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Heteroclinic Connection of Periodic Solutions of Delay Differential Equations

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Abstract For a certain class of delay equations with piecewise constant nonlinearities we prove the existence of a rapidly oscillating stable periodic solution and a rapidly oscillating unstable periodic solution. Introducing an appropriate Poincaré map, the dynamics of the system may essentially be reduced to a two dimensional map, the periodic solutions being represented by a stable and a hyperbolic fixed point. We show that the two dimensional map admits a one dimensional invariant manifold containing the two fixed points. It follows that the delay equations under consideration admit a one parameter family of rapidly oscillating heteroclinic solutions connecting the rapidly oscillating unstable periodic solution with the rapidly oscillating stable periodic solution.

Keywords Delay differential equations · Periodic solutions · Heteroclinic connection

Mathematics Subject Classifications (2000) 34K17 · 34K19

1 Introduction

In this paper we consider differential equations with constant delay of the form

$$\dot{x} = \mu(-x + f(x(t-1))) \tag{1}$$

where f is a piecewise constant function. The *solution of* (1) *with initial condition* $x_0 \in C([-1, 0])$ is the function $x \in C([-1, \infty))$ satisfying

- $x|_{[-1,0]} = x_0$
- x is piecewise differentiable on $t > 0$ and on every differentiable piece Eq. 1 is fulfilled.

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More specifically, we assume that the nonlinearity f in (1) is a piecewise constant, symmetric, positive feedback function of the form

$$f(x) = \begin{cases} -b, & \text{for } x \in I_1 = (-\infty, -1] \\ -a, & \text{for } x \in I_2 = (-1, 0) \\ 0, & \text{for } x = 0 \\ a, & \text{for } x \in I_3 = (0, 1) \\ b, & \text{for } x \in I_4 = [1, \infty) \end{cases} \tag{2}$$

with parameters $a > b > 1$.

Note: If we have two initial functions $z_1, z_2 \in C([-1, 0])$ with $z_1(0) = z_2(0)$ such that $f(z_1(t)) = f(z_2(t))$ for every $t \in [-1, 0]$, then the solutions x_1 and x_2 of (1) with $x_i|_{[-1,0]} = z_i$ are identical on $t > 0$. We have $f(z_1(t)) = f(z_2(t))$ if and only if $z_1(t)$ and $z_2(t)$ are in the same interval I_i or if $z_1(t) = z_2(t) = 0$.

We remark that as f is piecewise constant $f(x(t))$ is so too. If $f(x(t)) = c \in \mathbb{R}$ on some $[s_1 - 1, s_2 - 1]$, then the delay equation (1) on the translated interval $[s_1, s_2]$ is given by

$$\dot{x}(t) = -\mu x(t) + \mu \cdot c$$

and therefore has the solution

$$x(t) = c + (x(s_1) - c) \cdot e^{-\mu(t-s_1)}. \tag{3}$$

We say that x is an *exponential arc with limit c* in the interval $[s_1, s_2]$, or a *c-arc*, for short.

Therefore, solutions of (1) with f of type (2) consist of exponential arcs that are continuously glued together.

For the time ΔT needed for a c-arc to increase (resp. decrease) from $x = x_0$ to $x = x_1$, we get using (3) the following, often used formulas

$$e^{-\mu\Delta T} = \frac{x_0 - c}{x_1 - c} \qquad \text{or} \qquad \Delta T = \frac{1}{\mu} \log\left(\frac{x_1 - c}{x_0 - c}\right). \tag{4}$$

We will always consider the case of μ being sufficiently large. This will lead to rapidly oscillating solutions, i.e. solutions such that the distance between two roots is less than one. We begin by showing the existence of periodic solutions.

2 Existence of Periodic Solutions

We show that for suitable parameters $a > b > 1$ Eq. 1 admits an unstable as well as a stable rapidly oscillating periodic solution provided μ is sufficiently large.

To this end we restrict the set of initial functions to $\Phi = \{\varphi \in C([-1, 0])$ such that for times $-1 = t_1 < t_2 < \cdots < t_5 < 0$ and $v > 0$ with $t_5 < t_4 + v < -T$, where $T = \frac{1}{\mu} \log\left(\frac{a+1}{a}\right)$ the following conditions hold true

- φ is an a-arc on $[t_1, t_2]$ with $\varphi(t_1) = 0$
- $\varphi(t) \geq 1$ for $t \in [t_2, t_3]$ with $\varphi(t_2) = \varphi(t_3) = 1$
- φ is a $-a$-arc on $[t_3, t_4 + v]$ with $\varphi(t_4) = 0$, $\varphi(t_5) = -1 > \varphi(t_4 + v)$
- φ is a $-b$-arc for $t \in [t_4 + v, -T]$
- φ is a $-a$-arc for $t \in [-T, 0]$ $\}$.

We define the map $S : \Phi \to \mathbb{R}^2$ by $S(\varphi) = (u, v)$, where v is the number in the definition of Φ and $u = t_4 - t_1$ is the distance between the two roots.

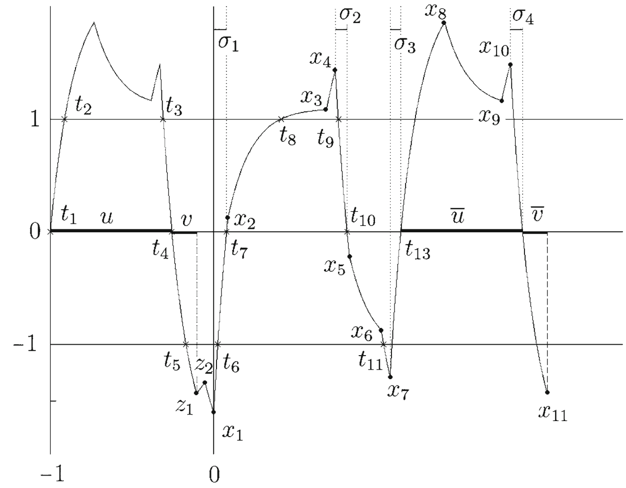

Fig. 1 Solution of type I

We first remark that for $\varphi \in \Phi$ the times t_1, t_2, t_3, t_4 and t_5 and the number $\varphi(0)$ are uniquely determined by $S(\varphi)$. To see this, note that $t_2 - t_1 = t_5 - t_4$ and $t_4 - t_3$ are fixed by a and μ. Using (4), we get $t_2 = -1 + \frac{1}{\mu}\log\left(\frac{a}{a-1}\right)$ and $t_4 - t_3 = \frac{1}{\mu}\log\left(\frac{a+1}{a}\right)$.

Thus, $S(\varphi)$ uniquely determines $f(\varphi(t))$ for $-1 \le t \le 0$ and $\varphi(0)$ and therefore also the solution x^φ of (1) for $t > 0$. We will later use this fact to reduce the problem of finding a periodic solution to the problem of finding a fixed point of a two dimensional map.

For a solution x with $x|_{[-1,0]} \in \Phi$, we define $-1 = t_1 < t_2 < \cdots$ as the consecutive times for which $x(t_i) \in \{-1, 0, 1\}$. Furthermore, we set $x_i = x(t_i + 1)$ and $T_i = t_{i+1} - t_i$.

We consider two different types of solutions. For both types we require the following properties (c.f. Fig. 1):

- $x|_{[-1,0]} \in \Phi$
- $x_2 \in (0, 1)$
- $x_3, x_4 > 1$
- $x_5 \in (-1, 0)$ (5)
- $x_7 < -1$
- $x_8, x_9, x_{10} > 1$
- $x_{11}, x_{12}, x_{13} < -1$

We say that a solution is of type I, if additionally to this $x_6 \in (-1, 0)$. A solution is of type II, if the conditions in (5) are fulfilled and $x_6 \le -1$.

The main result of this section is

Theorem 1 *Let $a > b > 1$ with $2 + b - ab > 0$. Then, the following assertions hold:*

1. *Let $p_i(a, b) > 0$ for $i = 1, \ldots, 3$ for the functions p_i defined in Table 1. Then, there exists a number $\mu_0 > 0$ such that for every $\mu > \mu_0$ there exists a unique unstable rapidly oscillating periodic solution of (1) of type I with initial conditions $\varphi_{unst} \in \Phi$ and $u \ge \frac{3}{4}$. Moreover, $S(\varphi_{unst}) = (u_{unst}, v_{unst})$ holds with $u_{unst} = 1 - \frac{1}{\mu}\log(U^1 + O(e^{-\mu/2}))$ and $v_{unst} = \frac{1}{\mu}\log(V^1 + O(e^{-\mu/2}))$, U^1 and V^1 defined in Table 1 (Fig. 2).*

Table 1 Constants and functions of Theorem 1

$d_1 = \left(a \cdot \frac{2+a+b}{a+1}\right)^2$	$d_2 = (a-b)a\frac{2+a+b}{a+1}$	$d_3 = \frac{a^2}{a+1} \cdot \left(2+a+b+\frac{2(ab-1)}{a-1}\right)$
$e_1 = \frac{a-1}{a+1} \cdot (2+a+b)^2$	$e_2 = (a-b)^2$	$e_3 = a \cdot \left(\frac{2(ab-1)}{a-1} - (a-b)\right)$
$U^1 := (e_1 - e_2)\frac{d_3}{d_2^2} - \frac{e_3}{d_2}$	$V^1 := \frac{d_3}{d_2}$	$g = \frac{2a(a+1)(ab-1)}{(a-1)(b-1)(2+a+b)^2}$
$h = \frac{a^2}{a-1}\left(-\frac{2(ab-1)(2+a+b)}{(a+1)^2} + \frac{2(a-b)(ab-1)}{(b-1)(a+1)} - \frac{(a-1)(2+a+b)^2}{(a+1)^2}\right)$		
$p_1(a,b) = a \cdot \frac{2+b-ab}{a^2-1} \cdot U^1 - (a-b) \cdot V^1 + a$		
$p_2(a,b) = 2a(ab-1) + a(a-1)(b-1) - (a-1)(b-1)(a-b)V^1 - \frac{a(a-1)(b-1)(2+a+b)U^1}{a+1}$		
$p_3(a,b) = \frac{2(ab-1)(b-1)}{(a+1)} - (a-1)^2 + \frac{(a-1)(b-1)(2+a+b)}{(a+1)} + \left(\frac{(a-1)^2(a-b)}{a} - \frac{(a-1)(b-1)(2+a+b)(a-b)}{a(a+1)}\right)V^1 + \left((a-1)(b-1) + \frac{(a-1)^2(b+1)}{a+1} - \frac{(a-1)(b-1)(2+a+b)^2}{(a+1)^2}\right)U^1$		

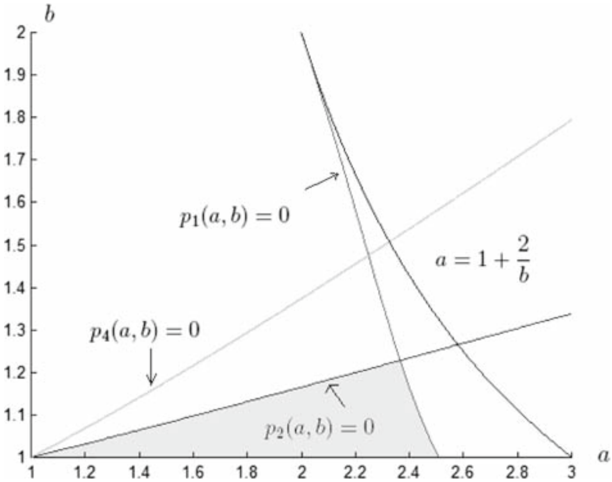

Fig. 2 Parameter for which an unstable rapidly oscillating periodic solution exists

2. *Let* $b < -a - 1 + \sqrt{3 + 4a + 2a^2}$ *and let* d_i, $i = 1, \ldots, 3$, h *and* g *be defined as in Table* 1. *Assume that*

$$d_2 g - d_3 > 0 \quad and \quad h > 0.$$

Then, there exists a number $\mu_0 > 0$ *such that for every* $\mu > \mu_0$ *there exists a unique stable rapidly oscillating periodic solutions of* (1) *of type II with* $u \geq \frac{1}{4}$. *The coordinates satisfy* $u_{stab} = \frac{1}{2} - \frac{1}{\mu}\log\left(\sqrt{(d_2 g - d_3)/h} + O(e^{-\mu/2})\right)$ *and* $v_{stab} = \frac{1}{\mu}\log(g + O(e^{-\mu/2}))$ (*Fig.* 3).

