario. Thus, it is an exogenous variable in our model. The trade volumes between each pair of countries are our final response variables.

Besides limited data quality and missing information about possible predictors, the main statistical challenge is the high number of dependent variables given a relatively small number of observations.

We address this challenge by modeling distortions by a simplification of the goal. Instead of asking: “How is the trade network formed each year and what determines its evolution?”, we ask the simpler question: “How does it respond to small changes in the production and demand of specific countries?”. Thus, we only need to distort the original network that is given by data. This network has been formed by complicated but realistic processes beyond our control. But we capture them indirectly and preserve the original dependence between the variables to a large extend.

The predictive quality of this approach is limited to relatively small shock sizes, small distortions and small time windows. For bigger shocks, the observed processes might change their nature and, on the long run, other variables beside production and demand might change and influence the evolution of the trade network. Only a deeper process understanding could solve this issue. But, with the data at hand, this is not feasible on the level of trade volumes between countries.

These considerations form the maxim of our modeling approach: We try to introduce as little distortion as possible to observed structures. Consequently, we model distortions as a cascade process that introduces changes to a network by relatively small successive local adjustments.

Our previously analyzed model of cascading export restrictions is limited in its scope, as it does not consider the following relevant aspects: (1) Countries cannot only reduce exports, they can also increase their imports. (2) Prices or other global variables influence trade decisions and can even introduce systemic feedback. (3) Countries value their trade relationships differently under changing economic conditions. (4) The trade networks of the different considered staples are highly coupled. This calls for multiplex perspective. Furthermore, it does not acknowledge the high uncertainty in a country’s trade decisions. Many factors can determine different crises response, for instance, human decisions. Most of these factors cannot be modeled deterministically.
We have presented preliminary approaches to address issues (1)-(3) for the international trade of maize.

As we have shown, our models are still inadequate, as they do not respect natural constraints that trade flows have to fulfill. As these constraints are linear, this issue can be solved by the formulation of a quadratic program that serves the joined estimation of parameters in models for changing demands, exports and trade volumes. To employ a multiplex view, the quadratic program could be complemented to consider further variables corresponding to other staples.

On the basis of the obtained models, a cascade process can be defined that successively changes the exports and imports of nations. These updated variables could again serve as input in a new estimation of affected weights.

Further possible steps in a thorough systemic risk analysis of international staple food trade are discussed in the next chapter, when we outline possible future research directions.

In summary, the goal is to estimate the response of the network representing the international trade of maize to a shock. This shock can originate from a production decrease or demand increase (e.g. caused by population growth) in specific countries of the world. Its size and location is an assumption in a stress test scenario. Thus, it is an exogenous variable in our model. The trade volumes between each pair of countries are our response variables.

Besides limited data quality and missing information about possible predictors, the main statistical challenge is the high number of dependent variables given a relatively small number of observations. The data quality issue is addressed in the end by comparing different reasonable scenarios to get an idea how sensitive our approach is to data distortions and our model assumptions.

The second statistical challenge is approached by a simplification of the goal. Instead of asking: "How is the trade network formed each year and what determines its evolution?", we ask the simpler question: "How does it respond to small changes in the production and demand of specific countries?". Thus, we only need to distort the original network that is given by data. This network has been formed by complicated but realistic processes beyond our control. But we capture them indirectly and preserve the original dependence between the variables to a large extend.

The predictive quality of this approach is certainly limited to relatively small shock sizes, small distortions and small time windows. For bigger shocks, the observed processes might change their nature and, on the long run, other variables beside production and demand might change and influence the evolution of the trade network. Only a deeper process understanding could solve this issue. But, with the data at hand, this is not feasible on the level of trade volumes between countries. Still, our general approach allows a trade dependence analysis that incorporates cascade effects.
Chapter 13

Summary and conclusions

13.1 Contributions

In this thesis, we have deepened the understanding of cascade processes following a generic modeling approach. This way, we have investigated one of the main sources of systemic risk and identified ample factors that constitute it.

We understand systemic risk as emergent property of a system that is comprised of many interacting agents, which are modeled by nodes in a network. Their interaction is characterized by the local load distribution of a failing node and lets us distinguish three different cascade model classes: the Constant Load (CL), the Load Redistribution (LRD), and the Overload Redistribution (OLRD) class.

They share some common overarching principles, which are of focus in Part I of this thesis. Also their differences, especially in relation to the role of hubs and leaves in the course of a cascade, are discussed in detail. Our main contribution beyond the comparison of different models and possible ways to reduce systemic risk is of methodological nature.

We thus summarize our main contributions in this respect. Afterwards we discuss in more detail our findings and insights that have been enabled by our methodological advancements.

13.1.1 Methodological contributions

Part I. We have derived a local tree approximation (LTA), also known as branching process approximation, for the cascade size evolution in infinitely large configuration model type random graphs. This analytical result is in perfect correspondence with simulation results. Furthermore, we have gained the insight that all cascade time steps are correctly computed. This is valid for all degree distributions with finite second moment and
arbitrary threshold distributions.

We have extended the class of constant load models that can be tackled by this approach. In the here provided form, it comprises also models where the distributed load can depend on properties of the failing node. Furthermore, several load redistribution and overload redistribution models can be covered by our derivations, including the LRD and OLRD LLSS variant. To our knowledge, this has not been possible before.

Employing a different perspective than the common generating function approach has paved the way for these results. We describe the load that a failed neighbor distributes by a random variable whose distribution is calculated iteratively. The heterogeneity of loads originates in a concept that we have introduced as damage diversification. As we found, this concept is core to the understanding of the role of hubs in a cascade, especially for LRD and OLRD models. Furthermore, it points to an additional option to reduce systemic risk.

Part II. We have extended our analytic local tree approximation (LTA) to a multiplex setting, in which we introduce a new type of link, a semi-dependency link. Interestingly, we find that the coupling between the layers can mainly be achieved by adapting the response function.

Similarly, we also have integrated systemic feedback in our iteration procedure.

