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Abstract: For each $\beta > 1$ we construct a family F_β of metric measure spaces which is closed under the operation of taking weak-tangents (i.e. blow-ups), and such that each element of F_β admits a $(1, P)$-Poincaré inequality if and only if $P > \beta$.

Keywords: Poincaré inequality; modulus

MSC: 31E05, 28A80

1 Introduction

Background

The abstract Poincaré inequality was introduced in [11] in the study of quasiconformal homeomorphisms of metric measure spaces where points can be connected by good families of rectifiable curves. The investigation of PI-spaces, i.e. metric measure spaces equipped with doubling measures and which admit a $(1, P)$-Poincaré inequality for some $P \in [1, \infty)$, has been object of intensive research.

One trend of investigation has focused on the infinitesimal structure of such spaces. For example, Cheeger [3] formulated a generalization of the classical Rademacher Differentiation Theorem which holds for PI-spaces and showed that in such spaces the infinitesimal geometry of Lipschitz maps is rather constrained. Moreover, this result has allowed to formulate a notion of *analytic dimension* and extend notions of differential geometry, like tangent and cotangent bundles, to a large class of nonsmooth spaces which includes Carnot groups [12], spaces with synthetic Ricci lower bounds [19], some inverse limit systems of cube complexes [4], and boundaries of certain Fuchsian buildings [2]. There are also more complicated examples which involve gluing constructions [9, 11]. However, the infinitesimal geometry of all these examples is rather special, in the sense that a generic tangent/blow-up is biLipschitz equivalent to a product of Carnot groups with an inverse limit systems of cube complexes as in [4]. In general, little is thus known about the infinitesimal structure of PI-spaces; nevertheless, recent progress on the topic has been achieved in [5], whose results imply that a version of *metric differentiation* holds of PI-spaces, and that for a typical blow-up (Y, ν) of a PI-space the measure ν admits a Fubini-like representation in terms of unit speed geodesics in Y.

Another line of investigation has focused on the study of the properties of the Poincaré inequality that depend on the exponent P. For $\Delta > 0$, a $(1, P)$-Poincaré inequality is stronger than a $(1, P + \Delta)$-Poincaré inequality in the sense that the former implies the latter; moreover, one can use gluing constructions to produce examples of spaces which admit a $(1, P)$-Poincaré inequality but not a $(1, P - \Delta)$-Poincaré inequality for some $\Delta > 0$. Intuitively, in a space admitting a $(1, P)$-Poincaré inequality any pair of points can be connected by a nice family of rectifiable curves, and the quality of these connections improves as P decreases.

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We mention two areas of research where understanding the exponent P is important. One is the study of quasiconformal maps. For example in [18] it is shown that if $\varphi : X \to Y$ is quasiconformal, where X and Y are metric measure spaces satisfying some regularity assumptions (in particular X is assumed to be Q-Ahlfors regular), if X admits a $(1, P)$-Poincaré inequality for $P \in [1, Q]$, so does Y. However, in [18] it is also shown that this is not the case if $P > Q$. A second area is the study of the regularity of minimizers and quasiminimizers of the P-Dirichlet energy (see for instance [15, 16]); in this setting it is usually necessary to assume a $(1, P - \Delta)$-Poincaré inequality for some $\Delta > 0$.

Given a doubling metric measure space (X, μ) we denote by $\mathrm{I}_{\mathrm{PI}}(X, \mu)$ the largest range of exponents $P \geq 1$ such that (X, μ) admits a $(1, P)$-Poincaré inequality. An open question in analysis, even for metric spaces which can be isometrically embedded in some Euclidean space, was whether $\mathrm{I}_{\mathrm{PI}}(X, \mu)$ is an open ray of the form (β, ∞). This question was answered in the affirmative in [14].

Main Result

As remarked above, as of today there is only one known class of models for the infinitesimal geometry of PI-spaces, i.e. biLipschitz deformations of products of Carnot groups and inverse limit systems of cube complexes as in [4]. At the same time a preliminary version of this paper appeared, B. Kleiner and the author have found other examples [17] whose *topological dimension* can be arbitrary but whose analytic dimension is 1.

The lack of sufficiently many examples for the infinitesimal geometry of PI-spaces makes difficult even to formulate reasonable conjectures about the infinitesimal geometric structure of such spaces. All the examples mentioned above and their blow-ups at generic points always admit a $(1, 1)$-Poincaré inequality; while at a conference at IPAM (2013) we learned from Le Donne of a question of Keith about whether a $(1, P)$-Poincaré inequality improves to a $(1, 1)$-Poincaré inequality by taking tangents. Specifically, it is easy to construct examples of $(1, p)$-PI spaces such that *some* tangent does not admit a $(1, p-\varepsilon)$-Poincaré inequality. For example, for $p = 2$ one can glue two copies of $\mathbb{R}^2$ at the origin and take on each copy the Lebesgue measure. However, in all known examples, at *a.e. point all* blow-ups admit a $(1, 1)$-Poincaré inequality.

In this work we answer Keith's question in the *negative* and produce new models for the infinitesimal geometry of a PI-space. In particular, in our examples it is *not* possible to improve the Poincaré inequality by passing to tangents.

Theorem 1.1. *There is a doubling metric space X such that, for each $P_c \in (1, \infty)$ there exists a doubling measure μ_{P_c} on X such that (X, μ_{P_c}) and any of its weak tangents admit a $(1, P)$-Poincaré inequality if and only if $P > P_c$. The space X has Assouad-Nagata dimension 1, and there is a Lipschitz function $\pi : X \to \mathbb{R}$ such that (X, μ_{P_c}) has a unique differentiability chart (X, π) (i.e. the analytic dimension is 1).*

An interesting feature of this example is that the measures $\{\mu_{P_c}\}_{P_c}$ can be taken *mutually singular*. The existence of $(1, 1)$-Poincaré inequalities for mutually singular measures was observed recently [23] in connection with the fact that Cheeger's differentiation theorem does *not* determine a canonical measure class on a metric space. In particular, in a PI-space there can be null sets which contain many differentiability points of a Lipschitz function, even a common differentiability point for each countable collection of Lipschitz functions.

Our examples are also of interest for two different reasons. One is that they show that there is not a strong connection between the exponent in the Poincaré inequality and the underlying metric geometry of X: by changing the measure class the optimal range of exponents for which the Poincaré inequality holds can be arbitrarily prescribed.

Secondly, our examples are connected to an attempt to answer in the *negative* the question of whether there are differentiability spaces (see [5] for details) whose infinitesimal geometry differs from that of PI-spaces. Roughly speaking, this question asks whether a Poincaré inequality is *necessary* (this is sometimes referred to as the "PI-rectifiability" conjecture/question stated in [5]), at the infinitesimal level, to have a Rademacher-like Theorem and a first-order calculus. The results in [5, 24] show that differentiability spaces share, on the infinitesimal level, similarities with PI-spaces. On the other hand, our examples allow to move the range of exponents towards ∞. The obstruction here is that degrading the range of exponents degrades the doubling constant and so it is not possible to get rid of the Poincaré inequality while keeping the measure doubling and having first-order calculus. In a subsequent work [21] we have generalized the examples discussed here to produce differentiability spaces which are PI-unrectifiable. For technical reasons, there we work with cube-complexes of dimension 3 instead of using graphs.