The proof of the first part of the theorem is organized as follows: In Sect. 2.1 we show that the essential dynamics of (1) with initial functions $\varphi \in \Phi$ may be reduced to a two-dimensional map Π. In Sect. 2.2 we show that Π admits a fixed point that induces a rapidly oscillating periodic solution. Finally, in Sect. 2.3, we show that the fixed point of Π is hyperbolic and thus the periodic solution is unstable.

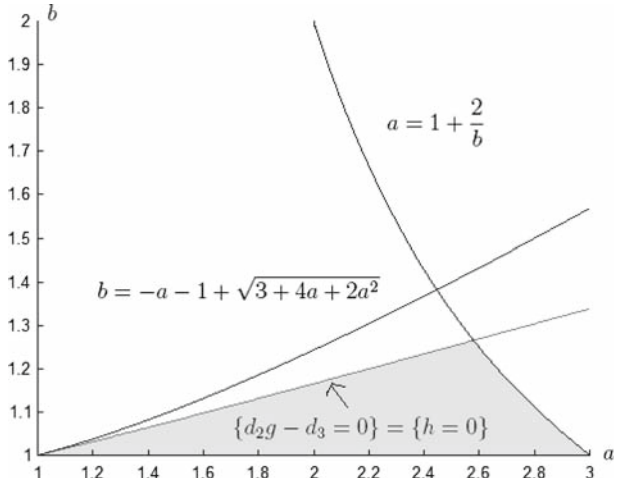

Fig. 3 Parameter for which a stable rapidly oscillating periodic solution exists

2.1 The Reduced Poincaré map

For a solution x of (1) and for $s > 0$ we define the translated function $\tau_s x$ by

$$\tau_s x : [-1, 0] \to \mathbb{R}$$
$$t \mapsto x(s + t).$$

If the solution x is of type I or II then for $s = t_{13} + 1$ the function $\tau_s x$ is in Φ too (c.f. Fig. 1). We can therefore define a Poincaré map

$$P : D \subset \Phi \to \Phi$$

with $P(z) = \tau_s x$ where x is the solution with initial function z, $s = t_{13} + 1$, and where D is the set of initial functions, leading to solutions of type I or II.

The problem of finding a periodic solution of (1) is reduced to the problem of finding a fixed point of P.

For $t > 0$, the solution x^z of (1) with initial function $z \in \Phi$ is uniquely determined by $S(z)$. Let us thus introduce the *reduced Poincaré map*

$$\Pi : D' \subset \mathbb{R}^2 \to \mathbb{R}^2$$

defined by $\Pi(u, v) = (\overline{u}, \overline{v}) = S(P(z))$, where $D' = S(D)$ is the set of parameters corresponding to initial conditions in $D \subset \Phi$. We do not have to concern ourselves with the explicit structure of D' and thus of D, as we will later work in small neighbourhoods of certain points, which we will show to lie in D' for the considered parameters a and b. In Sect. 3, we will even show that under suitable assumptions on the parameters a and b a large rectangle is contained in D' in which we will work.

For every fixed point (u^*, v^*) of Π there is a corresponding fixed point of P generating a rapidly oscillating periodic solution of (1). In the remaining part of this section we derive explicit formulas for the components of Π and expand them with respect to exponentially small terms in μ.

For the map Π we get (c.f. Fig. 1)

$$\begin{aligned}\overline{u} &= t_{10} + 1 + \sigma_4 - t_{13} = t_{10} + 1 + \sigma_4 - (t_7 + 1 + \sigma_3)\\ &= t_{10} - t_7 + \sigma_4 - \sigma_3 = t_4 + 1 + \sigma_2 - (t_1 + 1 + \sigma_1) + \sigma_4 - \sigma_3\\ &= u - \sigma_1 + \sigma_2 - \sigma_3 + \sigma_4,\end{aligned}$$

σ_1 being the time needed for an a-arc to increase from $x = x_1$ to $x = 0$, i.e.

$$\sigma_1 = \frac{1}{\mu} \log \left(\frac{a - x_1}{a} \right),$$

and similarly

$$\begin{aligned}\sigma_2 &= \frac{1}{\mu} \log \left(\frac{a + x_4}{a} \right)\\ \sigma_3 &= \frac{1}{\mu} \log \left(\frac{a - x_7}{a} \right)\\ \sigma_4 &= \frac{1}{\mu} \log \left(\frac{a + x_{10}}{a} \right).\end{aligned}$$

Therefore the first component of the Poincaré map is given by

$$\overline{u} = u + \frac{1}{\mu} \log \left(\frac{(a + x_4) \cdot (a + x_{10})}{(a - x_1) \cdot (a - x_7)} \right). \tag{6}$$

On the other hand, the second component $\overline{v}$ is the time needed for a $-a$-arc to decrease from $x = 0$ to $x = x_{11}$ and thus we have

$$\overline{v} = \frac{1}{\mu} \log \left(\frac{a}{a + x_{11}} \right). \tag{7}$$

To explicitly compute the map Π we first have to compute the values of x_i for $i = 1, \dots, 11$. For the moment we only consider solutions of type I. We will see later that for parameters a and b satisfying the assumptions of the theorem the resulting periodic solution, as well as all solutions generated by parameters close to the fixed point, are indeed of type I.

In this part, we only consider arguments (u, v) with $u > 3/4$. Then, for large μ, the terms of order $e^{-\mu u} = O(e^{-3\mu/4})$ are much smaller than the terms of order $e^{-\mu(1-u)}$ and it will be enough to compute Π up to terms of order $O(e^{-\mu u})$.

2.1.1 Computation of x_i

From Fig. 1, we see that the following holds for solutions of type I:

- x is an a-arc on the intervals $[t_1+1, t_2+1]$, $[t_3+1, t_4+1]$, $[t_7+1, t_8+1]$ and $[t_9+1, t_{10}+1]$.
- x is a b-arc on the intervals $[t_2 + 1, t_3 + 1]$ and $[t_8 + 1, t_9 + 1]$.
- x is a $-a$-arc on the intervals $[t_4 + 1, t_5 + 1]$, $[t_6 + 1, t_7 + 1]$ and $[t_{10} + 1, t_{11} + 1]$.
- x is a $-b$-arc on the interval $[t_5 + 1, t_6 + 1]$.

Furthermore, we can calculate the times T_i using formula (4)

- $T_1 = T_4 = \frac{1}{\mu} \log \left(\frac{a}{a-1} \right)$.
- $T_3 = T_6 = T_9 = \frac{1}{\mu} \log \left(\frac{a+1}{a} \right)$.
- Using $t_4 - t_1 = u$, we get $T_2 = u - \frac{1}{\mu} \log \left(\frac{a+1}{a-1} \right)$.

- $T_5 = t_6 - t_5 = 1 - u - T_4 + t_6 = 1 - u - \frac{1}{\mu}\log\left(\frac{a}{a-1}\right) + \frac{1}{\mu}\log\left(\frac{a-x_1}{a+1}\right)$ (Note that $1 - u = -t_4$).
- $T_7 = \frac{1}{\mu}\log\left(\frac{a}{a-x_2}\right) + \frac{1}{\mu}\log\left(\frac{b-x_2}{b-1}\right)$.

The time T_8 can be computed as follows

$$\begin{aligned} T_8 &= t_9 - t_8 = t_4 + 1 + \frac{1}{\mu}\log\left(\frac{a+x_4}{a+1}\right) - \left(\overbrace{t_2}^{=t_1+T_1} + 1 + \frac{1}{\mu}\log\left(\frac{b-x_2}{b-1}\right)\right) \\ &= t_4 - t_1 - T_1 + \frac{1}{\mu}\log\left(\frac{a+x_4}{a+1}\right) - \frac{1}{\mu}\log\left(\frac{b-x_2}{b-1}\right) \\ &= u - \frac{1}{\mu}\left(\log\left(\frac{a}{a-1}\right) - \log\left(\frac{a+x_4}{a+1}\right) + \log\left(\frac{b-x_2}{b-1}\right)\right), \end{aligned} \tag{8}$$

and similarly one gets

$$T_{10} = \frac{1}{\mu}\log\left(\frac{a}{a+x_5}\right) + 1 - u - \frac{1}{\mu}\log\left(\frac{a}{a-1}\right) + \frac{1}{\mu}\log\left(\frac{a-x_1}{a+1}\right) + \frac{1}{\mu}\log\left(\frac{a+x_6}{a-1}\right).$$

Therefore, one can compute the values of x_i using formula (3). We use the o-notation in a nonstandard way to denote terms of order $O(e^{-\mu u})$ having derivatives with respect to u and v of order $O(\mu e^{-\mu u})$.

We get

$$\begin{aligned} z_1 &= -a + a \cdot e^{-\mu v} \\ z_2 &= -b + \left(-(a-b)\cdot e^{-\mu(1-u-v)} + a \cdot e^{-\mu(1-u)}\right)\cdot e^{\mu T} \\ x_1 &= -a\frac{b+1}{a+1} - (a-b)\cdot e^{-\mu(1-u-v)} + a\cdot e^{-\mu(1-u)} \\ x_2 &= a + (x_1 - a)\cdot\frac{a-1}{a} = \frac{x_1(a-1)+a}{a} = 1 + \frac{a-1}{a}x_1 \\ x_3 &= b + o \\ x_4 &= a\cdot\frac{b+1}{a+1} + o \\ x_5 &= -\frac{2+b-ab}{a+1} + o \\ x_6 &= -b + \frac{2a(ab-1)}{(a-1)(a-x_1)}e^{-\mu(1-u)} + o \\ x_7 &= -a\frac{b+1}{a+1} + \frac{2a^2(ab-1)}{(a-1)(a+1)(a-x_1)}e^{-\mu(1-u)} + o \\ x_8 &= a + (x_7 - a)\cdot\frac{(b-1)(a-x_2)}{a(b-x_2)} \\ x_9 &= b + o \\ x_{10} &= a\cdot\frac{b+1}{a+1} + o \\ x_{11} &= -a + (x_{10} + a)\cdot\frac{a+1}{a-x_1}\cdot\frac{a+x_5}{a+x_6}\cdot e^{-\mu(1-u)}. \end{aligned} \tag{9}$$

For μ large enough we obviously have $x_3, x_4 > 1$ and $x_9, x_{10} > 1$. The assumption $2 + b - ab > 0$ implies $-\frac{2+b-ab}{a+1} \in (-1, 0)$ and thus $x_5 \in (-1, 0)$.

The remaining conditions a solution of type I has to fulfill do not hold for general arguments. We will later verify them for the solution generated by the fixed point of the Poincaré map.

