Comparing probabilistic and p-box input modelling in structural reliability analysis

Author(s):
Schöbi, Roland; Bruno Sudret, /

Publication Date:
2016

Permanent Link:
https://doi.org/10.3929/ethz-a-010671545

Rights / License:
In Copyright - Non-Commercial Use Permitted
Comparing probabilistic and p-box input modelling in structural reliability analysis

R. Schöbi & B. Sudret
Chair of Risk, Safety and Uncertainty Quantification, ETH Zurich, Switzerland

ABSTRACT: Structural reliability analysis aims at estimating failure probabilities with respect to a limit-state function accounting for uncertainty in the system’s basic variables. Typically, the uncertainty is modelled by probability theory. However, when only scarce datasets are available and knowledge is incomplete, a more general framework, such as probability-boxes, is more appropriate to describe the uncertainty. In this paper, we examine the effect of p-boxes on structural reliability analysis. Probability-boxes generally increase the complexity of the analysis and hence augment the computational costs. Using meta-models at different stages of the analysis reduces the total computational costs. In particular, advanced Kriging meta-models with adaptive experimental designs are used to estimate failure probabilities based on a small number of calls to the limit-state function. The effects are illustrated on two applications: an analytical toy function and a realistic engineering structure.

1 INTRODUCTION

Computer simulations, such as finite element model-based simulations, are a popular tool to analyze complex engineering structures and systems. At the same time, engineers are concerned with uncertainty, which is inherent to every structure and system. In this context, probabilistic analyses, such as structural reliability analysis, robust design optimization, and sensitivity analysis, have received much attention in the last two decades.

The cost for conducting such analyses depends greatly on (i) the computational model, (ii) the type of analysis, and (iii) the treatment of uncertainties. In this paper, we discuss structural reliability analysis, which typically requires repeated runs of the computational model to account for the uncertainties in model input parameters.

So far, structural reliability analysis has been developed mainly in the context of probability theory. However, a common situation in practice is to have too scarce datasets to develop a probabilistic input model. Then, the uncertainty in the input is composed of aleatory uncertainty (natural variability) and epistemic uncertainty (lack of knowledge due to limited data). In this situation, imprecise probabilities can characterize the uncertainty taking into account both types of uncertainty. Concepts to describe imprecise probabilities are e.g. evidence theory (Dempster, 1967; Shafer, 1976), possibility theory (Zadeh, 1999), and probability-boxes (Ferson & Ginzburg, 1996). In this paper, probability-boxes are examined in the context of structural reliability analysis.

As the general structure of p-boxes is more complex than a probabilistic variable, the computational costs for imprecise structural reliability analysis are higher than for a traditional probabilistic analysis. A popular strategy to reduce computational costs is to surrogate the
computational model with a meta-model, such as polynomial chaos expansions or Kriging (Sudret 2012, 2015). Hence, in this paper, Kriging meta-models are used at different stages of the imprecise structural reliability analysis to enhance the overall computational performance.

2 DESCRIPTION OF VARIABLES

2.1 Probability theory

Consider a random variable $X(\omega)$ in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\omega$ is an elementary event in the event space $\Omega$, equipped with the $\sigma$-algebra $\mathcal{F}$ and probability measure $\mathbb{P}$. Typically, a random variable is characterized by its cumulative distribution function (CDF) $F_X$, which describes the probability that $X \leq x$, i.e. $F_X(x) = \mathbb{P}(X \leq x)$. Thus, the probability measure $\mathbb{P}$ provides a single measure for the variability of variable $X$. This indicates that the variability is known and quantifiable.

2.2 Parametric probability-boxes

A parametric probability-box (p-box) is a more general framework accounting for aleatory and epistemic uncertainty. A parametric p-box is defined by a CDF the parameters of which are known within intervals:

$$X \sim F_X(x|\theta), \quad (1)$$

where $\theta_i \in [\theta_i^L, \theta_i^U]$, $i = 1, \ldots, n_\theta$ are the interval-valued distribution parameters. This representation has a clear distinction of aleatory and epistemic uncertainty: aleatory uncertainty is represented by the distribution function family, whereas epistemic uncertainty is represented by the interval-valued distribution parameters. Note that this construction resembles a Bayesian hierarchical model (Gelman, 2006), except that the distributions of the parameters $\theta_i$ are replaced by intervals.

