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Convergence rates of finite difference schemes for the linear advection and wave equation with rough coefficient

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We prove convergence rates of explicit finite difference schemes for the linear advection and wave equation in one space dimension with Hölder continuous coefficient. The obtained convergence rates explicitly depend on the Hölder regularity of the coefficient and the modulus of continuity of the initial data. We compare the theoretically established rates with the experimental rates of a couple of numerical examples.

Keywords: linear advection equation; linear wave equation; rough coefficient; finite difference scheme; convergence rate.

1. Introduction

Propagation of acoustic waves in a heterogeneous medium plays an important role in many applications, for instance in seismic imaging in geophysics and in the exploration of hydrocarbons (Ikelle & Amundsen, 2005; Biondi, 2006). This wave propagation is modeled by the linear wave equation:

$$\frac{1}{c^2(\mathbf{x})}\partial_{tt}^2 p(t,\mathbf{x}) - \Delta p(t,\mathbf{x}) = 0, \quad (t,\mathbf{x}) \in D_T, \tag{1.1a}$$

$$p(0,\mathbf{x}) = p_0(\mathbf{x}), \quad \mathbf{x} \in D, \tag{1.1b}$$

$$\partial_t p(0,\mathbf{x}) = p_1(\mathbf{x}), \quad \mathbf{x} \in D, \tag{1.1c}$$

where $D_T := [0,T] \times D$, $D \subset \mathbb{R}^d$, augmented with boundary conditions. Here, p is the acoustic pressure and the wave speed is determined by the coefficient $c^2 = c^2(\mathbf{x}) > 0$. The coefficient c encodes information about the material properties of the medium. As an example, the coefficient c represents various geological properties when seismic waves propagate in a rock formation. It is well known that the linear wave equation (1.1) can be rewritten as a first-order system of partial differential equations by defining $v(t,\mathbf{x}) := \partial_t p(t,\mathbf{x})$ and $\mathbf{u}(t,\mathbf{x}) := \nabla p(t,\mathbf{x})$, resulting in

$$\frac{1}{c^2(\mathbf{x})}\partial_t v(t,\mathbf{x}) - \operatorname{div}(\mathbf{u}(t,\mathbf{x})) = 0, \tag{1.2a}$$

$$\partial_t \mathbf{u}(t,\mathbf{x}) - \nabla v(t,\mathbf{x}) = 0, \quad (t,\mathbf{x}) \in D_T,$$

$$v(0,\mathbf{x}) = p_1(\mathbf{x}), \quad \mathbf{x} \in D, \tag{1.2b}$$

$$\mathbf{u}(0,\mathbf{x}) = \nabla p_0(\mathbf{x}), \quad \mathbf{x} \in D. \tag{1.2c}$$

The above system (1.2) is strictly hyperbolic (Gustafsson *et al.*, 1995) with wave speeds given by $\pm c$. Under the assumption that the coefficient $c^2 \in C^{0,\alpha} \cap L^\infty(D)$ for some $\alpha > 0$ and that it is uniformly positive on D, i.e., there exist constants $\underline{c}, \overline{c} > 0$ such that

$$0 < \underline{c} \le c^2(\mathbf{x}) \le \overline{c}, \quad \forall \mathbf{x} \in D, \tag{1.3}$$

and that the initial data $p_0 \in H^1(D)$ and $p_1 \in L^2(D)$, one can prove existence of a unique weak solution $p \in C^0([0,T]; H^1(D))$ with $\partial_t p \in C^0([0,T]; L^2(D))$ following classical energy arguments for linear partial differential equations. See, for instance, Lions & Magenes (1972, Chapter III, Theorems 8.1 and 8.2). A smoother coefficient c and more regular initial data p_0, p_1 result in a more regular solution (Lions & Magenes, 1972).

Even though equations (1.1) and (1.2) are linear, analytical solution formulae are in general not available due to the possibly complex geometry of the domain D, the heterogeneity of the coefficient, or boundary conditions. Consequently, solutions have to be approximated numerically. Among the most popular methods for the linear wave equation with inhomogeneous coefficient c are finite difference methods and finite element methods for which an extensive stability and convergence analysis is available (Gustafsson *et al.*, 1995; Larsson & Thomee, 2003; Kreiss & Lorenz, 2004). A key question is the rate at which numerical schemes converge to the exact solution as the discretization parameter goes to zero or the computational effort increases, since this allows us to estimate the computational work needed to get a certain desired quality of the approximation. If the underlying solution of the equation is smooth, this generally depends on the order of the truncation error which is determined by the order of the spatial and temporal discretization, that is, the order of the underlying difference operators (for finite difference schemes) or the dimension of the polynomial approximation spaces (for finite element methods). The smoothness of the solution again depends on the regularity of the coefficient and the initial data for the equation. If the coefficient c and the initial data p_0, p_1 are smooth, say $C^k(D)$ or $H^s(D)$ for some large enough Sobolev exponent s, then by regularity results for the linear wave equation (Lions & Magenes, 1972), the solution is also smooth, i.e., it belongs to $H^s(D_T)$ and the finite difference (resp. finite element) discretizations converge at the order of the underlying difference operators (resp. polynomial approximation spaces).

1.1 *Rough coefficients*

As noted above, the regularity of the solution to the wave equation (1.1) and the resulting (high) rate of convergence of numerical approximations rely on the smoothness of the coefficient c. Accordingly, most of the numerical analysis literature on the wave equation assumes a smooth coefficient c. However, this assumption is not always realized in practice. As noted before, the wave equation is heavily used to model seismic imaging in rock formations and other porous media (for instance, oil and gas reservoirs). Such media are very heterogeneous with sharp interfaces, strong contrasts and aspect ratios (Ikelle & Amundsen, 2005). Furthermore, the material properties of such media can only be determined by measurements. Such measurements are inherently *uncertain*. This uncertainty is modeled in a statistical manner by representing the material properties (such as rock permeability) as random fields. In particular, log-normal random fields are heavily used in modeling material properties in porous and other geophysically relevant media (Ikelle & Amundsen, 2005; Fouque *et al.*, 2007). Thus, the coefficient c is not smooth, not even continuously differentiable, see Fig. 1 for an illustration of coefficient c, where the rock permeability is modeled by a log-normal random field (the figure represents a single realization of the field). Closer inspection of the coefficients obtained in practice reveals that the material coefficient c is at most a Hölder continuous function, that is, $c \in C^{0,\alpha}$ for some $0 < \alpha < 1$. No further regularity can be assumed on the coefficient c representing material properties of most geophysical formations.

Given the above discussion, it is natural to search for numerical methods that can effectively and efficiently approximate the acoustic wave equation with rough (merely Hölder continuous) coefficients. In particular, one is interested in designing numerical methods that can be rigorously shown to converge to the underlying weak solution. Furthermore, one is also interested in obtaining a convergence rate for the discretization as the mesh parameters are refined. We remark that the issue of a convergence rate is not just of theoretical significance, it has, for example, profound implications on calculating complexity estimates for Monte Carlo and Multilevel Monte Carlo methods (see Mishra *et al.*, 2016) to solve the random (uncertain) PDE that results from considering the material coefficient as a random field (as is done in engineering practice).

A search through the literature revealed that there are not many results available concerning convergence rates for linear hyperbolic partial differential equations with rough coefficients. Jovanović *et al.* (1987) prove convergence rates for finite difference approximations of the hyperbolic problem

$$\partial_{tt}^2 u - \sum_{i,j=1}^{2} \partial_{x_i}\left(a_{ij}\partial_{x_j} u\right) + au = f, \quad \text{in } (0,T)\times D,$$

under the assumption that the solution u lies in the Sobolev space $H^\lambda((0,T)\times D)$ and the coefficients $a_{ij}(x) \in W^{\lambda-1,\infty}(D)$, $a(x) \in W^{\lambda-2,\infty}(D)$, with $2 < \lambda \leq 4$. It is shown that the approximations converge at rate $\Delta x^{\lambda-2}$ in the energy norm (a discrete version of the H^1-norm). In Jovanović (1992) this result is extended to coefficients $a_{ij}(x) \in W^{\lambda-1,2}(D)$, $a(x) \in W^{\lambda-2,2}(D)$.

Another related work is the article by Jovanović & Rohde (2005), where the authors establish error estimates for finite volume approximations of linear hyperbolic systems in multiple space dimensions for initial data with low regularity, however, under the assumption that the coefficients are smooth.

A survey of the available results on finite difference methods for hyperbolic equations whose coefficients and initial data have low regularity can be found in Chapter 4 of the book by Jovanović & Süli (2014).

Given this paucity of available results, we aim to contribute to the theory with the current paper. Since our ultimate goal is proving convergence rates for multidimensional wave e quations, we will start by analyzing rates for linear hyperbolic systems in one space dimension, in particular, we will consider the linear advection equation

$$\partial_t u(t,x) + \partial_x(a(x)u(t,x)) = 0, \quad (t,x) \in [0,T]\times D, \tag{1.4}$$

and the linear wave equation

$$\frac{1}{a(x)}\partial_{tt}^2 p(t,x) - \partial_{xx}^2 p(t,x) = 0, \quad (t,x) \in [0,T]\times D, \tag{1.5}$$

in one space dimension where the spatially heterogeneous coefficient a is positive and uniformly bounded, but has only little regularity, specifically, we assume $a \in C^{0,\alpha}(D)$, for some $0 < \alpha \leq 1$. We require the initial data u_0 for the transport equation to be Hölder continuous with exponent $\gamma > 0$, and for the wave equation, we assume that the derivatives $\partial_t p$ and $\partial_x p$ have moduli of continuity in L^2.

These equations can be seen as prototype models for equation (1.1) above for $a := c^2$.

Equation (1.4) also appears as a model for transport of pollutants in heterogeneous media (see, for example, Mendoza & McAlary, 1990; Elfeki *et al.*, 2012). Moreover, these two models are related to mixing in turbulent flow (Dimotakis, 2005).

In a first part (Section 2), we study the properties of equation (1.4) under the assumption that a is Hölder continuous, and propose a simple upwind scheme for the numerical approximation. We show that this scheme is stable under a linear Courant-Friedrichs-Lewy (CFL)-condition and converges, and then prove a convergence rate of the scheme in $L^1(D)$ and $L^2(D)$ depending explicitly on the Hölder regularity of the coefficient a and the initial data. To prove the rate, we show that the numerical approximations are approximately Hölder continuous in time, and use a variant of S. N. Kružkov's doubling of variables technique (Kružkov, 1970) combined with a type of Grönwall inequality for the L^2-case. We conclude the section with a couple of numerical experiments that confirm that the rates are indeed quite low, but higher than the theoretically established rate. This may indicate that our estimate is not sharp.

In the second part (Section 3), we show that the techniques from Section 2 can be used to establish a convergence rate for an upwind finite difference scheme for the first-order reformulation (1.2) of the linear wave equation (1.5) in one space dimension for Hölder continuous coefficient a, but under slightly stronger assumptions on the initial data, or given that the solution has a known modulus of continuity. Again, we conduct a couple of numerical experiments, and observe that the experimental rates are close to the theoretically derived ones. We conclude by summarizing the results and suggesting further research directions in Section 4.

2. Transport equation with Hölder continuous coefficient

The purpose of this section is to investigate the properties of the linear advection equation in one space dimension,

$$\partial_t u(t,x) + \partial_x(a(x)u(t,x)) = 0, \quad (t,x) \in D_T, \tag{2.1a}$$

$$u(0,x) = u_0(x), \quad x \in D, \tag{2.1b}$$

on the domain $D_T := (0,T] \times D$, for some finite time $T > 0$, a finite interval $0 \in D \subset \mathbb{R}$, periodic boundary conditions and $u_0 \in L^1(D)$ a given initial data. Alternatively, we could consider $D = \mathbb{R}$ and compactly supported initial data $u_0 \in L^1(\mathbb{R})$. We consider coefficients $a \in L^\infty(D)$ which are positive and bounded away from zero, that is

$$\overline{a} \geq a(x) \geq \underline{a} > 0, \quad \forall\, x \in D, \tag{2.2}$$

as well as Hölder continuous, $a \in C^{0,\alpha}(\overline{D})$, for some exponent $\alpha > 0$. As we will see in the following, it is more convenient to work with the variable $w := a(x)u$ instead of u and the equation it satisfies,

$$\partial_t\left(\frac{w(t,x)}{a(x)}\right) + \partial_x w(t,x) = 0, \quad (t,x) \in D_T, \tag{2.3a}$$

$$w(0,x) = w_0(x) := a(x)u_0(x), \quad x \in D. \tag{2.3b}$$

We assume that the initial data w_0 are Hölder continuous $C^{0,\gamma_\infty}(D)$ for some $\gamma_\infty > 0$:

$$|w_0|_{C^{0,\gamma_\infty}(D)} := \sup_{x\neq y\in D} \frac{|w_0(x) - w_0(y)|}{|x-y|^{\gamma_\infty}} \leq C < \infty. \tag{2.4}$$

We note that this implies in particular for any $x \in D$

$$\sup_{|h|\leq\sigma} |w_0(x+h) - w_0(x)| \leq C\,\sigma^{\gamma_\infty}.$$

We will see that some solution properties can also be obtained under the slightly weaker assumption

$$\int_D \sup_{|h|\leq\sigma} |w_0(x+h) - w_0(x)|^p \,\mathrm{d}x \leq \sigma^{p\gamma_p}. \tag{2.5}$$

In the following, we investigate to what extent the exponents α, γ_p influence the regularity of the solution w at a time $t > 0$ and the convergence rate of the finite difference scheme

$$\frac{D_t^+ w_j^n}{a_j} = -D_x^- w_j^n, \quad 1 \leq j \leq N_D,\ 0 \leq n \leq N_T$$

as the mesh is refined.

2.1 *Regularization of the coefficient*

Since the coefficient a is not differentiable, it is possible that the solution w is not differentiable in the classical sense either, and only weak solutions to equation (2.3a) can be defined. By a weak solution to (2.3), we mean a function $w = w(t,x) \in C^{0,\gamma}(D_T)$ for some $\gamma > 0$ satisfying (2.3) in the distributional sense, that is, for all smooth, periodic in x, test functions $\varphi \in C^\infty(D_T)$,

$$\int_{D_T} \frac{w}{a}\,\partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{D_T} w\,\partial_x\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_D \frac{w_0(x)}{a}\varphi(0,x)\,\mathrm{d}x = \int_D \frac{w(T,x)}{a}\varphi(T,x)\,\mathrm{d}x.$$

To deal with the possible nondifferentiability of the solution, we will in a first step regularize the coefficient a by convolving it with a smooth test function $\omega_\delta \in C_0^\infty(\mathbb{R})$, given as

$$\omega_\delta(x) = \frac{1}{\delta}\omega\left(\frac{x}{\delta}\right), \tag{2.6}$$

where $\delta > 0$ small enough, $\omega \in C_0^\infty(\mathbb{R})$ is an even function with the properties

$$0 \leq \omega \leq 1, \quad \omega(x) = 0 \text{ for } |x| \geq 1, \quad \int_{\mathbb{R}} \omega(x)\,\mathrm{d}x = 1.$$

We choose δ so small that ω_δ is compactly supported in D. Then we consider the solution w^δ of the equation

$$\partial_t\left(\frac{w^\delta(t,x)}{a^\delta(x)}\right) + \partial_x w^\delta(t,x) = 0, \quad (t,x) \in D_T, \tag{2.7a}$$

$$w^\delta(0,x) = w_0^\delta(x) := (w * \omega_\delta)(x), \quad x \in D, \tag{2.7b}$$

$$a^\delta(x) := (a * \omega_\delta)(x), \quad x \in D. \tag{2.7c}$$

The coefficient a^δ is smooth and therefore in particular Lipschitz continuous, and hence we can define classical solutions to (2.7) in a standard way, using the method of characteristics: we let η solve the ordinary differential equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\eta(t,x_0) &= a^\delta(\eta(t,x_0)), \quad (t,x_0) \in D_T, \\ \eta(0,x_0) &= x_0, \quad x_0 \in D. \end{aligned} \tag{2.8}$$

w^δ is constant along the characteristics η since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}w^\delta(t,\eta(t,x_0)) &= \partial_t w^\delta(t,\eta(t,x_0)) + \frac{\mathrm{d}}{\mathrm{d}t}\eta(t,x_0)\partial_x w^\delta(t,\eta(t,x_0)) \\ &= a^\delta(\eta(t,x_0))\left(\frac{\partial_t w^\delta(t,\eta(t,x_0))}{a^\delta(\eta(t,x_0))} + \partial_x w^\delta(t,\eta(t,x_0))\right) = 0. \end{aligned}$$

Thus the solution at time $t > 0$ is given by

$$w^\delta(t,\eta(t,x_0)) = w_0^\delta(x_0).$$

Therefore, if the initial data are in $L^\infty(\overline{D})$, the solution will be essentially bounded at any later time. Using this, we can derive Hölder continuity of the solution in time and space:

LEMMA 2.1 Assume that the coefficient a in (2.7) is bounded, i.e., it satisfies (2.2), and that w_0 is Hölder continuous with exponent γ_∞, as in (2.4). Then w^δ will be Hölder continuous in space and time with exponent γ_∞ for any time $t > 0$, independently of $\delta > 0$. In particular, we have

$$\begin{aligned} \sup_{x,t\in D_T,|h|\le\sigma} \frac{|w^\delta(t,x) - w^\delta(t+h,x)|}{h^{\gamma_\infty}} &\le \overline{a}^{\gamma_\infty}\|w_0\|_{C^{0,\gamma_\infty}}, \\ \sup_{x,t\in D_T,|h|\le\sigma} \frac{|w^\delta(t,x+h) - w^\delta(t,x)|}{h^{\gamma_\infty}} &\le \left(\frac{\overline{a}}{\underline{a}}\right)^{\gamma_\infty}\|w_0\|_{C^{0,\gamma_\infty}}. \end{aligned}$$

Proof. We consider the characteristics equation (2.8) once more and note that it is independent of the initial data w_0 of w. Hence any initial data will be propagated along the same characteristics, and we have for the difference $\widehat{w} := w_1^\delta - w_2^\delta$ corresponding to initial condition $\widehat{w}_0 := w_{0,1}^\delta - w_{0,2}^\delta$

$$\widehat{w}(t,\eta(t,x_0)) = \widehat{w}_0(x_0)$$

with η defined in (2.8). In particular, taking $w_{2,0}^\delta := w_1^\delta(h,\cdot)$ for some $h > 0$ (the case $h < 0$ is analogous), and omitting the index 1, this implies

$$\|w^\delta(t,\cdot) - w^\delta(t+h,\cdot)\|_{L^\infty} \le \|w_0^\delta - w^\delta(h,\cdot)\|_{L^\infty}.$$

The characteristics equation (2.8) implies for any $x \in D$,

$$x = \eta(h, x_0^x) = x_0^x + \int_0^h a^\delta(\eta(s, x_0^x))\,\mathrm{d}s$$

for some x_0^x. By the assumption on the boundedness of a^δ (2.2), we can bound x_0^x from above and below:

$$x - h\overline{a} \le x - \int_0^h a^\delta(\eta(s, x_0^x))\,\mathrm{d}s = x_0^x \le x, \tag{2.9}$$

and thus for any $x \in D$

$$\begin{aligned} |w_0^\delta(x) - w^\delta(h, x)| &= |w_0^\delta(x) - w_0^\delta(x_0^x)| \\ &\le \sup_{y \in [x - \overline{a}h, x]} |w_0^\delta(x) - w_0^\delta(y)| \\ &\le |w_0^\delta|_{C^{0,\gamma_\infty}} \overline{a}^{\gamma_\infty} h^{\gamma_\infty} \\ &\le |w_0|_{C^{0,\gamma_\infty}} \overline{a}^{\gamma_\infty} h^{\gamma_\infty} \end{aligned}$$

by the assumption on the initial data. Taking the supremum over all $x \in D$, we obtain

$$\|w^\delta(t, \cdot) - w^\delta(t + h, \cdot)\|_{L^\infty} \le \overline{a}^{\gamma_\infty} \|w_0\|_{C^{0,\gamma_\infty}} h^{\gamma_\infty},$$

and thus the Hölder continuity in time. To prove the Hölder continuity in space, we note that, by the characteristics equation (2.8) and the positivity of the coefficient a^δ, we have

$$w^\delta(x + h, t) = w^\delta(x, \tau^x),$$

for some $\tau^x < t$ such that

$$x + h = x + \int_{\tau^x}^t a^\delta(\eta(s, x))\,\mathrm{d}s \tag{2.10}$$

(the characteristics starting at (τ^x, x) and passing through $(t, x + h)$). This allows us to bound τ^x from below:

$$h = \int_{\tau^x}^t a^\delta(\eta(s, x))\,\mathrm{d}s \ge (t - \tau^x)\underline{a}$$

and therefore

$$\tau^x \ge t - h\underline{a}^{-1}. \tag{2.11}$$

This estimate is independent of x and δ. Hence

$$
\begin{aligned}
|w^\delta(x,t) - w^\delta(x+h,t)| &= |w^\delta(x,t) - w^\delta(x,\tau^x)| \\
&\le \sup_{\tau\in[t-h\underline{a}^{-1},t)} |w^\delta(x,t) - w^\delta(x,\tau)| \\
&\le \|w_0\|_{C^{0,\gamma_\infty}} \left(\frac{\overline{a}}{\underline{a}}\right)^{\gamma_\infty} h^{\gamma_\infty},
\end{aligned}
$$

which yields the Hölder continuity in space. □

REMARK 2.2 We note that the estimates in Lemma 2.1 are independent of $\delta > 0$ and the Hölder coefficient α of a. Moreover, if D is bounded, or the initial data w_0 have compact support, this Lemma implies that w^δ has a modulus of continuity in time,

$$\nu_t^p(w^\delta(t,\cdot),\sigma) := \sup_{|h|\le\sigma} \int_D |w^\delta(t+h,x) - w^\delta(t,x)|^p \,\mathrm{d}x \le C\sigma^{p\gamma_p}, \tag{2.12}$$

and a modulus of continuity of the same order in space:

$$\nu_x^p(w^\delta(t,\cdot),\sigma) := \sup_{|h|\le\sigma} \int_D |w^\delta(t,x+h) - w^\delta(t,x)|^p \,\mathrm{d}x \le C\sigma^{p\gamma_p} \tag{2.13}$$

with $\gamma_p = \gamma_\infty$ for all $p \in [1,\infty)$.

