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Proximity to ℓ_1 and Distortion in Asymptotic ℓ_1 Spaces

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Abstract

For an asymptotic ℓ_1 space X with a basis (x_i) certain asymptotic ℓ_1 constants, $\delta_\alpha(X)$ are defined for $\alpha < \omega_1$. $\delta_\alpha(X)$ measures the equivalence between all normalized block bases $(y_i)_{i=1}^k$ of (x_i) which are S_α-admissible with respect to (x_i) (S_α is the α^{th}-Schreier class of sets) and the unit vector basis of ℓ_1^k. This leads to the concept of the delta spectrum of X, $\Delta(X)$, which reflects the behavior of stabilized limits of $\delta_\alpha(X)$. The analogues of these constants under all renormings of X are also defined and studied. We investigate $\Delta(X)$ both in general and for spaces of bounded distortion. We also prove several results on distorting the classical Tsirelson's space T and its relatives.

1 Introduction

The first non-trivial example of what is now called an asymptotic ℓ_1 space was discovered by Tsirelson [26]. This space and its variations were extensively studied in many papers (see [8]). While the finite-dimensional asymptotic structure of these spaces is the same as that of ℓ_1, they do not contain an infinite-dimensional subspace isomorphic to ℓ_1, and thus their geometry is inherently different.

The idea of investigating the geometry of a Banach space by studying its asymptotic finite-dimensional subpaces arose naturally in recent studies related to problems of distortion, i.e. the stabilization of equivalent norms on infinite dimensional subspaces of a given Banach space. These ideas were further developed and precisely formulated in [18].

By a finite-dimensional asymptotic subspace of X we mean a subspace spanned by blocks of a given basis living sufficiently far along the basis. By an asymptotic ℓ_p space we mean a space all of whose asymptotic subspaces are ℓ_p^n, i.e. any n successive normalized blocks of the basis $\{e_i\}_{i=1}^\infty$ supported after e_n are C-equivalent to the unit vector basis of ℓ_p^n.

In this paper we introduce a concept which bridges the gap between this "first order" structure of an asymptotic ℓ_1 space and the global structure of its infinite-dimensional subspaces. This concept employs a hierarchy of families of finite subsets of $\mathbb{N}$ of increasing complexity, the Schreier classes $(\mathcal{S}_\alpha)_{\alpha<\omega_1}$ introduced in [1]. For $\alpha < \omega_1$ we define what it means for a normalized block basis to be $\mathcal{S}_\alpha$-admissible with respect to the basis (e_i), and then measure the equivalence constant

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between all such blocks and the standard unit vector basis of ℓ_1, obtaining the parameter $\delta_\alpha(e_i)$. These constants increase when passing to block bases and this leads us to define the Δ-spectrum of X, $\Delta(X)$, to be the set of all stabilized limits $\gamma = (\gamma_\alpha)$ of $(\delta_\alpha(e_i))$ as (e_i) ranges over all block bases of X.

We show that these concepts provide useful and efficient tools for studying the infinite dimensional and asymptotic structure of asymptotic ℓ_1 spaces. Indeed, even some first order asymptotic problems require a higher order analysis. The behavior of the Δ-spectrum of X has deep implications in regard to the distortability of X and its subspaces.

We now describe the contents of the paper in more detail.

Section 2 reviews concepts and results concerning distortion and asymptotic ℓ_1 spaces. We sketch the proof of the 2-distortability of Tsirelson's space in Proposition 2.7. This leads to a natural question as to whether the asymptotic structure of T can be distorted: can T be given an equivalent norm such that its asymptotic subspaces are closer to ℓ_1? Without resorting to the higher order analysis developed in subsequent sections we only obtain a partial solution (the complete solution is then provided in Section 5).

In Section 3 we define the Schreier families $\mathcal{S}_\alpha$ and establish some facts about their mutual relationship which are crucial for our later work.

Section 4 contains precise definitions of all the asymptotic ℓ_1 constants which we introduce in this paper. We also define the spectrum $\Delta(X)$. Elements $\gamma = (\gamma_\alpha)_{\alpha<\omega_1}$ of the spectrum satisfy $\gamma_\alpha\gamma_\beta \le \gamma_{\alpha+\beta}$ for all α, $\beta < \omega_1$ (Proposition 4.11). It follows that $\hat{\gamma}_\alpha = \lim_{n\to\infty} \gamma_{\alpha\cdot n}^{1/n}$ exists for all $\alpha < \omega_1$ and it is shown to equal $\ddot{\delta}_\alpha(Y)$ for some subspace $Y \subseteq X$ (Proposition 4.15). $\ddot{\delta}_\alpha(Y)$ is defined to be the largest of $\delta_\alpha((x_i), |\cdot|)$ as (x_i) ranges over all block bases of Y and $|\cdot|$ over all equivalent norms. The constants $(\ddot{\delta}_\alpha(X))_{\alpha<\omega_1}$ exhibit a remarkable regularity. They are constantly one until α reaches the spectral index of X, $I_\Delta(X)$; and then decrease geometrically to 0 as α reaches $I_\Delta(X)\cdot\omega$ (Theorem 4.23). An important tool in this section is the renorming result of Theorem 4.20.

Section 5 contains the calculation of asymptotic constants for various asymptotic ℓ_1 spaces. We consider T along with various other Tsirelson and mixed Tsirelson spaces. These and other examples show that there is potentially considerable variety in the spectrum of X despite the regularity conditions imposed when considering all renormings. In addition it is shown that for $\gamma \in \Delta(X)$ an appropriate block basis in X admits a lower T_{γ_1} block Tsirelson estimate.

The central theme of Section 6 is the following problem: Does there exist an asymptotic ℓ_1 Banach space of bounded distortion? In particular, is Tsirelson's space of bounded distortion? We apply our work to obtain some partial results in this and related directions. We consider the consequences of assuming that an asymptotic ℓ_1 space is of bounded distortion. In particular the asymptotic constants must behave in a geometric fashion (Theorem 6.8, Corollary 6.9, Propositions 6.12 and 6.13). Also, an asymptotic ℓ_1 space of bounded distortion bears a striking resemblance to a subspace of a Tsirelson-type space $T(\mathcal{S}_\alpha, \theta)$ for some $\alpha < \omega_1$ and $0 < \theta < 1$ (Theorem 6.10). Furthermore we show that a renorming of Tsirelson's space T for which there exists γ in the spectrum with $\gamma_1 = 1/2$ cannot distort T by more than a fixed constant (Theorem 6.2).

2 Preliminaries

In this paper we shall use certain notation and basic facts from Banach space theory, as presented in [16]. Furthermore, $X, Y, Z, \ldots$ shall denote separable infinite dimensional Banach spaces. By $Y \subseteq X$ we mean that Y is a closed infinite-dimensional linear subspace of X. By $S(X) = \{x \in X : \|x\| = 1\}$ we denote the unit sphere of X.

If (e_i) is a basic sequence and $F \subseteq \mathbb{N}$, $\langle e_i \rangle_F$ is the linear span of $\{e_i : i \in F\}$ and $[e_i]_F$ is the closure of $\langle e_i \rangle_{i \in F}$. For $F, G \subseteq \mathbb{N}$ the notation $F < G$ means that $\max F < \min G$ or either F or G is empty. $F < G$ are *adjacent intervals of* $\mathbb{N}$ if for some $k \le m < n$, $F = [k, m] = \{i \in \mathbb{N} : k \le i \le m\}$ and $G = [m+1, n]$. If $x \in \langle e_i \rangle$ and $x = \sum a_i e_i$ then $\text{supp}(x) = \{i : a_i \neq 0\}$ is the support of x with respect to (e_i) (w.r.t. (e_i)). For $x, y \in \langle e_i \rangle$, we write $x < y$ if $\text{supp}(x) < \text{supp}(y)$. By $(x_i) \prec (e_i)$ we shall mean that (x_i) is a block basis of (e_i). We say that Y is a *block subspace* of X, $Y \prec X$, if X has a basis (x_i) and $Y = [y_i]_{\mathbb{N}}$ for some $(y_i) \prec (x_i)$.

2.1 Distortion

If a Banach space $(X, \|\cdot\|)$ is given an equivalent norm $|\cdot|$ we define the distortion of $|\cdot|$ by

Definition 2.1

$$d(X, |\cdot|) = \inf_Y \sup \left\{ \frac{|x|}{|y|} : x, y \in S(Y, \|\cdot\|) \right\} ,$$

where the infimum is taken over all infinite-dimensional subspaces Y of X.

Remark 2.2 If X has a basis, then a standard approximation argument easily shows that in the above formula for $d(X, |\cdot|)$ it is sufficient to take the infimum over all block subspaces $Y \prec X$; and this is the form of the definition we shall always use.

The parameter $d(X, |\cdot|)$ measures how close $|\cdot|$ can be made to being a multiple of $\|\cdot\|$, by restricting to an infinite-dimensional subspace.

Definition 2.3 *For $\lambda > 1$, $(X, \|\cdot\|)$ is* λ-distortable *if there exists an equivalent norm $|\cdot|$ on X so that $d(X, |\cdot|) \ge \lambda$. X is* distortable *if it is λ-distortable for some $\lambda > 1$. X is* arbitrarily distortable *if it is λ-distortable for all $\lambda > 1$.*

Definition 2.4 *A space* $(X, \|\cdot\|)$ *is of* D-bounded distortion *if for all equivalent norms* $|\cdot|$ *on* X *and all* $Y \subseteq X$, $d(Y, |\cdot|) \leq D$. *A space* X *is of* bounded distortion *if it is of* D*-bounded distortion for some* $D < \infty$.

Let us mention a more geometric approach to distortion. A subset $A \subseteq X$ is called *asymptotic* if $\operatorname{dist}(A, Y) = 0$ for all infinite-dimensional subspaces Y of X, i.e. for all Y and $\varepsilon > 0$ there is $x \in A$ such that $\inf_{y \in Y} \|x - y\| < \varepsilon$. Given $\eta > 0$, consider the following property of X: there exist $A, B \subseteq S(X)$ and A^* in the unit ball of X^* such that: (i) A and B are asymptotic in X; (ii) for every $x \in A$ there is $x^* \in A^*$ such that $|x^*(x)| \geq 1/2$; (iii) for all $y \in B$ and $x^* \in A^*$, $|x^*(y)| < \eta$. It is well known and easy to see that if $d(X, |\cdot|) \geq \lambda$ for some equivalent norm $|\cdot|$ on X then in some $Y \subseteq X$ there exist such asymptotic (in Y) "almost biorthogonal" sets, with $\eta = 1/\lambda$. Conversely, given sets A, B and A^* as above, let $|x| = \|x\| + (1/\eta)\sup\{|x^*(x)| : x^* \in A^*\}$ for $x \in X$. Then $d(X, |\cdot|) \geq (1/2 + 1/4\eta)$.

A proof of the following simple proposition is left for the reader. Part b) was shown in [24].

Proposition 2.5 **a)** *Let* $(X, \|\cdot\|)$ *be of* D*-bounded distortion and let* $|\cdot|$ *be an equivalent norm on* X*. Then for all* $\varepsilon > 0$ *and* $Y \subseteq X$ *there exists* $Z \subseteq Y$ *and* $c > 0$ *so that* $|z| \leq c\|z\| \leq (D+\varepsilon)|z|$ *for all* $z \in Z$.

b) *Every Banach space contains either an arbitrarily distortable subspace or a subspace of bounded distortion.*

Note that if X has a basis then one may replace $Y \subseteq X$ and $Z \subseteq Y$, in Definition 2.4 and Proposition 2.5, and the definition of an asymptotic set, by $Y \prec X$ and $Z \prec Y$, respectively.

It was shown in [21], [22] that every X contains either a distortable subspace or a subspace isomorphic to ℓ_1 or c_0 (both of which are not distortable [11]). Currently no examples of distortable spaces of bounded distortion are known. It is known that such a space would for some $1 \leq p \leq \infty$ necessarily contain an asymptotic ℓ_p subspace (defined below for $p = 1$) with an unconditional basis and must contain ℓ_1^n's uniformly ([19], [17], [24]).

In light of these results it is natural to focus the search for a distortable space of bounded distortion on asymptotic ℓ_1 spaces with an unconditional basis.

2.2 Asymptotic ℓ_1 Banach spaces

Several definitions of asymptotic ℓ_1 spaces appear in the literature. We shall use the definition from [19].

Definition 2.6 *A space X with a basis (e_i) is an* asymptotic ℓ_1 space *(w.r.t. (e_i)) if there exists C such that for all n and all $e_n \le x_1 < \cdots < x_n$,*

$$\|\sum_1^n x_i\| \ge (1/C)\sum_1^n \|x_i\| \ .$$

The infimum of all C's as above is called the asymptotic ℓ_1 constant *of X.*

It should be noted that this definition depends on the choice of a basis: a space X may be asymptotic ℓ_1 with respect to one basis but not another. However when the basis is understood, the reference to it is often dropped.

In [18] a notion of asymptotic structure of an arbitrary Banach space was introduced; in as much as we shall not use it here, we omit the details. This led, in particular, to a more general concept of asymptotic ℓ_1 spaces; and spaces satisfying Definition 2.6 above were called there "stabilized asymptotic ℓ_1". Several connections between the "MMT-asymptotic structure" of a space [18] and the "stabilized asymptotic structure" of its subspaces can be proved; for instance, an MMT-asymptotic ℓ_1 space contains an asymptotic ℓ_1 space in the sense of Definition 2.6.

Before proceeding we shall briefly consider the prime example of an asymptotic ℓ_1 space not containing ℓ_1, namely Tsirelson's space T [26]. Our discussion will motivate our subsequent definitions. The space T is actually the dual of Tsirelson's original space. It was described in [10] as follows.

Let c_{00} be the linear space of finitely supported sequences. T is the completion of $(c_{00}, \|\cdot\|)$ where $\|\cdot\|$ satisfies the implicit equation

$$\|x\| = \max\left(\|x\|_\infty, \sup\left\{\frac{1}{2}\sum_{i=1}^n \|E_i x\| : n \in \mathbb{N} \text{ and } n \le E_1 < \cdots < E_n\right\}\right) \ .$$

In this definition the E_i's are finite subsets of $\mathbb{N}$. $E_i x$ is the restriction of x to the set E_i. Thus if $x = (x(j))$ then $E_i x(j) = x(j)$ if $j \in E_i$ and 0 otherwise. Of course it must be proved that such a norm exists. The unit vector basis (e_i) forms a 1-unconditional basis for T and T is reflexive. If $e_n \le x_1 < \cdots < x_n$ w.r.t. (e_i) then $\|\sum_1^n x_i\| \ge \frac{1}{2}\sum_1^n \|x_i\|$ and so T is asymptotic ℓ_1 with constant less than or equal to 2. The next proposition is the best that can currently be said about distorting T. The proof, which we sketch, is illustrative.

Proposition 2.7 *T is $(2-\varepsilon)$-distortable for all $\varepsilon > 0$.*

Proof. *(Sketch)* Let $\varepsilon > 0$ and choose n so that $1/n < \varepsilon$. Define for $x \in T$,

$$|x| = \sup\left\{\sum_{i=1}^n \|E_i x\| : E_1 < \cdots < E_n\right\} \ .$$

Clearly, $\|x\| \le |x| \le n\|x\|$ for $x \in T$ (in fact, for $n \le x$, $|x| \le 2\|x\|$). Let $(x_i) \prec (e_i)_n^\infty$. For any $k > n$ some normalized sequence $(y_i)_1^k \prec (x_i)_k^\infty$ is equivalent to the unit vector basis of ℓ_1^k, with the equivalence constant as close to 1 as we wish. Thus if $y=(1/k)\sum_1^k y_i$, then $\|y\|\approx 1$. Also if $E_1<\cdots<E_n$ then setting $I=\{i : E_j\cap \mathrm{supp}(y_i)\neq\emptyset \text{ for at most one } j\}$ and $J=\{1,\ldots,k\}\setminus I$ we have that $|J|\le n$ and

$$\begin{aligned}\sum_1^n \|E_j y\| &\le \frac{1}{k}\Big(\sum_{i\in I}\|y_i\| + \sum_{i\in J}\sum_j \|E_j y_i\|\Big)\\ &\le \frac{1}{k}\Big(\sum_{i\in I}\|y_i\| + \sum_{i\in J}2\|y_i\|\Big)\\ &\le \frac{1}{k}(k-|J|+2|J|)\le 1+\frac{n}{k}.\end{aligned}$$

Thus $\inf\{|x| : \|x\|=1,\ x\in\langle x_i\rangle\}=1$.

Now let $z=(2/n)\sum_1^n z_i\in\langle x_i\rangle_n^\infty$ where $z_1<\cdots<z_n$ and each z_i is an $\ell_1^{k_i}$-average of the sort just considered. Here k_{i+1} is taken very large depending on $\max\mathrm{supp}(z_i)$ and ε. Since $\|z_i\|\approx 1$, it follows that $|z|\ge(2/n)\sum\|z_i\|\approx 2$. Yet, if $m\le E_1<\cdots<E_m$, and i_0 is the smallest i such that $e_m<\max\mathrm{supp}(z_i)$, then the growth condition for k_i implies that k_i is much larger than m for $i_0<i\le n$. Hence by the argument above

$$\begin{aligned}\frac{1}{2}\sum_1^m\|E_j z\| &= \frac{1}{n}\sum_1^m\Big\|E_j\Big(\sum_1^n z_i\Big)\Big\|\\ &\le \frac{1}{n}\Big(\sum_{j=1}^m\|E_j z_{i_0}\| + \sum_{i=i_0+1}^n\sum_{j=1}^m\|E_j z_i\|\Big) \qquad (1)\\ &\le \frac{1}{n}\Big(2\|z_{i_0}\| + \sum_{i=i_0+1}^n(1+n/k_i)\Big)\lesssim\frac{n+1}{n}<1+\varepsilon\ .\end{aligned}$$

By the definition of the norm we get $\|z\|\le 1+\varepsilon$. This implies $\sup\{|z| : \|z\|=1,\ z\in\langle x_i\rangle\}>2/(1+\varepsilon)$. □

Later we shall say that a sequence $(y_i)_1^k$ is $\mathcal{S}_1$-admissible w.r.t. (x_i) if $x_k\le y_1<\cdots<y_k$. In the above proof we needed to consider an admissible sequence of admissible sequences; what we shall later call $\mathcal{S}_2$-admissible.

Inequality (1) obviously shows that the asymptotic ℓ_1 constant of T is greater than or equal, and hence equal, to 2. Furthermore, if $X\prec T$, then X is an asymptotic ℓ_1 space with constant again equal to 2. In other words, passing to a block basis of T does not improve the asymptotic ℓ_1 constant. Vitali Milman asked the question what would happen if in addition we renormed? The above technique gives that the constant cannot be improved too much.

Proposition 2.8 *If $|\cdot|$ is any equivalent norm on $X\prec T$ then X is asymptotic ℓ_1 with constant at least $\sqrt{2}$.*

Proof. *(Sketch)* Let $X\prec T$ and consider an equivalent norm $|\cdot|$ on X so that $(X,|\cdot|)$ is asymptotic ℓ_1 with constant θ. By multiplying $|\cdot|$ by a constant and passing to a block subspace of X if necessary we may assume that $\|\cdot\| \geq |\cdot|$ on X and for all $Y \prec X$ there exists $y \in Y$ with $\|y\| = 1$ and $|y| \approx 1$. Given n, choose $z_1 < z_2 < \cdots < z_n$ w.r.t. X so that $z_i = (1/k_i)\sum_1^{k_i} z_{i,j}$ where $z_{i,1} < \cdots < z_{i,k_i}$ in X and $\|z_{i,j}\| = 1 \approx |z_{i,j}|$. Here k_{i+1} is again large depending upon z_i.

Let $z = (2/n)\sum_1^n z_i$. Then as before we obtain $|z| \leq \|z\| \lesssim 1 + (1/n)$. On the other hand, $|z| \geq (2/n\theta)\sum_1^n |z_i| \gtrsim (2/n\theta^2)n = 2/\theta^2$. Hence $2/\theta^2 \lesssim 1$. □

Remark 2.9 For any $0 < \theta < 1$ Tsirelson's space T_θ is defined by the implicit equation analogous to the definition of T, in which the constant $1/2$ is replaced by θ. The properties of T remain valid for T_θ as well, with appropriate modification of the constants involved.

These results indicate that it could be of advantage to consider the ℓ_1-ness of sequences which are $\mathcal{S}_2$-admissible with respect to a basis or even $\mathcal{S}_n$-admissible. We do so in this paper and we shall obtain the best possible improvement of Proposition 2.8 in Theorem 5.2 (see also Remark 5.3). Of course the beautiful examples of Argyros and Deliyanni [3] of arbitrarily distortable mixed Tsirelson spaces (described below) also show the need for consideration of such notions when studying asymptotic ℓ_1 spaces. Our point here is that these are needed even to answer $\mathcal{S}_1$-admissibility questions.

3 The Schreier families $\mathcal{S}_\alpha$

Let $\mathcal{F}$ be a set of finite subsets of $\mathbb{N}$. $\mathcal{F}$ is *hereditary* if whenever $G \subseteq F \in \mathcal{F}$ then $G \in \mathcal{F}$. $\mathcal{F}$ is *spreading* if whenever $F = (n_1, \cdots, n_k) \in \mathcal{F}$, with $n_1 < \cdots < n_k$ and $m_1 < \cdots < m_k$ satisfies $m_i \geq n_i$ for $i \leq k$ then $(m_1, \ldots, m_k) \in \mathcal{F}$. $\mathcal{F}$ is *pointwise closed* if $\mathcal{F}$ is closed in the topology of pointwise convergence in $2^{\mathbb{N}}$. A set $\mathcal{F}$ of finite subsets of $\mathbb{N}$ having all three properties we call *regular*. If $\mathcal{F}$ and $\mathcal{G}$ are regular we let

$$\mathcal{F}[\mathcal{G}] = \left\{ \bigcup_1^n G_i : n \in \mathbb{N},\, G_1 < \cdots < G_n,\, G_i \in \mathcal{G} \text{ for } i \leq n,\, (\min G_i)_1^n \in \mathcal{F} \right\} .$$

Note that this operation satisfies the natural associativity condition $(\mathcal{F}[\mathcal{G}_1])[\mathcal{G}_2] = \mathcal{F}[\mathcal{G}_0]$, where $\mathcal{G}_0 = \mathcal{G}_1[\mathcal{G}_2]$.