Using the expressions in (9) for the values x_i, we can now compute $\overline{u}$ and $\overline{v}$. We get for the first component of Π

$$\begin{aligned}
\overline{u} &= u + \frac{1}{\mu}\log\left(\tfrac{(a+x_4)\cdot(a+x_{10})}{(a-x_1)\cdot(a-x_7)}\right)\\
&= u + \frac{1}{\mu}\log\left(\left(a + a\cdot\tfrac{b+1}{a+1} + o\right)\cdot\left(a + a\cdot\tfrac{b+1}{a+1} + o\right)\right)\\
&\quad - \frac{1}{\mu}\log\left((a-x_1)\cdot\left(a + a\cdot\tfrac{b+1}{a+1} - \tfrac{2a^2(ab-1)}{(a^2-1)\cdot(a-x_1)}e^{-\mu(1-u)} + o\right)\right)\\
&= u + \frac{1}{\mu}\log\left(\left(a\cdot\tfrac{2+a+b}{a+1}\right)^2 + o\right)\\
&\quad -\frac{1}{\mu}\log\left(\left(a\cdot\tfrac{2+a+b}{a+1}\right)^2 + \tfrac{a(a-b)(2+a+b)}{a+1}e^{-\mu(1-u-v)}\right.\\
&\quad \left. -\frac{a^2\cdot\left(2+a+b+\tfrac{2(ab-1)}{a-1}\right)}{a+1}e^{-\mu(1-u)} + o\right)\\
&= u + \frac{1}{\mu}\log\left(d_1 + o\right) - \frac{1}{\mu}\log\left(d_1 + d_2\cdot e^{-\mu(1-u-v)} - d_3\cdot e^{-\mu(1-u)} + o\right), \qquad (10)
\end{aligned}$$

by the definition of the constants in Table 1.

The second component is given by

$$\begin{aligned}
\overline{v} &= \frac{1}{\mu}\log\left(\frac{a}{a+x_{11}}\right)\\
&= 1 - u - \frac{1}{\mu}\log\left(\frac{a+1}{a}\cdot\frac{x_{10}+a}{a-x_1}\cdot\frac{a+x_5}{a+x_6}\right).
\end{aligned}$$

Using $a + x_5 = \frac{a-1}{a}\cdot(a+x_4)$ and the expressions for x_1, x_4, x_6 and x_{10} from (9), we get

$$\begin{aligned}
\overline{v} &= 1 - u - \frac{1}{\mu}\log\left(\tfrac{a+1}{a}\right) - \frac{1}{\mu}\log\left(\left(a + a\tfrac{b+1}{a+1} + o\right)\cdot\tfrac{a-1}{a}\left(a + a\tfrac{b+1}{a+1} + o\right)\right)\\
&\quad + \frac{1}{\mu}\log\left((a-x_1)\cdot\left(a - b + \tfrac{2a(ab-1)}{(a-1)(a-x_1)}e^{-\mu(1-u)} + o\right)\right)\\
&= 1 - u - \frac{1}{\mu}\log\left(\tfrac{a-1}{a+1}\cdot(2+a+b)^2 + o\right)\\
&\quad + \frac{1}{\mu}\log\left(a(a-b)\tfrac{2+a+b}{a+1} + (a-b)^2e^{-\mu(1-u-v)}\right.\\
&\quad \left. + a\cdot\left(\frac{2(ab-1)}{a-1} - (a-b)\right)e^{-\mu(1-u)} + o\right)\\
&= 1 - u - \frac{1}{\mu}\log\left(e_1 + o\right) + \frac{1}{\mu}\log\left(d_2 + e_2\cdot e^{-\mu(1-u-v)} + e_3e^{-\mu(1-u)} + o\right). \qquad (11)
\end{aligned}$$

Using these expressions, we can now prove the existence of a fixed point. Note that the formulas for $\overline{u}$ and $\overline{v}$ depend on the assumption $2 + b - ab > 0$. In the following this condition will always be assumed without stating it every time.

2.2 Existence of a Fixed Point of Π

We need to solve the equations

$$(u, v) = (\overline{u}, \overline{v}).$$

The first equation $\overline{u} = u$ leads to

$$u = \overline{u} = u + \frac{1}{\mu}\log(d_1 + o) - \frac{1}{\mu}\log\left(d_1 + d_2 \cdot e^{-\mu(1-u-v)} - d_3 \cdot e^{-\mu(1-u)} + o\right)$$

and therefore to

$$d_2 \cdot e^{-\mu(1-u-v)} - d_3 \cdot e^{-\mu(1-u)} + o = 0$$

or

$$e^{\mu v} = \underbrace{\frac{d_3}{d_2}}_{=:V^1} + o. \tag{12}$$

Hence for a fixed point (u, v), the second component v is of order $O(1/\mu)$.

The second equation $v = \overline{v}$ leads to a condition for u, namely

$$v = \overline{v} = 1 - u - \frac{1}{\mu}\log(e_1 + o) + \frac{1}{\mu}\log\left(d_2 + e_2 \cdot e^{-\mu(1-u-v)} + e_3 e^{-\mu(1-u)} + o\right)$$

or equivalently

$$\frac{1}{\mu}\log(e_1 + o) - (1 - u - v) = \frac{1}{\mu}\log\left(d_2 + e_2 \cdot e^{-\mu(1-u-v)} + e_3 e^{-\mu(1-u)} + o\right).$$

Taking exponentials and multiplying by $e^{\mu(1-u)}$ we get

$$(e_1 + o) \cdot e^{\mu v} = d_2 e^{\mu(1-u)} + e_2 \cdot e^{\mu v} + e_3 + o.$$

Finally, using $e^{\mu v} = d_3/d_2 + o$ yields

$$e^{\mu(1-u)} = \underbrace{(e_1 - e_2)\frac{d_3}{d_2^2} - \frac{e_3}{d_2}}_{=:U^1} + o.$$

Claim We have $U_1 > 0$ for all $3 \geq a > b > 1$.

Proof Computing U^1 results in

$$U^1(a, b) = \frac{a}{(2 + a + b)(a - b)(a^2 - 1)} z(a, b),$$

with

$$z(a, b) = 16 + 16b - 32ab - 15ab^2 - 21a^2 b + 8b^2 - 8a^2 + 3a^3 + 4a^4 + b^3 + 4a^3 b \\ + 12a^2 b^2 + a^4 b + 11a^3 b^2 - 4b^3 a + 3a^2 b^3 + a^5.$$

Thus, we have $U^1(a,b) > 0$ if and only if $z(a,b) > 0$. We use $z(1,1) = 0$ and estimate the derivatives of z by

$$\begin{aligned}\partial_a z(a,b) = &\underbrace{33a^2b^2 + 9a^2 - 42ab}_{>0} + \underbrace{4a^3b - 4b^3}_{>0} \underbrace{+16a^3 - 16a}_{>0} \\ &\underbrace{+12a^2b + 24ab^2 + 6b^3a + 5a^4 - 32b - 15b^2}_{>(12+24+6+5)ab-47b^2>0} \\ > &\ 0\end{aligned}$$

Furthermore, $\partial_b z(a,b) > 0$ for $a > b > 1$, which may be seen as follows.

$$\begin{aligned}\partial_b z(a,b) = &\ 16 - 32a - 30ab - 21a^2 + 16b + 3b^2 + 4a^3 + 24a^2b + a^4 + 22a^3b \\ &-12ab^2 + 9a^2b^2\end{aligned}$$

and thus $\partial_b z(1,1) = 0$. On the other hand, again for $a > b > 1$

$$\begin{aligned}\partial_a(\partial_b z)(a,b) &= -32 - 30b - 42a + 12a^2 + 42ab + 4a^3 + 66a^2b - 12b^2 + 18ab^2 \\ &= \underbrace{42ab - 42a}_{>0} + \underbrace{66a^2b + 18ab^2 + 12a^2 + 4a^3 - 32 - 30b - 12b^2}_{>(66+18+12+4)ab-(32+30+12)b^2>0} > 0\end{aligned}$$

and similarly

$$\begin{aligned}\partial_b(\partial_b z)(a,b) &= -30a + 16 + 6b + 24a^2 + 22a^3 - 24ab + 18a^2b \\ &> 16 + 6b + (24 + 22 + 18)a^2 - (30 + 24)a^2 \\ &> 0.\end{aligned}$$

Thus, $\partial_b z(a,b)$ is increasing with respect to a and b which implies

$$\partial_b z(a,b) > \partial_b z(1,1) = 0 \quad \text{for all } \ a > b > 1.$$

This proves the claim, as $z(a,b) > z(1,1) = 0$ for all $3 > a > b > 1$. □

Therefore, the fixed point equations are well defined and equivalent to

$$\begin{aligned}u &= 1 - \frac{1}{\mu}\log\left(U^1 + o\right) =: g_1(u,v) \\ v &= \frac{1}{\mu}\log\left(V^1 + o\right) =: g_2(u,v).\end{aligned} \tag{13}$$

We can now conclude the existence and local uniqueness of a fixed point of Π. First of all, the function $g = (g_1, g_2)$ maps a small neighbourhood of $(1 - \frac{1}{\mu}\log(U^1), \frac{1}{\mu}\log(V^1))$ onto itself if μ is large enough. As U^1 and V^1 are constants, the derivatives of g_1 and g_2 are of order $O(e^{-\mu u})$ and the map g is contracting. According to the fixed point theorem of Banach, there exists a unique fixed point (u_{unst}, v_{unst}) in this neighbourhood.

Furthermore, these calculations show that each fixed point of Π with $u > 3/4$ is in a small neighbourhood of $(1 - \frac{1}{\mu}\log(U^1), \frac{1}{\mu}\log(V^1))$, which shows the uniqueness of the unstable fixed point in the set $u \geq \frac{3}{4}$ for large enough numbers μ.

Next, we have to show that a fixed point computed as in (13) generates a solution of type I, i.e. that all conditions (5) are satisfied and $x_6 \in (-1, 0)$. Then, by continuity, the same holds for arguments close to the fixed point. We use the results of (9) to compute the different values of x_i in the fixed point. We have to verify that

- $x_1 < -1$
- $x_2 \in (0, 1)$
- $x_6 \in (-1, 0)$
- $x_7 < -1$
- $x_8, x_9, x_{10} > 1$ (or simply $x_8 > 1$ as the other inequalities then follow)
- $x_{11}, x_{12}, x_{13} < -1$ (or simply $x_{11} < -1$)

for the solution x generated by the fixed point (u_{unst}, v_{unst}) of Π. As we have $\overline{v} = v$, it follows that $z_1 = x_{11}$. One can therefore simply show that $z_1 < -1$ instead of the last condition.

Inserting the coordinates of the fixed point into the expressions of x_i, we get conditions for a and b. It is allowed to use the computed expressions to consecutively check the conditions as we use the structure of the solution one time unit earlier.

Simplifying those condition shows that for $a > b > 1$ and $a < \frac{2+b}{b}$ and for μ large enough (dependent on a and b)

- $z_1 < -1$ for all such a and b
- $x_1 < -1$ if $p_1 > 0$
- $x_6 \in (-1, 0)$, if $p_2 > 0$
- $x_7 < -1$ for all such a and b
- $x_8 > 1$, if $p_3 > 0$.

Therefore, the fixed point (u_{unst}, v_{unst}) induces a rapidly oscillating periodic solution of type I if all conditions in the theorem are fulfilled.