REMARK 2.3 (Entropy identity) Subtracting a constant k from equation (2.7a) and then multiplying by a regularized version of the sign function, we obtain the L^1-norm conservation property of the equation:

$$\int_D \frac{|w^\delta(t,x) - k|}{a^\delta(x)} \,\mathrm{d}x = \int_D \frac{|w_0^\delta(x) - k|}{a^\delta(x)} \,\mathrm{d}x. \tag{2.14}$$

Similarly, by multiplying with $\operatorname{sgn}(w^\delta - k)|w^\delta - k|^{p-1}$, $1 \le p < \infty$, we obtain conservation of L^p-norms:

$$\int_D \frac{|w^\delta(t,x) - k|^p}{a^\delta(x)} \,\mathrm{d}x = \int_D \frac{|w_0^\delta(x) - k|^p}{a^\delta(x)} \,\mathrm{d}x. \tag{2.15}$$

The moduli of continuity of Lemma 2.1 and Remark 2.2 are independent of $\delta > 0$, which implies that the sequence of solutions $\{w^\delta\}_{\delta>0}$ is relatively compact in L^p, $p \in [1,\infty)$ (by Kolmogorov's compactness theorem (Holden & Risebro, 2011, Theorem A.5)), and due to the compact embedding of the Hölder spaces $C^{0,\beta_1}(D_T) \subset\subset C^{0,\beta_2}(D_T)$ for $\beta_2 < \beta_1$ and bounded domains, also relatively compact in $C^{0,\gamma}(D_T)$, for any $\gamma < \gamma_\infty$, and thus the limit function $w := \lim_{\delta\to 0} w^\delta \in L^p(D_T) \cap C^{0,\gamma_\infty}(D_T)$, $p \in [1,\infty)$ is a weak solution of (2.3) with the same moduli of continuity in space and time. Moreover, the limit w satisfies the entropy identities (2.14) and (2.15).

2.2 *Approximation by an upwind scheme*

In order to compute numerical approximations to (2.3), we choose $\Delta x > 0$ such that $N_D := |D|/\Delta x \in \mathbb{N}$ and discretize the spatial domain by a grid with gridpoints $x_{j+1/2} := (j+1/2)\Delta x$, $j \in \{0, 1, \ldots, N_D\}$ and cell centers $x_j := j\Delta x$, $1 \leq j \leq N_D$. Furthermore, we let

$$0 < \Delta t := \theta \Delta x \leq \frac{\Delta x}{\overline{a}} \tag{2.16}$$

and set $t^n := n\Delta t$, $0 \leq n \leq N_T$, where N_T is such that $t^{N_T} = T$. We define the averaged quantities

$$a_j = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} a(x)\,\mathrm{d}x, \quad 1 \leq j \leq N_D, \tag{2.17}$$

and

$$w_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} w_0(x)\,\mathrm{d}x, \quad 1 \leq j \leq N_D. \tag{2.18}$$

Moreover, we denote, for a function $\sigma : D_T \to \mathbb{R}$, its approximation by $\sigma_j^n \approx \sigma(t^n, x_j)$, $j = 0, \ldots, N_D$, $n = 0, \ldots, N_T$ defined on the grid,

$$D_t^+ \sigma_j^n := \frac{1}{\Delta t}(\sigma_j^{n+1} - \sigma_j^n), \quad D_x^\pm \sigma_j^n = \pm\frac{1}{\Delta x}(\sigma_{j\pm 1}^n - \sigma_j^n), \quad D_x^c \sigma_j^n = \frac{1}{2\Delta x}(\sigma_{j+1}^n - \sigma_{j-1}^n). \tag{2.19}$$

Then we define approximations w_j^n by

$$\frac{D_t^+ w_j^n}{a_j} = -D_x^- w_j^n, \quad 1 \leq j \leq N_D,\ 0 \leq n \leq N_T. \tag{2.20}$$

Letting $u_j^n := w_j^n / a_j$, this is equivalent to

$$D_t^+ u_j^n = -D_x^-(a_j u_j^n), \quad 1 \leq j \leq N_D,\ 0 \leq n \leq N_T,$$

which will yield an approximation to the solution $u(t,x)$ of equation (2.1).

2.2.1 *Estimates on the numerical approximation.*

LEMMA 2.4 (Properties of the upwind scheme (2.20)) The approximations w_j^n, $1 \leq j \leq N_D$, $n = 0, \ldots, N_T$ defined by the numerical scheme (2.20) have the following properties:

(i) Maximum principle:

$$\sup_{1 \leq j \leq N_D, 1 \leq n \leq N_T} |w_j^n| \leq \sup_{1 \leq j \leq N_D} |w_j^0|. \tag{2.21}$$

(ii) Discrete entropy inequality in L^1:

$$\frac{|w_j^{n+1}-k|}{a_j}-\frac{|w_j^n-k|}{a_j}+\frac{\Delta t}{\Delta x}\big(|w_j^n-k|-|w_{j-1}^n-k|\big)\le 0. \tag{2.22}$$

Discrete entropy inequality in L^2:

$$\frac{|w_j^{n+1}-k|^2}{a_j}-\frac{|w_j^n-k|^2}{a_j}+\frac{\Delta t}{\Delta x}\big(|w_j^n-k|^2-|w_{j-1}^n-k|^2\big)\le 0. \tag{2.23}$$

(iii) Bound on the discrete L^1 and L^2-norms:

$$\Delta x\sum_{j=1}^{N_D}\frac{|w_j^n|}{a_j}\le\Delta x\sum_{j=1}^{N_D}\frac{|w_j^0|}{a_j}, \tag{2.24}$$

$$\Delta x\sum_{j=1}^{N_D}\frac{(w_j^n)^2}{a_j}\le\Delta x\sum_{j=1}^{N_D}\frac{(w_j^0)^2}{a_j}, \tag{2.25}$$

for all $n=1,\dots,N_T$.

Proof. Writing (2.20) as

$$\frac{w_j^{n+1}}{a_j}=\left(1-\frac{a_j\Delta t}{\Delta x}\right)\frac{w_j^n}{a_j}+\frac{a_j\Delta t}{\Delta x}\frac{w_{j-1}^n}{a_j}:=\mathcal{H}(w_{j-1}^n,w_j^n), \tag{2.26}$$

and taking the CFL-condition (2.16) into account, we immediately see that w_j^{n+1}/a_j is a convex combination of w_j^n/a_j and w_{j-1}^n/a_j, and thus the approximations satisfy the maximum principle (2.21). To obtain the discrete version (2.22) of a Kružkov entropy inequality for the quantities w_j^n, we denote

$$a\wedge b=\min\{a,b\}\quad\text{and}\quad a\vee b=\max\{a,b\},$$

and compute for a constant $k\in\mathbb{R}$,

$$\begin{aligned}\mathcal{H}(w_{j-1}^n\vee k,w_j^n\vee k)-&\mathcal{H}(w_{j-1}^n\wedge k,w_j^n\wedge k)\\ &=\left(\frac{1}{a_j}-\frac{\Delta t}{\Delta x}\right)(w_j^n\vee k-w_j^n\wedge k)+\frac{\Delta t}{\Delta x}(w_{j-1}^n\vee k-w_{j-1}^n\wedge k)\\ &=\left(\frac{1}{a_j}-\frac{\Delta t}{\Delta x}\right)|w_j^n-k|+\frac{\Delta t}{\Delta x}|w_{j-1}^n-k|\\ &=\frac{|w_j^n-k|}{a_j}-\frac{\Delta t}{\Delta t}\big(|w_j^n-k|-|w_{j-1}^n-k|\big).\end{aligned} \tag{2.27}$$

Moreover, we note that by (2.26) and thanks to the CFL-condition (2.16),

$$\mathcal{H}(w_{j-1}^n \vee k, w_j^n \vee k) \geq \mathcal{H}(w_{j-1}^n, w_j^n) = \frac{w_j^{n+1}}{a_j},$$
$$\mathcal{H}(w_{j-1}^n \vee k, w_j^n \vee k) \geq \mathcal{H}(k,k) = \frac{k}{a_j},$$

thus

$$\mathcal{H}(w_{j-1}^n \vee k, w_j^n \vee k) \geq \frac{w_j^{n+1} \vee k}{a_j},$$

and similarly

$$\mathcal{H}(w_{j-1}^n \wedge k, w_j^n \wedge k) \leq \frac{w_j^{n+1} \wedge k}{a_j}.$$

Combining this with (2.27), we obtain (2.22). Now we simply need to multiply the expression (2.22) by Δx, sum it over $j = 1, \ldots, N_D$ and set $k = 0$ to obtain the bound on the discrete $L^1(D)$-norm of w_j^n (2.24). To prove the L^2-entropy inequality, we note that the difference scheme (2.20) is equivalent to

$$\frac{D_t^+\left(w_j^n - k\right)}{a_j} = -D_x^-\left(w_j^n - k\right) \tag{2.28}$$

for any constant $k \in \mathbb{R}$, and then multiply both sides of equation (2.20) by $(w_j^n - k)$. Subsequently, we use that

$$ab = \frac{1}{2}(a^2 + b^2 - (a-b)^2), \quad a, b \in \mathbb{R}, \tag{2.29}$$

once for the left-hand side and once for the right-hand side to get (we write $\widehat{w}_j^n := w_j^n - k$ for convenience)

$$\frac{1}{2a_j}\left((\widehat{w}_j^{n+1})^2 + (\widehat{w}_j^n)^2 - (\widehat{w}_j^{n+1} - \widehat{w}_j^n)^2\right) = \frac{(\widehat{w}_j^n)^2}{a_j} + \frac{\Delta t}{2\Delta x}\left(-(\widehat{w}_j^n)^2 + (\widehat{w}_{j-1}^n)^2 - (\widehat{w}_j^n - \widehat{w}_{j-1}^n)^2\right).$$

Rearranging terms and using (2.28) for the difference $\widehat{w}_j^{n+1} - \widehat{w}_j^n$, this reads

$$\frac{(\widehat{w}_j^{n+1})^2}{2a_j} = \frac{(\widehat{w}_j^n)^2}{2a_j} + \frac{\Delta t}{2\Delta x}\left(-(\widehat{w}_j^n)^2 + (\widehat{w}_{j-1}^n)^2\right) + \frac{\Delta t}{2\Delta x}\left(\frac{a_j \Delta t}{\Delta x} - 1\right)(\widehat{w}_j^n - \widehat{w}_{j-1}^n)^2$$

from which the claim follows using the CFL-condition (2.16). Summing the discrete L^2-inequality over j and using induction over n, we furthermore obtain (2.25). □

Thanks to the linearity of the transport equation, we obtain the following corollary of Lemma 2.4:

COROLLARY 2.5 Let w_j^n denote the approximations computed by the scheme (2.20), (2.18) for initial data $w_0 \in L^1(D) \cap L^2(D) \cap L^\infty(\overline{D})$ and v_j^n another approximation computed by (2.20) for initial data $v_0 \in L^1(D) \cap L^2(D) \cap L^\infty(\overline{D})$. Then we have

$$\sup_{1\le j\le N_D} |w_j^n - v_j^n| \le \sup_{1\le j\le N_D} |w_j^0 - v_j^0| \le \|w_0 - v_0\|_\infty,$$
$$\Delta x \sum_j a_j^{-1}|w_j^n - v_j^n| \le \Delta x \sum_j a_j^{-1}|w_j^0 - v_j^0| \le \|(w_0 - v_0)/a\|_{L^1(D)},$$
$$\Delta x \sum_j a_j^{-1}|w_j^n - v_j^n|^2 \le \Delta x \sum_j a_j^{-1}|w_j^0 - v_j^0|^2 \le \|(w_0 - v_0)/\sqrt{a}\|^2_{L^2(D)}, \tag{2.30}$$

for all $1 \le n \le N_T$.

Proof. This follows from the fact that the differences $r_j^n := w_j^n - v_j^n$ satisfy (2.20) due to the linearity of the scheme, together with Lemma 2.4. □

Defining the piecewise constant approximations

$$w_{\Delta x}(t,x) := w_j^n, \quad (t,x) \in [t^n, t^{n+1}) \times [x_{j-1/2}, x_{j+1/2}), \tag{2.31}$$

this corollary enables us to show that the piecewise constant function $w_{\Delta x}$ has a modulus of continuity in time:

LEMMA 2.6 The piecewise constant functions $w_{\Delta x}$ defined in (2.31) have a modulus of continuity in time if the initial data w_0 satisfy (2.5) for $p = 1$:

$$\sup_{|h|\le\sigma} \int_D |w_{\Delta x}(t+h,x) - w_{\Delta x}(t,x)|\,\mathrm{d}x \le C(\sigma + \Delta x)^{\gamma_1} \tag{2.32}$$

or (2.5) for $p = 2$:

$$\sup_{|h|\le\sigma} \int_D |w_{\Delta x}(t+h,x) - w_{\Delta x}(t,x)|^2\,\mathrm{d}x \le C(\sigma + \Delta x)^{2\gamma_2}. \tag{2.33}$$

If the initial data are Hölder continuous with exponent γ_∞, the solution is approximately Hölder continuous in time with the same exponent, i.e.,

$$\sup_{|h|\le\sigma} \|w_{\Delta x}(t+h,x) - w_{\Delta x}(t,x)\|_{L^\infty} \le C(\sigma + \Delta x)^{\gamma_\infty}.$$

Proof. We observe that

$$\begin{aligned} w_j^k &= \left(1-\lambda a_j\right) w_j^{k-1} + \lambda a_j w_{j-1}^{k-1} \\ &= \left(1-\lambda a_j\right)\left\{\left(1-\lambda a_j\right) w_j^{k-2} + \lambda a_j w_{j-1}^{k-2}\right\} + \lambda a_j \left\{\left(1-\lambda a_{j-1}\right) w_{j-1}^{k-2} + \lambda a_{j-1} w_{j-2}^{k-2}\right\} \\ &= \cdots = \sum_{\ell=j-k}^{j} \lambda_{\ell,j}^k w_\ell^0, \end{aligned}$$

where we have denoted $\lambda := \Delta t/\Delta x$ and $\lambda_{\ell,j}^k$ is inductively defined as follows:

DEFINITION 2.7 We let $s_\ell := \lambda a_\ell$ and $v_\ell = (1-\lambda a_\ell)$ for all $\ell \in \mathbb{Z}$. Then we define $\lambda_{\ell,j}^k$ recursively by

$$\lambda_{\ell,j}^k = \begin{cases} 1, & \ell = j,\ k=0, \\ 0, & k+\ell < j \text{ or } \ell > j, \\ s_\ell \lambda_{\ell,j-1}^{k-1} + v_\ell \lambda_{\ell,j}^{k-1}, & \text{otherwise}. \end{cases} \tag{2.34}$$

REMARK 2.8 An explicit expression for the coefficients $\lambda_{\ell,j}^k$ for the third case is given by

$$\lambda_{j-\ell,j}^k = \prod_{n=j-\ell+1}^{j} s_n \sum_{\substack{m_i \geq 0, \\ \sum_{i=1}^{l+1} m_i = k-\ell}} v_j^{m_1} \cdot \ldots \cdot v_{j-\ell}^{m_{\ell+1}}.$$

CLAIM 2.9 The coefficients $\lambda_{\ell,j}^k$ satisfy

$$\sum_{\ell=j-k}^{k} \lambda_{\ell,j}^k = 1 \tag{2.35}$$

and $\lambda_{\ell,j}^k \geq 0$.

Proof. Equation (2.35) follows by induction and using that $v_\ell + s_\ell = 1$ for any $\ell \in \mathbb{Z}$. That the $\lambda_{\ell,j}^k$ are non-negative follows from the CFL-condition (2.16). □

Thus we have

$$\begin{aligned} \int_D |w_{\Delta x}(k\Delta t, x) - w_{\Delta x}(0,x)|\,\mathrm{d}x &= \Delta x \sum_j |w_j^k - w_j^0| \\ &= \Delta x \sum_j \left| \sum_{\ell=j-k}^{j} \lambda_{\ell,j}^k w_\ell^0 - w_j^0 \right| \end{aligned}$$

$$
\begin{aligned}
&\leq \Delta x \sum_j \sum_{\ell=j-k}^{j} \lambda_{\ell,j}^k |w_\ell^0 - w_j^0| \\
&\leq \Delta x \sum_j \sum_{\ell=j-k}^{j} \lambda_{\ell,j}^k \max_{j-k\leq m\leq j} |w_m^0 - w_j^0| \\
&= \Delta x \sum_j \max_{j-k\leq m\leq j} |w_m^0 - w_j^0| \\
&\leq \int_D \sup_{|h|\leq (k+1)\Delta x} |w_0(x+h) - w_0(x)|\,\mathrm{d}x \\
&\leq C(\Delta x k)^{\gamma_1},
\end{aligned}
$$

where we used Claim 2.9 for the first inequality and the third equality, and the assumption on the initial data (2.5) in the last inequality. Similarly, we compute in the L^2-setting,

$$
\begin{aligned}
\int_D |w_{\Delta x}(k\Delta t, x) - w_{\Delta x}(0,x)|^2\,\mathrm{d}x &= \Delta x \sum_j |w_j^k - w_j^0|^2 \\
&= \Delta x \sum_j \left| \sum_{\ell=j-k}^{j} \lambda_{\ell,j}^k w_\ell^0 - w_j^0 \right|^2 \\
&\leq \Delta x \sum_j \sum_{\ell_1=j-k}^{j} \sum_{\ell_2=j-k}^{j} \lambda_{\ell_1,j}^k \lambda_{\ell_2,j}^k |w_{\ell_1}^0 - w_j^0||w_{\ell_2}^0 - w_j^0| \\
&\leq \Delta x \sum_j \sum_{\ell_1=j-k}^{j} \sum_{\ell_2=j-k}^{j} \lambda_{\ell_1,j}^k \lambda_{\ell_2,j}^k \max_{j-k\leq m\leq j} |w_m^0 - w_j^0|^2 \\
&= \Delta x \sum_j \max_{j-k\leq m\leq j} |w_m^0 - w_j^0|^2 \\
&\leq \int_D \sup_{|h|\leq (k+1)\Delta x} |w_0(x+h) - w_0(x)|^2\,\mathrm{d}x \\
&\leq C(\Delta x k)^{2\gamma_2}.
\end{aligned}
$$

Then applying Corollary 2.5, we conclude. The approximate Hölder continuity in time follows in a very similar way, instead of summing over j, we take the maximum over all j. □

REMARK 2.10 If w_0 has bounded variation, obtaining that the solution has bounded variation, and so L^1-moduli of continuity in time and space are much easier. Similarly, if the initial data have a modulus of continuity of $\gamma_2 = 1$ in L^2, obtaining a rate is easier. The argument is similar to the one which will be outlined for the linear wave equation in Section 3, but less technical.

2.3 *A convergence rate in* L^1

In this section, we will prove a rate of convergence in L^1 of the numerical scheme to the limit of solutions of (2.7) as $\delta \to 0$. Our approach is based on the doubling of variables technique developed by Kružkov (1970) for nonlinear scalar conservation laws. The reason why we take this approach is the possible nondifferentiability of the solutions with which the doubling of variables technique can deal well. For this purpose, we use the test function ω_δ from (2.6), let $0 < \nu < \tau < T$ and $\epsilon_0, \epsilon_1 > 0$ such that $0 < 2\epsilon_0 < \min\{\nu, T-\tau\}$ and $\Delta t, \Delta x < \min\{\epsilon_0, \epsilon_1\}$, and define the function $\Omega : D_T^2 \to \mathbb{R}$ by

$$\Omega(t,s,x,y) = \mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_0}(t-s)\omega_{\epsilon_1}(x-y). \tag{2.36}$$

We note that the (smooth) solution to (2.7) satisfies

$$\int_{D_T}\left(\frac{|w^\delta(s,y)-k|}{a^\delta(y)}\partial_s\Omega + |w^\delta(s,y)-k|\partial_y\Omega\right)\mathrm{d}y\,\mathrm{d}s = 0, \tag{2.37}$$

whereas the approximations $w_{\Delta x}$ satisfy, by the discrete entropy inequality (2.22)

$$\int_{D_T}\left(\frac{|w_{\Delta x}(t,x)-\ell|}{a_{\Delta x}(x)}D_t^-\Omega + |w_{\Delta x}(t,x)-\ell|D_x^+\Omega\right)\mathrm{d}x\,\mathrm{d}t \geq 0, \tag{2.38}$$

where we have denoted

$$a_{\Delta x}(x) = a_j, \quad x \in [x_{j-1/2}, x_{j+1/2}).$$

THEOREM 2.11 Let $a \in C^{0,\alpha}(D)$ satisfy (2.2). Denote $w := \lim_{\delta\to 0} w^\delta$ the solution of (2.3) and $w_{\Delta x}$ the numerical approximation computed by scheme (2.20) and defined in (2.31). Assume that the initial data $w_0 \in L^1(D)$ and are Hölder continuous with exponent $\gamma_\infty > 0$. Then $w_{\Delta x}(t,\cdot)$ converges to the solution $w(t,\cdot)$, $0 < t < T$, at (at least) the rate

$$\|(w - w_{\Delta x})(t,\cdot)\|_{L^1(D)} \leq C\,\Delta x^{(\gamma_\infty\alpha)/(\gamma_\infty\alpha+2-\gamma_\infty)} + C\|(w_0 - w_{\Delta x}(0,\cdot))\|_{L^1(D)}, \tag{2.39}$$

where C is a constant depending on $\underline{a}$, $\overline{a}$, $\|a\|_{C^{0,\alpha}}$ and T, but not on Δx.