2.3 Free probability-boxes

An even more general formulation is the free p-box, which describes $X$ by a lower and upper boundary curve to the CDF, denoted by $F_X$ and $\overline{F}_X$, respectively (Ferson & Ginzburg, 1996; Ferson & Hajagos, 2004). In comparison to the parametric p-box, the free p-box does not assume any knowledge on the shape of the true distribution function as long as the true CDF lies within the p-box boundaries: $F_X(x) \leq F_X(x) \leq \overline{F}_X(x)$ for all $x \in \mathcal{D}_X$. When $F_X(x) = F_X(x) = \overline{F}_X(x)$ for all $x \in \mathcal{D}_X$, the epistemic uncertainty vanishes and the p-box reduces to a “regular” random variable.

In the context of the theory of evidence (Dempster, 1967; Shafer, 1976), the boundaries of the free p-box can be interpreted as the belief and plausibility measure, respectively. The belief describes the minimum amount of probability that must be associated to the event $\{X(\omega) < x\}$, whereas the plausibility measures the maximum amount of probability that might be associated to the same event $\{X(\omega) < x\}$.

3 STRUCTURAL RELIABILITY ANALYSIS

3.1 Limit-state function

The limit-state function is defined as a deterministic mapping from the $M$-dimensional input space to a one-dimensional output space:

$$G: x \in \mathcal{D}_X \subset \mathbb{R}^M \rightarrow y = G(x) \in \mathbb{R}, \quad (2)$$
where \( \mathbf{x} \) is the \( M \)-dimensional input vector and \( y \) is the scalar output. The sign of \( g(\mathbf{x}) \) indicates whether a realization \( \mathbf{x} \in \mathcal{D}_X \) corresponds to a safe system \( (g(\mathbf{x}) > 0) \) or a failed system \( (g(\mathbf{x}) \leq 0) \). It is assumed that the limit-state function is a black box. Hence, the governing equations are not directly and analytically available, such as in the case of assessing the performance through finite element models.

3.2 Failure probability for probabilistic input

3.2.1 Definition failure probability

In the context of probability theory and hence probabilistic random variables, the failure probability \( P_f \) is defined as the probability that a realization \( \mathbf{x} \in \mathcal{D}_X \) corresponds to a failed system in terms of the limit-state function \( g(\mathbf{x}) \):

\[
P_f = \mathbb{P}(g(\mathbf{X}) \leq 0) = \int_{\mathcal{D}_f} f_\mathbf{X}(\mathbf{x}) \, d\mathbf{x},
\]

where \( \mathcal{D}_f = \{ \mathbf{x} \in \mathcal{D}_X; g(\mathbf{x}) \leq 0 \} \) is the failure domain and \( f_\mathbf{X}(\mathbf{x}) \) is the joint probability density function of the input vector \( \mathbf{X} \).

3.2.2 Monte Carlo simulation

Due to the generally complex shape of the failure domain and the black box property of the limit-state function, the integration in Eq. (3) cannot be solved analytically. Monte Carlo simulation (MCS) allows one to numerically estimate the failure probability. Considering a sufficiently large set of samples of \( \mathbf{X} \), denoted by \( \mathcal{S} = \{ \mathbf{x}_1, ..., \mathbf{x}_n \} \), the failure probability can be estimated by:

\[
P_f = \frac{n_f}{n} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{g(\mathbf{x}) \leq 0}(\mathbf{x}_i),
\]

where \( n_f \) is the number of samples \( \mathbf{x}_i \) in the failure domain \( \mathcal{D}_f \), \( n = |\mathcal{S}| \) is the total number of samples and \( \mathbb{I}_{g(\mathbf{x}) \leq 0} \) is the indicator function for failure such that \( \mathbb{I} = 1 \) for \( g(\mathbf{x}) \leq 0 \) and \( \mathbb{I} = 0 \) otherwise. Note that the use of \( \mathbb{I} \) transforms the structural reliability analysis into a classification problem where only the sign of the limit-state function is relevant.

3.2.3 Adaptive-Kriging Monte Carlo simulation

A disadvantage of MCS is its reliance on a large number of model evaluations. Hence, when the limit-state function is an expensive-to-evaluate function, the total computational costs are large and MCS may become intractable. A workaround is the use of meta-models to approximate the expensive-to-evaluate limit-state function with a much simpler function.

