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Criteria for choosing Helmholtz or quarter-wave dampers to suppress combustion instabilities

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Abstract

The choice between different kinds of damping systems for the reduction of thermoacoustic instabilities is a complex task, which is mostly driven by past experience. This paper provides guidelines for the damping system designer in order to choose between quarter-wave and Helmholtz resonator design as function of the constraints on the available damping volume and purge mass flow. Both resonator types are modelled as damped harmonic oscillators with vortex shedding at the resonator mouth as the main dissipation mechanism. First, the properties of the stand-alone resonators are compared. Then, a model for the thermoacoustic behavior of combustion chamber equipped with the dampers is considered. A parametric study of the system linear stability is proposed using the Routh-Hurwitz criterion. It is shown that for given constraints and within the range of validity of the model, Helmholtz dampers have better stabilizing capabilities than quarter wave dampers.

Keywords: Helmholtz resonator, Quarter-wave resonator, Growth rate reduction, Dampers, Coupled enclosure

1. Introduction

The suppression of thermoacoustic instabilities is a key need to achieve efficient and clean combustion. The use of passive damping devices such as Helmholtz or quarter-wave resonators is a cost-effective option to prevent these combustion instabilities [1]. In this context, the choice between quarter wave and Helmholtz resonators to reduce the dynamic pressure in combustion chambers is often guided by past experience of the manufacturers. One finds examples of gas turbine combustion chambers equipped with Helmholtz resonators [2–4] or with quarter-wave resonators [5–7]. Also, the acoustic liners used in jet engines have Helmholtz-like features [8, 9] and in the field of rocket engines there are many examples of combustors equipped with quarter-wave resonator rings [10–15].

In the following study the impedance of both resonator types is modelled using a damped harmonic oscillator analogy, assuming that the vortex shedding at the resonator mouth due to the purge flow is the most contributing dissipation mechanism. Since the resonators used in rocket
engines applications are not purged with cooling flow (the main mechanism being the dissipation of acoustic energy in the boundary layers [16]), the scope of this study is limited to gas turbine and jet engines applications. In a first part the properties of the stand-alone resonators are compared for certain damping volume and purge mass flow constraints. In practice, the latter quantities are crucial because 1) the available space is often limited and the damper compactness is always of great concern and 2) the allowable purge flow results from the trade-off between the mitigation of hot-gas-ingestion risk [17] and the acceptable amount of air that will not participate to the combustion process and that will impact the engine efficiency and emissions. In a second part the resonator impedance model is included in the analytical model of a resonator coupled to an enclosure with an unstable acoustic eigenmode. This allows the extraction of analytical criteria providing direct comparison of the linear damping for different possible resonator geometries, with constraints on the damping volume and the purge mass flow, through the assessment of the system total transfer function behavior and stability limits of the coupled system. Those criteria will help making an enlightened choice of the damping device to be used for specific constraints.

2. Compared properties of the stand-alone resonators

2.1. Geometric parameters

The aim of this study is to compare quarter-wave (QW) and Helmholtz (HH) resonators having the same volume $V$ and the same resonance frequency $\omega_j$. In this whole study the influence of the end-corrections at the damper mouth will be left out, since it greatly simplifies the equations and does not change significantly the qualitative answer as to which damper geometry should be chosen, as can be seen in Appendix A. The length of the QW resonator is determined by its resonance frequency only:

$$L = \frac{\pi c}{2 \omega_j},$$

which then gives its cross-section $A = V/L$ for a fixed volume $V$. For a HH resonator of same total volume, the volume of the back cavity $V_v$ is defined as the total resonator volume $V$ minus the neck volume $al$ which is equivalent to:

$$V_v = AL - al,$$

since the QW and HH resonators have the same volume $V$. Tuning both resonators at the same frequency gives:

$$\frac{\omega_j}{c} = \frac{\omega_Q}{c} = \frac{\pi}{2L} = \sqrt{\frac{a}{(AL - al)}l} = \frac{\omega_H}{c},$$

which gives the following $2^{nd}$ order equation for $l$ if we define the aspect ratio $\alpha = a/A$ between HH neck cross-section and QW cross-section:

$$l^2 - \frac{L}{\alpha}l + \frac{4}{\pi^2}L^2 = 0,$$

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This only gives physical solutions if \( \alpha < \pi/4 \simeq 0.78 \). We notice here that \( l \) is independent of \( V \), it depends solely on the speed of sound \( c \), the resonance frequency \( \omega_j \) and the newly defined aspect ratio \( \alpha \). One can re-express the HH resonator neck cross-section as function of the problem parameters as follows:

\[
a = \alpha A = \frac{2\alpha V \omega_j}{\pi c}.
\]  

(5)

Four aspect ratio are considered to illustrate the outcome of this study, namely: \( \alpha = 1 \), defined as the QW resonator, \( \alpha = 0.75 \), close to the HH resonator limit case, and \( \alpha = 0.5 \) and \( 0.3 \) which give HH resonator shapes closer to those found in practical applications. Example geometries for those aspect ratio values \( \alpha \) are presented Fig. 1.