2.3 Hyperbolicity of the Fixed Point

It remains to show that the periodic solution is unstable. We will show that the fixed point of the Poincaré map is unstable and therefore the periodic solution is so too. Here we use that the periodic solution as well as all solutions with initial conditions close to it are of type I and thus their behavior may be studied by the use of the reduced Poincaré map. To complete the proof of the first assertion of Theorem 1 we prove the following lemma.

Lemma 1 *Let f be of the form (2) with $a > b > 1$ and $a < 1 + 2/b$.*

Let $\Pi(u, v) = (\overline{u}, \overline{v})$ be the reduced Poincaré map defined in (10) and (11).

Then, there exists $\mu_0 > 0$ such that for all $\mu > \mu_0$ the eigenvalues λ_1 and λ_2 of the Jacobian $D\Pi(u, v)$ satisfy

$$|\lambda_1| > 1 > |\lambda_2|,$$

for all $u > 3/4$ and $v > 0$ with $1 - u - v > 0$.

Proof Using (10), we get

$$\overline{u} = u + \frac{1}{\mu}\log(d_1 + o) - \frac{1}{\mu}\log\left(\underbrace{d_1 + d_2 \cdot e^{-\mu(1-u-v)} - d_3 \cdot e^{-\mu(1-u)}}_{=:n_1(a,b,u,v)} + o\right)$$

and (11) leads to

$$\overline{v} = 1 - u - \frac{1}{\mu}\log(e_1 + o) + \frac{1}{\mu}\log\left(\underbrace{d_2 + e_2 \cdot e^{-\mu(1-u-v)} + e_3 e^{-\mu(1-u)}}_{=:n_2(u,v)} + o\right).$$

Therefore, the derivatives are given by

$$\begin{aligned}
\partial_u \overline{u} &= 1 - \frac{d_2 \cdot e^{-\mu(1-u-v)} - d_3 e^{-\mu(1-u)}}{n_1} + o = \frac{d_1}{n_1} + o \\
\partial_v \overline{u} &= -\frac{d_2 \cdot e^{-\mu(1-u-v)}}{n_1} + o \\
\partial_u \overline{v} &= -1 + \frac{e_2 \cdot e^{-\mu(1-u-v)} + e_3 e^{-\mu(1-u)}}{n_2} + o = -\frac{d_1}{n_2} + o \\
\partial_v \overline{v} &= \frac{e_2 \cdot e^{-\mu(1-u-v)}}{n_2} + o.
\end{aligned} \tag{14}$$

Again o denotes an expression of order $O(e^{-\mu u})$ with derivatives of order $O(\mu e^{-\mu u})$.

It easily follows from $1 < b < a < 3$ that $e_3 > 0$ and as the other constants are naturally positive, we have $n_2 = d_2 + e_2 \cdot e^{-\mu(1-u-v)} + e_3 \cdot e^{-\mu(1-u)} > 0$ for all arguments u and v. Furthermore

Claim 1 $d_1 + d_2 - d_3 > 0$ for all $a > b > 1$.

Proof A short calculation shows

$$d_1 + d_2 - d_3 = \frac{a}{(a+1)^2}(2+a+b) \cdot (2a + a^2 - b) - \frac{2a^2(ab-1)}{a^2-1}.$$

To estimate the last term, we use that $(b+1) \cdot (a-1) = ab + a - b - 1 > ab - 1$ and get

$$\begin{aligned}
d_1 + d_2 - d_3 &> \frac{a}{(a+1)^2}(2+a+b) \cdot (a + a^2) - \frac{2a^2(a-1)(b+1)}{(a-1)(a+1)} \\
&> \frac{a^2}{a+1}(2+a+b-2(b+1)) = \frac{a^2}{a+1}(a-b) > 0
\end{aligned}$$

which proves Claim 1. □

Therefore

$$n_1 = e^{-\mu(1-u)} \left[d_1 \cdot e^{\mu(1-u)} + d_2 e^{\mu v} - d_3 + o \right] > e^{-\mu(1-u)}(d_1 + d_2 - d_3) > 0$$

and the Poincaré map given by (10) and (11) is well defined for all $u > 3/4$, $v > 0$ with $1 - u - v > 0$.

Using these estimates and the expressions for the derivatives in (14), we may obtain estimates for the determinant and the trace of the Jacobian and finally estimates of the eigenvalues.

Claim 2 For all parameters $a > b > 1$ with $a < 1 + 2/b$ we have

$$\det(D\Pi(u, v)) = o.$$

Proof We get

$$\begin{aligned}
\det(D\Pi(u, v)) &= \partial_u \overline{u} \cdot \partial_v \overline{v} - \partial_v \overline{u} \cdot \partial_u \overline{v} \\
&= \frac{d_1}{n_1} \cdot \frac{e_2 \cdot e^{-\mu(1-u-v)}}{n_2} - \left(-\frac{d_2 \cdot e^{-\mu(1-u-v)}}{n_1} \right) \cdot \left(-\frac{d_2}{n_2} \right) + o \\
&= \frac{d_1 e_2 - d_2^2}{n_1 n_2} + o = o, \text{ as } \quad d_1 e_2 - d_2^2 = 0.
\end{aligned}$$

□

Next, we prove

Claim 3 For all $a > b > 1$ with $a < \frac{2+b}{b}$

$$\mathrm{Tr}(D\Pi(u,v)) > 1 + \frac{d_2}{n_1 n_2} \cdot a(a-b)e^{-\mu(1-u)} + o \quad \text{holds.}$$

Proof

$$\begin{aligned}
\mathrm{Tr}(D\Pi(u,v)) &= \partial_u \overline{u} + \partial_v \overline{v} \\
&= 1 - \frac{d_2 \cdot e^{-\mu(1-u-v)} - d_3 e^{-\mu(1-u)}}{n_1} + \frac{e_2 \cdot e^{-\mu(1-u-v)}}{n_2} + o \\
&= 1 + \frac{e^{-\mu(1-u)}}{n_1 n_2} \Big((-d_2 e^{\mu v} + d_3) \cdot \Big(d_2 + e_2 e^{-\mu(1-u-v)} + e_3 e^{-\mu(1-u)} \Big) \\
&\quad + e_2 e^{\mu v} \cdot \Big(d_1 + d_2 e^{-\mu(1-u-v)} - d_3 e^{-\mu(1-u)} \Big) \Big) + o \\
&= 1 + \frac{e^{-\mu(1-u)}}{n_1 n_2} \Big(\underbrace{(-d_2^2 + e_2 d_1)}_{=0} e^{\mu v} + d_2 d_3 \Big) + \frac{e^{-2\mu(1-u)}}{n_1 n_2} \\
&\quad \cdot \left(-d_2 e_2 e^{2\mu v} - d_2 e_3 e^{\mu v} + d_3 e_2 e^{\mu v} + d_3 e_3 + d_2 e_2 e^{2\mu v} - d_3 e_2 e^{\mu v} \right) + o \\
&= 1 + \frac{1}{n_1 n_2} e^{-\mu(1-u)} \cdot \Big(d_2 d_3 - d_2 e_3 e^{-\mu(1-u-v)} + d_3 e_3 e^{-\mu(1-u)} \Big) + o
\end{aligned}$$

We now use the following claim which can be shown by simple computations.

Claim 4 We have $d_3 - e_3 > a(a-b)$.

Using this result, we get for μ large enough

$$\begin{aligned}
\mathrm{Tr}(D\Pi(u,v)) &> 1 + \frac{1}{n_1 n_2} d_2 \underbrace{(d_3 - e_3)}_{>a(a-b)} \cdot e^{-\mu(1-u)} + \underbrace{\frac{d_3 e_3}{n_1 n_2} e^{-2\mu(1-u)}}_{>0} + o \\
&> 1 + \frac{d_2}{n_1 n_2} \cdot a(a-b) e^{-\mu(1-u)} + o.
\end{aligned}$$

which proves Claim 3. □

We can now estimate the eigenvalues λ_1 and λ_2 of the Jacobian by

$$\begin{aligned}
\lambda_{1,2} &= \frac{\mathrm{Tr}(D\Pi)}{2} \pm \frac{1}{2} \cdot \sqrt{\mathrm{Tr}(D\Pi)^2 - 4 \underbrace{\det(D\Pi)}_{=o}} \\
&= \frac{\mathrm{Tr}(D\Pi)}{2} \pm \frac{1}{2} (\mathrm{Tr}(D\Pi) + o).
\end{aligned}$$

Thus, one eigenvalue is $\lambda_1 = \mathrm{Tr}(D\Pi) + o > 1 + d_2 \cdot (a-b)e^{-\mu(1-u)} + o > 1$ (for μ large enough). The other one is of order $\lambda_2 = o$ and therefore $|\lambda_2| < 1$ holds for μ large enough.

This proves the hyperbolicity of the fixed point, i.e. Lemma 1. □

The first part of the theorem follows immediately, as the periodic solution corresponding to an unstable fixed point of the Poincaré map is unstable too.

The second part, the existence of a stable periodic solution is proved similarly. (These calculations can be seen in [6, 91ff].)

There is however one more step. One finds a stable fixed point of the according reduced Poincaré map that gives rise to a periodic solution with initial condition $z^\star$. Then, one must show that the stability extends to more general initial conditions. More precisely, there exists a C^0-neighbourhood U of $z^\star$ such that all solutions to initial conditions in U converge orbitally to the periodic solution (see proof of Theorem 3 in [7]).

If the solution is of type II, the value of x_{13} is different and the Poincaré map is given by

$$\begin{aligned}\overline{u} &= u + \frac{1}{\mu}\Big[\log\Big(d_1 + he^{-\mu u} + O\left(e^{-\mu(1-v)}\right)\Big)\\ &\quad - \log\Big(d_1 + (d_2 e^{\mu v} - d_3)e^{-\mu(1-u)} + O(e^{-\mu})\Big)\Big] \qquad (15)\\ \overline{v} &= \frac{1}{\mu}\log\Big(g + \text{const.}\cdot e^{-\mu u} + O\left(e^{-\mu(1-v)} + e^{-2\mu u}\right)\Big), \qquad (16)\end{aligned}$$

where the constants are defined as in Table 1. It turns out that for the fixed point u is very close to $1/2$ and therefore expressions involving $e^{-\mu u}$ and $e^{-\mu(1-u)}$ have to be considered equally. Moreover, for the periodic solution we have again $v = O(1/\mu)$.

Note that the expression of $\overline{u}$ is similar to the one in the first case. Merely the terms of order $O(e^{-\mu u})$ or higher are different. One finds that the fixed point is stable and has coordinates as stated in the theorem. Again one can conclude the uniqueness of the fixed point in the considered domain, using the fixed point theorem of Banach.

3 Heteroclinic Connection

We wish to show that there exists an invariant manifold, containing both the stable and unstable fixed point from Theorem 1. This establishes an infinity of heteroclinic connections from the unstable fixed point to the stable one. We use the following theorem of Nipp and Stoffer [5] about attracting invariant manifolds.

We consider a function

$$P = (F, G) : X \times Y \ni (x, y) \mapsto (F(x, y), G(x, y)) \in B_x \times B_y$$

with B_x and B_y Banach spaces, $X \subset B_x$ and $Y \subset B_y$ open subsets, fulfilling the following hypotheses.

Hypothesis HM The functions $F \in C^0(X \times Y, B_x)$, $G \in C^0(X \times Y, B_y)$ have the following properties.