As a third contribution of Part III, we have moved our focus to the study of finite networks. Our derivations of the full cascade size distribution in finite fully connected and star-shaped networks rely solely on combinatorial arguments and the high symmetry of the considered networks. From a methodological perspective, these derivations differ completely from the earlier results. They apply to a relatively general class of cascade processes and are valid for arbitrary threshold distributions.

Part III. Because of the unsymmetrical finite network structure that is defined by the data on international trade of staple food, we cannot describe the cascade risk by analytic means. Instead, we have employed Monte-Carlo simulations to study systemic risk. The transfer and adaptation of cascade models to describe the evolution of the international trade of staples, as presented in this thesis, is a methodological advancement in food trade research. Furthermore, we have defined a set of new systemic risk indicators that measure different aspects of the cascade evolution. These could be used in an assessment of the shock resilience of international staple trade.
13.1.2 Main insights

Modeling systemic risk under network uncertainty

In Part I, we have focused on average properties to measure cascade effects on undirected random graph ensembles defined by prescribed degree and threshold distributions. Two influences on the amplification of failures can be investigated in this setting: the heterogeneity in the diversification behavior of nodes and heterogeneity in their robustnesses. From a modeling perspective, random graph ensembles reflect network uncertainty, i.e. missing information about a given, fixed network structure or quickly changing network topologies.

In our studies, we have extended the analysis of the three cascade model classes identified by Lorenz et al. (2009). However, investigations presented by Lorenz et al. (2009) were limited to fully connected, and partially regular, topologies. Because of this simplification, the different exemplary models in each class were indistinguishable in their cascade dynamics. In this thesis, we introduce heterogeneous degrees to the study of these model classes. With this extension, we were able to reveal differences between the exemplary models for all three classes.

Considering the phase diagrams with respect to the studied threshold parameters, we observe sudden regime shifts for Constant Load (CL) and Load Redistribution (LRD) models, and in principle low systemic risk for Overload Redistribution (OLRD) models. This is similar to the results presented in Lorenz et al. (2009). However, the impact of the overall connectivity, the role of hubs and leaves in the course of a cascade, and consequently their endowment with different thresholds, vary widely among the models. Especially correlations between threshold and degree distributions lead to strong deformations of the observed phase diagrams.

The trade-off between cascade exposure and damage diversification. Common to all cascade processes on undirected networks is that failures of nodes with high degrees have the potential to cause many subsequent failures. Thus, in a systemic risk reduction scheme, their failure probabilities usually need to be reduced or their failures mitigated.

A failure mitigation effect is present in most of our studied models, except the Constant Load Exposure Diversification model (CL ED). Usually, the damage that a failing node inflicts is shared between its functional network neighbors. Consequently, if the count of neighbors is higher, each of them receives a smaller amount of load. At the same time, hubs face a high failure risk because each additional link introduces a possible way to receive additional load in case of a neighbor’s failure. Still, the existence of hubs can be advantageous for a system. On the one hand they provide connectivity in times of ”normal” system operation. On the other hand, if they fail in the course of a cascade, their failure cuts many cascade paths and has hence the potential to decrease the total cascade size.
This effect is particularly prominent in Constant Load Damage Diversification (CL DD) models, where the damage that a node failure inflicts is constant over time.

**Time of failure.** However, in Load Redistribution (LRD) and Overload Redistribution (OLRD) models, nodes accumulate load over time and distribute it (at least partially) further in case of their failure. If they distribute a high amount of load, their failure causes many subsequent failures with a high probability. As in these models nodes accumulate load over the course of a cascade, their early failures can be beneficial for mitigating large cascades.

Because of the accumulation of load in these models, we have observed much higher systemic risk for the Local Load Sharing with load Conservation (LLSC) model than for the Local Load Sharing with load Shedding (LLSS) variant in case of Load Redistribution (LRD). In case of the former, the network interconnectivity increases over time, since neighbors of failing nodes become connected. Thus, also the degrees of nodes increase and the cascade process develops similarly as on a fully connected topology. In contrast, Local Load Sharing with Load Conservation (LLSC) incorporates less systemic risk in case of the Overload Redistribution models. Here, only small amounts of overload are distributed, such that a high connectivity, i.e. good damage diversification, provides protection for the neighborhood of failing nodes with a high degree.

In general, the time of a node’s failure is determined by its threshold. Consequently, different threshold allocation schemes can influence who fails earlier: hubs or leaves.

**The interplay between system robustness and topology.** Although system performance is not explicitly modeled, we have assumed that it is implicitly reflected in the given degree and threshold distributions. In case that the overall system performance depends only on these two distributions and not on the microscopic structure of a network or the combination of threshold and degree of single nodes, a reallocation of thresholds would not change the overall system performance. However, such a reallocation can have substantial impact on systemic risk. Here, we have studied three different reallocation schemes.

Random failures (rf) have served as our benchmark scheme, since a random allocation of thresholds, independent of a node’s degree, is the simplest and most reasonable assumption. Furthermore, hubs and leaves have no a-priori different failure risk. In contrast, the central failures (cf) scheme leads to the early failure of hubs, as nodes with a higher degree receive smaller thresholds. The reverse image we have studied by considering the peripheral failures (pf) scheme, where hubs receive higher thresholds than leaves.

One of the two extreme scenarios, peripheral failures (pf) or central failures (cf), usually leads to a reduction of systemic risk, while the other supports a further amplification of failures. We find a general tendency that for model variants which expose the system to

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1This choice maximizes the entropy, if no information about the correlation between threshold and degree distribution is available.
lower systemic risk than other models in the same class, central failures lead to smaller cascades. And vice versa, peripheral failures (pf) are better suited to reduce the cascade risk originating in models that show higher cascade risk for random failures (rf).

Our explanation for this observation is that models showing lower systemic risk in case of random failures usually do so because of a beneficial damage diversification effect. Central failures (cf) support this effect even further and lead to an early failure of hubs. In contrast, peripheral failures (pf) prevent the failure of hubs. This hinders the amplification of cascades if the failure of hubs are the most problematic events.

However, in direct comparison, the best combinations of cascade model variant and threshold allocation, for instance, CL ED pf and DD cf, show no consistently better results than the others. Usually, one combination reduces the cascade risk in regions of high systemic risk, while the other one exposes the system to lower risk for most other parameters. This poses a system design question, which we have discussed in detail in Chapter 4. The answer depends on the preferences and risk aversion of the respective stakeholders.