In a [22] we also modify these examples to obtain PI-spaces whose analytic dimension can increase by passing to tangents. Specifically, one can have PI-spaces which are purely 2-unrectifiable and have analytic dimension 1, but at generic points there are tangents biLipschitz equivalent to $\mathbb{R}^2$ with the Euclidean metric.

Recent interesting examples of spaces which admit $(1, P)$-Poincaré inequalities but not $(1, P-\Delta)$-Poincaré inequalities have been constructed in [6]: these examples show that the minimal P-weak upper gradient depends on the choice of the exponent P (i.e. if one has a $(1, P)$-Poincaré inequality but not a $(1, P-\Delta)$-Poincaré, the minimal P-weak upper gradient and the minimal $(P-\Delta)$-weak upper gradient can be different). One may check that this is not the case for our examples; this is unavoidable in the context of having examples whose properties are stable under passing to blow-ups as discussed in [25]. Note that the examples in [6] are rectifiable, and so do not provide new infinitesimal geometries.

Overview

We observed that to produce new examples for the infinitesimal geometry of PI-spaces one might consider an inverse limit of square complexes where the gluing locus has 0 1-capacity [4, Example 11.13]. However, such examples would have analytic and Assouad-Nagata dimension 2, and would not give access to the full range of exponents P_c. Moreover, the arguments in [4] would not carry over and one would have to resort to modulus estimates.

We thus decided to obtain X as an asymptotic cone of a metric graph G so that the stability under blow-up would be already built in the model. Note that one might also realize X as an inverse limit of a system of metric measure graphs, but it *would not* satisfy the same axioms as the inverse systems in [4]. Specifically, Axiom (2) in [4], i.e. the requirement that simplicial projections are open, would fail and the analysis in [4] would not carry over.

In Section 2 we first explain how G is obtained from the graph G and then focus on the construction of G and corresponding measure μ_G in function of some parameters. The choices for the weights on the measure will produce the different measures μ_{P_c}. We then make a study of the shape of balls. Note that in passing information from G to X we take advantage of a discretization procedure in [7].

We point out that the definition of G is somewhat technical and that the starting point of our research were explorations of the geometry of G in C++ and Python. Specifically, it is not hard to translate Definition 2.30 into a `Graph` class and then use Dijkstra's shortest path algorithm to verify the results in Subsections 2.3 and 2.4. To help the reader's intution we have added informal Remarks 2.31, 2.32, 2.35 and 2.40 to give a friendlier account of G.

In Section 3 we construct *good* quasigeodesics that connect pairs of points in G. For convenience, we focus on the construction of walks. To help the reader we have added an informal discussion in Remark 3.3.

Section 4 contains the technical part of the paper. We establish modulus estimates to prove/disprove the Poincaré inequality in G for a given choice of P. In this section we also recall the definition of modulus and a "geometric" characterization of the Poincaré inequality in terms of random curves.

Some parts of the construction of random curves are rather technical so we provide an overview of our approach at the beginning of Subsection 4.2, and have added informal Remarks 4.25 and 4.60.

In Section 5 we complete the proof of Theorem 1.1.

Notational conventions

We use the convention $a \approx b$ to say that $a/b, b/a \in [C^{-1}, C]$ where C is a universal constant; when we want to highlight C we write $a \approx_C b$. We similarly use notations like $a \lesssim b$ and $a \gtrsim_C b$. In the following C often denotes an unspecified universal constant (that can change from line to line) which can be explicitly estimated. We use the notation $E[\varphi]$ to denote the expectation of the random variable φ. The notation $B(A, r)$ denotes a ball of radius R centred on the set A, i.e. the set of points p at distance $< r$ from the set A.

2 Construction of the Example

2.1 Construction of X given the graph G

In this subsection we explain how to obtain the example X given the metric graph G, which is to be constructed in the next subsection. More precisely, we will assume that (G, μ_G) is a doubling metric graph, where μ_G restricts to a multiple of Lebesgue measure on each edge, and we will obtain (X, μ_X) taking asymptotic cones. Note that the construction of G will automatically yield the following kind of “self-similarity”: there will be distinguished scales $\sigma_n = 8^n$ such that, having fixed $q \in G$ and $p \in G$, for each $s > 0$ one can find n such that $\bar{B}_G(p, \sigma_n R)$ contains an isometric copy B_s of $\bar{B}_G(q, s)$.

The main point of this subsection is then Theorem 2.15 which links the range of exponents for which the Poincaré inequality holds for an asymptotic cone (X, μ_X) of (G, μ_G) to the range of exponents for which the Poincaré inequality holds for (G, μ_G). As the set of asymptotic cones of G is closed under passing to tangents 2.5, this will imply that the range of exponents for which (X, μ_X) admits a Poincaré inequality is the same for all its tangents.

A couple of notational remarks: we will often deal with balls of different spaces, and so at times we add a subscript to them to distinguish the space to which they belong. Given a metric space X, we will use λX to denote X with the metric rescaled by the factor $\lambda > 0$.

Definition 2.1 (Asymptotic cone)**.** An **asymptotic cone** of a metric measure space (X, μ) is a measured pointed Gromov-Hausdorff limit of a sequence of rescalings:

$$\left(\lambda_n^{-1} X, \frac{\mu}{\mu\left(B_X(p_n, \lambda_n)\right)}, p_n\right) \tag{2.2}$$

where $\lim_{n\to\infty} \lambda_n = \infty$. Note that $B_X(p_n, \lambda_n)$ denotes a ball of radius λ_n in X, that is a ball of radius 1 in $\lambda_n^{-1} X$. The set of asymptotic cones of (X, μ) will be denoted by as-Con(X, μ). Note that it would be more appropriate to say that as-Con(X, μ) is a set of equivalence classes of metric spaces under measure-preserving isometries, but we will avoid such subtleties in the following discussion.

Definition 2.3. A **weak tangent** (Y, ν, q) of a metric measure space (X, μ_X) is a measured pointed Gromov-Hausdorff limit of a sequence of rescalings:

$$\left(\lambda_n X, \frac{\mu_X}{\mu_X\left(B_X(p_n, \lambda_n^{-1})\right)}, p_n\right) \tag{2.4}$$

where $\lim_{n\to\infty} \lambda_n = \infty$. The set of weak tangents of (X, μ_X) will be denoted by w-Tan(X, μ_X).

In the case of (G, μ_G) the fact that asymptotic cones exist and that the corresponding measures are doubling with uniformly bounded doubling constants follows from a standard compactness argument.

Lemma 2.5. *The set of asymptotic cones* as-Con(G, μ_G) *is closed under the operation of taking weak tangents, i.e. whenever* $(X, \mu_X, p) \in$ as-Con(G, μ_G) *one has* w-Tan$(X, \mu_X) \subset$ as-Con(G, μ_G).

Proof. On the metric level, the proof is straighforward using that one can approximate a weak tangent $(Y, \mu_Y, q) \in$ w-Tan(X, μ_X) by rescaling an approximating sequence for (X, μ_X, p). There is, however, an issue with normalization of balls which is addressed in the following lemma. □

From now on we will assume that (G, μ_G) satisfies the following property, which we call **the measure-continuity of balls**, which will be proved after we analyze the metric-measure structure of (G, μ_G). For the moment we ask the reader to take the following Lemma (the proof is at the end of subsection 2.5) for granted so that we can complete the construction.