Kriging (a.k.a. Gaussian process modelling) is a meta-modelling technique that considers the limit-state function to be a realization of a Gaussian process (Santner et al., 2003):

\[
g^{(K)}(\mathbf{x}) = \mathbf{\beta}^T \mathbf{f}(\mathbf{x}) + \sigma^2 Z(\mathbf{x}, \omega),
\]

where \( \mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), ..., f_p(\mathbf{x})] \) are regression functions, \( \mathbf{\beta} \) is a vector of coefficients. \( \mathbf{\beta}^T \mathbf{f}(\mathbf{x}) \) and \( \sigma^2 \) are the mean and standard deviation of the Gaussian process respectively. \( Z(\mathbf{x}, \omega) \) is a zero-mean, unit-variance stationary Gaussian process, which is characterized by an autocorrelation function \( R(|\mathbf{x} - \mathbf{x}'|; \mathbf{\rho}) \) and its hyper-parameters \( \mathbf{\rho} \).

Due to the black box properties of the limit-state function, the Kriging model is trained with a set of input realizations \( \mathcal{X} = \{ \mathbf{x}^{(i)}; i = 1, ..., N \} \) and the corresponding responses of the limit-state function \( \mathcal{Y} = \{ \mathbf{y}^{(i)} = g(\mathbf{x}^{(i)}); i = 1, ..., N \} \). The Kriging parameters are then obtained by generalized least-squares solution (Santner et al., 2003):

\[
\mathbf{\beta}(\mathbf{\rho}) = (\mathbf{F}^T \mathbf{R}^{-1} \mathbf{F})^{-1} \mathbf{F}^T \mathbf{R}^{-1} \mathbf{Y},
\]

\[
\sigma_Y^2(\mathbf{\rho}) = \frac{1}{N} (\mathbf{Y} - \mathbf{F} \mathbf{\beta})^T \mathbf{R}^{-1} (\mathbf{Y} - \mathbf{F} \mathbf{\beta}),
\]
where $\mathbf{R}_{ij} = R\left(\mathbf{x}^{(i)} - \mathbf{x}^{(j)}; \mathbf{\rho}\right)$ is the correlation matrix and $F_{ii} = f_i(\mathbf{x}^{(i)})$, $i = 1, \ldots, N$, $j = 1, \ldots, N$, and $l = 1, \ldots, p$. In the case of unknown correlation parameters $\mathbf{\rho}$, their values can be estimated by e.g. maximum likelihood estimate.

Having defined the parameters of the Kriging model, the prediction value of the limit-state function at a point $\mathbf{x} \in \mathcal{D}$ is a Gaussian variable with the following mean value and variance:

$$
\mu_f(\mathbf{x}) = \mathbf{f}(\mathbf{x})^T \mathbf{\beta} + \mathbf{r}(\mathbf{x})^T \mathbf{R}^{-1}(\mathbf{y} - \mathbf{F} \mathbf{\beta}),
$$

$$
\sigma_f^2(\mathbf{x}) = \sigma_f^2(1 - \mathbf{r}(\mathbf{x})^T \mathbf{R}^{-1} \mathbf{r}(\mathbf{x}) + \mathbf{u}(\mathbf{x})^T (\mathbf{F}^T \mathbf{R}^{-1} \mathbf{F})^{-1} \mathbf{u}(\mathbf{x}))
$$

where $r_l(\mathbf{x}) = R\left(|\mathbf{x} - \mathbf{x}^{(l)}|; \mathbf{\rho}\right)$ and $\mathbf{u}(\mathbf{x}) = \mathbf{F}^T \mathbf{R}^{-1} \mathbf{r}(\mathbf{x}) - \mathbf{f}(\mathbf{x})$.

Based on the trained Kriging model, MCS can be performed to estimate the failure probability. By the nature of Kriging, however, the meta-model is accurate close to the points of the experimental design $\mathbf{X}$, which are generally speaking not optimal to estimate the failure probability. Thus, adaptively enriching the experimental design in a guided way can improve the accuracy of the failure probability estimate. Adaptive-Kriging-Monte Carlo simulation (AK-MCS) (Echard et al., 2011) uses the prediction variance in Eq. (9) as a local error estimate for the enrichment strategy. The main steps of AK-MCS are given here (see details in Schöbi et al. (2016)):

(i) Generate a small initial experimental design $\mathbf{X}$ and compute the corresponding limit-state function values $y^{(i)} = g(\mathbf{x}^{(i)})$.