Proof. Inserting $w_{\Delta x}(t,x)$ for k and $w^\delta(s,y)$ for ℓ in (2.37) and (2.38), integrating the respective equations over $(t,x) \in D_T$ and $(s,y) \in D_T$, respectively, and adding up, we have (for convenience, we will omit writing the arguments of $w_{\Delta x} = w_{\Delta x}(t,x)$, $w^\delta = w_{\Delta x}(s,y)$, $a^\delta = a^\delta(y)$ and $a_{\Delta x} = a_{\Delta x}(x)$ in the following)

$$\int_{D_T^2}\left(|w_{\Delta x} - w^\delta|\left(\frac{D_t^-\Omega}{a_{\Delta x}} + \frac{\partial_s\Omega}{a^\delta}\right) + |w_{\Delta x} - w^\delta|\left(D_x^+\Omega + \partial_y\Omega\right)\right)\mathrm{d}\underline{z} \geq 0, \tag{2.40}$$

where $\mathrm{d}\underline{z} := \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}s$. We have

$$\begin{aligned} D_t^-\Omega &= D_t^-\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_0}(t-\Delta t-s) + \mathbf{1}_{[\nu,\tau)}(t)D_t^-\omega_{\epsilon_0}(t-s) \\ &= D_t^-\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_0}(t-\Delta t-s) - \mathbf{1}_{[\nu,\tau)}(t)D_s^+\omega_{\epsilon_0}(t-s), \end{aligned} \tag{2.41}$$

so that we can rewrite equation (2.40) as

$$\begin{aligned} A+B+D+E := &\int_{D_T^2} \frac{|w_{\Delta x}-w^\delta|}{a_{\Delta x}} \mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_1}\left(\partial_s\omega_{\epsilon_0} - D_s^+\omega_{\epsilon_0}\right)\,\mathrm{d}\underline{z} \\ &+ \int_{D_T^2} \frac{|w_{\Delta x}-w^\delta|}{a_{\Delta x}a^\delta} \mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_1}\partial_s\omega_{\epsilon_0}\left(a_{\Delta x}-a^\delta\right)\,\mathrm{d}\underline{z} \\ &+ \int_{D_T^2} |w_{\Delta x}-w^\delta|\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_0}\left(\partial_y\omega_{\epsilon_1} + D_x^+\omega_{\epsilon_1}\right)\,\mathrm{d}\underline{z} \\ &+ \int_{D_T^2} \frac{|w_{\Delta x}-w^\delta|}{a_{\Delta x}} \omega_{\epsilon_1}\omega_{\epsilon_0}(t-s-\Delta t)D_t^-\mathbf{1}_{[\nu,\tau)}(t)\,\mathrm{d}\underline{z} \\ &\geq 0. \end{aligned} \tag{2.42}$$

We note that

$$D_t^-\mathbf{1}_{[\nu,\tau)}(t) = \frac{1}{\Delta t}\mathbf{1}_{[\nu,\nu+\Delta t)}(t) - \frac{1}{\Delta t}\mathbf{1}_{[\tau,\tau+\Delta t)}(t), \tag{2.43}$$

which means that we can rewrite (2.42) as

$$\begin{aligned} &\frac{1}{\Delta t}\int_{D_T^2} \frac{|w_{\Delta x}-w^\delta|}{a_{\Delta x}} \omega_{\epsilon_1}\omega_{\epsilon_0}(t-s-\Delta t)\mathbf{1}_{[\tau,\tau+\Delta t)}(t)\,\mathrm{d}\underline{z} \\ &\quad\leq \frac{1}{\Delta t}\int_{D_T^2} \frac{|w_{\Delta x}-w^\delta|}{a_{\Delta x}} \omega_{\epsilon_1}\omega_{\epsilon_0}(t-s-\Delta t)\mathbf{1}_{[\nu,\nu+\Delta t)}(t)\,\mathrm{d}\underline{z} + A + B + D. \end{aligned} \tag{2.44}$$

We begin by estimating the term A. To do so, we note that

$$\partial_s\omega_{\epsilon_0} - D_s^+\omega_{\epsilon_0} = \frac{1}{\Delta t}\int_0^{\Delta t} (\xi - \Delta t)\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)\,\mathrm{d}\xi \tag{2.45}$$

and that

$$\frac{1}{\Delta t}\int_0^{\Delta t}\int_{D_T^2} \frac{|w_{\Delta x}-w^\delta(t,y)|}{a_{\Delta x}} \mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_1}(\xi-\Delta t)\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)\,\mathrm{d}\underline{z}\,\mathrm{d}\xi = 0, \tag{2.46}$$

because $\frac{|w_{\Delta x}-w^\delta(t,y)|}{a_{\Delta x}}\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_1}(\xi-\Delta t)$ is constant with respect to s and $\omega_{\epsilon_0}(t+\xi-\cdot)$ is compactly supported in the domain. Therefore we can rewrite the term A as

$$A = \frac{1}{\Delta t}\int_0^{\Delta t}\int_{D_T^2} \frac{(|w_{\Delta x}-w^\delta| - |w_{\Delta x}-w^\delta(t,y)|)}{a_{\Delta x}} \mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_1}(\xi-\Delta t)\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)\,\mathrm{d}\underline{z}\,\mathrm{d}\xi, \tag{2.47}$$

and bound in the following way, using triangle inequality,

$$
\begin{aligned}
|A| &\leq \frac{1}{\Delta t}\int_0^{\Delta t}\int_{D_T^2}\frac{|w^\delta - w^\delta(t,y)|}{a_{\Delta x}}\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_1}|\xi-\Delta t|\left|\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)\right|\,\mathrm{d}\underline{z}\,\mathrm{d}\xi\\
&\leq \frac{1}{\Delta t\underline{a}}\int_0^{\Delta t}\int_\nu^\tau\int_{D_T}|w^\delta - w^\delta(t,y)|\,\mathrm{d}y\,|\xi-\Delta t|\left|\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)\right|\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\xi\\
&\leq \frac{1}{\Delta t\underline{a}}\int_0^{\Delta t}|\xi-\Delta t|\int_\nu^\tau\sup_{s\in[t-\epsilon_0,t+2\epsilon_0]}\int_D|w^\delta - w^\delta(t,y)|\,\mathrm{d}y\int_0^T\left|\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)\right|\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\xi\\
&\leq \frac{C\Delta t}{\underline{a}\,\epsilon_0^2}\int_\nu^\tau\sup_{s\in[t-\epsilon_0,t+2\epsilon_0]}\int_D|w^\delta - w^\delta(t,y)|\,\mathrm{d}y\,\mathrm{d}t\\
&\leq \frac{C\Delta t T}{\underline{a}\,\epsilon_0^{2-\gamma_\infty}},
\end{aligned}
\tag{2.48}
$$

where we have used Lemma 2.1 for the last inequality. We proceed to estimating the term B:

$$
B := \int_{D_T^2}\frac{|w_{\Delta x} - w^\delta|}{a_{\Delta x}a^\delta}\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_1}\partial_s\omega_{\epsilon_0}\left(a_{\Delta x}-a^\delta\right)\,\mathrm{d}\underline{z}.
$$

Similarly, to the case of term A, we use that

$$
\int_{D_T^2}\frac{|w_{\Delta x} - w^\delta(t,y)|}{a_{\Delta x}a^\delta}\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_1}\partial_s\omega_{\epsilon_0}\left(a_{\Delta x}-a^\delta\right)\,\mathrm{d}\underline{z} = 0.
$$

Thus subtracting this from B, we can bound term B using triangle inequality,

$$
\begin{aligned}
|B| &\leq \int_{D_T^2}\frac{|w^\delta - w^\delta(t,y)|}{a_{\Delta x}a^\delta}\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_1}|\partial_s\omega_{\epsilon_0}|\left|a_{\Delta x}-a^\delta\right|\,\mathrm{d}\underline{z}\\
&\leq \frac{1}{\underline{a}^2}\int_{D_T^2}|w^\delta - w^\delta(t,y)|\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_1}|\partial_s\omega_{\epsilon_0}|\left(\left|a_{\Delta x}-a^\delta(x)\right|+\left|a^\delta(x)-a^\delta\right|\right)\,\mathrm{d}\underline{z}\\
&\leq \frac{\|a_{\Delta x}-a^\delta\|_\infty + \sup_{|x-y|\leq 2\epsilon_1}|a^\delta(x)-a^\delta(y)|}{\underline{a}^2}\int_{D_T^2}|w^\delta - w^\delta(t,y)|\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_1}|\partial_s\omega_{\epsilon_0}|\,\mathrm{d}\underline{z}\\
&\leq C\frac{\Delta x^\alpha+\delta^\alpha+\epsilon_1^\alpha}{\underline{a}^2}\int_\nu^\tau\sup_{s\in[t-\epsilon_0,t+\epsilon_0]}\int_D|w^\delta(s,y)-w^\delta(t,y)|\,\mathrm{d}y\int_0^T|\partial_s\omega_{\epsilon_0}|\,\mathrm{d}s\,\mathrm{d}t\\
&\leq C\frac{\Delta x^\alpha+\delta^\alpha+\epsilon_1^\alpha}{\underline{a}^2}\epsilon_0^{\gamma_\infty-1}\\
&\leq C\frac{\delta^\alpha+\epsilon_1^\alpha}{\epsilon_0^{1-\gamma_\infty}},
\end{aligned}
\tag{2.49}
$$

since we assumed $\Delta x < \epsilon_1$. We continue to bound term D. First, we observe that

$$
\partial_y\omega_{\epsilon_1} + D_x^+\omega_{\epsilon_1} = \partial_y\omega_{\epsilon_1} - D_y^-\omega_{\epsilon_1} = \frac{1}{\Delta x}\int_0^{\Delta x}(\Delta x-\xi)\partial_{yy}\omega_{\epsilon_1}(x-y+\xi)\,\mathrm{d}\xi \tag{2.50}
$$

and that

$$\frac{1}{\Delta x}\int_0^{\Delta x}(\Delta x-\xi)\int_{D_T^2}|w_{\Delta x}-w^\delta(s,x)|\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_0}\partial_{yy}\omega_{\epsilon_1}(x-y+\xi)\,\mathrm{d}\underline{z}\,\mathrm{d}\xi=0,$$

since ω_{ϵ_1} is compactly supported. Thus, we can estimate D by

$$\begin{aligned}|D|&\le\frac{1}{\Delta x}\int_0^{\Delta x}|\Delta x-\xi|\int_{D_T^2}|w^\delta-w^\delta(s,x)|\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_0}\left|\partial_{yy}\omega_{\epsilon_1}(x-y+\xi)\right|\mathrm{d}\underline{z}\,\mathrm{d}\xi\\&\le\frac{1}{\Delta x}\int_0^{\Delta x}|\Delta x-\xi|\int_\nu^\tau\int_0^T\sup_{|h|\le2\epsilon_1}\int_D|w^\delta(s,y)-w^\delta(s,y+h)|\omega_{\epsilon_0}\int_D\left|\partial_{yy}\omega_{\epsilon_1}\right|\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\xi\\&\le C\frac{\Delta x}{\epsilon_1^2}\int_\nu^\tau\int_0^T\sup_{|h|\le2\epsilon_1}\int_D|w^\delta(s,y)-w^\delta(s,y+h)|\,\mathrm{d}x\omega_{\epsilon_0}\,\mathrm{d}s\,\mathrm{d}t\\&\le C\frac{\Delta x\,T}{\epsilon_1^{2-\gamma_\infty}}.\end{aligned}\tag{2.51}$$

It remains to relate the terms forming E to the L^1-norm of the differences $w^\delta(t,\cdot)-w_{\Delta x}(t,\cdot)$. We have

$$\begin{aligned}E_2&:=\frac{1}{\Delta t}\int_{D_T^2}\frac{|w_{\Delta x}-w^\delta|}{a_{\Delta x}}\omega_{\epsilon_1}\omega_{\epsilon_0}(t-s-\Delta t)\mathbf{1}_{[\tau,\tau+\Delta t)}(t)\,\mathrm{d}\underline{z}\\&=\frac{1}{\Delta t}\int_\tau^{\tau+\Delta t}\int_0^T\int_{D^2}\frac{|w_{\Delta x}-w^\delta|}{a_{\Delta x}}\omega_{\epsilon_1}\,\mathrm{d}x\,\mathrm{d}y\,\omega_{\epsilon_0}(t-s-\Delta t)\,\mathrm{d}s\,\mathrm{d}t\end{aligned}$$

and can rewrite

$$\|(w_{\Delta x}-w^\delta)/a_{\Delta x}\|_{L^1(D)}(\tau)=\frac{1}{\Delta t}\int_{D_T^2}\frac{|w_{\Delta x}(\tau,x)-w^\delta(\tau,x)|}{a_{\Delta x}}\omega_{\epsilon_1}\omega_{\epsilon_0}(t-s-\Delta t)\mathbf{1}_{[\tau,\tau+\Delta t)}(t)\,\mathrm{d}\underline{z},$$

so that

$$\begin{aligned}&\left|E_2-\|(w_{\Delta x}-w^\delta)/a_{\Delta x}\|_{L^1(D)}(\tau)\right|\\&\quad=\left|\frac{1}{\Delta t}\int_{D_T^2}\omega_{\epsilon_1}\mathbf{1}_{[\tau,\tau+\Delta t)}(t)\frac{1}{a_{\Delta x}}\left(|w_{\Delta x}-w^\delta|-|w_{\Delta x}(\tau,x)-w^\delta(\tau,x)|\right)\omega_{\epsilon_0}(t-s-\Delta t)\,\mathrm{d}\underline{z}\right|\\&\quad\le\frac{1}{\Delta t\underline{a}}\int_{D_T^2}\omega_{\epsilon_1}\mathbf{1}_{[\tau,\tau+\Delta t)}(t)\omega_{\epsilon_0}(t-s-\Delta t)\left(|w_{\Delta x}-w_{\Delta x}(\tau,\cdot)|+|w^\delta-w^\delta(\tau,x)|\right)\,\mathrm{d}\underline{z}\\&\quad\le\frac{1}{\Delta t\underline{a}}\int_{[0,T]^2}\mathbf{1}_{[\tau,\tau+\Delta t)}(t)\omega_{\epsilon_0}(t-s-\Delta t)\int_D|w_{\Delta x}-w_{\Delta x}(\tau,\cdot)|\,\mathrm{d}x\,\mathrm{d}\underline{z}\\&\qquad+\frac{1}{\Delta t\underline{a}}\int_{D_T^2}\omega_{\epsilon_1}\mathbf{1}_{[\tau,\tau+\Delta t)}(t)\omega_{\epsilon_0}(t-s-\Delta t)\left(|w^\delta-w^\delta(\tau,y)|+|w^\delta(\tau,y)-w^\delta(\tau,x)|\right)\,\mathrm{d}\underline{z}\end{aligned}$$

$$\leq \frac{1}{\Delta t \underline{a}} \int_{\tau}^{\tau+\Delta t} \sup_{|h|\leq 2\epsilon_0} \int_D |w_{\Delta x} - w_{\Delta x}(t+h,\cdot)| \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \frac{1}{\Delta t \underline{a}} \int_{[0,T]^2} \mathbf{1}_{[\tau,\tau+\Delta t)}(t)\omega_{\epsilon_0}(t-s-\Delta t) \int_D |w^\delta - w^\delta(\tau,\cdot)| \,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t$$
$$+ \frac{1}{\Delta t \underline{a}} \int_{D^2} \omega_{\epsilon_1} |w^\delta(\tau,y) - w^\delta(\tau,x)| \,\mathrm{d}x\,\mathrm{d}y$$
$$\leq C\epsilon_0^{\gamma_\infty} + C\epsilon_1^{\gamma_\infty}. \tag{2.52}$$

In a similar way, defining

$$E_1 := \frac{1}{\Delta t} \int_{D_T^2} \frac{|w_{\Delta x} - w^\delta|}{a_{\Delta x}} \omega_{\epsilon_1}\omega_{\epsilon_0}(t-s-\Delta t)\mathbf{1}_{[\nu,\nu+\Delta t)}(t)\,\mathrm{d}\underline{z}$$
$$= \frac{1}{\Delta t} \int_{\nu}^{\nu+\Delta t} \int_0^T \int_{D^2} \frac{|w_{\Delta x} - w^\delta|}{a_{\Delta x}} \omega_{\epsilon_1} \,\mathrm{d}x\,\mathrm{d}y\,\omega_{\epsilon_0}(t-s-\Delta t)\,\mathrm{d}s\,\mathrm{d}t$$

we obtain

$$\left| E_1 - \|(w_0^\delta - w_{\Delta x}(0,\cdot))/a_{\Delta x}\|_{L^1} \right| \leq C(\epsilon_1^{\gamma_\infty} + \epsilon_0^{\gamma_\infty} + \nu^{\gamma_\infty}). \tag{2.53}$$

Thus, combining the estimates (2.42), (2.48), (2.49) and (2.51)–(2.53), we have

$$\|w_{\Delta x} - w^\delta\|_{L^1}(\tau)$$
$$\leq C\|w_0^\delta - w_{\Delta x}(0,\cdot)\|_{L^1} + C\left(\epsilon_1^{\gamma_\infty} + \epsilon_0^{\gamma_\infty} + \nu^{\gamma_\infty} + \Delta t \epsilon_0^{\gamma_\infty - 2} + (\delta^\alpha + \epsilon_1^\alpha)\epsilon_0^{\gamma_\infty - 1} + \Delta x \epsilon_1^{\gamma_\infty - 2}\right).$$

We let $\delta \to 0$,

$$\|w_{\Delta x} - w\|_{L^1}(\tau) \leq C\left(\|w_0 - w_{\Delta x}(0,\cdot)\|_{L^1} + \epsilon_1^{\gamma_\infty} + \epsilon_0^{\gamma_\infty} + \nu^{\gamma_\infty} + \Delta t \epsilon_0^{\gamma_\infty - 2} + \epsilon_1^\alpha \epsilon_0^{\gamma_\infty - 1} + \Delta x \epsilon_1^{\gamma_\infty - 2}\right). \tag{2.54}$$

We choose in this last expression $\nu = 3\epsilon_0$, $\epsilon_1 = \epsilon_0^{1/\alpha}$ and $\epsilon_1 = \Delta x^{1/(\alpha\gamma_\infty + 2 - \gamma_\infty)}$ to obtain the rate. □

REMARK 2.12 Note that the above lemma implies a rate of convergence in $L^1(D)$ of at least $\min\{\alpha, {}^{(\gamma_\infty\alpha)}/_{(\gamma_\infty\alpha+2-\gamma_\infty)}\}$ for the variable $u_{\Delta x} = w_{\Delta x}/a_{\Delta x}$.

2.4 *A convergence rate in L^2*

The main ideas for proving a rate of convergence in L^2 are similar to those in Section 2.3, an additional tool involved is a type of Grönwall inequality, moreover the whole procedure is a bit more technical. We start by noting that the (smooth) solution to (2.7) satisfies

$$\int_{D_T} \left(\frac{|w^\delta(s,y) - k|^2}{a^\delta(y)} \partial_s \Omega + |w^\delta(s,y) - k|^2 \partial_y \Omega \right) \mathrm{d}y\,\mathrm{d}s = 0, \tag{2.55}$$

whereas the approximations $w_{\Delta x}$ satisfy, by the discrete entropy inequality (2.23),

$$\int_{D_T} \left(\frac{|w_{\Delta x}(t,x) - \ell|^2}{a_{\Delta x}(x)} D_t^- \Omega + |w_{\Delta x}(t,x) - \ell|^2 D_x^+ \Omega \right) \mathrm{d}x\, \mathrm{d}t \geq 0. \tag{2.56}$$

Then we can prove

THEOREM 2.13 Let $a \in C^{0,\alpha}(\mathbb{R})$ satisfy (2.2). Denote w the solution of (2.3) obtained as the limit as $\delta \to 0$ in (2.7) and $w_{\Delta x}$ the numerical approximation computed by scheme (2.20) and defined in (2.31). Assume that the initial data $w_0 \in L^1(D)$ are Hölder continuous with exponent $\gamma_\infty > 0$. Then $w_{\Delta x}(t,\cdot)$ converges to the solution $w(t,\cdot)$, $0 < t < T$, at (at least) the rate

$$\|(w - w_{\Delta x})(\tau,\cdot)\|_{L^2(D)} \leq C\, \Delta x^{(\gamma_\infty \alpha)/(\gamma_\infty \alpha + 2 - \gamma_\infty)} + C\|w_0 - w_{\Delta x}(0,\cdot)\|_{L^2(D)}, \tag{2.57}$$

where C is a constant depending on $\underline{a}$, $\overline{a}$, $\|a\|_{C^{0,\alpha}}$ and T, but not on Δx.