Lemma 2.6 (Measure-continuity of balls)**.** *Let* $(X, \mu, p) \in$ as-Con(G, μ_G) *and consider a sequence of rescalings:*

$$\left(\lambda_n^{-1} G, \underbrace{\frac{\mu_G}{\mu_G\left(B_X(p_n, \lambda_n)\right)}}_{\nu_n}, p_n\right) \to (X, \mu, p). \tag{2.7}$$

Then for each $t \geq 0$ one has:

$$\lim_{n\to\infty} \nu_n\left(B_G(p_n, \lambda_n t)\right) = \mu\left(B_X(p, t)\right). \tag{2.8}$$

Now we assume that (G, μ_G) admits Poincaré inequalities and introduce the following notation for the range of exponents for which the Poincaré inequality holds.

Definition 2.9 (Range of PI-exponents)**.** Assume that (Y, ν) is a doubling metric measure space which admits Poincaré inequalities. We denote by $\mathrm{I}_{\mathrm{PI}}(Y, \nu)$ the set of those $q \in [1, \infty)$ such that (Y, ν) admits a $(1, q)$-Poincaré inequality. By [14] $\mathrm{I}_{\mathrm{PI}}(Y, \nu)$ is either an open ray $(q_{\mathrm{critic}}, \infty)$ or the whole $[1, \infty)$.

We will now use a discretization procedure of Gill and Lopez [7] that allows to compare PI spaces and graphs. We rephrase their result in a slightly more general context, where there is more freedom in the choice of the approximating graph; the proof is omitted being a straightforward generalization of their argument.

Theorem 2.10. *Let H be a connected graph whose metric is a constant multiple of the length metric. For $\varepsilon > 0$ and $C_0 > 0$ consider a subset V of vertices of H which is an ε-separated net and $C_0\varepsilon$-dense. Assume that for some $C_1 > 0$ there is a C_1-biLipschitz embedding $F : V \to X$ such that $F(V)$ is $C_1\varepsilon$-dense in X. Let μ_X be a doubling measure on X with constant C_2. Let μ_H be a doubling measure on H which restricts to a multiple of arclength on each edge and such that one has, for some $C_3 > 0$:*

$$\mu_H\left(B_H(v, r)\right) \approx_{C_3} \mu_X\left(B_X(F(v), r)\right) \quad (\forall (v, r) \in V \times [\varepsilon, \infty)). \tag{2.11}$$

Then $\mathrm{I}_{\mathrm{PI}}(X, \mu_X) \subset \mathrm{I}_{\mathrm{PI}}(H, \mu_H)$; moreover, if $C_X(P)$ denotes the constant of the $(1, P)$-Poincaré inequality in (X, μ_X), then the corresponding constant $C_H(P)$ in (H, μ_H) satisfies:

$$C_H(P) \leq C(C_0, C_1, C_2, C_3, C_X(P), \varepsilon). \tag{2.12}$$

Since we work with pointed measured Gromov-Hausdorff convergence we however need a *local* version of Theorem 2.10.

Corollary 2.13. *In Theorem 2.10, assume that V is not $C_0\varepsilon$-dense in the whole of H, but that V now lies in a ball $\bar{B}_H(h, R)$ with $R > 0$ in which it is $C_0\varepsilon$-dense. Assume also that $F(V)$ contains a $C_1\varepsilon$-dense set in a ball $B_X(x, C_1^{-1}R)$. Furthermore, assume that X is geodesic. Then the conclusion of Theorem 2.10 holds replacing (H, μ_H) with:*

$$\left(\bar{B}_H(h, \tilde{C}^{-1}R), \mu_H \llcorner \bar{B}_H(h, \tilde{C}^{-1}R)\right), \tag{2.14}$$

where $\tilde{C}$ depends only on C_0, C_1, C_2, and ε.

Proof. One can reduce this local case to the *global* one, Theorem 2.10, by recalling that if (X,μ) is geodesic and admits a $(1,P)$-Poincaré inequality with exponent $C(P)$, there is a $C_1(C(P))$ such that for each $R>0$ the metric measure space $(\bar{B}(x,R),\mu\llcorner\bar{B}(x,R))$ admits a $(1,P)$-Poincaré inequality with constant C_1 (see [8]). □

We are now ready for the crucial result linking Poincaré inequalities on (G,μ_G) and (X,μ_X). Note that in the construction of G the vertices will get orders $l\in\mathbb{N}\cup\{0\}$ and to each l there will be associated a characteristic scale $\sigma_l=8^l$ (note that scales go up in l! $\sigma_l\nearrow\infty$ as $l\to\infty$) such that the set of vertices V_l of order $\geq l$ form a maximal σ_l-net in G. This implies that V_l is σ_l-dense in G and that each pair of vertices in V_l is at a distance $\geq\sigma_l$.

We will also use another property of G, a kind of self-similarity, that will follow immediately from its construction. Having fixed $q\in G$ and $p\in G$, for each $s>0$ we can find $n\geq N(s)$ such that $\bar{B}_G(p,\sigma_nR)$ contains an isometric copy B_s of $\bar{B}_G(q,s)$ and such that the measures $\mu_G\llcorner B_s$ and $\mu_G\llcorner\bar{B}_G(q,s)$ agree up to a multiple.

Theorem 2.15. *Let* $(X,\mu_X,p)\in\text{as-Con}(G,\mu_G)$*; then:*

$$\mathrm{I}_{\mathrm{PI}}(X,\mu_X)=\mathrm{I}_{\mathrm{PI}}(G,\mu_G). \tag{2.16}$$

Proof. `Step 1:` $\mathrm{I}_{\mathrm{PI}}(X,\mu_X)\subset\mathrm{I}_{\mathrm{PI}}(G,\mu_G)$.

Let

$$\left(\lambda_n^{-1}G,\underbrace{\frac{\mu_G}{\mu_G\left(B_G(p_n,\lambda_n)\right)}}_{\nu_n},p_n\right)\to(X,\mu_X,p) \tag{2.17}$$

and assume that $P\in\mathrm{I}_{\mathrm{PI}}(X,\mu_X)$, $C(P)$ being the corresponding constant. Choose $N(n)$ such that:

$$1\leq\frac{\lambda_n}{\sigma_{N(n)}}\leq 8 \tag{2.18}$$

and pass to a subsequence such that $\lim_{n\to\infty}\frac{\lambda_n}{\sigma_{N(n)}}$ exists. Therefore, up to rescaling the metric on X by a factor in $[1/8,1]$ we can assume that:

$$(\underbrace{\sigma_{N(n)}^{-1}G}_{G_n},\nu_n,p_n)\to(X,\mu_X,p); \tag{2.19}$$

note also that (X,μ_X) is geodesic being a limit of geodesic metric spaces. Fix $\varepsilon,R>0$; for $n\geq D_0(R,\varepsilon)$ we can assume that the Gromov-Hausdorff distance between $B_{G_n}(p_n,R)$ and $B_X(p,R)$ is at most $\frac{\varepsilon}{3}$. Now the vertices of order $\geq l$ in G form a maximal σ_l-net which becomes a maximal $\sigma_l\sigma_{N(n)}^{-1}$-net in G_n; for each n we choose $N_\varepsilon(n)\leq N(n)$ such that:

$$\varepsilon\leq\sigma_{N_\varepsilon(n)}\sigma_{N(n)}^{-1}\leq 8\varepsilon. \tag{2.20}$$