**Extensions and limitations of modeling assumptions**

We have challenged the modeling assumptions used in Part I by three different extensions of our previous model setting.

**Multiplex.** Modeling a complex system as network usually means a significant simplification of the structure. Different components are associated with similar nodes, while their various interactions are subsumed in one type of link.

We have therefore extended our model complexity by an additional second network layer, coupled to the original one. Furthermore, we have introduced another type of link, a *semi-dependency link*, characterizing this coupling. Via this link, cascades can propagate in an asymmetric way. While a failure in the main layer leads to the immediate failure of the corresponding node in the second layer, the other way around, only the threshold in the original layer is reduced.

This setup is motivated by an example, where firms face the decision to split their business in a core and subsidiary business. This scenario is best represented by the constant load exposure diversification model. Clearly, both strategies, leaving the networks as it is, as well as the split into subsidiary businesses, lead to a certain exposure to cascades. We quantify this exposure by an analytic LTA of the average final cascade size.

Which decision incorporates the lower systemic risk crucially depends on the coupling strength of the two layers. For lower coupling, a split of businesses is advantageous, if the core layer is relatively robust. Interestingly though, we observe a sudden regime shift for increasing coupling strength. At a certain point, cascades lead to an almost complete system break-down via a feedback mechanism between both layers. We provide an estimate
Chapter 13. Summary and conclusions

for this critical point.

As the phase diagrams, which we observe, differ qualitatively significantly from our results on single layers, we conclude that the multiplex nature of a system definitely needs to be regarded in a realistic systemic risk model.

**Systemic Feedback.** Previously, we have assumed that load is only distributed locally to functional network neighbors. However, often global information about the system state is available and nodes respond accordingly. We have studied an exemplary Constant Load Exposure Diversification model in which the downward trend in the course of a cascade is reinforced. Interestingly, this set-up leads to extremely similar results as a Load Redistribution model on a fully connected network structure. In fact, the systemic feedback mechanism that we have investigated could equivalently be interpreted as additional global distribution of load.

From this and other findings presented in this thesis, we see that the model class boundaries are often not sharply defined. The above example even pinpoints to a possible interpolation between the CL and the LRD class. We conclude that macro state observations are obviously not sufficient to deduce dynamics on the micro level.

**Finite networks.** In the previous chapters, we have studied the average cascade size in infinitely large random graph ensembles. However, real world systems are usually finite and the cascade outcomes are variable. We have derived a closed form solution for the probability distribution of the final cascade size for fully connected and star-shaped networks of arbitrary finite size $N$. With our approach, CL as well as LRD models can be captured. However, models of OLRD type introduce an additional challenge that we cannot solve yet.

Remarkably, despite the high symmetry of the considered network topologies and the threshold distributions, we find asymmetric and broad cascade size distributions for several parameters. Even for fully connected networks consisting of $N = 10^7$ nodes, we still find bi-modal shapes. We conclude that in this setting, the average is a bad measure of cascade size, as it is only a poor representation of the actual systemic risk. This finding is of special interest, as it is common practice to report only the average cascade risk in empirical studies.

Interestingly, these results question our conclusions from the previous chapters to some extent. Before, the CL Exposure Diversification (ED) seemed to outperform other constant load models on average when nodes with a higher degree receive higher thresholds (pf). Yet, in the presence of hubs, as in star-shaped networks, we find a tendency towards bi-modal cascade size distribution for the ED model. In contrast, the Damage Diversification (DD) variant leads more reliably to small systemic risk.

We conclude that a thorough systemic risk analysis needs to incorporate the shape of the full cascade size distribution instead of focusing on one-dimensional summary measures,
as for instance, the average.

Modeling systemic risk in a data driven approach

Ultimately, we study models, like the ones introduced in this thesis, for the purpose to better understand real-world systems. Specifically, we study the international trade of staple foods.

The final goal is to describe the reorganization of the international trade of staples as response to changing productions and demands. We have approached this goal by first steps towards cascade model improvements that regard our insights from the previous chapters. Unfortunately, we cannot transfer our findings par for par, as we have adjusted the model dynamics to fit our purpose. However, we have developed an intuition which general aspects are worth a consideration.

In our study of this particular real-world system, we have first identified the hubs and leaves and the basic properties of the topological structure. Moreover, we have investigated the allocation of thresholds. As we have additional information about the yearly time evolution of the networks, we have also observed trends in the network changes.

Descriptive data analysis. We have analyzed and compared the yearly evolution of four coupled international trade networks regarding the four major internationally traded staples: maize, rice, soy, and wheat.

All these networks show increasing interconnectivity. This can be partially explained by a growing production and for maize, rice, and soy even a growing production per head. However, the share of traded goods versus the total production still increases for all staples except maize. Especially, soy is traded to a large extend: 34 % of the whole production in 2013.

This trade is facilitated by a growing number of trade links between countries. However, we observe that the distribution of weights, i.e. trade volumes, is right skewed. Thus, a significant share of all trade is constituted by a relatively small number of links. The weight heterogeneity is highest for soy and still seems to be further increasing. Interestingly, wheat is traded much more equally, as can be observed by low link weight heterogeneity.

There are further differences in the stability of these big trade volumes over time. Remarkably, we find a structural change in 2013 for several staples. Most interestingly, the biggest trade volumes shifted. For instance, the maize exports from the USA to Japan declined, which in turn is partially compensated by exports from Brazil to Japan. We hypothesize that this decline can be mainly attributed to an increase in demand of the USA.

However, the possible causal explanations are manifold. We test the implications of several assumptions in the models we have developed in Chapter 11 and Chapter 12. These
models attribute a topological change to a distortion of the production or the demand distributions. We also present a more advanced model, in which demands depend also on the production distribution at least indirectly via prices.

