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Journal Article

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Publication date:
2015-08

Permanent link:
https://doi.org/10.3929/ethz-b-000123131

Originally published in:
Historia Mathematica 42(3), https://doi.org/10.1016/j.hm.2015.01.001

Citrabhānu's Twenty-One Algebraic Problems in Malayalam and Sanskrit

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Abstract: This paper studies the Sanskrit and Malayalam versions of Citrabhānu's Twenty-One problems: a discussion of quadratic and cubic problems from 16th-century Kerala. It reviews the differences in the approaches of the two versions, highlighting the distinction between the Sanskrit indeterminate integer-remainder arithmetical techniques and the Malayali fixed point iterations. The paper concludes with some speculations on the possible transmission of algebraic knowledge between Kerala and the west.

1. Introduction

The object of this paper is an early 16th-century south Indian algebraic text, Citrabhānu's *Twenty-One Problems* (*Ekaviṃśati Praśnottara*), which survives in two versions: one in Malayalam and the other in Sanskrit. The text presents rules for deriving the values of two unknowns, given any pair from the following list: the sum of the two unknowns, their difference, their product, the sum of their squares, the difference of their squares, the sum of their cubes, and the difference of their cubes. The title of the text refers to the twenty-one pairs of data that the above seven quantities can yield.

The *Twenty-One Problems* is a product of late medieval Kerala mathematics, an Indian mathematical culture that reached impressive results, and has gained renown for foreshadowing some of the deepest results of early European calculus (see Joseph 2009a for a general overview). The Sanskrit version of the text is a commentary included in Śaṅkara and Nārāyaṇa's 16th-century commentary on the famous 12th-century Līlāvatī.[1] It is this Sanskrit version that attributes the treatise to the 16th-century Kerala astronomer Citrabhānu (see Joseph 2009a, p. 21 for brief biographical details). The Malayalam version survives, as far as I know, in a single manuscript included in a bundle of astronomical writings.[2]

[1] This commentary was edited by Sarma (1975, pp. 109-129) based on four surviving manuscripts, and the relevant section was analyzed and partially translated by Hayashi and Kusuba (1998) and by Mallayya (2011). A partial Sanskrit copy of the rules and examples (without the solutions of the examples and the commentary, terminating with problem 17) survives as manuscript 22250C in Trivandrum University's Oriental Research Institute Manuscript Library. The palm leaves are partly damaged, and the handwriting is difficult to follow, but the text is almost identical to the relevant portions of Sarma's (1975) edition.

[2] The manuscript, C 541D in Trivandrum University's Oriental Research Institute Manuscript Library (formerly manuscript D 217 in the Madras Government Oriental Manuscripts Library), is mentioned by Sarma (1972, p. 74) and Ulloor (1990, Vol. 3, pp. 255-256). The *mana* is indicated as *kudallur meledathe sūryasiddhānta vyākhyā*. The manuscript is relatively well preserved and clearly written. The copyist betrays his weaknesses as a scribe in the opening Sanskrit praise of Gaṇeśa and in the use of Malayali numerals (numerals are missing in several places, sometimes replaced by blank spaces, sometimes miscopied). This might indicate that the scribe was a relative novice, that he was not an experienced mathematician, and/or that he used a corrupt source.

It is not clear which of the two versions precedes the other, and whether the Malayalam version should also be attributed to Citrabhānu. A comparison of the rules (see section 2) seems to fit better with the hypothesis that the Malayalam version represents an earlier stage in thinking about the 21 problems than the Sanskrit one, but I can't rule out the possibility that the Malayalam version is a later simplification of an original closer to the extant Sanskrit source.

The *Twenty-One Problems* continues earlier discussions that concern only the first five quantities above, forming linear and quadratic combinations.[3] The novelty of the treatise is, obviously, the inclusion of cubic sums and differences. Most rules for solving problems related to cubic equations are not presented in what we would consider a closed form; as we will see below, these rules depend on indeterminate integer arithmetic with remainders in the Sanskrit version and on fixed point iterations in the Malayalam one. While they share the same algebraic foundations, the two versions manifest very different kinds of mathematical practice. This and other differences are the focus of this paper.

The next section will compare the indeterminate arithmetic with remainders and the fixed point iteration approaches of the two versions. The third section will discuss other differences in the approaches of the two versions. The fourth section will speculate on the possible connections between the *Twenty-One Problems* and the solution of cubic equations in 16th-century Italy. Selected translations from the Malayalam are included in an appendix.

2. Sanskrit integer arithmetic vs. Malayalam fixed point iteration

Table 1 summarizes the rules proposed in the two versions for solving the 21 problems. To mitigate our tendency to anachronism, I did not use modern algebraic notation for the unknowns and the seven terms of the problems. Instead, I use the abbreviations *small* and *large* for the unknowns, and *sum*, *dif*, *prod*, sum_2 (sum of squares), dif_2 (difference of squares), sum_3 (sum of cubes) and dif_3 (difference of cubes) for their respective combinations. For the arithmetic operations relating these quantities (summation, product, power and their inverses) I use modern notations.

The choice of my abbreviated terms is repeated throughout this text as well as in the appended translation, and should be justified. Indeed, each of the abbreviated terms stands for a single and fixed word or compound. Most of these terms are Sanskrit in origin, but have been naturalized

[3] See Hayashi and Kusuba (1998, section 3) and Kusuba (2010) for Sanskrit references. A Malayalam collection of all ten linear and quadratic problems is available in the Gaṇita Yukti Bhāṣā (Sarma et al. 2008, chapter 2), but this chapter may postdate our sources. In the Tantrasaṅgraha there is a section dealing with a set of five related trigonometric quantities, showing how to derive any pair of quantities from the other three (Ramasubramanian and Sriram 2010, 200-228). This section is also entitled "ten problems", but these are not the same problems we are dealing with here. Nevertheless, the dedicatory Sanskrit verse preceding the Malayalam version refers to the ten questions of the Tantrasaṅgraha as its model, and the Malayalam introduction uses a similar combinatorial procedure to count them (see the translation in the appendix).

into Malayalam like many other scientific and lay terms, while other terms in the Malayalam version are Dravidian (such as *kūṭuka* and *kaḷayuka* for addition and subtraction). The mathematical Sanskrit of the twenty-one problems tends to be quite formulaic, and the Malayalam equivalent further strengthens this tendency. The vocabulary in the Malayalam version is very limited, and unlike the Sanskrit version, uses very few synonyms for technical terms (the use of both *kaḷayuka* and *vāṅṅuka* for subtraction stands out as an exception). There's a tendency to more or less repeat phrases that express similar sequences of operations (as in the closing lines of the rules, which say that the unknowns can be derived by halving the sum and difference of *sum* and *dif*).