2.2. Impedance modelling as harmonic oscillator

In this section Helmholtz and quarter-wave resonators are modelled in the form of second order damped harmonic oscillators. One starts with the Helmholtz resonator which is sketched in Fig. 2.
Figure 3: Reflection coefficient $R$ of the different resonators (same volume and same mass flow): (a) modulus, (b) phase, and (c) quality factor as function of area ratio $\alpha$.

2, and an expression for the damper impedance is derived. Assuming plane wave propagation, the transition between back cavity and neck being in $x = 0$, and using the boundary condition $\hat{u}_1(-L_v) = 0$ one can extract the full damper impedance in the vicinity of the neck:

$$Z_d = \frac{\hat{p}_2(l)}{\hat{u}_2(l)} = \frac{\hat{p}_2(l) - \hat{p}_1(0)}{\hat{u}_2(l)} + \frac{\hat{p}_1(0)}{\hat{u}_1(0^+)}$$  \hspace{1cm} (6)

with $k$ the wave number, $Z_n$ the neck impedance, $\sigma = a/A_v$ the neck to back-cavity aspect ratio and considering that the acoustic velocity is the same across the resonator neck (i.e. compact neck): $\hat{u}_1(0^+) = \hat{u}_2(l) = \hat{u}_n$ (in practice, this assumption is quite strong in the case $\alpha = 0.75$ for which the neck length is relatively long). If one assumes that the losses are dominated by the coherent vortex shedding at the resonator mouth, the neck impedance can be modelled using the $(l - \zeta)$ model [18–20] as follows:

$$Z_n = \frac{\hat{p}_2(l) - \hat{p}_1(0)}{\hat{u}_n} = \rho c \frac{i\sigma}{\tan k L_v} \left( i k l + \zeta \frac{\bar{u}}{c} \right),$$  \hspace{1cm} (7)

with $\zeta$ the pressure loss term, and $\bar{u}$ the mean velocity through the neck. Injecting this into the expression for $Z_d$ gives:

$$Z_d = \rho c \left( i k l + \zeta \frac{\bar{u}}{c} \right) - \rho c \frac{i\sigma}{\tan k L_v},$$  \hspace{1cm} (8)

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with $L_v$ the length of the damper back cavity. For the QW resonator case, replacing the neck impedance only by a model of the pressure loss induced by the sudden expansion at the exit of the damper, one gets a similar expression with $l = 0$, $\bar{u}$ the mean velocity through the damper, $\sigma = 1$ and $L_v = L$.

2.2.1. Helmholtz resonator (HH)

For simplicity only the subscript “H” is used in the following equations. In this case $kL_v \ll 1$, thus we approximate $\tan kL_v \simeq kL_v$. Also replacing $\bar{u}$ by $\dot{\bar{m}}/\rho a$ with $\dot{\bar{m}}$ the mass flow gives the resonator impedance:

$$Z_H = \rho l \frac{s^2 + s \frac{\zeta \dot{\bar{m}}}{\rho a} + \omega_H^2}{s},$$

(9)

where $s = i\omega$ is the Laplace variable and $\omega_H = c\sqrt{\sigma/L_v}$. The HH resonator quality factor is defined as follows:

$$q_H = \frac{\omega_H \rho la}{\zeta \dot{\bar{m}}},$$

(10)

which finally gives:

$$Z_H = \rho l \left( \frac{\omega_H}{q_H} + s \left( 1 + \frac{\omega_H^2}{s^2} \right) \right).$$

(11)

2.2.2. Quarter-wave resonator (QW)

For simplicity only the subscript “Q” is used in the following equations. For the QW resonator Eq. (8) becomes:

$$Z_Q = \rho c \zeta \frac{\bar{u}}{c} - \rho c \frac{i}{\tan kL}.$$ 

(12)