**Cascading export restrictions.** As a first approach to better understand the reorganization of the trade networks as response to local shocks, we have investigated a simple model for cascading export restrictions. This model originates in the studied Overload Redistribution models, but has been adapted in several ways to match our application better. Given the results of this model, in a second step we have constructed a directed weighted network in which each link corresponds to the outcome of a shock scenario. The resulting network can be interpreted as a higher order representation of trade dependencies, while only the first order is given by the original trade network. We argue that such a cascade dependency network gives valuable insights about more realistic economic dependencies than direct trade relationships, as it also acknowledges the capabilities of countries to mitigate shocks. This representation furthermore addresses the issue of information loss introduced by systemic risk summary measures, as it provides an overview about the heterogeneity of cascade outcomes.

In contrast to the analysis presented in the previous chapters, temporal information is provided as well. Because of this, we have been able to investigate the evolution of the final fraction of failed nodes for years 1992 to 2013. We have complemented the earlier presented static measures by several systemic risk indicators that capture different aspects of the evolving cascade.

We have found that the number of countries, that are affected by cascades, increased in the considered time period. This can be mainly attributed to the growing trade interconnectivity. As result, at the end of the observation period shocks are shared by a high number of countries in relatively equal proportions. Also, the fraction of the population that would be affected by the shock scenarios we have considered is rather small. We have interpreted this as a sign of good risk diversification. Our findings are also robust with respect to our empirical observations of structural changes of the largest trades in 2012/2013.

Let us note that these conclusions do not take into account the vulnerability of the system to trade deprivations. Indeed, trade distortions usually come along with a loss of the benefits associated with canceled trades.

We have found that the reorganization of the networks due to trade restrictions is considerable. Often, cascades evolve over long times and comprise a high number of export withdrawals. Except for soy trade, the fraction of trade that is possibly canceled increases over time. This diminishing trade has especially negative consequences for trade intermediaries. Our model has proven useful in the identification of such trade intermediaries. These are characterized by by a high cascade betweenness centrality, which measures a
node’s vulnerability to trade losses.

**Adding model complexity.** Several assumptions in our cascading export restrictions model are critical in a realistic risk scenario. The following aspects need to be revised in next steps: 1) Import increases are not possible in the mitigation of shocks. 2) Countries cannot respond to the measurements of the system state. Prices or the overall production amount do not influence trade decisions. 3) Preferences of countries concerning their trade partners are not reflected adequately. Some links might break down completely, while others are strengthened in response to shocks. 4) Food trade networks are studied separately. In reality, strong couplings between the networks are observed, as consumers can substitute one staple by another in response to price increases. 5) Price elasticities are not considered. However, people adjust their demand accordingly.

We have given an outline, how these challenges can be addressed and have discussed first steps towards the considered model improvements.

## 13.2 Future research

**Modeling Framework.** We see ample of opportunities to continue our research in the setting of random graph ensembles. On the basis of our methodological contribution, we could explore how the cascade outcome depends on other threshold distributions or on the average of the degree distribution. We could ask whether the existence of a cascade window, as encountered by Gleeson and Cahalane (2007) for the Constant Load Exposure Diversification model, is in fact a relatively general property of cascade processes. Furthermore, we could study the impact of the relation between the fraction of hubs and leaves on systemic risk. Thus, we would optimize the degree distribution under some constraints, for instance a fixed average degree.

A challenge that we might be able to solve in future is to provide an analytic description of the average cascade size corresponding to the Local Load Sharing with load Conservation (LLSC) models, i.e. a fiber bundle model with local load sharing. A combination of our branching process approximation for the Local Load Sharing with load Shedding (LLSS) variant and advancements for cascading processes in clique based random graphs by Hackett and Gleeson (2013) seems to be a promising first step.

In the first two parts of this thesis, we have measured systemic risk by the (average) fraction of failed nodes. This does not penalize for a loss of connectivity. Conversely, we have regarded early disconnections to be positive, if they cut possible path ways that a cascade process can take. However, the functionality of real world systems depends often critically on the connectedness of its components. Consequently, we would ask in future for the size distribution of its connected components. Similarly, we could measure system performance explicitly in different ways and study its evolution in the course of a cascade.
Furthermore, we could introduce additional model complexity while preserving analytic tractability. For instance, interventions that increase the robustness of specific nodes or possible recoveries of failed nodes with time delay are often studied in the context of cascade processes and considered as practically relevant.

In principle, we could also analyze cascade processes where the load distribution mechanisms of our classes are mixed or study multiplexes with layers corresponding to different load distribution mechanisms. However, we would object arbitrary extensions of the presented generic models, especially because the number of options seems to be infinite.

Instead, we would focus on extensions which have or could have an equivalent in an application setting. We would be guided by three questions: 1) How does systemic risk emerge? 2) How do specific topological network or process features impact the evolution of cascades? 3) How can systemic risk be reduced?

In an economic networks setting, answers to the above questions could be based on agent based models in which nodes maximize, for instance, individual utility functions. On this basis they could make decisions about their allocation of resources. This research direction would especially be interesting, as it studies the implications of a systemic perspective, where the system safety is not necessarily aligned with the interests of individual nodes. This would suggest to test several incentive schemes to reduce systemic risk to compensate for the utility loss that specific nodes would encounter in this situation. In the context of financial networks, several measures are considered that organize the system with a macro viewpoint, for instance, taxation of systemic risk increasing trades or the introduction of a central counter party.

**Multiplex.** Ample extension options exist for studies of cascade processes on multiplexes. We only focus on possible immediate next steps based on our investigations.

First and foremost, from an analytic point of view, it would be straight forward to introduce semi-dependency links between a number of network layers where nodes according to other load distribution mechanisms. Furthermore, we could also handle heterogeneous coupling strengths and allow that only a fraction $q$ of all nodes could be coupled to nodes in another layer. We would expect that we could also observe a sudden regime shift when varying $q$. Additionally, a higher number of interlayer links starting from single nodes are a possibility.

Especially interesting for the application that has motivated our model, we could allow a mixture of business splits and business merges. Moreover, it would be more realistic to consider that certain properties of nodes in one layer preferably attract nodes with certain properties in another layer. As example, nodes with high degrees in one layer have a higher chance to connect to nodes that also have a high degree in the other layer to increase risk diversification in both layers.

