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Wronski's Infinities

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Abstract: This paper interprets Hoene Wronski's (1776-1853) use of actual infinities in his mathematical work. The interpretation places this usage, which undermined Wronski's acceptance as mathematician, in his contemporary mathematical and philosophical context, and in the context of his own socio-political-philosophical project.

Keywords: Infinity; philosophy of mathematics; German idealism; 19th century mathematics; infinitesimal calculus

1. Introduction

Józef Maria Hoëné Wronski (1776-1853) was born in Poland, studied in Germany, spent a decade in Marseille, and from 1810 onward lived mostly in Paris. Today he is commemorated through the "Wronskian" determinant, but his intellectual work encompassed mathematics, physics, philosophy and religion, and resulted in a large number of thick publications.¹

From the very beginning of his stay in Paris, Wronski managed to antagonize the mathematical establishment in a way that led to his total marginalization. If it weren't for a small circle of followers who promoted his work, and a rekindling of interest in his mathematical work from the 1870s to around 1900, Wronski may have not been remembered at all.

Given Wronski's social position, this paper belongs to the genre of "losers' history" – a history of marginalized scientific persons and ideas. Unless rehabilitated, such losers are considered as ranging from "unfortunately misguided" to "charlatans and quacks". The fact of having lost is simply attributed to their being wrong. I can't deny that Wronski was wrong in many ways, both according to the terms of his contemporaries and to our own, but he also had some interesting and good ideas, which did not receive due credit.

¹ A comprehensive collection of Wronski's publications and manuscripts is freely available online from the Wielkopolska Biblioteka Cyfrowa.

The main point of this paper, however, is not why Wronski lost (to this question there are several easy mathematical and sociological answers that will come to light by the end of the paper). The main point here is to isolate one element of Wronski's losing mathematics – his use of actual infinities – and account for it. Wronski's use of infinity, which challenged the standards of his contemporaries, was deliberate and purposeful. Wronski insisted on it against a strong opposition.² What I want to explain is why Wronski chose to treat mathematical infinities as he did, even though he knew this could undermine his acceptance. The answer to this question will turn out to reflect on the relationship between mathematics, philosophy and society, and this paper will study this interaction in order to contribute to an understanding of the contexts of 19th century European mathematics.³

The historiography of the philosophy of mathematics usually focuses on empiricist, rationalist and conventionalist approaches (Brouwer's intuitionism and its relation to Kant and to Husserl is a rare exception). Moreover, the technical issues involved in the philosophy of mathematics tend to set it apart from other branches of philosophy. But an analysis of Wronski's work engages the suppressed relations between the philosophy of mathematics and such sweeping trends as German Idealism, which traditional historiography ignores. It suggests that a wider look at mathematics and its philosophy, beyond the "winners" of the Parisian Polytechnique, Weierstrass' Berlin and Hilbert's Goettingen, might expose a richer mathematical thinking and a richer thinking about mathematics, which may open new paths for contemporary thought as well.

² So strong indeed, that Poncelet wrote in an 1818 letter to his former teacher, Olry Terquem: "wanting to base all my research on the admission of the principle of continuity in geometry [as opposed to algebraic, finitist approaches] and of all the metaphysical consequences that it entails, I am afraid ... to contradict the ordinarily received ideas and thus fail to obtain the assent of the enlightened men whom I want to be my very judges. The example of Wronski scares and intimidates me in more than one way, without, however, giving me reason to believe that my work has anything in common with the delusions of the philosophy from across the Rhine" (Poncelet 1864, 538). Only much later would Poncelet dare to write that Wronski's critique of Lagrange was "not without reason" (Poncelet 1864, 566).

³ A note to those who know Wronski's work: this paper conspicuously leaves aside Wronski's Universal Law of Creation. I allow myself this glaring oversight, because I intend to treat Wronski's notion of foundations – mathematical and philosophical – in a subsequent paper.

Since information about Wronski is not widely available, I include a brief biography in Appendix A and a brief summary of Wronski’s mathematics in Appendix B. Those who do not know Wronski are advised to read these appendices first. Wronski’s philosophy is surveyed comprehensively in Warrain (1933-38) and d’Arcy (1970), and summarily in Murawski (2006). I do not provide a general summary here, but some elements of this philosophy will surface as the paper unfolds.

The paper will proceed as follows. In section 2 we will review two uses of actual infinity by Wronski. We will claim that Wronski makes bolder endorsement of actual infinities and place them in more foundational positions than do most of his contemporaries (most notably, his direct audience of the Parisian Academy of Science). In section 3 we will give the Kantian background to Wronski’s philosophy, and show its limits in accounting for Wronski’s use of infinities. In section 4 we’ll frame Wronski’s position with respect to his socio-political environment and to post Kantian thinking. This framing will articulate the philosophical/political problem that Wronski tries to resolve, and explain why existing philosophies of mathematics were not sufficient for the task. In sections 5 and 6 we will explain Wronski’s own architecture of mathematical infinities. In section 7 we will motivate this organization of mathematical ideas by placing them within a more general political and philosophical context.

2. Wronski’s use of actual infinities

Here are two examples for Wronski’s “outrageous” use of actual infinity.

In his first major publication, the *Introduction à la Philosophie des Mathématiques* (INT 11-17),⁴ Wronski introduces the logarithm as the function that satisfies the equation: $x = a^{\log_a(x)}$. Taking the m-th root on both sides, one obtains

$$x^{1/m} = \left(\sqrt[m]{a}\right)^{\log_a(x)} = \left(1 + \left(\sqrt[m]{a} - 1\right)\right)^{\log_a(x)}.$$

Applying the binomial formula results in

$$x^{1/m} - 1 = \frac{\log_a(x)}{1} \left(\sqrt[m]{a} - 1\right) + \frac{\log_a(x)}{1} \cdot \frac{\log_a(x) - 1}{2} \left(\sqrt[m]{a} - 1\right)^2 + \text{etc.}$$

Now, “when the arbitrary quantity m is infinitely large, the second member [right hand side] of the

⁴ A list of abbreviations for works by Wronski cited in this paper is available at the end of this paper.

last equality reduces to its first term, and one finally obtains $\log_a(x) = \frac{x^{1/\infty} - 1}{\infty\sqrt[a]{a} - 1}$.” Moreover, since

both the numerator and denominator are zeros, Wronski multiplies them by infinity to render them finite while maintaining their ratio. Defining e to be the number for which $\infty\left(\infty\sqrt[e]{e} - 1\right) = 1$ and setting $a=e$, one obtains

$$(1) \quad e = \left(1 + \frac{1}{\infty}\right)^{\infty} \quad \text{and} \quad \log_e(x) = \infty(x^{1/\infty} - 1).$$

The beginning of this maneuver is not new. In fact, it is taken (as Wronski explicitly acknowledges) from Halley (1695). It is also similar to the maneuvers applied by the main figure of 18th century mathematics: Leonhard Euler (1787[1748], 93 ff; analyzed in Jahnke 2003, 115). But, as we’ll see below, this traditional maneuver was not welcome among many of Wronski’s addressees: the Parisian academy, supporters of Lagrange’s algebraic analysis and those involved in the attempts to suppress infinitesimals at the polytechnic. Wronski was obviously aware of the objections that could be raised to his use of actual infinity, and anticipated them in a footnote that promised to deal with them “rigorously” elsewhere (the best reference would be his 1814 *Philosophie de l’Infini*).

But the element which is most “reactionary” here is the final definition of e . In the part borrowed from Halley, the use of infinity takes an argument involving finite magnitudes to an extreme. It fits into a culture where infinity was a somewhat volatile entity to be handled with care, or perhaps even a useful fiction, to be controlled by the pragmatics of consistency and of computational and empirical verification (see Ferraro 2004). One could go *through* infinity to obtain finite result, or use it to simplify complicated expression (e.g. the binomial expansion above) in succinct terms (the definition of the logarithm above). But when it comes to Wronski’s definition of e , infinity is cast in a founding, rather than accessory role. In Wronski’s own words, this definition is “the primitive theoretical expression, the idea or primary conception of the number called the *base of the natural logarithms*” (INT 14). It is not just *one* useful identity among others, as it was for Wronski’s predecessors; it is *the* fundamental defining expression of this quantity. As we shall see, this is precisely the role that Wronski assigns to actual infinity.

In the next example (TECI 72-78), Wronski goes considerably farther in his use of actual infinities. Here the task is to expand the function $x^{-m} - 1$ as a product of linear terms, as one does with polynomials. If this function were a polynomial, we’d simply have to multiply all the terms $x - y_j$,

where the y_j 's solve the equation $x^{-m} - 1 = 0$ (namely, are unit roots of order m). But if we do that here we'd get $(x - y_0)(x - y_1)(x - y_2) \dots (x - y_{m-1}) = x^m - 1$, rather than the desired function $x^{-m} - 1$.

So, in order to complete the decomposition, Wronski has to break the quotient $\frac{x^{-m} - 1}{x^m - 1}$ into a

product of further linear terms. Since this quotient (which equals $-x^{-m}$) has only infinite roots, these roots must be, according to Wronski, all quantities of the form

$$(2) y_{m+k} = \infty(\cos.a + \sqrt{-1}\sin.a).$$

We obtain:

$$\frac{x^{-m} - 1}{x^m - 1} = \Phi \cdot (x - y_m)(x - y_{m+1})(x - y_{m+2}) \dots (x - y_\infty),$$

where Φ is a numerical scaling coefficient. In

other words,

$$(3) x^{-m} - 1 = \Phi \cdot \{(x - y_0)(x - y_1)(x - y_2) \dots (x - y_{m-1}) \times (x - y_m)(x - y_{m+1})(x - y_{m+2}) \dots (x - y_\infty)\},$$

where the first m values of y_j are m -th unit roots, the other y_j 's are all the possible infinite roots from (2), and Φ can be derived by substituting any value for x on both sides of (3), and comparing the results.