If $N = (n_1, n_2, \ldots)$ is a subsequence of $\mathbb{N}$ then $\mathcal{F}(N) = \{(n_i)_{i\in F} : F \in \mathcal{F}\}$. If $\mathcal{F}$ is regular and M is a subsequence of N then, since $\mathcal{F}$ is spreading, $\mathcal{F}(M) \subset \mathcal{F}(N)$. If $\mathcal{F}$ is regular and $n \in \mathbb{N}$ we define $[\mathcal{F}]^n$ by $[\mathcal{F}]^1 = \mathcal{F}$ and $[\mathcal{F}]^{n+1} = \mathcal{F}\Big[[\mathcal{F}]^n\Big]$. Finally, if F is a finite set, $|F|$ denotes the cardinality of F.

Definition 3.1 [1] *The* Schreier classes *are defined by* $\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$, $\mathcal{S}_1 = \{F \subseteq \mathbb{N} : \min F \geq |F|\} \cup \{\emptyset\}$*; for* $\alpha < \omega_1$, $\mathcal{S}_{\alpha+1} = \mathcal{S}_1[\mathcal{S}_\alpha]$, *and if* α *is a limit ordinal we choose* $\alpha_n \uparrow \alpha$ *and set*

$$\mathcal{S}_\alpha = \{F : \text{for some } n \in \mathbb{N},\ F \in \mathcal{S}_{\alpha_n} \text{and } F \geq n\} .$$

It should be noted that the definition of the $\mathcal{S}_\alpha$'s for $\alpha \geq \omega$ depends upon the choices made at limit ordinals but this particular choice is unimportant for our purposes. Each $\mathcal{S}_\alpha$ is a regular class of sets. It is easy to see that $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \cdots$ and $\mathcal{S}_n[\mathcal{S}_m] = \mathcal{S}_{m+n}$, for $n, m \in \mathbb{N}$, but this fails for higher ordinals. However we do have

Proposition 3.2 a) *Let $\alpha < \beta < \omega_1$. Then there exists $n \in \mathbb{N}$ so that if $n \leq F \in \mathcal{S}_\alpha$ then $F \in \mathcal{S}_\beta$.*

b) *For all $\alpha, \beta < \omega_1$ there exists a subsequence N of $\mathbb{N}$ so that $\mathcal{S}_\alpha[\mathcal{S}_\beta](N) \subseteq \mathcal{S}_{\beta+\alpha}$.*

c) *For all $\alpha, \beta < \omega_1$ there exists a subsequence M of $\mathbb{N}$ so that $\mathcal{S}_{\beta+\alpha}(M) \subseteq \mathcal{S}_\alpha[\mathcal{S}_\beta]$.*

We start with an easy formal observation.

Lemma 3.3 *Let $\mathcal{F}$and $\mathcal{G}$ be sets of finite subsets of $\mathbb{N}$ and let $\mathcal{G}$ be spreading. Assume that there exists a subsequence N of $\mathbb{N}$ so that $\mathcal{F}(N) \subseteq \mathcal{G}$. Then for all subsequences L of $\mathbb{N}$ there exists a subsequence L' of L with $\mathcal{F}(L') \subseteq \mathcal{G}$.*

Proof. Let $L = (l_i)$. Let $N = (n_i)$ such that $\mathcal{F}(N) \subseteq \mathcal{G}$. Since $\mathcal{G}$ is spreading, any $L' = (l'_i) \subseteq (l_i)$ such that $l'_i \geq n_i$ for all i satisfies the conclusion (for instance one can take $L' = (l_{n_i})$). □

Proof of Proposition 3.2. a) We proceed by induction on β. If $\beta = \gamma + 1$ then $\alpha \leq \gamma$ and so we may choose n so that if $n \leq F \in \mathcal{S}_\alpha$ then $F \in \mathcal{S}_\gamma \subseteq \mathcal{S}_\beta$. If β is a limit ordinal and $\beta_n \uparrow \beta$ is the sequence used in defining $\mathcal{S}_\beta$, choose n_0 so that $\alpha < \beta_{n_0}$. Choose $n \geq n_0$ so that if $n \leq F \in \mathcal{S}_\alpha$ then $F \in \mathcal{S}_{\beta_{n_0}}$. Thus also $F \in \mathcal{S}_\beta$.

b) We induct on α. Since $\mathcal{S}_0[\mathcal{S}_\beta] = \mathcal{S}_\beta$, the assertion is clear for $\alpha = 0$. If $\alpha = \gamma + 1$, then $\mathcal{S}_\alpha[\mathcal{S}_\beta] = \mathcal{S}_1[\mathcal{S}_\gamma[\mathcal{S}_\beta]]$ and $\mathcal{S}_{\beta+\alpha} = \mathcal{S}_1[\mathcal{S}_{\beta+\gamma}]$. Thus we can take N to satisfy $\mathcal{S}_\gamma[\mathcal{S}_\beta](N) \subseteq \mathcal{S}_{\beta+\gamma}$.

If α is a limit ordinal we argue as follows. First, by Lemma 3.3, the inductive hypothesis implies that for every $\alpha' < \alpha$ and every subsequence L of $\mathbb{N}$ there exists a subsequence N of L with $\mathcal{S}_{\alpha'}[\mathcal{S}_\beta](N) \subseteq \mathcal{S}_{\beta+\alpha'}$. Let $\alpha_n \uparrow \alpha$ and $\gamma_n \uparrow \beta + \alpha$ be the sequences of ordinals used to define $\mathcal{S}_\alpha$ and $\mathcal{S}_{\beta+\alpha}$, respectively.

Choose subsequences of $\mathbb{N}$, $L_1 \supseteq L_2 \supseteq \cdots$ so that $\mathcal{S}_{\alpha_k}[\mathcal{S}_\beta](L_k) \subseteq \mathcal{S}_{\beta+\alpha_k}$. If $L_k = (\ell_i^k)_{i=1}^\infty$ we let L be the diagonal $L = (\ell_k) = (\ell_k^k)_{k=1}^\infty$. It follows that if $F \in \mathcal{S}_{\alpha_k}[\mathcal{S}_\beta](L)$ and $F \geq \ell_k$ then $F \in \mathcal{S}_{\alpha_k}[\mathcal{S}_\beta](L_k)$ and so $F \in \mathcal{S}_{\beta+\alpha_k}$. For each k choose $\bar{n}(k)$ so that $\beta + \alpha_k < \gamma_{\bar{n}(k)}$. Using a) choose $j(1) < j(2) < \cdots$ so that if $j(k) \leq F \in \mathcal{S}_{\beta+\alpha_k}$ then $F \in \mathcal{S}_{\gamma_{\bar{n}(k)}}$. Let $N = (n(k))_{k=1}^\infty$ be a subsequence of $\mathbb{N}$ with $n(k) \geq \ell_k \vee j(k) \vee \bar{n}(k)$ for all k. Then if $F \in \mathcal{S}_\alpha[\mathcal{S}_\beta](N)$ there exists k so that $n_k \leq F \in \mathcal{S}_{\alpha_k}[\mathcal{S}_\beta](N)$ and so $\ell_k \leq F \in \mathcal{S}_{\beta+\alpha_k}$ and $j(k) \leq F \in \mathcal{S}_{\gamma_{\bar{n}(k)}}$, whence since $\bar{n}(k) \leq F$ we have $F \in \mathcal{S}_{\beta+\alpha}$.

c) As in b) we induct on α. The cases $\alpha = 0$ and $\alpha = \gamma + 1$ are trivial. Thus assume that α is a limit ordinal. Let $\alpha_r \uparrow \alpha$ and $\bar{\gamma}_r \uparrow \beta + \alpha$ be the sequences defining $\mathcal{S}_\alpha$ and $\mathcal{S}_{\beta+\alpha}$ respectively.

We may write $(\bar\gamma_r) = (\bar\gamma_1, \dots, \bar\gamma_{n_0-1}, \beta+\gamma_{n_0}, \beta+\gamma_{n_0+1}, \dots)$ where $\bar\gamma_i < \beta$ if $i < n_0$. By a) there exists m_0 so that if $m_0 \le F \in \bigcup_1^{n_0-1} \mathcal{S}_{\bar\gamma_i}$ then $F \in \mathcal{S}_\beta$. We shall take later $M = (m_i)_1^\infty$ where $m_1 \ge m_0$. By the inductive hypothesis and Lemma 3.3. choose sequences $L_{n_0} \supseteq L_{n_0+1} \supseteq \cdots$ so that $\mathcal{S}_{\beta+\gamma_k}(L_k) \subseteq \mathcal{S}_{\gamma_k}[\mathcal{S}_\beta]$ for $k \ge n_0$ and $m_0 \le L_{n_0}$. If $L_k = (\ell_i^k)_i$ set $L = (\ell_k)$ where $\ell_k = \ell_k^k$ for $k \ge n_0$, and $m_0 \le \ell_1 < \cdots < \ell_{k-1} < \ell_k < \cdots$. Thus if $k \ge n_0$, $\ell_k \le F \in \mathcal{S}_{\beta+\gamma_k}(L)$ implies that $F \in \mathcal{S}_{\gamma_k}[\mathcal{S}_\beta]$. Also for $k < n_0$, $\ell_k \le F \in \mathcal{S}_{\bar\gamma_k}$ implies that $F \in \mathcal{S}_\beta$. For $k \ge n_0$ choose $\bar m(k)$ so that $\beta + \gamma_k < \beta + \alpha_{\bar m(k)}$. By a) there exists $n(k)$ so that $n(k) \le F \in \mathcal{S}_{\gamma_k}[\mathcal{S}_\beta]$ implies that $F \in \mathcal{S}_{\alpha_{\bar m(k)}}[\mathcal{S}_\beta]$ for all $k \ge n_0$. Finally we choose $M = (m(k))$ where $m(k) = \ell_k$ for $k < n_0$ and $m(k) \ge \ell_k \vee \bar m(k) \vee n(k)$ for $k \ge n_0$. Thus if $F \in \mathcal{S}_{\beta+\alpha}(M)$ then $F \in \mathcal{S}_\beta$ or else there exists $k \ge n_0$ with $m(k) \le F \in \mathcal{S}_{\beta+\gamma_k}(M)$. Hence $\ell_k \le F \in \mathcal{S}_{\gamma_k}[\mathcal{S}_\beta]$ and so $n(k) \le F \in \mathcal{S}_{\alpha_{\bar m(k)}}[\mathcal{S}_\beta]$. Since $F \ge \bar m(k)$ we get that $F \in \mathcal{S}_\alpha[\mathcal{S}_\beta]$. $\square$

Corollary 3.4 *For all $\alpha < \omega_1$ and $n \in \mathbb{N}$ there exist subsequences M and N of $\mathbb{N}$ satisfying $[\mathcal{S}_\alpha]^n(N) \subseteq \mathcal{S}_{\alpha\cdot n}$ and $\mathcal{S}_{\alpha\cdot n}(M) \subseteq [\mathcal{S}_\alpha]^n$.*

Proof. This is easily established by induction on n using Proposition 3.2. For example, if $[\mathcal{S}_\alpha]^n(P) \subseteq \mathcal{S}_{\alpha\cdot n}$ and $\mathcal{S}_\alpha[\mathcal{S}_{\alpha\cdot n}](L) \subseteq \mathcal{S}_{\alpha\cdot(n+1)}$, let $N = (p_{l_i})$ (here $P = (p_i)$ and $L = (l_i)$). Then $[\mathcal{S}_\alpha]^n(N) \subseteq \mathcal{S}_{\alpha\cdot n}(L)$ and so $[\mathcal{S}_\alpha]^{n+1}(N) = \mathcal{S}_\alpha[\mathcal{S}_\alpha]^n(N) \subseteq \mathcal{S}_\alpha[\mathcal{S}_{\alpha\cdot n}](L) \subseteq \mathcal{S}_{\alpha\cdot(n+1)}$. $\square$

Remark 3.5 The Schreier family $\mathcal{S}_\alpha$ has been used in [1] to construct an interesting subspace S_α of $C(\omega^{\omega^\alpha})$ as follows. S_α is the completion of c_{00} under the norm

$$\|x\| = \sup\{|\sum_{i\in E} x(i)| : E \in \mathcal{S}_\alpha\} .$$

The unit vector basis is an unconditional basis for S_α. The space S_α does not embed into $C(\omega^{\omega^\beta})$ for any $\beta < \alpha$.

The next important proposition is a slight generalization of a result in [3] and is a descendent of results in [5].

Proposition 3.6 *Let $\beta < \alpha < \omega_1$, $\varepsilon > 0$ and let M be a subsequence of $\mathbb{N}$. Then there exists a finite set $F \subseteq M$ and $(a_j)_{j\in F} \subseteq \mathbb{R}^+$ so that $F \in \mathcal{S}_\alpha(M)$, $\sum_{j\in F} a_j = 1$ and if $G \subseteq F$ with $G \in \mathcal{S}_\beta$ then $\sum_{j\in G} a_j < \varepsilon$.*

Proof. We proceed by induction on α. The result is clear for $\alpha = 1$. Let $M = (m_i)$. We choose $1/k < \varepsilon$, $F \subseteq M$, $F > m_k$, $|F| = k$ and let $a_j = 1/k$ if $j \in F$.

If α is a limit ordinal let $\alpha_n \uparrow \alpha$ be the sequence used to define $\mathcal{S}_\alpha$. Choose n so that $\beta < \alpha_n$. Applying the induction hypothesis to β, α_n and $\{m \in M : m \ge m_n\}$ yields the result.

If $\alpha = \gamma + 1$ we may assume (by Proposition 3.2) that $\beta = \gamma$. If γ is a limit ordinal let $\gamma_n \uparrow \gamma$ be the sequence used to define $\mathcal{S}_\gamma$. Choose k so that $1/k < \varepsilon/2$. Choose sets $F_i \subseteq M$ with $m_k \le F_1 < \cdots < F_k$ along with scalars $(a_j)_{j\in\bigcup_1^k F_i} \subseteq \mathbb{R}^+$ and $n_1 < \cdots < n_k$ satisfying the following:

1) $\sum_{j\in F_i} a_j = 1$ for $i \le k$

2) $F_i \in \mathcal{S}_{\gamma_{n_i}}(M)$ and $m_{n_i} < F_i$ for $1 \le i \le k$

3) $\sum_{j\in G} a_j < 1/2^i$ if $G \subseteq F_{i+1}$ with $G \in \mathcal{S}_{\gamma_\ell}$ whenever $\ell \le \max F_i$ for $1 \le i < k$.

Let $F = \bigcup_1^k F_i$. Then $F \subseteq M$ and $F \in \mathcal{S}_{\gamma+1}(M)$. For $j \in F$ set $b_j = k^{-1}a_j$. Then $(b_j)_{j\in F} \subseteq \mathbb{R}^+$, $\sum_{j\in F} b_j = 1$ and if $G \in \mathcal{S}_\gamma$, $G \subseteq F$ then $\sum_{j\in G} b_j < \varepsilon$. Indeed there exists n with $n \le G \in \mathcal{S}_{\gamma_n}$. Thus if $i_0 = \min\{i : G \cap F_i \ne \emptyset\}$ then $n \le \max F_{i_0}$ and so by 3),

$$\sum_{j\in G} b_j = \sum_{i=i_0}^{k}\Big(\sum_{j\in F_i\cap G} b_j\Big) \le \frac{1}{k}\left(1 + \frac{1}{2^{i_0}} + \cdots + \frac{1}{2^k}\right) < \varepsilon\ .$$

If $\gamma = \eta + 1$ we again choose $1/k < \varepsilon/2$ and sets $m_k \le F_1 < \cdots < F_k$, $F_i \in \mathcal{S}_\gamma(M)$, along with $(a_j)_{j\in F_i} \subseteq \mathbb{R}^+$, $\sum_{j\in F_i} a_j = 1$ so that if $G \in \mathcal{S}_\eta$ then $\sum_{j\in G\cap F_{i+1}} a_j < (1/2^i \max(F_i))$ for $1 \le i < k$. As above we set $F = \bigcup_1^k F_i$ and let $b_j = k^{-1}a_j$ if $j \in F_i$. Thus $F \in \mathcal{S}_\alpha(M)$ and if $G \in \mathcal{S}_\gamma$, $G \subseteq F$, write $G = \bigcup_{s=1}^p G_s$ where $p \le G_1 < \cdots < G_p$ and $G_i \in \mathcal{S}_\eta$ for each i. Then if $i_0 = \min\{i : G \cap F_i \ne \emptyset\}$, since $\max(F_{i_0}) \ge p$,

$$\sum_{j\in G} b_j \le \frac{1}{k} + \sum_{i=i_0+1}^{k} \frac{1}{2^i}\,\frac{p}{k}\,\frac{1}{\max(F_i)} \le \frac{1}{k} + \frac{1}{k}\sum_{i=i_0+1}^{k} 2^{-i} < \varepsilon\ .$$

□

Definition 3.7 *Let $\varepsilon > 0$ and $\beta < \alpha < \omega_1$. If (e_i) is a normalized basic sequence, M is a subsequence of $\mathbb{N}$ and F and $(a_i)_{i\in F}$ are as in Proposition 3.6, we call $x = \sum_{i\in F} a_i e_i$ an* $(\alpha,\beta,\varepsilon)$-average of $(e_i)_{i\in M}$. *If (x_i) is a normalized block basis of (e_i) and F and $(a_i)_{i\in F}$ are as in Proposition 3.6 for $M = (\min \operatorname{supp}(x_i))$, we call $x = \sum_{i\in F} a_i x_i$ an* $(\alpha,\beta,\varepsilon)$-average of (x_i) w.r.t. (e_i).

The Schreier families are large within the set of all classes of pointwise closed subsets of $[\mathbb{N}]^{<\omega}$. Our next two propositions show that they are in a sense the largest among all regular classes of a given complexity. To make this concept precise we consider the index $I(\mathcal{F})$ defined as follows. Let $D(\mathcal{F}) = \{F \in \mathcal{F} :$ there exist $(F_n) \subseteq \mathcal{F}$ with $1_{F_n} \to 1_F$ pointwise and $F_n \ne F$ for all $n\}$, $D^{\alpha+1}(\mathcal{F}) = D(D^\alpha(\mathcal{F}))$ and $D^\alpha(\mathcal{F}) = \bigcap_{\beta<\alpha} D^\beta(\mathcal{F})$ when α is a limit ordinal. Then

$$I(\mathcal{F}) = \inf\{\alpha < \omega_1 : D^\alpha(\mathcal{F}) = \{\emptyset\}\}\ .$$

$\mathcal{F}$ is a countable compact metric space in the topology of pointwise convergence and so $I(\mathcal{F})$ must be countable, see e.g. [14], p. 261-262.

Remark 3.8 The Cantor-Bendixson index of $\mathcal{F}$ (under the topology of pointwise convergence) is $I(\mathcal{F})+1$. This is because $\emptyset$ corresponds to the 0 function and one needs one more derivative to get $\emptyset : D^{I(\mathcal{F})+1}(\mathcal{F}) = \emptyset$, which defines the Cantor-Bendixson index.

Now we have ([1])

Proposition 3.9 *For $\alpha < \omega_1$, $I(\mathcal{S}_\alpha) = \omega^\alpha$.*

Proof. We induct on α. The result is clear for $\alpha = 0$. If the proposition holds for α it can be easily seen that for $n \in \mathbb{N}$, $D^{\omega^\alpha \cdot n}(\mathcal{S}_{\alpha+1}) = \{F :$ there exists $k \in \mathbb{N}$, $k > n$, with $F = \bigcup_1^{k-n} F_i$, $k \le F_1 < \cdots < F_{k-n}$, and $F_i \in \mathcal{S}_\alpha$ for $i \le k-n\}$. Hence $I(\mathcal{S}_{\alpha+1}) = \omega^{\alpha+1}$. The case where α is a limit ordinal is also easily handled. □

Proposition 3.10 *If $\mathcal{F}$ is a regular set of finite subsets of $\mathbb{N}$ with $I(\mathcal{F}) \le \omega^\alpha$ then there exists a subsequence M of $\mathbb{N}$ with $\mathcal{F}(M) \subseteq \mathcal{S}_\alpha$.*

This proposition is a special case of more complicated statements (Proposition 3.12 and Remark 3.13) below. First let us recall (see e.g., [20]) that every ordinal $\beta < \omega_1$ can be uniquely written in Cantor normal form as

$$\beta = \omega^{\alpha_1} \cdot n_1 + \omega^{\alpha_2} \cdot n_2 + \cdots + \omega^{\alpha_j} \cdot n_j$$

where $(n_i)_1^j \subseteq \mathbb{N}$ and $\omega_1 > \alpha_1 > \cdots > \alpha_j \ge 0$.

Definition 3.11 *If $(\alpha_i)_1^j$ are countable ordinals and $(n_i)_1^j \subseteq \mathbb{N}$, by $((\mathcal{S}_{\alpha_1})^{n_1}, \ldots, (\mathcal{S}_{\alpha_j})^{n_j})$ we denote the class of subsets of $\mathbb{N}$ that can be written in the form*

$$E_1^1 \cup \cdots \cup E_{n_1}^1 \cup E_1^2 \cup \cdots \cup E_{n_2}^2 \cup \cdots \cup E_1^j \cup \cdots \cup E_{n_j}^j$$

where $E_1^1 < E_2^1 < \cdots < E_{n_j}^j$ and $E_i^k \in \mathcal{S}_{\alpha_k}$ for all $i \le n_k$ and $k \le j$.