Furthermore, we have not studied different threshold allocation schemes in the multiplex
setting yet. Also, the threshold of a node could imply a preference for establishing links
to nodes with a similar threshold, or to nodes with a higher threshold to acquire more
capital.

In summary, we could improve our model in such a way that it reflects several strategies
of firms concerning their merging and business split decisions.

**Finite networks and volatility.** As we have found, a thorough systemic risk analysis
in finite systems needs to consider the full final cascade size probability distribution. This
insight will influence all of our future work. Furthermore it has sparked a new research
question: What are the main reasons for high uncertainty, i.e. rather broad distributions,
in the cascade outcome? Moreover, is there a point in time at which we can narrow down
the cascade outcome with higher certainty?

As extension of our framework models we could also consider stochastic cascade dynamics
as developed by Lorenz et al. (2009) and estimate the implications of this additional
stochastic element. How does this increase the uncertainty of the cascade outcome?

We would start in each case from a network motif based analysis and look for similar
patterns. Additionally, we would be interested in the question how our findings correspond
to adjacency matrix properties of the considered network structure and, of course, on
properties of a given threshold distribution.

For this analysis, but also in practice, it might be valuable to have risk measures at
hand to summarize relevant properties of the cascade size distribution. Although such
a summary might be inadequate in all possible cases, it would be worthwhile to identify
certain classes of processes or sets of parameters, where such measures carry meaningful
information. These measures would need to reflect criteria that we relate with low systemic
risk. This would allow us to look for topological improvements that reduce systemic risk in
this sense. For instance, starting from a given network structure we could redirect certain
links in a Greedy fashion.

**Resilience of international food trade networks.** This research direction we find
especially promising to continue with, as it relates to one of the millennium development
goals, the guarantee of world wide food security.

There are a multitude of problems that could and need to be worked on. For instance,
only the modeling of possible shocks to the production system is a demanding task tackled
by the joined effort of many researchers. Also the identification of the stakeholders in the
world food system and their interests or strategies that could enter an agent based model
are a challenge, which is often addressed in case studies.

Our complex systems approach requires strong simplifications of these tasks on the macro
level, but has to capture at the same the most important aspects. We rely on deepening
the understanding of the mentioned issues. Our models have to be successively improved
to capture as many realistic aspects as possible. Moreover, we can investigate further how international trade is constituted and find ways to control for effects like changing trade agreements, subsidies, import taxation etc.

Another issue is, that we make strong causal assumptions about the propagation of cascades and the value of trade relationships for the involved countries. These call for a rigorous statistical validation. First, we have to focus on our model selection. In a next step, we would formulate and implement a linear program (LP) to fit an adequate model for the link weights. It has to respect that the link weights have to fulfill constraints so that the import and export of countries are correctly matched. Furthermore, we need to take care of over-fitting also in our models for country strategies, for instance by cross-validation. Theoretically, the resulting model parameters are stochastic and introduce variability in our cascade outcomes. This variability needs to be reported as well. Moreover, it helps us judge the quality of our model. For instance, we could try to explain the trade network evolution with our model by comparing our model results with given data. Thus we would measure the change in our predictors, i.e the production amount, population, gdp of countries, etc., and use these as model input and describe the change of the international staple trade networks by a cascade process. This would be bases on a multiplex view that we have to employ further and incorporate substitution possibilities of different types of crops. Additionally, we would seek to explain also the creation or existence as well as deletion of links and not only the trade volume along existing links.

Moreover, it might be possible to infer to some degree the usage of quantities traded along certain links with the help of the usage data in countries. This would allow us to develop more realistic shock scenarios and risk measures. For instance, if crops for feed are canceled, the population is not immediately at risk of starving, but the meat production industry faces a difficult time.

On the long run, we can also consider further predictor variables that reflect what we believe is the relevant information for economic actors: the comparison of alternatives. However, the alternatives that we observe in yearly aggregates might not have been real alternatives at the time a trade decision was made. Thus, we might require a higher time resolution of constituting trades than we have access to at the moment.

Most critical for a thorough estimation of systemic risk is that we lack data about stocks on the country level. These stocks would naturally define the real thresholds of countries and also be a valuable source of information in the definition of suitable systemic risk measures. Without this information we have to focus on the quantification of losses that countries can face in several shock scenarios. Further reasonable proxies for this information can still be incorporated in future work. Moreover, we can utilize our data on stock changes fully to understand which shock sizes are critical for a country.

Finally, the implications of our analysis for food security have to be understood.
As we have shown, our work has paved the way for a thorough systemic risk analysis of international food trade. Starting from generic cascade models, we have successively adjusted our models to match our application scenario better. These model improvements are built upon an deepened understanding of generic cascade models, which apply to general systems comprising interdependent components. Our methodological advancements have laid the foundation for further explorations of cascade effects in interdependent systems.
Appendix A

Supplementary material to Chapter 3

A.1 Existence of a fixed point

We are going to prove with the help of the Knaster-Tarski Theorem that the function $G$ in Equation (3.4) attains a fixed point. Recall that for any $q \in [0, 1]^c$ the $k-th$ component $G_k(q)$ of $G(q)$ is defined as:

$$G_k(q) := \mathbb{P}\left(F_{\Theta(k)} \leq \sum_{j=1}^{k-1} L_j(q)\right) = \sum_x F_{\Theta(k)}(x) p^{(k-1)}(x),$$

or

$$G_k(q) = \int_0^\infty F_{\Theta(k)}(x) p^{(k-1)}(x) \, dx,$$

where $p_L(q)$ is defined analogously to Equ. (3.1) as:

$$p_L(q)(0) = 1 - \sum_{d=1}^c \frac{q_d p(d) d}{z} + \sum_{d=1}^c \frac{q_d p(d) d}{z} p_W(d,k)(0),$$

$$p_L(q)(l) = \sum_{d=1}^c \frac{q_d p(d) d}{z} p_W(d,k)(l) \quad \text{for} \ l \neq 0.$$

We need to show that $G$ is monotone with respect to a partial ordering and maps the complete lattice $[0, 1]^c$ onto itself.