Wronski notes that the infinite roots expressed in (2) must include all possible values of the argument a , as well as all possible infinities: $\infty, 2\infty, 3\infty$ until $\infty\infty$. This yields altogether ∞^2 infinite values for the roots y_m, y_{m+1} , etc in (3). Note also that by substituting an infinitesimal value for m , one can obtain from this and from (1) a new infinitary formula for the logarithm.⁵

This move can be understood as analogous (in its intention, if not in the consistency of its end result) to the introduction of multiple and imaginary roots in order to decompose real polynomials into as many factors as their degree, and to the introduction of the imaginary point at infinity in projective geometry so as to homogenize claims concerning parallel and non-parallel lines. However, unlike in the case of (1), the use of infinities here is not grounded in common 18th century practice, and would appear far more suspicious to almost everyone involved in the mathematical scene.

⁵ On top of the obvious objections that would arise here concerning the determination and use of actual infinities, there's another problem that Wronski misses: given this representation, the expression for $-x^{-m}$ ends up independent of m ! This shows that Wronski's algebraic manipulations were not always properly thought through. My point, however, is not to evaluate the end result, but explain the motivation that led Wronski to go there.

This point should be qualified carefully, because experts who saw drafts of this paper reacted in ways that ranged from claiming that actual infinities had already been in common use to stating that they were hardly ever endorsed – and they were all correct. It is clear that many of Wronski's predecessors and contemporaries used infinities in their discussions and calculations, but almost all of them (including Leibniz and Euler) commented at one point or another that this magnitude doesn't have to be taken too seriously – that it could be considered as a shortcut or a useful fiction (the Aristotelian tradition and its objection to an actual infinity plays a role here). Wronski, on the other hand, found infinity to be tenable both ontologically and mathematically, and insisted on its founding, rather than accessory role. With respect to those such as Wallis and Fontenelle (the latter's 1728 *Elémens de la Géométrie de l'Infini* attempts a systematic calculus of infinities), who elaborated unreserved calculations with infinities (and whose arithmetic approach to infinities has been falling out of grace by the beginning of the 19th century), Wronski's calculations go several steps further in integrating infinities into series and product calculations. Compared to the more recent clearly articulated endorsements of actual infinities, such as Schultz' use of infinitely large surfaces in his attempt to prove the parallel postulates, Wronski's approach takes a distinct old-fashioned arithmetic turn, and does not discriminate between infinitely large and small magnitudes, accepting them both (see Schubring 1982 for a discussion of Schultz). This is possibly why Wronski does not refer to Schultz.

His unorthodox approach does not mean that Wronski was a quack. He did make many mistakes and many over-generalized, extravagant claims (even more than was typical among his often over-generalizing and extravagant contemporaries), but he was at the same time an able mathematician who advanced some original contributions. Moreover, Wronski does acknowledge that formulas such as (3) may “perhaps never present any use in the application of the science, but they're important for the science itself, and especially for its philosophy” (TECI, 77-78). Here the philosophical point is the general equivalence that, as we shall discuss below, “should take place”, according to Wronski's conceptions, between infinite products and infinite series in the representation of functions: since $x^{-m} - 1$ has a power series representation, it should also have an infinite product representation, and these representations should be interchangeable according to general rules.

We see that Wronski used infinity in a way that had roots in the practice and ideas of his predecessors, but he also went beyond these common practices and reorganized these ideas in a way

that would antagonize the very people he sought to impress. Moreover, he insisted on his use of infinities, even against criticism that practically ruined him. This puzzling state of affairs defines my task for the rest of this paper: try and explain the political and philosophical motivation for Wronski's insistence on his aberrant use of infinities.

3. Wronski as a Kantian

In order to understand Wronski's approach, let's start from his commentary on the formula for the logarithm (1). Wronski explains that functions like the logarithm are such that their "NATURE (and not only VALUE) can be constructed only through a progressive generation, more and more exact, but rigorously indefinite, that is by a generation that is at the same time finite at every step and indefinite as a whole" (REFII, 298).

This echoes Wronski's earliest work on mathematics, the *Philosophie ou Législation des Mathématiques* (LEG), begun at the summer of 1804 and published posthumously. This work presents a philosophical categorization of various kinds of magnitudes, culminating in a reformulation of one of Kant's antinomies of pure reason: the contradiction between the apparently well founded claims that the universe is infinite and that it is bounded in time and space. LEG is Wronski's most Kantian work, and since he saw his work as a continuation of Kant's, this is a good place to start our analysis.

Wronski's analysis is based on a distinction between the Kantian faculties of understanding and reason. In Wronski's terms, reason generates concepts, while the understanding uses these concepts to impose an organized unity on our objective intuitions or empirical experiences. Based on this division of intellectual labor, some magnitudes, to be named "intellectual", can be contained by the understanding, while others, to be termed "ideal," cannot (LEG 114).⁶ According to this definition, discrete finite quantities (integers and rational numbers) belong under the term "intellectual", as they are used for putting together our concrete intuitions and empirical experiences. But when we advance to irrational quantities, imaginary numbers and infinities, we also go beyond our discretely articulated empirical experiences, and so exceed the realm of the understanding.

⁶ Wronski clarifies that this division concerns the magnitudes themselves, not their concepts.

But Wronski does not clump all ideal magnitudes together. He further divides them into “intellectually generated,” namely those whose process of generation can be contained by the understanding, as opposed to “ideally generated” (LEG 111–112). An ideal magnitude that is intellectually generated is ideal as a given magnitude, but its process of generation is accessible to the understanding (LEG 116–17). This is precisely the situation of the logarithm according to the above analysis: when considering the binomial expansion above, each step in its generation (each term in the series) is finite and containable by the understanding, but its ideal final result (an irrational quantity) is a product of reason. The same applies to other irrational quantities. In contrast, ideally generated ideal magnitudes are magnitudes that cannot be generated by an indefinite accumulation of finite magnitudes, namely, imaginary numbers and actual infinities, which have no power series approximations. These latter magnitudes are pure creations of reason in their generation as well as in their end result.

This account links a traditional distinction between magnitudes with the new, Kantian, system of faculties. A specification of the structure of this system is provided in *La philosophie de la Tecnie*:

the faculty of Understanding produces REAL or FINITE quantities, which are, in a way, the matter of Algorithmics [Wronski’s preferred title for analysis]; and the faculty of Reason establishes, by means of INDEFINITE quantities, an IDEAL link within real or finite quantities, shaping in this way, so to speak, the form⁷ of Algorithmics. Understanding provides a DISCONTINUOUS SUMMATION for the generation of quantities, and reason introduces an INDEFINITE TRANSITION or CONTINUITY within this generation (TECI, 2).

Indeed, according to Wronski, if we remained in the realm of the understanding, we would miss not only actual infinities, but also series and irrational numbers (TECI 241, INF 41) as well as continuity and curved lines (INF 43-44), because our understanding can handle only discrete observations.

The architecture that allows reason to form the matter provided by the understanding requires some mediation. Indeed,

the concept of a finite quantity is a product of the UNDERSTANDING, which, under the conditions of time [empirical experience] that are proper to it, introduces an intellectual unity or a signification in the being that is set opposite to knowledge. ... the idea of infinity,

⁷ Wronski writes “matter” here, but this is obviously a scribal error.

is a product of REASON, which, in itself, is outside the conditions of time, and consequently inapplicable or transcendent with respect to the constitutive use which we make of knowledge for the cognition of Being (INF 34; cf. INT 167 for a similar treatment of imaginary numbers).

How is it possible, then, for infinity to apply to our empirical experience? The Kantian architecture provides us with a mediating faculty for this precise purpose:

used regulatively by the influence of the [faculty of] JUDGEMENT for the conditions of time that are foreign to it, this product of reason, the idea of infinity, transformed thus into an idea of the indefinite, serves to link our conceptions of quantity the idea of the indefinite, by an application of the conditions of time, serves, at least regulatively, in the immanent sphere of our cognition by introducing the last unity or the last signification not into the object of knowledge, into being, but into the very functions of knowledge, relative to the cognition of quantity (INF 34).⁸

What we have here is a transcendent concept of infinity in reason, an immanent concept of the finite in the understanding, and a regulative application of the infinite-turned-indefinite, applied by an act of judgment, in order to make sense of our intuition of the world. All this is quite in line with the Kantian solution to the antinomy concerning the in/finitude of the universe. According to this solution, we have no experience of a complete, infinite universe, only of concrete finite pieces. But our thinking is also subject to the postulate that any step toward the end of the universe can be followed by another, and that any reason in the chain that leads to the origin of the universe should have another reason behind it. These regulative postulates put together our finite experiences into a unified indefinite scheme. This indefiniteness of the universe solves Kant's antinomy and provides the initial scaffolding for Wronski's infinities.