(ii) Train a Kriging model $g^{(K)}$ based on the experimental design.

(iii) Generate a large set $\mathcal{S} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ and predict the response of $g^{(K)}$: $\mu_f(\mathbf{x}_i)$ and $\sigma_f^2(\mathbf{x}_i)$.

(iv) Estimate the failure probability based on the current meta-model, as well as a confidence interval $[\tilde{\rho}_f^-, \tilde{\rho}_f^+]$:

$$
\tilde{\rho}_f^- = \mathbb{P}(\mu_f(\mathbf{x}) \leq 0),
$$

$$
\tilde{\rho}_f^\pm = \mathbb{P}(\mu_f(\mathbf{x}) \mp 2\sigma_f(\mathbf{x}) \leq 0)
$$

(v) Check for convergence:

$$
\frac{\tilde{\rho}_f^+ - \tilde{\rho}_f^-}{\tilde{\rho}_f^-} \leq \varepsilon_p
$$

If it is not fulfilled, continue with step (vi), otherwise stop the adaptive design algorithms and return the last meta-model $g^{(K)}$. Note that a value of $\varepsilon_p = 5\%$ leads to accurate results at a reasonable cost (Schöbi et al., 2016).

(vi) Enrich the experimental design by a single sample $\mathbf{x}^* \in \mathcal{S}$ by maximizing the probability of misclassification:

$$
\mathbf{x}^* = \arg \max_{\mathbf{x} \in \mathcal{S}} \Phi\left(-\frac{|\mu_f(\mathbf{x})|}{\sigma_f(\mathbf{x})}\right),
$$

where $\Phi$ is the CDF of a standard normal variable. Then, $\mathbf{X} \leftarrow \{\mathbf{X}, \mathbf{x}^*\}$.

(vii) Compute the corresponding limit-state function value $y^* = g(\mathbf{x}^*)$ and add it to the vector $y$. Return to step (ii).

After the termination of the adaptive design algorithm, the failure probability is estimated with the last meta-model $g^{(K)}$ (see Eq. (10)).

3.3 Failure probability for parametric p-boxes

In contrast to random variables, parametric p-boxes have interval-valued distribution parameters. Hence, the distribution for the structural reliability analysis is not unique. In order to obtain the extreme values for the failure probability, two optimization operations are necessary:

$$
P_f = \min_{\boldsymbol{\theta}_l \in \mathcal{G}^{l}} \int_{D_f} f_X(x|\boldsymbol{\theta}) \, dx,
$$
\[
\overline{\bar{f}} = \max_{\bar{\theta} \in [\bar{\theta}_i, \bar{\theta}_j]} \int_{\Delta_f} f_X(x|\theta) \, dx,
\]

where \( i = 1, \ldots, n_\theta \) and \( f_X(x|\theta) \) is the conditional joint probability density function of the input variables.

Hence, the estimation of the boundary values of the failure probability consist of estimating (i) conditional failure probabilities, denoted by \( \overline{\bar{f}}_f \), and (ii) the optimization on \( \bar{\theta} \), as shown in Figure 1. The conditional failure probabilities can be estimated as discussed in Section 3.2.3 with the help of AK-MCS, whereas the optimization can be done by e.g. the efficient global optimization (EGO) method (Jones et al. 1998).

Figure 1: Failure probability estimation for parametric p-boxes

EGO is an optimization method that is also based on Kriging and consists of an algorithm similar to AK-MCS: an adaptive experimental design enrichment strategy facilitates the search for the optimal point in the parameter space. The main steps are summarized here briefly:

(i) Generate a small initial experimental design \( \mathcal{T} = \{\tau^{(i)}, \ldots, \tau^{(k)}\} \in \mathcal{D}_\theta \) in the distribution parameter space and compute the corresponding conditional failure probability \( \mathcal{P}^{(i)} = \mathcal{P}_f(\tau^{(i)}) \) by AK-MCS.

(ii) Train a Kriging model \( \mathcal{M}^{(K)} \) based on the experimental design.