In this case $kL \simeq \pi/2$, which allows us to approximate $1/\tan kL$ by $-kL + \frac{\pi}{2} \frac{\omega}{\omega_Q}$, considering the operated frequencies $\omega$ are close to that of the resonator. Using this Taylor expansion to model the QW resonator is quite uncommon in the damper literature although it allows to suppress the tangent and therefore greatly simplify the derivation and conclusions. Replacing $\bar{u}$ by $\dot{\bar{m}}/\rho A$ with $\dot{\bar{m}}$ the mass flow gives the resonator impedance:

$$Z_Q = \rho L \frac{s^2 + s \frac{\zeta \dot{\bar{m}}}{\rho LA} + \omega_Q^2}{s},$$

(13)

where $s = i\omega$ is the Laplace variable and $\omega_Q = \frac{c \pi}{2L}$ the damper resonance frequency. The QW resonator quality factor is defined as follows:

$$q_Q = \frac{\omega_Q \rho A L}{\zeta \dot{\bar{m}}},$$

(14)

which finally gives:

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Z_Q = \rho L \left( \frac{\omega_Q}{q_Q} + s \left( 1 + \frac{\omega_Q^2}{s^2} \right) \right). \tag{15}

It is interesting to consider the equivalent resistance, mass and stiffness of these dampers. Those values are presented for both resonator types in Table (1). One can remark that the resonance frequency defined as $\sqrt{K/m}$ indeed gives in each case the expected frequency, defined to be equal to $\omega_j$. The effective mass of the QW resonator corresponds to $\simeq 40\%$ of the mass of air inside the tube. For low values of $\alpha$, this is still bigger than the HH effective mass which corresponds to the mass of air in the neck. The stiffness is therefore also greater in the case of the QW than in the case of the HH resonator.

<table>
<thead>
<tr>
<th></th>
<th>$R$ [kg.s$^{-1}$]</th>
<th>$m$ [kg]</th>
<th>$K$ [kg.s$^{-2}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>QW</td>
<td>$\zeta \in \frac{4}{\pi^2} \rho LA$</td>
<td>$\rho \frac{c^2 A}{L}$</td>
<td></td>
</tr>
<tr>
<td>HH</td>
<td>$\zeta \in \rho a$</td>
<td>$\frac{\rho c^2 a^2}{V_v}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Equivalent resistance $R$, mass $m$ and stiffness $K$ for both resonator geometries

Another important quantity to consider is the corresponding reflection coefficient defined as $\mathcal{R} = (Z_d - \rho c)/(Z_d + \rho c)$ and shown Fig. 3, along with the quality factor as function of the aspect ratio $\alpha$. Notice the non-existence of the curve for $\pi/4 < \alpha < 1$, which is where one cannot define a HH resonator with same resonance frequency and same volume as a QW resonator. Since all curves have been plotted for the same mass flow, the behavior of the quality factor is straightforward and directly comes from its definition in Eqs. (10) and (14) as well as $a = \alpha A$: the lower $\alpha$, the lower the reference cross-section, and the lower $Q$.

The behavior of the reflection coefficient can thus be explained: the lower $\alpha$, the lower the HH neck cross-section and thus the higher the velocity, which explains why the sharpest peak is in the case of the QW resonator (i.e. highest cross-section means lowest velocity).

3. Interaction between resonator and acoustic mode of a cavity

3.1. Acoustically coupled model

Now that the properties of the stand-alone resonator have been highlighted for the different geometries, one is interested in their behavior when coupled to a chamber of volume $V_c$ with an unstable mode of growth rate $\nu_j$ at frequency $\omega_j$. Since we just want a clear comparison between the different geometry types and do not want to add additional parameters like the number of dampers installed which would bias the comparison, we assume there is one single resonator perfectly tuned to the frequency $\omega_j$. It has a constraint on its volume $V$ which will be compared to the volume $V_c$. This is a pre-print version. Presented at International Symposium: Thermoacoustic Instabilities in Gas Turbines and Rocket Engines, May 30 - June 02, 2016 Munich, Germany.
of the chamber $V_c$. The second constraint is on the mass flow going through the damper. In real engine applications, this mass flow would have to be compared to the total mass flow going through the engine. Since the chamber mass flow is not explicitly given in the model, the resonator mass flow is normalized by $\rho V_c \omega_j$. $\rho V_c$ corresponds to the total mass of air inside the chamber, so that the ratio $\dot{m}/\rho V_c \omega_j$ describes the fraction of the mass of air in the chamber supplied during one oscillation by the purge flow through the damper.