In fact, the very structure of the Malayalam language is as if designed for formulaic presentation, often fitting what logicians call *Reverse Polish Notation*: a sequence of arguments, then an operation, the result of which can in turn be used as an argument of a further operation. An example is the following word-by-word translation of the passage expressing the formula $dif = dif_2^2 / ((4dif_3 - dif^3)/3)$ (rule 20, the Malayalam transcript is slightly amended):

> *dif* cubed [*antaratte ghaniccu*]
>
> dif_3 by 4 multiplied [*ghanāntaratte nālil perukkiy-*]
>
> from that subtract [*atiṃkannu vāṅṅi*]
>
> by three divide [*mūnil haricc-*]
>
> by that dif_2 squared divide, thereby [*atu koṇṭu varggāntaravarggatte hariccāl*]
>
> exactly-being *dif* comes [*sūkṣmamāyiṭṭuḷḷa antaram varum*]

Due to these characteristics, I find that my use of abbreviated terms is faithful to the original practice.

Table 1: Summary of rules in the Sanskrit and Malayalam versions

Given quantities	Sanskrit rule	Malayalam rule
1. *sum*, *dif*	$large = (sum + dif)/2$ $small = (sum - dif)/2$	$large = (sum + dif)/2$ $small = (sum - dif)/2$
2. *sum*, *prod*	$dif = (sum^2 - 4prod)^{1/2}$	$dif = (sum^2 - 4prod)^{1/2}$
3. *sum*, sum_2	$sum^2 - sum_2 = 2prod$	$sum^2 - sum_2 = 2prod$
4. *sum*, dif_2	$dif = dif_2 / sum$	$dif = dif_2 / sum$

5. *sum*, sum_3	$prod = (sum^3 - sum_3) / 3sum$	$prod = (sum^3 - sum_3) / 3sum$
6. *sum*, dif_3	$dif = 4dif_3 / 3sum^2$, provided that the remainder equals dif^3	$dif = (4dif_3 - dif^3) / 3sum^2$, iterated from the initial value $dif = 4dif_3 / 3sum^2$ until a stable result is obtained
7. *dif*, *prod*	$sum = (dif^2 + 4prod)^{1/2}$	$sum = (dif^2 + 4prod)^{1/2}$
8. *dif*, sum_2	$prod = (sum_2 - dif^2) / 2$	$prod = (sum_2 - dif^2) / 2$
9. *dif*, dif_2	$sum = dif_2 / dif$	$sum = dif_2 / dif$
10. *dif*, sum_3	$sum = (4sum_3)^{1/3}$ provided that the remainder equals $3dif^2 \times sum$	$sum = (4sum_3 - 3dif^2 \times sum)^{1/3}$, iterated from the initial value $sum = (4sum_3 - 3dif^3)^{1/3}$ until a stable result is obtained
11. *dif*, dif_3	$prod = (dif_3 - dif^3) / 3dif$	$sum = ((4dif_3 - dif^3) / 3dif)^{1/2}$
12. *prod*, sum_2	$sum = (sum_2 + 2prod)^{1/2}$ $dif = (sum_2 - 2prod)^{1/2}$	$sum = (sum_2 + 2prod)^{1/2}$ $dif = (sum_2 - 2prod)^{1/2}$
13. *prod*, dif_2	$sum_2 = (dif_2{}^2 + 4prod^2)^{1/2}$	$sum_2 = (dif_2{}^2 + 4prod^2)^{1/2}$
14. *prod*, sum_3	$dif_3 = (sum_3{}^2 - 4prod^3)^{1/2}$	$dif_3 = (sum_3{}^2 - 4prod^3)^{1/2}$
15. *prod*, dif_3	$dif = dif_3 / 3prod$, provided that the remainder equals dif^3	$sum_3 = (dif_3{}^2 + 4prod^3)^{1/2}$
16. sum_2, dif_2	Omitted (but alluded to in rules 1 and 13)	$large = ((sum_2 + dif_2) / 2)^{1/2}$ $small = ((sum_2 - dif_2) / 2)^{1/2}$
17. sum_2, sum_3	$small = sum_3 / sum_2$, provided that the remainder equals $(sum_2 - small^2)dif$	$dif = ((4sum_3 - sum^3) / 3sum)^{1/2}$ $sum = (2sum_2 - dif^2)^{1/2}$, iterated from the initial value $sum=(2sum_2)^{1/2}$ until a stable result is obtained
18. sum_2, dif_3	$dif = 2dif_3 / 3sum_2$, provided that the excess (the number missing from the	$dif = (4dif_3 - dif^3) / 3(2sum_2 - dif^2)$, iterated from the initial value $dif = (4 dif_3) / 3(2sum_2)$ until a stable

	denominator to get an equality) equals dif^3	result is obtained
19. dif_2, sum_3	$large = sum_3 / dif_2$, provided that the remainder equals $(large^2 - sum_2)\times sum$	$sum = (4sum_3 - 3dif^2\times sum)^{1/3}$ $dif = dif_2 / sum$, iterated from the initial value $sum = (4sum_3 - 3dif_2)^{1/3}$ until a stable result is obtained
20. dif_2, dif_3	$dif = 3dif_2 / 4dif_3$, provided that the excess (the number missing from the denominator to get an equality) equals dif^4	$dif = dif_2^2 / ((4dif_3 - dif^3)/3)$, iterated from the initial value $dif = dif_2^2 / ((4dif_3)/3)$ until a stable result is obtained
21. sum_3, dif_3	Omitted (but alluded to in rule 1)	$large = ((sum_3 + 3dif) / 2)^{1/3}$ $small = ((sum_3 - 3dif) / 2)^{1/3}$

2a. The Sanskrit rules: Integer arithmetic

Let's begin by describing the rule for solving problem 10 in the Sanskrit version. The data here are *dif* and sum_3. The Sanskrit version reads:

> The cube root of four times sum_3 is *sum*, if it is made in such a way that that [i.e., the *sum* obtained], multiplied by the square of *dif* and by three, can be subtracted from the remainder (Hayashi and Kusuba 1998, 9; my abbreviations inserted into the translation).