Proof. As in the L^1-case, we insert $w_{\Delta x}(t,x)$ for k and $w^\delta(s,y)$ for ℓ in (2.55) and (2.56), and integrate the respective equations over $(t,x) \in D_T$ and $(s,y) \in D_T$, respectively. Then adding up, we have (for convenience, we will again omit writing the arguments of $w_{\Delta x} = w_{\Delta x}(t,x)$, $w^\delta = w_{\Delta x}(s,y)$, $a^\delta = a^\delta(y)$ and $a_{\Delta x} = a_{\Delta x}(x)$ in the following)

$$\int_{D_T^2} \left(|w_{\Delta x} - w^\delta|^2 \left(\frac{D_t^- \Omega}{a_{\Delta x}} + \frac{\partial_s \Omega}{a^\delta} \right) + |w_{\Delta x} - w^\delta|^2 \left(D_x^+ \Omega + \partial_y \Omega \right) \right) \mathrm{d}\underline{z} \geq 0, \tag{2.58}$$

where $\mathrm{d}\underline{z} := \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t\, \mathrm{d}s$. By (2.41), we can rewrite equation (2.58) as

$$\begin{aligned}
&\mathrm{A} + \mathrm{B} + \mathrm{D} + \mathrm{E} \\
&\quad := \int_{D_T^2} \frac{|w_{\Delta x} - w^\delta|^2}{a_{\Delta x}} \mathbf{1}_{[\nu,\tau)}(t) \omega_{\epsilon_1} \left(\partial_s \omega_{\epsilon_0} - D_s^+ \omega_{\epsilon_0} \right) \mathrm{d}\underline{z} \\
&\qquad + \int_{D_T^2} \frac{|w_{\Delta x} - w^\delta|^2}{a_{\Delta x} a^\delta} \mathbf{1}_{[\nu,\tau)}(t) \omega_{\epsilon_1} \partial_s \omega_{\epsilon_0} \left(a_{\Delta x} - a^\delta \right) \mathrm{d}\underline{z} \\
&\qquad + \int_{D_T^2} |w_{\Delta x} - w^\delta|^2 \mathbf{1}_{[\nu,\tau)}(t) \omega_{\epsilon_0} \left(\partial_y \omega_{\epsilon_1} + D_x^+ \omega_{\epsilon_1} \right) \mathrm{d}\underline{z} \\
&\qquad + \int_{D_T^2} \frac{|w_{\Delta x} - w^\delta|^2}{a_{\Delta x}} \omega_{\epsilon_1} \omega_{\epsilon_0} (t - s - \Delta t) D_t^- \mathbf{1}_{[\nu,\tau)}(t)\, \mathrm{d}\underline{z} \\
&\quad \geq 0,
\end{aligned} \tag{2.59}$$

and by (2.43), this is equivalent to

$$\frac{1}{\Delta t}\int_{D_T^2}\frac{|w_{\Delta x}-w^{\delta}|^2}{a_{\Delta x}}\omega_{\epsilon_1}\omega_{\epsilon_0}(t-s-\Delta t)\mathbf{1}_{[\tau,\tau+\Delta t)}(t)\,\mathrm{d}\underline{z}$$
$$\leq \frac{1}{\Delta t}\int_{D_T^2}\frac{|w_{\Delta x}-w^{\delta}|^2}{a_{\Delta x}}\omega_{\epsilon_1}\omega_{\epsilon_0}(t-s-\Delta t)\mathbf{1}_{[\nu,\nu+\Delta t)}(t)\,\mathrm{d}\underline{z}+\mathrm{A}+\mathrm{B}+\mathrm{D}. \tag{2.60}$$

We start with the term A. To do so, we use again (2.45) and that, similarly to (2.46), it holds,

$$\frac{1}{\Delta t}\int_0^{\Delta t}\int_{D_T^2}\frac{|w_{\Delta x}-w^{\delta}(t,y)|^2}{a_{\Delta x}}\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_1}(\xi-\Delta t)\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)\,\mathrm{d}\underline{z}\,\mathrm{d}\xi=0. \tag{2.61}$$

Hence we can rewrite the term A as

$$\mathrm{A}=\frac{1}{\Delta t}\int_0^{\Delta t}\int_{D_T^2}\frac{(|w_{\Delta x}-w^{\delta}|^2-|w_{\Delta x}-w^{\delta}(t,y)|^2)}{a_{\Delta x}}\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_1}(\xi-\Delta t)\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)\,\mathrm{d}\underline{z}\,\mathrm{d}\xi, \tag{2.62}$$

and bound in the following way, using triangle and Cauchy–Schwarz inequality,

$$|\mathrm{A}|\leq\frac{1}{\Delta t}\int_0^{\Delta t}\int_{D_T^2}\frac{|w^{\delta}-w^{\delta}(t,y)||2w_{\Delta x}-w^{\delta}-w^{\delta}(t,y)|}{a_{\Delta x}}\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_1}|\xi-\Delta t|\left|\partial_{ss}\omega_{\epsilon_0}\right|\mathrm{d}\underline{z}\,\mathrm{d}\xi,$$
$$\leq\frac{1}{\Delta t}\int_0^{\Delta t}\int_{\nu}^{\tau}\int_0^T|\xi-\Delta t|\left(\int_{D^2}\frac{|w^{\delta}-w^{\delta}(t,y)|^2}{a_{\Delta x}}\omega_{\epsilon_1}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}$$
$$\times\left[\left(\int_{D^2}\frac{|w_{\Delta x}-w^{\delta}(t,y)|^2}{a_{\Delta x}}\omega_{\epsilon_1}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}+\left(\int_{D^2}\frac{|w_{\Delta x}-w^{\delta}|^2}{a_{\Delta x}}\omega_{\epsilon_1}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}\right]\left|\partial_{ss}\omega_{\epsilon_0}\right|\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\xi$$
$$\leq C\frac{2\Delta t}{\epsilon_0^2}\int_{\nu}^{\tau}\sup_{s\in[t-2\epsilon_0,t+2\epsilon_0]}\left(\int_{D^2}\frac{|w^{\delta}-w^{\delta}(t,y)|^2}{a_{\Delta x}}\omega_{\epsilon_1}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}\sup_{s\in[t-2\epsilon_0,t+2\epsilon_0]}\left(\int_{D^2}\frac{|w_{\Delta x}-w^{\delta}|^2}{a_{\Delta x}}\omega_{\epsilon_1}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}\mathrm{d}t$$
$$\leq C\frac{\Delta t}{\sqrt{\underline{a}}\epsilon_0^{2-\gamma_\infty}}\int_{\nu}^{\tau}\sup_{s\in[t-2\epsilon_0,t+2\epsilon_0]}\left(\int_{D^2}\frac{|w_{\Delta x}-w^{\delta}|^2}{a_{\Delta x}}\omega_{\epsilon_1}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}\mathrm{d}t.$$

We denote

$$\kappa(t)=\int_0^T\int_{D^2}\frac{|w^{\delta}(s-\Delta t,y)-w_{\Delta x}(t,x)|^2}{a_{\Delta}(x)}\omega_{\epsilon_1}\omega_{\epsilon_0}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$

and observe that

$$\int_{\nu}^{\tau}\sup_{s\in[t-2\epsilon_0,t+2\epsilon_0]}\left(\int_{D^2}\frac{|w_{\Delta x}(t,x)-w^{\delta}(s,y)|^2}{a_{\Delta x}(x)}\omega_{\epsilon_1}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\mathrm{d}t$$
$$\leq\int_{\nu}^{\tau}\left\{\sup_{s\in[t-2\epsilon_0,t+2\epsilon_0]}\left(\int_{D^2}\frac{|w^{\delta}(t,y)-w^{\delta}(s,y)|^2}{a_{\Delta x}}\omega_{\epsilon_1}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\right.$$

$$
\begin{aligned}
&\quad + \left(\int_{D^2} \frac{|w_{\Delta x} - w^\delta(t,y)|^2}{a_{\Delta x}} \omega_{\epsilon_1} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2} \Bigg\} \, \mathrm{d}t \\
&\le CT\epsilon_0^{\gamma_\infty} + \int_\nu^\tau \left(\int_{D^2} \frac{|w_{\Delta x}(t,x) - w^\delta(t,y)|^2}{a_{\Delta x}} \omega_{\epsilon_1} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2} \mathrm{d}t \\
&\le CT\epsilon_0^{\gamma_\infty} + \int_\nu^\tau \Bigg\{ \left(\int_0^T \int_{D^2} \frac{|w_{\Delta x}(t,x) - w^\delta(s-\Delta t,y)|^2}{a_{\Delta x}} \omega_{\epsilon_1}\omega_{\epsilon_0} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s \right)^{1/2} \\
&\quad + \left(\int_0^T \int_{D^2} \frac{|w^\delta(t,y) - w^\delta(s-\Delta t,y)|^2}{a_{\Delta x}} \omega_{\epsilon_1}\omega_{\epsilon_0} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}s \right)^{1/2} \Bigg\} \, \mathrm{d}t \\
&\le CT\epsilon_0^{\gamma_\infty} + \int_\nu^\tau \sqrt{\kappa(t)} \, \mathrm{d}t. \qquad (2.63)
\end{aligned}
$$

Therefore, the term A can be bounded as

$$
|A| \le C \frac{\Delta t}{\epsilon_0^{2-\gamma_\infty}} \left\{ \epsilon_0^{\gamma_\infty} + \int_\nu^\tau \sqrt{\kappa(t)} \, \mathrm{d}t \right\}. \qquad (2.64)
$$

We proceed to estimating the term B:

$$
B := \int_{D_T^2} \frac{|w_{\Delta x} - w^\delta|^2}{a_{\Delta x} a^\delta} \mathbf{1}_{[\nu,\tau)}(t) \omega_{\epsilon_1} \partial_s \omega_{\epsilon_0} \left(a_{\Delta x} - a^\delta \right) \, \mathrm{d}\underline{z}.
$$

We again use that

$$
\int_{D_T^2} \frac{|w_{\Delta x} - w^\delta(t,y)|^2}{a_{\Delta x} a^\delta} \mathbf{1}_{[\nu,\tau)}(t) \omega_{\epsilon_1} \partial_s \omega_{\epsilon_0} \left(a_{\Delta x} - a^\delta \right) \, \mathrm{d}\underline{z} = 0,
$$

which admits us to rewrite the term B and estimate as follows:

$$
\begin{aligned}
|B| &\le \int_{D_T^2} \frac{|w^\delta - w^\delta(t,y)||2w_{\Delta x} - w^\delta - w^\delta(t,\cdot)|}{a_{\Delta x} a^\delta} \mathbf{1}_{[\nu,\tau)}(t) \omega_{\epsilon_1} |\partial_s \omega_{\epsilon_0}| \left| a_{\Delta x} - a^\delta \right| \mathrm{d}\underline{z} \\
&\le \frac{1}{\underline{a}} \int_\nu^\tau \int_0^T \int_{D^2} \frac{|w^\delta - w^\delta(t,y)||2w_{\Delta x} - w^\delta - w^\delta(t,\cdot)|}{a_{\Delta x}} \omega_{\epsilon_1} |\partial_s \omega_{\epsilon_0}| \left(\left| a_{\Delta x} - a^\delta(x) \right| + \left| a^\delta(x) - a^\delta \right| \right) \mathrm{d}\underline{z} \\
&\le \frac{1}{\underline{a}} \left(\|a_{\Delta x} - a^\delta\|_\infty + \sup_{|x-y|\le 2\epsilon_1} |a^\delta(x) - a^\delta(y)| \right) \\
&\quad \times \int_\nu^\tau \int_0^T \int_{D^2} \frac{|w^\delta - w^\delta(t,y)||2w_{\Delta x} - w^\delta - w^\delta(t,\cdot)|}{a_{\Delta x}} \omega_{\epsilon_1} |\partial_s \omega_{\epsilon_0}| \, \mathrm{d}\underline{z}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\frac{\Delta x^{\alpha}+\delta^{\alpha}+\epsilon_1^{\alpha}}{\underline{a}}\int_{\nu}^{\tau}\int_0^T\int_{D^2}\frac{|w^{\delta}-w^{\delta}(t,y)||2w_{\Delta x}-w^{\delta}-w^{\delta}(t,\cdot)|}{a_{\Delta x}}\omega_{\epsilon_1}|\partial_s\omega_{\epsilon_0}|\,\mathrm{d}\underline{z}\\
&\leq C\frac{\Delta x^{\alpha}+\delta^{\alpha}+\epsilon_1^{\alpha}}{\underline{a}^{3/2}}\int_{\nu}^{\tau}\sup_{s\in[t-\epsilon_0,t+\epsilon_0]}\left(\int_{D^2}|w^{\delta}-w^{\delta}(t,y)|^2\omega_{\epsilon_1}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\\
&\quad\times\sup_{s\in[t-\epsilon_0,t+\epsilon_0]}\left(\int_{D^2}\frac{|w_{\Delta x}-w^{\delta}|^2}{a_{\Delta x}}\omega_{\epsilon_1}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\int_0^T|\partial_s\omega_{\epsilon_0}|\,\mathrm{d}s\,\mathrm{d}t\\
&\leq C\frac{\Delta x^{\alpha}+\delta^{\alpha}+\epsilon_1^{\alpha}}{\epsilon_0^{1-\gamma_\infty}}\int_{\nu}^{\tau}\sup_{s\in[t-\epsilon_0,t+\epsilon_0]}\left(\int_{D^2}\frac{|w_{\Delta x}-w^{\delta}|^2}{a_{\Delta x}}\omega_{\epsilon_1}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\mathrm{d}t\\
&\leq C\frac{\Delta x^{\alpha}+\delta^{\alpha}+\epsilon_1^{\alpha}}{\epsilon_0^{1-\gamma_\infty}}\left(\epsilon_0^{\gamma_\infty}+\int_{\nu}^{\tau}\sqrt{\kappa(t)}\,\mathrm{d}t\right),
\end{aligned}\tag{2.65}
$$

where we have used (2.63) for the last inequality. We continue to estimate the term D. We have using (2.50)

$$
\begin{aligned}
D&=\int_{D_T^2}|w_{\Delta x}-w^{\delta}|^2\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_0}\left(\partial_y\omega_{\epsilon_1}+D_x^+\omega_{\epsilon_1}\right)\,\mathrm{d}\underline{z}\\
&=\frac{1}{\Delta x}\int_0^{\Delta x}\int_{D_T^2}|w_{\Delta x}-w^{\delta}|^2\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_0}(\Delta x-\xi)\partial_{yy}\omega_{\epsilon_1}(x-y+\xi)\,\mathrm{d}\underline{z}\,\mathrm{d}\xi\\
&=\frac{1}{\Delta x}\int_0^{\Delta x}\int_{D_T^2}\left(|w_{\Delta x}-w^{\delta}|^2-|w_{\Delta x}-w^{\delta}(s,x)|^2\right)\mathbf{1}_{[\nu,\tau)}(t)\omega_{\epsilon_0}(\Delta x-\xi)\partial_{yy}\omega_{\epsilon_1}(x-y+\xi)\,\mathrm{d}\underline{z}\,\mathrm{d}\xi.
\end{aligned}
$$

Hence we can estimate the term D by

$$
\begin{aligned}
|D|&\leq\frac{1}{\Delta x}\int_0^{\Delta x}\int_{\nu}^{\tau}\int_{D_T}\int_D|w^{\delta}-w^{\delta}(\cdot,x)||2w_{\Delta x}-w^{\delta}-w^{\delta}(\cdot,x)|\omega_{\epsilon_0}|\Delta x-\xi|\left|\partial_{yy}\omega_{\epsilon_1}\right|\,\mathrm{d}\underline{z}\,\mathrm{d}\xi\\
&\leq 2\int_0^{\Delta x}\int_{\nu}^{\tau}\sup_{|h|\leq 2\epsilon_1}\left(\int_{D_T}|w^{\delta}(\cdot,x+h)-w^{\delta}(\cdot,x)|^2\omega_{\epsilon_0}\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac12}\\
&\quad\times\sup_{|h|\leq 2\epsilon_1}\left(\int_{D_T}|w_{\Delta x}-w^{\delta}(\cdot,x+h)|^2\omega_{\epsilon_0}\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac12}\int_D\left|\partial_{yy}\omega_{\epsilon_1}\right|\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}\xi\\
&\leq C\frac{\Delta x}{\epsilon_1^{2-\gamma_\infty}}\int_{\nu}^{\tau}\sup_{|h|\leq 2\epsilon_1}\left(\int_{D_T}|w_{\Delta x}-w^{\delta}(\cdot,x+h)|^2\omega_{\epsilon_0}\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac12}\mathrm{d}t.
\end{aligned}
$$

We have

$$
\int_{\nu}^{\tau}\sup_{|h|\leq 2\epsilon_1}\left(\int_{D_T}|w_{\Delta x}(t,x)-w^{\delta}(s,x+h)|^2\omega_{\epsilon_0}\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2}\mathrm{d}t
$$

$$
\begin{aligned}
&\le \int_\nu^\tau \Bigg\{ \sup_{|h|\le 2\epsilon_1} \left(\int_{D_T} |w^\delta(s,x) - w^\delta(s,x+h)|^2 \omega_{\epsilon_0} \,\mathrm{d}x\,\mathrm{d}s \right)^{1/2} \\
&\quad + \sup_{|h|\le 2\epsilon_1} \left(\int_{D_T} |w_{\Delta x}(t,x) - w^\delta(s,x)|^2 \omega_{\epsilon_0} \,\mathrm{d}x\,\mathrm{d}s \right)^{1/2} \Bigg\} \,\mathrm{d}t \\
&\le C\epsilon_1^{\gamma_\infty} + \int_\nu^\tau \left(\int_{D_T} |w_{\Delta x}(t,x) - w^\delta(s,x)|^2 \omega_{\epsilon_0}\omega_{\epsilon_1} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \right)^{1/2} \mathrm{d}t \\
&\le C\epsilon_1^{\gamma_\infty} + \int_\nu^\tau \Bigg\{ \left(\int_D \int_{D_T} |w^\delta(s,x) - w^\delta(s-\Delta t,y)|^2 \omega_{\epsilon_0}\omega_{\epsilon_1} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \right)^{1/2} \\
&\quad + \left(\int_D \int_{D_T} |w_{\Delta x}(t,x) - w^\delta(s-\Delta t,y)|^2 \omega_{\epsilon_0}\omega_{\epsilon_1} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \right)^{1/2} \Bigg\} \,\mathrm{d}t \\
&\le C(\epsilon_1^{\gamma_\infty} + \epsilon_0^{\gamma_\infty}) + C\overline{a} \int_\nu^\tau \sqrt{\kappa(t)}\,\mathrm{d}t, \qquad (2.66)
\end{aligned}
$$

and consequently,

$$
|D| \le C \frac{\Delta x}{\epsilon_1^{2-\gamma_\infty}} \left(\epsilon_1^{\gamma_\infty} + \epsilon_0^{\gamma_\infty} + \int_\nu^\tau \sqrt{\kappa(t)}\,\mathrm{d}t \right). \qquad (2.67)
$$

Summing up, equation (2.60) becomes

$$
\frac{1}{\Delta t} \int_\tau^{\tau+\Delta t} \kappa(t)\,\mathrm{d}t \le \frac{1}{\Delta t} \int_\nu^{\nu+\Delta t} \kappa(t)\,\mathrm{d}t + \mathrm{M}_1 + \mathrm{M}_2 \int_\nu^\tau \sqrt{\kappa(t)}\,\mathrm{d}t, \qquad (2.68)
$$

where

$$
\mathrm{M}_1 = C\left(\frac{\Delta t}{\epsilon_0^{2-2\gamma_\infty}} + \frac{\delta^\alpha + \epsilon_1^\alpha}{\epsilon_0^{1-2\gamma_\infty}} + \frac{\Delta x}{\epsilon_1^{2-2\gamma_\infty}} + \frac{\Delta x\,\epsilon_0^{\gamma_\infty}}{\epsilon_1^{2-\gamma_\infty}} \right), \quad \mathrm{M}_2 = C\left(\frac{\Delta t}{\epsilon_0^{2-\gamma_\infty}} + \frac{\delta^\alpha + \epsilon_1^\alpha}{\epsilon_0^{1-\gamma_\infty}} + \frac{\Delta x}{\epsilon_1^{2-\gamma_\infty}} \right).
$$

We choose ν and τ such that $\tau/\Delta t, \nu/\Delta t \in \mathbb{N}$, i.e., $\nu = N_1\Delta t$ and $\tau = N_2\Delta t$ for some $N_1, N_2 \in \mathbb{N}$ and notice that

$$
\int_{k\Delta t}^{(k+1)\Delta t} \sqrt{\kappa(t)}\,\mathrm{d}t \le \Delta t \sqrt{\frac{1}{\Delta t} \int_{k\Delta t}^{(k+1)\Delta t} \kappa(t)\,\mathrm{d}t} := \Delta t\, X_k.
$$

Hence we can rewrite equation (2.68) as

$$
X_{N_2}^2 \le X_{N_1}^2 + \mathrm{M}_1 + \Delta t \mathrm{M}_2 \sum_{i=N_1}^{N_2-1} X_i.
$$

Now we use the following simple adaption of Dragomir (2003, Theorem 5, page 4):

LEMMA 2.14 Let $X_0 \in \mathbb{R}_{\geq 0}, X_k \geq 0, k = 1, \ldots, N$ for some $N \in \mathbb{N}$ satisfy

$$X_k^2 \leq X_0^2 + C_1 + C_2 \sum_{i=0}^{k} X_i, \tag{2.69}$$

for all $k \in \{1, \ldots, N\}$, for some $C_1, C_2 \geq 0$. Then

$$X_k \leq X_0 + \sqrt{C_1} + C_2 k.$$

Using this lemma with $C_1 = \mathrm{M}_1$ and $C_2 = \Delta t \mathrm{M}_2$, we obtain the estimate

$$X_{N_2} \leq X_{N_1} + \sqrt{\mathrm{M}_1} + \Delta t \mathrm{M}_2 (N_2 - N_1) = X_{N_1} + \sqrt{\mathrm{M}_1} + T\mathrm{M}_2. \tag{2.70}$$