The coupling between a resonator and an enclosure is a topic which has been largely studied in previous work. The interested reader may consult Morse and Ingard \cite{18} for the derivation of the model, later applied to a cavity coupled to a single resonator by Fahy and Schofield \cite{21} and to a resonator array by Cummings \cite{22} and Li and Cheng \cite{23}. Bellucci \cite{20} investigated the case of Helmholtz damper used to stabilize linearly unstable acoustic eigenmodes, which was a solid basis for the subsequent investigation of Noiray and Schuermans \cite{24}. The standing waves in the cavity can be expressed with an orthonormal basis $\psi$ constituted of the different acoustic eigenmodes of the enclosure. The pressure at each location within the enclosure can be expressed in the frequency domain by:

$$p(\omega, x) = \sum_{i=1}^{\infty} \eta_i(\omega) \psi_i(x)$$  \hspace{1cm} (16)

One can approximate the modal amplitude for an unstable eigenmode $\psi_j$ as follows \cite{24}:

$$\eta_j(s) = \frac{s \rho c^2}{s^2 - 2\nu_j s + \omega_j^2} \frac{1}{V_c \Lambda_j} \left( \gamma - 1 \right) \frac{1}{\rho c^2} \int_V \hat{Q}_N(s, x) \psi_j^*(x) dV$$ 

$$- \int_{S_d} \eta_j(s) \frac{|\psi_j(x)|^2}{Z_d(s, x)} dS,$$  \hspace{1cm} (17)

with $\omega_j$ the angular frequency of the unstable mode, $\nu_j$ its growth rate, $V_c$ the cavity volume, $S_d$ the surface of the enclosure walls equipped with dampers, $Z_d$ the damper impedance, $\hat{Q}_N$ the fluctuating heat release and $\Lambda_j$ the spatial norm of the mode $j$ defined as:

$$\Lambda_j = \frac{1}{V} \int_V |\psi_j|^2 dV.$$  \hspace{1cm} (18)

The volumetric acoustic source over the whole cavity is defined as:

$$\hat{Q}_N = \frac{\gamma - 1}{-2\nu_j V_c \Lambda_j} \int_V \hat{Q}_N(s, x) \psi_j^*(x) dV,$$  \hspace{1cm} (19)

which results from the mode-shape-weighted non-coherent heat release fluctuations. We also define the transfer function of the enclosure without dampers:

$$\mathcal{H}_{wod} = \frac{-2\nu_j s}{s^2 - 2\nu_j s + \omega_j^2}.$$  \hspace{1cm} (20)

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Using the “low-frequency hypothesis” (damper neck considered acoustically compact) and assuming the general case with $N$ similar dampers of impedance $Z_d$ installed at locations $x_k$, the second integral in the right hand side of Eq. (17) becomes:

$$S_d \sum_{k=1}^{N} \frac{\psi_j^2(x_k)}{Z_d} \eta_j.$$ (21)

This yields:

$$\eta_j = H_{\mathrm{wood}} Q_N - H_{\mathrm{wood}} S_d \sum_{k=1}^{N} \frac{\psi_j^2(x_k)}{V_c \Lambda_j} \frac{\rho_c}{Z_d} \eta_j.$$ (22)

In our case since a single resonator is used then $N = 1$ and the sum simplifies to a single term $\psi_j^2(x)$ with $x$ the location of the damper. For the HH resonator, the damper covered surface $S_d$ corresponds to the neck surface $a$, and the top and bottom of Eq. (22) are multiplied by the volume of the damper back cavity (i.e. without the neck) $V_c$ to make the damper resonance frequency $\omega_H^{(2)} = \frac{\psi_j^2}{\Lambda_j} a$ appear. This gives:

$$\eta_j = H_{\mathrm{wood}} Q_N - H_{\mathrm{wood}} \frac{\varepsilon_H^{(2)} \omega_H^{(2)} S}{s^2 + s \frac{\omega_H^{(2)}}{\eta_j}} + \frac{s}{\Lambda_j} \eta_j.$$ (23)

with $\varepsilon_H$ a damping efficiency factor, depending on the volume ratio and the location of the damper $x$ in regard to the mode shape, defined as:

$$\varepsilon_H^{(2)} = \frac{V_c \psi_j^2(x)}{V_c \Lambda_j}.$$ (24)