Mallayya's less literal but more readable form states:

> The *sum* will be that cube root extracted from four times sum_3 so that the residue left after subtracting the cube of that obtained from four times the sum_3 is equal to the product of thrice that root and the square of *dif* (Mallayya 2011, 110; my abbreviations inserted into the translation).

This rule is based on the identity: $4sum_3 = sum^3 + 3sum\times dif^2$.[4] Unlike quadratic identities, cubic identities were not solved in closed form by the Indian authors, so *sum* was not expressed as a combination of arithmetic operations on sum_3 and *dif*. Instead, the identity was rearranged into the form $sum = \sqrt[3]{4sum_3 - 3sum\times dif^2}$, and the rule states that *sum* equals the cubic root of

[4] In Anachronistic notation, this would read: $4(x^3 + y^3) = (x+y)^3 + 3(x+y)(x-y)^2$

$4sum_3$, provided that the remainder equals the product of this cubic root (*sum*) and $3dif^2$. The term "cubic root" refers here not to the possibly irrational root, but to an integer standing for the root. The term "remainder", which is to equal $3sum\times dif^2$, refers to the difference between $4sum_3$ and that integer-root cubed.

The accompanying example has sum_3=11375 and *dif*=5. The cubic root of 4×11,375 = 45,500 is approximately 35. The difference between 45,500 and 35^3 is 2,625, which indeed equals $3\times 35\times 5^2$.

Note that this rule is exceptional in the Sanskrit version. The other rules that do not provide a closed formula for the solution are stated in terms of integer quotients rather than roots. If the rule had followed the pattern of the other Sanskrit rules without closed formulas, we would be told that *sum* is the quotient of $4sum_3$ and $3dif^2$, provided that the remainder is the quotient (*sum*) cubed (this deviation will be considered below).

The Sanskrit version does not offer a deterministic procedure (closed formula) for finding the quotients satisfying the conditions on the remainders. One can only assume that a process of trial and error is called for, starting from the integer best approximating the result of the division. This generates an implicit underdetermination of the unknown quantity. In fact, as observed by Hayashi and Kusuba (1998, pp. 6, 15), in some cases the determination of unknowns is not unique, and in others impossible. These facts are not alluded to in the Sanskrit text.

Since the Sanskrit rules do not provide a deterministic procedure, and do not always have unique solutions, I refer to them as *indeterminate*. Note, however, that my use of this term is slightly unusual for the Indian context. In Indian mathematics, indeterminate arithmetic usually refers to equations of the form $ax + by = c$ or $x^2 - ay^2 = b$, where all parameters and unknowns are positive integers. These equations can be solved by the *kuṭṭaka* and *cakravāla* methods respectively (Raghavan 2008), and, if solvable, have cyclic solutions. Here, on the other hand, I consider equations of the form $ax + b = f(x)$, where f is a polynomial or a composition of polynomials and roots. In such cases the number of solutions is limited in advance, and a plurality of integer solutions is rare. The former indeterminate techniques are not relevant for these latter equations.

Unlike the classical Indian indeterminate problems, the problems we're dealing with here do make sense outside an integer setting. But it is important to note that the solutions of the twenty-one problems are tacitly assumed to be integers. In the Sanskrit version, this is manifest not only by the fact that in all examples the solutions are actually integers, but also by the language used to discuss remainders. Indeed, the rules sometimes state that the remainder should be divisible by some quantity *without a remainder*.

For example, in rule 17 we are told that $small = sum_3 / sum_2$, provided that the remainder equals $(sum_2 - small^2)\times dif$. More precisely, the text states that if the remainder is divided by $sum_2 - small^2$ (which equals $large^2$), then the result will be *dif*. In the accompanying example it is stated that: "The remainder [of the above division] can be divided by the difference of the given sum of squares [sum_2] and the square of the quotient [$small^2$], which amounts to the square of the large [$large^2$], *without remainder*" (Sarma 1975, p. 121, my translation and emphasis).[5] This statement can only make sense in an integer context. The same is echoed in the discussion of rule 19.

This integer setting is in line with the tradition of the Līlāvatī, whose arithmetic part (as opposed to the geometric part) displays only rational solutions, even in the context of quadratic equations. In fact, the Līlāvatī's chapter 19 bundles together indeterminate quadratic Diophantine equations with the treatment of a couple of the quadratic problems that appear in the twenty-one problems.

I also note that in most examples accompanying the rules, the term "quotient" refers to the integer best approximating the result of the division (or root, in rule 10). Only in the examples accompanying rules 17 and 19 are the quotients different from the integer best approximating the result of the division (otherwise, the conditions on the remainders would not be met). This reflects a tacit expectation that the two unknowns are relatively close to each other. Indeed, when this is the case, the prescribed remainders are necessarily smaller than the respective divisors. In the proof of rule 15, *small* is actually stated to be larger than *dif*, supporting this interpretation. This may also explain the unique use of cubic root instead of division in rule 10: if the division suggested above would have been used (stating that *sum* is the quotient $4sum_3/3dif^2$ with a remainder equal to the quotient, namely *sum*, cubed), this remainder (sum^3) would be larger than the divisor ($3dif^2$).

The most significant differences between the identities that underlie the Sanskrit and Malayalam versions can be found in rules 17-20. The Malyalam rules 17 and 19 do not easily lend themselves to a reformulation in terms of division with remainder. This may explain why the Sanskrit version replaces them with rules that are exceptional in that they involve the unknowns directly, rather than just some of their seven combinations.

As for rules 18 and 20, the quotients in the Malayalam version – $4dif_3/3(2sum_2)$ in rule 18 and $dif_2^2/((4dif_3)/3)$ in rule 20 – are corrected in the *denominator* by the unknown term so as to equal the sought quantity *dif*. The Sanskrit version rearranges the rules so as to move the correction terms to the enumerator, so that these corrections can be considered as the remainder or excess of an integer division. This rearrangement leaves a trace in the Sanskrit proof of rule 20, which actually goes through the Malayali rule. This lends some support to the hypothesis that the Malayalam version represents an earlier stage in dealing with the 21-problems (rather than a later simplification of an original close to the extant Sanskrit text) – but this evidence is far from conclusive.