Next, we relate the L^2-norm of the difference $(w^\delta - w_{\Delta x})(\tau)$ to X_{N_2}. Indeed,

$$\begin{aligned}
&\left| \|(w^\delta - w_{\Delta x})/\sqrt{a_{\Delta x}}\|_{L^2}(\tau) - X_{N_2} \right| \\
&\quad \leq \left(\frac{1}{\Delta t} \int_\tau^{\tau+\Delta t} \int_{D_T} \int_D \frac{|w^\delta(\tau, x) - w^\delta(s - \Delta t, y)|^2}{a_{\Delta x}} \omega_{\epsilon_1} \omega_{\epsilon_0} \, \mathrm{d}\underline{z} \right)^{1/2} \\
&\quad \leq C(\epsilon_0^{\gamma_\infty} + \epsilon_1^{\gamma_\infty}).
\end{aligned}$$

In a similar way, we can show

$$\left| \|(w_0^\delta - w_{\Delta x}(0, \cdot))/\sqrt{a_{\Delta x}}\|_{L^2} - X_{N_1} \right| \leq C(\epsilon_0^{\gamma_\infty} + \epsilon_1^{\gamma_\infty} + \nu^{\gamma_\infty}),$$

and therefore, with (2.70),

$$\|w^\delta - w_{\Delta x}\|_{L^2}(\tau) \leq C \left(\|w_0^\delta - w_{\Delta x}(0, \cdot)\|_{L^2} + \epsilon_1^{\gamma_\infty} + \epsilon_0^{\gamma_\infty} + \nu^{\gamma_\infty} + \sqrt{\mathrm{M}_1} + T\mathrm{M}_2 \right).$$

Letting $\delta \to 0$ and inserting the definitions of M_1 and M_2, this is

$$\begin{aligned}
\|w - w_{\Delta x}\|_{L^2}(\tau) \leq C \Bigg(&\|w_0 - w_{\Delta x}(0, \cdot)\|_{L^2} + \epsilon_1^{\gamma_\infty} + \epsilon_0^{\gamma_\infty} + \nu^{\gamma_\infty} + \frac{\Delta t^{1/2}}{\epsilon_0^{1-\gamma_\infty}} \\
&+ \frac{\epsilon_1^{\alpha/2}}{\epsilon_0^{1/2-\gamma_\infty}} + \frac{\Delta x^{1/2}}{\epsilon_1^{1-\gamma_\infty}} + \frac{\Delta x^{1/2}\, \epsilon_0^{\gamma_\infty/2}}{\epsilon_1^{1-\gamma_\infty/2}} + \frac{\Delta t}{\epsilon_0^{2-\gamma_\infty}} + \frac{\epsilon_1^{\alpha}}{\epsilon_0^{1-\gamma_\infty}} + \frac{\Delta x}{\epsilon_1^{2-\gamma_\infty}} \Bigg).
\end{aligned}$$

Now we can choose $\nu = 3\epsilon_0$, $\epsilon_0 = \epsilon_1^\alpha$ and $\epsilon_1 = \Delta x^{1/(\alpha\gamma_\infty + 2 - \gamma_\infty)}$ to balance the errors and finally obtain (2.57). □

Proof of Lemma 2.14. Define $Y_k := X_0^2 + C_1 + C_2 \sum_{i=0}^{k} X_i$. Then by (2.69), $X_k^2 \leq Y_k$. Moreover, subtracting the expression for Y_{k-1} from the expression for Y_k, we have

$$Y_k - Y_{k-1} = C_2 X_k \leq C_2 \sqrt{Y_k} \leq C_2 \left(\sqrt{Y_k} + \sqrt{Y_{k-1}} \right).$$

Since $Y_k - Y_{k-1} = (\sqrt{Y_k} - \sqrt{Y_{k-1}})(\sqrt{Y_k} + \sqrt{Y_{k-1}})$, we can divide both sides of the above equation by $\sqrt{Y_k} + \sqrt{Y_{k-1}}$ to obtain

$$\sqrt{Y_k} - \sqrt{Y_{k-1}} \leq C_2.$$

Using induction over k, we obtain

$$\sqrt{Y_k} \leq \sqrt{Y_0} + C_2 k.$$

Hence

$$X_k \leq \sqrt{Y_k} \leq \sqrt{Y_0} + C_2 k = \sqrt{X_0 + C_1} + C_2 k,$$

by the definition of Y_k. Using that $\sqrt{a^2 + b^2} \leq |a| + |b|$, this proves the claim. □

2.5 *Experimental rates for the advection equation*

In this section, we run a few numerical experiments to compare the theoretically established rates with experimentally observed ones. As a model coefficient a, we choose a sample (single realization) of a log-normally distributed random field, which was generated using a spectral Fast Fourier Transform (FFT) method (Pardo-Iguzquiza & Chica-Olmo, 1993; Chiles & Delfiner, 1997; Ravalec *et al.*, 2000; Müller *et al.*, 2013) from a given covariance operator $\hat{c}$ which we assume to be log-normal, so that the covariance operator completely determines the law of $\hat{c}$. It is easy to check that this coefficient a is uniformly positive, bounded from above and Hölder continuous with exponent $1/2$. See Fig. 1 for an illustration of the coefficient. For the function w_0, we choose the product of the Lipschitz continuous hat

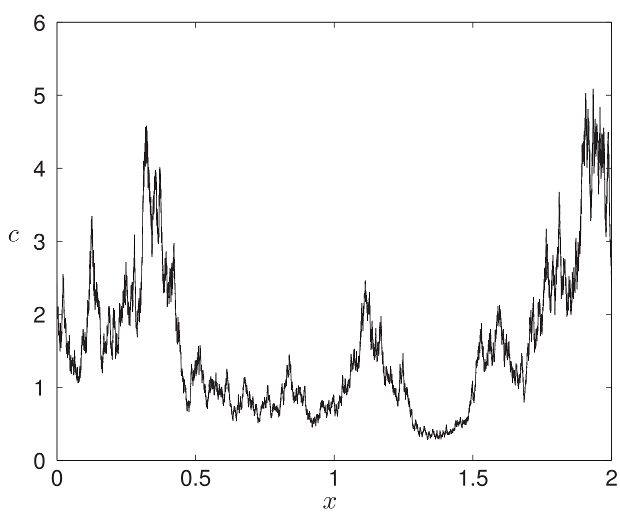

FIG. 1. The coefficient a used for the numerical experiments for the transport equation (2.3).

function

$$h(x) = \begin{cases} 1 + 2(x - 1/2), & x \in [0, 0.5), \\ 1 - 2(x - 1/2) & x \in [0.5, 1), \\ 1 + 2(x - 3/2), & x \in [1, 1.5), \\ 1 - 2(x - 3/2), & x \in [1.5, 2), \end{cases}$$

with a modification of the Weierstrass function for different parameters γ:

$$f^{\gamma}(x) = \sum_{n=1}^{\infty} 2^{-\gamma n} \cos(2^n \pi x), \quad \gamma \in (0, 1), \tag{2.71}$$

that is, we add a constant $f_0 > 0$ such that f^{γ} becomes strictly positive, and truncate f^{γ} after $N = 400$ terms,

$$\widetilde{f}^{\gamma}(x) = \sum_{n=1}^{400} 2^{-\gamma n} \cos(2^n \pi x) + f_0, \quad \alpha \in (0, 1), \tag{2.72}$$

and define $w_{0,\gamma}(x) = h(x)\widetilde{f}^{\gamma}(x)$. It can be shown that (2.71) is nowhere differentiable, but Hölder continuous with exponent γ. As a computational domain, we take $D = [0, 2]$ with periodic boundary conditions. We run experiments up to time $T = 1$ with CFL-number $\theta = 0.4/\bar{a}$ with initial data $w_{0,\gamma}$ for $\gamma = 1/2, 1/4, 1/8$ and for $w_{0,0}(x) := h(x)$.

To approximate the coefficient, we interpolate (2.72) and a on a grid with mesh width $\Delta x = 2^{-14}$ and average it to obtain an approximation on the coarser grids. The reference solution has been computed on a grid with $N_x = 2^{14}$ mesh points. We have used the following approximation for the numerical convergence rate

$$r^m = \frac{1}{N_{\exp} - 1} \sum_{k=1}^{N_{\exp}-1} \frac{\log \mathcal{E}^m_{\Delta x_k} - \log \mathcal{E}^m_{\Delta x_{k-1}}}{\log 2}, \quad m = 1, 2, \tag{2.73}$$

where $\Delta x_k = 2^{-k} \Delta x_0$ and $\mathcal{E}^m_{\Delta x_k}$, the relative distance of the approximation with gridsize Δx_k to the reference solution in the discrete L^m-norm, that is,

$$\mathcal{E}^m_{\Delta x_k} = 100 \times \frac{\sum_{j=1}^{N_x} |u_{\Delta x_k}(T, x_j) - u_{\Delta x_{\text{ref}}}(T, x_j)|^m}{\sum_{j=1}^{N_x} |u_{\Delta x_{\text{ref}}}(T, x_j)|^m}. \tag{2.74}$$

We used $\Delta x_0 = 1/16$ ($N_{x,0} = 32$) and $N_{\exp} = 6$. In Fig. 2, we have plotted the Weierstrass function and the reference solution for Hölder exponent $\gamma = 1/2$. Interestingly, the variable w seems to be much smoother at time $T = 1$ than initially, and is also much smoother than the variable u. This is probably due to the diffusion in the scheme. In Table 1 the experimentally observed rates are computed for initial data $w_{0,\gamma}$ and $w_{0,0}$ for $\gamma = 2^{-k}$, $k = 2, 4, 8$. We notice that the experimental rates for this example are low, but better than what we obtain from the theoretical estimates. This can be due to the fact that we compute the errors with respect to a reference solution computed by the same scheme. Moreover, other

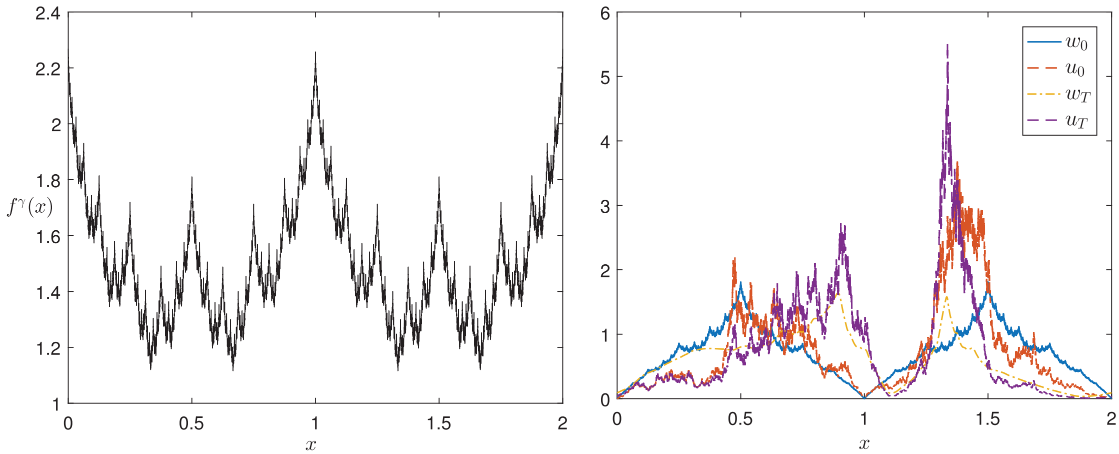

FIG. 2. Left: Approximation of Weierstrass function (2.72) for $\gamma = 1/2$. Right: Approximation of (2.3) by scheme (2.20) at time $T = 0$ and $T = 1$, $N_x = 2^{14}$, $\gamma = 1/2$.

TABLE 1 *Experimental rates*

γ	r_u^1	r_w^1	r_u^2	r_w^2
1	0.6018	0.5598	0.6468	0.5829
1/2	0.5170	0.4554	0.5400	0.4996
1/4	0.4412	0.3816	0.4678	0.4356
1/8	0.4550	0.3970	0.4810	0.4484

examples of initial data might give lower rates. However, we do not know whether the rates (2.39) and (2.57) are sharp.

3. A convergence rate for the wave equation in one space dimension

The techniques from the last section can be used to prove a rate of convergence for approximate solutions to the acoustic wave equation in one space dimension with rough coefficient under some assumptions. Defining $u := \partial_x p$ and $v := \partial_t p$, the second-order wave equation

$$\frac{1}{a(x)}\partial_{tt}^2 p(t,x) - \partial_{xx}^2 p(t,x) = 0, \quad (t,x) \in D_T,$$

$D_T := [0,T] \times D$, where $D = [d_L, d_R]$, $-\infty < d_L < d_R < \infty$, can be rewritten as

$$\begin{aligned} \partial_t u(t,x) - \partial_x v(t,x) &= 0, \\ \frac{1}{a(x)}\partial_t v(t,x) - \partial_x u(t,x) &= 0, \quad (t,x) \in D_T. \end{aligned} \tag{3.1}$$

For simplicity, let us assume that $D = [0,2]$ with periodic boundary conditions.

3.1 *Numerical approximation of* (3.1) *by a finite difference scheme*

In order to compute numerical approximations to (3.1), we choose $\Delta x > 0$ and discretize the spatial domain by a grid with gridpoints $x_{j+1/2} := (j + 1/2)\Delta x$, $j \in \{0, 1, \ldots, N_x\}$, where $N_x \in \mathbb{N}$ is such that $N_x \Delta x = |D|$. Similarly, let Δt denote the time step and $t^n = n\Delta t$ with $n = 0, 1, \ldots, N$ denote the nth time level with $N\Delta t = T$.

We define the averaged quantities

$$a_j = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} a(x)\,\mathrm{d}x, \quad j = 1, \ldots, N_x, \tag{3.2}$$

and

$$\left(u_j^0, v_j^0\right) = \frac{1}{\Delta x}\left(\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\,\mathrm{d}x, \int_{x_{j-1/2}}^{x_{j+1/2}} v_0(x)\,\mathrm{d}x\right), \quad j = 1, \ldots, N_x. \tag{3.3}$$

We recall (2.19) and define approximations to (3.1) by the finite difference scheme:

$$D_t^+ u_j^n = D_x^c v_j^n + \frac{\Delta x}{2} D_x^+ D_x^- u_j^n, \tag{3.4a}$$

$$\frac{D_t^+ v_j^n}{a_j} = D_x^c u_j^n + \frac{\Delta x}{2} D_x^+ D_x^- v_j^n, \quad j \in \mathbb{Z}, n = 1, \ldots, N, \tag{3.4b}$$

with the time step Δt being chosen such that the CFL-condition,

$$2\Delta t \max_j \left\{\max\left\{2a_j + 1, a_j/4 + 5/4\right\}\right\} \leq \Delta x \tag{3.5}$$

is satisfied.

Moreover, for any $k, l \in \mathbb{R}$, we define the discrete entropy (energy) function and flux

$$\eta_j^n := \frac{|u_j^n - k|^2}{2} + \frac{|v_j^n - \ell|^2}{2a_j}, \quad q_j^n := -(u_j^n - k)(v_j^n - \ell).$$

Furthermore, we will for technical reasons need the following difference quotients: we denote for $\gamma \in (0, 1]$ and a discrete quantity σ_j^n defined on the grid,

$$D_{\gamma,t}^{\pm}\sigma_j^n = \mp\frac{\sigma_j^n - \sigma_j^{n\pm 1}}{\Delta t^\gamma}, \quad D_{\gamma,x}^{\pm}\sigma_j^n = \mp\frac{\sigma_j^n - \sigma_{j\pm 1}^n}{\Delta x^\gamma}, \quad D_{\gamma,x}^{c}\sigma_j^n = \frac{\sigma_{j+1}^n - \sigma_{j-1}^n}{2\Delta x^\gamma}. \tag{3.6}$$

When $\gamma = 1$, the above-defined quantities reduce to the usual finite differences (2.19). The scheme (3.4) satisfies the following properties:

LEMMA 3.1 Assume $a \in C^{0,\alpha}(D)$ and $u_0, v_0 \in L^2(D)$. Then the numerical approximations u_j^n and v_j^n defined by (3.4), (3.2) and (3.3) have the following properties:

(i) Discrete entropy inequality:

$$D_t^+\eta_j^n + D_x^c q_j^n \le \frac{\Delta x\,(\Delta t - \Delta x)}{2} D_x^-\left(D_x^+\left(u_j^n - k\right) D_x^+\left(v_j^n - l\right)\right) + \frac{\Delta x}{4} D_x^+ D_x^-\left(\left(u_j^n - k\right)^2 + \left(v_j^n - l\right)^2\right). \tag{3.7}$$

(ii) Bounds on the discrete L^2-norms:

$$\Delta x \sum_j \left((u_j^n)^2 + \frac{1}{a_j}(v_j^n)^2 \right) \le \Delta x \sum_j \left((u_j^0)^2 + \frac{1}{a_j}(v_j^0)^2 \right) \le \|u_0\|_{L^2}^2 + \left\|a^{-1/2} v_0\right\|_{L^2}^2 . \tag{3.8}$$

(iii) For any function $w = w(x)$, define the L^2 modulus of continuity in space as γ if,

$$\nu_x^2(w,\sigma) := \sup_{\delta \le \sigma} \int_{\mathbb{R}} |w(x+\delta) - w(x)|^2 \, \mathrm{d}x \le C\,\sigma^{2\gamma}. \tag{3.9}$$

If we also assume that the initial data u_0 and v_0 have moduli of continuity in $L^2(D)$,

$$\nu_x^2(u_0,\sigma) \le C\sigma^{2\gamma}, \quad \nu_x^2(v_0,\sigma) \le C\,\sigma^{2\gamma},$$

for some $\gamma > 0$, the approximations satisfy,

$$\Delta x \sum_j \left(\left|D_{\gamma,t}^+ u_j^n\right|^2 + \frac{1}{a_j}\left|D_{\gamma,t}^+ v_j^n\right|^2 \right) \le C,$$
$$\Delta x \sum_j \left(\left|D_{\gamma,x}^c u_j^n\right|^2 + \left|D_{\gamma,x}^c v_j^n\right|^2 + \frac{\Delta x^2}{4}\left(\left|D_{\gamma,x}^+ D_x^- u_j^n\right|^2 + \left|D_{\gamma,x}^+ D_x^- v_j^n\right|^2\right) \right) \le C, \tag{3.10}$$

for all $n = 0, \ldots, N_T$, where C is a constant, depending on a and the initial data u_0 and v_0.

Proof. By linearity, it is sufficient to prove (3.7) for $k = l = 0$. We shall use the following identities

$$u_j^n D_t^+ u_j^n = \frac{1}{2} D_t^+\left(u_j^n\right)^2 - \frac{\Delta t}{2}\left(D_t^+ u_j^n\right)^2, \tag{3.11}$$

$$u_j^n D_x^+ D_x^- u_j^n = \frac{1}{2} D_x^+ D_x^-\left(u_j^n\right)^2 - \frac{1}{2}\left(\left(D_x^- u_j^n\right)^2 + \left(D_x^+ u_j^n\right)^2\right), \tag{3.12}$$

$$D_x^-\left(D_x^+ u_j^n D_x^+ v_j^n\right) = \left(D_x^+ D_x^- u_j^n\right) D_x^c v_j^n + \left(D_x^+ D_x^- v_j^n\right) D_x^c u_j^n, \tag{3.13}$$

$$u_j^n D_x^c v_j^n + v_j^n D_x^c u_j^n = D_x^c\left(u_j^n v_j^n\right) - \frac{\Delta x^2}{2} D_x^-\left(D_x^+ u_j^n D_x^+ v_j^n\right). \tag{3.14}$$

Multiplying (3.4a) by u_j^n and (3.4b) by v_j^n, we get

$$\frac{1}{2} D_t^+ \left(u_j^n\right)^2 - \frac{\Delta t}{2}\left(D_t^+ u_j^n\right)^2 = u_j^n D_x^c v_j^n + \frac{\Delta x}{4} D_x^+ D_x^- \left(u_j^n\right)^2 - \frac{\Delta x}{4}\left(\left(D_x^- u_j^n\right)^2 + \left(D_x^+ u_j^n\right)^2\right)$$

$$\frac{1}{2a_j} D_t^+ \left(v_j^n\right)^2 - \frac{\Delta t}{2a_j}\left(D_t^+ v_j^n\right)^2 = v_j^n D_x^c u_j^n + \frac{\Delta x}{4} D_x^+ D_x^- \left(v_j^n\right)^2 - \frac{\Delta x}{4}\left(\left(D_x^- v_j^n\right)^2 + \left(D_x^+ v_j^n\right)^2\right).$$

Adding these two equations

$$\begin{aligned} D_t^+ \eta_j^n &= D_x^c \left(u_j^n v_j^n\right) - \frac{\Delta x^2}{2} D_x^- \left(D_x^+ u_j^n D_x^+ v_j^n\right) \\ &+ \frac{\Delta x}{4} D_x^+ D_x^- \left(\left(u_j^n\right)^2 + \left(v_j^n\right)^2\right) \\ &- \frac{\Delta x}{4}\left(\left(D_x^- u_j^n\right)^2 + \left(D_x^+ u_j^n\right)^2 + \left(D_x^- v_j^n\right)^2 + \left(D_x^+ v_j^n\right)^2\right) \\ &+ \frac{\Delta t}{2}\left[\underbrace{\left(D_x^c v_j^n + \frac{\Delta x}{2} D_x^+ D_x^- u_j^n\right)^2 + a_j \left(D_x^c u_j^n + \frac{\Delta x}{2} D_x^- D_x^+ v_j^n\right)^2}_{K}\right]. \end{aligned}$$