This gives the following transfer function:

$$\frac{\eta_j}{Q_N} = \frac{-2 \nu_j s \left( s^2 + s \frac{\omega_H^{(2)}}{\eta_j} + \omega_H^{(2)} \right)}{(s^2 - 2 \nu_j s + \omega_j^2) \left( s^2 + s \frac{\omega_H^{(2)}}{\eta_j} + \omega_H^{(2)} \right) + s^2 \omega_H^{(2)} \varepsilon_H^{(2)}}.$$ (25)

In the case of the Helmholtz resonator, it is indeed the volume of the damper back cavity and not the damper total volume which plays a role in the efficiency factor. For the case of a QW resonator, one can extract the same transfer function by replacing the subscripts “$H$” with “$Q$” and with the damping efficiency factor now defined as:

$$\varepsilon_Q^{(2)} = \left( \frac{4}{\pi^2} \right) \frac{V_c \psi_j^2(x)}{V_c \Lambda_j}.$$ (26)

Depending on the shape of the mode, the spatial norm of the mode can take values comprised between $1/8 < \Lambda_j < 1/2$. One also has: $0 < \psi_j^2(x) < 1$ depending on whether the resonator is placed close to a pressure anti-node or not. For simplification purposes, in the following study the assumption is made that $\psi_j^2(x)/\Lambda_j = 1$. The new damping efficiency factors are $\varepsilon_Q^{(2)} = (4/\pi^2)(V/V_c)$ and $\varepsilon_H^{(2)} = V_c/V_c$.

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3.2. Routh-Hurwitz criterion

With the transfer function of the coupled system defined in Eq. (25), one can use the Routh-Hurwitz criterion to determine that for a damper tuned to the unstable frequency ($\omega_H = \omega_j$) the system is stable when:

\[
\begin{align*}
\omega_H q_H & \geq 2 \nu_j & \text{RH1} \\
\omega_j^2 \varepsilon_H^2 & \geq 2 \nu_j \omega_H q_H & \text{RH2}
\end{align*}
\]

Replacing the subscripts “$H$” with “$Q$” gives the equivalent Routh-Hurwitz (RH) criteria for the QW resonator. We are now going to reformulate both Routh-Hurwitz criteria as function of the real constraints (volume $V$ and mass flow $\dot{m}$) in order to build stability maps.

3.2.1. Routh-Hurwitz criterion for fixed volume

For a given volume $V$, using the expressions for $\varepsilon_Q$ and $q_Q$, the Routh-Hurwitz criteria for the QW resonator becomes:

\[
\begin{align*}
\dot{m} & \geq \frac{2 \rho V \nu_j}{\zeta} & \text{RH1} \\
\dot{m} & \leq \frac{2 \omega_j^2 V^2 \rho}{\pi^2 V \zeta} \frac{1}{\nu_j} & \text{RH2}
\end{align*}
\]

In the same manner, using the expressions for $\varepsilon_H$ and $q_H$, the Routh-Hurwitz criteria for the...
Figure 5: Stability of the system as function of unstable mode growth rate $\nu_j/\omega_j$ and normalized volume $V/V_c$ for fixed normalized mass flow constraint $\dot{m}/\rho V_c \omega_j$.

HH resonator gives:

\[
\begin{align*}
\dot{m} & \geq \frac{2\rho a_l}{\zeta} \nu_j & \text{RH1} \\
\dot{m} & \leq \frac{\omega_j^2 V_c a_l \rho}{2 V_c \zeta} \frac{1}{\nu_j} & \text{RH2}
\end{align*}
\]  

(29)

Using those new formulations one can plot the stability map of the coupled system for fixed damping volume as function of the unstable mode growth rate $\nu_j/\omega_j$ and the normalized purge flow $\dot{m}/\rho V_c \omega_j$, which can be seen in Fig. 4. For certain ranges of mass flow, one or the other geometry is best fitted to stabilize an unstable mode of the cavity. However since the resonators with the lowest values of $\alpha$ are able to stabilize an unstable mode with higher growth rate, and the mass flow is usually restrained by an upper limit only, if one has the possibility to reduce the purge mass flow then the HH resonator with the lowest $\alpha$ possible would seem to be the best choice, provided that it can still be described by a linear ($\ell - \zeta$) model.