(iii) Search for the optimal point for enriching the experimental design for the minimization and maximization, respectively:

\[
\tau^*_{\text{min}} = \arg\max_{\bar{\theta} \in [\bar{\theta}_i, \bar{\theta}_j]} \text{EI}_{\text{min}}(\theta),
\]

\[
\tau^*_{\text{max}} = \arg\max_{\bar{\theta} \in [\bar{\theta}_i, \bar{\theta}_j]} \text{EI}_{\text{max}}(\theta),
\]

where the expected improvement (EI) is defined as follows:

\[
\text{EI}_{\text{min}}(\theta) = \left( \mathcal{P}_{\text{min}} - \hat{\mu}_f(\theta) \right) \Phi \left( \frac{\mathcal{P}_{\text{min}} - \hat{\mu}_f(\theta)}{\sigma_{\mathcal{P}_f(\theta)}} \right) + \sigma_{\mathcal{P}_f(\theta)} \Phi \left( \frac{\mathcal{P}_{\text{min}} - \hat{\mu}_f(\theta)}{\sigma_{\mathcal{P}_f(\theta)}} \right),
\]

\[
\text{EI}_{\text{max}}(\theta) = \left( \hat{\mu}_f(\theta) - \mathcal{P}_{\text{max}} \right) \Phi \left( \frac{\hat{\mu}_f(\theta) - \mathcal{P}_{\text{max}}}{\sigma_{\mathcal{P}_f(\theta)}} \right) + \sigma_{\mathcal{P}_f(\theta)} \Phi \left( \frac{\hat{\mu}_f(\theta) - \mathcal{P}_{\text{max}}}{\sigma_{\mathcal{P}_f(\theta)}} \right),
\]

where \( \mathcal{P}_{\text{min}} = \min_i \mathcal{P}^{(i)} \) and \( \mathcal{P}_{\text{max}} = \max_i \mathcal{P}^{(i)} \), \( \Phi(\cdot) \) and \( \phi(\cdot) \) are the CDF and PDF of a standard normal variable, respectively.

(iv) Check the convergence of the additive algorithm for the minimization and maximization procedure, respectively, e.g.:

\[
\text{EI}_{\text{min}}(\tau^*_{\text{min}}) \leq 0.001,
\]
EI_{\text{max}}(\tau_{\text{max}}^*) \leq 0.001. \quad (21)

If the convergence criterion is fulfilled, return the optimal failure probability: \( P_f = \min_{i} p^{(i)}, \quad \bar{P}_f = \max_{i} p^{(i)}. \) Otherwise, continue with step (v).

(v) Add the best next sample(s) to the experimental design and compute the conditional failure probability for \( \tau_{\text{min}}, \tau_{\text{max}}. \)

Note that there are a number of vectors \( \theta \) where AK-MCS computations are required to estimate conditional failure probabilities. When doing so, any previous evaluations of the limit-state function \( G \) can be used in the initial experimental design of the next AK-MCS analysis to speed up convergence and to limit the total number of limit-state function evaluations.

3.4 Failure probability for free p-boxes

In the case of free p-boxes, the structural reliability analysis problem can be recast as two structural reliability analyses as previously discussed in Schöbi & Sudret (2015). Considering a random vector \( C \) of length \( M \) following a uniform distribution in the unit-hypercube domain \([0,1]^M\), the boundaries of the limit-state function p-box are obtained by:

\[
\bar{Y} = G(c) = \min_{x_i \in [\bar{x}_i, \bar{x}_i]} G(x), \quad (22)
\]

\[
\underline{Y} = \bar{G}(c) = \max_{x_i \in [\underline{x}_i, \bar{x}_i]} \bar{G}(x), \quad (23)
\]

where \( \bar{x}_i = F_{X_i}^{-1}(c_i), \underline{x}_i = F_{X_i}^{-1}(c_i) \) and \( i = 1, ..., M \), assuming that \( C_i \) describes the CDF value of \( X_i \) and that the input variables are statistically independent. Then, the extreme values of the failure probability are obtained by \( P_f = \mathbb{P}(\bar{Y} \leq 0) = \mathbb{P}(\bar{G}(C) \leq 0) \) and \( \bar{P}_f = \mathbb{P}(\bar{Y} \leq 0) = \mathbb{P}(\bar{G}(C) \leq 0) \). Hence, the extreme failure probabilities can be estimated as a function of the probabilistic vector \( C \), allowing for random sampling MCS and AK-MCS.

Considering Eq. (22) and (23), the main contributors to the total computational costs are (i) the expensive-to-evaluate limit-state function \( G \), (ii) the \( M \)-dimensional optimization operations in the model conversion, and (iii) the estimation of the boundaries of the failure probability by Monte Carlo sampling. In order to limit the total computational costs, Schöbi & Sudret (2016) proposed to implement a two-level meta-modelling scheme which is sketched in Figure 2.