Lett $V(n;\varepsilon)$ be the set of vertices of G_n whose order in G is at least $N_\varepsilon(n)$ and which are contained in $B_{G_n}(p_n,R)$. Then $V(n;\varepsilon)$ is an ε-separated net in $B_{G_n}(p_n,R)$ and is also 8ε-dense there. Thus the cardinality of $V(n;\varepsilon)$ is uniformly bounded in n and $V(n;\varepsilon)\to W$ in the Hausdorff sense where W is a $\frac{2}{3}\varepsilon$-separated net in $B_X(p,R)$ in which it is also 6ε-dense. Therefore for $n\geq D_0(R,\varepsilon)$ we find an L-biLipschitz map:

$$F_n:V(n;\varepsilon)\to W, \tag{2.21}$$

where L does not depend on ε or n. Now, as the cardinalities of $V(n;\varepsilon)$ and W are uniformly bounded, for $n\geq D_1(R,\varepsilon)$ we can assume that the sets $V(n;\varepsilon)$ and W have the same cardinality and write $V(n;\varepsilon)=\{v_\alpha^{(n)}\}_{\alpha\in A}$ and $W=\{w_\alpha\}_{\alpha\in A}$ so that $F_n(v_\alpha^{(n)})=w_\alpha$ for each $\alpha\in A$. We now use a variation on the argument of Lemma 2.6 (where we take balls not centred on the basepoints) to conclude that for each $r\in[\varepsilon,R]$ one has:

$$\nu_n\left(B_{G_n}(v_\alpha^{(n)},R)\right)\to\mu_X\left(B_X(w_\alpha,R)\right); \tag{2.22}$$

so for $n \geq D_2(R, \varepsilon)$ we can assume that:

$$\nu_n \left(B_{G_n}(v_\alpha^{(n)}, R)\right) \approx_{1+\varepsilon} \mu_X \left(B_X(w_\alpha, R)\right). \tag{2.23}$$

We now apply Corollary 2.13 and find $C_{\mathrm{cut}} = C_{\mathrm{cut}}(\varepsilon)$ such that

$$\left(\bar{B}_{G_n}(p_n, R/C_{\mathrm{cut}}), \nu_n \llcorner \bar{B}_{G_n}(p_n, R/C_{\mathrm{cut}})\right) \tag{2.24}$$

admits a $(1, P)$-Poincaré inequality with constant $C_{\mathrm{PI}} = C(C(P), \varepsilon)$. By rescaling back we conclude that:

$$\left(\bar{B}_G(p_n, \sigma_{N(n)}R/C_{\mathrm{cut}}), \mu_G \llcorner \bar{B}_G(p_n, \sigma_{N(n)}R/C_{\mathrm{cut}})\right) \tag{2.25}$$

admits a $(1, P)$-Poincaré inequality with constant C_{PI}. Fix a basepoint $q \in G$. For each $s > 0$ we can find $n \geq D_3(s)$ such that $\bar{B}_G(p_n, \sigma_{N(n)}R/C_{\mathrm{cut}})$ contains an isometric copy B_s of $\bar{B}_G(q, s)$ and such that the measures $\mu_G \llcorner B_s$ and $\mu_G \llcorner \bar{B}_G(q, s)$ agree up to a multiple. Thus

$$\left(\bar{B}_G(q, s), \mu_G \llcorner B(G, q, s)\right) \tag{2.26}$$

admits a $(1, P)$-Poincaré inequality with constant C_{PI}; as C_{PI} does not depend on s we conclude by letting $s \to \infty$.

`Step 2:` $\mathrm{I}_{\mathrm{PI}}(G, \mu_G) \subset \mathrm{I}_{\mathrm{PI}}(X, \mu_X)$.

This follows from the stability of the Poincaré inequality under measured pointed Gromov-Hausdorff convergence, see [13]. □

2.2 Construction of G

Because of Theorem 2.15 we can focus on the construction of G with the goal of constructing measures μ_G for which we can pin down exactly $\mathrm{I}_{\mathrm{PI}}(G, \mu_G)$. We start with choosing some parameters (for a more general construction refer to the arXiv version [20]).

Definition 2.27 (Parameters, symbols and scales)**.** Here are the parameters used in the construction:

(P1) The integer 8.
(P2) The set of symbols $\mathrm{Symb}_1 = \{\{\emptyset\}, \{\spadesuit\}\}$.
(P3) The set of symbols $\mathrm{Symb}_2 = \{\{\emptyset\}, \{\lozenge\}\}$.

The symbol $\{\emptyset\}$ will be called the **end symbol** and will be used as a “stop-letter” in labels attached to edges of G. The symbol $\{\spadesuit\}$ which we will call the **gluing symbol** will be used to affect the dynamics by which G is connected and to tune the range of exponents for which the Poincaré inequality holds. Finally, the symbol $\{\lozenge\}$ is introduced just to get Symb_2 have two distinct elements and so it does not deserve a name.

For $k \geq 0$ we finally introduce the scales $\sigma_k = 8^k$.

We now introduce the labels for the edges of G.

Definition 2.28 (Labels and orders)**.** Let Λ (resp. Θ) denote the set of labels on Symb_1 (resp. Symb_2), i.e. the infinite strings $\lambda = \{\lambda(n)\}$ (resp. $\theta = \{\theta(n)\}$) where $\lambda(n) \in \mathrm{Symb}_1$ (resp. $\theta(n) \in \mathrm{Symb}_2$) and $\lambda(n)$ (resp. $\theta(n)$) is eventually the end symbol.

We now regard $\mathbb{R}$ as a graph whose vertices are the elements of $\mathbb{Z}$; using the scales σ_k we associate to each $m \in \mathbb{Z}$ an **order** $\mathrm{ord}(m)$ by the formula:

$$\mathrm{ord}(m) = \begin{cases} 0 & \text{if } m = 0 \\ \max\{k : \sigma_k \text{ divides } |m|\} & \text{otherwise.} \end{cases} \tag{2.29}$$

Note that if none of the $\{\sigma_k\}_k$ divides $|m|$, then by formula (2.29) $\mathrm{ord}(m) = 0$ as we convene that the max over an empty set of natural integers is 0.

We now define the graph G and introduce a specific terminology for some of its vertices.

Definition 2.30. Consider the graph $\mathbb{R} \times \Lambda \times \Theta$ and a vertex $v = (m, \lambda, \theta)$. Recall that we regard $\mathbb{R}$ as a graph whose vertices are the elements of $\mathbb{Z}$ and therefore $\mathbb{R}\times\Lambda\times\Theta$ is a countable union of disjoint graphs isomorphic to $\mathbb{R}$ (with vertices the elements of $\mathbb{Z}$ and edges of the form $[j, j + 1]$ for $j \in \mathbb{Z}$). As v is a vertex of $\mathbb{R} \times \Lambda \times \Theta$ recall also that $m \in \mathbb{Z}$.