Proposition 3.12 *Let $\mathcal{F}$ be a regular set of finite subsets of $\mathbb{N}$ with*

$$I(\mathcal{F}) = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_{k-1}} \cdot n_{k-1} + \omega^{\alpha_k} \cdot n_k,$$

in Cantor normal form. Then there exists a subsequence M of $\mathbb{N}$ so that $\mathcal{F}(M) \subseteq ((\mathcal{S}_{\alpha_k})^{n_k}, \ldots, (\mathcal{S}_{\alpha_1})^{n_1})$.

Remark 3.13 The conclusion of the proposition holds even if $I(\mathcal{F}) < \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k$. Indeed this follows from the fact that if $\alpha < \beta$ and $\mathcal{F}(N) \subseteq \mathcal{S}_\alpha$, and $N = (n_i)$, then there exists $r \in \mathbb{N}$ so that $\mathcal{F}((n_i)_{i \ge r}) \subseteq \mathcal{S}_\beta$ (by Proposition 3.2(a)).

Proof of Proposition 3.12. We induct on $I(\mathcal{F})$. If $I(\mathcal{F}) = 1$ then $\mathcal{F}$ contains only singletons $\{n\}$ and so $\mathcal{F}(\mathbb{N}) \subseteq ((\mathcal{S}_0)^1)$.

Assume the proposition holds for all classes with index $< \beta$, and let $I(\mathcal{F}) = \beta$. For $j \in \mathbb{N}$ set $\mathcal{F}_j = \{F \in \mathcal{F} : \{j\} \cup F \in \mathcal{F} \text{ and } j < F\}$. Each $\mathcal{F}_j$ is regular.

Case 1. β is a successor Let $\beta = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_{k-1}} \cdot n_{k-1} + n_k$ with $n_k > 0$ ($\alpha_k = 0$ here). For every j, $I(\mathcal{F}_j) \leq \beta - 1$. Thus there exists $N_j \subseteq \mathbb{N}$ such that

(1) $\mathcal{F}_j(N_j) \subseteq ((\mathcal{S}_0)^{n_k - 1}, \ldots, (\mathcal{S}_{\alpha_1})^{n_1})$,

with the convention that $(\mathcal{S}_0)^0 = \emptyset$. Let $N_j = (n_i^j)_{i=1}^{\infty}$ and choose $N = (m_j)$ so that $m_j \geq n_j^1 \vee \cdots \vee n_j^j$ for all j.

We shall show that

$$\mathcal{F}(N) \subseteq ((\mathcal{S}_0)^{n_k}, \ldots, (\mathcal{S}_{\alpha_1})^{n_1}) \ .$$

Indeed, let $F \in \mathcal{F}$ and let $\min F = j$ and $G = F \setminus \{j\}$. Then

$$(m_i)_{i \in F} = \{m_j\} \cup \{m_i\}_{i \in G} \in ((\mathcal{S}_0), (\mathcal{S}_0)^{n_k - 1}, \ldots, (\mathcal{S}_{\alpha_1})^{n_1}) \ ,$$

by (1), the choice of N and the fact that each $\mathcal{S}_\alpha$ is spreading.

Case 2. β is a limit ordinal.

Let $\beta = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k$. Note that $\alpha_k > 0$. We have $I(\mathcal{F}_j) < \beta$ for all j.

Case 2.1. α_k is a successor.

Pick $p_j \uparrow \infty$ such that for every j,

$$I(\mathcal{F}_j) \leq \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot (n_k - 1) + \omega^{\alpha_k - 1} \cdot p_j \ .$$

By induction there exist subsequences N_j with

$$\mathcal{F}_j(N_j) \subseteq ((\mathcal{S}_{\alpha_k - 1})^{p_j}, (\mathcal{S}_{\alpha_k})^{n_k - 1}, \ldots, (\mathcal{S}_{\alpha_1})^{n_1}) \ .$$

Let $N_j = (n_i^j)$ and $N = (m_j)$ where $m_j \geq n_j^1 \vee \cdots \vee n_j^j \vee (p_j + 1)$ for all j. If $F \in \mathcal{F}$ with $\min F = j$ and $G = F \setminus \{j\}$ then

$$(m_i)_{i \in G} = H_1 \cup \cdots \cup H_{p_j} \cup H$$

where $H_1 < \cdots < H_{p_j} < H$; and $H_1, \ldots, H_{p_j} \in \mathcal{S}_{\alpha_k - 1}$, and $H \in ((\mathcal{S}_{\alpha_k})^{n_k - 1}, \ldots, (\mathcal{S}_{\alpha_1})^{n_1})$.

Since $\{n_j\} \in \mathcal{S}_0 \subseteq \mathcal{S}_{\alpha_k - 1}$ and $m_j \geq p_j + 1$, we have, $\{m_j\} \cup H_1 \cup \cdots \cup H_{p_j} \in \mathcal{S}_{\alpha_k}$. Thus

$$(m_i)_{i \in F} = \{m_j\} \cup H_1 \cup \cdots \cup H_{p_j} \cup H \in ((\mathcal{S}_{\alpha_k}), (\mathcal{S}_{\alpha_k})^{n_k - 1}, \ldots, (\mathcal{S}_{\alpha_1})^{n_1}) \ .$$

Case 2.2. α_k is a limit ordinal.

Let $\gamma_\ell \uparrow \alpha_k$ be the sequence of ordinals defining $\mathcal{S}_{\alpha_k}$. Set

$$\eta_\ell = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot (n_k - 1) + \omega^{\gamma_\ell} \cdot 1$$

so that $\eta_\ell \uparrow \beta$. Choose $\ell_j \uparrow \infty$ so that $I(\mathcal{F}_j) < \eta_{\ell_j}$. As above choose N_j so that $\mathcal{F}_j(N_j) \subseteq ((\mathcal{S}_{\gamma(\ell_j)})^1, (\mathcal{S}_{\alpha_k})^{n_k-1}, \dots, (\mathcal{S}_{\alpha_1})^{n_1}$. By Proposition 3.2(a) there exists $r_j \in \mathbb{N}$ so that $r_j \le H \in \mathcal{S}_{\gamma(\ell_j)+1}$ implies $H \in \mathcal{S}_{\gamma(\ell_{j+1})}$. Set $N_j = (n_i^j)$ and choose $N = (m_j)$ with $m_j \ge n_j^1 \vee \cdots \vee n_j^j \vee r_j \vee \ell_{j+1}$ for all j.

If $F \in \mathcal{F}$ with $\min F = j$ and $G = F \setminus \{j\}$ then $(m_i)_{i \in G} = H_1 \cup H_2$ where $H_1 < H_2$, $H_1 \in \mathcal{S}_{\gamma(\ell_j)}$ and $H_2 \in ((\mathcal{S}_{\alpha_k})^{n_k-1}, \dots, (\mathcal{S}_{\alpha_1})^{n_1})$. Now $\{m_j\} \cup H_1 \in \mathcal{S}_{\gamma(\ell_j)+1}$ and by $m_j \ge r_j$ we have $\{m_j\} \cup H_1 \in \mathcal{S}_{\gamma(\ell_{j+1})}$. Also $m_j \ge \ell_{j+1}$ so $\{m_j\} \cup H_1 \in \mathcal{S}_{\alpha_k}$. □

Remark 3.14 The proof of Proposition 3.12 is due to Denny Leung and Wee Kee Tang. They pointed out that our original proof was nonsense and supplied the argument given. We thank them for permission to reproduce it here.

In addition Denny Leung [15] has independently discovered a heirarchy of sets similar to that of the Schreier classes.

Corollary 3.15 *Let $\mathcal{F}$ be a pointwise closed class of finite subsets of $\mathbb{N}$. Then there exist $\alpha < \omega_1$ and a subsequence M of $\mathbb{N}$ so that $\mathcal{F}(M) \subseteq \mathcal{S}_\alpha$.*

Proof. Let $\mathcal{R}$ be the regular hull of $\mathcal{F}$; that is, $\mathcal{R} = \{G :$ there exists $F = (n_1, \dots, n_k) \in \mathcal{F}$ with $G \subseteq (m_i)_1^k$ for some $m_1 < \cdots < m_k$ with $m_i \ge n_i$ for $i \le k\}$.

Clearly, $\mathcal{R}$ is hereditary and spreading. We check that it is also pointwise closed, and hence the corollary follows from Proposition 3.10. Let $G_n \to G$ pointwise for some $(G_n) \subseteq \mathcal{R}$. If $|G| < \infty$ then G is an initial segment of G_n for large n and so $G \in \mathcal{R}$. It remains to note that $|G| = \infty$ is impossible. If $G = (n_1, n_2, \dots)$ then for all k, $(n_1, \dots, n_k)$ is a subset of some spreading of some set $F_k \in \mathcal{F}$. In particular $|\{n \in F_k : n \le n_j\}| \ge j$ for $1 \le j \le k$. Thus any limit point of $(F_k)_{k=1}^\infty$ is infinite which contradicts the hypotheses that $\mathcal{F}$ is pointwise closed and consists of finite sets. □

Remark 3.16 R. Judd [12] has recently proved the following dichotomy result for Schreier sets.

Theorem 3.17 *Let $\mathcal{F}$ be a hereditary family of subsets of $\mathbb{N}$ and let $\alpha < \omega_1$. Then either there exists a subsequence M of $\mathbb{N}$ so that $\mathcal{S}_\alpha(M) \subseteq \mathcal{F}$ or there exist subsequences M and N of $\mathbb{N}$ so that $\mathcal{F}[M](N) \subseteq \mathcal{S}_\alpha$, where $\mathcal{F}[M] = \{F \subset M : F \in \mathcal{F}\}$.*

For some other interesting properties of the Schreier classes we refer the reader to [4] and [2].

4 Asymptotic constants and $\Delta(X)$

Asymptotic constants considered in this paper will be determined by the Schreier families $\mathcal{S}_\alpha$; nevertheless it should be noted that they can be introduced for a very general class of families of finite subsets of $\mathbb{N}$.

Definition 4.1 *If $\mathcal{F}$ is a regular set of finite subsets of $\mathbb{N}$, a sequence of sets $E_1 < \cdots < E_k$ is* $\mathcal{F}$-admissible *if $(\min(E_i))_{i=1}^k \in \mathcal{F}$. If (x_i) is a basic sequence in a Banach space and $(y_i)_1^k \prec (x_i)$, then $(y_i)_1^k$ is* $\mathcal{F}$-admissible *(w.r.t. (x_i)) if $(\operatorname{supp}(y_i))_1^k$ is $\mathcal{F}$-admissible, where $\operatorname{supp}(y_i)$ is taken w.r.t. (x_i). We use a short form* α-admissible *to mean $\mathcal{S}_\alpha$-admissible.*

The next definition was first introduced in [25] for asymptotic ℓ_p spaces with $1 \le p < \infty$.

Definition 4.2 *Let $\mathcal{F}$ be a regular set of finite subsets of $\mathbb{N}$. For a basic sequence (x_i) in a Banach space X we define $\delta_{\mathcal{F}}(x_i)$ to be the supremum of $\delta \ge 0$ such that whenever $(y_i)_1^k \prec (x_i)$ is $\mathcal{F}$-admissible w.r.t. (x_i) then*

$$\|\sum_{i=1}^k y_i\| \ge \delta \sum_{i=1}^k \|y_i\| \ .$$

If X is a Banach space with a basis (e_i) we write $\delta_{\mathcal{F}}(X)$ for $\delta_{\mathcal{F}}(e_i)$. For $\alpha < \omega_1$, we set $\delta_\alpha(x_i) = \delta_{\mathcal{S}_\alpha}(x_i)$ and $\delta_\alpha(X) = \delta_{\mathcal{S}_\alpha}(X)$.

Remark 4.3 Note that $\delta_{\mathcal{F}}(x_i)$ is equal to the supremum of all $\delta' \ge 0$ such that $\|y\| \ge \delta' \sum \|E_i y\|$, for all $y \in \langle x_i \rangle$ and all adjacent $\mathcal{F}$-admissible intervals $E_1 < \cdots < E_k$ such that $\bigcup E_i \supseteq \operatorname{supp}(y)$. Here the support of y and restrictions $E_i y$ are understood to be w.r.t. (x_i). Indeed, clearly $\sup \delta' \ge \delta_{\mathcal{F}}(x_i)$. Conversely, given $(y_i)_1^k \prec (x_i)$ $\mathcal{F}$-admissible we set $y = \sum y_i$ and we let $(E_1, \ldots, E_k)$ be adjacent intervals such that $E_i \supseteq \operatorname{supp}(y_i)$ and $\min E_i = \min \operatorname{supp}(y_i)$ for all i.

In as much as distortion problems involve passing to block subspaces and renormings, it is natural to make two more definitions.

Definition 4.4 *Let $\mathcal{F}$ be a regular set of finite subsets of $\mathbb{N}$ and let (e_i) be a basis for X.*

$$\begin{aligned}\dot{\delta}_{\mathcal{F}}(X) &= \dot{\delta}_{\mathcal{F}}(e_i) = \sup\{\delta_{\mathcal{F}}(x_i) : (x_i) \prec (e_i)\} \quad \textit{and} \\ \ddot{\delta}_{\mathcal{F}}(X) &= \ddot{\delta}_{\mathcal{F}}(e_i) = \sup\{\dot{\delta}_{\mathcal{F}}((e_i), |\cdot|) : |\cdot| \text{ is an equivalent norm on } X\} \ .\end{aligned}$$

We write $\dot{\delta}_{\mathcal{S}_\alpha}(X) = \dot{\delta}_\alpha(X)$ and $\ddot{\delta}_{\mathcal{S}_\alpha}(X) = \ddot{\delta}_\alpha(X)$.

The asymptotic constants provide a measurement of closeness of block subspaces of X to ℓ_1. Clearly X is asymptotic ℓ_1 w.r.t. (e_i) if and only if $\delta_1(X) > 0$. The asymptotic ℓ_1 constant of X is then equal to $\delta_1(X)^{-1}$. On the other hand we also have

Proposition 4.5 *X contains a subspace isomorphic to* ℓ_1 *if and only if* $\dot{\delta}_\alpha(X) > 0$ *for all* $\alpha < \omega_1$.

Proof. This follows from Bourgain's ℓ_1 index of a Banach space X which we recall now. For $0 < c < 1$, $\mathcal{T}(X, c)$ is the tree of all finite normalized sequences $(x_i)_1^k \subseteq X$ satisfying $\|\sum_1^k a_i x_i\| \geq c\sum_1^k |a_i|$ for $(a_i)_1^k \subseteq \mathbb{R}$. The order on the tree is $(x_i)_1^k \leq (y_i)_1^n$ if $k \leq n$ and $x_i = y_i$ for $i \leq k$. For ordinals $\beta < \omega_1$ we define $\mathcal{D}^\beta(\mathcal{T}(X, c))$ inductively by $\mathcal{D}^1(\mathcal{T}(X, c)) = \{(x_i)_1^k \in \mathcal{T}(X, c) : (x_i)_1^k$ is not maximal$\}$. $\mathcal{D}^{\beta+1}(\mathcal{T}(X, c)) = \mathcal{D}^1(\mathcal{D}^\beta(\mathcal{T}(X, c)))$ and $\mathcal{D}^\beta(\mathcal{T}(X, c)) = \bigcap_{\gamma<\beta} \mathcal{D}^\gamma(\mathcal{T}(X, c))$ if β is a limit ordinal. The index $\mathcal{I}(X)$ is defined by $\mathcal{I}(X) = \sup_{0<c<1} \inf\{\beta : \mathcal{D}^\beta(\mathcal{T}(X, c)) = \emptyset\}$, where the infimum is set equal to ω_1 if no such β exists. Bourgain showed that for a separable space X, $\mathcal{I}(X) < \omega_1$ if and only if X does not contain a subspace isomorphic to ℓ_1 [6].

Now observe that if $\mathcal{F}$ is a regular set of finite subsets of $\mathbb{N}$ then $D(\mathcal{F}) = \{F \in \mathcal{F} : F \cup \{k\} \in \mathcal{F}$ for some $F < k\}$. It follows that if $\delta_\mathcal{F}(x_i) > 0$ for some basic sequence (x_i) in X then $\mathcal{I}(X) \geq I(\mathcal{F})$. Hence by Proposition 3.9, if $\dot{\delta}_\alpha(X) > 0$ for every $\alpha < \omega_1$ then $\mathcal{I}(X) = \omega_1$, hence X contains a subspace isomorphic to ℓ_1. The converse implication is obvious. □

Other facts about Bourgain's ℓ_1 index can be found in [13].

The next lemma collects some simple observations about the asymptotic constants.

Lemma 4.6 *Let* (e_i) *be a basis for* X *and let* $(x_i) \prec (e_i)$. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be regular classes of finite subsets of* $\mathbb{N}$.

a) $\delta_\mathcal{F}(e_i) \leq \delta_\mathcal{F}(x_i)$ *and* $\dot{\delta}_\mathcal{F}(x_i) \leq \dot{\delta}_\mathcal{F}(e_i)$;

b) $\delta_\mathcal{F}(e_i) \leq \dot{\delta}_\mathcal{F}(e_i) \leq \ddot{\delta}_\mathcal{F}(e_i)$;

c) $\inf_n \delta_n(e_i) > 0$ *iff* $(e_i/\|e_i\|)$ *is equivalent to the unit vector basis of* ℓ_1;

d) $\ddot{\delta}_\mathcal{F}(e_i) = \sup_{(x_i)\prec(e_i)} \sup\{\dot{\delta}_\mathcal{F}((x_i), |\cdot|) : |\cdot|$ is an equivalent norm on $[x_i]_{i\in\mathbb{N}}\}$;

e) $\delta_{\mathcal{F}[\mathcal{G}]}(x_i) \geq \delta_\mathcal{F}(x_i)\delta_\mathcal{G}(x_i)$.

Proof. a) and b) are immediate; the first part of a) uses that $\mathcal{F}(M) \subseteq \mathcal{F}$. c) follows from the fact that $\bigcup_{n=1}^\infty \mathcal{S}_n$ contains all finite subsets of $\{2, 3, \ldots\}$. d) is true because if $Y \subseteq X$ and $|\cdot|$ is an equivalent norm on Y then $|\cdot|$ can be extended to an equivalent norm on X. For e) notice that if $(y_i)_i^k$ is $\mathcal{F}[\mathcal{G}]$-admissible w.r.t. (x_i), then it can be blocked in a $\mathcal{F}$-admissible way into successive blocks each of which consists of $\mathcal{G}$-admissible vectors (w.r.t. (x_i)). This directly implies the inequality. □

The most important situation for the study of the constants δ_α is when the whole sequence $(\delta_\alpha)_{\alpha<\omega_1}$ is stabilized on a nested sequence of block subspaces. This leads to the concept of the Δ-spectrum of X to be all possible stabilized limits of δ_α's of block bases. We formalize it in the following definition.

Definition 4.7 *Let X be a Banach space and let $\gamma = (\gamma_\alpha)_{\alpha<\omega_1} \subseteq \mathbb{R}$. We say that a basic sequence (x_i) in X* Δ-stabilizes γ *if there exist $\varepsilon_n \downarrow 0$ so that for every $\alpha < \omega_1$ there exists $m \in \mathbb{N}$ so that for all $n \geq m$ if $(y_i) \prec (x_i)_n^\infty$ then $|\delta_\alpha(y_i) - \gamma_\alpha| < \varepsilon_n$.*

Let X have a basis (e_i). The Δ-spectrum *of X, $\Delta(X)$, is defined to be the set of all γ's so that there exists $(x_i) \prec (e_i)$ such that (x_i) Δ-stabilizes γ. By $\ddot{\Delta}(X)$ we denote the set of all γ's so that (x_i) Δ-stabilizes γ for some $(x_i) \prec (e_i)$, under some equivalent norm $|\cdot|$ on $[x_i]_{i\in\mathbb{N}}$.*

Remark 4.8 It is important to note that the asymptotic constants $\delta_\alpha(y_i)$ considered here and appearing in the definition of the spectrum $\Delta(X)$ refer to the admissibility with respect to the block basis (y_i) itself. It is sometimes convenient, however, to consider asymptotic constants that keep a reference level for admissibility fixed when passing to block bases. Precisely, if (e_i) is a basis in X and $(x_i) \prec (e_i)$, we define $\delta_{\mathcal{F}}((x_i),(e_i))$ as the supremum of $\delta \geq 0$ such that whenever $(y_i)_1^k \prec (x_i)$ is $\mathcal{F}$-admissible w.r.t. (e_i) then $\|\sum_1^k y_i\| \geq \delta \sum_1^k \|y_i\|$. Clearly, $\delta_{\mathcal{F}}(e_i) \leq \delta_{\mathcal{F}}((x_i),(e_i)) \leq \delta_{\mathcal{F}}(x_i)$. We can then define the spectrum $\Delta(X,(e_i))$ by replacing $\delta_\alpha(y_i)$ by $\delta_{\mathcal{S}_\alpha}((y_i),(e_i))$, in Definition 4.7 above. Let us also note that it has been proved in [2] that these two concepts of spectrum actually coincide and $\Delta(X,(e_i)) = \Delta(X)$.

Remark 4.9 The definition of $\mathcal{S}_\alpha$ for $\alpha \geq \omega_0$ depended upon certain choices made at limit ordinals. It follows that the constants $\delta_\alpha(e_i)$ also depend upon the particular choice of $\mathcal{S}_\alpha$. However $\Delta(X)$ is independent of the choice of each $\mathcal{S}_\alpha$. Indeed, this follows from a consequence of Propositions 3.9 and 3.10. If $\mathcal{S}_\alpha$ and $\bar{\mathcal{S}}_\alpha$ are two choices for the Schreier class then there exist subsequences of $\mathbb{N}$, M and N such that $\mathcal{S}_\alpha(N) \subseteq \bar{\mathcal{S}}_\alpha$ and $\bar{\mathcal{S}}_\alpha(M) \subseteq \mathcal{S}_\alpha$. We also deduce that the constants $\dot{\delta}_\alpha$ and $\ddot{\delta}_\alpha$ are independent of the particular choice of $\mathcal{S}_\alpha$.