But if we ended our description of Wronski's philosophy here, we would provide a very partial and biased account. The middle ground of "indefiniteness" or "potential infinity" was a popular means of approaching the problem of infinity at least since Aristotle, and was enough to account for most uses of infinity in applied mathematics, which could be rearticulated as limits in the intuitive sense of Newton and d'Alembert. But not all of Wronski's infinities (or those of his predecessors and

⁸ Note here the contrast between links that have to do with our objective intuitions of being (applied by the understanding) and links that concern the sphere of our knowledge rather than objective intuitions (applied by an act of judgment).

contemporaries) were reducible to indefinite approximation. Indeed, (3) uses actual infinities, which are not mediated by any form of indefiniteness or approximation, and may therefore appear cut off from any being in the world and from the mathematics that is concerned with such being.⁹ Therefore, to understand Wronski's trajectory beyond Kant, we need to continue analyzing Wronski's philosophical choices in terms of their intellectual and socio-political context.

4. Wronski in context, part I

The positions and trends to be presented in this section have, of course, a long history, and many of them are not new. I will present them here in reference to revolutionary Paris and post Kantian philosophy, because these are the contexts which are most relevant for Wronski's thought. The presentation reflects the division as seen by Wronski, which is, of course, a very partial point of view, verging on a caricature.

The use of actual infinities in European calculus was never taken for granted. Indeed, the conjunction of French mathematics and enlightenment brought up some interesting reflections on infinities. If we take as our tokens Condillac and d'Alembert, we see them both inclined toward a sensualist ontology; but since the senses alone do not give us enough for building a discursive order, they both call reason to our aid. For Condillac, numbers begin from concrete representations of objects by fingers. But then, as numbers are abstracted, they lose their footing in objects and are perceived "in the names that have become the signs of the numbers" (Condillac, *Langue des Calculs*, quoted in Schubring 2005, 260). This suggests a nominalist perception of higher mathematical entities, subject to Condillac's program of a "langue bien faite" (well formed language).

D'Alembert's philosophical framework was more oriented toward reflection than language. Reason was to constitute a distinct faculty involved in the study of nature. Indeed, "although no calculation

⁹ Note that for Wronski an indefinitely increasing or decreasing sequence does not approximate the infinitely large or infinitely small respectively. The former is clear, because the difference between any term in an increasing sequence and the infinitely large is infinite. The latter case is less obvious for us, because we think of the difference between terms of a decreasing sequence and their limit at zero. But Wronski (like his contemporaries) thought in terms of the ratio between decreasing finite terms and the infinitely small (which is not a zero!), and this ratio remains infinite.

proper is possible except by numbers, nor any magnitude measurable except by extension . . . , we arrive through the continual generalization of our ideas at the principal part of mathematics”. But along with this abstract generalization, mathematics “retraces its steps, reconstitutes anew its perceptions themselves, and, little by little and by degrees, produces from them the concrete beings that are the immediate and direct objects of our sensations”. (d’Alembert 1995, 20-21). We see that, for d’Alembert, while experience must be complemented by reason, the work of reason must be retraceable, in turn, to empirical experience (Daston 1979).

This anchoring of reason to experience was a philosophical principle that protected the liberal thinkers of the enlightenment from submission to the absolute authority of the *ancien regime*. Now I do not claim that there's a necessary relation between anchoring reason to experience and liberation from authority. Indeed, one could use the anchoring of reason to experience in order to claim that without recourse to the experience of revelation as reported in the scriptures, reason can only provide a limited understanding of our world. My claim is that a contingent relation between anchoring reason to experience and liberation from absolute authority can be discerned in revolutionary France.

Given this superposition, D’Alembert’s rejection of infinitesimals in favor of limits in his *Encyclopédie* entry (Ewald 1996, 123-130) and Lagrange’s attempt to circumvent limits and infinities in his *Théorie des Fonctions Analytiques* (1797) were highly influential in the mathematical institutions of revolutionary Paris, and particularly in the newly established Polytechnic. This does not mean that there is a necessary relation between rejecting infinitesimals and liberalism. Indeed, Lagrange did appeal to infinities outside his attempt to provide a new, finitary foundation to analysis, and Cauchy, an avid monarchist, struck a compromise between the infinitely small and the indefinite in his *Cours d'Analyse*, rather than fully endorse a classical infinitesimal presentation (Schubring 2005).¹⁰ But in France, in the decades around the revolution, some correlation did exist between the philosophy of the French *idéologues*, rejection of

¹⁰ In fact, rejecting infinitesimals needn’t have anything to do with liberalism, as demonstrated by the critiques of Hobbes against Wallis and of Berkeley against Newton. Both Hobbes and Berkeley rejected infinitesimals based on a sensualist ontology (if it can’t be sensed, it needn’t be considered part of reality). But to complement this meager ontology and enable social order, they required, respectively, consent or revelation, which in turn established absolute sovereign authority. Indeed, Berkeley’s *The Analyst* opens with an explicit controversy over authority (Ewald 1996, 62). For a detailed study of the politics of infinitesimals in the 17th century see Alexander (2014).

infinitesimals, and revolutionary positions.

Indeed, outside the circles of the Parisian academy of science and of the polytechnic limits and algebraic analysis were considered cumbersome and ill-suited for the education of engineers. As elucidated by Schubring (2005), this practical argument was overdetermined by conflicts between center and periphery and between republican elites and a military and bureaucratic establishment that remained entangled with what was left of the old order. By 1812 the tables have turned. The infinitely small was reinstated as the basic concept in teaching differential calculus even in the Polytechnic (Schubring 2005, 295-306).

This tension is embodied in the work of Carnot. The earlier versions of his *Réflexions sur la Métaphysique du Calcul Infinitésimal* (an unpublished memoir from 1785 and a short publication from 1797, see Schubring 2005, ch. V for an analysis of Carnot's work; for a comparison of Carnot and Wronski see Guerraggio and Panza 1985, Blay 1998, 131-163) focus on his theory for replacing infinitesimals by independent variables, whereas the final edition (1813) makes bolder endorsements of infinitesimals, and prefers their practical strength over their foundational weakness:

“In conclusion, it is not in the explanation of the principles that one can shew the advantage of the Infinitesimal Analysis over all others. All of them are for the most part equally clear in their principles: but it is not equally easy to apply them to particular questions. The principal difficulty is to put problems into equations, which, on the other hand, in the Infinitesimal Analysis is very easy” (Carnot 1832[1813], 127-128).

Now Wronski's mathematical and philosophical ideas had been developed in Marseilles in the first decade of the 19th century. His early publications, which appeared in the period of the polytechnic's transition (between 1810 and the publication of Cauchy's 2009[1821] attempted synthesis of infinitesimals and limits) affirmed the role of infinitesimals, first against Lagrange's analytic method (LAG), and then against Carnot (INF). Indeed, for Wronski, Carnot's pragmatic approach was not enough. For him infinitesimals were the one true foundation of the calculus. But to understand this position, we must look at Wronski's more general philosophical and political motivation.

Indeed, as we will see, Wronski sought to resolve a deep contradiction between what he called “liberals” (who champion independent thinking while anchoring reason in empirical experience)

and “illiberals” (who take upon faith the transcendent principles exposed by the Christian revelation and submit themselves unquestioningly to clergy and absolute sovereign authority). This is, of course, Wronski’s very schematic view of the political divide, which is not an objective description. As a loose approximation, however, it is not entirely unfounded. Typically, the champions of enlightenment (even while using infinitesimals) tended to use them as convenient shortcuts, whereas resorting to actual infinities as foundations tended to carry with it absolutist overtones. As we shall see, Wronski's motivation for his oversimplification of political divisions was to place himself in the position of a unifying mediator.

Choosing to set actual infinity as a foundational element places Wronski, at least in terms of his own analysis, in opposition to the republican establishment and in coalition with the restoration of infinitesimals and with other restorative trends. This choice is motivated by the fact that France of the revolution and enlightenment was dominated by those he saw as “liberals”. As a countermeasure, this domination had to be balanced by a bold revindication of infinitesimals. Only the sound affirmation of actual, constitutive infinity would counter the prevalence of the “materialist” philosophy spread by the encyclopedists (including d’Alembert), which Wronski considered as limited to the level of the understanding and responsible for the mathematical contradictions and perceived stagnation of the time (e.g. LEG 73; REFII lxi-lxii; LOI 36-38). This opposition was further intensified as the polemic between Wronski and members of the academy forced him into an ever more antagonistic stance.

To further understand Wronski’s mathematical choices in the context of his program to reconcile liberals and illiberals (or immanence and transcendence, or finitude and infinity – all analogies that he explicitly made himself, e.g. PROD 2-15, 25-37) into a political and philosophical unity that integrated their best features, let’s compare these choices to those of another contemporary Kantian philosopher of mathematics, Jakob Friedrich Fries (1773-1843).

Fries’ *Die Mathematische Naturphilosophie* (1822; see also Herrmann 2000, ch. 3 and the overview in Schubring 1990) was published after Wronski had already printed the first round of his major mathematical works, but serves to delineate his position. Fries’ treatment of infinities derives from two approaches. The first is a Kantian notion of indefiniteness, and the second is what we could in hindsight call a formalist approach, influenced by Hindenburg’s combinatorial school (Jahnke 1990, Teil B.; 1993; Séguin 2005; Noble de la Torre 2011).