(a) $G(x, y) \in Y$ holds for all $(x, y) \in X \times Y$.
(b) For every $\overline{x} \in X$, $y \in Y$ there is $x \in X$ such that $F(x, y) = \overline{x}$.
(c) There are nonnegative constants Γ_{11}, L_{12}, L_{21} and L_{22} such that for $x, x_1, x_2 \in X$, y, $y_1, y_2 \in Y$, the functions F and G satisfy

$$\begin{aligned}|F(x_1, y) - F(x_2, y)| &\geq \Gamma_{11}|x_1 - x_2|\\ |F(x, y_1) - F(x, y_2)| &\leq L_{12}|y_1 - y_2|\\ |G(x_1, y_1) - G(x_2, y_2)| &\leq L_{21}|x_1 - x_2| + L_{22}|y_1 - y_2|.\end{aligned}$$

Hypothesis HMA There is $y^* \in Y$ such that the function $G(\cdot\,, y^*) : X \to Y$ is bounded.

Under these Hypothesises it is shown that there exists an attractive outflowing invariant manifold for the map P provided the following two conditions are satisfied.

Condition CM

$$2\sqrt{L_{12}L_{21}} < \Gamma_{11} - L_{22}\,. \tag{17}$$

Condition CMA

$$L_{22} + \Delta < 1 \text{ where } \Delta = \frac{2L_{12}L_{21}}{\Gamma_{11} - L_{22} + \sqrt{(\Gamma_{11} - L_{22})^2 - 4L_{12}L_{21}}}.$$

Remark If $\Gamma_{11} \le 1$ then condition CM implies condition CMA.

Proof We have $\Delta = \frac{2L_{12}L_{21}}{\Gamma_{11}-L_{22}+\sqrt{(\Gamma_{11}-L_{22})^2-4L_{12}L_{21}}} < \frac{2L_{12}L_{21}}{\Gamma_{11}-L_{22}}$ and thus by CM

$$\Gamma_{11} - L_{22} = \frac{(\Gamma_{11} - L_{22})^2}{\Gamma_{11} - L_{22}} > \frac{4L_{12}L_{21}}{\Gamma_{11} - L_{22}} \ge 2\Delta.$$

Therefore $L_{22} + \Delta \le \Gamma_{11} - \Delta < 1$, which proves the remark.

In [5] the following existence result for an attractive outflowing invariant manifold of the map P is proved.

Theorem 2 *Let the map P satisfy Hypotheses HM and HMA. Moreover, assume that the constants $\Gamma_{11}, L_{12}, L_{21}, L_{22}$ satisfy Conditions CM and CMA.*

Then, there exists a bounded function $s : X \to Y$ such that the following assertions hold.

(i) *The set $M = \{(x, y) \mid x \in X,\ y = s(x)\}$ is an outflowing invariant manifold of the map P, i.e., $P(M) \cap (X \times Y) = M$.*
The function s satisfies the invariance equation

$$G(x, s(x)) = s(F(x, s(x))) \tag{18}$$

for all x with $F(x, s(x)) \in X$.

(ii) *The function s is bounded and uniformly λ-Lipschitz with*

$$\lambda = \frac{2L_{21}}{\Gamma_{11} - L_{22} + \sqrt{(\Gamma_{11} - L_{22})^2 - 4L_{12}L_{21}}}.$$

(iii) *The invariant manifold M is uniformly attractive with attractivity constant*

$$\chi_A := L_{22} + \Delta < 1, \quad \Delta = \frac{2L_{12}L_{21}}{\Gamma_{11} - L_{22} + \sqrt{(\Gamma_{11} - L_{22})^2 - 4L_{12}L_{21}}} = L_{12}\lambda,$$

i.e., if $(x_0, y_0) \in X \times Y$ and $(x_1, y_1) := P(x_0, y_0) \in X \times Y$ then the inequality

$$|y_1 - s(x_1)| \le \chi_A\, |y_0 - s(x_0)|$$

holds.

(iv) *If the set $\Lambda \subset X \times Y$ is bounded with respect to y and outflowing invariant under the map P then Λ is contained in M, i.e., if $\Lambda \subset X \times Y_1$ with bounded $Y_1 \subset Y$ such that $\Lambda \subset P(\Lambda)$ then $\Lambda \subset M$.*

We wish to apply this theorem to our problem such that both the stable and the unstable fixed points lay within the domain of P. For P we would like to choose the reduced Poincaré map Π, but as we will see later, we have to change Π on a small set in order to fulfill all requirements. We will proceed as follows:

- We first choose a possible domain $(u, v) \in X_1 \times Y_1$ such that condition HM (a) is fulfilled for $P = \Pi$ if we neglect the terms of order o.

- We find estimates for the constants L_{ij} and Γ_{11} for the function Π on the set $X_1 \times Y_1$.
- We broaden the domain using a small neighbourhood of $X_1 \times Y_1$ for $X \times Y$. We extend the functions F and G on this set. We will see that we may still use $G = \overline{v}$, but we have to change $F = \overline{u}$ on the set $(X \times Y) \setminus (X_1 \times Y_1)$ to fulfill HM (b).
- We adapt the constants L_{12} and Γ_{11} for the new function F and the new domain and check that for suitable parameters a and b the necessary conditions still hold.
- Finally, we show that indeed for all parameters in this domain the corresponding solution is either of type I or of type II.

Remark The step of showing that the considered arguments lead indeed to solutions of type I or II may be done last, as all computations can be done formally for any arguments, simply using the expressions (10) and (11) if $x_6 > -1$ respectively (15) and (16) if $x_6 \leq -1$. However there is no geometrical meaning to this, unless the considered arguments lead to solutions of type I or II.

3.1 Initial Choice of the Domain

We use again the nonstandard notation of o for terms of order $O(e^{-\mu u})$ with derivatives of order $O(\mu e^{-\mu u})$.

We first have to choose a possible domain $X_1 \times Y_1$. We will choose X_1 and Y_1 (and also later X and Y) as intervals. As we want the fixed points to be in the domain, we set $X_1 = [u_{stab}, u_{unst}]$.

Choosing the domain for v we have to ensure that v_{unst} and v_{stab} lie in Y_1 and that $\overline{v}$ maps the set $X_1 \times Y_1$ onto Y_1.

From Sect. 2, we know that

$$e^{\mu v_{stab}} = g + o \quad \text{and} \quad e^{\mu v_{unst}} = V^1 + o.$$

We only consider the case where a and b are such that $g > V^1$ and therefore $v_{stab} > v_{unst}$.

We first consider the map $\overline{v}$ for solutions of type I, i.e. if $x_6 > -1$. Using (11), we get

$$\overline{v} = 1 - u - \frac{1}{\mu} \log(e_1 + o) + \frac{1}{\mu} \log\left(d_2 + e_2 \cdot e^{-\mu(1-u-v)} + e_3 e^{-\mu(1-u)} + o\right).$$

Thus, we have

$$\begin{aligned} e^{\mu\overline{v}} &= e^{\mu(1-u)} \frac{1}{e_1 + o} \cdot \left(d_2 + e_2 \cdot e^{-\mu(1-u-v)} + e_3 e^{-\mu(1-u)} + o\right) \\ &= \frac{1}{e_1 + o} \cdot \left(d_2 e^{\mu(1-u)} + e_2 \cdot e^{\mu v} + e_3 + o\right). \end{aligned} \tag{19}$$

Obviously, the map $\overline{v}$ is monotone decreasing in u and increasing in v and we find for solutions with $x_6 > -1$

$$\overline{v}(u, v) \geq \overline{v}(u_{unst}, v_{unst}) = v_{unst} \text{ for } u \leq u_{unst} \text{ and } v \geq v_{unst}.$$

We may thus use v_{unst} as the lower bound of Y_1 and set $Y_1 = [v_{unst}, v_2]$ with the upper bound v_2 yet to be determined.

To get a possible upper bound v_2 we make the following considerations:

Using the results of (9), the condition $x_6 > -1$ leads after a small calculation to

$$\begin{aligned} 2a(ab-1)e^{-\mu(1-u)} > \frac{a(a-1)(b-1)}{a+1}(2+a+b) + (a-1)(b-1) \\ \cdot \left((a-b)e^{-\mu(1-u-v)} - ae^{-\mu(1-u)}\right) \end{aligned}$$

or equivalently

$$0 < 2a(ab-1) + a(a-1)(b-1) - (a-1)(b-1)(a-b)e^{\mu v} \tag{20}$$
$$- \frac{a(a-1)(b-1)(2+a+b)}{a+1} e^{\mu(1-u)} + O\left(e^{-\mu u} e^{\mu(1-u)}\right) + o. \tag{21}$$

Thus, for solutions of type I, we conclude $1 - u = O(1/\mu)$, since otherwise the fourth term would tend to $-\infty$ for $\mu \to \infty$, while all other terms remain bounded (remember $v = O(1/\mu)$ and $u \geq u_{stab} = \frac{1}{2} + O(\frac{1}{\mu})$).

Therefore, the term $O(e^{\mu(1-u)}e^{-\mu u})$ has order $o = O(e^{-\mu u})$ and $x_6 > -1$ implies

$$\frac{a(a-1)(b-1)(2+a+b)}{a+1} e^{\mu(1-u)} < 2a(ab-1) + a(a-1)(b-1) - (a-1)(b-1)(a-b)e^{\mu v} + o.$$

Using $e^{\mu v} \geq e^{\mu v_{unst}} = V^1 + o$, we get the estimate

$$e^{\mu(1-u)} \leq \underbrace{\frac{(a+1)\cdot\left(2a(ab-1) + a(a-1)(b-1) - (a-1)(b-1)(a-b)\cdot V^1\right)}{a(a-1)(2+a+b)(b-1)}}_{=:U^2} + o. \tag{22}$$

Inserting the values of the constants and using (19), we get

$$e^{\mu\overline{v}} \leq \frac{a+1}{(a-1)(2+a+b)^2} \cdot \left[a(a-b)\frac{2+a+b}{a+1} U^2 + (a-b)^2 e^{\mu v} + a \cdot \left(\frac{2(ab-1)}{a-1} - (a-b) \right) + o \right].$$

For a possible upper bound v_2 of the domain Y_1, we find the condition

$$\left(1 - \frac{(a+1)(a-b)^2}{(a-1)(2+a+b)^2}\right) e^{\mu v_2} \leq \frac{a+1}{(a-1)(2+a+b)^2} \cdot \left(\frac{a(a-b)(2+a+b)}{a+1} U^2 + a \cdot \left(\frac{2(ab-1)}{a-1} - (a-b) \right) + o \right),$$

or as $\frac{(a+1)(a-b)^2}{(2+a+b)^2(a-1)} < 1$

$$e^{\mu v_2} \leq \underbrace{\frac{(a+1)\cdot\left(\frac{a(a-b)(2+a+b)}{a+1} U^2 + a\frac{2(ab-1)}{a-1} - a(a-b)\right)}{\left(1 - \frac{(a+1)(a-b)^2}{(a-1)(2+a+b)^2}\right)(2+a+b)^2(a-1)}}_{=:V^2} + o.$$

For the moment we neglect the terms of order o and set $v_2 := \frac{1}{\mu}\log(V^2)$ and thus $Y_1 := [v_{unst}, v_2]$.

As seen, the map $\overline{v}$ is increasing in v and thus $\overline{v}(u, v) \leq \overline{v}(u, v_2) \leq v_2 + o$ for all $(u, v) \in X_1 \times Y_1$ with $x_6 > -1$.

It remains to consider the set of parameters leading to solutions with $x_6 \leq -1$. Here $\overline{v}$ is given by $\overline{v} = \frac{1}{\mu}\log(g + o)$. We consider only the case $\frac{1}{\mu}\log(g) < v_2$. Then, for μ large enough, we have $\overline{v}(u, v) \in [v_{unst}, v_2] = Y_1$ for all arguments u and v with $x_6 \leq -1$, which shows that v_2 is a good initial choice as upper bound of Y_1.