We can summarize by saying that the Sanskrit version aims to adapt the twenty-one rules to a framework of indeterminate integer quotients (and in one case, root) with remainder/excess, and

[5] labdha-vargasyoddiṣṭa-varga-yogasya ca viśleṣeṇa mahad-vargākhyena tac-cheṣo niśśeṣaṃ hartuṃ śakyate.

seems to tacitly assume as default that the unknowns should be relatively close and that remainders should be relatively small with respect to divisors.

2b. The Malayalam rules: Fixed point iteration

The Malayalam version is not only less committed to integers (one of the examples goes through a fractional interim result), but offers means to find the correct answer through *fixed point iteration*. This term refers to solving the equation $x = f(x)$ by choosing an initial guess x_0 and reiterating the substitution $x_{n+1} = f(x_n)$ until convergence is reached.

Rule 10 provides the most elaborate example. Given sum_3 and *dif*, we wish to derive *sum*. The relevant identity is: $sum = \sqrt[3]{4sum_3 - 3dif^2 \times sum}$. The right hand side requires that we know the *sum* in order to derive the *sum* on the left hand side. To break this vicious circle, the author instructs us to take *dif* instead of *sum* as our initial estimate. Given *dif*=2, sum_3=152, and our initial estimate of *sum*=*dif*=2, we get $sum = \sqrt[3]{4\times152 - 3\times2^2\times2} = \sqrt[3]{584} \approx 8¼$. Now we take our new estimate for the sum, and reiterate the procedure. We get $sum = \sqrt[3]{4\times152 - 3\times2^2\times8¼} = \sqrt[3]{509} \approx 8$. Finally, if we insert *sum*=8 into our expression, we obtain $sum = \sqrt[3]{4\times152 - 3\times2^2\times8} = \sqrt[3]{512} = 8$, and we reach our fixed point solution.[6] Rules 17 and 19 involve a slightly more intricate process, where the unknown *sum* and unknown *dif* are iterated alternately, because the derivation of one depends on the other.

The text does not discuss in any way the conditions required for the iterations to convergence. The initial estimates are also more or less arbitrary. With the exception of rules 10 and 19, the initial estimates fit the assumption *dif*=0 (namely, in three cases, ignoring terms involving the unknown *dif* in the formula for the sought quantity, and in one case replacing *sum* by $\sqrt{2sum_2}$). This makes sense, if our default expectation is that the two unknowns are relatively close. Rule 19 assumes that *dif*=1 (or, more precisely, replaces the product of unknowns $dif^2\times sum$ by the known dif_2). Rule 10 is exceptional in assuming that the unknown *small* is null (replacing the unknown *sum* by the known *dif*). In both cases the assumption *dif*=0 would have served just as well. It seems more plausible to explain the choice of initial estimates thus: "render unknown quantities null, as long as they occur as isolated added or subtracted terms; if they occur multiplied by another quantity, replace them by apparently similar given quantities".

Iterative fixed point methods have a long history, going back at least as far the Greeks (possibly even the Babylonians). To this very day, the most famous fixed point iteration is probably the

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[6] The last step (the verification) is absent from the treatment of this example, but present in all other examples. On the other hand, in all other examples the correct result is reached already in the first, rather than the second iteration. The possibility of further iterations is explicitly mentioned in the statements of rules 6 and 10.

approximation of the square root of a number *A* via the iteration $x_{n+1} = (x_n + A/x_n)/2$ (Heath 1921, vol. II, p. 323).

Indian mathematics was well acquainted with iterations. In fact, Divakaran (2010) offers a reading of the entire history of Indian mathematical practice through the framework of recursive iteration. As for fixed point iterations, Indian astronomy does not only use, but even names them: *asakṛt* (not once) or *aviśeṣa* (without difference, referring to the identity of the "input" and "output" when the procedure converges; this latter name is explicitly used in the Malayalam version). These methods are used in Indian astronomy not only to solve intractable trigonometric equations such as $x = a + b \cdot \sin(x)$, but also quadratic equations such as $x = \sqrt{(a+x)^2 + b}$, which had known analytic solutions (Plofker 1996, 2002, 2011).

Plofker (2011) points out that fixed point iterative approximation was reserved to astronomical contexts (*jyotiṣa*) and absent from arithmetical contexts (*gaṇita*). She conjectures that this difference had something to do with standards of certainty: such methods are not guaranteed to converge. We should not forget, however, that the Sanskrit rules used here do not necessarily lead to a correct solution either. Nevertheless, the Sanskrit rules are each described in terms of a single fundamental arithmetical operation – division or root extraction with remainder/excess – rather than in terms of an iterative process. Since the Malayalam text explicitly uses the term *aviśeṣa*, it is clear that this is a case of importing knowledge from astronomy to mathematics. I have not found in the literature mentions of other uses of *aviśeṣa* in strictly mathematical contexts, so it may well be a unique feature.

What we see here is that standards of solution may vary not only between disciplines (astronomy and mathematics), but even inside the mathematical discipline, between languages. The Malayalam version is all about *aviśeṣa*, whereas the Sanskrit is strictly about indeterminate integer arithmetic with a remainder/excess. Since in Kerala the production of scientific Sanskrit and Malayalam texts was closely related, knowledge was probably not shaped differently depending on its producers, but according to its readership. This observation is in line with van Buitenen's (1996) recommendation to study the relation of Sanskrit and vernacular cultures in south India, a recommendation reiterated by Pollock (2011) in the scientific context.

3. Other differences between the Sanskrit and the Malayalam

Looking at Table 1, it is obvious that both versions stem from the same tradition. The organization of problems and kind of algebraic identities used are indeed the same. But the practice that these two versions represent is very different.

In the quadratic problems, both versions use the same rules, which are rearrangements of the following simple quadratic identities:[7]
$sum^2 = sum_2 + 2prod$
$dif^2 = sum_2 - 2prod$
$sum^2 - dif^2 = 4prod$
$sum^2 + dif^2 = 2sum_2$
$dif_2 = sum \times dif$

In the cubic problems, the Malayalam version depends only on the following three identities:
$sum^3 = sum_3 + 3prod \times sum$ (rule 5)
$4sum_3 = sum^3 + 3dif^2 \times sum$ (rules 6, 11)
$4dif_3 = dif^3 + 3sum^2 \times dif$ (rules 10, 17, 19).[8]
For rules 18 and 20, the last identity is used as well, but *sum* is replaced by $2sum_2 - dif^2$ and by dif_2/dif respectively, in order to express *sum* in terms of the quantities given by the problem.