Fries' Kantian approach appears clearly in such statements as "the infinite is the indefinite, and an infinite greatness or smallness can never be considered as a given totality" (Fries 1822, 258), which is brought as a solution for the Kantian antinomies and for some contradictions that Fries diagnoses in the mathematics of his contemporaries (including Schultz, who endorsed an actual infinite expanse despite his close association with Kant, see Schubring 1982). This approach gave a philosophical dimension to the understanding of calculus in terms of limits as developed (independently of Kantian philosophy, of course) by such mathematicians as D'Alembert, Cauchy and Bolzano.

Fries' "formalist" approach to infinities, on the other hand, distinguishes between syntactic operations (combinations of signs that follow arbitrary rules) and arithmetic operations (operations that deal with magnitudes and follow their inherent Kantian axioms). The former depends on order, and the latter on measure, and so the syntactic approach is independent of the arithmetic one (Fries 1822, 68). Operations with infinities and infinitesimals are strictly syntactic (Fries 1822, 280, 294), so infinite series generally have only a syntactic meaning, except where they converge arithmetically. Fries explicitly associated the syntactic approach with Hindenburg's combinatorial school. This school was not popular in France, but it had its presence there through such figures as Kramp and Arbogast and even Lagrange's earlier work (1774), which had helped to inspire it. Fries further associated the syntactic approach with the creative imagination, which for him belonged to reason (Jahnke 1990, 229; 1993, 268).

At first sight, Wronski, who was strongly influenced by both Kant and the combinatorial school, might be expected to have affirmed ideas similar to those of Fries.¹¹ However, the Kantian indefinite was too closely associated with the mathematics of limits (and therefore with D'Alembert and the enlightenment's anchoring of reason to the understanding) for Wronski to endorse it as foundational. The syntactic approach, on the other hand, was a nominalist-formalist approach that all but cut the cord between reason and understanding; related notions such as "useful fiction" or "arbitrary signs" reeked of such enlightened philosophers as Condillac and Condorcet, and left reason with no constitutive role.

¹¹ Schubring (1981) points out a further analogy between Fries and Wronski in terms of their attempt to set pure mathematics and applied mathematics apart. This approach helped professionalize mathematics in Germany, whereas its rejection seems to have contributed to the decline of mathematics in France.

Neither of the above paths was sufficiently opposed to the French Academy of Sciences, and neither accentuated enough the cleavage between liberal immanent finitists and illiberal transcendent infinitists, for Wronski to accept them as foundational. Indeed, Wronski's project of articulating, and then resolving, the gap between liberal/illiberal or immanent/transcendent or finite/infinite required an infinity that was more than an indefinite or a syntactic element. It required, first, an infinity that could stand in clear opposition to enlightened conceptions of magnitude, and, second, the means to reconcile this infinity with the rationalist-sensualist approach associated with Wronski's "liberals".

5. Beyond Kant's Infinity

Already in 1804 Wronski wrote of Kant that he "stopped at the RELATIVE, and did not ascend as far as the ABSOLUTE infinity" (LEG 123; here, "relative" and "absolute" infinities being the above "intellectually generated" and "ideally generated" ideal magnitudes respectively). Indeed, Kant proposed a transcendental solution to the antinomy of reason: since the understanding can only contain intellectually generated, relative infinities, the role of reason is reduced to a regulative role, that of ruling over the indefinite formation of intellectually generated, relative infinities. But the constitutive role of reason, establishing an actual, absolute infinity – Kant set aside.¹²

According to Wronski, this was no accident. Wronski believed that reason necessarily hides its role as constitutive of absolute infinities (Wronski uses here the French term "subreption"). Indeed, once reason constituted absolute infinity, this general notion shouldn't contain anything that's not also containable in one of its concrete manifestations, that is, in some relative infinity (this assumption is an important component of Wronski's philosophy, as we shall see in section 7 below). But then, once contained in some relative infinity, it becomes accessible to the understanding in such a way that reason reveals itself in a merely regulative role (LEG 130-131). So reason never presents itself to the understanding as constitutive, and in every encounter with the understanding the infinite is necessarily reduced to an indefinite.

But despite this appearing-disappearing act of reason, the existence of an absolute, actual infinity¹³

¹² Wronski did recognize that the transcendent level, where the absolute subsists, is available in Kant, but laments that it is only to be found in his critique of practical reason, rather than that of pure reason (Prod 58 fn.).

¹³ Since the above analysis only shows that absolute infinity would necessarily be hidden *if* it actually existed, Wronski

is the cornerstone of Wronski's philosophy in general and his philosophy of mathematics in particular. Indeed, in discussing mathematical infinite powers, for example, Wronski says that they are "possible only by the application of *reason* to functions of the understanding ... and are therefore *transcendent* functions, or conceptions of reason, ideas proposed by this supreme intellectual faculty" (INT 12; cf. REFI 345-346 for other transcendent infinities). Note that Wronski is talking here about transcendence (that which is independent of experience) rather than the transcendental (that which regulatively conditions experience). Wronski's infinities go beyond the transcendental and the indefinite.

As we saw, an intellectually generated infinity is one that would be contained in the understanding by an indefinite process of approximation, while an ideally generated infinity could not be contained by the understanding at all. This means that such an infinity does exist, but is completely invisible to the understanding. How is this expressed in actual mathematical experience? Wronski's answer is that an actual infinitely small magnitude would be seen as zero by the understanding. This is why two quantities differing by an actual infinitesimal, which are genuinely different from the point of view of reason, must be regarded as rigorously equal from the point of view of the understanding, that is, in empirical experience. Here is how Wronski puts it:

[T]he laws of infinitesimal calculus are purely subjective, that is, they are only rules for the generation of the cognition of the quantity, and not objective laws of the actual relations of quantities [I]t is true ... that two quantities A and B that differ only by an INDEFINITELY small quantity C are rigorously equal. Indeed, the idea of the infinitesimal quantity C being only a rule for the generation of the cognition of quantities of the [finite] order ... of the quantities A and B, and not yet a cognition acquired or engendered by some quantity (because ... the indefinitely small quantity C ... has no objective reality in the sphere of magnitude where the quantities A and B are to be found), it is clear that the relation between the quantities A and B in question, considered in its objective reality, is not changed at all by the purely subjective influence of the infinitesimal quantity C. (INF 38-39)

More succinctly put, "WITHIN THE CONDITIONS OF TIME, one could NEVER attain [an] indefinitely small difference" (INF 20).

provides an independent proof of its existence and our ability to grasp it, which won't be discussed here. Very briefly, Wronski derives this from our very doubt in our ability to grasp infinity – only if we have some understanding of what it is, could we have grounds for doubting our ability to grasp it.

In the above quotation there are two levels of articulation: subjective and objective. In Wronski's jargon "subjective" designates that which pertains to knowledge, but not necessarily to an individual knower. We see that this subjective level is "generative": it creates our conception of quantities by means of the infinitely small. The objective level relates to the world of phenomena. At this level, the infinitely small can only be rendered as zero. The infinitely small, which exists only at the former level, and cannot be approximated or represented in the latter, is therefore transcendent. It can give rise to indefinite approximations (e.g., of an irrational number by a power series of rational terms), but is never given or approximated as such.¹⁴

In his polemic against Carnot concerning the foundations of the calculus, Wronski asserted that Carnot's attempt to ground infinitesimal calculus algebraically, in terms of independent variables rather than infinitesimals, was logically flawed due to a vicious circle (reconstructed in Guerraggio and Panza 1985), which was covered over by the alleged reduction of calculus to Carnot's purely analytic principle: "two non-arbitrary quantities cannot differ except by a non-arbitrary quantity" (INF 22; "non-arbitrary" means here determined quantities without "arbitrarily small" or other variable components; the term "analytic" is used here in its Kantian sense: an a-priori logical truth, derivable from the concepts of the relevant terms). But Wronski objected that a purely analytic principle cannot expand our mathematical knowledge beyond the realm of understanding, which contains only rational numbers. To use Kantian language, mathematics includes a synthetic element (an element that cannot be derived from mere definitions and logical laws), and therefore depends on the constructive introduction of an element foreign to the understanding. Carnot's theory, as well as any other theory that is reducible to finite, rational quantities, lacks such an element.

Wronski's alternative founding principle (which is borrowed from Euler, but is given a new philosophical meaning) states that two quantities differing by an infinitesimal are really equal. This principle is properly generative: it carries over from reason something foreign to the realm of the understanding (INF 24), and this foreignness can account for the mathematical leap of knowledge beyond the understanding's world of straight lines and rational numbers, a leap inherent in the infinitesimal calculus. This is why, for Wronski, the metaphysical account of the foundation of calculus by the actual infinities of reason is superior to all other accounts: no other account explains how mathematics can exceed the realm of finite rational quantities available to the understanding

¹⁴ The fact that a sequence approximates a number does not mean that their difference approximates an infinitely small magnitude; see footnote 9 above.

(recall that for Wronski, even the incommensurability of the side and the diagonal of a square requires an infinitary reasoning, as well as the continuous variation of any curve that is not a straight line, as they are not accessible to discrete observation).

According to Wronski, the manifestation of the generative power of infinity is explicit in infinite expansions, such as Taylor's theorem (and Wronski's various generalizations), where a function is expressed by means of its derivatives, that is, by quotients of its differentials, which are infinitely small entities. Indeed, this generative-infinitary character is why, in function theory, Wronski prefers infinite sums to finite partial sums complemented by an error term (even though he did not ignore the problem of divergence):

this completely useless admission of a complementary quantity by which mathematicians wanted to form the idea of series proves irrefutably that they see in [series] only an ordinary simple summation ...; and that they still confuse with this summation ... the superior generation of quantities in terms of [infinite] series, which constitutes a totally different and heterogeneous algorithm (TECII 290).