We say that the vertex v is a **gluing point** of order t if $\mathrm{ord}(m) = t > 1$ and at least some symbol in $\{\lambda(j)\}_{j<t}$ is **not** the gluing symbol. We say that v is a **socket point** of order t if $\mathrm{ord}(m) = t$ and $\lambda(j)$ is the gluing symbol for $j < t$. Note that a vertex with $\mathrm{ord}(m) = 1$ is **always** a socket point.

The graph G is obtained from $\mathbb{R}\times\Lambda\times\Theta$ by gluing pairs of vertices $(m_1, \lambda_1, \theta_1), (m_2, \lambda_2, \theta_2) \in (\mathbb{Z}\times\Lambda\times\Theta)^2$ if either on the following conditions **(Gluing)** or **(Socket)** holds:

Gluing:
- $(m_1, \lambda_1, \theta_1), (m_2, \lambda_2, \theta_2)$ are gluing points;
- $m_1 = m_2$ and $\theta_1 = \theta_2$;
- $\lambda_1(j) = \lambda_2(j)$ for $j \neq \mathrm{ord}(m_1)$;

Socket:
- $(m_1, \lambda_1, \theta_1), (m_2, \lambda_2, \theta_2)$ are socket points;
- $m_1 = m_2$;
- $\lambda_1(j) = \lambda_2(j)$ and $\theta_1(j) = \theta_2(j)$ for $j \neq \mathrm{ord}(m_1)$.

Remark 2.31. The previous definition of G gives a precise mathematical account of the gluing scheme of vertices, and we used it to define data structures representing finite subgraphs of G and their geodesics while we were exploring the connectivity properties of G in C++ and Python. In this remark we give a more intuitive description of G to help the reader's intuition.

The first step in the construction is to take countably many graphs isomorphic to $\mathbb{R}$ (where the vertices are the elements of $\mathbb{Z}$) and index them by pairs $(\lambda, \theta) \in \Lambda\times\Theta$. These graphs are just lines, and we can think of this union as a bunch of disjoint lines carrying labels and whose points can be represented by triples (t, λ, θ) where t is a "continuous" degree of freedom (the "horizontal direction") and λ and θ are discrete degrees of freedom. In order to keep the set of these lines countable we impose the restriction that labels λ and θ are sequences of symbols that eventually end in the end symbol $\{\emptyset\}$.

The second step is to glue the lines together to obtain a connected graph. Intuitively we can think of moving from a point (t, λ, θ) to a point (s, λ', θ'), and the task becomes to change t to s, λ to λ' and θ to θ'. Changing t to s does not pose a challenge as one can travel along the horizontal direction. To change $\lambda = \{\lambda(j)\}_{j\in\mathbb{N}}$ we change each of the symbols $\lambda(j)$ at a time. We first focus on the case $j > 1$; then to change $\lambda(j)$ to $\lambda'(j)$ it is sufficient to reach a gluing point (or a socket point if it happens that the first $(j - 1)$ entries of λ are $\{\spadesuit\}$) traveling along the horizontal direction a distance $\lesssim \sigma_j$. For the case $j = 1$ the situation is similar but we always reach a socket point of order 1. Essentially the intuition is that changing $\lambda(j)$ is "easy".

On the other hand, to change $\theta(j)$ to $\theta(j')$ we must reach a socket point w of order j. If $j > 1$ we cannot just move horizontally, because the λ-label of w is restricted to have its first $(j - 1)$-entries equal to the gluing symbol $\{\spadesuit\}$. Thus, socket points occur more sparsely, and unless we already have $\lambda(i) = \{\spadesuit\}$ for all $i < j$ we must first modify some of the labels in $\{\lambda(i)\}_{i<j}$. Essentially, the intuition is that changing $\theta(j)$ is "hard" and this will pose an obstruction to the existence of Poincaré inequalities. Note however, that the maximal length needed to reach a socket point of order j is still $\lesssim \sigma_j$.

Finally, for $j = 1$ socket points are not hard to reach as the restriction of their λ-label becomes vacuous. We classify them as "socket points" just because they can be used to change both $\lambda(1)$ and $\theta(1)$.

We make G a metric graph by considering the length metric where each edge has length 1. Points in G are then equivalence classes $[(t, \lambda, \theta)]$ of points $(t, \lambda, \theta) \in \mathbb{R} \times \Lambda \times \Theta$. The quotient map $\mathbb{R} \times \Lambda \times \Theta \to G$ will be denoted by Q. The Q-image of a gluing point (resp. a socket point) will be called a **gluing point** (resp. a **socket point**) of G. Note that the projection $\mathbb{R} \times \Lambda \to \mathbb{R}$ induces a 1-Lipschitz map $\pi : G \to \mathbb{R}$.

Remark 2.32. Continuing the informal discussion in Remark 2.31, we observe that the vertices of G can be classified in 3 categories. Let $v = [(m, \lambda, \theta)]$ be such a vertex. If $\mathrm{ord}(m) = j$ and if for $i < j$ some $\lambda(i)$ does not equal $\{\spadesuit\}$, then v is a gluing point of order j and has valence $2 \times \#\,\mathrm{Symb}_1$. If $\mathrm{ord}(m) = j$ and if for all $i < j$ one has $\lambda(i) = \{\spadesuit\}$, then v is a socket point of order j and has valence $2 \times \#\,\mathrm{Symb}_1 \times \#\,\mathrm{Symb}_2$. All the remaining vertices are those corresponding to the case $\mathrm{ord}(m) = 0$ and have valence 2. Finally note that G is a graph where no edge starts and ends at the same point, simply because we never glue together two vertices (m, λ, θ) and (m', λ', θ') of $\mathbb{R} \times \Lambda \times \Theta$ when $m \neq m'$. In particular, each inclusion $\mathbb{R} \times \{\lambda\} \times \{\theta\}$ in G is an isometry.

To analyze the shape of balls in G the following definitions are useful.

Definition 2.33. To the sequence of scales $\{\sigma_k\}$ we associate the discretized logarithm $\lg : [0, \infty) \to \mathbb{N}$ as follows:

$$\lg(p) = \begin{cases} 0 & \text{if } |p| < \sigma_1 \\ \{\max k : \sigma_k \leq |p|\} & \text{otherwise.} \end{cases} \tag{2.34}$$

Note that each vertex $v \in G$ has the form $[(k, \lambda, \theta)]$ where $k \in \mathbb{Z}$, and $\mathrm{ord}(k)$ will be called the **order** of v.

Remark 2.35. In analyzing the structure of G the scales σ_k will play a crucial role. A first immediate consequence of the construction is that if v had order k and w has order k' then $d(v, w') \geq \sigma_{\min(k,k')}$. Another immediate consequence is that the set of vertices V_l of order $\geq l$ is σ_l-dense in G; more succintly, V_l is a maximal σ_l-net.

2.3 Construction of walks

To analyze the metric structure of G and prove Poincaré inequalities we will work with walks instead of paths.