The partial ordering on $[0, 1]^c$ is defined by the following: For any two vectors $x, y \in [0, 1]^c$ holds $x \leq y$, if and only if $x_i \leq y_i$ holds for all their components $i \in I$.

First, we note that we can see immediately from the definition of $G_k(q)$ that $G$ is onto. Since $p_L(q)$ is a probability distribution for all $q \in [0, 1]^c$, also $G_k(q)$ is a probability so that we naturally have $G_k(q) \in [0, 1]$ for all $k$. 
It is left to show that $G$ is monotone with respect to the introduced partial ordering.

We recall that $G$ maps a vector $v \in [0,1]^c$, whose $k$-th entry $v_k$ can be interpreted as conditional failure probability of a neighbor with degree $k$, to another vector $G(v)$ that we interpreted as well as conditional failure probabilities, but one iteration further in a cascade.

Let’s assume that we have two vectors $v, w \in [0,1]^c$ with $v \leq w$. Without loss of generality we further assume that they only differ in one component $d$, i.e. $\Delta := w_d - v_d$. If they differ in more than one component, we can argue as we proceed separately for each component.

From the definition of $p_{\mathbf{L}(q)}$ we can deduce the $k$-dependent relation

$$p_{\mathbf{L}(w)}(l) = p_{\mathbf{L}(v)}(l) + \Delta \frac{p(d)d}{z} p_{W(d,k)}(l)$$

for $l > 0$, while for $l = 0$ we observe with

$$p_{\mathbf{L}(w)}(0) = p_{\mathbf{L}(v)}(0) - \Delta \frac{p(d)d}{z} \left(1 - p_{W(d,k)}(0)\right)$$

a decrease in probability mass (if $\Delta > 0$). Thus, the inflicted loss distribution by a single neighbor is shifted for $w$ towards higher losses. Consequently, the same holds for its convolution $p_{\mathbf{L}(w)}^{k-1}$, the distribution of the sum of losses inflicted by $k - 1$ independent neighbors. Since $F_{\Theta(k)}$ is monotonously increasing as cumulative distribution function we can conclude from Equations this argument that $G_k(v) \leq G_k(w)$ for all $k$, as was to be shown.

The essence of the proof is that the increase of the failure probability of a neighbor with degree $k$ also increases the probability that a loss is inflicted (i.e. load is distributed) to any of its neighbors. This higher probability of a loss can only increase the failure risk (of each node or neighbor).

### A.2 Approximation of the convolution

For the DD it is necessary but computational expensive to calculate the convolution $p_{\mathbf{L}^{nk}(k,t-1)}^{s(k-1)}$. where

$$p_{\mathbf{L}^{nk}(k,t-1)} \left(\frac{1}{d}\right) = p_{\mathbf{L}^{nk}(t-1)} \left(\frac{1}{d}\right) = \mathbb{P}(s_{nb}(t-1) = 1 \mid d) \frac{dp(d)}{z}$$

denotes the probability of a loss $1/k$ caused by the failure of one single neighbor with degree $k$ before or at time $t - 1$. Thus, the loss $\lambda_{nb}(k,t-1) = \sum_{j=1}^{k-1} L_j^{nb}(t-1)$ that a focal neighbor faces at time $t - 1$ follows the distribution $p_{\mathbf{L}^{nk}(t-1)}^{s(k-1)}$. This distribution is discrete.
so that we can consider a sum instead of an integral as in Equ.(3.2).

However, the number of values \( l \) with nonzero probability mass that \( \lambda^{nb}(k, t - 1) \) attains, which are of relevant size for good accuracy, as well as the number of accumulation points grows exponentially with \( k - 1 \). But, in the end we are only interested in calculating the failure probability \( p_{L^{nb}(t-1)}^{s(k-1)}(l)F_{\Theta}(l) \) of a node. So it suffices to compute \( p_{L^{nb}(t-1)}^{s(k-1)} \) accurately on an interval \([0, b]\) \( \subset \mathbb{R} \), where the threshold distribution \( F_{\Theta} \) is effectively smaller than 1.

For \( l \in \mathbb{R} \setminus [0, b] \) outside of this interval (which means that \( l > b \) since \( l \geq 0 \)) we consider the summand

\[
\sum_{l>b} p_{L^{nb}(t-1)}^{s(k-1)}(l)F_{\Theta}(l) \simeq 1 - \sum_{l\leq b} p_{L^{nb}(t-1)}^{s(k-1)}(l),
\]

since we can approximate \( F_{\Theta}(l) \) by 1 for \( l > b \).

We vary the parameters \( \mu \) and \( \sigma \) of the threshold distribution between 0 and 1 so that we can safely set \( b = 5 \). Next, we partition \([0, b]\) into \( J \) small intervals \( I_j = \mathbb{R}_{j-1}^{j} h, jh \) of length \( h \), with \( j = 1, \ldots, J \). For small enough \( h \) (here \( h = 10^{-5} \)) it is numerically precise enough to assume the approximation of \( p_{L^{nb}(t-1)}^{s(k-1)} \) to be constant on each \( I_j \) or having its probability mass in an interval \( I_j \) concentrated in one point in \( I_j \).

### A.2.1 Convolution with the help of FFT

In the standard (and faster) algorithm that we use, we simply bin \( p_{L^{nb}(t-1)} \) for DD to \([0, b]\) by

\[
\hat{p}_{L^{nb}(t-1)}(jh) := \begin{cases} 
\sum_{d: (j-1)h < d \leq jh} \left( \Pr(s_{nb}(t-1) = 1 \mid d) \frac{dp(d)}{2} \right) & j = 1, \ldots, J, \\
0 & \text{otherwise}.
\end{cases}
\]

Then, we apply the Fast Fourier Transformation (FFT) (Frigo and Johnson, 2005; Ruckdeschel and Kohl, 2014), take the \((k-1)\)-th power of the resulting distribution and transform it back to obtain \( \hat{p}_{L^{nb}(t-1)}^{s(k-1)} \).