The following stabilization argument shows that $\Delta(X)$ is always non-empty.

Proposition 4.10 *Let X be a Banach space with a basis (e_i). Then there exists $\gamma = (\gamma_\alpha)_{\alpha<\omega_1}$ and $(x_i) \prec (e_i)$ so that (x_i) Δ-stabilizes γ. In particular, $\Delta(X) \neq \emptyset$.*

Proof. Fix $\varepsilon_n \downarrow 0$. If $[e_i]_{i\in\mathbb{N}}$ contains ℓ_1, then, since ℓ_1 is not distortable, we can choose a normalized sequence $(x_i) \prec (e_i)$ with $\|\sum_n^\infty a_i x_i\| \geq (1-\varepsilon_n)\sum |a_i|$ for all (a_i); thus the proposition follows with $\gamma_\alpha = 1$ for all α.

If $[e_i]_{i\in\mathbb{N}}$ does not contain ℓ_1 then by Proposition 4.5, $\dot{\delta}_\alpha(e_i) > 0$ for at most countably many α's.

Fix an arbitrary $\alpha < \omega_1$. It follows from Lemma 4.6 that if $(y_i) \prec (e_i)$ then $\dot{\delta}_\alpha((y_i)_n^\infty) = \dot{\delta}_\alpha(y_i)$ for all n. Since $\dot{\delta}_\alpha(y_i) \leq \dot{\delta}_\alpha(z_i)$ whenever $(y_i) \prec (z_i)$, by a standard argument we can stabilize $\dot{\delta}_\alpha$. That is, given $(w_i) \prec (e_i)$ we can find $(z_i) \prec (w_i)$ so that

$$\gamma_\alpha \equiv \dot{\delta}_\alpha(z_i) = \dot{\delta}_\alpha(y_i) \quad \text{for all } (y_i) \prec (z_i) \ .$$

(To do this, construct $(w_i) \succ (z_i^{(1)}) \succ (z_i^{(2)}) \succ \ldots$ such that $\dot{\delta}_\alpha(z_i^{(k+1)}) \le \inf\{\dot{\delta}_\alpha(y_i) : (y_i) \prec (z_i^{(k)})\} + 2^{-k}$, for every k, and set $z_i = z_i^{(i)}$ for all i.)

Now choose by induction $(z_i) \succ (x_i^{(1)}) \succ (x_i^{(2)}) \succ \ldots$ such that

$$|\delta_\alpha(x_i^{(n+1)}) - \dot{\delta}_\alpha(x_i^{(n)})| = |\delta_\alpha(x_i^{(n+1)}) - \gamma_\alpha| \le \varepsilon_n \quad \text{for all } n ,$$

and let $x_i = x_i^{(i)}$ for all i. Then $|\delta_\alpha((x_i)_n^\infty) - \gamma_\alpha| < \varepsilon_n$ for all n. If $(y_i) \prec (x_i)_n^\infty$ then $\delta_\alpha((x_i)_n^\infty) \le \delta_\alpha(y_i) \le \dot{\delta}_\alpha(y_i) = \gamma_\alpha$.

Then using this and a diagonal argument for the countably many α's so that $\dot{\delta}_\alpha(e_i) > 0$ we obtain the proposition. □

Our next proposition collects some basic facts about the Δ-spectrum.

Proposition 4.11 *Let X have a basis (e_i).*

a) *$\Delta(X) \neq \emptyset$ and if $\gamma \in \Delta(X)$ then $\gamma_\alpha \in [0,1]$ for $\alpha < \omega_1$.*

b) *X contains ℓ_1 iff there exists $\gamma \in \Delta(X)$ with $\gamma_1 = 1$.*

c) *If $\gamma \in \Delta(X)$ then $\gamma_\alpha \ge \gamma_\beta$ if $\alpha \le \beta < \omega_1$.*

d) *If $\gamma \in \Delta(X)$ and $\alpha, \beta < \omega_1$, then $\gamma_\alpha\gamma_\beta \le \gamma_{\beta+\alpha}$.*

e) *If $\gamma \in \Delta(X)$ and $\alpha < \omega_1$, $n \in \mathbb{N}$ then $\gamma_{\alpha\cdot n} \ge (\gamma_\alpha)^n$.*

f) *If $\gamma \in \Delta(X)$ then γ is a continuous function of α.*

g) *$\ddot{\delta}_\alpha(X) = \sup\{\gamma_\alpha : \gamma \in \ddot{\Delta}(X)\}$.*

Proof. We have already seen the non-trivial part of a) and one implication in b). Next, e) follows immediately from d) while f) and g) follow from the relevant definitions, using c) to get f).

To complete b) note that if $\gamma_1 = 1$ then $\gamma_\alpha = 1$, for all $\alpha < \omega_1$ (for $\alpha = \beta + 1$ this follows from d) and for α a limit ordinal—from f)). Thus by Proposition 4.5, X contains ℓ_1.

c) Let $\gamma \in \Delta(X)$ and $\alpha \le \beta < \omega_1$. For $n \in \mathbb{N}$ let $\mathbb{N}_n = (n, n+1, \ldots)$. Let (x_i) stabilize γ. Given $m \in \mathbb{N}$ choose $n \ge m$ by Proposition 3.2 so that $\mathcal{S}_\alpha(\mathbb{N}_n) \subseteq \mathcal{S}_\beta(\mathbb{N}_m)$. It follows that $\delta_\alpha((x_i)_n^\infty) \ge \delta_\beta((x_i)_m^\infty)$. Letting $m \to \infty$ we get $\gamma_\alpha \ge \gamma_\beta$.

d) Let (y_i) be basic. By Proposition 3.2 there exists M with $\mathcal{S}_{\beta+\alpha}(M) \subseteq \mathcal{S}_\alpha[\mathcal{S}_\beta]$. It follows that $\delta_{\beta+\alpha}((y_i)_M) \ge \delta_{\mathcal{S}_\alpha[\mathcal{S}_\beta]}(y_i)$. By Lemma 4.6 we see that $\delta_{\mathcal{S}_\alpha[\mathcal{S}_\beta]}(y_i) \ge \delta_\alpha(y_i)\delta_\beta(y_i)$. Thus $\delta_{\beta+\alpha}((y_i)_M) \ge \delta_\alpha(y_i)\delta_\beta(y_i)$. Using this for $(y_i) = (x_i)_{i\in\mathbb{N}_n}$ where (x_i) stabilizes γ, we obtain that $\gamma_{\beta+\alpha} \ge \gamma_\alpha\gamma_\beta$. □

Remark 4.12 It is often useful to note that the constants δ_n satisfy conditions c) and d) for natural numbers. If $m, n \in \mathbb{N}$ and $m \le n$ then $\delta_m(x_i) \ge \delta_n(x_i)$ and $\delta_{m+n}(x_i) \ge \delta_m(x_i)\delta_n(x_i)$, hence also $\delta_{mn}(x_i) \ge (\delta_n(x_i))^m$ (because $\mathcal{S}_m \subseteq \mathcal{S}_n$ and $\mathcal{S}_n[\mathcal{S}_m] = \mathcal{S}_{m+n}$).

It is well known that the supermultiplicativity property d) of sequences $\gamma \in \Delta(X)$ formally implies a "sub-power-type" behavior of γ, which we shall find useful in various situations. This depends on an elementary lemma. For two sequences $(b_n), (c_n) \subseteq (0,1]$ we shall write $c_n \ll b_n$ to denote that $\lim_n b_n/c_n = \infty$.

Lemma 4.13 *Let $(b_n) \subseteq (0,1]$ satisfy $b_{n+m} \ge b_n b_m$ for all $n, m \in \mathbb{N}$. Then $\lim_n b_n^{1/n}$ exists and equals $\sup_n b_n^{1/n}$. Moreover, for every $0 < \xi < \lim_n b_n^{1/n}$ we have $\xi^n \ll b_n$.*

Proof. Let $a_n = \log(b_n^{-1})$. Then $a_n \ge 0$ and $a_{n+m} \le a_n + a_m$ for all n, m. It suffices to prove that $a_n/n \to a \equiv \inf_m\{a_m/m\}$. Given $\varepsilon > 0$ choose k with $|a_k/k - a| < \varepsilon$. For $n > k$, $a_n/n - a < a_n/n - a_k/k + \varepsilon$. Setting $n = pk + r$, $0 \le r < k$ and using $a_{p_k} \le pa_k$ we obtain

$$\begin{aligned}\frac{a_n}{n} - \frac{a_k}{k} + \varepsilon &\le \frac{a_{pk} + a_r}{n} - \frac{a_k}{k} + \varepsilon \le \frac{pa_k}{pk + r} + \frac{a_r}{n} - \frac{a_k}{k} + \varepsilon \\ &\le \frac{pa_k}{pk} - \frac{a_k}{k} + \frac{a_r}{n} + \varepsilon = \frac{a_r}{n} + \varepsilon\ .\end{aligned}$$

The first part of the lemma follows. The moreover part can be easily proved by contradiction. □

We have an immediate corollary.

Corollary 4.14 *Setting $\widehat{\gamma}_\alpha = \lim_n(\gamma_{\alpha\cdot n})^{1/n}$ for $\alpha < \omega_1$ we have that for every $0 < \xi < \widehat{\gamma}_\alpha$, $\xi^n \ll \gamma_{\alpha\cdot n} \le \widehat{\gamma}_\alpha^n$, for all $\alpha < \omega_1$ and $n \in \mathbb{N}$.*

Setting $\widehat{\delta} = \lim_n(\delta_n(x_i))^{1/n}$, for a basic sequence (x_i), we have that for every $0 < \xi < \widehat{\delta}$, $\xi^n \ll \delta_n(x_i) \le \widehat{\delta}^n$ for all $n \in \mathbb{N}$.

There is an interesting connection between the constants $\ddot{\delta}_\alpha(X)$ which allow for renormings of a given space X, and the supermultiplicative behavior of $\gamma \in \Delta(X)$, in particular of $\widehat{\gamma}_\alpha$, which involved the original norm only.

Proposition 4.15 *Let X have a basis (e_i) and let $\gamma \in \Delta(X)$. Then there exists $(y_i) \prec (e_i)$ so that (y_i) Δ-stabilizes γ and so that for all $\alpha < \omega_1$, $\ddot{\delta}_\alpha(y_i) = \lim_n(\gamma_{\alpha\cdot n})^{1/n} \equiv \widehat{\gamma}_\alpha$.*

The argument is based on the following renorming result which we shall use again.

Proposition 4.16 *Let Y be a Banach space with a bimonotone basis (y_i). Let $\alpha < \omega_1$ and $n \in \mathbb{N}$. Then there exists an equivalent bimonotone norm $\| \cdot \|$ on Y with $\delta_\alpha((y_i), \| \cdot \|) \ge \left(\delta_{[\mathcal{S}_\alpha]^n}(y_i)\right)^{1/n}$.*

Proof. Denote the original norm on Y by $|\cdot|$ and set $\theta = \delta_{[\mathcal{S}_\alpha]^n}(y_i)$. For $0 \le j \le n$ define a norm $|\cdot|_j$ on Y by

$$|y|_j = \sup\Big\{\theta^j \sum_1^\ell |E_i y| \ : \ (E_i y)_1^\ell \text{ is } [\mathcal{S}_\alpha]^j\text{-admissible w.r.t. } (y_i) \\ \text{and } E_1 < \cdots < E_\ell \text{ are adjacent intervals}\Big\} .$$

Here we take $[\mathcal{S}_\alpha]^0 = \mathcal{S}_0$ so that $|y|_0 = |y|$. For $0 \le j \le n$ we have $|y|_j \ge \theta^j |y|$ and $|y| \ge \theta^{n-j}|y|_j$. The former inequality follows trivially from the definition of $|\cdot|_j$ and the latter from the fact that any $[\mathcal{S}_\alpha]^j$-admissible family is $[\mathcal{S}_\alpha]^n$-admissible and the definition of θ.

Set $\|y\| = \frac{1}{n}\sum_0^{n-1} |y|_j$ for $y \in Y$. Then $\|\cdot\|$ is an equivalent norm on Y.

Let $(x_s)_1^r$ be α-admissible w.r.t. (y_i). First observe that $|\sum_1^r x_s| \ge \theta \sum_1^r |x_s|_{n-1}$. Indeed, arbitrary $[\mathcal{S}_\alpha]^{n-1}$-admissible decompositions for each x_s can be put together to give a $[\mathcal{S}_\alpha]^n$-admissible decomposition for $\sum_1^r x_s$, thus the estimate follows from the definition of $|\cdot|_{n-1}$ and the fact that $\delta_{[\mathcal{S}_\alpha]^n}(y_i) = \theta^n$. To be more precise, for $1 \le s \le r$ choose adjacent intervals of integers $E_1^s < \cdots < E_{k(s)}^s$ so that $(E_j^s)_1^{k(s)}$ is $[\mathcal{S}_\alpha]^{n-1}$ admissible and

$$|x_s|_{n-1} = \theta^{n-1} \sum_{j=1}^{k(s)} |E_j^s x_s| .$$

Let $F_j^s = E_j^s$ if $j < k(s)$ and $F_{k(s)}^s = [\min E_{k(s)}^s, \min E_1^{s+1})$ if $s < r$ and $F_{k(r)}^r = E_{k(r)}^r$. Then $F_1 < \cdots < F_{k(1)}^1 < \cdots < F_{k(r)}^r$ are $[\mathcal{S}_\alpha]^n$-admissible adjacent intervals of $\mathbb{N}$ and so

$$\begin{aligned} \Big|\sum_{l=1}^r x_l\Big| &\ge \theta^n \sum_{s=1}^r \sum_{j=1}^{k(s)} |F_j^s (\sum_{l=1}^r x_l)| \\ &\ge \theta \sum_{s=1}^r \theta^{n-1} \sum_{j=1}^{k(s)} |E_j^s(x_s)| = \theta \sum_{s=1}^r |x_s|_{n-1} , \end{aligned}$$

(since $|F_{k(s)}^s(\sum_{l=1}^r x_l)| = |F_{k(s)}^s(x_s + x_{s+1})| \ge |E_{k(s)}^s(x_s)|$ if $s < r$, using that the norm is monotone).

Similarly, $|\sum_1^r x_s|_{j+1} \ge \theta \sum_1^r |x_s|_j$ for $j = 1, 2, \ldots, n-2$, by the definitions of $|\cdot|_{j+1}$ and $|\cdot|_j$. Thus

$$\Big\|\sum_1^r x_s\Big\| = \frac{1}{n}\sum_{j=0}^{n-1}\Big|\sum_{s=1}^r x_s\Big|_j \ge \frac{\theta}{n}\sum_{s=1}^r |x_s|_{n-1} + \frac{1}{n}\Big(\sum_{j=0}^{n-2} \theta \sum_{s=1}^r |x_s|_j\Big) .$$

Thus $\|\sum_1^r x_s\| \ge \theta \sum_1^r \|x_s\|$. □

Proposition 4.17 *Let X be an asymptotic ℓ_1 space and let $\gamma \in \Delta(X)$. If $(e_i) \prec X$ Δ-stabilizes γ then for all $\varepsilon_i \downarrow 0$ there exists $(x_i) \prec (e_i)$ and an equivalent norm $|\cdot|$ on $[x_i]$ satisfying*

a) *For all n and $x \in \langle x_i \rangle_n^\infty$ we have*

$$\|x\| \le |x| \le (2 + \varepsilon_n)\|x\| .$$

b) *(x_i) is bimonotone for $|\cdot|$.*

c) *(x_i) Δ-stabilizes $\bar\gamma \in \Delta(X,|\cdot|)$ with $\bar\gamma_\alpha \geq \gamma_\alpha$ for all $\alpha < \omega_1$.*

Proof. We may assume that $[e_i]$ does not contain ℓ_1. Thus by Rosenthal's theorem [23] there exists $(x_i) \prec (e_i)$ which is normalized and weakly null. By passing to a subsequence of (x_i) we may assume that for all $n < m$ and $(a_i)_1^m \subseteq \mathbb{R}$, $\|\sum_1^n a_i x_i\| \leq (1+\bar\varepsilon_n)\|\sum_1^m a_i x_i\|$, where $\bar\varepsilon_n = \varepsilon_n/2$.

Define the norm $|\cdot|$ for $x \in X$ by

$$|x| = \sup\{\|Ex\| : E \text{ is an interval}\}\ .$$

Passing to a block basis of (x_i) we may assume that (x_i) Δ-stabilizes some $\bar\gamma \in \Delta(X,|\cdot|)$. For $x = \sum_{i=n}^m a_i x_i$ with $|x| = \|Fx\|$ we have

$$\|x\| \leq |x| \leq \|\sum_{i=n}^{\max F} a_i x_i\| + \|\sum_{i=n}^{\min F - 1} a_i x_i\| \leq 2(1+\bar\varepsilon_n)\|x\| = (2+\varepsilon_n)\|x\|\ .$$

Thus a) holds and b) is immediate. It remains to check c). Fix $\alpha < \omega_1$ and $m \in \mathbb{N}$. Let $x_m < y_1 < \cdots < y_\ell$ (w.r.t. $(x_i)_m^\infty$) where $(y_i)_1^\ell$ is α-admissible w.r.t. $(x_i)_m^\infty$ and hence w.r.t. $(e_i)_m^\infty$. Choose intervals $E_1 < \cdots < E_\ell$ such that $|y_i| = \|E_i y_i\|$ for $i \leq \ell$ and $E_i \subseteq [\min \operatorname{supp}(y_i), \max \operatorname{supp}(y_i)]$. Define $(F_i)_1^\ell$ to be adjacent intervals so that $\min F_i = \min E_i$. Thus $F_i = [\min E_i, \min E_{i+1}) \subseteq \mathbb{N}$ for $i < \ell$ and $F_\ell = E_\ell$. Let $F = \bigcup_1^\ell F_i$. Then, by Remark 4.3,

$$\begin{aligned}\Big|\sum_1^\ell y_i\Big| &\geq \Big\|F(\sum_1^\ell y_i)\Big\| \geq \delta_\alpha((e_i)_m^\infty)\sum_{j=1}^\ell \|F_j(\sum_1^\ell y_i)\| \\ &\geq \delta_\alpha((e_i)_m^\infty)(1+\bar\varepsilon_m)^{-1}\sum_{j=1}^\ell \|F_j y_j\| = \delta_\alpha((e_i)_m^\infty)(1+\bar\varepsilon_m)^{-1}\sum_{j=1}^\ell |y_j|\ .\end{aligned}$$

It follows that $\delta_\alpha((x_i)_m^\infty, |\cdot|) \geq \delta_\alpha((e_i)_m^\infty)(1+\bar\varepsilon_m)^{-1}$. Letting $m \to \infty$ we obtain $\bar\gamma_\alpha \geq \gamma_\alpha$. □

Remark 4.18 It is worth noting the following. Let (e_i) be a basic sequence in X Δ-stabilizing $\gamma \in \Delta(X)$. Then there exists $(x_i) \prec (e_i)$ and an equivalent monotone norm $|\cdot|$ on $[x_i]$ so that (x_i) Δ-stabilizes $\gamma \in \Delta(X,|\cdot|)$. Furthermore $\Big||x| - \|x\|\Big| < \varepsilon_n$ for $x \in \langle x_i\rangle_n^\infty$ and some $\varepsilon_n \downarrow 0$. Assuming as we may that $[e_i]$ does not contain ℓ_1, this is accomplished by taking (x_i) to be a suitable weakly null block basis of (e_i) and setting $|\sum a_i x_i| = \sup_n \|\sum_1^n a_i x_i\|$.

A similar argument yields

Proposition 4.19 *Let $\mathcal{F}$ be a regular set of finite subsets of $\mathbb{N}$ and let (e_i) be a basis for X. Given $\varepsilon > 0$ and $\varepsilon_i \downarrow 0$ there exists an equivalent norm $|\cdot|$ on some block subspace $[x_i] \subseteq X$ satisfying a) and b) of Proposition 4.17 and $\delta_{\mathcal{F}}((x_i),|\cdot|) \geq \delta_{\mathcal{F}}(e_i) - \varepsilon$.*

As a corollary to these propositions we obtain

Theorem 4.20 *Let Y be a Banach space with a basis (y_i). Let $\alpha < \omega_1$, $n \in \mathbb{N}$, $\varepsilon > 0$ and $\theta^n = \delta_{[\mathcal{S}_\alpha]^n}(y_i)$. Then there exists an equivalent norm $|||\cdot|||$ on $X = [x_i] \prec Y$ with $\delta_\alpha((x_i), |||\cdot|||) \ge \theta - \varepsilon$.*

Proof of Proposition 4.15. Let $(x_i) \prec (e_i)$ Δ-stabilize γ (for the original norm $\|\cdot\|$). We may assume that $X' = [x_i]$ does not contain ℓ_1. It follows that there exists $\alpha_0 < \omega_1$ so that $\ddot{\delta}_\beta(X') = 0 = \gamma_\beta$ for all $\beta > \alpha_0$. Also from Lemma 4.6, $\ddot{\delta}_\alpha(z_i) \le \ddot{\delta}_\alpha(w_i)$ if $(z_i) \prec (w_i) \prec (e_i)$; moreover, $\ddot{\delta}_\alpha((z_i)_n^\infty) = \ddot{\delta}_\alpha(z_i)$ for all $n \in \mathbb{N}$. We can therefore stabilize the $\ddot{\delta}_\alpha$'s (as in the proof of Proposition 4.10) to find $(y_i) \prec (x_i)$ so that for all $\alpha \le \alpha_0$, $\ddot{\delta}_\alpha(y_i) = \ddot{\delta}_\alpha(z_i)$ if $(z_i) \prec (y_i)$. Of course (y_i) still Δ-stabilizes γ. We shall prove that $\ddot{\delta}_\alpha(y_i) = \lim_n(\gamma_{\alpha\cdot n})^{1/n}$.