This approach sheds light on Wronski's obsession with all sorts of series expansions: these are not just means of calculation or expression, but representations of the genuine generation of mathematical quantities by from transcendent infinities.

Because of this generative role, actual infinity, while not attainable in empirical experience, "can, in this domain of time [empirical experience], be used as are experimental quantities and in conjunction with them, and ... lead ... to very important and eminently correct results" (REFI 346).

A relatively standard application of this conception of the conjunctive use of finite and infinite quantities is the derivation of a formula for π (INT 26, with slight amendments). Defining π as the

number for which $e^{\sqrt{-1}\pi} = 1$ and taking the fourth root, one gets $e^{\frac{\sqrt{-1}\pi}{4}} = \sqrt{-1} = \frac{1 + \sqrt{-1}}{1 - \sqrt{-1}}$ and

therefore $\pi = \frac{4}{\sqrt{-1}} \left\{ \log(1 + \sqrt{-1}) - \log(1 - \sqrt{-1}) \right\}$. Using the definition of the logarithm from (1)

above, one gets

$$(4) \quad \pi = 4 \frac{\infty}{\sqrt{-1}} \left\{ (1 + \sqrt{-1})^{\infty} - (1 - \sqrt{-1})^{\infty} \right\}.$$

Wronski then develops the two binomials according to the standard binomial formula, and, neglecting higher infinitesimals to form the identity

$$\binom{1/\infty}{n} = \frac{(1/\infty)(1/\infty-1)\dots(1/\infty-n+1)}{n!} = (-1)^{n-1} \frac{1}{n^\infty},$$

obtains

$$(5) \pi = 4 \frac{\infty}{\sqrt{-1}} \frac{2\sqrt{-1}}{\infty} \left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right) = 8 \left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right).$$

But for Wronski these are not just *any* formulas for π . (4) is the philosophical generation of π by reason: a finite combination of ideal magnitudes and operations (actual infinities and powers). (5), on the other hand, is the best representation of π by the understanding: an indefinite series that can be used to approximate π using only addition and subtraction. These two formulas represent the two levels of mathematical cognition, and demonstrate how they are related.

A more idiosyncratic use of actual infinities that derives from Wronski's philosophy concerns an alleged paradox, discovered by Kramp (INT 202). Kramp defines factorials (*facultés*, a generalization of what is now called factorials) as:

$$a^{nr} = a(a+r)(a+2r)\dots(a+(n-1)r).$$

According to this definition, for any integer n one has

$$a^{nr} = a^n 1^{\frac{nr}{a}}; (-a)^{n-r} = (-a)^n 1^{\frac{nr}{a}}$$

And therefore

$$\frac{a^{nr}}{(-a)^{n-r}} = \frac{a^n}{(-a)^n}$$

This operation of factorial is extended to non integer values of n by means of power series representations and extension of algebraic identities. In turn, these power series and identities lead to such paradoxes as

$$\tan(m\pi) = \frac{m^{\frac{1}{2}}}{(-m)^{\frac{1}{2}-1}} = \frac{m^{\frac{1}{2}}}{(-m)^{\frac{1}{2}}},$$

Which are clearly false. This led Kramp to cast doubts on some established identities and techniques.

Wronski's solution was to go back to the infinitary generation of the relevant quantities (namely, the expressions that Wronski takes, following Kramp's manipulations and his own foundational hierarchy, to be their "philosophical" definitions, which, like (4) above represent their generation in reason), according to which $(-a)^n$ is a product of infinitely many terms of the form

$$\left(1 + \frac{\pi\sqrt{-1}}{\infty}\right) \left(1 + \frac{\log(a)}{\infty}\right).$$

According to Wronski's analysis, building on that of Kramp, the left (complex) term of this product does not appear in the decomposition of $(-a)^{n/r}$ into an infinite product. The missing infinitesimal complex term is supposedly the reason that the identity

$$a^{n/r} = a^n 1^{n/r/a}$$

does not extend to negative a and r . Therefore, this infinitary-generative analysis circumvents the ensuing contradiction (I do not enter into details, because Wronski's analysis is inconsistent; but for the purpose of explaining Wronski's ideology, his wrong analyses are as good as his correct ones).¹⁵ The point of this example is to show how the transcendent, infinitary, generative level reveals ideal elements that are hidden at the immanent level, and can serve to explain what appears paradoxical from the point of view of the understanding.

6. Universalizing transcendent and immanent magnitudes by acts of will

To further understand the relations between transcendent infinities and immanent mathematical practice, let's consider Wronski's distinction between theory and *techne*:

The Theory of Algorithmics [Wronski's preferred term for analysis] has as its object the *nature* or the *construction* of ... quantities. – The theory is a *speculation* where the *understanding* in general dominates: its object ... consists of *that which is* in the essence or construction of algorithmic quantities.¹⁶ The *Techne* is a sort of *action* where the *will* dominates: its object ... consists in *that which should be done* to arrive at the evaluation of algorithmic quantities. ... While the different quantities produced by the Theory of Algorithmics are given immediately by [summation, product, power, logarithms and

¹⁵ Indeed, in Wronski's analysis $1^{n/r/a}$ is decomposed differently when r and a are replaced by $-r$ and $-a$. The correct explanation for the apparent contradiction from a modern point of view would be the multiple values of fractional powers.

¹⁶ It seems that Wronski is concerned here with how the *concepts* of magnitudes are contained in the understanding (that is as concepts, not as the magnitudes themselves); otherwise ideal magnitudes would simply be foreign to the understanding.

exponents, trigonometric functions, additive and multiplicative differences and differentials, congruences and equations] nothing hinders the concept of generating the same quantities by means of other algorithms than those which give them their primitive determination, namely [infinite series]. But this secondary generation of algorithmic quantities, foreign to the primitive generation given by the algorithms forming the Theory of Algorithmics, cannot be a product of the understanding, which is precisely the faculty of the above named primitive generation. ... The secondary generation ... can only be a product of the will, an end or a goal (INT 206-208; see also TECI 17-18, REFI 56).

This relation between theory and techne or understanding and will is manifest in equations (4) and (5). In (4) we are presented with the theoretic nature of π : a finite combination of infinite magnitudes and higher arithmetical operations (such as powers). In (5) we have the evaluation of π , namely its elementary construction by an indefinite combination of finite magnitudes and the most basic arithmetic operations (addition and subtraction). The point is precisely to bridge the gap between the two levels: the understanding's finite manipulation of given theoretical elements (which may include elements not containable by the understanding as quantities, such as infinite magnitudes) and the will's technical power of indefinite construction.¹⁷

To understand the concept of "will" above, we should look at its more general definition. The will is not only the faculty that transforms finite combinations of mathematical elements (or theoretical expressions, like (4)) into indefinite summations (or technical expression, like (5)). The will is in charge of any passage from individual to universal:

Individuality manifests itself immediately in the effective existence of objects ..., whereas their universality is only a necessary postulate of the very possibility of that existence. From that follows that the former, individuality, forms an immediate object of our cognition, and that the latter, universality, can only become an object of our cognition mediately, by means of first

¹⁷ There's a change of point of view here in Wronski's own work, which may be confusing. If we look at the quantities themselves, then (4) involves ideal quantities generated by reason, whereas (5) involves only finite quantities and summations. From this point of view, (5) is a simpler expression. But from the point of view of putting together the concepts, without looking into their content, (4) is a finite combination, which is therefore accessible to the understanding, while (5) is an indefinite combination, which the understanding can never contain in its entirety. So from this point of view, (4) turns out to be simpler.

becoming a goal or an end of speculation, that is an object of our will (REFI 46-47).¹⁸

In other words, to exist in knowledge (the subjective level) means to exist under some universal. To exist in the world (objectively) means to exist as an individual. Putting together individuals under universals is not something given, but the constructive project of knowledge, and the faculty which implements this project is called the will. In mathematics, the ability to express all mathematical functions in terms of series would unite the various individual quantities under a universal form. This is precisely the project that Wronski terms “*techne*”: providing general means for producing power series and infinite product representation for all mathematical quantities.

Given this formulation, it makes sense to place mathematics in the grand schematic articulation of philosophical knowledge under the title “Accomplishment of *Nature*” (REFI 530), that is, the project of doing what’s necessary to bring elementary, immediate given beings to their full development in universal knowledge. Mathematics in general, and its *techne* in particular, is the task of *creating* a universal systematization of magnitudes, whereby reason forms being through acts of will.

In Wronski’s system, the absolute infinity of reason is not just the Kantian tool that regulates our understanding, but a constative product that relegates to our will the task of applying it so as to unify and reform our world. Reason’s infinity generates magnitudes that the understanding can handle as given concepts (even if it can’t approximate them as quantities), and then charges the mathematician’s will with the task of systematizing the resulting conceptual reality in terms accessible to finite empirical intuition.

The use of actual infinities in decomposing a function into an indefinite product of linear terms, as in (3), or an indefinite sum, as in (5), may initially appear as an arbitrary game of signs; in fact, however, this is the fulfillment of the will’s task of turning distinct magnitudes into a universal system, where all functions can be expressed as infinite products of linear terms and infinite sums of

¹⁸ This notion of will seems to be Kantian (standing with respect to theoretical understanding as practical reason stands with respect to pure reason), rather than influenced by Kant's later followers, for whom the concept of will tends to associate with autonomy and self determination rather than with universal laws. Wronski's early usage of "will" is too early to be influenced by Schopenhauer, but we can't rule out influence by Fichte, Schelling or Hegel. This influence clearly grows in Wronski's later writings.

powers. The will actively constructs such secondary (indefinite additive or multiplicative) descriptions of those magnitudes generated by reason, which finite understanding was able to handle as combinations of given concepts, but was too poor to systematize and approximate without external intervention.