We only calculate convolutions of maximal power 50. If a node has more than 50 failed neighbors, we assume it fails with probability 1. This is numerically accurate enough for the calculation of the final fraction of failed nodes \( g \) and the correct shape of \( \Pr(s = 1 \mid k) \).
Figure A.1: The studied degree distributions: Poisson distribution with parameter $\lambda = 8$ and cutoff degree $c = 50$ (blue), scale free distribution with exponent $\gamma = 3$ and maximal degree $c = 200$ (red), and degree distribution measured from a snapshot of the Italian interbank market in 1st October, 2002 (black), in log-log scale.

A.3 Systemic risk for further topologies

We calculate the final fraction of failed nodes for Poisson random graphs and scale free networks with degree distributions

$$p_P(k) := \frac{1}{S_P} \frac{\lambda^k}{k!}, \quad p_S(k) := \frac{1}{S_S} \frac{1}{k^\gamma}$$

for $k \in \{1, \cdots, c\}$ with normalizing constants

$$S_P := \sum_{k=1}^{c} \frac{\lambda^k}{k!} \quad \text{and} \quad S_S := \sum_{k=1}^{c} \frac{1}{k^{\gamma}}$$

and additionally also for a degree distribution measured from a snapshot of the Italian interbank market in 1st October, 2002 as published in Ref. (De Masi et al., 2006) and shown in the second row of Fig. A.2. The last case we present as illustration of a real world system. Fig. A.1 depicts the studied degree distributions.

We do not claim a deeper financial interpretation, since this would require to incorporate empirical weights as well. This is outside the scope of this paper, which studies the effect of basic diversification strategies. We observe that the DD variant leads to lower systemic risk for most threshold parameters. Still, close to the sudden regime shift, the ED variant can expose the system to lower risk. This can also be observed for the degree distribution measured from interbank lending data. For other threshold parameters, the DD variant is especially effective for systemic risk reduction in comparison with ED.

The consistently observed regime shift vanishes in case of both model variants for small connectivity. Scale free random graphs with $z = 1.36$ show only continuous changes of the
Figure A.2: Phase diagrams for the final fraction of failed nodes $\varrho$ calculated numerically for different degree distributions, diversification variants, and their differences. The thresholds $\Theta$ are normally distributed with mean $\mu$ and standard deviation $\sigma$ ($\Theta \sim N(\mu, \sigma^2)$). The fraction of initially failed nodes is given by $F_\Theta(0)$. The darker the color the higher is the systemic risk.

First row: Poisson distribution with parameter $\lambda = 8$ and cutoff degree $c = 50$ for the ED (left) and DD (right). The middle shows their difference $\varrho^{(ED)} - \varrho^{(DD)}$.

Second row: Degree distribution of the Italian interbank lending network at 1st October, 2002 for the ED (left) and DD (right). The middle shows their difference $\varrho^{(ED)} - \varrho^{(DD)}$.

Third row: Scale free distribution with exponent $\gamma = 3$ and maximal degree $c = 200$ for the ED (left) and DD (right). The middle shows their difference $\varrho^{(ED)} - \varrho^{(DD)}$.

final fraction of failed nodes with respect to the threshold parameters. Interestingly, the random graphs are still connected, as a giant connected component emerges in the limit $N \to \infty$ in the configuration model. We can deduce this from the fact that the second moment $< K^2 >$ is $< K^2 > = 4.89$. So, we have $< K^2 > - 2z > 0$ and the existence of a giant connected component follows (Newman, 2010).

### A.4 Systemic risk for net exposures

Last, we demonstrate that our analytic approach is not limited to undirected networks.
In many real world applications, only a net exposure leads to a loss. For example, netting agreements in financial interbank lending networks or flow cancellations lead to effective weights

$$w_{ji}^{\text{eff}} := \max\{w_{ji} - w_{ij}, 0\}.$$  

Thus, there exists only an exposure in the direction of the higher weight between two nodes $i$ and $j$.

In case of the DD variant, the effective link points from a node with higher degree $k_j$ to a node with lower degree $k_i$. We have $w_{ji}^{\text{eff}} = 1/k_j - 1/k_i$ and $w_{ij}^{\text{eff}} = 0$, if $k_j \geq k_i$.

Consequently, Equation (3.1) specializes to:

$$\mathbb{P}\left(L_{D\text{Deff}}^{nb}(k, t-1) = 0\right) = (1 - \pi(t-1)) + \sum_{d=k}^{c} \frac{\mathbb{P}(s_{nb}(t-1) = 1 \mid d)p(d)\,d}{z}$$

$$\mathbb{P}\left(L_{D\text{Deff}}^{nb}(k, t-1) = \frac{1}{d} - \frac{1}{k}\right) = \frac{\mathbb{P}(s_{nb}(t-1) = 1 \mid d)p(d)\,d}{z}$$

for $d < k$ and $\mathbb{P}\left(L_{D\text{Deff}}^{nb}(k, t-1) = l\right)$ for all other $l$.

In this setting, the final cascade size $\varrho$ is smaller than for the undirected case, since in total the exposures are smaller as can be seen in Fig. A.3.
Appendix B

Supplementary material to Chapter 9

Proof of Equation (9.2) for fully connected networks

The probabilistic inclusion-exclusion principle Szpankowski (2001) states that for events $A_1, \ldots, A_n$ in an arbitrary probability space, at least one of these events occurs with probability:

$$\mathbb{P} \left( \bigcup_{j=1}^{n} A_j \right) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{I_j} \mathbb{P} \left( \bigcap_{l \in I_j} A_l \right)$$

where $I_j = \{l_1, \ldots, l_j\}$ is a set containing exactly $j$ distinct indices $l_i \in \{1, \ldots, n\}$. In case that the probability $q_j = \mathbb{P} \left( \bigcap_{l \in I_j} A_l \right)$ only depends on the number $j$ of events that are intersected, we have

$$\mathbb{P} \left( \bigcup_{j=1}^{n} A_j \right) = \sum_{j=1}^{n} (-1)^{j} \binom{n}{j} q_j,$$

as there exist $\binom{n}{j}$ different subsets of $\{1, \ldots, n\}$ with $j$ elements.