We can estimate K as follows:

$$\begin{aligned} K &\le \frac{1}{2}\left(\left(D_x^- u_j^n\right)^2 + \left(D_x^+ u_j^n\right)^2 + a_j\left(D_x^- v_j^n\right)^2 + a_j\left(D_x^+ v_j^n\right)^2\right) \\ &\quad + \Delta x \left(D_x^+ D_x^- u_j^n D_x^c v_j^n + a_j D_x^+ D_x^- v_j^n D_x^c u_j^n\right) + \frac{\Delta x^2}{4}\left(\left(D_x^+ D_x^- u_j^n\right)^2 + a_j\left(D_x^+ D_x^- v_j^n\right)^2\right) \\ &\le \left(D_x^- u_j^n\right)^2 + \left(D_x^+ u_j^n\right)^2 + a_j\left(D_x^- v_j^n\right)^2 + a_j\left(D_x^+ v_j^n\right)^2 \\ &\quad + \Delta x \left(D_x^+ D_x^- u_j^n D_x^c v_j^n + a_j D_x^+ D_x^- v_j^n D_x^c u_j^n\right) \\ &= \left(D_x^- u_j^n\right)^2 + \left(D_x^+ u_j^n\right)^2 + a_j\left(D_x^- v_j^n\right)^2 + a_j\left(D_x^+ v_j^n\right)^2 \\ &\quad + \Delta x D_x^+ \left(D_x^- u_j^n D_x^- v_j^n\right) + \Delta x \left(a_j - 1\right) D_x^+ D_x^- v_j^n D_x^c u_j^n \\ &\le \left(D_x^- u_j^n\right)^2 + \left(D_x^+ u_j^n\right)^2 + a_j\left(D_x^- v_j^n\right)^2 + a_j\left(D_x^+ v_j^n\right)^2 \\ &\quad + \Delta x D_x^+ \left(D_x^- u_j^n D_x^- v_j^n\right) \\ &\quad + \frac{1}{2}\left|a_j - 1\right|\left(\left(\left|D_x^+ v_j^n\right| + \left|D_x^- v_j^n\right|\right)^2 + \frac{1}{4}\left(D_x^- u_j^n + D_x^+ u_j^n\right)^2\right) \end{aligned}$$

$$\leq \Delta x D_x^+\left(D_x^- u_j^n D_x^- v_j^n\right) + \left(1 + \frac{1}{4}\left|a_j - 1\right|\right)\left(D_x^- u_j^n\right)^2 + \left(1 + \frac{1}{4}\left|a_j - 1\right|\right)\left(D_x^+ u_j^n\right)^2$$
$$+ \left(a_j + \left|a_j - 1\right|\right)\left(D_x^- v_j^n\right)^2 + \left(a_j + \left|a_j - 1\right|\right)\left(D_x^+ v_j^n\right)^2.$$

This implies that

$$\begin{aligned}
D_t^+\eta_j^n + D_x^c q_j^n \leq\ & \frac{\Delta x\left(\Delta t - \Delta x\right)}{2} D_x^-\left(D_x^+ u_j^n D_x^+ v_j^n\right) \\
&+ \frac{\Delta x}{4} D_x^+ D_x^-\left(\left(u_j^n\right)^2 + \left(v_j^n\right)^2\right) \\
&+ \frac{1}{2}\left(\left(1 + \frac{1}{4}\left|a_j - 1\right|\right)\Delta t - \frac{\Delta x}{2}\right)\left(D_x^- u_j^n\right)^2 \\
&+ \frac{1}{2}\left(\left(1 + \frac{1}{4}\left|a_j - 1\right|\right)\Delta t - \frac{\Delta x}{2}\right)\left(D_x^+ u_j^n\right)^2 \\
&+ \frac{1}{2}\left(\left(a_j + \left|a_j - 1\right|\right)\Delta t - \frac{\Delta x}{2}\right)\left(D_x^- v_j^n\right)^2 \\
&+ \frac{1}{2}\left(\left(a_j + \left|a_j - 1\right|\right)\Delta t - \frac{\Delta x}{2}\right)\left(D_x^+ v_j^n\right)^2.
\end{aligned}$$

If Δt satisfies the CFL-condition (3.5), the four last terms above are nonpositive and (3.7) follows. The L^2 bound (3.8) also follows upon summing over j and multiplying by Δx.

By the linearity of the equation, (3.8) also holds for the difference of two approximations computed by (3.4a) and (3.4b), thus in particular for $D_{\gamma,t}^+ u_j^n$ and $D_{\gamma,t}^+ v_j^n$. Hence, using the handy equality

$$\begin{aligned}
&\sum_j\left(\left|D_t^+ u_j^n\right|^2 + \frac{1}{a_j^2}\left|D_t^+ v_j^n\right|^2\right) \\
&\quad = \sum_j\left(\left|D_x^c u_j^n\right|^2 + \left|D_x^c v_j^n\right|^2 + \frac{\Delta x^2}{4}\left(\left|D_x^+ D_x^- u_j^n\right|^2 + \left|D_x^+ D_x^- v_j^n\right|^2\right)\right),
\end{aligned} \tag{3.15}$$

the CFL-condition (3.5), (3.8) implies

$$\begin{aligned}
&\Delta x \sum_j\left(\left(D_{\gamma,t}^+ u_j^n\right)^2 + \frac{1}{a_j}\left(D_{\gamma,t}^+ v_j^n\right)^2\right) \\
&\leq \Delta x \sum_j\left(\left(D_{\gamma,t}^+ u_j^0\right)^2 + \frac{1}{a_j}\left(D_{\gamma,t}^+ v_j^0\right)^2\right) \\
&\leq \max\{1, \overline{a}\}\Delta x \sum_j\left(\left(D_{\gamma,t}^+ u_j^0\right)^2 + \frac{1}{a_j^2}\left(D_{\gamma,t}^+ v_j^0\right)^2\right) \\
&= \max\{1, \overline{a}\}\Delta x \Delta t^{2-2\gamma}\sum_j\left(\left(D_x^c u_j^0\right)^2 + \left(D_x^c v_j^0\right)^2\right.
\end{aligned} \tag{3.16}$$

$$+\frac{\Delta x^2}{4}\left(D_x^+D_x^-u_j^0\right)^2+\left(D_x^+D_x^-v_j^0\right)^2\Bigg)$$

$$\le \max\{1,\overline{a}\}\Delta x\,\theta^{2-2\gamma}\sum_j\Bigg(\left(D^c_{\gamma,x}u_j^0\right)^2+\left(D^c_{\gamma,x}v_j^0\right)^2$$

$$+\frac{\Delta x^2}{4}\left(\left(D^+_{\gamma,x}D_x^-u_j^0\right)^2+\left(D^+_{\gamma,x}D_x^-v_j^0\right)^2\right)\Bigg)$$

$$\le 2\theta^{2-2\gamma}\max\{1,\overline{a}\}\Delta x\sum_j\left(\left(D^+_{\gamma,x}u_j^0\right)^2+\left(D^+_{\gamma,x}v_j^0\right)^2\right)=:C(\alpha,u_0,v_0),$$

where we have set $\theta=\Delta t/\Delta x$. Applying (3.15) once more, we also obtain the second equation in (3.10),

$$\Delta x\sum_j\left(\left(D^c_{\gamma,x}u_j^n\right)^2+\left(D^c_{\gamma,x}v_j^n\right)^2+\frac{\Delta x^2}{4}\left(\left(D^+_{\gamma,x}D_x^-u_j^n\right)^2+\left(D^+_{\gamma,x}D_x^-v_j^n\right)^2\right)\right)$$

$$=\theta^{2\gamma-2}\Delta x\sum_j\left(\left(D^+_{\gamma,t}u_j^n\right)^2+\frac{1}{a_j^2}\left(D^+_{\gamma,t}v_j^n\right)^2\right)\le C(\alpha,u_0,v_0). \tag{3.17}$$

□

Defining

$$u_{\Delta x}(t,x)=u_j^n,\quad (t,x)\in[t^n,t^{n+1})\times[x_{j-1/2},x_{j+1/2}), \tag{3.18a}$$

$$v_{\Delta x}(t,x)=v_j^n,\quad (t,x)\in[t^n,t^{n+1})\times[x_{j-1/2},x_{j+1/2}), \tag{3.18b}$$

$$a_{\Delta x}(x)=a_j,\quad x\in[x_{j-1/2},x_{j+1/2}), \tag{3.18c}$$

we have that a subsequence of $(u_{\Delta x},v_{\Delta x})_{\Delta x>0}$ converges weakly to a weak solution of (3.1). If $\gamma=1$, equation (3.10) implies that u and v have an L^2-modulus of continuity of $\gamma=1$ in space and time, and therefore by Ladyzhenskaya's theorems of interpolation of finite difference approximations (Ladyzhenskaya, 1985, Lemmas 3.1 and 3.2, Theorem 3.2) we get a strongly convergent subsequence to limit functions $u,v\in H^1(D_T)\cap \mathrm{Lip}([0,T];L^2(D))$. The limit functions satisfy the entropy inequality

$$\partial_t\eta(u-k,v-\ell,a)+\partial_x q(u-k,v-\ell)\le 0,\ \text{ in the sense of distributions,} \tag{3.19}$$

where

$$\eta(u,v,a):=\frac{u^2}{2}+\frac{v^2}{2a},\quad q(u,v):=-uv, \tag{3.20}$$

which follows from (3.8) in the limit $\Delta x\to 0$. They are therefore unique among solutions satisfying the entropy inequality (thanks to the linearity of the equation, we can insert another solution $(\widetilde{u},\widetilde{v})$ for (k,ℓ)).

3.2 *Convergence rate for the one-dimensional wave equation*

In the last section, we showed that the numerical scheme (3.4) converges to the weak solution of the one-dimensional wave equation. However, the key question is the rate at which the approximate solutions converge to the exact solution as the mesh is refined, i.e., $\Delta x \to 0$. The answer to this question is provided in the following theorem,

THEOREM 3.2 Let $a \in C^{0,\alpha}(\overline{D})$ satisfy $\infty > \overline{a} \geq a(x) \geq \underline{a} > 0$ for all $x \in D$. Denote by (u, v) the solution of (3.1) and $(u_{\Delta x}, v_{\Delta x})$ the numerical approximation computed by the scheme (3.4) and defined in (3.18). Assume that the initial data $u_0, v_0 \in L^2(D)$ and that $u, v, u_{\Delta x}, v_{\Delta x}$ have moduli of continuity

$$\begin{aligned} \nu_x^2(u(t,\cdot),\sigma) &\leq C\,\sigma^{2\gamma}, \quad \nu_x^2(v(t,\cdot),\sigma) \leq C\,\sigma^{2\gamma}, \\ \nu_x^2(u_{\Delta x}(t,\cdot),\sigma) &\leq C\,\sigma^{2\gamma}, \quad \nu_x^2(v_{\Delta x}(t,\cdot),\sigma) \leq C\,\sigma^{2\gamma}. \end{aligned} \tag{3.21}$$

Then the approximation $(u_{\Delta x}(t,\cdot), v_{\Delta x}(t,\cdot))$ converges to the solution $(u(t,\cdot), v(t,\cdot))$, $0 < t < T$, and we have the estimate on the rate

$$\begin{aligned} &\|(u - u_{\Delta x})(t,\cdot)\|_{L^2(D)} + \|(v - v_{\Delta x})(t,\cdot)/a\|_{L^2(D)} \\ &\quad \leq C\left(\|u_0 - u_{\Delta x}(0,\cdot)\|_{L^2(D)} + \|(v_0 - v_{\Delta x}(0,\cdot))/a\|_{L^2(D)} + \Delta x^{(\alpha\gamma)/(2(\alpha\gamma+1-\gamma))}\right), \end{aligned} \tag{3.22}$$

where C is a constant depending on c and T, but not on Δx.

REMARK 3.3 If the initial data u_0, v_0 have moduli of continuity

$$\nu_x^2(u_0,\sigma) \leq C\sigma^2, \quad \nu_x^2(v_0,\sigma) \leq C\sigma^2,$$

it follows from Lemma 3.1 that u, v have moduli of continuity in space and time with $\gamma = 1$.

Proof. We let $\phi \in C_0^2((0,T) \times D)$ and define

$$\Lambda_T(u, v, k, \ell, \phi) := \int_{D_T} \left(\left(\frac{(u-k)^2}{2} + \frac{(v-\ell)^2}{2a}\right)\partial_t\phi - (u-k)(v-\ell)\partial_x\phi\right) \mathrm{d}x\,\mathrm{d}t. \tag{3.23}$$

The above definition is similar to the one used in Section 2.4—an adaptation of the Kružkov doubling of variables technique (Holden & Risebro, 2011) in our current L^2 setting. We will use special test functions in Λ_T: we recall the definition of the mollifier ω_ϵ in (2.6) and define for some $0 < \nu < \tau < T$,

$$\psi^\mu(t) := H_\mu(t-\nu) - H_\mu(t-\tau), \quad H_\mu(t) = \int_{-\infty}^{t} \omega_\mu(\xi)\,\mathrm{d}\xi.$$

We define $\Omega : D_T^2 \to \mathbb{R}$ by

$$\Omega(t,s,x,y) = \psi^\mu(t)\omega_{\epsilon_0}(t-s)\omega_{\epsilon_1}(x-y). \tag{3.24}$$

We choose ν and τ such that $0 < \epsilon_0 < \min\{\nu, T-\tau\}$ and $0 < \mu < \min\{\nu-\epsilon_0, T-\tau-\epsilon_0\}$. We note that

$$\partial_t \Omega + \partial_s \Omega = \partial_t \psi^\mu \omega_{\epsilon_1}\omega_{\epsilon_0}, \quad \partial_x \Omega + \partial_y \Omega = 0.$$

We assume without loss of generality $\Delta x \le \min\{\epsilon_1, \epsilon_0, \nu\}$. By the entropy inequality (3.19), we have for the solution (u, v) of (3.1) that $\Lambda_T(u, v, u_{\Delta x}(s, y), v_{\Delta x}(s, y), \phi) \ge 0$ for all $(s, y) \in D_T$ and test functions $\phi \in C_0^2((0, T) \times D)$. By (3.7), we have on the other hand that

$$\begin{aligned}
&\int_{D_T} \left(\left(\frac{(u_{\Delta x} - u(t,x))^2}{2} + \frac{(v_{\Delta x} - v(t,x))^2}{2a} \right) D_s^- \phi - (u_{\Delta x} - u(t,x))(v_{\Delta x} - v(t,x)) D_y^c \phi \right) \mathrm{d}y\, \mathrm{d}s \\
&\quad \ge \int_{D_T} (v_{\Delta x} - v(t,x))^2 \left(\frac{1}{2a} - \frac{1}{2a_{\Delta x}} \right) D_s^- \phi \,\mathrm{d}y\, \mathrm{d}s \\
&\quad - \frac{\Delta x^2}{2}(\theta - 1) \int_{D_T} \left(D_y^+(u_{\Delta x} - u) D_y^+(v_{\Delta x} - v) \right) D_y^+ \phi \,\mathrm{d}y\, \mathrm{d}s \\
&\quad + \frac{\Delta x}{4} \int_{D_T} (D_y^+(v_{\Delta x} - v(t,x))^2 + D_y^+(u_{\Delta x} - u(t,x))^2) D_y^+ \phi \,\mathrm{d}y\, \mathrm{d}s,
\end{aligned} \tag{3.25}$$

where $D_s^- \phi$ and $D_y^+ \phi$ are defined by

$$D_s^\pm \phi(s, y) = \mp \frac{\phi(s, y) - \phi(s \pm \Delta t, y)}{\Delta t}, \quad D_y^\pm \phi(s, y) = \mp \frac{\phi(s, y) - \phi(s, y \pm \Delta x)}{\Delta x}. \tag{3.26}$$

Adding $\Lambda_T(u, v, u_{\Delta x}(s, y), v_{\Delta x}(s, y), \phi) \ge 0$ and (3.25), choosing Ω as a test function and integrating over D_T (we abbreviate $\mathrm{d}\underline{z} := \mathrm{d}y\, \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}t$), we obtain

$$\begin{aligned}
&\underbrace{\int_{D_T^2} \left(\frac{(u_{\Delta x} - u)^2}{2} + \frac{(v_{\Delta x} - v)^2}{2a} \right) \left(\partial_t \Omega + D_s^- \Omega \right) \mathrm{d}\underline{z}}_{\mathrm{A}} \\
&\quad \underbrace{- \int_{D_T^2} (u_{\Delta x} - u)(v_{\Delta x} - v) \left(\partial_x \Omega + D_y^c \Omega \right) \mathrm{d}\underline{z}}_{\mathrm{B}} \\
&\quad \ge \underbrace{\int_{D_T^2} (v_{\Delta x} - v)^2 \left(\frac{1}{2a(x)} - \frac{1}{2a_{\Delta x}(y)} \right) D_s^- \Omega \,\mathrm{d}\underline{z}}_{\mathrm{D}} \\
&\quad \underbrace{+ \frac{\Delta x^2}{2}(\theta - 1) \int_{D_T^2} D_y^- \left[D_y^+ (u_{\Delta x} - u) D_y^+ (v_{\Delta x} - v) \right] \Omega \,\mathrm{d}\underline{z}}_{\mathrm{E}} \\
&\quad \underbrace{- \frac{\Delta x}{4} \int_{D_T^2} ((v_{\Delta x} - v(t,x))^2 + (u_{\Delta x} - u(t,x))^2) D_y^- D_y^+ \Omega \,\mathrm{d}\underline{z}}_{\mathrm{F}}.
\end{aligned} \tag{3.27}$$

We rewrite the term A as

$$\begin{aligned}\mathrm{A} &= \int_{D_T^2} \eta(u-u_{\Delta x}, v-v_{\Delta x}, a)(\partial_t \Omega + D_s^- \Omega)\,\mathrm{d}\underline{z} \\ &= \underbrace{\int_{D_T^2} \eta(u-u_{\Delta x}, v-v_{\Delta x}, a)\partial_t \psi^\mu \omega_{\epsilon_1}\omega_{\epsilon_0}\,\mathrm{d}\underline{z}}_{\mathrm{A}_1} \\ &\quad + \underbrace{\int_{D_T^2} \eta(u-u_{\Delta x}, v-v_{\Delta x}, a)\psi^\mu\,\omega_{\epsilon_1}\left(\partial_t\omega_{\epsilon_0} + D_s^-\omega_{\epsilon_0}\right)\mathrm{d}\underline{z}}_{\mathrm{A}_2}.\end{aligned}$$

The term A_1 can be written as

$$\mathrm{A}_1 = \int_{D_T^2} \eta(u-u_{\Delta x}, v-v_{\Delta x}, a)\omega_\mu(t-\nu)\omega_{\epsilon_1}\omega_{\epsilon_0}\,\mathrm{d}\underline{z} - \int_{D_T^2} \eta(u-u_{\Delta x}, v-v_{\Delta x}, a)\omega_\mu(t-\tau)\omega_{\epsilon_1}\omega_{\epsilon_0}\,\mathrm{d}\underline{z}.$$

Introducing λ as

$$\lambda(t) = \int_0^T \int_{D^2} \eta\left(u_{\Delta x}(s,y) - u(t,x), v_{\Delta x}(s,y) - v(t,x), a(x)\right) \times \omega_{\epsilon_1}(x-y)\omega_{\epsilon_0}(t-s)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s, \tag{3.28}$$

we have that

$$\mathrm{A}_1 = \int_0^T \lambda(t)\omega_\mu(t-\nu)\,\mathrm{d}t - \int_0^T \lambda(t)\omega_\mu(t-\tau)\,\mathrm{d}t,$$

so that (3.27) implies

$$\int_0^T \lambda(t)\omega_\mu(t-\nu)\,\mathrm{d}t + |\mathrm{A}_2| + |B| + |D| + |E| + |F| \geq \int_0^T \lambda(t)\omega_\mu(t-\tau)\,\mathrm{d}t. \tag{3.29}$$

Our task is now to overestimate $|\mathrm{A}_2|$, $|B|$, $|D|$, $|E|$ and $|F|$.