Definition 2.36 (Walks)**.** A **walk** on G is a finite string on vertices and edges $W = \{w_0\, e_1\, w_1 \cdots e_l\, w_l\}$ where w_{i-1} and w_i are the endpoints of e_i for $1 \leq i \leq l$. In the following we will often suppress the edges from the notation, i.e. simply write $W = \{w_0\, w_1 \cdots w_l\}$; we will also say that W is a walk from w_0 to w_l and that l is the length of W, which we will denote by $\mathrm{len}\, W$. The starting point $\mathrm{str}\, W$ of W is w_0 and the end point $\mathrm{end}\, W$ of W is w_l. Two walks W_1, W_2 with $\mathrm{end}\, W_1 = \mathrm{str}\, W_2$ can ba concatenated to obtain a walk $W_1 \star W_2$.

We say that a walk W from x to y is **geodesic** if $\mathrm{len}\, W = d(x, y)$. This notion can be also extended to the case in which x and / or y are not vertices of G. In this case a **geodesic walk** from x to y is a geodesic walk from a vertex w_x to a vertex w_y such that:

$$d(x, w_x) < 1 \tag{2.37}$$

$$d(y, w_y) < 1 \tag{2.38}$$

$$d(x, y) = d(x, w_x) + \mathrm{len}\, W + d(y, w_y); \tag{2.39}$$

note that (2.39) implies $\mathrm{len}\, W = d(w_x, w_y)$. A walk $W = \{w_0\, w_1 \cdots w_l\}$ is **monotone increasing** (resp. **decreasing**) if for $0 \leq i \leq l - 1$ one has $\pi(w_{i+1}) > \pi(w_i)$ (resp. $\pi(w_{i+1}) < \pi(w_i)$).

Remark 2.40. We have preferred to introduce walks because they are more convenient than parametrized paths to describe the construction of quasigeodesics and random curves that we present later in the paper. Specifically, the following Lemmas 2.41, 2.42, and 2.47 will be used to build quasigeodesics in Section 3 and to prove the Poincaré inequality in Section 4.

In working with walks, it is important to keep track of the labels of their vertices and edges. Recall that, except for countably many points of G, the fibre $Q^{-1}(x)$ is a singleton; the points x for which $\#Q^{-1}(x) > 1$ are either gluing points or socket points. Note also that if x is neither a gluing point nor a socket point, the labels $\lambda_x \in \Lambda$ and $\theta_x \in \Theta$ are well-defined as $x = \big[\big(\pi(x), \lambda, \theta\big)\big]$ for unique $\lambda = \lambda_x$ and $\theta = \theta_x$. In particular, if e is an edge, all points in e, except possibly one of the vertices, have the same labels λ_e and θ_e.

On the other hand, for gluing or socket points we can still say something about their labels. If x is a gluing point of order k, then x is a vertex of G of the form $\left[(\pi(x), \lambda, \theta)\right]$ where: θ is uniquely defined, and $\lambda(l)$ is uniquely defined for $l \neq k$. If x is a socket point of order k, then it is a vertex of G of the form $\left[(\pi(x), \lambda, \theta)\right]$ where: $\lambda(l)$ is the gluing symbol for $l < k$, $\lambda(l)$ is uniquely defined for $l > k$, and $\theta(l)$ is uniquely defined for $l \neq k$. Therefore, if x is either a gluing point or a socket point, at most one entry of each label $\lambda(l)$ and / or $\theta(l)$ is not uniquely defined; in this case we will sometimes make an arbitrary choice and still write $\lambda_x(l)$ or $\theta_x(l)$.

Finally, in connection with the valence of the vertices, note that if x is a gluing point $Q^{-1}(x)$ has cardinality $\#\,\mathrm{Symb}_1 = 2$, and if x is a socket point $Q^{-1}(x)$ has cardinality $\#\,\mathrm{Symb}_1 \times \#\,\mathrm{Symb}_2 = 4$. Sometimes we will say that λ is the Λ-label of an edge or vertex and that θ is the Θ-label of an edge or vertex.

In discussing walks that pass through socket points of G, it will be convenient to have defined a partial order on the set of labels Λ as one must first modify the values of the label λ to reach a socket pont. We say that $\lambda < \tilde{\lambda}$ if there are integers $1 \leq k_1 \leq k_2$ such that: $\lambda(j) = \tilde{\lambda}(j)$ for $j < k_1$ and $j > k_2$, and for some $j \in [k_1, k_2]$ the entry $\tilde{\lambda}(j)$ is not the gluing symbol, and $\lambda(j) = \{\spadesuit\}$ for $j \in [k_1, k_2]$. A walk $W = \{w_0\, e_1\, w_1 \cdots e_l\, w_l\}$ is **label nondecreasing** (resp. **nonincreasing**) if for $1 \leq i \leq l-1$ one has $\lambda_{e_{i+1}} \geq \lambda_{e_i}$ (resp. $\lambda_{e_{i+1}} \leq \lambda_{e_i}$).

In the following lemma we construct walks that reach a gluing (or sometimes a socket point) moving only horizontally. They will be used to change the value of the label λ.

Lemma 2.41. *Let $(p, k) \in G \times \mathbb{N}$, and let (λ, θ) denote the labels of one of the edges e incident to p. Then there is a constant C such that there are monotone walks W_+ and W_- satisfying:*

1. *$W_\pm$ is a walk from p to $v_\pm$, where either $v_\pm$ is a gluing point if some $\{\lambda(j)\}_{j<k}$ is not the gluing symbol, or is a socket point of order k;*
2. *$\pm(\pi(v_\pm) - \pi(p)) \in [\sigma_k, C\sigma_k]$;*
3. *$\operatorname{len} W_\pm \in [\sigma_k, C\sigma_k]$;*
4. *All edges in $W_\pm$ have the same labels (λ, θ).*

Proof. We just build W_+. Because p is incident to an edge with label (λ, θ) we have $p \in Q(\mathbb{R} \times \{\lambda\} \times \{\theta\})$, and thus we can find a monotone increasing walk $W_0 \subset Q(\mathbb{R} \times \{\lambda\} \times \{\theta\})$ which starts at p, has length $\operatorname{len} W_0 \in [\sigma_k, 2\sigma_k]$, and ends at a vertex w_0 with $\operatorname{ord}(w_0) = 0$. There is a uniform constant $C \geq 1$ such that the set $\mathbb{R} \cap [\pi(w_0), \pi(w_0) + C\sigma_k]$ contains an integer t with $\operatorname{ord}(t) = k$. Let v_+ be the vertex of $Q(\mathbb{R} \times \{\lambda\} \times \{\theta\})$ which projects to t. Then, if all the symbols $\{\lambda(j)\}_{j \leq k-1}$ equal $\{\spadesuit\}$, v_+ is a socket point of order k; otherwise v_+ is a gluing point of order k. Let $W_1 \subset Q(\mathbb{R} \times \{\lambda\} \times \{\theta\})$ be a monotone increasing walk starting at w_0 and ending at v_+. Then W_+ is obtained by concatenating W_0 and W_1. □

In the following lemma we describe a walk to reach a socket point of a given order k. This walk has to satisfy several technical assumptions that we need later in the paper. Some key properties are bounds on the length (2), the fact that the θ-label is constant (3), and restrictions (7)–(8) on the time we move in a region where a portion of the values of the λ-label is $\{\spadesuit\}$.