Now that the initial domain is fixed, we proceed by estimating the constants $\Gamma_{11}, L_{12}, \ldots$ on $X_1 \times Y_1$.

3.2 Estimates for the Constants Γ_{11} and L_{ij}

As Π is continuous and piecewise differentiable, it is enough to find estimates for the derivatives.

For the moment we only consider arguments $(u, v) \in X_1 \times Y_1$. Therefore, we have $V^1 + o \leq e^{\mu v} \leq V^2 + o$.

We first consider the part of the domain for which $x_6 \geq -1$. As seen in (22), we have $U^1 + o \leq e^{\mu(1-u)} \leq U^2 + o$. This allows us to find estimates for the derivatives.

Using (14), we get

$$\begin{aligned}\partial_u \overline{u} &= 1 - \frac{d_2 \cdot e^{\mu v} - d_3}{d_1 + (d_2 \cdot e^{\mu v} - d_3)e^{-\mu(1-u)}} e^{-\mu(1-u)} \\ &= 1 - \frac{x}{d_1 + x} + o,\end{aligned}$$

for $x(u, v) = (d_2 \cdot e^{\mu v} - d_3)e^{-\mu(1-u)}$. This derivative is minimal, if $\frac{x}{d_1+x}$, or equivalently, x is maximal.

Obviously, $x(u, v)$ is increasing with respect to v. Furthermore, it can easily be checked that $d_2 \cdot e^{\mu v_2} - d_3 > 0$ for μ large enough and thus $u \mapsto x(u, v_2)$ is increasing. Thus

$$x(u, v) \leq x(u, v_2) \leq x(u_{unst}, v_2) = \frac{d_2 V^2 - d_3}{U^1} + o$$

for any $(u, v) \in X_1 \times Y_1$ and we get

$$\partial_u \overline{u} \geq \underbrace{1 - \frac{(d_2 \cdot V^2 - d_3)\frac{1}{U^1}}{d_1 + (d_2 \cdot V^2 - d_3)\frac{1}{U^1}}}_{:=\Gamma_{11}^{(1)}} + o.$$

Similarly, we can find estimates for the other derivatives. We get

$$\begin{aligned}
|\partial_v \overline{u}| &\leq \underbrace{\max\left\{\frac{d_2 \cdot V^1}{d_1 + (d_2 \cdot V^1 - d_3)\frac{1}{U^1}} \cdot \frac{1}{U^1}, \frac{d_2 \cdot V^2}{d_1 + (d_2 \cdot V^2 - d_3)\frac{1}{U^1}} \cdot \frac{1}{U^1}\right\}}_{=:L_{12}} + o \\
|\partial_u \overline{v}| &\leq \underbrace{1 - \frac{e_2 \cdot V^1 + e_3}{e_1 + (e_2 V^1 + e_3)\frac{1}{U^2}} \cdot \frac{1}{U^2}}_{=:L_{21}} + o \\
|\partial_v \overline{v}| &\leq \underbrace{\frac{e_2 \cdot V^2}{e_1 + (e_2 V^2 + e_3)\frac{1}{U^1}} \cdot \frac{1}{U^1}}_{=:L_{22}} + o.
\end{aligned}$$

Note that as $\Gamma_{11}^{(1)} < 1$, by the remark following condition CMA, it is enough to show condition CM.

Let us now consider the case $x_6 \leq -1$. As remarked, the Poincaré map is given by

$$\begin{aligned}\overline{u} &= u + \frac{1}{\mu}\left[\log\left(d_1 + he^{-\mu u} + O\left(e^{-\mu(1-v)}\right)\right)\right.\\ &\quad \left. - \log(d_1 + (d_2 e^{\mu v} - d_3)e^{-\mu(1-u)} + O(e^{-\mu}))\right]\\ \overline{v} &= \frac{1}{\mu}\log\left(g + \text{const.}\cdot e^{-\mu u} + O\left(e^{-\mu(1-v)} + e^{-2\mu u}\right)\right).\end{aligned}$$

The expression for $\overline{u}$ differs from the one in the first case only by terms of order $O(e^{-\mu u})$. But such terms can be neglected as certainly $u > \frac{1}{4}$. Therefore, we can again estimate the derivatives of $\overline{u}$ by $\Gamma_{11}^{(1)}$.

Furthermore, the derivatives of $\overline{v}$ are of order $O(e^{-\mu u})$ and have no influence on the choice of the constants L_{21} and L_{22}.

The functions $\overline{u}$ and $\overline{v}$ are continuous and piecewise differentiable. Therefore, for every $\delta > 0$ there exists μ_0 such that for $\mu > \mu_0$ the estimates in HM (c) are satisfied in the domain $X_1 \times Y_1$ with constants

$$\Gamma_{11}^{(1)} - \delta, \quad L_{12} + \delta, \quad L_{21} + \delta, \quad L_{22} + \delta.$$

We will see that the first constant has to be modified later, as we cannot use $\Pi = (F, G)$ on the whole domain $X \times Y$.

3.3 Expansion of the Domain

As the map $\overline{v}$ is continuous on the whole domain, condition HMA is fulfilled for all considered parameters a and b, as long as we choose a bounded domain. Thus, we only have to show conditions HM (a) and (b). For this, we have to expand the domain (Fig. 4). For $\varepsilon > 0$ small we use the ansatz

$$X \times Y = \left(u_{stab} - k_1\frac{\varepsilon}{\mu}, u_{unst} + k_2\frac{\varepsilon}{\mu}\right) \times \left(v_{unst} - k_3\frac{\varepsilon}{\mu}, v_2 + k_4\frac{\varepsilon}{\mu}\right), \tag{23}$$

with $0 < k_i \leq 1$ independent of ε and μ. We determine the constants k_i such that conditions HM (a) and (b) are fulfilled for a suitable continuation of Π onto $X \times Y$.

As remarked, we can still use $G(u, v) = \overline{v}(u, v)$.

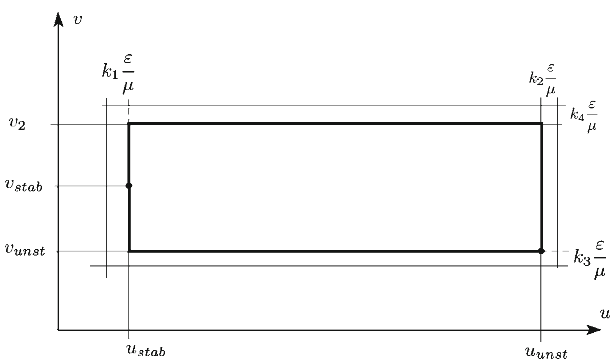

Fig. 4 Expanded domain

Condition HM (a) requires

$$G(u, v) \in Y \quad \text{for all } (u, v) \in X \times Y.$$

On the set of arguments (u, v) with $x_6 \leq -1$, we have $\overline{v} = v_{stab} + o$.

As we assumed that $v_{unst} < v_{stab} < v_2$, we have $v_{unst} < \overline{v}(u, v) < v_2$ for all (u, v) with $x_6 \leq -1$, if μ is sufficiently large.

It remains to consider arguments (u, v) with $x_6 > -1$. We have chosen v_2 in such a way that $\overline{v}(u, v_2) \leq v_2 + o$ for all arguments with $u \leq u_{unst}$ that lead to solutions with $x_6 > -1$. As $\overline{v}$ is decreasing in u, this estimate holds also for $u \leq u_{unst} + \frac{\varepsilon}{\mu}$.

As the derivative

$$\partial_v \overline{v} = \frac{e_2 e^{-\mu(1-u-v)}}{e_1 + e^{-\mu(1-u)}(e_2 e^{\mu v} + e_3)} + o$$

is continuous and in (0, 1) for all (u, v), the maximum principle implies that for every compact set $K \subset [0, 1] \times [0, 1]$ there exists a constant $C < 1$ with $0 < \partial_v \overline{v} \leq C$ for all $(x, y) \in K$.

In particular, for every $\varepsilon > 0$ small enough, there exists $C < 1$ with

$$\partial_v \overline{v}(u, v) \leq C \text{ for all } (u, v) \in \left(u_{stab} - \frac{\varepsilon}{\mu}, u_{unst} + \frac{\varepsilon}{\mu}\right) \times \left(v_{stab} - \frac{\varepsilon}{\mu}, v_2 + \frac{\varepsilon}{\mu}\right).$$

Therefore, we have for $0 \leq t \leq k_4 \frac{\varepsilon}{\mu}$

$$\begin{aligned}\overline{v}(u, v_2 + t) &= \overline{v}(u, v_2) + \int_0^t \partial_v \overline{v}(u, v_2 + s) ds \\ &\leq v_2 + o + t \cdot C \leq v_2 + o + \frac{k_4 \varepsilon}{\mu} \cdot C < v_2 + k_4 \frac{\varepsilon}{\mu}\end{aligned}$$

if μ is large enough. This shows that independently of the choice of k_4, the upper estimate of the condition $\overline{v}(u, v) \in Y$ is fulfilled. We use $k_4 = 1$.

We now find a condition for the ratio k_3/k_2 that implies the lower bound of $\overline{v}(u, v) \in Y$, i.e.

$$\overline{v}(u, v) \geq v_{unst} - \frac{k_3 \varepsilon}{\mu},$$

for all $u \leq u_{unst} + k_2 \frac{\varepsilon}{\mu}$ and $v \geq v_{unst} - k_3 \frac{\varepsilon}{\mu}$.

By the monotonicity of $\overline{v}$, we have for all $u \leq u_{unst} + \frac{k_2 \varepsilon}{\mu}$ and $v \geq v_{unst} - \frac{k_3 \varepsilon}{\mu}$

$$\overline{v}(u, v) \geq \overline{v}\left(u_{unst} + \frac{k_2 \varepsilon}{\mu}, v_{unst} - \frac{k_3 \varepsilon}{\mu}\right),$$

so it remains to show $\overline{v}\left(u_{unst} + \frac{k_2 \varepsilon}{\mu}, v_{unst} - \frac{k_3 \varepsilon}{\mu}\right) \geq v_{unst} - \frac{k_3 \varepsilon}{\mu}$. For this we use the explicit values of the derivatives at the unstable fixed point.