3.2.2. Routh-Hurwitz criterion for fixed mass flow

If the fixed constraint is now the purging mass flow $\dot{m}$, one has to “reverse” the previous RH criteria to make the resonator volume appear. For the QW resonator this is straight forward:

\[
\begin{align*}
V & \leq \frac{\dot{m} \zeta}{2 \rho} \frac{1}{\nu_j} & \text{RH1} \\
V & \geq \frac{\pi}{\omega_j} \sqrt{\frac{V_c \dot{m}}{2 \rho} \frac{1}{\nu_j}} & \text{RH2}
\end{align*}
\]  

(30)

For the HH resonator, keeping in mind that $l$ does not depend on $V$ and making use of Eq. (5)
Figure 6: (a) Modulus and (b) root locus of the coupled system transfer function \( \frac{\eta_j}{Q_N} \) as function of the mass flow \( \dot{m} \) for the Helmholtz resonator for \( \alpha = 0.75 \), fixed volume \( V/V_c \) and fixed instability growth rate \( \nu_j/\omega_j \). (c) Corresponding measurement points on the stability map coloured with \[ \frac{\eta_j}{Q_N} \] yields:

\[
\begin{align*}
V & \leq \frac{\dot{m}_c \pi c}{4 \rho \alpha \nu_j} \\
V^2 & \geq \frac{\nu_j / \omega_j}{\omega_j^3 \alpha l \rho} \nu_j \left( 1 - \frac{2 \alpha \omega_j l}{\pi c} \right)^{-1}
\end{align*}
\]

One can plot in a similar manner the stability map of the system for fixed mass flow as function of the unstable mode growth rate \( \nu_j/\omega_j \) and the damper to cavity volume ratio \( V/V_c \), which can be seen in Fig. 5. From this figure it is clearly again the lowest values of \( \alpha \) that allow the stabilization of an unstable mode with highest growth rate. However, this is done at the cost of increasing the resonator volume, which is not always possible in reality. One has to find a compromise between resonator volume and achievable growth rate reduction.

3.3. Influence of the parameters on the total transfer function

The previously built stability maps give a “boolean” information about the stability of the system depending on the growth rate of the unstable mode \( \nu_j/\omega_j \), however they do not give any information about the influence of the parameters on the coupled system transfer function. The influence of the resonator damping ratio has been studied in [25] for the HH resonator, but not associated to the real constraints on \( V \) and \( \dot{m} \) or the stability limits from the Routh-Hurwitz criterion. The \( \| \cdot \|_2 \) of the transfer function has been computed for a few points within the stable domain in order to give an energy content-scaled stability map. The \( \| \cdot \|_2 \) is computed as follows:

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Figure 7: (a) Modulus and (b) root locus of the coupled system transfer function $\frac{\eta_j}{Q_N}$ as function of the volume $V$ for the Helmholtz resonator for $\alpha = 0.75$, fixed purging mass flow $\dot{m}/\rho V$, and fixed instability growth rate $\nu_j/\omega_j$. (c) Corresponding measurement points on the stability map coloured with $||\cdot||^2$.

\[ ||\eta_j||_{Q_N} = \frac{\eta_j}{Q_N} \text{ as function of } \nu_j/\omega_j. \]

\[ ||\cdot||^2 = \int_{-\infty}^{\infty} \left| \frac{\eta_j}{Q_N}(\omega) \right|^2 \, d\omega. \]

For fixed resonator geometry (fixed volume and HH resonator $\alpha = 0.75$ in this case), the $||\cdot||^2$ coloured stability map is shown in Fig. 6(c). For a given instability growth rate $\nu_j/\omega_j$, if one operates the resonator with a purging mass flow close to that of the second Routh-Hurwitz criterion (upper limit of the stability domain), the $||\cdot||^2$ tends to infinity since it is on the stability limit, and one can see in Fig. 6(a) that the transfer function modulus diverges at the resonance frequency of the unstable mode $\omega_j$. This is consistent with the transfer function modulus at $\omega_j$ for a perfectly tuned resonator which obviously tends to infinity if one is at the RH2 limit from Eq. (27):

\[ \eta_j/Q_N(\omega_j) = \frac{1}{1 - \omega_j^2 \frac{\epsilon}{\nu_j^2}}. \]

Lowering the mass flow progressively, one can see on the root locus in Fig. 6(b) that the two poles initially at the same frequency $\omega_j$ but different growth rate get closer together until a point where they split left and right from the eigenfrequency of the unstable mode without dampers. All the while the value of the transfer function at resonance, which in this case also corresponds to the $||\cdot||^\infty$ gets lower and lower. The “mode splitting” point corresponds to the point where the coupled system has the lowest total growth rate $\nu_t$, which however does not correspond to the lowest energy content of the system (which would be where the $||\cdot||^2$ is lowest).