Representing functions as series was a common feature of much of 18th century mathematics. Wronski gives this project a philosophical motivation in terms of the various faculties of cognition and the task of organizing knowledge under universals. Due to this motivation, Wronski sought to bring all series representation techniques known at the time under a single form: his so called “universal law” (see Appendix B).

But to meet this challenge of universalization, the will must not be subject to the limitations of finite calculations: universality is more important than convergence. Therefore, Wronski first affirms that both convergent and divergent series are acceptable. Indeed, given a function F , the coefficients of its Taylor series $F(x) = A_0 + A_1x + A_2x^2 + \dots$ (as well as its generalizations studied by Wronski) can be determined by such iterative processes as:

$$A_0 = F(0); A_1 = \frac{F(dx) - A_0}{dx}; A_2 = \frac{F(dx) - A_0 - A_1dx}{dx^2} \dots,$$

where dx are infinitesimals. The coefficients of the series are assumed to fully characterize the function F , because it is evident that no other function “can produce the same sequence of quantities A_0, A_1, A_2 , etc.”, and therefore regardless of whether it is “convergent or divergent, this series always presents, in the law of the generation of the coefficients \dots , a rigorous and complete determination of the quantity $F(x)$ (TECII 288-289).¹⁹ According to this view, even in cases where, as in (3), coefficients turn out to be infinite, they still represent the given function, and may be related to finite convergent sums (see also Wagner 2012).

¹⁹ While today this claim is considered false, it was held true by Lagrange (1797, 37) (who qualifies it as cast in doubt by “some geometers” who remain unnamed). Even after Cauchy’s famous example of $\exp(-1/x^2)$, which generates the same Taylor coefficients as 0 when expanded around 0, some mathematicians remained unconvinced. Poisson rejected the claim by allowing the order of the derivative to increase as x decreases (Grattan-Guinness 1990, 735-736); Martin Ohm rejected it because the derivatives at zero were not well defined in the complex plain; and Pringsheim and du Bois-Reymond later explained why, in the framework of 18th century algebraic analysis, the derivatives at zero of this function were not well defined, and, therefore, Cauchy’s example did not work (Jahnke 1990, 302-306).

Problems arise, however, if we start from a series, and try to go back to the function. If the series does not converge, then “the hypothetic function F_x , corresponding to the general value of the series, will not exist effectively unless [it is transformed into a series] that could be convergent”. Therefore “a series formed arbitrarily does not always have a general value F_x corresponding to all values of the variable x ” (TEC II 294-295). Indeed, the two volumes of TEC devote hundreds of pages to transforming divergent series into convergent ones and to extending domains of convergence. These transformations include change of variables, the use of weighted partial sums of coefficients (in modern terms: convolutions), various decompositions and continued fractions.

This approach shows how wrong it is to view Wronski as some sort of mindless formalist, even when he allows himself to play around with quantity concepts that the understanding cannot even gradually approximate. Wronski’s work is always about the connection between extravagant infinitary maneuvers and a concern for the level of bottom line calculation. His claims are often too general, and his techniques do not always guarantee the convergence that they purport to bring, but he still considers convergence to be an important concern. For Wronski, mathematical techne is all about willing a connection – a connection which is *not* given in advance – between a constative infinity and finite worldly experience. Where will fails in this task, our knowledge is either false or not completely formed. Wronski’s system does not use infinities “opportunistically” to obtain empirically relevant results; it is rather founded on infinities, and seeks to reform the world by reorganizing it under their generative foundation.

In fact, this vision is not limited to mathematics. The creative aspect of reason is applicable to the entire philosophical project:

[T]he true definition of philosophy, unknown to this day, consists in that the object of philosophy is the CREATION OF THE TRUTH, and not, as is still believed universally today, the KNOWLEDGE OF created TRUTHS, knowledge which is properly only the special object of the sciences (REFI 517; see also Warrain 1933, vol. 1, 91-92).

7. Wronski in context, Part II

Now that we’ve placed Wronski’s infinity in the context of his mathematical project, it’s time to zoom out to a wider ideological context. Indeed, Wronski repeatedly explains that his mathematical work is subordinate to his general philosophical goal, and that the point of the former is merely to

prefigure the fruitfulness and validity of the latter.

As stated above, Wronski's philosophy is intertwined with a social project: bridging the gap between what he considered as the liberal and illiberal parties. The liberal party consists of champions of free thinking and reason – but they anchor reason to the conditions of empirical experience, and therefore to finite immanence. The illiberals endorse the Christian revelation upon faith; they therefore accept transcendence and infinity, but neglect to support their faith with reason. Wronski's socio-politico-philosophical project is a project of reconciliation, where

it is by means of philosophy itself, namely by transcendent philosophy, that the party of cognition [liberals] will come to realize the ideal of God in the PRIMITIVE IDENTITY between the non-I and the I, constituting the absolute True, and ... it is by the way of religion, namely of accomplished christianity, that the party of sentiment [illiberals] will too come to realize the ideal of the immortality of the soul in the FINAL IDENTITY of the I and non-I, constituting the absolute Good; in such a way that, having arrived in the same regions of the absolute, where the true and the good become identical, all antagonism between these social parties will have to cease (PROL 465).

In the “final identity” above, “I” is created man and “Non-I” the creator God (the use of I and non-I here and below is explicitly linked to Fichte). What illiberals miss is the creative power of man's reason, whose discovery is independent of (Christian) revelation of an infinite absolute. Wronski wants to bring illiberals to acknowledge and synthesize their own rational creative power so as to advance the process of man's becoming immortal and one with God.²⁰ The creative, universalizing project of the will described in section 6 is a means to this goal. Mathematics provides a pristine example of how man can apply constitutive infinitary reason and create a unified world. Such an example, Wronski hoped, would convince illiberals in the promise of Wronski's larger philosophical project.

In the “primitive identity” above, “I” is subjective knowledge and “Non-I” objective reality. What liberals miss is the fact that constructs of reason are not only in reason, but are actual realities. Wronski wants to bring liberals to deduce the necessary existence of that which they confine to the realm of subjective speculation: actual infinities. Again, mathematics is a pristine example. The

²⁰ Note that for Wronski, what man creates is also destined to be created, so creativity should not be confused with open ended freedom; man might lose his way and fail to advance, but his advance has only one direction.

foundational project described in section 5 is meant to convince that an actual infinity must be hypothesized as a generative element in mathematics in order to understand mathematics correctly. If he convinced liberals of this truth, they would be more likely to accept the reality of transcendent infinities outside mathematics as well. The two projects follow different paths (backward analysis vs. forward synthesis), but when they are completed, the gap between liberals and illiberals will close.

In the above discourse concerning the will, self creation, and the identity of knowledge and being, we can hear the obvious echoes of German idealism. Wronski refers repeatedly in his later work to such thinkers as Fichte, Schelling, Krause, Hegel and F. Schlegel (e.g. PROL 112-177, see also d'Arcy 1970). According to Wronski, each of them has an affinity to some aspect of his own complete system, but always in a very partial or distorted way.

An interesting point of comparison here is Hegel. In his analysis of infinity in the *Science of Logic*, Hegel rejected both actual infinity and Kantian indefiniteness as insufficient expressions of the concept of infinity. For Hegel, the true infinity is manifest in a contradiction which he finds expressed in (or reconstructs from what he takes as) Newton's and Lagrange's conceptions of the differential. When one considers the differential quotient as the quotient of two evanescent quantities, this differential quotient is not contained in any of the finite quotients involved in the indefinite process of approximation; nor do we ever reach a quotient of actual infinitely small quantities, because the evanescent quantities disappear when the differential quotient is "reached". It is the inherent contradiction of a ratio that is obtained only when the relata disappear – or of a ratio without relata – which is, for Hegel, the true expression of the concept of infinity.²¹

Wronski did not comment directly on Hegel's concept of the infinite, but he did comment on his general philosophical project. For Wronski, Hegel's work developed only one of two necessary directions: that of advancing reason beyond empirical reality towards the absolute (the "final identity" above). Since Hegel failed to identify this development of subjective reason with an already given transcendent external reality (an originary absolute), his move was fundamentally

²¹ See Lacroix (2000) for a continental interpretation of Hegel's notion of infinity, and Pinkard (1981) for an analytic approach; Wolff (1986) provides a thorough analysis of Hegel's understanding of mathematical entities in the context of the work of Lagrange and Cauchy, which shows some further parallels with Wronski's early work, but these parallels exceed the scope of this paper.

partial. Wronski's own mathematical approach was meant, like Hegel's, to convert "liberals" to transcendent reason, but unlike Hegel (at least as perceived by Wronski), he insisted on maintaining an identity between the absolute goal of the process and a foundational absolute reality. Indeed, Wronski wrote that his mathematical task was, among other things, to "retrieve for this sublime idea [infinity] the creative role in mathematics" (PROD 85), whereas for Hegel the point seems to have been only to lead us via mathematics from immanent, empirical perception to an idea of the absolute.