![Figure 2: Failure probability estimation for free p-boxes](image)

The first-level meta-model surrogates the original limit-state function \( G \) with the final Kriging model of an AK-MCS analysis. As the input is defined by p-boxes in the \( D_X \) domain, an auxiliary...
probabilistic input vector $\bar{X}$, characterized by its CDF $\bar{F}_{X_i} = \frac{1}{2}(F_{X_i} + \bar{F}_{X_i})$, is used to model the limit-state surface in $\bar{G}$. The resulting meta-model is denoted as $\bar{G}^{(K)}$. On the second level, two AK-MCS analyses are used to estimate efficiently the boundary values of the failure probability based on the limit-state functions $\bar{G}$ and $\bar{G}$, respectively, as a function of the probabilistic input vector $C$. For further details, it is referred to Schöbi & Sudret (2016).

### 3.5 Combination and comparison

Sections 3.3 and 3.4 dealt with the case where either parametric or free p-boxes are present. However, if both types of p-boxes appear in a problem setting for different variables, the corresponding concepts are combined to one.

Denote the variables modelled by parametric p-boxes as $X^{(p)}$ and the ones modelled by free p-boxes as $X^{(f)}$, such that $X = \{X^{(p)}, X^{(f)}\}$ and $|X^{(p)}| + |X^{(f)}| = M$. Then, for a given $\theta \in \mathcal{D}_{\Theta}$, the boundaries of the failure probability can be estimated by the two-level approach described in Section 3.4 in the context of free p-boxes. The resulting boundary values of the failure probability are conditional on the parameters $\theta$, i.e. $P_{f|\theta}$ and $\bar{P}_{f|\theta}$. In a second stage, the minimum value of $P_{f|\theta}$ and the maximum value of $\bar{P}_{f|\theta}$ are obtained by applying EGO twice in the parameter space $\mathcal{D}_{\Theta}$, as described in Section 3.2.

The procedure is summarized in Figure 3. It consists of three levels of meta-modelling: a Kriging model of $G$ (defined in the auxiliary input space $\mathcal{D}_{\bar{X}}$), Kriging models of $\bar{G}$ and $\bar{G}$ (as a function of $C$), and finally two Kriging models for optimization (for $P_{f|\theta}$ and $\bar{P}_{f|\theta}$ in $\mathcal{D}_{\Theta}$). Note that evaluations of the true limit-state function $G$ appear only for $G^{(K)}$. The evaluations of $G$ can be recycled in the different iterations of EGO as discussed in Section 3.3.

![Figure 3](image)

**Figure 3**: Failure probability estimation for mixed free and parametric p-boxes at the same time

### 4 Application

#### 4.1 Hat function

The hat function is an analytical function defined as (Schöbi et al., 2016):

$$g(x) = 20 - (x_1 - x_2)^2 - 8(x_1 + x_2 - 4)^3,$$

(24)
where \( x_i, i = 1, 2 \) is modelled by different types of p-boxes depending on the following cases: (a) both parametric p-boxes, (b) both free p-boxes, and (c) \( X_1 \) as a parametric p-box and \( X_2 \) as a free p-box. In all cases, the parametric p-box of \( X_i \) is defined as Gaussian variables with mean value \( \mu_{X_i} \in [0.5, 1.5] \) and standard deviation \( \sigma_{X_i} = 1.0 \). The boundary CDFs of the free p-boxes are given by \( F_{X_i}(x_i) = F_{X_i}(x_i|\mu = 1.5, \sigma = 1.0) \) and \( F_{X_i}(x_i) = F_{X_i}(x_i|\mu = 0.5, \sigma = 1.0) \). Note that the input variables are assumed to be independent. Moreover, failure is defined as \( g(x) \leq 0 \) and hence \( P_f = P(g(X) \leq 0) \). Figure 4 illustrates the limit-state surface \( G(x) = 0 \) and the boundary curves for the parametric and free p-boxes of \( X \).

![Figure 4: Hat function – limit-state surface and p-box of the input parameters \( X_1 \) and \( X_2 \)](image)

For all three cases, the Kriging meta-models in AK-MCS use Gaussian correlation functions, a constant trend \( f(x) = 1 \) (i.e. ordinary Kriging), and an initial experimental design of \( N_0 = 12 \) Latin-hypercube samples. The settings of EGO are the same, except an initial experimental design of \( N_0 = 4 \) samples due to the expected smoothness of the map \( \theta \rightarrow P_f|\theta \).