From this point on the coupled system resonates at two frequencies symmetrically opposite from...
the initial resonance frequency. Further lowering the mass flow again, the value of the transfer function at $\omega_j$ decreases, while its value at the mode splitting frequencies increases. The optimum damping can then be defined either as that where the infinite norm of the system is the lowest, or where the $|| \cdot ||_2$ is lowest. Lowering the mass flow even more worsens the system response again, with the $|| \cdot ||_2$ and transfer function modulus diverging again, although not at the expected unstable frequency $\omega_j$.

The same study can be made for same geometry and fixed mass flow and given unstable mode growth rate $\nu_j/\omega_j$, which is shown in Fig. 7. In this case however it is by increasing the resonator volume (from RH2 limit value to RH1) that one can trigger mode splitting of the coupled system. From both these studies one can observe the following:

- The stability limits are well defined by the Routh-Hurwitz criterion: close to the RH2 stability limit, the system response diverges at the expected resonance frequency. However, near the stability limit corresponding to criteria RH1, the diverging mode splitting occurs, which means the system response diverges not at the expected resonance frequency, but at two frequencies symmetrical with respect to $\omega_j$.

- A general rule in damping design is that the bigger the resonator volume, the better. One can see that this is not true for a fixed mass flow: increasing the resonator volume indeed increases the damping efficiency factor $\varepsilon_H$, but also can bring the system closer to its stability limit RH1 thus inducing diverging mode splitting. Therefore, “the bigger the better” rule is only provided that the mass flow is adequately set.

- One can also notice from Figs. 6(c) and 7(c) that for a given growth rate of the instability $\nu_j/\omega_j$, the ideal resonator purging mass flow (resp. resonator volume) regarding $|| \cdot ||_2$ is not necessarily that at which one would be able to stabilize the highest growth rate: the best $|| \cdot ||_2$ is achieved for a bit lower purging mass flow (resp. higher volume) than expected.

In general the study of the transfer function tends to show that if a resonator has been designed for certain constraints on its volume and mass flow (for example so that one would be able to achieve the stabilization of the highest growth rate), then the practical conditions in which the resonator is used should not differ from the design values if one wants to avoid diverging system response, either at the expected resonance frequency $\omega_j$ or in the more dangerous case of mode splitting, at symmetrically opposite frequencies from $\omega_j$.

### 3.4. Determination of the ideal damper geometry

As seen in Figs. 4 and 5, for a given geometry (i.e. fixed aspect ratio $\alpha$) there are certain volume and mass flow constraints which give the best results in terms of stabilization of the system. Combining those two figures, one can define for each fixed volume $V$ the mass flow $\dot{m}$ at which the best geometry “switches” from one to the other, as well as the mass flow at which the best damping is achieved (i.e. coloured points at the edge of the stability domain). Hence one

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can build a map of the best resonator geometry as function of the constraints $V$ and $\dot{m}$, which is presented Fig. 8. If one finds itself under the threshold for which HH resonator geometries are the best available, one could tune the aspect ratio parameter $\alpha$ to design a damper which achieves the stabilization of the highest growth rate exactly at the point given by the constraints. From this figure, one can notice that there indeed are some volume and mass flow constraints for which the QW resonator is the best option. However, this map does not give information about how much damping the resonator achieves.

Therefore, the curves corresponding to this best achievable damping for the four studied resonator geometries have been traced and coloured with the maximum stabilizable growth rate $\nu_{j,\text{lim}}$ in Fig. 9. For a fixed volume and fixed mass flow constraints, one would for example have the QW resonator as the best geometry. However, reducing the mass flow (which is in practice always possible) would steer into choosing a Helmholtz resonator with low aspect ratio $\alpha$ which would achieve better damping of the unstable mode. This is however only valid if the flow velocity through the neck is high enough so that modelling the dissipation with the $(l - \zeta)$ model is valid and that this linear damping mechanism dominates the other sources of acoustic dissipation – for instance, vortex shedding at damper mouth triggered by nonlinear 2nd order terms, acoustic boundary layers, non-linear effects due to for example grazing flow at the mouth.