To estimate the term A_2, we recall (2.45) and observe that (cf. (2.46))

$$\frac{1}{\Delta t}\int_0^T \int_0^{\Delta t} \eta(u(t,x) - u_{\Delta x}(t,y), v(t,x) - v_{\Delta x}(t,y), a)(\xi - \Delta t)\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)\,\mathrm{d}\xi\,\mathrm{d}s = 0,$$

since all the terms in the integrand except $\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)$ are independent of s. Therefore, subtracting this term from A_2, we obtain,

$$\mathrm{A}_2 = \underbrace{\frac{1}{2\Delta t}\int_{D_T^2}\int_0^{\Delta t}(u_{\Delta x}(t,y)-u_{\Delta x})(2u-u_{\Delta x}-u_{\Delta x}(t,y))\psi^\mu\,\omega_{\epsilon_1}(\xi-\Delta t)\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)\,\mathrm{d}\xi\,\mathrm{d}\underline{z}}_{\mathrm{A}_{2,1}}$$

$$+\underbrace{\frac{1}{2\Delta t}\int_{D_T^2}\int_0^{\Delta t}\frac{1}{a}(v_{\Delta x}(t,y)-v_{\Delta x})(2v-v_{\Delta x}-v_{\Delta x}(t,y))\psi^\mu\,\omega_{\epsilon_1}(\xi-\Delta t)\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)\,\mathrm{d}\xi\,\mathrm{d}\underline{z}}_{\mathrm{A}_{2,2}}.$$

We will outline the estimates for the term $\mathrm{A}_{2,1}$, the term $\mathrm{A}_{2,2}$ is estimated similarly. By the triangle and Hölder's inequality

$$\begin{aligned}
|\mathrm{A}_{2,1}| &\leq \frac{1}{2\Delta t}\int_{D_T^2}\int_0^{\Delta t}|u_{\Delta x}(t,y)-u_{\Delta x}(s,y)|\big(|u(t,x)-u_{\Delta x}(s,y)|+|u(t,x)-u_{\Delta x}(t,y)|\big)\\
&\quad\times\psi^\mu\,\omega_{\epsilon_1}\,|\xi-\Delta t|\,|\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)|\,\mathrm{d}\xi\,\mathrm{d}\underline{z}\\
&\leq \frac{1}{2\Delta t}\int_0^{\Delta t}\int_0^T\int_0^T\left(\int_{D^2}|u_{\Delta x}(t,y)-u_{\Delta x}(s,y)|^2\omega_{\epsilon_1}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\\
&\quad\times\left\{\left(\int_{D^2}|u(t,x)-u_{\Delta x}(s,y)|^2\omega_{\epsilon_1}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\right.\\
&\quad\left.+\left(\int_{D^2}|u(t,x)-u_{\Delta x}(t,y)|^2\omega_{\epsilon_1}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\right\}\\
&\quad\times\psi^\mu\,|\xi-\Delta t|\,|\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)|\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\xi\\
&\leq \frac{1}{2\Delta t}\int_0^{\Delta t}\int_0^T\sup_{\substack{0\leq s\leq T\\|t-s|<2\epsilon_0}}\left(\int_{D^2}|u_{\Delta x}(t,y)-u_{\Delta x}(s,y)|^2\omega_{\epsilon_1}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\\
&\quad\times\left\{\sup_{\substack{0\leq s\leq T\\|t-s|<2\epsilon_0}}\left(\int_{D^2}|u(t,x)-u_{\Delta x}(s,y)|^2\omega_{\epsilon_1}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\right.\\
&\quad\left.+\left(\int_{D^2}|u(t,x)-u_{\Delta x}(t,y)|^2\omega_{\epsilon_1}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\right\}\\
&\quad\times\psi^\mu\,|\xi-\Delta t|\int_0^T|\partial_{ss}\omega_{\epsilon_0}(t-s+\xi)|\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\xi\\
&\leq \frac{C}{\Delta t\,\epsilon_0^{2-\gamma}}\int_0^{\Delta t}\int_0^T\left\{\sup_{\substack{0\leq s\leq T\\|t-s|<2\epsilon_0}}\left(\int_{D^2}|u(t,x)-u_{\Delta x}(s,y)|^2\omega_{\epsilon_1}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\right.\\
&\quad\left.+\left(\int_{D^2}|u(t,x)-u_{\Delta x}(t,y)|^2\omega_{\epsilon_1}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2}\right\}\psi^\mu\,|\xi-\Delta t|\,\mathrm{d}t\,\mathrm{d}\xi
\end{aligned} \tag{3.30}$$

$$\leq \frac{C\Delta t}{\epsilon_0^{2-\gamma}} \int_0^{\Delta t} \int_0^T \Bigg\{ \sup_{\substack{0\leq s\leq T \\ |t-s|<2\epsilon_0}} \left(\int_{D^2} |u(t,x) - u_{\Delta x}(s,y)|^2 \omega_{\epsilon_1} \,\mathrm{d}y\,\mathrm{d}x \right)^{1/2}$$

$$+ \left(\int_{D^2} |u(t,x) - u_{\Delta x}(t,y)|^2 \omega_{\epsilon_1} \,\mathrm{d}y\,\mathrm{d}x \right)^{1/2} \Bigg\} \psi^{\mu} \,\mathrm{d}t,$$

where we used the moduli of continuity for $u_{\Delta x}$, viz. (3.21), in the penultimate inequality and that $\Delta t \leq \epsilon_0$. Furthermore, in a similar way as we did for the advection equation in (2.63), one can show that

$$\int_0^T \sup_{\substack{0\leq s\leq T \\ |t-s|<\epsilon_0}} \left(\int_{D^2} |u_{\Delta x}(s,y) - u(t,x)|^2 \,\omega_{\epsilon_1} \,\mathrm{d}y\,\mathrm{d}x \right)^{1/2} \psi^{\mu} \,\mathrm{d}t$$

$$\leq CT\epsilon_0^{\gamma} + \int_0^T \left(\int_0^T \int_{D^2} |u_{\Delta x}(s,y) - u(t,x)|^2 \,\omega_{\epsilon_1}\omega_{\epsilon_0} \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \right)^{1/2} \psi^{\mu} \,\mathrm{d}t \tag{3.31}$$

using the triangle inequality and similarly

$$\int_0^T \sup_{\substack{0\leq s\leq T \\ |t-s|<\epsilon_0}} \left(\int_{D^2} \frac{1}{a} |v_{\Delta x}(s,y) - v(t,x)|^2 \,\omega_{\epsilon_1} \,\mathrm{d}y\,\mathrm{d}x \right)^{1/2} \psi^{\mu} \,\mathrm{d}t$$

$$\leq \frac{CT\epsilon_0^{\gamma}}{\underline{a}} + \int_0^T \left(\int_0^T \int_{D^2} \frac{1}{a} |v_{\Delta x}(s,y) - v(t,x)|^2 \,\omega_{\epsilon_1}\omega_{\epsilon_0} \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \right)^{1/2} \psi^{\mu} \,\mathrm{d}t. \tag{3.32}$$

Using λ, cf. (3.28), (3.30) can be bounded as

$$\left|\mathrm{A}_{2,2}\right| \leq C\Delta t\, \epsilon_0^{2\gamma-2} + \frac{C\Delta t}{\epsilon_0^{2-\gamma}} \int_0^T \sqrt{\lambda(t)}\, \psi^{\mu} \,\mathrm{d}t$$

and so, using a similar argument for the term $\mathrm{A}_{2,1}$

$$|A_2| \leq C\Delta t\, \epsilon_0^{2\gamma-2} + \frac{C\Delta t}{\epsilon_0^{2-\gamma}} \int_0^T \sqrt{\lambda(t)}\, \psi^{\mu} \,\mathrm{d}t. \tag{3.33}$$

In order to bound the term B, we use

$$\partial_x \Omega + D_y^c \Omega = \frac{-1}{4\Delta x} \int_0^{\Delta x} (\xi - \Delta x)^2 \left[\partial_{yyy}\Omega(t,s,x,y-\xi) + \partial_{yyy}\Omega(t,s,x,y+\xi) \right] \mathrm{d}\xi$$

$$= \frac{1}{4\Delta x} \int_0^{\Delta x} (\xi - \Delta x)^2 \left[\partial_{xxx}\Omega(t,s,x,y-\xi) + \partial_{xxx}\Omega(t,s,x,y+\xi) \right] \mathrm{d}\xi$$

and that

$$\frac{1}{4\Delta x}\int_0^{\Delta x}\int_{D_T^2}(\xi-\Delta x)^2\,(u_{\Delta x}-u(t,y))\,(v_{\Delta x}-v(t,y))$$
$$\times\left[\partial_{xxx}\omega_{\epsilon_1}(x-y+\xi)+\partial_{xxx}\omega_{\epsilon_1}(x-y-\xi)\right]\omega_{\epsilon_0}\psi^{\mu}\,\mathrm{d}\xi\,\mathrm{d}\underline{z}=0,$$

since all the terms in the integrand, except $\left[\partial_{xxx}\omega_{\epsilon_1}(x-y+\xi)+\partial_{xxx}\omega_{\epsilon_1}(x-y-\xi)\right]$, are independent of x. We subtract this term from B and add and subtract the term

$$\frac{1}{4\Delta x}\int_0^{\Delta x}\int_{D_T^2}(\xi-\Delta x)^2\,(u_{\Delta x}-u(t,y))\,(v_{\Delta x}-v(t,x))$$
$$\times\left[\partial_{xxx}\omega_{\epsilon_1}(x-y+\xi)+\partial_{xxx}\omega_{\epsilon_1}(x-y-\xi)\right]\omega_{\epsilon_0}\psi^{\mu}\,\mathrm{d}\xi\,\mathrm{d}\underline{z},$$

so that

$$\begin{aligned}
B=&\frac{1}{4\Delta x}\int_0^{\Delta x}\int_{D_T^2}(\xi-\Delta x)^2\,(u(t,y)-u(t,x))\,(v_{\Delta x}-v(t,x))\\
&\times\left[\partial_{xxx}\omega_{\epsilon_1}(x-y+\xi)+\partial_{xxx}\omega_{\epsilon_1}(x-y-\xi)\right]\omega_{\epsilon_0}\psi^{\mu}\,\mathrm{d}\xi\,\mathrm{d}\underline{z}\\
&+\frac{1}{4\Delta x}\int_0^{\Delta x}\int_{D_T^2}(\xi-\Delta x)^2\,(u_{\Delta x}-u(t,y))\,(v(t,y)-v(t,x))\\
&\times\left[\partial_{xxx}\omega_{\epsilon_1}(x-y+\xi)+\partial_{xxx}\omega_{\epsilon_1}(x-y-\xi)\right]\omega_{\epsilon_0}\psi^{\mu}\,\mathrm{d}\xi\,\mathrm{d}\underline{z}\\
:=&B_1+B_2.
\end{aligned}$$

We start by bounding B_1,

$$\begin{aligned}
|B_1|\le&\frac{1}{4\Delta x}\int_0^{\Delta x}\int_{D_T^2}(\xi-\Delta x)^2|u(t,y)-u(t,x)|\,|v_{\Delta x}-v(t,x)|\\
&\times|\partial_{xxx}\omega_{\epsilon_1}(x-y+\xi)+\partial_{xxx}\omega_{\epsilon_1}(x-y-\xi)|\,\omega_{\epsilon_0}\psi^{\mu}\,\mathrm{d}\xi\,\mathrm{d}\underline{z}\\
\le&\frac{1}{4\Delta x}\int_0^{\Delta x}\int_{D_T}\left(\int_{D_T}|u(t,y)-u(t,x)|^2\omega_{\epsilon_0}\,\mathrm{d}y\,\mathrm{d}s\right)^{1/2}\\
&\times\left(\int_{D_T}|v_{\Delta x}-v(t,x)|^2\omega_{\epsilon_0}\,\mathrm{d}y\,\mathrm{d}s\right)^{1/2}(\xi-\Delta x)^2\\
&\times|\partial_{xxx}\omega_{\epsilon_1}(x-y+\xi)+\partial_{xxx}\omega_{\epsilon_1}(x-y-\xi)|\,\psi^{\mu}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\xi\\
\le&\frac{1}{4\Delta x}\int_0^{\Delta x}\int_0^T\sup_{\substack{x\text{ s.t.}\\|x-y|\le 3\epsilon_1}}\left(\int_{D_T}|u(t,y)-u(t,x)|^2\omega_{\epsilon_0}\,\mathrm{d}y\,\mathrm{d}s\right)^{1/2}\\
&\times\sup_{\substack{x\text{ s.t.}\\|x-y|\le 3\epsilon_1}}\left(\int_{D_T}|v_{\Delta x}-v(t,x)|^2\omega_{\epsilon_0}\,\mathrm{d}y\,\mathrm{d}s\right)^{1/2}(\xi-\Delta x)^2
\end{aligned}$$

$$
\begin{aligned}
&\times \int_D |\partial_{xxx}\omega_{\epsilon_1}(x-y+\xi) + \partial_{xxx}\omega_{\epsilon_1}(x-y-\xi)| \,\mathrm{d}x\, \psi^\mu \,\mathrm{d}t\,\mathrm{d}\xi \\
&\le \frac{C}{{\epsilon_1}^{3-\gamma}\Delta x}\int_0^{\Delta x}\int_0^T \sup_{\substack{x \text{ s.t.}\\ |x-y|\le 3\epsilon_1}} \left(\int_{D_T} |v_{\Delta x} - v(t,x)|^2 \omega_{\epsilon_0}\,\mathrm{d}y\,\mathrm{d}s\right)^{1/2} (\xi-\Delta x)^2 \psi^\mu \,\mathrm{d}t\,\mathrm{d}\xi \\
&\le \frac{C\Delta x^2}{{\epsilon_1}^{3-\gamma}}\int_0^T \sup_{\substack{x \text{ s.t.}\\ |x-y|\le 3\epsilon_1}} \left(\int_{D_T} |v_{\Delta x} - v(t,x)|^2 \omega_{\epsilon_0}\,\mathrm{d}y\,\mathrm{d}s\right)^{1/2} \psi^\mu \,\mathrm{d}t,
\end{aligned}
$$

where we have used that ω_{ϵ_1} is compactly supported in $[-\epsilon_1, \epsilon_1]$, and where C is a constant depending on the L^2-norms and the moduli of continuity of the initial data and on T. Using that (cf. (3.31) and (2.66))

$$
\begin{aligned}
&\int_0^T \sup_{\substack{x \text{ s.t.}\\ |x-y|\le 3\epsilon_1}} \left(\int_{D_T} |u_{\Delta x}(s,y) - u(t,x)|^2\, \omega_{\epsilon_0}\,\mathrm{d}y\,\mathrm{d}s\right)^{1/2} \psi^\mu \,\mathrm{d}t \\
&\qquad \le CT\epsilon_1^\gamma + \int_0^T \left(\int_0^T\int_{D^2} |u_{\Delta x}(s,y) - u(t,x)|^2\, \omega_{\epsilon_1}\omega_{\epsilon_0}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2} \psi^\mu \,\mathrm{d}t,
\end{aligned} \tag{3.34}
$$

and analogously,

$$
\begin{aligned}
&\int_0^T \sup_{\substack{x \text{ s.t.}\\ |x-y|\le 3\epsilon_1}} \left(\int_{D^T} |v_{\Delta x}(s,y) - v(t,x)|^2\, \omega_{\epsilon_0}\,\mathrm{d}y\,\mathrm{d}s\right)^{1/2} \psi^\mu \,\mathrm{d}t \\
&\qquad \le CT\epsilon_1^\gamma + \int_0^T \left(\int_0^T\int_{D^2} |v_{\Delta x}(s,y) - v(t,x)|^2\, \omega_{\epsilon_1}\omega_{\epsilon_0}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2} \psi^\mu \,\mathrm{d}t,
\end{aligned} \tag{3.35}
$$

for B_1, we obtain the estimate

$$
|B_1| \le \frac{C\Delta x^2}{\epsilon_1^{3-2\gamma}} + \frac{C\Delta x^2}{\epsilon_1^{3-\gamma}}\int_0^T \sqrt{\lambda(t)}\,\psi^\mu \,\mathrm{d}t. \tag{3.36}
$$

Similarly,

$$
\begin{aligned}
|B_2| &\le \frac{1}{4\Delta x}\int_0^{\Delta x}\int_{D_T^2} (\xi-\Delta x)^2 |u_{\Delta x} - u(t,y)|\,|v(t,y) - v(t,x)| \\
&\quad \times |\partial_{xxx}\omega_{\epsilon_1}(x-y+\xi) + \partial_{xxx}\omega_{\epsilon_1}(x-y-\xi)|\, \omega_{\epsilon_0}\psi^\mu \,\mathrm{d}\xi\,\mathrm{d}\underline{z} \\
&\le \frac{C\Delta x^2}{\epsilon_1^{3-\gamma}}\int_0^T \left(\int_{D_T} |v_{\Delta x}(s,y) - v(t,y)|^2 \omega_{\epsilon_0}\,\mathrm{d}y\,\mathrm{d}s\right)^{1/2} \psi^\mu \,\mathrm{d}t.
\end{aligned}
$$

Using (3.35), we find, as for B_1,

$$
|B_2| \le \frac{C\Delta x^2}{\epsilon_1^{3-2\gamma}} + \frac{C\Delta x^2}{\epsilon_1^{3-\gamma}}\int_0^T \sqrt{\lambda(t)}\,\psi^\mu \,\mathrm{d}t, \tag{3.37}
$$

and therefore

$$|B| \leq \frac{C\Delta x^2}{\epsilon_1^{3-2\gamma}} + \frac{C\Delta x^2}{\epsilon_1^{3-\gamma}} \int_0^T \sqrt{\lambda(t)}\, \psi^\mu \, \mathrm{d}t. \tag{3.38}$$

We proceed to bound the term D. Observing that

$$\int_{D_T^2} (v(t,x) - v_{\Delta x}(t,y))^2 \left(\frac{1}{2a(x)} - \frac{1}{2a_{\Delta x}(y)} \right) D_s^- \Omega \, \mathrm{d}\underline{z} = 0,$$

we can rewrite D as

$$D = \int_{D_T^2} \left((v(t,x) - v_{\Delta x}(t,y))^2 - (v(t,x) - v_{\Delta x}(s,y))^2 \right) \left(\frac{1}{2a(x)} - \frac{1}{2a_{\Delta x}(y)} \right) D_s^- \Omega \, \mathrm{d}\underline{z}.$$

Noting that,

$$D_s^- \Omega(t,s,x,y) = \frac{1}{\Delta t} \int_0^{\Delta t} \partial_s \Omega(t, s-\xi, x, y) \, \mathrm{d}\xi, \tag{3.39}$$

this becomes

$$\begin{aligned} D = \frac{1}{\Delta t} \int_{D_T^2} \int_0^{\Delta t} & (2v(t,x) - v_{\Delta x}(t,y) - v_{\Delta x}(s,y)) \\ & \times (v_{\Delta x}(t,y) - v_{\Delta x}(s,y)) \frac{a_{\Delta x}(y) - a(x)}{2a(x)a_{\Delta x}(y)} \partial_s \Omega \, \mathrm{d}\xi \, \mathrm{d}\underline{z}, \end{aligned}$$

which can be bounded by

$$\begin{aligned} |D| \leq & \frac{1}{2\underline{a}\Delta t} \sup_{|x-y|<\epsilon_1} |a(x) - a_{\Delta x}(y)| \\ & \times \int_{D_T^2} \int_0^{\Delta t} \frac{1}{c} |2v(t,x) - v_{\Delta x}(t,y) - v_{\Delta x}(s,y)| \, |v_{\Delta x}(t,y) - v_{\Delta x}(s,y)| \, |\partial_s \Omega| \, \mathrm{d}\xi \, \mathrm{d}\underline{z} \\ \leq & \frac{C(\epsilon_1 + \Delta x)^\alpha}{2\underline{a}\,\epsilon_0} \sup_{t\in(0,T)} \nu_t^2(v_{\Delta x}(t,\cdot), \epsilon_0)^{1/2} \\ & \times \int_0^T \sup_{\substack{0\leq s\leq T \\ |t-s|<\epsilon_0}} \left(\int_{D^2} \frac{1}{a} |v_{\Delta x}(t,y) - v(s,x)|^2 \, \omega_{\epsilon_1} \, \mathrm{d}y \, \mathrm{d}x \right)^{1/2} \psi^\mu \, \mathrm{d}t \\ \leq & \frac{C(\epsilon_1 + \Delta x)^\alpha}{2\underline{a}\epsilon_0^{1-2\gamma}} + \frac{C(\epsilon_1 + \Delta x)^\alpha}{2\underline{a}\epsilon_0^{1-\gamma}} \int_0^T \sqrt{\lambda(t)}\, \psi^\mu \, \mathrm{d}t, \end{aligned} \tag{3.40}$$

where we have used (3.32) for the last inequality. For the term E, we note that it can be written

$$E = \frac{\Delta x^2}{2} (\theta - 1) \int_0^T \int_{D_T} D_y^- \left[D_y^+ u_{\Delta x} D_y^+ v_{\Delta x} \right] \int_D \omega_{\epsilon_1}(x-y) \, \mathrm{d}x \, \omega_{\epsilon_0} \psi^\mu \, \mathrm{d}y \, \mathrm{d}s \, \mathrm{d}t,$$

so that

$$E = \frac{\Delta x^2}{2}(\theta - 1)\int_0^T \int_{D_T} D_y^- \left[D_y^+ u_{\Delta x} D_y^+ v_{\Delta x}\right] \omega_{\epsilon_0}\psi^\mu \, \mathrm{d}y\, \mathrm{d}s\, \mathrm{d}t, \tag{3.41}$$

$$= \frac{\Delta x^3}{2}(\theta - 1)\int_0^T \int_0^T \sum_j D_y^- \left[D_y^+ u_{\Delta x}(s, x_j) D_y^+ v_{\Delta x}(s, x_j)\right] \omega_{\epsilon_0}\psi^\mu \, \mathrm{d}s\, \mathrm{d}t.$$

$$= 0.$$

In order to estimate the term F, we use that

$$D_x^+ D_x^- \phi(x) = \frac{1}{2\Delta x^2}\int_{-\Delta x}^0 \int_0^{\Delta x} \phi''(x + \eta + \xi)\, \mathrm{d}\xi\, \mathrm{d}\eta, \tag{3.42}$$

and that

$$\frac{1}{8\Delta x}\int_{-\Delta x}^0 \int_0^{\Delta x} \int_{D_T^2} ((v_{\Delta x} - v(t,y))^2 + (u_{\Delta x} - u(t,y))^2)\partial_{xx}^2 \omega_{\epsilon_1}(x - y - \eta - \xi)\, \omega_{\epsilon_0}\psi^\mu \, \mathrm{d}\underline{z}\, \mathrm{d}\xi\, \mathrm{d}\eta = 0,$$

since all the terms in the integrand, but $\partial_{xx}^2 \omega_{\epsilon_1}(x - y - \eta - \xi)$ are independent of x. We subtract this term from F to find

$$F = \underbrace{\frac{1}{8\Delta x}\int_{-\Delta x}^0 \int_0^{\Delta x} \int_{D_T^2} (v - v(t,y))(v + v(t,y) - 2v_{\Delta x})\partial_{xx}^2 \omega_{\epsilon_1}(x - y - \eta - \xi)\, \omega_{\epsilon_0}\psi^\mu \, \mathrm{d}\underline{z}\, \mathrm{d}\xi\, \mathrm{d}\eta}_{\mathrm{F}_1}$$

$$+ \underbrace{\frac{1}{8\Delta x}\int_{-\Delta x}^0 \int_0^{\Delta x} \int_{D_T^2} (u - u(t,y))(u + u(t,y) - 2u_{\Delta x})\partial_{xx}^2 \omega_{\epsilon_1}(x - y - \eta - \xi)\, \omega_{\epsilon_0}\psi^\mu \, \mathrm{d}\underline{z}\, \mathrm{d}\xi\, \mathrm{d}\eta}_{\mathrm{F}_2}.$$