The results are summarized in Table 1 in terms of failure probability and number of samples in each meta-model. \( N_1 \) denotes the number of calls to the exact limit-state function \( G \). \( N_2 \) and \( N_3 \) refer to algorithm-specific statistics, i.e. the number of calls to \( \{G, G\} \) (for free p-boxes) and the number of samples in EGO (for parametric p-boxes) respectively. In order to get statistically significant results, 50 independent replications of the same analysis are done with different initial experimental designs. Note that the reference value for the failure probability is obtained by Monte Carlo-based Importance Sampling with \( n = 10^6 \) samples.

<table>
<thead>
<tr>
<th>( P_{f, \text{ref}} )</th>
<th>( \mathbb{E}[\hat{\beta}_f] )</th>
<th>( \text{Cov}[\hat{\beta}_f] )</th>
<th>( \mathbb{E}[N_1] )</th>
<th>( \mathbb{E}[N_2] )</th>
<th>( \mathbb{E}[N_3] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10.1 \cdot 10^{-3} )</td>
<td>( 10.0 \cdot 10^{-3} )</td>
<td>( 1.3% )</td>
<td>( 20.1 )</td>
<td>( - )</td>
<td>( - )</td>
</tr>
<tr>
<td>( [1.28, 52.3] \cdot 10^{-3} )</td>
<td>( [1.31, 52.1] \cdot 10^{-3} )</td>
<td>( [8.0%, 6.6%] )</td>
<td>( 24.5 )</td>
<td>( [25.6, 26.1] )</td>
<td>( [7.5, 5.68] )</td>
</tr>
<tr>
<td>( [1.22, 52.9] \cdot 10^{-3} )</td>
<td>( [1.21, 53.1] \cdot 10^{-3} )</td>
<td>( [3.8%, 2.3%] )</td>
<td>( 20.1 )</td>
<td>( [25.0, 30.4] )</td>
<td>( [5.0, 5.0] )</td>
</tr>
<tr>
<td>( [1.24, 52.6] \cdot 10^{-3} )</td>
<td>( [1.26, 52.6] \cdot 10^{-3} )</td>
<td>( [3.7%, 1.2%] )</td>
<td>( - )</td>
<td>( - )</td>
<td>( - )</td>
</tr>
</tbody>
</table>

As seen in Table 1, the boundary values of the failure probability are estimated accurately in all cases by means of the mean value of the 50 replications (\( \mathbb{E}[\hat{\beta}_f] \)), compared to the reference value. The coefficient of variation \( \text{Cov}[\cdot] \) (computed from the 50 replications) has low values which indicates reliable results when changing the initial experimental design. Although the limit-state function is non-monotone, the failure probability intervals across the three p-box cases are almost coinciding. Hence, the regions of non-monotonicity of the limit-state surface are away from the probability mass responsible for failure (see the limit-state surface in Figure 4).
Moreover, the number of model evaluations is summarized in Table 1. The total number of calls to the limit-state function \( g \), denoted by \( N_1 \), is low and comparable for all cases. \( N_2 \) and \( N_3 \) are low numbers too, which indicates that the Kriging meta-models are also efficient on the second (and third) level of the optimization processes. In fact, one or two samples are required in EGO to get convergence in a large number of repetitions in cases (a) and (c).

### 4.2 Truss structure

Consider a two-dimensional, truss structure subjected to vertical loads as sketched in Figure 5 (Blatman & Sudret, 2010). The quantity of interest is the deflection of the truss at mid-span, denoted by \( u \). The limit-state function is defined as \( g_{\text{truss}}(x) = 0.12m - u(x) \). The input vector consists of 10 independent variables describing loads \( (P_1, ..., P_6) \), material parameters \( (E_1, E_2) \), and geometrical quantities \( (A_1, A_2) \). The input marginals are defined in Blatman & Sudret (2010) in the probabilistic context. In this section, we represent an extension using p-boxes. In particular, the parametric p-boxes are defined by distributions of which the mean value and standard deviation are varied as \( \mu_{X_i}^{(p)} \in [\mu_{X_i} - \alpha, \mu_{X_i} + \alpha] \) and \( \sigma_{X_i}^{(p)} \in [\sigma_{X_i} - \alpha, \sigma_{X_i} + \alpha] \), respectively. Free p-boxes are defined by the boundary curves for the CDF:

\[
F_{X_i}(x_i) = \min_{\mu_{X_i}^{(p)}, \sigma_{X_i}^{(p)}} F_{X_i}(x_i | \mu_{X_i}^{(p)}, \sigma_{X_i}^{(p)}) \quad \text{and} \quad \bar{F}_{X_i}(x_i) = \max_{\mu_{X_i}^{(p)}, \sigma_{X_i}^{(p)}} F_{X_i}(x_i | \mu_{X_i}^{(p)}, \sigma_{X_i}^{(p)})
\]