The back-and-forth relation between the infinite and the finite was Wronski's main goal. This goal is expressed in formulas such as (1), (3) and (4-5) which link finite combinations of infinite magnitudes with indefinite combinations of finite magnitudes; this goal is expressed in the foundational role given to infinity in Wronski's system; and, finally, it is expressed in the universalizing act of will, which reaches its zenith with Wronski's "universal law": the expansion of any function as an infinite series of any sequence of functions. This supreme law implemented infinitary reasoning in order to couple any mathematical magnitude with an infinite series that could, in principle, be converted into a convergent expression (REFI 11), thus creating a reality that unifies objective experience with reason's generative capacity.

Given the privileged status of mathematics in his contemporary system of knowledge, Wronski's reform of mathematics was supposed to be metonymic of his general socio-political-philosophical reform, and convince the most serious scientists that his path were to bear fruit.

8. Conclusion

Wronski's project may have been noble, but he failed miserably in promoting it. There are several reasons for this failure. First, Wronski insisted on mixing philosophy and mathematics in a way that was not accessible to contemporary mathematicians. Indeed, his early mathematical work repeatedly refers to the Kantian philosophy not yet disseminated in France, while using some of its concepts without properly introducing them. Second, Wronski's work tended to have an esoteric dimension, withholding proofs from a community that was wary of unsubstantiated claims. Third, Wronski's writing style was repetitive, self congratulatory, dismissive and alienating. Finally, the intellectual scene, especially around the academy, was highly polemical, and professional discords were often overdetermined by generation, class and ideological gaps (see Ehrhardt 2010, 2011 for examples). Wronski, as a foreigner, was on the disadvantaged end of these gaps. He often felt that his origin

turned people against him, and that he was without “national protection” (REFII 204-205).

On top of these “communication” issues, the very substance of Wronski’s work was unappealing to the Parisian high mathematics scene. First, neither Wronski’s philosophical style (German idealism) nor his mathematical style (influenced by Hindenburg’s combinatorial school) were of the kind popular in France. Second, as observed in Appendix B, Wronski was out of touch with contemporary mathematical developments, especially Fourier series and Cauchy’s work. With all these odds turned against him, Wronski still dared direct attacks against too many powerful enemies. The oppressive repercussions of these attacks are summarized in Appendix A.

So, where does this loser fit in a history of mathematics? Whig histories of mathematical infinity tell the story of a problematic notion, plagued by inherent contradictions, which was finally tamed by Weierstrass’ epsilon-delta calculus and Zermello-Fraenkel’s axiomatic set theory. Other histories do exist, but most of them (notable exceptions notwithstanding) tend to concentrate on the heroes and capitals of mathematical production. This “heroic” focus obscures the role that mathematical concepts played in other scenes, such as the scientific periphery, mid level education and popular science.

This paper presented one “loser’s” approach to infinity. This loser promoted a transcendent notion of actual infinity, which, through acts of will, was to unite transcendent knowledge and immanent being into one absolute reality. This was not only a mathematical project, but also a political project. For Wronski, the huge divide between transcendent and immanent reflected the political divide between liberals and illiberals; the creative act of bridging this divide was as political as it was philosophical, mathematical and religious. This conglomerate of scientific, political, philosophical and religious unification of the infinite and the finite was christened by Wronski “messianism”. The project of messianism was the project of *creating* a new, salvific reality through rational investigation, led by the investigation of mathematics and its philosophy.

Only when Wronski’s mathematical work is set in this kind of context, can we make sense of his insistence on treating infinity in a way that antagonized the very people he sought to impress and cost him his reputation and career. Given that this is the case here, more general questions open up, such as how mathematics related to philosophy and society beyond the context of 19th century European mathematical “heroes”, and which roles mathematics played in the various scenes of knowledge/power/culture where German Idealism thrived before the advent of 20th century

modernism. An exploration of these questions may shed new light on the foundations of mathematics debate of the early 20th century – a debate which is usually presented in abstract philosophical terms and in the context of its most recent history, but which may prove to be linked with earlier foundational projects anchored in the early 19th century and the philosophical trends of the time. I hope that this paper makes a contribution to the project of answering such questions.

Appendix A: A short biography:

Józef Maria Hoene (later Hoëné Wronski) was born near Poznan in 1776.²² His father was a Czech architect. We know little about his upbringing, but he was well educated, joined the army at the age of 16, and took part in the failed Polish uprising of 1794. Wronski was taken prisoner by the Russians and then joined their army, where he reached the rank of lieutenant colonel. In 1797 he left Eastern Europe for good to join an exile Polish legion in the west. He spent a couple of years studying in Germany, and reached the legion in Marseilles in 1800.

Wronski then moved on to scientific work as astronomer in the local observatory and was officially involved in scientific associations in Marseilles. He started publishing his review of Kantian philosophy already in 1803, but quickly discontinued the publication. By 1810 he had accumulated enough philosophical ideas and mathematical work to travel to Paris and make a submission to the *Académie des Sciences*. The report written by Lagrange and Lacroix commended some of his achievements, but pointed out some lack of clarity and missing proofs. Wronski published a rather aggressive response in the *Moniteur* of November 21, 1810.²³

²² Wronski included a short autobiography in his REFII, and another manuscript was published posthumously in his LOI. Dickstein (1896) wrote his only significant biography, which is in Polish, so I could only consult it via the dubious mediation of computer generated translations. Brief biographies are available in Phili (1996), Murawski (2005) and Pragacz (2008).

²³ According to the Procès Verbaux des Séances de l'Académie, on August 13, 1810 Wronski presented to the academy two papers: "Premiers Principes des Méthodes Analytiques" and "Examen Analytique du Micromètre Géodésique". The former was reported on by Lagrange and Lacroix on October 15 (reprinted in the *Moniteur* of November 15, 1810). I could not find a report on the latter.

In 1811 Wronski published his *Introduction à la Philosophie des Mathématiques* (INT)²⁴, which attempted a reorganization of contemporary mathematics based on his (then unstated and perhaps underdeveloped) philosophical program. Gergonne published a mixed critique in the second volume of his *Annales* (pp. 65-58), seeing promise in the work, but lamenting the lack of access to many of its underlying philosophical ideas, and encouraging clearer publications.²⁵ In 1812 Wronski published a so-called “refutation” of Lagrange’s theory of Analytic Functions (LAG) and a general solution for all polynomial equations by means of roots of the coefficients.²⁶ According to Wronski, the counterattack following the former publication cost him a stipend he had begun receiving from the Russian court. The latter publication resulted in a series of critiques by Gergonne (1812-1813a,b,c, 1818-1819), and an unpleasant exchange with an anonymous respondent (who according to Wronski’s allusion was Biot) in the *Moniteur* of November 22, 1812.

Following these polemics, Wronski describes a reality where students he had tutored were deliberately failed, whereas a student that he had tutored, but was examined as the student of another tutor, passed (REFII 103). This form of persecution left Wronski destitute. He was granted the opportunity of presenting a course on Kantian philosophy at the Sorbonne, but had to decline, because he was so poor he couldn’t even afford proper shoes. Moreover, he reports that he could not afford to get medical help for his dying son, and that his works were sold for the weight of their paper. In 1831, Galois would write that he did not want to end up “compared with Wronski or these indefatigable men who every year find a new solution to the squaring of the circle” (Neumann 2011, 249).

In 1814 Wronski was introduced to Arson, a person who made a quick fortune and decided to leave

²⁴ A list of abbreviations for works by Wronski cited in this paper is available at the end of this paper.

²⁵ On June 10, 1811, Wronski presented to the academy his introduction to the philosophy of mathematics (INT).

Lacroix was assigned to review it, but a week later announced that it had nothing to do with the previous work, and, being based on a metaphysics that Lacroix had not studied, he could not review it.

²⁶ On September 9, 1811, Wronski submitted the “réfutation de la Théorie des Fonction” (which would later be published in LAG). Legendre and Arago wrote a condemning report on November 11. Servois (1814), too, reacted fiercely. On may 11, 1812, Wronski submitted his “Résolutions Générales des Equations” (later appended to INT). A week later Poisson reports on it, but the report itself (which, according to LOI 32, dismissed the possibility of a solution) is not cited.

his business ventures to pursue science and philosophy. Their relationship resulted in several mathematical publications, but also a public polemic over alleged debts that Arson refused to pay. The notoriety of the affair was exacerbated by a question that Wronski publicly presented to Arson: did he, or did he not, disclose to Arson the secret of the absolute. If the answer were “no”, then Wronski would forgo the alleged debt. Arson, who portrays Wronski as a truly wise man, but also as someone we might today call bipolar,²⁷ could not bring himself to state an unequivocal “no”. He thought Wronski may be deliberately causing a scandal in order to increase his own fame. There were also allegations concerning a “secret society” meddling in the affair. The repercussions of the whole drama included several pamphlets published by the protagonists, coverage in popular daily newspapers,²⁸ and a novel by Balzac (*La Recherche de l’Absolu*).

Following this affair Wronski took up a challenge set by the British Board of Longitudes to construct measuring devices. His instruments were confiscated by customs, and his presentation of a reform of mathematics and astronomy was considered to be too theoretical and vague to merit the Board’s attention. This resulted in further disputes and allegations of plagiarism against the Board’s secretary, Thomas Young.²⁹ Later Wronski would make plagiarism allegations against Poisson and others as well (e.g. REFI 18, REFII cci).