Note that if $|\cdot|$ is an equivalent norm on $[y_i]$ and $\bar{\gamma} \in \Delta((y_i), |\cdot|)$ then $\lim_n(\bar{\gamma}_{\alpha\cdot n})^{1/n} = \lim_n(\gamma_{\alpha\cdot n})^{1/n}$. Indeed if $(z_i) \prec (y_i)$ Δ-stabilizes $\bar{\gamma}$ in $|\cdot|$ then since (z_i) Δ-stabilizes γ in $\|\cdot\|$ and the norms are equivalent, we obtain $c\bar{\gamma}_\beta \le \gamma_\beta \le d\bar{\gamma}_\beta$ for all $\beta < \omega_1$ and for some constants c, $d > 0$. Thus

$$\bar{\gamma}_\alpha \le \sup_n(\bar{\gamma}_{\alpha\cdot n})^{1/n} = \lim_n(\gamma_{\alpha\cdot n})^{1/n} \ .$$

By Proposition 4.11 we obtain that $\ddot{\delta}_\alpha(y_i) \le \lim_n(\gamma_{\alpha\cdot n})^{1/n}$.

Fix $\theta < \lim_n(\gamma_{\alpha\cdot n})^{1/n}$. Thus there exists n_0 with $\theta^{n_0} < \gamma_{\alpha\cdot n_0}$. Choose $(z_i) \prec (y_i)$ with $\theta^{n_0} < \delta_{\alpha\cdot n_0}(z_i)$. By Corollary 3.4 there exists M so that $[\mathcal{S}_\alpha]^{n_0}(M) \subseteq \mathcal{S}_{\alpha\cdot n_0}$, which yields $\delta_{\alpha\cdot n_0}(z_i) \le \delta_{[\mathcal{S}_\alpha]^{n_0}}((z_i)_M)$. So letting $(w_i) = (z_i)_M$ we have $\delta_{[\mathcal{S}_\alpha]^{n_0}}(w_i) > \theta^{n_0}$. By Theorem 4.20 there exists an equivalent norm $|||\cdot|||$ on $[w_i']_{\mathbb{N}}$, for some $(w_i') \prec (w_i)$ with $\delta_\alpha((w_i'), |||\cdot|||) > \theta$. The reverse inequality, $\ddot{\delta}_\alpha(y_i) \ge \lim_n(\gamma_{\alpha\cdot n})^{1/n}$, follows. □

As we will see in later sections, some further regularity properties of sequences $\gamma \in \Delta(X)$ are closely related to distortion properties of the space X, and they may or may not hold in general. In contrast, the sequences $(\ddot{\delta}_\alpha)$ which allow for renorming display a complete power type behavior. In fact, we will give a comprehensive description of behavior of such sequences in Theorem 4.23 below.

In the result that follows we shall be particularly interested in part c).

Proposition 4.21 *Let X have a basis (e_i). Let $\alpha < \omega_1$ and $n \in \mathbb{N}$.*

a) $\ddot{\delta}_{[\mathcal{S}_\alpha]^n}(X) = (\ddot{\delta}_\alpha(X))^n$

b) $\ddot{\delta}_{[\mathcal{S}_\alpha]^n}(X) = \ddot{\delta}_{\alpha\cdot n}(X)$

c) $\ddot{\delta}_{\alpha\cdot n}(X) = (\ddot{\delta}_\alpha(X))^n$

Proof. c) will follow from a) and b).

a) Since for any equivalent norm $|\cdot|$ on X we have $\delta_{[\mathcal{S}_\alpha]^n}((y_i),|\cdot|) \geq (\delta_\alpha((y_i),|\cdot|))^n$ (Lemma 4.6, e)), the inequality $\ddot{\delta}_{[\mathcal{S}_\alpha]^n}(X) \geq (\ddot{\delta}_\alpha(X))^n$ follows from g) of proposition 4.11. To see the reverse inequality let $|\cdot|$ be an equivalent norm on X, and let $(y_i) \prec (e_i)$ and $\theta > 0$ satisfy $\delta_{[\mathcal{S}_\alpha]^n}((y_i),|\cdot|) > \theta^n$. By Theorem 4.20 there exist $(x_i) \prec (y_i)$ and an equivalent norm $\|\cdot\|$ on $[x_i]_{i\in\mathbb{N}}$ such that $\delta_\alpha((x_i),\|\cdot\|) > \theta$. This completes the proof.

b) As we have shown earlier, whenever $(y_i) \prec (e_i)$ and $|\cdot|$ is an equivalent norm, by Corollary 3.4 there exists a subsequence M such that $\delta_{[\mathcal{S}_\alpha]^n}((y_i)_M,|\cdot|) \geq \delta_{\alpha\cdot n}((y_i),|\cdot|)$. It follows that $\ddot{\delta}_{[\mathcal{S}_\alpha]^n}(X) \geq \ddot{\delta}_{\alpha\cdot n}(X)$. The reverse inequality follows by choosing N with $\mathcal{S}_{\alpha\cdot n}(N) \subseteq [\mathcal{S}_\alpha]^n$. □

Let us introduce the following natural and convenient definition.

Definition 4.22 *Let X be an asymptotic ℓ_1 space. The* spectral index of X, $I_\Delta(X)$, *is defined to be*

$$I_\Delta(X) = \inf\{\alpha < \omega : \ddot{\delta}_\alpha(X) < 1\} \ .$$

Theorem 4.23 *If X is an asymptotic ℓ_1 space not containing ℓ_1, then $I_\Delta(X) = \omega^\alpha$ for some $\alpha < \omega_1$. If $I_\Delta(X) = \alpha_0$ and $\ddot{\delta}_{\alpha_0}(X) = \theta$ then $\ddot{\delta}_{\alpha_0\cdot n+\beta}(X) = \theta^n$ for all $n \in \mathbb{N}$ and $\beta < \alpha_0$. Finally, $\ddot{\delta}_\beta(X) = 0$ for all $\alpha_0 \cdot \omega \leq \beta < \omega_1$.*

Proof. For the proof of the first statement, it suffices to show that if $\beta < I_\Delta(X)$ then for all $n \in \mathbb{N}$, $\beta \cdot n < I_\Delta(X)$ ([20], Thm. 15.5). But by Proposition 4.21, $\ddot{\delta}_{\beta\cdot n}(X) = (\ddot{\delta}_\beta(X))^n = 1$, so $\beta \cdot n < I_\Delta(X)$.

Now let $\alpha_0 = \omega^\alpha$ for some α and assume that $\ddot{\delta}_{\alpha_0}(X) = \theta$ for some $0 < \theta < 1$. Fix $\beta < \alpha_0$. We first show that for any $\varepsilon > 0$ we can find $(y_i) \prec X$ and an equivalent norm $\|\cdot\|$ on $[y_i]_\mathbb{N}$ with $\delta_\beta((y_i),\|\cdot\|) > 1-\varepsilon$ and $\delta_{\alpha_0}((y_i),\|\cdot\|) > \theta-\varepsilon$. Indeed, let $\theta' = \theta-\varepsilon$ and choose by Proposition 4.17 $(x_i) \prec X$ and an equivalent bimonotone norm $|\cdot|$ on X so that $\delta_{\alpha_0}((x_i),|\cdot|) > \theta'$. Given $m \in \mathbb{N}$ we can choose a subsequence N of $\mathbb{N}$ so that $\mathcal{S}_{\alpha_0}\Big[[\mathcal{S}_\beta]^j\Big](N) \subseteq \mathcal{S}_{\beta\cdot j+\alpha_0} = \mathcal{S}_{\alpha_0}$ for $j = 0,1,\ldots,m$; this follows from Proposition 3.2, Corollary 3.4 and the fact that $\beta\cdot m + \omega^\alpha = \omega^\alpha$. Let $(y_i) = (x_i)_N$ and $a^m \equiv \delta_{[\mathcal{S}_\beta]^m}((y_i),|\cdot|)$. Note that since $[\mathcal{S}_\beta]^m(N) \subseteq \mathcal{S}_{\alpha_0}$ then $a^m \geq \theta'$ and so $a \geq (\theta')^{1/m}$. For $y \in [y_i]_\mathbb{N}$ and $0 \leq j \leq m$ set

$$|y|_j = \sup\{a^j \sum_1^\ell |E_i y| \ \ : \ \ (E_i y)_1^\ell \text{ is } [\mathcal{S}_\beta]^j\text{-admissible w.r.t. } (y_i) \text{ and } E_1 < \cdots < E_k \text{ are adjacent intervals}\} \ .$$

It can be checked by a straightforward calculation, using the choice of N and that (y_i) is monotone for $|\cdot|$, that $\delta_{\alpha_0}((x_i),|\cdot|_j) \geq \delta_{\alpha_0}((y_i),|\cdot|) > \theta'$ for $j = 0,\ldots,m$. For $y \in [y_i]_\mathbb{N}$ set $\|y\| =$

$\frac{1}{m}\sum_{j=0}^{m-1}|y|_j$. Then $\delta_{\alpha_0}((y_i),|\!|\!|\cdot|\!|\!|) > \theta'$ and from the proof of Proposition 4.16, $\delta_\beta((y_i),|\!|\!|\cdot|\!|\!|) \geq a > (\theta')^{1/m}$. Taking m such that $(\theta')^{1/m} \geq 1-\varepsilon$ we get what we wanted.

Now by Proposition 3.2 there exists a subsequence M of $\mathbb{N}$ with $\mathcal{S}_{\alpha_0+\beta}(M) \subseteq \mathcal{S}_\beta[\mathcal{S}_{\alpha_0}]$. It follows that

$$\delta_{\alpha_0+\beta}((y_i)_M,|\!|\!|\cdot|\!|\!|) > (1-\varepsilon)\theta' = (1-\varepsilon)(\theta-\varepsilon)\ .$$

Hence $\ddot{\delta}_{\alpha_0+\beta}(X) = \theta$.

The case of general n is proved similarly, replacing α_0 by $\alpha_0\cdot n$ above and recalling (Proposition 4.21) that $\ddot{\delta}_{\alpha_0\cdot n}(X) = (\ddot{\delta}_{\alpha_0}(X))^n$. The last statement is obvious. □

5 Examples–Tsirelson Spaces

Our primary source of examples of asymptotic ℓ_1 spaces with various behaviors of asymptotic constants is the class of mixed Tsirelson spaces introduced by Argyros and Deliyanni in [3].

Definition 5.1 *Let $I \subseteq \mathbb{N}$ and for $n \in I$ let $\mathcal{F}_n$ be a regular family of finite subsets of $\mathbb{N}$. Let $(\theta_n)_{n\in I} \subseteq (0,1)$ satisfy $\sup_{n\in I}\theta_n < 1$. The* mixed Tsirelson space $T(\mathcal{F}_n,\theta_n)_{n\in I}$ *is the completion of c_{00} under the implicit norm*

$$\|x\| = \max\left(\|x\|_\infty, \sup_{n\in I}\sup\left\{\theta_n\sum_{i=1}^k\|E_ix\| : (E_i)_{i=1}^k \textit{ is } \mathcal{F}_n\textit{-admissible}\right\}\right)\ .$$

It is shown in [3] that such a norm exists. It is also proved that if I is finite or if $\theta_n \to 0$, then $T(\mathcal{F}_n,\theta_n)_{n\in I}$ is a reflexive Banach space, in which the standard unit vectors (e_i) form a 1-unconditional basis. In [3] it is proved that for an appropriate choice of θ_n and $\mathcal{F}_n$ the space $T(\mathcal{F}_n,\theta_n)_{n\in\mathbb{N}}$ is arbitrarily distortable. Deliyanni and Kutzarova [9] proved a result that illustrates the possible complexity these spaces can possess. They proved that a mixed Tsirelson space may uniformly contain ℓ_∞^n's in all subspaces. Notice that the Tsirelson space T satisfies $T = T(\mathcal{S}_n,2^{-n})_{n\in\mathbb{N}} = T(\mathcal{S}_1,2^{-1})$. For $0<\theta<1$ we denote the θ-Tsirelson space by $T_\theta = T(\mathcal{S}_1,\theta)$.

Theorem 5.2 *Let (e_i) denote the unit vector basis for T.*

a) *If $(x_i) \prec (e_i)$ then for all n, $\delta_n(x_i) = 2^{-n}$ and $\ddot{\delta}_n(x_i) = 2^{-n}$.*

b) *For all $\gamma \in \Delta(T)$, $\gamma_n = 2^{-n}$ for $n \in \mathbb{N}$ and $\gamma_\alpha = 0$ for $\alpha \geq \omega$.*

c) *For all $\gamma \in \ddot{\Delta}(T)$, $\gamma_n \leq 2^{-n}$ for $n \in \mathbb{N}$.*

d) *$I_\Delta(X) = 1$ for all $X \prec T$.*

Remark 5.3 Condition a) immediately implies that for an arbitrary equivalent norm $|\cdot|$ on T and $(x_i) \prec (e_i)$, we have $\delta_1((x_i), |\cdot|) \le 1/2$. Since the asymptotic ℓ_1 constant is equal to δ_1^{-1}, this improves the constant in Proposition 2.8 from $\sqrt{2}$ to 2.

Remark 5.4 For T_θ we have $\delta_n(T_\theta) = \ddot{\delta}_n(T_\theta) = \theta^n$ for $n \in \mathbb{N}$; and all other equalities and inequalities from Theorem 5.2 hold with appropriate modifications. Also, clearly, $I_\Delta(T_\theta) = 1$.

Proof of Theorem 5.2. a) By definition of the norm $\|\cdot\|$ for T, $\delta_n(e_i) \ge 2^{-n}$ and so if $(x_i) \prec (e_i)$ then $\delta_n(x_i) \ge 2^{-n}$ as well.

We next show that there exists $C < \infty$ so that $\delta_m(x_i) \le C2^{-m}$ for all m. This will yield the equality for δ_n. Indeed if for some n, $\delta_n(x_i) = A/2^n$ where $A > 1$ then since $\delta_{nk}(x_i) \ge (\delta_n(x_i))^k$ (Remark 4.12), we would have that $C2^{-nk} \ge \delta_{nk}(x_i) \ge A^k 2^{-nk}$ for all k, which is impossible.

First we consider the case $(x_i) = (e_i)_{i\in M}$ where M is a subsequence of $\mathbb{N}$. Let $\varepsilon > 0$, $n \in \mathbb{N}$ and let $x = \sum_{i\in F} a_i e_i$ be an $(n, n-1, \varepsilon)$-average of $(e_i)_{i\in M}$ (see Proposition 3.6 and Notation 3.7). Thus $\|x\| \ge 2^{-n}$. Iterating the definition of the norm in T yields that $\|x\| = \sum_{i=1}^n 2^{-i} \sum_{j\in F_i} a_j$ where $(F_i)_{i=1}^n$ partitions F into sets with $F_i \in \mathcal{S}_i$ for $i \le n$. Thus if $\varepsilon < 2^{-n}$,

$$\|x\| = \|\sum_{i\in F} a_i e_i\| \le \sum_{i=1}^{n-1} 2^{-i}\varepsilon + 2^{-n} \sum_{j\in F_n} a_j \le 2/2^n = 2/2^n \sum_{i\in F} \|a_i e_i\| \ .$$

Hence $\delta_n((e_i)_M) \le 2/2^n$.

If (x_i) is normalized with $(x_i) \prec (e_i)$ then by [7] (see also [8]), there exists a subsequence M such that (x_i) is D-equivalent to $(e_i)_{i\in M}$, where D is an absolute constant (we let $m_i = \min \operatorname{supp}(x_i)$, and then $M = (m_i)$). Thus $\delta_n(x_i) \le D\delta_n((e_i)_M) \le 2D/2^n$.

To get the equality for $\ddot{\delta}_n$ we first observe that for any equivalent norm $|\cdot|$ on T there is a constant C' (depending on $|\cdot|$) such that $\delta_n((x_i), |\cdot|) \le C'\delta_n(x_i)$, and then we follow the previous argument.

b) is immediate from the first part of a); and c) and d) follow from the second part of a). □

Remark 5.5 For the subsequence $M = (m_i)$ above one could take any $m_i \in \operatorname{supp}(x_i)$ for all i. In the space T_θ, any normalized block basis is D-equivalent to $(e_i)_M$ as well, with the equivalence constant $D = c\theta^{-1}$, where c is an absolute constant. The choice of a subsequence M is the same as indicated above (for $\theta = 1/2$).

The next example illustrates Theorem 4.23.

Example 5.6 *Let $\alpha < \omega_1$ and let $X = T(\mathcal{S}_{\omega^\alpha}, \theta)$. Then*

a) $\ddot{\delta}_{\omega^\alpha \cdot n}(X) = \theta^n$ *for* $n \in \mathbb{N}$

b) $I_\Delta(X) = \omega^\alpha$

Proof. a) Let $(x_i) \prec X$ be a normalized block basis that Δ-stabilizes $\gamma \in \Delta(X)$. Let $n \in \mathbb{N}$ and let $\varepsilon > 0$. Choose N by Corollary 3.4 so that $[\mathcal{S}_{\omega^\alpha}]^n \supseteq \mathcal{S}_{\omega^\alpha \cdot n}(N)$ and also $[\mathcal{S}_{\omega^\alpha}]^{n-1}(N) \subseteq \mathcal{S}_{\omega^\alpha \cdot (n-1)}$. Choose $x = \sum_F a_i x_i$ to be an $(\omega^\alpha \cdot n, \omega^\alpha \cdot (n-1), \varepsilon)$ average of $(x_i)_N$ w.r.t. (e_i), the unit vector basis of X. Clearly $\|x\| \geq \theta^n$. As in T, $\|x\|$ is calculated by a tree of sets where the first level of sets is $\mathcal{S}_{\omega^\alpha}$-admissible, the second level is $[\mathcal{S}_{\omega^\alpha}]^2$-admissible and so on.

If we stop this tree after $n-1$ levels, discarding sets which stopped before then and shrinking those sets which split the support of some x_i we obtain for some $(E_i x)_1^\ell$ being $\omega^\alpha \cdot (n-1)$-admissible,

$$\|x\| \leq \theta^{n-1} \sum_1^\ell \|E_i x\| + \varepsilon \ .$$

The next level of splitting may indeed split the supports of some of the x_j's. However since those x_j's have not yet been split the contribution of $a_j x_j$ to the next level of sets is at most $a_j \theta^{-1}$. Thus we obtain

$$\|x\| \leq \theta^n \Big(\sum a_j \theta^{-1}\Big) + \varepsilon = \theta^{n-1} + \varepsilon \ .$$

It follows that $\gamma_{\omega^\alpha \cdot n} \leq \theta^{n-1} = \frac{1}{\theta}(\theta^n)$.

Thus, just as in the case of T, $\gamma_{\omega^\alpha \cdot n} = \theta^n$. Indeed, if $\gamma_{\omega^\alpha \cdot n_0} > \theta^{n_0}$ then

$$\gamma_{\omega^\alpha \cdot n_0 k} \geq (\gamma_{\omega^\alpha \cdot n_0})^k > \frac{1}{\theta}\theta^{n_0 k}$$

for large enough k (Proposition 4.11), which is a contradiction.

Similarly if $\gamma \in \ddot{\Delta}(X)$ then for some C, $\gamma_{\omega^\alpha \cdot n} \leq C\theta^n$ and so $\gamma_{\omega^\alpha \cdot n} \leq \theta^n$ for all n. This yields that $\ddot{\delta}_{\omega^\alpha \cdot n}(X) = \theta^n$.

b) The argument in Proposition 4.23(b) yields this result: for $\beta < \omega^\alpha$ and $\varepsilon > 0$ there exists (x_i) and $|\!|\!| \cdot |\!|\!|$ with $\delta_\beta((x_i), |\!|\!| \cdot |\!|\!|) > 1 - \varepsilon$. □

Before we pass to further examples, let us note a fundamental and useful connection between the spectrum $\Delta(X)$ and a lower estimate for the norm on some block subspace.

Proposition 5.7 *Let X be an asymptotic ℓ_1 space and let $(z_i) \prec X$ be a normalized bimonotone block basis Δ-stabilizing some $\gamma \in \Delta(X)$ with $0 < \gamma_1 < 1$. Let (e_i) be the unit vector basis of $T_{\gamma_1} \equiv T(\mathcal{S}_1, \gamma_1)$. Then for all $\varepsilon > 0$ there exists a subsequence (x_i) of (z_i) satisfying for all $(a_i) \subseteq \mathbb{R}$*

$$\|\sum a_i x_i\| \geq (1-\varepsilon)\|\sum a_i e_i\|_{T_{\gamma_1}} \ .$$

Proof. We shall prove the proposition in the case where $\gamma_1 = 1/2$ (and so $T_{\gamma_1} = T$). We shall describe below the argument in a general case, but the reader is advised to first test the special case when $\delta_1(z_i) = 1/2$ (when $\varepsilon_n = 0$ for all n and the m_i's can be omitted.) Choose integers $m_i \uparrow \infty$ so that $\sum_1^\infty 2^{-m_i} < \varepsilon$ and then choose $\varepsilon_n \downarrow 0$ to satisfy, for all $k \in \mathbb{N}$,

$$\prod_1^k \Big(\frac{1}{2} - \varepsilon_{n(i)}\Big) \quad > \quad (1-\varepsilon)2^{-k} \text{ whenever } (n(i))_{i=1}^k \subseteq \mathbb{N}$$
$$\text{satisfy for every } j,\ |\{i : n(i) = j\}| \le m_j\ . \tag{2}$$

Let (x_i) be a subsequence of (z_i) which satisfies: for all n, if $x_n \le y_1 < \cdots < y_n$ w.r.t. (x_i) then $\|\sum_1^n y_i\| > (\frac{1}{2} - \varepsilon_n)\sum_1^n \|y_i\|$. Such a sequence exists since (z_i) Δ-stabilizes γ with $\gamma_1 = 1/2$.