Let

$$|\partial_u \overline{u}(u_{unst}, v_{unst})| = K_{11} = \frac{d_1}{d_1 + (d_2 \cdot V^1 - d_3)/U^1} + o = 1 + o$$
$$|\partial_v \overline{u}(u_{unst}, v_{unst})| = K_{12} = \frac{d_2 \cdot V^1}{d_1 U^1 + d_2 \cdot V^1 - d_3} + o$$
$$|\partial_u \overline{v}(u_{unst}, v_{unst})| = K_{21} = \frac{e_1}{e_1 + (e_2 \cdot V^1 + e_3)/U^1} + o$$
$$|\partial_v \overline{v}(u_{unst}, v_{unst})| = K_{22} = \frac{e_2 \cdot V^1}{e_1 U^1 + e_2 \cdot V^1 + e_3} + o < 1.$$

By local continuity of the derivatives, we find for every $\delta > 0$ an $\varepsilon > 0$ such that for μ large enough and for all $(u, v) \in [u_{unst}, u_{unst} + \frac{\varepsilon}{\mu}] \times [v_{unst} - \frac{\varepsilon}{\mu}, v_{unst}]$ we have

$$\begin{aligned} |\partial_u \overline{u}(u, v)| &\geq 1 - \delta \\ |\partial_v \overline{u}(u, v)| &\leq K_{12} + \delta \\ |\partial_u \overline{v}(u, v)| &\leq K_{21} + \delta \\ |\partial_v \overline{v}(u, v)| &\leq K_{22} + \delta. \end{aligned} \tag{24}$$

Thus

$$\begin{aligned} \overline{v}\left(u_{unst} + \frac{k_2\varepsilon}{\mu}, v_{unst} - \frac{k_3\varepsilon}{\mu}\right) &= \overline{v}(u_{unst}, v_{unst}) + \overline{v}\left(u_{unst}, v_{unst} - \frac{k_3\varepsilon}{\mu}\right) - \overline{v}(u_{unst}, v_{unst}) \\ &\quad + \overline{v}\left(u_{unst} + \frac{k_2\varepsilon}{\mu}, v_{unst} - \frac{k_3\varepsilon}{\mu}\right) - \overline{v}\left(u_{unst}, v_{unst} - \frac{k_3\varepsilon}{\mu}\right) \\ &\geq v_{unst} - (K_{22} + \delta)\frac{k_3\varepsilon}{\mu} - (K_{21} + \delta)\frac{k_2\varepsilon}{\mu}, \end{aligned}$$

and it is enough to demand that

$$(K_{22} + \delta)\frac{k_3\varepsilon}{\mu} + (K_{21} + \delta)\frac{k_2\varepsilon}{\mu} < \frac{k_3\varepsilon}{\mu}$$

or equivalently

$$\frac{k_3\varepsilon}{k_2\varepsilon} > \frac{K_{21} + \delta}{1 - K_{22} - \delta}.$$

As $\delta > 0$ can be chosen arbitrarily small (ε and μ have to be chosen accordingly), we can fulfill condition HM (a) if

$$\frac{k_3}{k_2} > \frac{K_{21}}{1 - K_{22}}.$$

Finally, we may set $k_1 = 1$ as no further restrictions are imposed by HM (a). It remains to adapt the first component of Π such that condition HM (b) is satisfied, i.e. that $X \subset F(X, v)$ for any $v \in (v_{stab} - k_3\frac{\varepsilon}{\mu}, v_2 + \frac{\varepsilon}{\mu})$.

We cannot use $F = \overline{u}$, since the map $v \mapsto \overline{u}(u_{unst}, v)$ is strictly decreasing in v and therefore $\overline{u}(u_{unst}, v_2) < u_{unst}$. Thus, by continuity, $\overline{u}(u, v_2) < u_{unst}$ for all $u_{unst} \leq u \leq u_{unst} + \frac{\varepsilon}{\mu}$, if ε is small. In addition $\overline{u}$ is increasing in u on the whole domain and thus the former inequality holds for all $u \in X$. This shows that for small choices of ε the set X is not contained in $F(X, v_2)$. Therefore, we need to change Π for $u > u_{unst}$. On the other hand (u_{stab}, v_{stab}) is attractive and we cannot guarantee that for every v we have $u_{stab} \in \overline{u}([u_{stab}, u_{unst}], v)$

which may lead to the same problem. To be on the safe side, we also change F on $u < u_{unst}$. We set

$$F(u, v) = \vartheta(u) \cdot \overline{u}(u, v) + (1 - \vartheta(u)) \cdot u, \tag{25}$$

where ϑ is a differentiable function with $0 \leq \vartheta(u) \leq 1$ and

$$\vartheta(u) = \begin{cases} 0, & \text{in a neighbourhood of } u_{stab} - \frac{\varepsilon}{\mu} \\ 1, & \text{in a neighbourhood of } [u_{stab}, u_{unst}] \\ 0, & \text{in a neighbourhood of } u_{unst} + \frac{k_2\varepsilon}{\mu}. \end{cases} \tag{26}$$

For every $\delta > 0$, $\varepsilon > 0$ we can find a function ϑ of the form (26) such that

$$\begin{array}{ll} 0 \leq \vartheta'(u) \leq (1+\delta) \cdot \frac{\mu}{\varepsilon} & \text{for } u \in [u_{stab} - \frac{\varepsilon}{\mu}, u_{stab}] \\ 0 \geq \vartheta'(u) \geq -(1+\delta) \cdot \frac{\mu}{k_2\varepsilon} & \text{for } u \in [u_{unst}, u_{unst} + \frac{k_2\varepsilon}{\mu}]. \end{array} \tag{27}$$

For this choice of F, we clearly have

$$X \subset F(X, v)$$

for every $v \in (v_{unst} - \frac{k_3\varepsilon}{\mu}, v_2 + \frac{k_4\varepsilon}{\mu})$, i.e. condition HM (b) is satisfied.

However, we need to adapt the constants Γ_{11} and L_{12} to our new choice of F, by estimating the derivatives of F on $[u_{stab} - \frac{\varepsilon}{\mu}, u_{stab}]$ and $[u_{unst}, u_{unst} + \frac{k_2\varepsilon}{\mu}]$.

We have

$$\partial_u F(u, v) = \partial_u \overline{u} \cdot \vartheta(u) + (1 - \vartheta(u)) + \vartheta'(u) \cdot (\overline{u} - u). \tag{28}$$

Let $\delta > 0$ be fixed and ϑ be defined as above.

We begin by considering $u \in [u_{unst}, u_{unst} + \frac{k_2\varepsilon}{\mu}]$.

First of all, let $v \geq v_{unst}$.

Then, again by monotonicity of $\overline{u}$

$$\overline{u}(u_{unst}, v) - u_{unst} \leq \overline{u}(u_{unst}, v_{unst}) - u_{unst} = 0.$$

This implies

$$\begin{aligned} \overline{u}(u, v) - u &= \overline{u}(u, v) - \overline{u}(u_{unst}, v) + \underbrace{\overline{u}(u_{unst}, v) - u_{unst}}_{\leq 0} - (u - u_{unst}) \\ &\leq (\overline{u} - id_u)(u, v) - (\overline{u} - id_u)(u_{unst}, v) \end{aligned}$$

with $id_u(u, v) = u$.

Furthermore, for $v \geq v_{unst}$, we have $d_2 e^{\mu v} - d_3 \geq d_2 e^{\mu v_{unst}} - d_3 \geq o$. Therefore

$$\partial_u \overline{u} = 1 - \frac{d_2 e^{\mu v} - d_3}{n_3} + o < 1 + o$$

and thus $\partial_u(\overline{u} - u) = \partial_u \overline{u} - 1 \leq o$.

This leads to

$$\overline{u}(u, v) - u \leq \overline{u}(u_{unst}, v) - u_{unst} + |u - u_{unst}| \cdot o \leq o \cdot \frac{k_2\varepsilon}{\mu}.$$

This gives us a lower estimate of the derivative of F on the set $v \geq v_{unst}$ (using that $\partial_u \overline{u}(u,v) > \Gamma_{11}^{(1)} - \delta$ for ε sufficiently small and $1 \geq \Gamma_{11}^{(1)}$) of the form

$$\begin{aligned}\partial_u F(u,v) &= \partial_u \overline{u}(u,v) \cdot \vartheta(u) + 1 - \vartheta(u) + \vartheta'(u) \cdot (\overline{u}(u,v) - u)\\ &> \Gamma_{11}^{(1)} - \delta - (1+\delta) \cdot \frac{\mu}{k_2\varepsilon} \cdot \frac{k_2\varepsilon}{\mu} \cdot o\\ &= \Gamma_{11}^{(1)} - \delta + o.\end{aligned}$$

So on this part of the domain, we can again use $\Gamma_{11}^{(1)}$.

For the set $v \leq v_{unst}$ we use $\partial_u \overline{u} = 1 - \frac{d_2 e^{\mu v} - d_3}{n_3} + o \geq 1 + o$, again by monotonicity. Therefore, we get

$$\begin{aligned}\partial_u F(u,v) &= \partial_u \overline{u} \cdot \vartheta(u) + 1 - \vartheta(u) + \vartheta'(u) \cdot (\overline{u}(u,v) - u)\\ &> 1 + o + \vartheta'(u) \cdot (\overline{u}(u,v) - u).\end{aligned}$$

It remains to estimate $\overline{u} - u$ on $[u_{unst}, u_{unst} + \frac{k_2\varepsilon}{\mu}] \times [v_{unst} - \frac{k_3\varepsilon}{\mu}, v_{unst}]$. We can again use (24), to see

$$\begin{aligned}\overline{u}(u,v) - u &= \overline{u}(u,v) - \overline{u}(u, v_{unst}) + \overline{u}(u, v_{unst}) - \overline{u}(u_{unst}, v_{unst}) - (u - u_{unst})\\ &\leq (K_{12} + \delta) \cdot |v - v_{unst}| + (\overline{u} - id_u)(u, v_{unst}) - (\overline{u} - id_u)(u_{unst}, v_{unst})\\ &\leq (K_{12} + \delta) \cdot k_3 \frac{\varepsilon}{\mu} + \delta \cdot k_2 \frac{\varepsilon}{\mu} + o\end{aligned}$$

as $K_{11} = 1 + o$, and thus $|\partial_u \overline{u} - 1| < \delta + o$ on the considered domain.

All in all we have

$$\begin{aligned}\partial_u F(u,v) &> 1 + o - (1+\delta) \cdot \frac{\mu}{k_2} \left(\frac{k_3}{\mu} \cdot (K_{12} + \delta) + \delta \cdot \frac{k_2}{\mu} \right)\\ &= 1 - \frac{k_3}{k_2} \cdot K_{12} + o + \text{const.} \cdot \delta.\end{aligned}$$

As seen before, we need $\frac{k_3}{k_2} > \frac{K_{21}}{1 - K_{22}}$ for condition HM (a) to be satisfied. If we choose $k_3/k_2 = \frac{K_{21}}{1-K_{22}} + \text{const} \cdot \delta$, we get the estimate

$$\partial_u F(u,v) \geq 1 - \frac{K_{21} K_{12}}{1 - K_{22}} + o - \text{const} \cdot \delta,$$

with a positive constant.

Here δ can be chosen arbitrarily small (we simply have to adapt ε). We thus set

$$\Gamma_{11}^{(2)} = 1 - \frac{K_{21} K_{12}}{1 - K_{22}}.$$

Finally, we treat the case $u \in [u_{stab} - \frac{\varepsilon}{\mu}, u_{stab}]$.

Here, $\partial_u \overline{u} = 1 + o$ and $\partial_v \overline{u} = o$ which leads to

$$\partial_u F(u,v) > 1 + o + \frac{o}{\varepsilon},$$

due to (28). For given $\delta > 0$, the choice of $\varepsilon > 0$ as above is independent of μ. Using $\Gamma_{11}^{(1)} < 1$, we can thus find $\mu_0 > 0$ such that $\partial_u F(u,v) > \Gamma_{11}^{(1)}$ for all $\mu > \mu_0$ on $u \in [u_{stab} - \frac{\varepsilon}{\mu}, u_{stab}]$.

Finally for the derivative with respect to v we can use the same estimate as before because

$$|\partial_v F(u,v)| = |\vartheta(u) \cdot \partial_v \overline{u}(u,v)| \leq |\partial_v \overline{u}(u,v)|.$$

Therefore, we only have to adapt the constant Γ_{11} and we set

$$\begin{aligned} \Gamma_{11}(\delta) &= \min\left(\Gamma_{11}^{(1)}, \Gamma_{11}^{(2)}\right) - \delta, \; L_{12}(\delta) = L_{12} + \delta \\ L_{21}(\delta) &= L_{21} + \delta, \qquad\qquad\qquad L_{22}(\delta) = L_{22} + \delta. \end{aligned} \tag{29}$$

All in all, we have shown the following: For every $\delta > 0$ there exists an $\varepsilon > 0$ such that for F defined as in (25) and $G(u, v) = \overline{v}(u, v)$ the conditions HM (b) and HM (c) are fulfilled on $X \times Y$.