To prove Equation (9.2) we need to define appropriate events $A_j$ so that $p_n = \mathbb{P} \left( \bigcup_{j=1}^{n} A_j \right)$ and $q_j = F(\lambda[n-j])^j p_{n-j}$, where we associate index $j$ with $j = n - l$. Then Equation (9.2) follows immediately from the stated principle.

First, we recall that $p_n$ is the probability that $n$ fully connected nodes fail in a cascade process considering only these $n$ nodes (so that none of the failures is additionally influenced by other nodes in the network). There are several threshold configurations that lead to this outcome. All of them are of the form that some $n - j$ nodes have failed first, while the remaining $j$ nodes fail because of the $n - j$ failures before.
Each $A_j$ subsumes events where the same $j$ specific nodes $i_1, \ldots, i_j$ have thresholds smaller than $\lambda[n-j]$ and the remaining $n-j$ nodes failed in a cascade without considering $i_1, \ldots, i_j$. Indeed, the union of all $A_j$ coincides with all cascades of size $n$ and each $A_j$ occurs with probability $q_j$. 
Appendix C

Supplementary material to Chapter 10

C.1 List of countries considered in the international food trade networks

<table>
<thead>
<tr>
<th>ISO 3166-1 alpha-3 codes</th>
<th>Official country name</th>
</tr>
</thead>
<tbody>
<tr>
<td>USA</td>
<td>United States of America</td>
</tr>
<tr>
<td>CAN</td>
<td>Canada</td>
</tr>
<tr>
<td>ATG</td>
<td>Antigua and Barbuda</td>
</tr>
<tr>
<td>ARG</td>
<td>Argentine Republic</td>
</tr>
<tr>
<td>BHS</td>
<td>Commonwealth of the Bahamas</td>
</tr>
<tr>
<td>BRB</td>
<td>Barbados</td>
</tr>
<tr>
<td>BMU</td>
<td>Bermuda</td>
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<tr>
<td>BOL</td>
<td>Republic of Bolivia</td>
</tr>
<tr>
<td>BRA</td>
<td>Federative Republic of Brazil</td>
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<tr>
<td>ABW</td>
<td>Aruba</td>
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<td>BLZ</td>
<td>Belize</td>
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<td>CHL</td>
<td>Republic of Chile</td>
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<td>COL</td>
<td>Republic of Colombia</td>
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<td>CRI</td>
<td>Republic of Costa Rica</td>
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<td>CUB</td>
<td>Republic of Cuba</td>
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<td>DMA</td>
<td>Commonwealth of Dominica</td>
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<td>SLV</td>
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<td>GTM</td>
<td>Republic of Guatemala</td>
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<td>Cooperative Republic of Guyana</td>
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<td>Republic of Honduras</td>
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<td>Jamaica</td>
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<td>PER</td>
<td>Republic of Peru</td>
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<td>Saint Vincent and the Grenadines</td>
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<tr>
<td>NOR</td>
<td>Kingdom of Norway</td>
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<td>Portuguese Republic</td>
</tr>
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### C.2 Data summary statistics

Additional to Table 10.1 for Maize, we provide the corresponding data referring to rice, soy and wheat.

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**Table C.2:** Summary of country variables relating to rice. Values are rounded to two digits. The minimum values of stockVar are $-11.32$ (1992-1999), $-9.43$ (2000-2007), and $-5.22$ (2008-2011).
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Table C.3: Summary of country variables relating to soybeans. Values are rounded to two digits. The minimum values of stockVar are $-11.0$ (1992-1999), $-6.0$ (2000-2007), and $-4.7$ (2008-2011).

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<td></td>
</tr>
<tr>
<td>imp/10^6tonnes</td>
<td>0.00 0.38</td>
<td>10.97</td>
<td></td>
</tr>
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11.3 [Maize] Cascade dependencies for international trade of maize in 2010. Each link corresponds to a shock scenario where the country that passes its color to the link is shocked initially. Only links with large weights are visible. The link width is proportional to the demand deficit of the country in which the link ends after \(T = 5 \cdot 10^4\) cascade steps. Node colors follow the scheme introduced in Fig. 10.1, which encodes regional proximity. The bar size of each country represents its weighted degree, i.e. the sum of all in-coming and out-going edge weights in the cascade dependency network. Shock scenario in case country \(i\) is shocked initially: (a) shock\(_i\) = \(sa^{(m)}(2010)\), (b) shock\(_i\) = 0.25 max \(\left(\prod_i^{(m)}(2010), \text{dem}_i^{(m)}(2010)\right)\).

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<td>$E$</td>
<td>Link/edge set of a graph/network $G = (V, E)$.</td>
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<td>$G$</td>
<td>Graph/network $G = (V, E)$ with node/vertex set $V$ consisting of $N$ nodes and link/edge set $E$.</td>
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<td>$N$</td>
<td>Number of nodes in a graph/network $G = (V, E)$. $N :=</td>
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<td>$T$</td>
<td>Time. End of cascade process. $T \leq N - 1$.</td>
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<td>Node/vertex set of a graph/network $G = (V, E)$.</td>
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<td>$\varrho_N$</td>
<td>Fraction of failed nodes in a single network of size $N$. Cascade size. Measure of systemic risk. Time dependent. If not stated otherwise $\varrho_N(T)$ is meant.</td>
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<td>In-degree of node $i \in V$. Number of links in $G$ that point to $i$.</td>
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<td>Load that a load distributes at failure. Time dependent.</td>
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<td>$l_{ij}$</td>
<td>Load that a failing node $i$ distributes to its neighbor $j$. Time dependent.</td>
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<td>$s_i$</td>
<td>Binary state of node $i \in V$. $s_i \in {0, 1}$. $s_i = 1$ indicates $i$ is failed and $s_i = 0$ that $i$ is functional. It is time dependent. If not stated otherwise $s_i(T)$ is meant.</td>
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<td>Weight of link $(i, j)$ in case of Constant Load Damage Diversification model. $w^{(DD)}_{ji} = 1/k_j$.</td>
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<td>$\lambda_i$</td>
<td>Load that node $i$ carries, fragility, time dependent. If no time dependence is explicitly stated, the final $\lambda_i(T)$ is meant.</td>
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<td>$\lambda_0$</td>
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