4. Conclusion

The different resonator geometries have been modelled as harmonic oscillators, and this impedance model has been included in a resonator-enclosure coupling analytical model. The stability of the system has been studied, and the triggering of the mode splitting in the cavity-resonator coupled system has also been highlighted and put into relation with the Routh-Hurwitz criterion. Our
Figure 9: Loci of the best achievable damping for different values of aspect ratio $\alpha$ on the previous map, coloured as function of the highest stabilizable growth rate $\nu_{j,\text{lim}}/\omega_j$

Figure 10: Sketch of a Helmholtz resonator dimensions with end-correction

study is limited to a single perfectly tuned resonator and to the assumption that the main dissipation mechanism is the coherent vortex shedding at the resonator mouth associated with first order resistive contribution in the linearized unsteady Bernoulli equation. From this study, one can conclude that even if for certain volume and mass flow constraints the stability map designates the QW resonator as the best geometry, since the HH resonator with low aspect ratio $\alpha$ is able of achieving stabilization of unstable modes with higher growth rate, and since in practice the mass flow can always be lowered, then one should clearly favour HH resonator design. One should however remember that the impedance model used in this study is only valid under certain assumptions: in the HH resonator case, the neck velocity must be high enough so that the non-linear effects are kept negligible, whereas in the QW case one has to ensure that the friction losses in the boundary layers over the length of the resonator do not dominate the damping mechanism.

Appendices

A. Influence of the end-correction

In order to see how the end-correction would influence the qualitative results of Figs. 4 and 5, the previous calculations have been made including the effect of an idealized end-correction in the
model. Note that depending on the damper geometry, these end corrections can vary significantly. The calculations of damper impedance and coupled system transfer function are not detailed, only the computation of the damper geometrical parameters is shown here. The end-correction coefficient used is the one given by Lord Rayleigh [26] for a flanged pipe: $8/3\pi$. As in the previous section the case studied is a single ideally placed and perfectly tuned resonator. The dampers geometric parameters are determined as function of the constraints so that they are tuned to unstable eigenmode angular frequency $\omega_j$.

In the case of a QW resonator, its effective length must verify: $L_{\text{eff}} = \pi c/2\omega_j$. $L_{\text{eff}}$ can now be defined as function of the physical length $L$ and the volume $V$. The end-correction is only applied at the damper exit in the QW case:

$$L_{\text{eff}} = L + \frac{8}{3\pi} \sqrt{\frac{A}{\pi}} = L + \alpha \sqrt{\frac{V}{L\pi}},$$ (34)

with $A = V/L$ the cross-section of the QW resonator. It yields a 3rd order polynomial with the physical length of the QW resonator as a root. Once the physical length $L$ and resonator cross-section $A$ have been determined, one can calculate the corresponding HH resonator neck cross-section $a$ with the chosen aspect ratio $\alpha$ as was done before: $a = \alpha A$. In the case of the HH resonator the geometrical parameters are determined as shown in Fig. 10: the effective neck length is the physical length plus an end-correction on both sides $l_{\text{eff}} = l + l_{e,1} + l_{e,2}$, and the back cavity effective volume is the total physical volume allowed $V$ which is the same as that of the QW resonator ($V = AL$) minus the volume of the neck elongated of one end-correction on the inside $V_c = V - a(l + l_{e,1})$. One defines $l_{e,1} = l_{e,2} = 8/3\pi \sqrt{a/\pi}$. One can again equalize the resonance frequencies:

Figure 11: Comparison of the limit points of the stability domain as function of aspect ratio $\alpha$ with and without end-correction: (a) limit mass flow and (b) limit growth rate for fixed volume. (c) limit resonator volume and (d) limit growth rate for fixed mass flow.

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\frac{\omega_Q}{c} = \frac{\pi}{2L_{\text{eff}}} = \sqrt{\frac{\alpha}{(AL - a(l_{c,1})L_{\text{eff}})} = \frac{\omega_H}{c},}
\]

and calculate the physical length of the HH resonator neck as function of the chosen aspect ratio \(\alpha\). The rest of the calculations are very similar to those done in Secs. 2.2 and 3.1 and are not reproduced here. The results of the stability maps from Figs. 4 and 5 with and without end-correction are compared by comparing the coordinates of the limit points (represented by coloured circles) of the stability domain for each damper geometry, to be seen Fig. 11. They are also computed in both cases for \(\alpha\) values which were not displayed on Figs. 4 and 5. From this figure we can confirm the initial assumption that the inclusion of the end-correction in the model induces close to no change in the qualitative criteria derived to choose between different resonator geometries. It should however be taken into account when one wants to design the actual resonator geometry.

References


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