A few years after returning from England, Wronski tried his luck in Belgium for a short period, but found little success. He published little throughout the 1820s, but still retained a few friends, and managed to attract a few more followers. During the 1830-40s he tried to promote a reform of locomotion, replacing railways by vehicles with their own tracks (as in modern tanks). This project attracted the attention of, among others, the Ministry of Public Works, but ended without results. At the same time Wronski developed and published his philosophy and political analysis, and was again attacked by *Académiciens*, in the *Montieur* of June 5, 1844.

In the early 50s, shortly before his death in 1853, Wronski toured Germany, trying to disseminate his ideas. In 1852 Adolphe Constant managed to print a series of favorable reports on Wronski’s

²⁷ A police report from around that time says of Wronski “this is not a dangerous madman” (Phili 1996, 300).

²⁸ *L’homme Gris* no. 5 from 1817, *La Quotidienne* of May 24 and July 9 1818, and later, when Wronski went back to publishing, in *Le Figaro* of August 21, 1831.

²⁹ On October 10, 1822, Wronski submits a series of manuscripts concerning his polemic against the British Board of Longitudes, but the Academy decides not to review them.

philosophy in the *Moniteur* (issues 23,26,30,48). The most notable cultural figures who made Wronski's acquaintance and felt indebted to him were the musician Camille Durutte, the physicist Yvon Villarceau and the mystic Eliphas Levi.

Appendix B: A review of Wronski's mathematical work

Wronski wrote profusely on mathematics,³⁰ philosophy, politics, religion and physics, and did not refrain from commenting on many other scientific subjects. Given his biography, it is easy to dismiss him as a quack, but at least in mathematics he did make some genuine contributions, which could have made an impact, had he integrated better with the mathematical establishment. It should be noted that while he was highly knowledgeable in the mathematics of the turn of the century, it seems that he did not keep up with developments. Cauchy, for example, is hardly ever mentioned in Wronski's works, and never in a substantial way. Neither is there any attempt to engage with Fourier series, which could be highly relevant to Wronski's own work.

Wronski's most notable mathematical contribution is his "Supreme Law": a formula for the coefficients of the development of any function F by any sequence of functions $\Omega_1, \Omega_2, \Omega_3, \dots$. His proof is based on taking the system of equations formed by the equation $F = a_1\Omega_1 + a_2\Omega_2 + a_3\Omega_3 + \dots$ and its term-by-term derivatives of all finite orders. Implicitly assuming that these term-by-term derivations work, Wronski uses clever combinatorial determinant-like sums to derive formulas for the coefficients a_i (TECHI). This treatment led Muir (1890) to christen the determinant known today as "Wronskian" by that name. Wronski's universal law attracted the attention of Cayley (1873), who reproved some version of it. Charles Lagrange (1884, 1884a) then provided a classical treatment with an error term and conditions of convergence, and Banach (1939) provided a modern, functional analytic treatment.

It should be noted that while this formula subsumed not only Taylor's formula as a special case, but also a vast number of other contemporary methods (a feat acknowledged by Lacroix and Lagrange), it necessarily failed where term-by-term derivation did not work. This failure characterizes the most common Fourier series, which were on the rise at the time, and which Wronski almost never considered. Perhaps he meant to provide an alternative to Fourier series by his "higher trigonometric

³⁰ The best summary of Wronski mathematics is available in Dickstein (1892-1896).

functions”. These functions are sine and cosine like functions formed by choosing n -th roots of unity ρ and σ , and extracting from the series expansion of $e^{\rho x}$ the series of those terms which are a real multiple of σ . These functions were discussed as tools for analyzing celestial mechanics, but never explicitly as an alternative to Fourier series.

Wronski explored many variations of expansions of function as series of other functions, and attempted to provide methods that would guarantee convergence (TECHI, TECHII). Of particular interest is his “supreme method”. Here, having constructed the approximation $a_1\Omega_1+a_2\Omega_2+\dots+a_n\Omega_n$ for the function F , the function Ω_{n+1} is chosen so as to render zero the Wronskian determinant of F and $\Omega_1, \Omega_2, \dots, \Omega_n$. The point here is of course not to find the trivial solution $\Omega_{n+1} = F - a_1\Omega_1 - a_2\Omega_2 - \dots - a_n\Omega_n$, but an optimal approximate solution from within a restricted collection of functions. Such expansion would reflect, according to Wronski, not only the value of the function, but also its nature.

Wronski’s next step was to solve differential equations by extending a theorem of Lagrange that provided a power series representation for a function given implicitly by a functional or differential equation. This was Wronski’s so called “universal problem” (REFI, Hanegraeff 1856). This approach led to the following method. Given a differential equation, one constructs a homotopy between the given equation and a linear equation with constant coefficients. Then, starting from the known solution of the linear equation, one expands the solution of the original equation as a power series in terms of the homotopy parameter. This method is described by West (1886) in a book that evolved from a series of papers by West on Wronski’s mathematics in the *Journal des Mathématiques Pures et Appliquées*.

Here should be noted Wronski’s concern with providing exhaustive calculations of the coefficients of his various theorems and techniques, rather than leaving the practitioner to calculate them ad-hoc. To do that at the required level of generality, Wronski devised a complex apparatus of combinatorial sums enabling him to state general recursive and closed formulas for his coefficients.

Wronski’s next contribution concerns polynomial equations. His INT (111-112) includes a non constructive argument that’s meant to show that all polynomial equations can be solved by applying the basic arithmetical operations to roots of the coefficients (the reasoning there was refuted in Servois 1814, 163-166). An appendix includes a conversion of a polynomial equation of degree n to a system of n equations in $n-1$ symmetric variables, which is supposed to be reducible to a single

polynomial of degree $n-1$ in one variable (even though the degree of the previous system of equations is hyperexponential in n). The initial conversion seems plausible (based on my very cursory reading), while the subsequent reduction is false, and was refuted by Ruffini (1818) and independently by Torriani (1819). In 1827 (CAN 59) Wronski acknowledged that the degree of his “reduced” equation may be higher than that of the original equation, but later he seems to state that his solution does work after all (REFI 24, REFII cci, cciv).

Another approach (which does work) to reducing polynomial equations is based on the above mentioned series expansion of a function given implicitly (REFIII). Here the construction of the coefficients of the reduced equation requires an indefinite approximation. A third approach uses recursive calculations with the coefficients of the equation in order to approximate solutions (CAN, REFIII). This method is based on a method of Daniel Bernoulli. Bernoulli noted that under some circumstances, if one substitutes recursively into the equation $x_n = a_1 x_{n-1} + a_2 x_{n-2} + \dots + a_m x_{n-m}$, one would get a result of the order of c^n , where c is the largest root of the polynomial equation $x^n = a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_m x^{n-m}$. Wronski generalized this method so as to solve all polynomial equations, but withheld the proof. The proof was reconstructed independently by Hanegraeff (1854), Bukaty (1878), de Montessus (1905) and Lascoux (1990).

Wronski tried to provide a general solution of congruence equations of the form $x^m = a \pmod{b}$ as well (REFI). Here the treatment is based on Gauss’ solution of linear congruence equations, and aims at reducing the range of searching for solutions, rather than actually providing them. Wronski’s unproven statements here are more general than his methods can actually deliver. Hanegraeff (1860) and Bukaty (1873) provide reconstructions.

Another achievement, this time a popular mathematical artifact, was Wronski’s compact logarithm tables published in 1827 (CAN). These were based on the first terms of a series expansion of the logarithm. The trick was to give, instead of the full logarithm, a simple way to pick up from a single sheet of paper two or three blocks of digits that compose the desired logarithm. These tables were studied by Bushaw (1983), who commended their ingenuity.

Wronski also planned some computation machines, and ventured to provide an alternative probability theory, which rejected the independence of subsequent steps in a game (e.g., dice throws, roulette wheel turns, stock price increments). Instead, he claimed that subsequent steps tended to “even out”: if a certain result was obtained, its chances of recurrence diminished. As odd as this may

sound today, this approach in fact builds on a previous attempt by no less than d'Alembert (Daston 1979).

I cannot give an adequate review or evaluation of Wronski's physics (and as far as I know no such review was ever attempted). Wronski's attempted reforms covered terrestrial and celestial mechanics as well as hydrodynamics, and were applied to locomotion, the study of the shape of the earth, the study of tides, and astronomy. He dabbled with the idea of *perpetum mobile*, and claimed that Leverrier's discovery of Neptune could not have been made except by his methods "since there are no others" (REFIII, 6).

Villarceau (1882) wrote that Wronski's physics is not fully coherent, but does contain useful ideas. A group of scholars (Ksawery Jankowski, Godofredo García, Paulin Chomicz and P.P. Demiańczuk) found Wronski's physical theory relevant as late as the 1930s, and tried to build on it. Some discussion and references are available in the 1938 issue of *Acta Astronomica* (Ser. C, Vol 3).

Abbreviations of cited publications by Wronski

INT	Hoëné de Wronksi 1811
LAG	Hoëné Wronksi 1812
INF	Hoëné Wronksi 1814
TECI	Hoëné Wronksi 1815
TECII	Hoëné Wronksi 1816-7
CAN	Hoëné Wronksi 1827
PROD	Hoëné Wronksi 1831
PROL	Hoëné Wronksi 1842-3
REFI	Hoëné Wronksi 1847
REFII	Hoëné Wronksi 1848
REFIII	Hoëné Wronksi 1848
LEG	Hoëné Wronksi 1879
LOI	Hoëné Wronksi 1890

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