Let $x = \sum_1^\ell a_i x_i$ and assume that $\|\sum_1^\ell a_i e_i\|_T = 1$. We shall show that $\|x\| > (1-\varepsilon)^2$. If $\|\sum a_i e_i\|_T = |a_j|$ for some j then $\|x\| = 1$. Otherwise for some 1-admissible family of sets, $\|\sum a_i e_i\|_T = \frac{1}{2}\sum_{j=1}^n \|E_j(\sum a_i e_i)\|_T$. Accordingly we have that (here is where the bimonotone assumption is used)

$$\|x\| > \Big(\frac{1}{2} - \varepsilon_i\Big)\sum_{j=1}^n \|E_j x\|$$

where $i = \min(\operatorname{supp} E_1 x)$. We then repeat the step above for each $E_j x$. Ultimately we obtain for some $J \subseteq \mathbb{N}$,

$$1 = \|\sum a_i e_i\|_T = \sum_{i\in J} 2^{-\ell(i)}|a_i|$$

where $\ell(i) =$ the number of splittings before we stop at $|a_i|$. We follow the same tree of splittings in getting a lower estimate for $\|x\|$ with one additional proviso. Each splitting of Ex in $\langle x_i\rangle$ will introduce a factor of $(\frac{1}{2} - \varepsilon_n)$ for some n. A given factor $(\frac{1}{2} - \varepsilon_n)$ may be repeated a number of times. If any $(\frac{1}{2} - \varepsilon_n)$ is repeated m_n times we shall discard the corresponding set $\|Ex\|$ at that instant. By virtue of (2) we thus obtain that $\|x\| \ge \sum_{i\in I}(1-\varepsilon)2^{-\ell(i)}|a_i|$ where $I \subseteq J$ and $a_i x_i$ belonged to a discarded set for $i \in J\setminus I$. However the contribution of the discarded sets to $\|\sum a_i e_i\|_T$ is at most $\sum_{n=1}^\infty 2^{-m_n} < \varepsilon$ since from our construction for any given n (where $(\frac{1}{2} - \varepsilon_n)$ is repeated m_n times) we will discard at most one set, something of the form $2^{-k}\|Ex\|_T$ where $k \ge m_n$. It follows that $\|x\| > (1-\varepsilon)(\|x\|_T - \varepsilon) = (1-\varepsilon)^2$. □

The proof also yields the following block result.

Corollary 5.8 *Let (z_i) be a bimonotone basic sequence in a Banach space X which Δ-stabilizes $\gamma \in \Delta(X)$ where $0 < \gamma_1 < 1$. Let (e_i) be the unit vector basis of T_{γ_1}. Then for all $\varepsilon > 0$ there exists a subsequence (x_i) of (z_i) satisfying for all $(y_j)_1^k \prec (x_i)$ if $m_j = \min(\operatorname{supp}(y_i))$ w.r.t. (x_i) then*

$$\Big\|\sum_1^k y_i\Big\| \ge (1-\varepsilon)\Big\|\sum_1^k \|y_j\| e_{m_j}\Big\|_{T_\gamma}\ .$$

Remark 5.9 We can remove the bimonotone assumption on the norm if we have that for some $\varepsilon_n \downarrow 0$, $\|y_0 + \sum_1^m y_i\| \ge (\gamma_1 - \varepsilon_n)\sum_1^m \|y_i\|$, whenever $z_n \le y_0 \le z_m < y_1 < \cdots < y_m$. Without either this assumption or the bimonotone property we obtain a slightly weaker result.

Theorem 5.10 *Let X be an asymptotic ℓ_1 space and let $(z_i) \prec X$ be a basic sequence Δ-stabilizing some $\gamma \in \Delta(X)$, with $0 < \gamma_1 < 1$. Then for all $\varepsilon > 0$ there exists a normalized $(x_i) \prec (z_i)$ satisfying for all $(a_i) \subseteq \mathbb{R}$*

$$\| \sum a_i x_i \| \geq \frac{1}{2}(1-\varepsilon) \| \sum a_i e_i \|_{T_{\gamma_1}} \ .$$

Moreover if $(y_i)_1^k \prec (x_i)$ with $m_j = \min(\mathrm{supp}(y_i))$ w.r.t. (x_i) then one has

$$\| \sum_1^k y_i \| \geq \frac{1}{2}(1-\varepsilon) \Big\| \sum_1^k \|y_i\| e_{m_i} \Big\|_{T_{\gamma_1}} \ .$$

Proof. By Proposition 4.17 there exists a $\|\cdot\|$-normalized $(x_i) \prec (z_i)$ and a bimonotone norm $|\cdot|$ on $[x_i]$ with $\|x\| \leq |x| \leq (2+\varepsilon)\|x\|$ for $x \in [x_i]$ and such that (x_i) Δ-stabilizes $\bar{\gamma} \in \Delta(X, |\cdot|)$ with $\bar{\gamma}_1 \geq \gamma_1$. We may thus assume that (x_i) satisfies the conclusion of Corollary 5.8 for $|\cdot|$ and ε' such that $(1-\varepsilon')/(2+\varepsilon') = \frac{1}{2}(1-\varepsilon)$. Thus if $(y_i)_1^k$ is as in the statement of the theorem,

$$\Big\| \sum_1^k y_i \Big\| \geq \frac{1}{2+\varepsilon'} \Big| \sum_1^k y_i \Big| \geq \frac{1-\varepsilon'}{2+\varepsilon'} \Big\| \sum_1^k |y_i| e_{m_i} \Big\|_{T_{\bar{\gamma}_1}} \geq \frac{1}{2}(1-\varepsilon) \Big\| \sum_1^k \|y_i\| e_{m_i} \Big\|_{T_{\gamma_1}} \ .$$

□

The following can be proved by an argument similar to that in Proposition 5.7.

Proposition 5.11 *Let X be an asymptotic ℓ_1 space and let $(z_i) \prec X$ be a normalized bimonotone block basis Δ-stabilizing $\gamma \in \Delta(X)$. Let $\alpha < \omega_1$ with $0 < \gamma_\alpha < 1$ and let $\varepsilon > 0$. Then there exists a subsequence (x_i) of (z_i) satisfying the following: if $(y_i)_1^k \prec (z_i)$ with $\min(\mathrm{supp}(y_i)) = m_i$ (w.r.t. (x_i)) then*

$$\Big\| \sum_1^k y_i \Big\| \geq (1-\varepsilon) \Big\| \sum_1^k \|y_i\| e_{m_i} \Big\|_{T(\mathcal{S}_\alpha, \gamma_\alpha)} \ .$$

The next example is a space X for which the sequences of asymptotic constants $(\delta_\alpha(X))$ and $(\ddot{\delta}_\alpha(X))$ are "essentially" the same as for Tsirelson's space T; still, X and T have no common subspaces–no subspace of X is isomorphic to a subspace of T. It is worth noting that X also has the property that the sequence $\ddot{\delta} = (\ddot{\delta}_\alpha(X))$ does not belong to $\ddot{\Delta}(X)$.

Example 5.12 *Let $0 < c < 1$ and let $X = T(\mathcal{S}_n, c2^{-n})_{n \in \mathbb{N}}$. Then*

a) *$\ddot{\delta}_n(X) = 2^{-n}$ for all n*

b) *For all $\gamma \in \ddot{\Delta}(X)$, $\gamma_n < 2^{-n}$ for all n.*

c) *No subspace of X embeds isomorphically into T.*

Before verifying these assertions we first require some observations.

The norm of $x \in X$, if not equal to $\|x\|_\infty$, is computed by a tree of sets, the *first level* being $(E_i)_1^\ell$ where for some j, $(E_i)_1^\ell$ is j-admissible and

$$\|x\| = \frac{c}{2^j} \sum_{i=1}^{\ell} \|E_i x\| \ .$$

For each i, if $\|E_i x\|$ does not equal $\|E_i x\|_\infty$, then we split $\|E_i x\|$ into a second level of sets m_i-admissible for some m_i, and so on. If every set keeps splitting then after k steps we obtain an expression of the form

$$c^k \sum_{s=1}^{r} 2^{-n(s)} \|F_s x\| \ . \tag{3}$$

Of course some sets may stop splitting, in which case if we carry on for k-steps, we only obtain a lower estimate for $\|x\|$. Consider the case where $(x_i) \prec X$ and $x \in \langle x_i \rangle$. We set $\|x\|_{\mathcal{T}_k,(x_i)}$ to be the largest of the expressions of the form (3) obtained by splitting k-times (a k level tree of sets, where $(F_s)_1^r$ is the k^{th}-level), subject to the additional constraint that for all i and s, F_s does not split x_i. Thus $F_s x_i$ is either x_i or 0.

Lemma 5.13 *Let $(x_i) \prec X$, $\varepsilon > 0$ and $k \in \mathbb{N}$. Then there exists $x \in \langle x_i \rangle$ with $\|x\| = 1$ such that $\|x\|_{\mathcal{T}_k,(x_i)} > 1 - \varepsilon$.*

Proof. Assume without loss of generality that $\|x_i\| = 1$ for $i \in \mathbb{N}$. We call $x \in [x_i]$ an (n, ε)-*normalized* average (of (x_i) w.r.t. (e_i)) if $x = \sum_{i\in F} a_i x_i / \| \sum_{i \in F} a_i x_i\|$, where $\sum_{i\in F} a_i x_i$ is an $(n, n-1, c\varepsilon/2^n)$-average of (x_i) w.r.t. (e_i). Thus $(x_i)_{i\in F}$ is n-admissible w.r.t. (e_i) and if $G \subseteq F$ satisfies $(x_i)_{i\in G}$ is $(n-1)$-admissible then $\sum_G a_i < c\varepsilon/2^n$. Also $\sum_{i\in F} a_i = 1$ and $a_i > 0$ for $i \in F$. (We can always find such vectors by Proposition 3.6.) Note that if $(x_i)_{i\in G}$ is $(n-1)$-admissible and if we write x in the form $x = \sum_{i\in F} b_i x_i$ (for some $b_i > 0$), then $\sum_G b_i < (c\varepsilon/2^n)(2^n/c) = \varepsilon$ (since $\|\sum_{i\in F} a_i x_i\| \geq c/2^n$).

We first indicate how to find x satisfying $\|x\| = 1$ and $\|x\|_{\mathcal{T}_1,(x_i)} > 1 - \varepsilon$. Let $\varepsilon_i = 2^{-(i+1)}\varepsilon$ so that $\sum_1^\infty \varepsilon_i = \varepsilon/2$. Let

$$\| \cdot \|_n = \sup\Big\{ c2^{-n} \sum_{j=1}^{\ell} \|E_j x\| : (E_j x)_{j=1}^{\ell} \text{ is } n\text{-admissible}\Big\} \ .$$

and observe that for all x, $\lim_n \|x\|_n = 0$. Let $n_1 = 1$ and choose $(y_i^1) \prec (x_i)$ and $n_j \uparrow \infty$ by induction so that each y_j^1 is an (n_j, ε_j)-normalized average of (x_i) and for all j, $\|\sum_{i=1}^{j} y_i^1\|_m < \varepsilon_{j+1}$ if $m \geq n_{j+1}$. Then we choose y^2 to be an $(n, \varepsilon/2)$-normalized average of (y_i^1) where $n \in \mathbb{N}$ is not important but we may assume that $y^2 = \sum_F b_i y_i^1$ where $n < n_{\min F}$.

We have $1 = \|y^2\|$ and so by the definition of the norm in X, there exists j such that $1 = \|y^2\|_j = c/2^j \sum_{s=1}^{\ell} \|E_s(y^2)\|$ where $(E_s y^2)_1^\ell$ is j-admissible. We claim that by somewhat altering the E_s's we can ensure, by losing no more than ε, that the sets E_s do not split any of the x_i's. Indeed if $1 \le j < n$, then $G = \{i \in F : E_s \text{ splits } y_i^1 \text{ for some } s\} \in \mathcal{S}_j$. Since $j < n$, $\sum_{s\in G} b_s < \varepsilon/2$ and thus by shrinking the offending sets E_s to avoid splitting y_i's we obtain the desired sets. If $n \le j < n_{\min F}$ then if we fix $i \in F$ and consider $G_i = \{r : E_s$ splits one or more of the x_r's in the support of $y_i\}$ we get that, by similarly shrinking the offending E_s's so as to not split such an x_r, and letting $\widetilde{E}_s$ be the new sets, that

$$\frac{c}{2^j}\sum \|\widetilde{E}_s y^2\| > 1 - \sum_{i\in F} b_i\varepsilon_i > 1-\varepsilon\ .$$

Finally if $F = (k_1,\ldots,k_r)$ and $n_{k_p} \le j < n_{k_{p+1}}$ then

$$\Big\|\sum_{\substack{i\in F\\ i<k_p}} b_i y_i\Big\|_j < \varepsilon_{k_p} \quad\text{and}\quad b_{k_p} < \varepsilon/2$$

so we first discard the E_s's which intersect $\operatorname{supp}(\sum_{i\le k_p} b_i y_i)$. Then arguing as above we shrink the remaining E_s's so as to not split any x_i. We obtain

$$\frac{c}{2^j}\sum \|\widetilde{E}_s y^2\| \ge 1 - \varepsilon_{k_p} - \varepsilon/2 - \sum_{\substack{i\in F\\ i>k_p}} b_i\varepsilon_i > 1-\varepsilon\ .$$

This proves the lemma in the case $k = 1$. For the general case we continue as above letting (y_i^2) be (n_i^2, ε_i)-normalized averages of (y_i^1), etc. If $x = y_1^{k+1}$ then x satisfies the lemma for k. We omit the tedious calculations. □

Proof of the assertions in Example 5.12. By Proposition 4.16, since $\delta_n(X) \ge c2^{-n}$, we have $\ddot{\delta}_1(X) \ge 2^{-1}$. If there exists $\gamma \in \ddot{\Delta}(X)$ with $\gamma_1 \ge 2^{-1}$ then by Theorem 5.10 there exists $(x_i) \prec (e_i)$ and $d > 0$ so that for all $(y_j)_1^\ell \prec (x_i)$ if $m_j = \min(\operatorname{supp}(y_j))$ w.r.t. x_i, then

$$\Big\|\sum_1^\ell y_j\Big\| \ge d\Big\|\sum_1^\ell \|y_j\| e_{m_j}\Big\|_T\ . \tag{4}$$

Fix an arbitrary k. By Lemma 5.13 there exists $x \in \langle x_i\rangle$ with $\|x\| = 1$ and $\|x\|_{\mathcal{T}_{k},(x_i)} > 1/2$. Thus there exists a k-level tree of sets whose final level is $(E_1,\ldots,E_r)$ so that $c^k\sum_{s=1}^r 2^{-n(s)}\|E_s x\| > 1/2$. Following the same partition scheme in T and using (4) for $y_s = E_s x$ we get (with $m_s = \min(\operatorname{supp}(E_s x))$),

$$d^{-1} = d^{-1}\|x\| \ge \Big\|\sum_{s=1}^r \|E_s x\| e_{m_s}\Big\|_T \ge \sum_{s=1}^r 2^{-n(s)}\|E_s x\| > \frac{1}{2}(c^{-k})\ .$$

Since $c < 1$, this is impossible for large enough k. This proves b) for $n = 1$ and that $\ddot{\delta}_1(X) = 2^{-1}$. Then Proposition 4.21 yields $\ddot{\delta}_n(X) = 2^{-n}$ for all n.

The remainder of b) easily follows from the proof of Proposition 4.16. Indeed assume that some $\gamma \in \ddot{\Delta}(X)$ satisfies $\gamma_n = 2^{-n}$, for some $n > 1$. By Proposition 4.17 there is $(y_i) \prec X$ and an equivalent bimonotone norm $|\cdot|$ on $[y_i]$ such that (y_i) Δ-stabilizes $\bar{\gamma} \in \ddot{\Delta}(X, |\cdot|)$ and $\bar{\gamma}_n = 2^{-n}$. By passing to a subsequence we may assume that for some sequence $\varepsilon_n \downarrow 0$, for all m,

$$\Big|\sum_1^k x_i\Big| \geq 2^{-n}(1-\varepsilon_m)\sum_1^k |x_i|$$

if $(x_i)_1^k \prec (y_i)_m^\infty$ and (x_i) is n-admissible w.r.t. $(y_i)_1^\infty$. Let $|\!|\!|\cdot|\!|\!|$ be the norm constructed in the proof of Proposition 4.16 for $\alpha = 1$ and $\theta = 1/2$. If $y_r \leq x_1 < \cdots < x_r$ then

$$\Big|\sum_1^r x_s\Big| \geq (1/2)(1-\varepsilon_r)\sum_1^r |x_s|_{n-1} \ .$$

The remaining estimates remain true and, as in the proof of Proposition 4.16, we obtain

$$|\!|\!|\sum_1^r x_s|\!|\!| \geq (1/2)(1-\varepsilon_r)\sum_1^r |\!|\!|x_s|\!|\!| \ .$$

Thus $\gamma_1 = 1/2$ which is impossible.

If c) were not true, then, by Theorem 5.2 b), a subspace Y of X isomorphic to a subspace of T would admit a renorming for which $\gamma_1(Y) = 1/2$, in contradiction to b). □

Remark 5.14 The above example X yields the following. There exists $(x_i) \prec (e_i)$ and a sequence of equivalent norms $|\!|\!|\cdot|\!|\!|_j$ so that for all k on $[x_i]_k^\infty$, $\|x\| \geq |\!|\!|x|\!|\!|_j \geq c^2\|x\|$ if $j \geq k$ and furthermore $\delta_1(|\!|\!|\cdot|\!|\!|_j, (x_i)) > \frac{1}{2} - \varepsilon_j$ for some $\varepsilon_j \to 0$. Yet $\gamma_1 < \frac{1}{2}$ for all $\gamma \in \ddot{\Delta}(X)$. To see this one needs only choose (x_i) so that on $[x_i]_k^\infty$, $\|x\| = \sup_{\ell \geq k}\|x\|_\ell$. This can be accomplished by taking each x_j to be an iterated $j+1$-normalized average of (e_i) (as in lemma 5.13). Then set $|\!|\!|x|\!|\!|_j = (1/j)\sum_1^j \|x\|_i$. Since $\|x\| \geq \|x\|_i \geq c\|x\|_j \geq c^2\|x\|$ on $\langle x_s\rangle_j^\infty$, $\|x\| \geq |\!|\!|x|\!|\!|_j \geq c^2\|x\|$.

We mention one other example, taken from [2]. First suppose that $X = T(\mathcal{S}_n, \overline{\theta}_n)_{n\in\mathbb{N}}$ where $1 > \sup_n \overline{\theta}_n$ and $\lim_{n\to\infty}\overline{\theta}_n = 0$. We shall call (θ_n) *regular* if for all $n, m \in \mathbb{N}$, $\theta_{n+m} \geq \theta_n\theta_m$. It is easy to verify that every such X has a regular representation, i.e., for some regular sequence (θ_n) we have $X = T(\mathcal{S}_n, \theta_n)_{\mathbb{N}}$. Thus $\lim_n \theta_n^{1/n}$ exists by Lemma 4.13.

Example 5.15 *Let $X = T(\mathcal{S}_n, \theta_n)_{\mathbb{N}}$ where $1 > \sup_n \theta_n$, $\theta_n \to 0$ and (θ_n) is regular. Let $\theta = \lim_n \theta_n^{1/n}$. Then*

a) *For all $Y \prec X$ we have $\ddot{\delta}_1(Y) = \theta$.*

b) *For all $Y \prec X$ and for all $n \in \mathbb{N}$, $\ddot{\delta}_n(Y) = \theta^n$ and $\ddot{\delta}_\omega = 0$.*

c) *For all $Y \prec X$, $I_\Delta(Y) = \begin{cases} \omega & \text{if } \theta = 1 \\ 1 & \text{if } \theta < 1 \end{cases}$*

d) *For all $Y \prec X$ and $j \in \mathbb{N}$ we have $\delta_j(Y) \leq \theta^j \sup_{n\geq j}\theta_n\theta^{-n} \vee \theta_j/\theta_1$. In particular, if $\theta_n\theta^{-n} \to 0$ then X is arbitrarily distortable.*

6 Renormings of T, and spaces of bounded distortion

Definition 6.1 *The distortion constant of a space X is defined by*

$$D(X) = \sup_{|\cdot|\sim\|\cdot\|} d(X, |\cdot|) \ .$$

So X is distortable iff $D(X) > 1$. Similarly, X is arbitrarily distortable iff $D(X) = \infty$. Finally, X is of bounded distortion iff there is $D < \infty$ such that $D(Y) \leq D$ for every subspace $Y \subseteq X$.

As we saw in Proposition 2.7, Tsirelson's space T satisfies $D(T) \geq 2$. Similarly one can show that $D(T_\theta) \geq \theta^{-1}$. However, not much more is known about distorting T. It is unknown if T is arbitrarily distortable, or at least whether it contains an arbitrarily distortable subspace; and, if not, what is $D(T)$ or at least a reasonable upper estimate for it. The interest in these questions lies in the fact that, as already mentioned, no examples are yet known of distortable spaces which are of bounded distortion.

From techniques developed earlier in this paper we easily get some information on asymptotic constants of equivalent norms on Tsirelson space. This should be compared with Theorem 5.2 where the constants for the original norm were established.