The Sanskrit version, however, derives rules 11 and 15 from
$dif^3 = dif_3 + 3prod \times dif$,
which adds another identity to the list. Note that rule 15 could be solved in a closed form by a quadratic identity (as in the Malayali text), but the Sanskrit version prefers an indeterminate rule based on a cubic identity with a remainder term.

For rules 17-20, the Sanskrit version applies the following identities respectively, further expanding the arsenal of identities:[9]
$sum_3 = small \times sum_2 + (sum_2 - small^2) \times dif$
$2dif_3 + dif^3 = 3sum_2 \times dif$
$sum_3 = large \times dif_2 + (small^2 - sum_2) \times sum$
$3dif_2^2 + dif^4 = 4\, dif_3 \times dif$

Above we suggested that this change was due to the organization of Sanskrit rules around integer arithmetic with remainder/excess instead of fixed point iteration. Another reason may be that the Sanskrit version aims to show off a wider variety of rules, whereas the Malayalam version attempts to reduce the problems to a smaller knowledge base. One should indeed be careful making such judgments, because stating that a formula is "the same as" or "derives from" another depends on what kind of differences one is allowed to ignore and on one's arsenal of derivations, and so contemporary algebraic thought may impose anachronistic notions of

[7] $(x \pm y)^2 = x^2 + y^2 \pm 2xy$; $(x+y)^2 - (x-y)^2 = 4xy$; $(x+y)^2 + (x-y)^2 = 2(x^2 + y^2)$; $x^2 - y^2 = (x+y)(x-y)$

[8] $(x \pm y)^3 = x^3 \pm y^3 \pm 3xy(x \pm y)$; $4(x^3 \pm y^3) = (x \pm y)^3 \pm 3(x \mp y)^2(x \pm y)$

[9] $x^3 + y^3 = y(x^2 + y^2) + ((x^2 + y^2) - y^2)(x - y)$; $2(x^3 - y^3) + (x - y)^3 = 3(x^2 + y^2)(x - y)$; $x^3 + y^3 = x(x^2 - y^2) + (x^2 - (x^2 + y^2))(x + y)$; $3(x^2 - y^2)^2 + (x - y)^4 = 4(x^3 - y^3)(x - y)$

sameness and derivability on Indian rules. However, following the arguments and notations of the texts themselves, I believe that my claim is still valid: the Sanskrit version constructs more variety into the rules.

The same applies to the examples that accompany the rules. All examples in the Malayalam version have the same solution: *large*=5, *small*=3; the Sanskrit version, on the other hand, uses eight different examples for the twenty-one problems (of which the example *large*=20, *small*=15 is repeated ten times).[10] Moreover, while the Malayalam rules describe all the steps necessary to find the two unknowns, the Sanskrit rules only explain how to derive a third quantity, which can lead to the unknowns by applying previous rules.

Another crucial difference between the Sanskrit and Malayalam versions is the inclusion of proofs. The Sanskrit rules are accompanied by *śloka*s that provide algebraic proofs for the quadratic rules and for some of the cubic rules (involving the citation of some basic algebraic identities and some algebraic manipulations), and geometric proofs for the other cubic rules (ingenious three-dimensional cut and paste arguments for all irreducible cubic rules, excluding rules 6 and 20, the latter being four-dimensional). The Malayalam version brings almost no proofs.[11]

At first sight, these differences might suggest the following conclusion: the Sanskrit version uses the twenty-one problems to show off knowledge and explore the reasoning behind it in a setting that connects integer arithmetic, algebra and geometry. The Malayalam version is meant to be a practical guide for solving the same set of twenty-one problems while presupposing a smaller knowledge base. At first sight, it seems to make sense to conjecture that the vernacular Malayalam was used for practical and accessible instruction, whereas the scientific Sanskrit was used for scholarly reflection.

But there's a catch: the cubic problems in the list had no practical value for Indian practitioners (this is possibly why the *Gaṇita Yukti Bhāṣā* brings only the ten quadratic problems). So according to the above interpretation, the Malayalam rendering would be a practice-oriented approach to a set of non-practical problems! The "practical vernacular vs. scholarly Sanskrit" theory also fails for the *Gaṇita Yukti Bhāṣā*, whose Malayalam version is better reasoned and earlier than its Sanskrit recension (Sarma 2009).

In fact, the smaller variety of formulas and examples may be a deliberate choice to help understand better the interconnections between the different problems. So the theory of "practical vernacular, scholarly Sanskrit" may be a simplistic and premature characterization. The

[10] It is curious that both the Malayalam version and the Sanskrit version exceptionally fail to accompany rule 9 by an example, whereas the manuscript 22250C in Trivandrum University's Oriental Research Institute Manuscript Library provides the example *large*=26, *small*=18.

[11] The exception being a cursory mention that some formula is the inverse of another known formula in rule 8: "Since *dif* squared and double *prod* added are sum_2, *prod* comes from subtracting *dif* squared from sum_2 and halving".

interaction of the vernacular and Sanskrit mathematical cultures of Kerala is therefore far from settled, and requires further exploration.

4. Transmission outside India?

The issue of transmission of knowledge from Kerala to Europe in the late Middle Ages and early modern period is unsettled. As Raju (2007, Ch. 7) has put it, the motive and opportunity were there, along with many mathematical analogies. But the efforts collected in Joseph (2009b) have not unearthed any hard proof of transmission.

This paper will not attempt to solve this problem, but suggest that the transmission could be looked for not only in the context of calculus, but also in that of algebra, in particular around cubic equations. I warn the reader, however, that this section is strictly speculative.

In order to establish transmission, we must first consider the question of dating. If Citrabhānu is an early 16th-century mathematician, his work is unlikely to be relevant for the solution of cubic equations in Italy by Dal Ferro before 1515. However, already in 1356, Narayaṇa's *Gaṇitakaumudī*, which discussed the quadratic problems from Citrabhānu's list in a chapter on miscellaneous problems, wrote: "the rest is useful for geometry. I will explain [it] there" (Hayashi and Kusuba 1998, p. 19), referring to an unidentified location. This suggests that the treatment of cubic equations by means of solid geometry may have a longer history, which did not find its way to the surviving Sanskrit canon before the 16th-century.

Before we reach Europe, we should proceed through the Arab world. The conic section based solution of cubic equations by Al-Khayyām (Woepcke 1851) and the foreshadowing of the Ruffini-Horner scheme by Al-Ṭūsī (Rashed 1986; Hogendijk 1989), are irrelevant both in terms of their form and their date (11th and 12th-century; see also Rashed 1994, Ch. 3).