Lemma 2.42. *Let $(p, k) \in G \times \mathbb{N}$ and let (λ, θ) be the labels of an edge incident to p. Then there is a universal constant C such that there are label nonincreasing monotone walks W_+ and W_- satisfying:*

1. *$W_\pm$ is a walk from p to $v_\pm$, where $v_\pm$ is a socket point of order k such that $\lambda(v_\pm; l) = \lambda(p; l)$ for $l > k$;*
2. *$\pm(\pi(v_\pm) - \pi(p)) \in [\sigma_k, C\sigma_k]$ and $\operatorname{len} W_\pm \in [\sigma_k, C\sigma_k]$;*
3. *The Θ-label equals θ along all the edges of $W_\pm$;*
4. *All the edges in $W_\pm|[0, 3\sigma_k/2]$ have the same label (λ, θ);*
5. *There are $(\tau_i)_{1 \leq i \leq k-1} \subset \mathbb{N} \cap [0, \operatorname{len} W_\pm]$ such that the map $i \mapsto \tau_i$ is strictly decreasing, $\tau_{k-1} \in [\frac{3\sigma_k}{2}, C\sigma_k]$;*
6. *The point w_{τ_i} is either a gluing point or a socket point of order i;*
7. *$\operatorname{len} W_\pm - \tau_i \in [\sigma_i, C\sigma_i]$;*

8. *Let e_l be an edge of $W_\pm$; if $l \in [0, \tau_{k-1}]$, $\lambda_{e_l} = \lambda_p$; if $l \in (\tau_{i+1}, \tau_i]$ $\lambda(e_l; j) = \lambda(p; j)$ for $j \le i$ or $j > k-1$ and $\lambda(e_l; j) = \{\spadesuit\}$ for $i+1 \le j \le k-1$; if $\lambda \in (\tau_1, \operatorname{len} W_\pm]$ $\lambda(e_l; j) = \{\spadesuit\}$ for $1 \le j \le k-1$ and $\lambda(e_l; j) = \lambda(p_0; j)$ for $j \ge k$.*

Proof. We focus on building W_+ which will be built as a concatenation of walks $\tilde{W}, W_{k-1}, W_{k-2}, \dots, W_0$.

Because p is incident to an edge with label (λ, θ) we have $p \in Q(\mathbb{R} \times \{\lambda\} \times \{\theta\})$, and thus we can find a monotone increasing walk $\tilde{W} \subset Q([\pi(p), \infty) \times \{\lambda\} \times \{\theta\})$ of length $\operatorname{len} \tilde{W} \in [\frac{3\sigma_k}{2}, 2\sigma_k]$ which starts at p and ends at a vertex $\tilde{v}$ with $\operatorname{ord}(\tilde{v}) = 0$.

Let $I = [\pi(\tilde{v}), \infty)$; in I we can find a sequence of integers:

$$t_{k-1} \le t_{k-2} \le \cdots \le t_1 \le t_0 \tag{2.43}$$

such that $\operatorname{ord}(t_i) = i$ for $i \ge 1$ and $\operatorname{ord}(t_0) = k$, and for some universal constant C one has $0 \le t_0 - t_{k-1} \le C\sigma_k$. To be explicit, let t_0 be an integer of order k in $[\pi(\tilde{v}) + \sigma_k, \pi(\tilde{v}) + 3\sigma_k]$ and let $t_i = t_0 - \sigma_i$ for $i \ge 1$.

In the following we will let:

$$\tau_i = t_i - \pi(p), \tag{2.44}$$

and we will introduce the auxiliary notation $\lambda^{(i)}$ for the label:

$$\lambda^{(i)}(j) = \begin{cases} \lambda(j) & \text{if } j \ge k \text{ or } j \le i \\ \{\spadesuit\} & \text{otherwise.} \end{cases} \tag{2.45}$$

Let v_{k-1} be the vertex of $Q([\pi(p), \infty) \times \{\lambda\} \times \{\theta\})$ with $\pi(v_{k-1}) = t_{k-1}$; then we let $W_{k-1} \subset Q([\pi(p), \infty) \times \{\lambda\} \times \{\theta\})$ be a monotone increasing walk which starts at $\tilde{v}$ and ends in v_{k-1}. We let $w_{\tau_{k-1}} = v_{k-1}$ and note that v_{k-1} is either a gluing or a socket point of order $k-1$.

For $i \ge 1$ the walk W_i is obtained from W_{i+1} as follows. The (backward) inductive assumption is that the last edge of W_{i+1} has label $(\lambda^{(i+1)}, \theta)$ and that the last vertex v_{i+1} of W_{i+1} is either a gluing or a socket point of order $i+1$. Note that then $v_{i+1} \in Q([\pi(p), \infty) \times \{\lambda^{(i)}\} \times \{\theta\})$; we now let v_i denote the vertex of $Q([\pi(p), \infty) \times \{\lambda^{(i)}\} \times \{\theta\})$ with $\pi(v_i) = t_i$. Therefore, by (2.45) v_i is either a gluing or a socket point of order i. The walk $W_i \subset Q([\pi(p), \infty) \times \{\lambda^{(i)}\} \times \{\theta\})$ is then defined as a monotone increasing walk starting at v_{i+1} and ending in v_i. We then let $w_{\tau_i} = v_i$.

We complete the construction by producing W_0 as follows; we let $\lambda^{(0)}$ be the label such that:

$$\lambda^{(0)}(j) = \begin{cases} \lambda(j) & \text{if } j \ge k \\ \{\spadesuit\} & \text{otherwise.} \end{cases} \tag{2.46}$$

We then let $v_+ = v_0$ be the vertex of $Q([\pi(p), \infty) \times \{\lambda^{(0)}\} \times \{\theta\})$ such that $\pi(v_0) = t_0$. The walk W_0 is then a monotone increasing walk joining v_1 to v_0.

We now explain how each property in the statement of this Lemma holds:

(1) because $v_+ = v_0$ is a socket point of order k as $\operatorname{ord}(t_0) = k$ and the label $\lambda^{(0)}$ has its first $k-1$ entries equal to $\{\spadesuit\}$;
(2) because we have $\operatorname{len} \tilde{W} \lesssim \sigma_k$, $\operatorname{len} W_i \lesssim \sigma_i$ for $i \ge 1$ and $\operatorname{len} W_0 \lesssim \sigma_1$;
(3) because the walks $\tilde{W}, W_{k-1}, W_{k-2}, \dots, W_0$ lie in $Q(\mathbb{R} \times \Lambda \times \{\theta\})$;
(4) because of how $\tilde{W}$ was constructed;
(5–7) because of how the t_i where chosen;
(8) because of how the labels $\lambda^{(i)}$ were chosen.

□

The next Lemma 2.47 is proven like Lemma 2.42; the proof is omitted as it looks like the specular image of the previous one. Note that this lemma is just the reverse situation in which we start from a socket point of order k and we want to move away from it modifying the first k-entries of the λ-label.