Surprisingly, it is not known if there exists $(x_i) \prec T$ and an equivalent norm $|\cdot|$ on $[x_i]$ with $\delta_1((x_i), |\cdot|) < 1/2$. Our next result shows that the class of equivalent norms for which $\delta_1 = 1/2$ cannot arbitrarily distort T.

Theorem 6.2 *There exists an absolute constant D with the following property. Let $X \prec T$ and let $|\cdot|$ be an equivalent norm on X such that for some $\gamma \in \Delta(X, |\cdot|)$, $\gamma_1 = 1/2$. Then $d(X, |\cdot|) \leq D$.*

Proof. Let (z_i) be a basic sequence in X Δ-stabilizing γ under $|\cdot|$ where $\gamma_1 = 1/2$. Let $\varepsilon > 0$. By passing to a block basis of (z_i) and multiplying $|\cdot|$ by a constant if necessary we may assume that $\|\cdot\|_T \geq |\cdot|$ on $[z_i]$ and for all $(w_i) \prec (z_i)$ there exists $w \in \langle w_i \rangle$ with $1+\varepsilon > \|w\|_T \geq |w| = 1$. Choose a normalized block basis (w_i) of (z_i) satisfying $1 + \varepsilon \geq \|w_i\|_T \geq |w_i| = 1$ for all i. Theorem 5.10 allows us to also assume that

$$|\sum a_i w_i| \geq (1/2 - \varepsilon) \|\sum a_i e_i\|_T \ .$$

There exists an absolute constant D_1 so that $(w_i/\|w_i\|_T)$ is D_1-equivalent to (e_{m_i}) in $\|\cdot\|_T$, where $m_i = \min \mathrm{supp}(w_i)$ w.r.t. (e_i), for each i [7]. Thus we have, for all $(a_i) \subseteq \mathbb{R}$,

$$(1+\varepsilon) D_1 \|\sum a_i e_{m_i}\|_T \geq \|\sum a_i w_i\|_T \geq |\sum a_i w_i| \geq (1/2 - \varepsilon)\|\sum a_i e_i\|_T \ . \quad (5)$$

Consider the subsequence (p_i) of $\mathbb{N}$ defined by induction by $p_1 = 1$ and $p_{i+1} = m_{p_i}$, for $i \geq 1$. There is a universal constant D_2 so that (e_{p_i}) is D_2-equivalent to $(e_{p_{i+1}})$ in $\|\cdot\|_T$ [7]. Also, on the subspace $[w_{p_i}]$ we have, by (5),

$$(1+\varepsilon) D_1 \|\sum a_i e_{p_{i+1}}\|_T \geq |\sum a_i w_{p_i}| \geq (1/2 - \varepsilon)\|\sum a_i e_{p_i}\|_T \ .$$

Thus the conclusion follows with $D = 2D_1D_2$. □

A natural question in light of the above results is whether one can quantify the distortion $d(X, |\cdot|)$ of an equivalent norm $|\cdot|$ on $X \prec T$ in terms of $\Delta(X, |\cdot|)$.

Problem 6.3 *Let $|\cdot|$ be an equivalent norm on T and let $(x_i) \prec T$ $(\Delta, |\cdot|)$-stabilize γ. Thus for some $c > 0$, $c2^{-n} \le \gamma_n \le 2^{-n}$ for all n. Does there exist a function $f(c)$ so that $d(X, |\cdot|) \le f(c)$?*

We shall give a suggestive partial answer to a weaker problem. First we note the following proposition.

Proposition 6.4 *For $n \in \mathbb{N}$ define the equivalent norm $\|\cdot\|_n$ on T by $\|x\|_n = \sup\{2^{-n}\sum_1^\ell \|E_ix\| : (E_ix)_1^\ell$ is n-admissible$\}$. Given $X \prec T$ and $\varepsilon_n \downarrow 0$ there exists $(x_i) \prec X$ so that for all n if $x \in \langle x_i\rangle_n^\infty$ then $\Big|\|x\| - \|x\|_n\Big| < \varepsilon_n\|x\|$.*

Proof. First note that if

$$\|\cdot\|_{\mathcal{S}_n} = \sup\Big\{\sum_{i\in E} |x(i)| : E \in \mathcal{S}_n\Big\}$$

then for all $x \in T$ we have $\|x\|_n \le \|x\| \le \|x\|_n + \|x\|_{\mathcal{S}_n}$. Indeed, if $\|x\| \ne \|x\|_\infty$ then $\|x\| = x^*(x)$ for some functional x^* (with $\|x^*\| = 1$) determined by the successive iterations of the implicit equation of the norm in T; in particular, $x^*(e_i) = \pm 2^{-n(i)}$ for all i. We may write $x^* = y^* + z^*$ where $z^*(e_i) = \pm 2^{-n(i)}$ if $n(i) \le n$ and 0 otherwise. Thus, since the support of z^* is n-admissible, $|z^*(x)| \le (1/2)\|x\|_{\mathcal{S}_n}$ and $|y^*(x)| \le \|x\|_n$. Furthermore, $\|x\|_{\mathcal{S}_n} \le 2^n\|x\|$. Since the Schreier space S_n is isomorphic to a subspace of $C(\omega^{\omega^n})$ (Remark 3.5), , it is c_0-saturated, i.e., every infinite-dimensional subspace contains a copy of c_0, and thus $\|\cdot\|_{\mathcal{S}_n}$ cannot be equivalent to $\|\cdot\|$ on any infinite-dimensional subspace of T. In particular we can chose $(x_i) \prec X$ so that for all $x \in \langle x_i\rangle_n^\infty$, $\|x\|_{\mathcal{S}_n} \le \varepsilon_n\|x\|$. The conclusion follows. □

Problem 6.5 *Let $|\cdot|$ be an equivalent norm on $X = [x_i] \prec T$. Let $(y_i) \prec (x_i)$, $C < \infty$ and suppose that for all n, if $y \in [y_i]_n^\infty$ then $C^{-1}|y|_n \le |y| \le C|y|_n$, where $|y|_n = \sup\{2^{-n}\sum_1^\ell |E_iy| : (E_iy)_1^\ell$ is n-admissible w.r.t. $(x_i)\}$. Does there exist a function $F(C)$ so that $d(Y, |\cdot|) \le F(C)$?*

Proposition 6.6 *Let $(y_i) \prec (x_i) \prec T$ and let $|\cdot|$ be an equivalent norm on $[x_i]$. Suppose that for all n and $y \in [y_i]_n^\infty$, $C^{-1}|y|_n \le |y| \le C|y|_n$ (where $|\cdot|_n$ is defined as above). Then for all $\varepsilon > 0$ there exists n_0 and an equivalent norm $|\!|\!|\cdot|\!|\!|$ on $[y_i]_{n_0}^\infty$ such that $C^{-1}|\!|\!|y|\!|\!| \le |y| \le C|\!|\!|y|\!|\!|$ for $y \in [y_i]_{n_0}^\infty$ and $\delta_1((y_i)_{n_0}^\infty, |\!|\!|\cdot|\!|\!|) > \frac{1}{2} - \varepsilon$.*

Proof. Choose n_0 so that $C^2/n_0 < \varepsilon$. On $[y_i]_{n_0}^\infty$ define $\|\!|y|\!\| = \frac{1}{n_0}\sum_1^{n_0} |y|_j$. Clearly the inequality between the norms hold. Let $p \in \mathbb{N}$ and let $(z_i)_1^p \prec [y_i]_{n_0}^\infty$ satisfy $y_{n_0+p} \le z_1 < \cdot < z_p$. Let $z = \sum_1^p z_i$. Then (see the proof of Proposition 4.16) $|z|_{j+1} \ge \frac{1}{2}\sum_{i=1}^p |z_i|_j$ for $j = 1, \ldots, n_0 - 1$. Hence

$$\|\!|z|\!\| \ge \frac{1}{n_0}\sum_{j=1}^{n_0-1}\frac{1}{2}\sum_{i=1}^p |z_i|_j = \frac{1}{2}\sum_{i=1}^p \|\!|z_i|\!\| - \frac{1}{2n_0}\sum_{i=1}^p |z_i|_{n_0} \ .$$

Now $|z_i|_{n_0} \le C|z_i| \le C^2\|\!|z_i|\!\|$ and so

$$\|\!|z|\!\| \ge \frac{1}{2}\sum_{i=1}^p \|\!|z_i|\!\| \left(1 - \frac{C^2}{2n_0}\right) > (1-\varepsilon)\frac{1}{2}\sum_{i=1}^p \|\!|z_i|\!\| \ ,$$

completing the proof. □

Finally, let us recall the following known [7] property of T. There exists an absolute constant D_1 so that if $x_1 < y_1 < x_2 < y_2 < \cdots$ are normalized in T then (x_i) is D_1-equivalent to (y_i). It turns out that equivalent norms on T that satisfy this property (with a fixed constant) cannot arbitrarily distort T. The result, in fact, holds in any space having this subsequence property.

Proposition 6.7 *There exists a function $f(D)$ satisfying the following. If $|\cdot|$ is an equivalent norm on $[x_i]_{\mathbb{N}} \prec T$ so that (y_i) is D-equivalent to (z_i) whenever $y_1 < z_1 < y_2 < \cdots$ is a normalized block basis of (x_i), then $d(X, |\cdot|) \le f(D)$.*

Proof. By passing to a block basis of (x_i) and scaling the norm $|\cdot|$ we may assume that there exists $d > 1$ so that for all $x \in [x_i]$, $d^{-1}\|x\| \le |x| \le \|x\|$; furthermore, in any block subspace Y of (x_i) there exist $y, z \in Y$ with $|y| = |z| = 1$ and $\|y\| \le 2$ and $\|z\| > d/2$. Choose a $|\cdot|$-normalized block basis of (x_i), $y_1 < z_1 < y_2 < \cdots$ with $\|z_i\| > d/2$ and $\|y_i\| \le 2$ for all i. There exists $w = \sum a_i z_i$ satisfying $|w| = 1$ and $\|w\| < 2$. Since (z_i) and (y_i) are D-equivalent for $|\cdot|$, $|\sum a_i y_i| > D^{-1}$. Also $(z_i/\|z_i\|_T)$ and $(y_i/\|y_i\|_T)$ are D_1-equivalent in T. Thus

$$\|\sum a_i y_i\|_T \le 2D_1\|\sum a_i z_i/\|z_i\|_T\,\|_T \le 4D_1/d\ \|\sum a_i z_i\|_T \le 8D_1/d\ .$$

Thus $D^{-1} \le 8D_1/d$ and so $d \le 8D_1 D \equiv f(D)$. □

We now turn to some results about spaces of bounded distortion.

Theorem 6.8 *Let X be an asymptotic ℓ_1 space. Let $\gamma \in \Delta(X)$ and let $(y_i) \prec X$ Δ-stabilize γ. If $Y = [y_i]$ is of D-bounded distortion then for any $\alpha < \omega_1$ and $n, m \in \mathbb{N}$,*

a) $D^{-1}(\ddot{\delta}_\alpha(Y))^n \le \gamma_{\alpha\cdot n} \le (\ddot{\delta}_\alpha(Y))^n$

b) $\gamma_{\alpha\cdot n}\gamma_{\alpha\cdot m} \le \gamma_{\alpha\cdot(n+m)} \le D^2\gamma_{\alpha\cdot n}\gamma_{\alpha\cdot m}$.

Proof. a) Let $\overline{\gamma} = (\overline{\gamma}_\alpha) \in \ddot{\Delta}(Y)$. Choose an equivalent norm $|\cdot|$ on Y and $(w_i) \prec (y_i)$ which $(\Delta, |\cdot|)$-stabilizes $\overline{\gamma}$. Let $\varepsilon > 0$. By passing to a block basis of (w_i) and scaling $|\cdot|$ we may suppose that

$$|w| \le \|w\| \le (D+\varepsilon)|w| \text{ for } w \in [w_i] \ .$$

Let $\alpha < \omega_1$ and $n \in \mathbb{N}$. We may assume that $\delta_{\alpha\cdot n}((w_i), |\cdot|) > \overline{\gamma}_{\alpha\cdot n} - \varepsilon$. Thus if $(x_s)_1^r$ is $\alpha \cdot n$-admissible w.r.t. (w_i),

$$\Big\| \sum_1^r x_s \Big\| \ge \Big| \sum_1^r x_s \Big| \ge (\overline{\gamma}_{\alpha\cdot n} - \varepsilon) \sum_1^r |x_s| \ge \frac{\overline{\gamma}_{\alpha\cdot n} - \varepsilon}{D+\varepsilon} \sum_1^r \|x_s\| \ .$$

It follows that $\gamma_{\alpha\cdot n} \ge \overline{\gamma}_{\alpha\cdot n}/D$ and so $\overline{\gamma}_{\alpha\cdot n} \le D\gamma_{\alpha\cdot n}$. Passing to the supremum over all $\overline{\gamma}_{\alpha\cdot n}$ and using Proposition 4.11 g), we get $\ddot{\delta}_{\alpha\cdot n}(Y) \le D\gamma_{\alpha\cdot n}$. Hence by Proposition 4.21,

$$D^{-1}(\ddot{\delta}_\alpha(Y))^n = D^{-1}\ddot{\delta}_{\alpha\cdot n}(Y) \le \gamma_{\alpha\cdot n} \le \ddot{\delta}_{\alpha\cdot n}(Y) = (\ddot{\delta}_\alpha(Y))^n \ .$$

b) Using part a) and Proposition 4.11 d),

$$\begin{aligned} \gamma_{\alpha\cdot n}\gamma_{\alpha\cdot m} &\le \gamma_{\alpha\cdot(n+m)} \le (\ddot{\delta}_\alpha(Y))^{n+m} \\ &= (\ddot{\delta}_\alpha(Y))^n(\ddot{\delta}_\alpha(Y))^m \le D^2\gamma_{\alpha\cdot n}\gamma_{\alpha\cdot m} \ , \end{aligned}$$

completing the proof. □

Combining the proposition with Theorem 4.23 we get a complete description, up to equivalence, of sequences γ from $\Delta(X)$, in spaces of D-bounded distortion. We leave the details to the reader.

Recall the notation $\widehat{\gamma}_\alpha = \lim_k (\gamma_{\alpha\cdot k})^{1/k}$, for $\alpha < \omega_1$ (Corollary 4.14). If $Y \prec X$ Δ-stabilizes γ, we may write $\widehat{\gamma}_\alpha(Y)$ to emphasise the subspace Y. By Proposition 4.15, $\widehat{\gamma}_\alpha(Y) = \ddot{\delta}_\alpha(Y)$. Therefore, by Proposition 2.5, we have an important sufficient condition for an asymptotic ℓ_1 space to contain an arbitrary distortable subspace.

Corollary 6.9 *Let X be an asymptotic ℓ_1 space. Let $\gamma \in \Delta(X)$ and let $(y_i) \prec X$ Δ-stabilize γ. If there exists $\alpha < \omega_1$ such that $\gamma_\alpha > 0$ and $\lim_n \gamma_{\alpha\cdot n}\widehat{\gamma}_\alpha(Y)^{-n} = 0$, then Y contains an arbitrarily distortable subspace.*

Let us present an alternative approach to Corollary 6.9, taken from [25], which is of independent interest. It is based on a construction of certain asymptotic sets in a general asymptotic ℓ_1 space.

An alternative proof of Corollary 6.9. (Sketch) Let $\gamma \in \Delta(X)$, let $Y = [y_i] \prec X$ Δ-stabilize γ and let (y_i^*) be the biorthogonal functionals in Y^*. Suppose that Y is of D-bounded distortion. Fix an arbitrary $\alpha < \omega_1$. We shall show that $(1/3D)(\widehat{\gamma}_\alpha(Y))^n \le \gamma_{\alpha\cdot(n-1)}$. By Proposition 4.15, this is slightly weaker than Theorem 6.8, but sufficient to imply Corollary 6.9.

Fix $n \in \mathbb{N}$. First we shall show that for all $\varepsilon > 0$, all normalized blocks $(x_i) \prec (y_i)$, and all $0 < \lambda < 1$, there is an $(\alpha\cdot n, \alpha\cdot(n-1), \varepsilon)$ average x of (x_i) w.r.t. (y_i) such that $\|x\| \ge \lambda(\widehat{\gamma}_\alpha(Y))^n \equiv \lambda'$.

This is done by blocking, in the spirit of James [11]. Fix m sufficiently large and pick $N \subseteq \mathbb{N}$ such that $[\mathcal{S}_{\alpha\cdot n}]^m(N) \subseteq \mathcal{S}_{\alpha\cdot(n\,m)}$ (Corollary 3.4) and that $\lambda\gamma_{\alpha\cdot(n\,m)} \le \delta_{\alpha\cdot(n\,m)}((x_i)_N)$ (this is possible by the Definition 4.7 of the Δ-spectrum). Pick $(z_i^{(1)}) \prec (x_i)_N$ such that for all i, $z_i^{(1)}$ is an $(\alpha\cdot n, \alpha\cdot(n-1), \varepsilon)$ average of $(x_i)_N$ w.r.t. (y_i). If for all i, $\|z_i^{(1)}\| < \lambda'$, then pick $(z_i^{(2)}) \prec (z_i^{(1)})$ such that for all i, $z_i^{(2)}$ is an $(\alpha\cdot n, \alpha\cdot(n-1), \varepsilon)$ average of $(z_i^{(1)}/\|z_i^{(1)}\|)$ w.r.t. (y_i). And keep going. Assume that after m steps we still had that $\|z_i^{(k)}\| < \lambda'$ for all i and all $k \le m$. Write $z_1^{(m)} = \sum_{j\in N} b_j x_j$; then $b_j \ge 0$ and let J be the set of all $j \in N$ such that $b_j > 0$. It is easily seen that $(x_j)_{j\in J}$ is $[\mathcal{S}_{\alpha\cdot n}]^m(N)$-admissible w.r.t. (y_i), hence also $(\alpha\cdot(n\,m))$–admissible w.r.t. (y_i). Moreover, our assumption on the norms of the $z_i^{(k)}$'s easily yields that $\sum b_j > (1/\lambda')^{m-1}$. Thus

$$\begin{aligned}(1/\lambda')^{m-1}\lambda\gamma_{\alpha\cdot(n\,m)} &\le \lambda\gamma_{\alpha\cdot(n\,m)}\sum_{j\in J}\|b_j x_j\| \\ &\le \delta_{\alpha\cdot(n\,m)}((x_i)_N)\sum_{j\in J}\|b_j x_j\| \le \|\sum_{j\in J} b_j x_j\| = \|z_1^{(m)}\| < \lambda' \ .\end{aligned}$$

It follows that $\lambda\gamma_{\alpha\cdot(n\,m)} < \lambda'^m$, hence $(\gamma_{\alpha\cdot(n\,m)})^{1/n\,m} < \lambda^{1/n-1/mn}\widehat{\gamma}_\alpha(Y)$, a contradiction, if m is large enough.

Now we shall define asymptotic sets $A, B \subseteq S(Y)$ and a set A^* in the unit ball of Y^* such that A^* 2-norms A and the action of A^* on B is small. By passing to a tail subspace of Y if necessary, we may assume without loss of generality that $\frac{3}{4}\gamma_{\alpha\cdot(n-1)} \le \frac{7}{8}\delta_{\alpha\cdot(n-1)}(Y)$. Fix $\varepsilon > 0$, quite small as determined at the end of this proof. Let A^* consist of all functionals in Y^* of the form $y^* = \frac{3}{4}\gamma_{\alpha\cdot(n-1)}\sum_{k\in K} w_k^*$, where $(w_k^*)_K \prec (y_i^*)$ is $(\alpha\cdot(n-1))$-admissible (w.r.t. (y_i^*)); and let A consist of all $y \in S(Y)$ that are 2-normed by A^*. The set A is asymptotic by the definition of the Δ-stabilization. Since Y Δ-stabilizes γ, it is not difficult to see that A is asymptotic in Y and that functionals from A^* have the norm not exceeding 1. Then B consists of all vectors of the form $x/\|x\|$, where x is an $(\alpha\cdot n, \alpha\cdot(n-1), \varepsilon)$ average w.r.t. (y_i) of some normalized $(x_i) \prec (y_i)$, such that $\|x\| \ge (1-\varepsilon)(\widehat{\gamma}_\alpha(Y))^n$. By the first part of this proof, B is asymptotic in Y. We will show that if $y^* \in A^*$ and $z \in B$, then $|y^*(z)| \le \frac{3}{4}\widehat{\gamma}_\alpha(Y)^{-n}(\gamma_{\alpha\cdot(n-1)} + \frac{7}{6}\varepsilon)/(1-\varepsilon) \equiv \eta$.

This is a direct consequence of the following estimate. If x is an $(\alpha\cdot n, \alpha\cdot(n-1), \varepsilon)$ average as above, and if $(E_k) \in \mathcal{S}_{\alpha\cdot(n-1)}$, and $E_k x$ denotes the restriction of x whose support w.r.t. (y_i) is E_k; then $\sum\|E_k x\| \le 1 + 7\varepsilon/6\gamma_{\alpha\cdot(n-1)}$. To see this, write x in the form $x = \sum_{i\in F} a_i x_i$ where $(x_i)_{i\in F}$ is $\alpha\cdot n$-admissible w.r.t. (y_i) and if $J \subseteq F$ satisfies $(x_i)_J$ is $\alpha\cdot(n-1)$-admissible then $\sum_G a_i < \varepsilon$. Also $\sum_{i\in F} a_i = 1$ and $a_i > 0$ for $i \in F$. Set $I = \{i : E_k \cap \operatorname{supp}(x_i) \ne \emptyset$ for at most one $k\}$ and $J = F \setminus I$; and for $i \in J$ let $K_i = \{k : E_k \cap \operatorname{supp}(x_i) \ne \emptyset\}$. Then it can be checked that $(x_i)_J$ is $\alpha\cdot(n-1)$-admissible, hence

$$\sum_k \|E_k x\| \le \sum_{i\in I} a_i\|x_i\| + \sum_{i\in J} a_i \sum_{k\in K_i}\|E_k x_i\| \le 1 + \varepsilon/\delta_{\alpha\cdot(n-1)}(Y) \le 1 + 7\varepsilon/6\gamma_{\alpha\cdot(n-1)} \ .$$

Now, if $y^* = \frac{3}{4}\gamma_{\alpha\cdot(n-1)} \sum_{k\in K} w_k^* \in A^*$ then letting $E_k = \text{supp}(w_k)$ for all k we get $|y^*(z)| \le \eta$, as required.