Al-Kāshī's work, however, dates from the 15th-century, and solves a cubic equation by means of procedure very close to a fixed point iteration (Aaboe 1954). Indeed the only difference is that Al-Kāshī reorganizes the fixed point iteration equations so as to determine the successive digits of the solution, rather than a sequence of convergent approximations (this follows and simplifies Al-Ṭūsī approach, which provided a different equation for each digit). Note that Indian astronomical fixed point techniques were adopted by some Arabic writers, but did not become part of the Arabic mainstream (Plofker 2002). Determining the meaning of these observations in terms of the plausibility and directionality of transmission will require more research.

In Italy, Even before the famous 16th-century solution of the cubic, some mathematicians were seriously engaged with cubic equations. One anonymous 14th-century treatise discusses the conversion of various kinds of cubic equations (with constant, linear or quadratic terms on either side of the equality sign) to a form that can be rendered anachronistically as $x^3 = px + q$ with positive p and q (Franci and Pancanti 1988, pp. 98-107; see also Høyrup 2009, pp. 13-16). The solution of such an equation is termed the "cubic root of the number q with the addition of the number p". The author provides the following example: to calculate the cubic root of 36, take the closest cubic number above, 64, and take its cubic root, 4. Now, the cube 64 is in excess of 28 over the given 36, so 4 is said to be the cubic root of 36 with an addition of 7=28/4. In modern terms, 4 solves the equation $x^3 = 36 + 7x$.

What does this have to do with the twenty-one problems? I note that both in the Italian treatment above and in rule 10 of the *Twenty-One Problems* we find a conjunction of the following three characteristics:

- a systematic treatment of cubic problems,
- the discussion of a cubic root with remainder/excess, and
- the discussion of this remainder or excess as a product of the approximate root with another quantity (this is the most nonstandard feature of the three).

I emphasize again that I do not claim that this conjunction is proof of transmission. I only suggest that if we suspect transmission, we might try to look for it around arithmetic-geometric discussions of cubic roots and cubic equations in Kerala and in Northern Italy.

Another conjunction that may suggest possible transmission relates rule 15 to the Italian solution of cubic equation. In rule 15 the givens are sum_3 and $prod$. The Malayalam provides a determinate solution based on a quadratic identity: $dif_3^2 = sum_3^2 - 4prod^3$. Now, from sum_3 and dif_3, the unknowns are easily reconstructed. The Sanskrit version offers an indeterminate solution based on the cubic identity: $dif^3 + 3prod \times dif = dif_3$. Each of the authors of the two versions obviously had access to both methods, as indicated by rules 5 and 14. So rule 15 presents a problem that can be solved in two ways: a determined quadratic solution and an indeterminate cubic solution.

The same conjunction appears again in the context of the 16th-century Italian solution of the equation $x^3 + px = q$ by Dal Ferro, his student Fior, and the latter's opponent, Tartaglia (Bortolotti 1928, p. 21). Indeed, this solution replaces the unknown x by a difference of two unknown quantities (*dif*). Then q turns out to be the difference of the cubes of the unknowns (dif_3) and p their product (*prod*). This reformulation turns the original equation into the identity $dif^3 + 3prod \times dif = dif_3$, which underlies the Sanskrit rule 15. The two unknowns are then extracted from the given q and p in a manner identical to the Malayali rule 15.

There has been speculation as to how the Italians derived their solution. I was of the opinion that the rule was derived algebraically, by substituting cubic binomials and residuals into the cubic equation, extending a practice that goes back at least as far as Fibonacci (Wagner 2010, section 4; cf. Vacca 1930; Katscher 2006). Tartaglia, however, insisted that his solution was a geometric one (e.g. Targtaglia 1546, Book IX, §23),[12] and the geometric diagrams of Cardano's *Ars Magna* display the same cut-and-paste reasoning as the Sanskrit proofs of Citrabhānu's rules.[13]

Again, we have a conjunction of the same methods around a single problem in the Italian and Indian cases. Again, this is not enough to establish transmission, but might suggest to us the mathematical topics around which we should look for evidence.

Acknowledgment

I'd like to thank Ophira Gmaliel and David Shulman for their help with questions of language. I would also like to express my gratitude to P.L. Shaji from the Trivandrum University's Oriental Research Institute Manuscript Library for his assistance with procuring manuscripts.

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[12] Tartaglia claims that he had solved the case $x^3+px^2=q$ in 1530, and only later, during his competition with Fior in 1535, solved the case $x^3+px=q$ (Targtaglia 1546, Book IX, §§14, 25). I found no reconstruction that would explain how the former (apparently more difficult case) could be solved while leaving the latter case unsolved. The only plausible conjecture I found (assuming we believe Tartaglia) is that he rewrote the former equation as $x^2(x+p)=q$, and found some geometric construction for determining the two unknowns (x and $x+p$) from their difference (p) and the first of their two mean proportionals ($q^{1/3} = (x^2(x+p))^{1/3}$). Indeed, given the Greek geometric methods for finding two mean proportionals (the standard ersatz for the famous problem of doubling the cube), one might be able reconstruct such a method. Still, I know of no such explicit method in Tartaglia's work.

[13] One more anecdote: Tartaglia includes a mnemonic poem for summarizing his solution of cubic equations (Targtaglia 1546, Book IX, §34). Such poems were rare in the Italian mathematical culture. This poem might be interpreted as echoing the Sanskrit style of presentation; however, internalist explanations are also available (Saiber 2014).

Appendix: Translations of the Malayalam Introduction and rules 6, 10, 17, 18, 19 and 20

My transcript of the Malayalam text is available here: https://www.academia.edu/3524735/Transcript_of_Citrabhanu_s_Twenty_One_Algebraic_Problems_in_Malayalam_16th_century_Kerala_

Translating Malayalam into English literally is practically impossible due to the very different syntax and morphology. I did try, however, to keep the translation of lexical and syntactic units consistent, even where it felt awkward, in order to allow the reader some access to the formal aspects of the text.

I translated number words as words and numerals as numerals, but note that the Malayali numerals, as used in this manuscript, are not arranged in a place-value system (symbols for tens, hundreds and thousands designate the decimal order of numerals, and zero is not used, so 304 is rendered as the sequence 3-100-4).