The integrals F_1 and F_2 are estimated in the same way, therefore we outline only the estimate of F_1.

$$|\mathrm{F}_1| \le \frac{1}{8\Delta x}\int_{-\Delta x}^0 \int_0^{\Delta x} \int_{D_T^2} |v - v(t,y)|\big(|v - v_{\Delta x}| + |v(t,y) - v_{\Delta x}|\big)|\partial_{xx}^2 \omega_{\epsilon_1}|\, \omega_{\epsilon_0}\psi^\mu \, \mathrm{d}\underline{z}\, \mathrm{d}\xi\, \mathrm{d}\eta$$

$$\le \frac{1}{8\Delta x}\int_{-\Delta x}^0 \int_0^{\Delta x} \int_0^T \sup_{\substack{x \text{ s.t.} \\ |x-y| \le 3\epsilon_1}} \left(\int_{D_T} |v - v(t,y)|^2 \omega_{\epsilon_0}\, \mathrm{d}y\, \mathrm{d}s\right)^{1/2}$$

$$\times \left\{ \sup_{\substack{x \text{ s.t.} \\ |x-y| \le 3\epsilon_1}} \left(\int_{D_T} |v - v_{\Delta x}|^2 \omega_{\epsilon_0}\, \mathrm{d}y\, \mathrm{d}s\right)^{1/2} \right.$$

$$\left. + \left(\int_{D_T} |v(t,y) - v_{\Delta x}|^2 \omega_{\epsilon_0}\, \mathrm{d}y\, \mathrm{d}s\right)^{1/2} \right\}$$

$$\times \int_D |\partial_{xx}^2 \omega_{\epsilon_1}|\, \mathrm{d}x\, \psi^\mu \, \mathrm{d}t\, \mathrm{d}\xi\, \mathrm{d}\eta$$

$$\leq \frac{C\Delta x}{\epsilon_1^{2-\gamma}} \int_0^T \Bigg\{ \sup_{\substack{x \text{ s.t.} \\ |x-y|\leq 3\epsilon_1}} \left(\int_{D_T} |v - v_{\Delta x}|^2 \omega_{\epsilon_0} \, \mathrm{d}y \, \mathrm{d}s \right)^{1/2}$$

$$+ \left(\int_{D_T} |v(t,y) - v_{\Delta x}|^2 \omega_{\epsilon_0} \, \mathrm{d}y \, \mathrm{d}s \right)^{1/2} \Bigg\} \psi^\mu \, \mathrm{d}t.$$

Using (3.35), we find

$$|\mathrm{F}_1| \leq \frac{C\Delta x}{\epsilon_1^{2-2\gamma}} + \frac{C\Delta x}{\epsilon_1^{2-\gamma}} \int_0^T \sqrt{\lambda(t)} \psi^\mu \, \mathrm{d}t,$$

and therefore

$$|F| \leq \frac{C\Delta x}{\epsilon_1^{2-2\gamma}} + \frac{C\Delta x}{\epsilon_1^{2-\gamma}} \int_0^T \sqrt{\lambda(t)} \, \psi^\mu \, \mathrm{d}t. \tag{3.43}$$

Referring to (3.29), we have established the following bounds

$$|\mathrm{A}_2| \leq C \left(\frac{\Delta x}{\epsilon_0^{2-2\gamma}} + \frac{\Delta x}{\epsilon_0^{2-\gamma}} \int_0^T \sqrt{\lambda(t)} \, \psi^\mu \, \mathrm{d}t \right),$$

$$|B| \leq C \left(\frac{\Delta x^2}{\epsilon_1^{3-2\gamma}} + \frac{\Delta x^2}{\epsilon_1^{3-\gamma}} \int_0^T \sqrt{\lambda(t)} \, \psi^\mu \, \mathrm{d}t \right),$$

$$|D| \leq C \left(\frac{\epsilon_1^\alpha}{\epsilon_1^{1-2\gamma}} + \frac{\epsilon_1^\alpha}{\epsilon_0^{1-\gamma}} \int_0^T \sqrt{\lambda(t)} \, \psi^\mu \, \mathrm{d}t \right),$$

$$|E| = 0,$$

$$|F| \leq C \left(\frac{\Delta x}{\epsilon_1^{2-2\gamma}} + \frac{\Delta x}{\epsilon_1^{2-\gamma}} \int_0^T \sqrt{\lambda(t)} \, \psi^\mu \, \mathrm{d}t \right),$$

where we have used that $\Delta t = C\Delta x$ and $\Delta x \leq \epsilon_1$. Hence,

$$\int_0^T \lambda(t)\omega_\mu(t-\tau) \, \mathrm{d}t \leq \int_0^T \lambda(t)\omega_\mu(t-\nu) \, \mathrm{d}t + \underbrace{C \left(\frac{\Delta x}{\epsilon_0^{2-2\gamma}} + \frac{\Delta x^2}{\epsilon_0^{3-2\gamma}} + \frac{\epsilon_1^\alpha}{\epsilon_0^{1-2\gamma}} + \frac{\Delta x}{\epsilon_1^{2-2\gamma}} \right)}_{\mathrm{M}_1}$$

$$+ \underbrace{C \left(\frac{\Delta x}{\epsilon_0^{2-\gamma}} + \frac{\Delta x^2}{\epsilon_1^{3-\gamma}} + \frac{{\epsilon_1}^\alpha}{\epsilon_0^{1-\gamma}} + \frac{\Delta x}{\epsilon_1^{2-\gamma}} \right)}_{\mathrm{M}_2} \int_0^T \sqrt{\lambda(t)} \, \psi^\mu \, \mathrm{d}t.$$

Sending μ to zero, we find

$$\lambda(\tau) \leq \lambda(\nu) + \mathrm{M}_1 + \mathrm{M}_2 \int_\nu^\tau \sqrt{\lambda(t)} \, \mathrm{d}t.$$

With an application of a Grönwall type inequality (Dragomir, 2003, Chapter 1, Theorem 4) we obtain the estimate

$$\lambda(\tau) \leq \left(\sqrt{\lambda(\nu) + \mathrm{M}_1} + (\tau - \nu)\mathrm{M}_2\right)^2 \leq 2\left(\lambda(\nu) + \mathrm{M}_1 + T^2\mathrm{M}_2^2\right). \tag{3.44}$$

By the triangle inequality, we have

$$\begin{aligned}
&\left|\left(\int_D \int_{D_T} |u_{\Delta x}(s,x) - u(t,y)|^2\, \omega_{\epsilon_1}\omega_{\epsilon_0}\, \mathrm{d}x\, \mathrm{d}s\, \mathrm{d}y\right)^{1/2} - \|u(t,\cdot) - u_{\Delta x}(t,\cdot)\|_{L^2(D)}\right| \\
&\quad \leq \left(\int_D \int_{D_T} |u_{\Delta x}(s,x) - u_{\Delta x}(t,y)|^2\, \omega_{\epsilon_1}\omega_{\epsilon_0}\, \mathrm{d}x\, \mathrm{d}s\, \mathrm{d}y\right)^{1/2} \\
&\quad \leq \left(\int_{D^2} |u_{\Delta x}(t,x) - u_{\Delta x}(t,y)|^2\, \omega_{\epsilon_1}\, \mathrm{d}x\, \mathrm{d}y\right)^{1/2} + \left(\int_{D_T} |u_{\Delta x}(t,x) - u_{\Delta x}(s,x)|^2\, \omega_{\epsilon_0}\, \mathrm{d}s\, \mathrm{d}x\right)^{1/2} \\
&\quad \leq C\,(\epsilon_0^\gamma + \epsilon_1^\gamma),
\end{aligned} \tag{3.45}$$

and similarly

$$\begin{aligned}
&\left|\left(\int_D \int_{D_T} \frac{1}{a(x)}\, |v_{\Delta x}(t,x) - v(s,y)|^2\, \omega_{\epsilon_1}\omega_{\epsilon_0}\, \mathrm{d}x\, \mathrm{d}s\, \mathrm{d}y\right)^{1/2} - \|(v - v_{\Delta x})(t,\cdot)/a\|_{L^2(D)}\right| \\
&\quad \leq C\,(\epsilon_0^\gamma + \epsilon_1^\gamma).
\end{aligned} \tag{3.46}$$

Moreover,

$$\begin{aligned}
&\|(u - u_{\Delta x})(\nu,\cdot)\|_{L^2(D)} + \|(v - v_{\Delta x})(\nu,\cdot)/a\|_{L^2(D)} \\
&\quad \leq \|u_{\Delta x}(\nu,\cdot) - u_{\Delta x}(0,\cdot)\|_{L^2(D)} + \|(v_{\Delta x}(\nu,\cdot) - v_{\Delta x}(0,\cdot))/a\|_{L^2(D)} \\
&\qquad + \|u_0 - u_{\Delta x}(0,\cdot)\|_{L^2(D)} + \|(v_0 - v_{\Delta x}(0,\cdot))/a\|_{L^2(D)} \\
&\qquad + \|u(\nu,\cdot) - u_0\|_{L^2(D)} + \|(v(\nu,\cdot) - v_0)/a\|_{L^2(D)} \\
&\quad \leq C(\nu + \Delta t)^\gamma + \|u_0 - u_{\Delta x}(0,\cdot)\|_{L^2(D)} + \|(v_0 - v_{\Delta x}(0,\cdot))/a\|_{L^2(D)}.
\end{aligned} \tag{3.47}$$

Write

$$e(\tau) = \|(u - u_{\Delta x})(\tau,\cdot)\|_{L^2(D)} + \|(v - v_{\Delta x})(\tau,\cdot)/a\|_{L^2(D)}.$$

Thus, combining (3.44)–(3.47), the definition of M_1 and M_2 and some basic calculus inequalities, we obtain

$$\begin{aligned}
\mathrm{e}^2(\tau) \leq C\Bigg(&\mathrm{e}^2(0) + \epsilon_1^{2\gamma} + \epsilon_0^{2\gamma} + \frac{\Delta x}{\epsilon_0^{2-2\gamma}} + \frac{\epsilon_1^\alpha}{\epsilon_0^{1-2\gamma}} + \frac{\Delta x^2}{\epsilon_0^{4-2\gamma}} \\
&+ \frac{\Delta x^4}{\epsilon_1^{6-2\gamma}} + \frac{\epsilon_1^{2\alpha}}{\epsilon_0^{2(1-\gamma)}} + \frac{\Delta x^2}{\epsilon_1^{3-2\gamma}} + \frac{\Delta x}{\epsilon_1^{2-2\gamma}} + \frac{\Delta x^2}{\epsilon_1^{4-2\gamma}}\Bigg).
\end{aligned} \tag{3.48}$$

Hence, choosing $\epsilon_1 = \epsilon_0^{1/\alpha}$ and $\epsilon_1 = \Delta x^{1/(2(\gamma\alpha+1-\gamma))}$,

$$e(\tau) \leq C\left(e(0) + \Delta x^{(\alpha\gamma)/(2(\alpha\gamma+1-\gamma))}\right). \qquad \square$$

REMARK 3.4 We note that for $\gamma = 1$, this reduces to a rate of $\Delta x^{1/2}$ independently of α.

3.3 *Numerical examples*

Next, we shall compare the above-derived convergence rates with the ones obtained in practice. To this end, we implement the finite difference scheme (3.4) and test it on a set of numerical test cases. For all the test cases, we use the interval $D = [0, 2]$ as the computational domain with periodic boundary conditions. We use again the sample of a log-normally distributed random field from Section 2.5 as a material coefficient a (cf. Fig. 1). We compute approximations at time $T = 1$ and test the scheme in this setting with different choices of initial data. We only test the case that the initial data v_0, u_0 have a moduli of continuity $\gamma = 1$, for which we could show in Lemma 3.1 that the solutions have the same moduli of continuity. In this case, Theorem 3.2 predicts a rate of convergence of $1/2$. Specifically, we run experiments with initial data

$$v_{0,1}(x) = \sin(2\pi x), \quad u_{0,1}(x) = \cos(2\pi x), \tag{3.49}$$

and with

$$v_{0,2}(x) = \begin{cases} -1 - 2(x - 1/2), & x \in [0, 0.5) \\ -1 + 2(x - 1/2) & x \in [0.5, 1), \\ -1 - 2(x - 3/2), & x \in [1, 1.5), \\ -1 + 2(x - 3/2), & x \in [1.5, 2), \end{cases} \qquad u_{0,2}(x) = \begin{cases} 1 + 2(x - 1/2), & x \in [0, 0.5) \\ 1 - 2(x - 1/2) & x \in [0.5, 1), \\ 1 + 2(x - 3/2), & x \in [1, 1.5), \\ 1 - 2(x - 3/2), & x \in [1.5, 2) \end{cases} \tag{3.50}$$

(note that $v_{0,2} = -u_{0,2}$). As a third set of initial data we take $v_{0,3} = v_{0,2}$ and for $u_{0,3}$ we take the composition of 30 random hat functions on $[0, 2]$, i.e.,

$$u_{0,3} = \sum_{j=1}^{30} h_j(x),$$

where h_j is given by

$$h_j(x) = \begin{cases} 0, & x \in [0, x_0] \cup (x_2, 2], \\ q\frac{x-x_0}{x_1-x_0}, & x \in (x_0, x_1], \\ q\frac{x_2-x}{x_2-x_1}, & x \in (x_1, x_2], \end{cases}$$

where $q \sim \mathcal{U}(-1, 1)$, $x_0 \sim \mathcal{U}(0, 1)$, $x_1 \sim \mathcal{U}(x_0, 2)$ and $x_2 \sim \mathcal{U}(x_1, 2)$ are samples of uniformly distributed random variables. The initial data $u_{0,3}$ and $v_{0,3}$ are pictured in Fig. 3 on the left, and the approximation of the linear wave equation by scheme (3.4) at time $T = 1$ on the right (on a grid with 2^{14} points).

The above-chosen initial data have moduli of continuity of $\gamma = 1$ in L^2 since they are contained in $H^1([0, 2])$. We ran $N_{\text{exp}} = 6$ experiments for each set of initial data for mesh resolutions

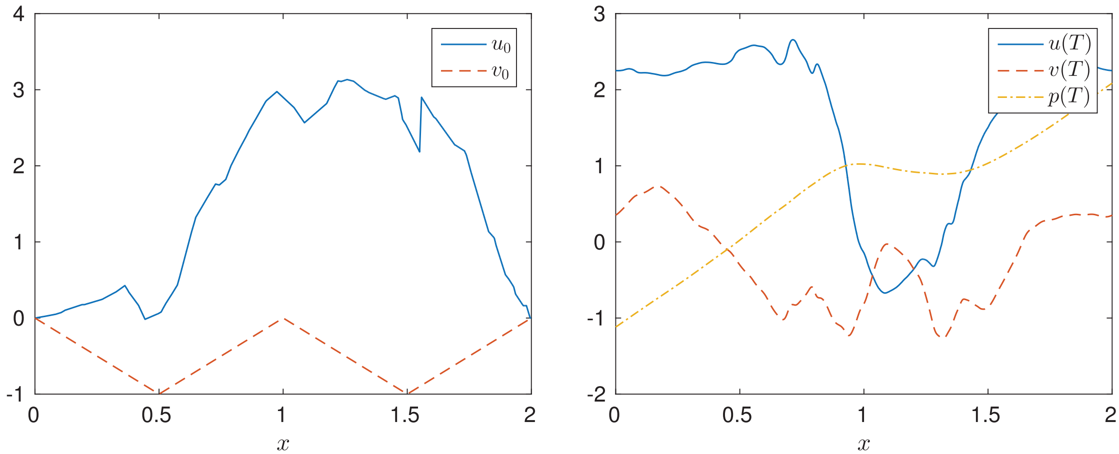

FIG. 3. Left: Initial data $u_{0,3}$ and $v_{0,3}$; Right: Solution at time $T = 1$ for initial data $u_{0,3}, v_{0,3}$.

TABLE 2 *Experimental rates, r_u^2 is the rate for u, r_v^2 the rate for v and r_p^2 the rate for p*

	r_u^2	r_v^2	r_p^2
$u_{0,1}, v_{0,1}$	0.8715	0.6968	0.9170
$u_{0,2}, v_{0,2}$	0.7424	0.7542	0.9214
$u_{0,3}, v_{0,3}$	0.5992	0.5868	0.9503

$\Delta x = 2^{-5}, \dots, 2^{-10}$ (i.e., $N_x = 2^6, \dots, 2^{11}$) and compute errors and rates as in (2.74), (2.73) for $m = 2$ against a reference solution computed on a grid with 2^{14} points. The obtained rates for the three sets of initial data are displayed in Table 2.

We observe that the rates are higher than the $1/2$ which the theory predicts, but not by much, as the example with initial data $u_{0,3}, v_{0,3}$ shows. Moreover, we are testing self-convergence, so the actual convergence rate could be slower. We have also computed convergence rates for the approximation of p, which we computed by integrating the approximation of v with a forward Euler scheme, i.e.,

$$p_j^{n+1} = p_j^n + \Delta t\, v_j^n,$$

for which we have not proved any theoretical results. We observe that the rate for this variable is higher, close to 1, which is probably due to the fact that p has more regularity than v and u, as it can be written as an integral of either of those.

4. Conclusions

Acoustic waves that propagate in a heterogeneous medium, for instance an oil and gas reservoir, are modeled using the linear wave equation (1.1) with a variable material coefficient c. Standard finite difference and finite element approximations converge to the solution as the mesh is refined. A rate of convergence for these approximations can be obtained based on the assumption that the underlying solution is smooth enough. This requires enough smoothness of the material coefficient (wave speed).

However in many practical situations of interest such as seismic wave imaging and hydrocarbon exploration, the material coefficient is not smooth, not even continuously differentiable. Moreover, the material coefficient (rock permeability) is usually modeled by a log-normal random field. Path-wise realizations of such fields are at most Hölder continuous. Thus, the design of numerical schemes that can approximate wave propagation in Hölder continuous media is a necessary first step in the efficient solution of the underlying uncertain PDE with a log-normal distributed material coefficient (Mishra *et al.*, 2016). We are not aware of rigorous numerical analysis results for discretizations of the wave equation with such rough coefficients apart from the works (Jovanović *et al.*, 1987; Jovanović, 1992; Jovanović & Süli, 2014) which require the coefficient to be in $W^{s,2}(D)$ for some $s \in (1, 3]$.

The current paper is therefore an attempt to design robust numerical approximations for the one-dimensional transport and the wave equation with rough, i.e., only Hölder continuous coefficients. For low enough Hölder exponent, this regularity requirement is less than the one in Jovanović *et al.* (1987), Jovanović & Süli (2014) and Jovanović (1992), and also our assumptions on the regularity of the solution are weaker. However, our results (so far) restrict to the one-dimensional case.

We propose upwind finite difference approximations and show that these approximations converge as the mesh is refined. Furthermore, we establish rigorous convergence rates of these approximations. The obtained rates explicitly depend on the Hölder exponent of the material coefficient, as well as the modulus of continuity in L^1 or L^2 of the initial data. The rates of convergence are obtained by a novel adaptation of the Kružkov doubling of variables technique from scalar conservation laws to our L^2 linear system setting. In particular, we prove that for coefficients which are Hölder continuous with exponent α and initial data that are Hölder continuous with exponent γ the solution of the transport equation, and its approximation have the same Hölder regularity and the approximations converge with rate at least $(\gamma\alpha)/(\gamma\alpha + 2 - \gamma)$ in L^1 and L^2 (cf. Theorems 2.11 and 2.13). For the wave equation, we could show that if the initial data have a modulus of continuity of $\gamma = 1$ in L^2, then the solution will inherit it. In this case, the finite difference approximations converge at rate of at least $1/2$. The numerical experiments demonstrate the near sharpness of this rate. We also show rates of convergence under the assumption that the numerical approximations have lower moduli of continuity; however, in this case, we cannot prove that the numerical approximations actually inherit those.

We conclude with a brief discussion on possible limitations and future extensions of our methods:

- We consider finite difference discretizations in the current paper. The formal order of accuracy of our three-point finite difference schemes is 1. One can argue that analogous to linear hyperbolic systems with smooth coefficients, one can obtain higher rates of convergence by designing schemes with a larger stencil (a higher formal order of accuracy). We find that prospect unlikely to hold in practice on account of the lack of smoothness of the coefficient. Furthermore, the irregularities of the coefficient are not localized. Hence, one cannot expect any localization of singularities in the solution and its derivatives. This is in marked contrast to nonlinear systems of conservation laws, where discontinuities such as shocks and contact discontinuities separate smooth parts of the flow. Thus, high-resolution finite difference schemes perform better than low order schemes for conservation laws. Such a situation does not hold for wave propagation in a rough medium. We expect that the low-order schemes presented here are not only simple, but also optimal in this case.
- We present the analysis only in one space dimension and for uniform grids. The extension of the finite difference scheme to the two- and three-dimensional wave equation is straightforward; however, it

is not easy to show that the solution computed in this way has a modulus of continuity, which is fundamental to obtaining convergence rates using our technique. In fact, we do not know currently if the approximations do have a modulus of continuity. Obtaining more insight into the regularity of the approximations is one of the objectives of our current research efforts. We would furthermore like to extend the method and convergence analysis to unstructured grids.

- The numerical experiments for the transport equation (Section 2.5) suggest that the rate from Theorem 2.11 may not be sharp. Consequently, we plan to experiment more in order to find out if the rate is sharp or not, and otherwise try to improve the estimate.
- We restrict ourselves to acoustic wave propagation in rough media in this paper. However, elastic wave propagation also involves media with material properties that lead to rough, Hölder continuous coefficients. The extension of these methods to such problems will be considered in a forthcoming paper. Another possible direction of research would be to prove a rate of convergence for numerical methods that approximate electromagnetic wave propagation in heterogeneous media. Possible extensions to nonlinear wave equations, and discontinuous and time-dependent coefficients will also be considered.

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