The parameter is set to \( \alpha = 0.05 \) for all loads and \( \alpha = 0.02 \) for the remaining variables.

![Figure 5: Sketch of the truss structure](image)

The same settings for structural reliability analysis are used as in Section 4.1, except the initial experimental design for AK-MCS and EGO are chosen to have \( N_0 = 12 \) samples. Three cases are investigated: (a) all variables are modelled by parametric p-boxes, (b) all variables are modelled by free p-boxes, and (c) loads \( P_1, ..., P_6 \) are modelled as free p-box, whereas \( E_1, E_2, A_1, A_2 \) are modelled by parametric p-boxes.

The proposed algorithms result in (a) \( \bar{P}_f \in [1.4 \cdot 10^{-4}, 1.0 \cdot 10^{-2}] \), (b) \( \bar{P}_f \in [1.1 \cdot 10^{-4}, 1.2 \cdot 10^{-2}] \), and (c) \( \bar{P}_f \in [1.1 \cdot 10^{-4}, 1.2 \cdot 10^{-2}] \), respectively. The ranges of failure probabilities are similar for all three cases. Hence, the decision on free or parametric p-boxes does not affect the failure probability considerably. The reason for this behavior lies the in the monotonicity of the limit-state function. As a result the input CDFs leading to the extreme failure probabilities are very similar for all three cases. A reference structural reliability analysis consisting of Importance Sampling analyses \( (n = 10^6) \) resulted in the following failure probabilities: (a) \( P_f \in [1.2 \cdot 10^{-4}, 1.1 \cdot 10^{-2}] \), (b) \( P_f \in [1.1 \cdot 10^{-4}, 1.2 \cdot 10^{-2}] \), and (c) \( P_f \in [1.1 \cdot 10^{-4}, 1.2 \cdot 10^{-2}] \). These values confirm the results obtained by the proposed approaches and show their accuracy in estimating boundary values of the failure probability.

The corresponding total number of model evaluations for the proposed approach is (a) \( N_1 = 12 + 274 = 286 \), (b) \( N_1 = 12 + 122 = 134 \), and (c) \( N_1 = 12 + 316 = 328 \), respectively. For all three cases, the number of model evaluations is small compared to the large candidate sample set of \( |S| = 10^6 \), which confirms the efficiency of the proposed approaches. Note that in cases (a) and (c), the number of model evaluations is larger than in case (b). The iterative refinement strategy for parametric p-boxes (see Figure 1) leads to larger \( N_1 \) compared to the sequential meta-
modelling strategy for free p-boxes (see Figure 2). When applying the sequential strategy for parametric p-boxes, \( N_1 \) would be identical to the case of free p-boxes.

5 CONCLUSION

In engineering practice, datasets are typically sparse due to high costs of data acquisition. Hence, apart from aleatory uncertainty (natural variability), also epistemic uncertainty (lack of knowledge) is present. Probability-boxes (p-boxes) are one class of methods to describe the uncertainty in variables taking into account both types of uncertainty. Two types of p-boxes are discussed in this paper: parametric and free p-boxes.

This paper discusses the estimation of failure probabilities in the context of p-boxes. P-boxes increase the complexity and hence the computational costs of imprecise structural reliability analyses. In order to reduce the computational effort, Kriging meta-models are applied at various stages of the failure probability estimation. The efficiency of the failure probability estimation is improved by applying adaptive experimental design algorithms such as AK-MCS and EGO. This allows for an efficient estimation of imprecise failure probabilities in the context of parametric and free p-boxes and when both types of p-boxes are present in the input vector.

The capabilities of the proposed algorithms are illustrated on a benchmark analytical function and the deflection of a structural truss. A small number of evaluations of the limit-state function is sufficient to estimate the bounds of the failure probability accurately.

REFERENCES