If the condition CM is satisfied for $\Gamma_{11}(0), L_{12}(0) \ldots$, then we can choose $\delta > 0$ small enough such that CM is also fulfilled for $\Gamma_{11}(\delta), L_{12}(\delta) \ldots$, and we can choose ε so small that condition HM (c) is fulfilled on $X \times Y$.

However, apart from the condition that CM imposes on a and b, we have to make sure that for all variables (u, v) in the domain the solution is either of type I or of type II, as only in these cases, the map Π truly represents the reduced Poincaré map.

Thus, it remains to show that for all $(u, v) \in \left(u_{stab} - \frac{k_1\varepsilon}{\mu}, u_{unst} + \frac{k_2\varepsilon}{\mu}\right) \times \left(v_{unst} - \frac{k_3\varepsilon}{\mu}, v_2 + \frac{k_4\varepsilon}{\mu}\right)$ the conditions in (5) are satisfied for ε small enough and μ sufficiently large.

This is done step by step, which allows to progressively use the structure of the solution a time unit earlier. Except for x_{11}, we can use the expressions computed in the first section. This is the only value that is different for solutions of type I and II. As we can choose ε arbitrarily small and as the functions F and G are continuous, we only have to show these conditions for $(u, v) \in [u_{stab}, u_{unst}] \times [v_{unst}, v_2]$, the rest follows by continuity.

In particular, we can use $e^{-\mu(1-u)} \leq \frac{1}{U^1} + o$ and $V^1 + o \leq e^{\mu v} \leq V^2 + o$. Then, one can see that for μ large enough, the conditions $x_3 > 1$ and $x_5 \in (-1, 0)$ are easily fulfilled if μ is large enough, if $a > b > 1$ and $2 + b - ab > 0$.

On the other hand, the other conditions in (5) do not hold for all such parameters a and b. Additional restrictions have to be imposed. More precisely

- $q_1 > 0$ implies $x_1 < -1$
- $q_2 > 0$ implies $x_2 \in (0, 1)$
- $q_3 > 0$ implies $x_7 < -1$
- $q_4 > 0$ implies $x_8 > 1$

for the numbers q_i defined in Table 2. Finally, we have to make sure that $x_{11} < -1$ for both cases. One can check that if $x_6 \leq -1$, then

$$x_{11} = -a + \left(a + a\frac{b+1}{a+1} + O(e^{-\mu u})\right) \cdot \frac{b-1}{a} \cdot \frac{a - \frac{2+b-ab}{a+1} + O(e^{-\mu u})}{b - \frac{2+b-ab}{a+1} + O(e^{-\mu u})} < -1,$$

for all $a > b > 1$ with $2 + b - ab > 0$ and $b < -a - 1 + \sqrt{3 + 4a + 2a^2}$. For $x_6 > -1$ we get the condition $q_5 > 0$, q_5 defined in Table 2. Thus, assuming additionally that $q_i > 0$, $i = 1, \ldots, 5$, shows that on the whole rectangle $X \times Y$ the corresponding solutions are either of type I and II assuming ε is small enough and thus Π represents the Poincaré map in coordinates (u, v). This leads to the main result of this paper

Theorem 3 *Let $a > b > 1$, with $2 + b - ab > 0$ and $b < -a - 1 + \sqrt{2a^2 + 4a + 3}$, satisfy the following additional conditions*

- *Condition CM holds for $\Gamma_{11}(0), L_{12}(0), L_{21}(0), L_{22}(0)$ defined as in (29).*
- *$d_2 g - d_3 > 0$ and $h > 0$.*
- *$q_i > 0$ for $i = 1, \ldots, 5$ for the constants q_i defined as in Table 2.*

Table 2 Value of the constants of Theorem 3

$$q_1 = \frac{ab-1}{a+1} \cdot U^1 + (a-b) \cdot V^1 - a \qquad q_2 = a \cdot \frac{2+b-ab}{a^2-1} U^1 - (a-b) \cdot V^2 + a$$

$$q_3 = \frac{a}{a+1}(2+a+b)U^1 + (a-b)V^1 - \frac{2a^2}{a-1} - a$$

$$q_4 = \begin{cases} k_1 + k_2 V^1 + k_3 U^1, & \text{falls } k_2 > 0 \\ k_1 + k_2 V^2 + k_3 U^1, & \text{falls } k_2 \leq 0 \end{cases}$$

$$q_5 = \frac{a(a-1)(a-b)(2+a+b)U^1 + (a^2-1)(a-b)^2 V^1 + 2(a^2+1)(ab-1) - a(a-1)(2+a+b)^2 - a(a^2-1)(a-b)}{a+1}$$

with constants

$$k_1 = \frac{2(ab-1)(b-1)}{(a+1)} - (a-1)^2 + \frac{(a-1)(b-1)(2+a+b)}{(a+1)}$$

$$k_2 = \frac{(a-1)^2(a-b)}{a} - \frac{(a-1)(b-1)(2+a+b)(a-b)}{a(a+1)}$$

$$k_3 = (a-1)(b-1) + \frac{(a-1)^2(b+1)}{a+1} - \frac{(a-1)(b-1)(2+a+b)^2}{(a+1)^2}$$

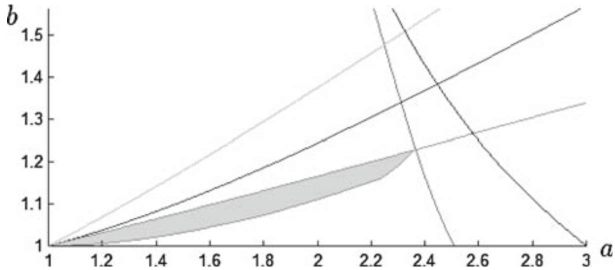

Fig. 5 Parameter for which all conditions of Theorem 3 are fulfilled

Then, there exists μ_0, so that for all $\mu > \mu_0$ the following holds:

1. *The delay differential equation (1) has a unique stable rapidly oscillating periodic solution of type II with $u > \frac{1}{4}$, induced by a stable fixed point (u_{stab}, v_{stab}) of the restricted Poincaré map Π. The estimates $u_{stab} = 1/2 + O(1/\mu)$ and $v_{stab} = O(1/\mu)$ hold.*
2. *There exists a unique unstable rapidly oscillating periodic solution of type I with $u > 3/4$, induced by an unstable fixed point (u_{unst}, v_{unst}) of the reduced Poincaré map Π. The estimates $u_{unst} = 1 - O(1/\mu)$ and $v_{unst} = O(1/\mu)$ hold.*
3. *The stable fixed point (u_{stab}, v_{stab}) and the unstable fixed point lie on a attracting invariant manifold of Π. There is a one parameter family of heteroclinic solutions connecting (u_{unst}, v_{unst}) with (u_{stab}, v_{stab}) (Fig. 5).*

Proof For $q_i > 0$, $i = 1, \ldots, 5$, all conditions in (5) are fulfilled for all $(u, v) \in X \times Y$ (defined as in (23)) for $\varepsilon > 0$ small enough, thus in particular for the fixed points. The existence and uniqueness of the rapidly oscillating periodic solutions follow from Theorem 1.

Furthermore, as seen in this section, the conditions of Theorem 3 allow us to apply theorem 2 to the function (F, G) with $G = \overline{v}$ and F defined as in (25).

Therefore, there exists an attractive invariant manifold. Since (u_{stab}, v_{stab}) is in the domain this point is also being attracted by the manifold under $P(u, v) = (F(u, v), G(u, v))$ (see Theorem 2(ii)). However (u_{stab}, v_{stab}) is fixed under P and thus lies on the invariant manifold. The same holds for the unstable fixed point.

For $u \in [u_{stab}, u_{unst}]$ the functions F and G are independent of the choice of ϑ, namely $(F(u, v), G(u, v)) = \Pi(u, v)$. We show that in this part of the domain also the invariant manifold is independent of the choice of ϑ. Let F_1 and F_2 be defined as in (25) for two

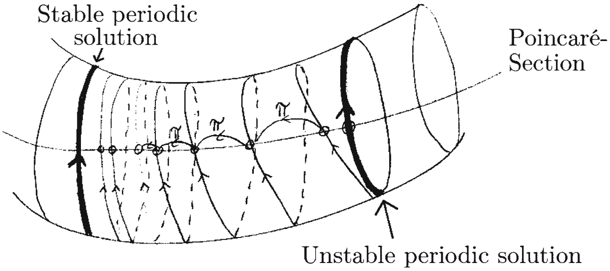

Fig. 6 A heteroclinic rapidly oscillating solution connecting the unstable and the stable periodic solution

functions ϑ_1 and ϑ_2 and let M_1 and M_2 be the corresponding invariant manifolds. Then, M_1 restricted to $[u_{stab}, u_{unst}] \times Y$ is also invariant under F_2. Using Theorem 2(iv)), we have $M_1 \cap ([u_{stab}, u_{unst}] \times Y) \subset M_2$. Interchanging M_1 and M_2, we get $M_2 \cap ([u_{stab}, u_{unst}] \times Y) \subset M_1$, implying that both invariant manifolds coincide in the significant part of the domain. □

Thus, we have proven the existence of a heteroclinic connection between the unstable and the stable rapidly oscillating periodic solution in the following sense (Fig. 6). Let φ_{unst} be the initial condition of the unstable periodic solution. Then, we have in addition to the normal behavior of a hyperbolic fixed point the following connection to the stable fixed point:

In every C^0-neighbourhood of φ_{unst}, there are initial conditions φ such that the corresponding rapidly oscillating solutions converge orbitally towards the stable rapidly oscillating periodic solution. The type of the solution changes from type I to type II. Other types of solutions do not occur.

We can construct a rapidly oscillating solution $x^*(t)$ of (1) defined on $(-\infty, \infty)$, by choosing an arbitrary initial condition φ^* on the invariant manifold connecting the stable and the unstable periodic solutions and extending the solution to $(-\infty, 0)$ by the use of Π^{-1}. This solution has the property that $x^*(t)$ converges orbitally towards

- the stable rapidly oscillating periodic solution for $t \to \infty$
- the unstable rapidly oscillating periodic solution for $t \to -\infty$

This solution shows the structure of a solution with coordinates on the invariant manifold between the fixed points, up to translations.

4 Outlook

Instead of considering piecewise constant functions, one could consider C^1 functions, as done in [7], satisfying

$$f(x) = \begin{cases} -b, & \text{for } x \in I_1 = (-\infty, -1-\varepsilon] \\ -a, & \text{for } x \in I_2 = [-1+\varepsilon, -\varepsilon] \\ a, & \text{for } x \in I_3 = [\varepsilon, 1-\varepsilon] \\ b, & \text{for } x \in I_4 = [1+\varepsilon, \infty). \end{cases}$$

This results in error terms of order ε to occur in all computations. But assuming ε to be small enough (also depending on the values of a and b), this would not change the result.

On the other hand the situation might get more complicated for more general feedback functions that are in some C^0 neighbourhood of the originally considered ones. However, due to the nature of the equation and its dependence of f, a similar result should be expected to hold.

Finally, for large values of μ, the existence of solutions with a similar structure, but with much smaller period can be shown by a rescaling argument as detailed in [7]. Showing stability of such solutions is more complicated (for an example of a similar case see [6, 33ff]). Related results for Delay equations may also be found in [1–4].

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