Malayali technical terms are included in square brackets when they first appear in this excerpt. I also include transcriptions of a few important sentences. The transcriptions reproduce the original spelling (which, among other things, does not distinguish between short and long variants of the vowels *e* and *o*). When transcribing isolated terms, I used nominative and infinitive forms of the relevant nouns and verbs instead of the original inflected forms. I accompany each rule by a short symbolic summary.

Introduction:

Now, before we begin to state the twenty-one problems and answers, we state their form. Having set the following seven: sum [*yogam*], difference [*antaram*], product [*ghātam*], sum of squares [*vargayogam*], difference of squares [*vargāntaram*], sum of cubes [*ghanayogam*] and difference of cubes [*ghanāntaram*] of two quantities [*rāśi*], knowing the quantities separately if we know two values [from among the seven] – this has twenty-one kinds.

To show this: when we put together the sum with the six [combinations] starting with difference, there are six kinds; when we put together the difference with the five starting with product, there are five kinds; when we put together the product with the four starting with sum of squares, there are four kinds; when we put together the sum of squares with the three starting with difference of squares, there are three kinds; when we put together the difference of squares with the two starting with sum of cubes, there are two kinds; when we put together the sum of cubes with the difference of cubes, there is one kind – thus twenty-one kinds. Now we show the previous kinds.

[From here on the words and compounds that designate the seven terms of the *Twenty-One Problems* are translated as *sum*, *dif*, etc., in order to keep in line with the notation above.]

Problem 6

Now the method for getting the quantities from knowing *sum* and dif_3. *dif* comes from multiplying [*perukkuka*] dif_3 by four, subtracting [*vāṅṅuka*] dif^3, and dividing [*harikkuka*] by sum^2 divided by three. Here, in order to subtract dif^3 without having *dif*, multiplying dif_3 itself by four and dividing by three and by sum^2 is one *dif*. This makes an approximation [*sthūlam*]. The exact [*sukṣmam*] *dif* comes from subtracting this approximate dif^3 from dif_3 multiplied by four, and dividing by sum^2 divided by three. The previous quantities come from halving [*arddhikkuka*] the sum [*kūṭuka*] and difference [*kaḷayuka*] of this *dif* and *sum*. If the result comes out approximate, the procedure must be done until it comes without difference [*aviśeṣam varuvoḷam kriya ceyyeṇam*, namely until the "output" *dif* is equal to the "input" *dif*].

Example: *sum* is eight, dif_3 is ninety eight. This multiplied by four is three hundred and ninety two. Dividing that by three comes to one hundred thirty and two thirds. Dividing this by sum^2, 64, comes close [*prāyeṇa*] to two. The remainder [*śeṣam*] from subtracting the cube [*ghanam*] of this two from the previous three hundred and ninety two is three hundred and eighty four. Dividing this by three is a hundred and twenty eight. The exact *dif* comes from dividing this by the sum^2, 64. The quantities come from halving the sum and difference of this *dif* and *sum*.

Rule:
$dif = (4{\times}dif_3 - dif^3)/3/sum^2$
Example:
$sum = 8, dif_3 = 98$
First iteration (dif = 0):
$dif = (4{\times}98)/3/8^2 = 130\frac{2}{3} \,/\, 64 \approx 2$
Second iteration ($dif = 2$):
$dif = (4{\times}98 - 2^3)/3/8^2 = 128/64 = 2$

Problem 10

Now the method for getting the quantities from knowing their sum_3 and *dif*. Having multiplied sum_3 by four, *sum* comes from taking the cubic root [*ghanamūlam*] of dif^2 multiplied by *sum* multiplied by three subtracted from sum_3 multiplied by four.

Here, in order to multiply dif^2 by *sum* without having *sum*, an approximate *sum* comes from taking the cubic root of the dif^2 multiplied by *dif* itself multiplied by three subtracted from four times sum_3. The exact *sum* comes from taking the cubic root of the dif^2 multiplied by the approximate *sum* multiplied by three and subtracted from the previous sum_3 multiplied by four.

Here, if it turns out that the exact sum does not come out, you must apply the *aviśeṣam* method [fixed point iteration] until an exact result comes out [*sūkṣmam varuvoḷḷam aviśeṣicc[u]* *koḷḷukayum veṇam*]. The quantities come from halving the sum and difference of this exact *sum* and *dif*.

Example: sum_3 is a hundred and fifty two, *dif* is two. The sum_3 multiplied by four is six hundred and eight. dif^2 is four. This multiplied by the *dif*, two, is eight. This eight multiplied by three is twenty four. Twenty four subtracted from six hundred and eight is five hundred and eighty four. Its cubic root is roughly [*ekadeṣam*] eight and a quarter. The dif^2, four, multiplied by this eight and a quarter is thirty three. This multiplied by three is ninety nine. This ninety nine subtracted from six hundred and eight is five hundred and nine. Its cubic root is eight itself. You can neglect its fractions because they measure a negligible amount [*atinr̲e avyavaṅṅḷ aṇu parimāṇaṅṅḷ-ākakoṇṭu upekṣikām*]. The quantities come from halving the sum and difference of *sum*, eight, and *dif*, two, [yielding 5 and 3].

Rule:
$sum = \sqrt[3]{(4 \times sum_3 - 3 \times dif^2 \times sum)}$
Example:
$dif = 2, sum_3 = 152$
First iteration ($sum = dif = 2$):
$$sum = \sqrt[3]{(4 \times 152 - 3 \times 2^2 \times 2)} = \sqrt[3]{584} \approx 8\frac{1}{4}$$
Second iteration ($sum = 8\frac{1}{4}$):
$$sum = \sqrt[3]{\left(4 \times 152 - 3 \times 2^2 \times 8\frac{1}{4}\right)} = \sqrt[3]{509} \approx 8$$

Problem 17

Now the method for knowing the quantities separately from knowing both sum_2 and sum_3. The root [*mūlikkuka*] of double [*eraṭṭikkuka*] sum_2 comes close to the *sum*. The remainder of subtracting the cube of this approximate *sum* from sum_3 multiplied by four, divided by the approximate *sum* divided by three, comes to dif^2. The precise *sum* comes from taking the root of this dif^2 subtracted from double sum^2. The remainder of subtracting this sum^3 from sum_3 multiplied by four, divided by the *sum* divided by three, comes to dif^2. The root of this dif^2 is *dif*. The former quantities come from halving the sum and difference of this *dif* and *sum*.