As mentioned in Section 2, Y is $(1/2 + 1/4\eta)$-distortable. Hence the assumption of D-bounded distortion implies $1/2 + 1/4\eta \le D$. Substituting the definition of η and taking $\varepsilon > 0$ sufficiently small we get the inequality $(1/3D)(\widehat{\gamma}_\alpha(Y))^n \le \gamma_{\alpha\cdot(n-1)}$, as promised. □

As we remarked earlier, the assumption of bounded distortion implies the existence of certain subspaces with a nice structure ([19], [17], [24]). We would like to identify more such regular subspaces in the class of asymptotic ℓ_1 spaces of bounded distortion.

Recall (Proposition 6.4) that in Tsirelson's space $T = T_\theta$, for all $\varepsilon_n \downarrow 0$ there exists $(x_i) \prec T$ so that for all n and all $x \in \langle x_i\rangle_n^\infty$ we have

$$(1+\varepsilon_n)^{-1}\|x\|_T \le \sup\{\theta^n \sum_1^\ell \|E_i x\|_T : (E_i)_1^\ell \text{ is } n\text{-admissible}\} \le \|x\|_T \ .$$

In any asymptotic ℓ_1 space with bounded distortion one can find a block basis that displays an isomorphic version of this phenomenon.

Theorem 6.10 *Let X be an asymptotic ℓ_1 space of D-bounded distortion not containing ℓ_1. There exist $(w_i) \prec X$, $\alpha = \omega^{\beta_0}$, $0 < \theta < 1$, and $(z_i) \prec (w_i)$ such that for every $k \in \mathbb{N}$ we have, for $z \in [z_i]_k^\infty$,*

$$\begin{aligned}(1/4\,D) \sup_{1\le n\le k} \sup\Big\{\theta^n \sum \|E_i z\| : (E_i) \text{ is } \alpha\cdot n\text{-admissible}\Big\} &\le \|z\| \\ &\le 4\,D \inf_{1\le n\le k} \sup\Big\{\theta^n \sum \|E_i z\| : (E_i) \text{ is } \alpha\cdot n\text{-admissible}\Big\} \ .\end{aligned}$$

(Here, for an interval E of $\mathbb{N}$ and $z = \sum a_i w_i \in [w_i]$, Ez denotes the restriction w.r.t. (w_i), i.e., $Ez = \sum_{i\in E} a_i w_i$.)

Proof. By Proposition 4.5, $\ddot{\delta}_\beta(X) > 0$ for at most countably many β's; write this set as (β_m). For an arbitrary $\beta < \omega_1$, it follows from Lemma 4.6 that if $(y_i) \prec (e_i)$ then $\ddot{\delta}_\beta((y_i)_n^\infty) = \ddot{\delta}_\beta(y_i)$ for all n; and that $\ddot{\delta}_\beta(z_i) \le \ddot{\delta}_\beta(y_i)$ whenever $(z_i) \prec (y_i)$. Letting, for example, $f(y_i) = \sum 2^{-m}\ddot{\delta}_{\beta_m}(y_i)$, by a standard induction argument, similar to that in Proposition 4.10, we can stabilize $f(y_i)$. That is, we can find $(y_i) \prec X$ such that $f(z_i) = f(y_i)$ for all $(z_i) \prec (y_i)$. Since $\ddot{\delta}_\beta(X) = 0$ implies $\ddot{\delta}_\beta(z_i) = 0$ for all $(z_i) \prec X$, the stabilization of f implies that we have, for all $(z_i) \prec (y_i)$,

$$\ddot{\delta}_\beta(z_i) = \ddot{\delta}_\beta(y_i) \qquad \text{for all } \beta < \omega_1 \ .$$

Let $\alpha = I_\Delta(y_i)$; by Theorem 4.23, $\alpha = \omega^{\beta_0}$ for some $\beta_0 < \omega_1$. Let $\theta = \ddot{\delta}_\alpha(y_i)$. Then $\ddot{\delta}_{\alpha\cdot n}(y_i) = \theta^n$ for $n \in \mathbb{N}$, by Proposition 4.21. By an inductive construction followed by a diagonal argument, using Proposition 4.17, we can find $(w_i) \prec (y_i)$ and equivalent bimonotone norms $|\cdot|_n$ on $[w_i]_n^\infty$ such that for all $(z_i) \prec (w_i)_n^\infty$ and $n \in \mathbb{N}$,

$$\delta_\alpha([z_i]_n^\infty, |\cdot|_n) \geq 2^{-1/n}\theta. \tag{6}$$

Notice that (6) is preserved if the norms involved are multiplied by constants. Therefore by scaling and the assumption of bounded distortion we may additionally ensure that $\|w\| \leq |w|_n \leq 2\,D\|w\|$ for $w \in [w_i]_n^\infty$ and all $n \in \mathbb{N}$.

Now, given any α-admissible family of intervals $(F_i)_1^k$ of $\mathbb{N}$, let $(G_i)_1^k$ be adjacent intervals such that $\min F_i = \min G_i$ for $i < k$ and let $G_k = F_k$. Since the norms $|\cdot|_n$ are bimonotone, $|F_i w|_n \leq |G_i w|_n$ for $w \in [w_i]_n^\infty$ and all $n \in \mathbb{N}$. In particular, by Remark 4.3 for $n \in \mathbb{N}$ and $w \in [w_i]_n^\infty$ we get

$$|w|_n \geq \delta_\alpha\Big([w_i]_n^\infty, |\cdot|_n\Big) \sum_{i=1}^k |G_i w|_n \geq \delta_\alpha\Big([w_i]_n^\infty, |\cdot|_n\Big) \sum_{i=1}^k |F_i w|_n \ .$$

Using this and the assumption (6) on δ_α's we easily get, for $n \in \mathbb{N}$ and $w \in [w_i]_n^\infty$,

$$\begin{aligned} 2\,D\|w\| \geq |w|_n &\geq \sup\Big\{\delta_\alpha^n \sum_{i=1}^k |E_i w|_n : (E_i) \text{ is } [\mathcal{S}_\alpha]^n\text{-admissible}\Big\} \\ &\geq (1/2)\sup\Big\{\theta^n \sum_{i=1}^k \|E_i w\| : (E_i) \text{ is } [\mathcal{S}_\alpha]^n\text{-admissible}\Big\}\ , \end{aligned}$$

where we have abbreviated $\delta_\alpha\Big([w_i]_n^\infty, |\cdot|_n\Big)$ to δ_α. Finally, using Corollary 3.4 and a diagonal argument, construct a subsequence $M = (m_i)$ of $\mathbb{N}$ such that setting $M_n = (m_i)_n^\infty$ we get $\mathcal{S}_{\alpha\cdot n}(M_n) \subseteq [\mathcal{S}_\alpha]^n$ for all n. Thus for $w \in [w_i]_{i \in M_n}$ replacing the supremum in the last formula by the supremum over $\mathcal{S}_{\alpha\cdot n}(M_n)$-admissible families and relabelling the subsequence by (w_i') we get, for $n \in \mathbb{N}$ and $w \in [w_i']_n^\infty$,

$$\|w\| \geq (1/4\,D)\sup\Big\{\theta^n \sum \|E_i w\| : (E_i) \text{ is } \alpha\cdot n\text{-admissible}\Big\}\ .$$

It should be noted that in this last estimate, the admissibility condition is understood with respect to the above subsequence (w_i') of (w_i) which indeed corresponds to the subsequence M of $\mathbb{N}$.

We relabel once more, denoting (w_i') simply by (w_i). Set $|\!|\!|w|\!|\!|_n = \sup\Big\{\theta^n \sum \|E_i w\| : (E_i)$ is $\alpha\cdot n$-admissible$\Big\}$, for $w \in [w_i]_n^\infty$ and $n \in \mathbb{N}$. These are equivalent norms on the subspaces where they are defined. Therefore stabilizing all norms $|\!|\!|\cdot|\!|\!|_n$ on a nested sequence of block subspaces, using the assumption of bounded distortion, and passing to a diagonal subspace we get $(z_i) \prec (w_i)$ and A_n such that $[z_i]_n^\infty \prec [w_i]_n^\infty$ and $A_n|\!|\!|z|\!|\!|_n \leq \|z\| \leq 2\,DA_n|\!|\!|z|\!|\!|_n$ for $z \in [z_i]_n^\infty$. Since for all $(z_i) \prec (w_i)$ we have $\delta_{\alpha\cdot n}([z_i], \|\cdot\|) \leq \ddot{\delta}_{\alpha\cdot n}(z_i) = \theta^n < 2\theta^n$, then for all $(z_i) \prec (w_i)$ and all $n \in \mathbb{N}$, there exists $v_n \in [z_i]_n^\infty$ such that $\|v_n\| \leq 1$ and $|\!|\!|v_n|\!|\!|_n \geq 1/2$. Hence $A_n \leq 2$, thus $\|z\| \leq 4\,D|\!|\!|z|\!|\!|_n$ on $[z_i]_n^\infty$.

We have shown that for all $k \in \mathbb{N}$, $\|z\| \geq (1/4\,D) \sup_{1\leq n\leq k} \|\!|z|\!\|_n$ on $[w_i]_k^\infty \succ [z_i]_k^\infty$; and $\|z\| \leq 4\,D \inf_{1\leq n\leq k} \|\!|z|\!\|_n$ on $[z_i]_k^\infty$. □

We would like to directly relate the norm of an asymptotic ℓ_1 space of bounded distortion with a norm in some Tsirelson space. While we were unable to obtain two-sided estimates we did obtain the following lower estimate.

Proposition 6.11 *Let X be an asymptotic ℓ_1 space of D-bounded distortion, $\alpha \prec \omega_1$ and suppose that $\ddot{\delta}_\alpha(Y) = \theta \in (0,1)$ for all $Y \prec X$. Let $\varepsilon_n \downarrow 0$. There exist $(w_i) \prec X$ so that for all n if $w \in [w_i]_n^\infty$ then $\|w\| \geq (1-\varepsilon_n)(D+\varepsilon_n)^{-1} \|\sum \|E_i w\| e_{p_i}\|_{T(\mathcal{S}_\alpha, \theta-\varepsilon_n)}$, whenever $E_1 < E_2 < \cdot$ are adjacent intervals, $E_i w$ denotes the restriction of w w.r.t. (w_i) and $p_i = \min E_i$.*

Note that the first paragraph of the proof of Theorem 6.10 shows how to choose a subspace X satisfying the above hypothesis in an asymptotic ℓ_1 space of bounded distortion.

Proof. Choose $(z_i) \prec X$ so that for all n there exists an equivalent bimonotone norm $|\cdot|_n$ on $[z_i]_n^\infty$ with $\delta_\alpha((z_i)_n^\infty, |\cdot|_n) > \theta - \varepsilon_n$. This can be done by Proposition 4.17 using that $\ddot{\delta}_\alpha(Z) = \theta$ for all $Z \prec X$. Hence by a diagonal argument, applying Corollary 5.8, we may assume also that if $z \in \langle z_i \rangle_n^\infty$ and $E_1' < \cdots < E_\ell'$ are adjacent intervals then

$$|z|_n \geq (1-\varepsilon_n) \Big\| \sum_1^\ell |E_i' z|_n e_{r_i} \Big\|_{T(\mathcal{S}_\alpha, \theta-\varepsilon_n)} \ ,$$

where $r_i = \min E_i'$ and $E_i' z$ is the restriction of z w.r.t. (z_i). Using that X is of D-bounded distortion and scaling $|\cdot|_n$ we may obtain $(w_i) \prec (z_i)$ so that for all n and $w \in \langle w_i \rangle_n^\infty$, $\|w\| \geq |w|_n \geq \frac{1}{D+\varepsilon_n} \|w\|$. We thus obtain for $w \in \langle w_i \rangle_n^\infty$,

$$\begin{aligned} \|w\| &\geq (1-\varepsilon_n) \Big\| \sum |E_i' w|_n e_{p_i} \Big\|_{T(\mathcal{S}_\alpha, \theta-\varepsilon_n)} \\ &\geq \frac{1-\varepsilon_n}{D+\varepsilon_n} \Big\| \sum \|E_i' w\| e_{r_i} \Big\|_{T(\mathcal{S}_\alpha, \theta-\varepsilon_n)} \ . \end{aligned}$$

Now given adjacent intervals $E_1 < E_2 < \cdots$, take intervals $E_1' < E_2' < \cdots$ such that for all $w \in \langle w_i \rangle_n^\infty$, and all i, the restriction $E_i w$ w.r.t. (w_i) coincides with the restriction $E_i' w$ with respect to (z_i). Then we have $r_i = \min E_i' \geq p_i = \min E_i$ for all i and since $\mathcal{S}_\alpha$ is invariant under spreading we easily get that $\|\sum a_i e_{r_i}\|_{T(\mathcal{S}_\alpha, \theta')} \geq \|\sum a_i e_{p_i}\|_{T(\mathcal{S}_\alpha, \theta')}$ for all (a_i) and all $0 < \theta' < 1$. Thus the final lower estimate follows. □

The following proposition generalizes the fact that for the Tsirelson space T_θ, $D(T_\theta) \geq \theta$.

Proposition 6.12 *Let X be an asymptotic ℓ_1 space. Then $\sup\{D(Y) : Y \prec X\} \geq \sup\{\gamma_1^{-1} : \gamma \in \Delta(X)\}$.*

Proof. Let $\gamma \in \Delta(X)$ and let $(x_i) \prec X$ Δ-stabilize γ. Thus for some $\varepsilon_n \downarrow 0$, all n and all $(y_i) \prec (x_i)_n^\infty$,

$$\gamma_1 \geq \delta_1(y_i) \geq \gamma_1 - \varepsilon_n \ .$$

For $n \in \mathbb{N}$ and $(y_i) \prec (x_i)$ define

$$\delta_1(n)(y_i) = \sup\Big\{\delta : \|y\| \geq \delta \sum_1^n \|E_i y\| : y \in [y_i], Ey \text{ is a restriction w.r.t.}(y_i),$$
$$E_1 < \cdots < E_{n_0} \text{ are adjacent intervals with } \bigcup E_i = \text{supp}(y)\Big\} \ .$$

Now observe that given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ and $(y_i) \prec (x_i)$ so that $\delta_1(n_0)(w_i) < \gamma_1 + \varepsilon$ for all $(w_i) \prec (y_i)$. Indeed, if not, we could, by a diagonal argument, produce $(y_i) \prec (x_i)$ with $\delta_1(y_i) \geq \gamma_1 + \varepsilon$.

On $[y_i]$ define the norm

$$|y| = \sup\Big\{\sum_1^{n_0} \|E_i y\| \ : \ E_1 y < \cdots < E_{n_0} y \text{ w.r.t. } (y_i) \text{ and}$$
$$E_1 < \cdots < E_{n_0} \text{ are adjacent intervals with } \bigcup E_i = \text{supp}(y)\Big\} \ .$$

Thus, by the choice of (y_i), for all $W \prec Y = [y_i]_{\mathbb{N}}$, there exists $w \in W$, $\|w\| = 1$ and $|w| > \frac{1}{\gamma_1+\varepsilon}$. Also by considering long ℓ_1^k-averages (see the proof of Proposition 2.7) there exists $x \in W$, $\|x\| = 1$ and $|x| < 1 + \varepsilon$. Thus $D(Y) \geq d(X, |\cdot|) \geq (1+\varepsilon)/(\gamma_1 + \varepsilon)$. □

More generally, we have

Proposition 6.13 *Let X be asymptotic ℓ_1 and suppose that $I_\Delta(X) = \alpha_0$. then*

$$\sup\{D(Y) : Y \prec X\} \geq \sup\{\gamma_{\alpha_0}^{-1} : \gamma \in \Delta(X)\} \ .$$

Proof. We may assume $\alpha_0 > 1$ by Proposition 6.12. Thus by Theorem 4.23, α_0 is a limit ordinal. Let $\alpha_n \uparrow \alpha_0$ be the ordinal sequence used in defining $\mathcal{S}_{\alpha_0}$. Let $\gamma \in \Delta(X)$, $\varepsilon > 0$. Then for some n_0, $\gamma_{\alpha_{n_0}} < \gamma_{\alpha_0} + \varepsilon$. Let (x_i) Δ-stabilize γ. Choose $(y_i) \prec (x_i)$ and an equivalent norm $|\cdot|$ on $[y_i]$ with $\delta_{\alpha_{n_0}}((y_i), |\cdot|) > 1 - \varepsilon$. By passing to a block basis of (y_i) and scaling $|\cdot|$ if necessary we may assume that for some D we have $\|\cdot\| \leq |\cdot| \leq D\|\cdot\|$ on $[y_i]$, and for all $W = [w_i] \prec Y$ there exists $w \in W$, $\|w\| = 1$ and $|w| < 1 + \varepsilon$. Since $\gamma_{\alpha_{n_0}} < \gamma_{\alpha_0} + \varepsilon$ there exists $z \in W$ with $\|z\| = 1$ and $\sum_1^\ell \|z_i\| \geq 1/(\gamma_{\alpha_0} + \varepsilon)$, for some decomposition $z = \sum_1^\ell z_i$ where $(z_i)_1^\ell$ is α_{n_0}-admissible w.r.t. (w_i). Hence $|z| \geq (1-\varepsilon)\sum |z_i| \geq (1-\varepsilon)\sum \|z_i\| \geq (1-\varepsilon)/(\gamma_{\alpha_0} + \varepsilon)$. Comparing the norms $|z|$ and $\|z\|$ we get $D(Y, |\cdot|) > (1-\varepsilon)(1+\varepsilon)/(\gamma_{\alpha_0} + \varepsilon)$. □

We have a simple corollary.

Corollary 6.14 *Let X be asymptotic ℓ_1 with $I_\Delta(X) = I_\Delta(Y) = \alpha_0$ for all $Y \prec X$. If $\ddot{\delta}_{\alpha_0}(X) = 0$ then no subspace of X is of bounded distortion.*

References

[1] D. Alspach and S. Argyros: *Complexity of weakly null sequences*, Diss. Math., **321** (1992), 1–44.

[2] G. Androulakis and E. Odell: *Distorting mixed Tsirelson spaces*, preprint.

[3] S. Argyros and I. Deliyanni: *Examples of asymptotically ℓ^1 Banach spaces*, preprint.

[4] S. Argyros, S. Merkourakis and A. Tsarpolias: *Convex unconditionality and sumability of weakly null sequences*, preprint.

[5] S. Bellenot: *The Banach space T and the fast growing hierarchy from logic*, Israel J. of Math., **47** (1984), 305–313.

[6] J. Bourgain: *On convergent sequences of continuous functions*, Bull. Soc. Math. Bel., **32** (1980), 235–249.

[7] P. G. Casazza, W. B. Johnson and L. Tzafriri: *On Tsirelson's space*, Israel J. Math., **47** (1984), 81–98.

[8] P. G. Casazza and T. J. Shura: *Tsirelson's Space*, Lecture Notes in Math., vol. 1363, Springer-Verlag, Berlin and New York, 1989.

[9] I. Deliyanni and D. Kutzarova: *On some asymptotic ℓ_1 Banach spaces*, preprint.

[10] T. Figiel and W. B. Johnson: *A uniformly convex Banach space which contains no ℓ_p*, Comp. Math., **29** (1974), 179–190.

[11] R. C. James: *Uniformly non-square Banach spaces*, Ann. of Math., **80** (1964), 542–550.

[12] R. Judd: in preparation.

[13] R. Judd and E. Odell: *Concerning the Bourgain ℓ_1 index of a Banach space*, preprint.

[14] K. Kuratowski: *Topology*, vol I, academic Press, New York 1966.

[15] D. Leung, *Compact subsets of $\mathcal{P}_{<\infty}(\mathbb{N})$ with applications to the embedding of symmetric sequence spaces into $C(\alpha)$*, preprint.

[16] J. Lindenstrauss and L. Tzafriri: *Classical Banach spaces*, vol I, Springer Verlag 1977.

[17] B. Maurey: *A remark about distortion*, Operator Theory: Advances and Applications, **77** (1995), 131–142.

[18] B. Maurey, V.D. Milman and N. Tomczak-Jaegermann: *Asymptotic infinite-dimensional theory of Banach spaces*, Operator Theory: Advances and Applications, **77** (1995), 149–175.

[19] V. Milman and N. Tomczak-Jaegermann: *Asymptotic ℓ_p spaces and bounded distortion*, (Bor-Luh Lin and W.B. Johnson, eds.), Contemporary Math., **144** Amer. Math. Soc.,(1993), 173–195.

[20] J. D. Monk: *Introduction to Set Theory*, McGraw-Hill, 1969.

[21] E. Odell and Th. Schlumprecht: *The distortion problem*, GAFA, **3** (1993), 201–207.

[22] E. Odell and Th. Schlumprecht: *The distortion problem*, Acta Math., **173** (1994), 259–281.

[23] H. Rosenthal: *A characterization of Banach spaces containing ℓ_1*, Proc. Nat. Acad. Sci. (U.S.A.) **71** (1974), 2411–2413.

[24] N. Tomczak-Jaegermann: *Banach spaces of type $p > 1$ have arbitrarily distortable subspaces*, preprint.

[25] N. Tomczak-Jaegermann: Notes privately distributed, December 1994.

[26] B. S. Tsirelson: *Not every Banach space contains ℓ_p or c_0*, Funct. Anal. Appl., **8** (1974), 138–141.

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