Example: sum_2 is 34 [the manuscript has 24], sum_3 is 152. This sum_2, 34, doubled, is 68. Its root is close to 8. The remainder of subtracting the 512, the cube of 8, from 608, the sum_3 multiplied by 4, is 96. Divided by three, it's 32. This thirty two divided by the *sum* eight is 4. This is dif^2 [the copyist misread this as five *dif*]. Subtracting this dif^2 from double dif_2, that is from 68, is 64. Its root is 8. 5 [and 3] come from halving the sum and difference of this 8 and *dif* [the manuscript has *sum*].

Rule:
$sum = \sqrt{2\times sum_2 - dif^2}$; $dif = \sqrt{(4\times sum_3 - sum^3)/3/sum}$
Example:
$sum_2 = 34$, $sum_3 = 152$
First iteration ($dif = 0$):
$sum = \sqrt{2\times 34} = \sqrt{68} \approx 8$; $dif = \sqrt{(4\times 152 - 8^3)/3/8} = 2$
Second iteration ($dif = 2$):
$sum = \sqrt{2\times 34 - 2^2} = \sqrt{64} = 8$

Problem 18

Now the method for knowing the quantities from knowing both sum_2 and dif_3. The approximate *dif* is the result [*phalam*] of dif_3 multiplied by four, divided by three, and divided by double sum_2. The exact *dif* comes from subtracting this *dif* cubed from the former dif_3 multiplied by four, dividing by three, and dividing by the remainder of subtracting dif^2 from double sum_2 [the manuscript is a little garbled here, probably noting that the latter remainder is sum^2]. The quantities come from halving the sum and difference of this and of *sum*.

Example: sum_2 is 34, dif_3 is 98. This 98 multiplied by four is 392. This divided by three is 131 [the manuscript has 151]. Dividing this by double sum_2, which is 68, is close to 2. Its cube is 8. The remainder of subtracting this eight from dif_3, 392, is 384. Dividing this by three is 128 [the manuscript has 118]. The result of dividing this by the remainder of subtracting dif^2, 4, from 68, the double sum_2 [the manuscript has dif_2], is 2. Subtracting the square [*varggikkuka*] of this two from 68, the double sum_2, is 64. Its root is 8. Halving the sum and difference of this *sum* and the *dif*, 5 and 3 [these numerals are garbled in the manuscript] are the quantities.

Rule:
$dif = (4\times dif_3 - dif^3)/3/(2\times sum_2 - dif^2)$
Example:
$sum_2 = 34$, $dif_3 = 98$
First iteration ($dif = 0$):
$dif = (4\times 98)/3/(2\times 34) \approx 131 / 68 \approx 2$
Second iteration ($dif = 2$):
$dif = (4\times 98 - 2^3)/3/(2\times 34 - 2^2) = 128/64 = 2$

Problem 19

Now the method for knowing the quantities from knowing both dif_2 and sum_3. The *sum* comes from taking the cubic root of the remainder of subtracting dif^2 multiplied by *sum* and multiplied by three from sum_3 multiplied by four.

Here, not having *sum* and dif^2, the approximate *sum* comes from taking the cubic root of dif_2 multiplied by three and subtracted from sum_3 multiplied by four. *dif* comes from dividing dif_2 by this approximate *sum*. The exact *sum* comes from taking the cubic root of the remainder of subtracting the square of the approximate *dif* multiplied by the approximate *sum* and multiplied by three from sum_3 multiplied by four. The exact *dif* comes from dividing dif_2 by this *sum*. The quantities come from halving the sum and difference of this *dif* and *sum*.

Example: dif_2 is 16, sum_3 is 152. Multiplied by 4, it's 608. The remainder of subtracting from this 48, the dif_2 multiplied by three, is 560. Its cubic root is close to 8. The result of dividing dif_2, 16, by this eight is 2. This is *dif*. The square of this *dif* is 4. Multiplying it by the approximate *sum*, 8, is 32. Multiplying this by three is 96. The remainder of subtracting this from 608, the sum_3 multiplied by 4, is 512. Taking its cubic root, 8, is the exact *sum*. Dividing dif_2, which is 16, by this exact *sum*, 2 is the exact *dif*. Halving the sum and difference of this *dif* and *sum*, 5 and 3 [the manuscript has 53] are the quantities.

Rule:
$sum = \sqrt[3]{4 \times sum_3 - 3 \times dif^2 \times sum}; \; dif = dif_2/sum$
Example:
$dif_2 = 16, sum_3 = 152$
First iteration ($dif^2 \times sum = dif_2 = 16$):
$sum = \sqrt[3]{4 \times 152 - 3 \times 16} = \sqrt[3]{560} \approx 8; \; dif = 16/8 = 2$
Second iteration ($sum = 8, dif = 2$):
$sum = \sqrt[3]{4 \times 152 - 3 \times 2^2 \times 8} = \sqrt{512} = 8$

Problem 20

Now the method for knowing the quantities from knowing both dif_2 and dif_3. Squaring dif_2, the result of dividing dif_3 multiplied by four and divided by three, by the squared dif_2 comes close to *dif*. The exact *dif* comes from subtracting this *dif* cubed from dif_2 multiplied by four, dividing by three, and having this divide the squared dif_2. Dividing dif_2 by this *dif* comes to *sum*. The quantities come from halving the sum and difference of this *sum* and *dif*.

Example: dif_2 is 16, dif_3 is 98. Multiplying this ninety eight by four is 392. Dividing this by three is 131 [the manuscript has 121]. The square of dif_2, which is 16, is 256. Dividing this by 131 is close to two. The cube of this two is 8. The remainder of this 8 subtracted from three hundred and ninety two, the dif_3 multiplied by 4, is 384. This divided by three is 128. Dividing 256, the square of sixteen, the dif_2, by this hundred and twenty eight, is 2. This is the exact *dif*. The result of dividing dif_2, sixteen, by this *dif* is 8. This is the exact *sum*. Halving the sum and difference of this *sum* and *dif*, the former quantities come to 5 and 3.

Rule:
$dif = {dif_2}^2/\left((4 \times dif_3 - dif^3)/3\right)$

Example:
$dif_2 = 16$, $dif_3 = 98$
First iteration ($dif = 0$):
$dif = 16^2/(4\times 98/3) \approx 256/131 \approx 2$
Second iteration ($dif = 2$):
$dif = 16^2/\big((4\times 98 - 2^3)/3\big) = 256/128 = 2$

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