Lemma 2.47. *Let $v \in G$ be a socket point of order k_0 and let λ be a label in Λ such that for $k \le k_0$ one has $\lambda(l) = \lambda_v(l)$ for $l > k$, $l \neq k_0$. Let θ be a label in Θ such that $\theta_v(j) = \theta(j)$ for $j \neq k_0$. Then there is a universal constant C such that there are label non-decreasing monotone walks W_+ and W_- satisfying:*

1. *$W_\pm$ is a walk from v to a vertex $p_\pm$ of order 0 such that $\lambda_{p_\pm} = \lambda$ and $\theta_{p_\pm} = \theta$;*
2. *$\pm(\pi(p_\pm) - \pi(v)) \in [\sigma_k, C\sigma_k]$ and $\operatorname{len} W_\pm \in [\sigma_k, C\sigma_k]$;*
3. *All edges of $W_\pm$ have Θ-label θ;*
4. *All the edges in $W_\pm|[\operatorname{len} W_\pm - \sigma_k/2, \operatorname{len} W_\pm]$ have the same labels;*
5. *There are $(\tau_i)_{1\le i\le k-1} \subset \mathbb{N}\cap[0, \operatorname{len} W_\pm]$ such that the map $i \mapsto \tau_i$ is strictly increasing, and $\tau_{k-1} \in [0, \operatorname{len} W_\pm - \frac{\sigma_k}{2}]$;*
6. *The point w_{τ_i} is either a gluing point or a socket point of order i;*
7. *$\tau_i \in [\sigma_i, C\sigma_i]$;*
8. *Let e_l be an edge of $W_\pm$; if $l \in [0, \tau_1]$, $\lambda(e_l; j) = \lambda(v; j)$ for $j \neq k_0$ and $\lambda(e_l; k_0) = \lambda(k_0)$; if $l \in (\tau_i, \tau_{i+1}]$ $\lambda(e_l; j) = \lambda(j)$ for $j \le i$ or $j > k-1$ and $\lambda(e_l; j) = \{\spadesuit\}$ for $i < j \le k-1$; if $\lambda \in (\tau_{k-1}, \operatorname{len} W_\pm]$ $\lambda_{e_l} = \lambda$.*

2.4 Comparison of balls and boxes

In the following it will be useful to replace balls by boxes because it is easier to estimate the measure of a box; given a Borel set $I \subset \mathbb{R}$, $k \in \mathbb{N} \cup \{0\}$ and a finite set $S_1 \times S_2 \subset \Lambda \times \Theta$, we define the **box** $\operatorname{Box}(I, S_1 \times S_2, k)$ as follows:

$$\left\{[(t, \lambda, \theta)] \in G : t \in I \text{ and } \exists(\tilde\lambda, \tilde\theta) \in S_1 \times S_2 : \forall l > k\ (\lambda(l), \theta(l)) = (\tilde\lambda(l), \tilde\theta(l))\right\}. \tag{2.48}$$

The following lemma shows that boxes and balls are uniformly comparable.

Lemma 2.49. *Let $x = [(t, \lambda, \theta)] \in G$ and $R > 0$. Let M be the highest order of an integer $m \in [t - R, t + R]$. If $M \le \lg(2R)$ let $S(x, R) = \{(\lambda, \theta)\}$. If $M > \lg(2R)$ let Ω_M be the set of those labels (λ', θ') obtained from (λ, θ) by making $(\lambda(M), \theta(M))$ arbitrary, and let $S(x, R) = \Omega_M$. Then there is a universal constant C depending only on* **(P1)–(P3)** *such that:*

$$\begin{aligned}\operatorname{Box}\left(\left[\pi(x) - R/2, \pi(x) + R/2\right], \{(\lambda, \theta)\}, \lg(R/C)\right) &\subset \bar B(x, R)\\ &\subset \operatorname{Box}\left(\left[\pi(x) - R, \pi(x) + R\right], S(x, R), \lg(2R)\right).\end{aligned} \tag{2.50}$$

Proof. If C is sufficiently large, using Lemmas 2.42, 2.47 we can find, for any label $(\tilde\lambda, \tilde\theta)$ such that:

$$(\tilde\lambda(j), \tilde\theta(j)) = (\lambda, \theta) \quad (\text{for } j > \lg(R/C)), \tag{2.51}$$

a path of length at most $R/2$ from x to a point $\tilde x$ such that:

$$\pi(\tilde x) = \pi(x) \tag{2.52}$$

$$\tilde x \in Q(\mathbb{R} \times \{\tilde\lambda\} \times \{\tilde\theta\}); \tag{2.53}$$

this implies the inclusion:

$$\operatorname{Box}\left(\left[\pi(x) - R/2, \pi(x) + R/2\right], \{(\lambda, \theta)\}, \lg(R/C)\right) \subset \bar B(x, R). \tag{2.54}$$

Let γ be a geodesic from x to $p \in \bar B(x, R)$; note that $\operatorname{len} \pi(\gamma) = \operatorname{len} \gamma$ and thus $\pi(\gamma(t)) \in [\pi(x) - R, \pi(x) + R]$ for each $t \in \operatorname{dom} \gamma$. Therefore, if $(\lambda(p; k), \theta(p; k)) \neq (\lambda(x; k), \theta(x; k))$, then $\pi(\gamma)$ passes through an integer t_k of order k. Assume that $k < M$ and let $t_M \in [\pi(x) - R, \pi(x) + R]$ have order M; as:

$$|t_k - t_M| \ge \sigma_k, \tag{2.55}$$

we conclude that $k \le \lg(2R)$. Therefore the inclusion

$$\bar B(x, R) \subset \operatorname{Box}\left(\left[\pi(x) - R, \pi(x) + R\right], S(x, R), \lg(2R)\right) \tag{2.56}$$

follows. □

2.5 Construction of measures

We now turn to the construction of the measure μ on G. One possibility is to take the pushforward under the quotient map $Q : \mathbb{R} \times \Lambda \times \Theta \to G$ of the measure which coincides with Lebesgue measure on each $\mathbb{R} \times \{\lambda\}$. For extra flexibility, in particular to produce mutually singular measures with different values of $\inf \mathrm{I}_{\mathrm{PI}}(X, \mu)$, we need to choose the weights Weight $= \{w_{\{\emptyset\}}, w_{\{\spadesuit\}}, w_{\{\lozenge\}}\}$ subject to the restrictions $w_s > 0$ and $w_{\{\emptyset\}} = 1$. The restriction $w_s > 0$ is needed to ensure the doubling condition, while $w_{\{\emptyset\}} = 1$ is needed as our labels end eventually in $\{\emptyset\}$. Thus we have just two parameters $w_{\{\spadesuit\}}$ and $w_{\{\lozenge\}}$ and in principle we might also set $w_{\{\lozenge\}} = 1$ and be left with one parameter $w_{\{\spadesuit\}}$. However, for extra flexibility, we allow $w_{\{\lozenge\}} \in (0, \infty)$.

For each $\lambda \in \Lambda$ and $\theta \in \Theta$ we denote by $w(\lambda)$, $w(\theta)$ the associated weights:

$$w(\lambda) = \prod_{n=1}^{\infty} w_{\lambda(n)}, \tag{2.57}$$

$$w(\theta) = \prod_{n=1}^{\infty} w_{\theta(n)}, \tag{2.58}$$

where the products in (2.57–2.58) are actually finite. We also use the notation $w((\lambda, \theta))$ for the product $w(\lambda)w(\theta)$.

Definition 2.59. We denote by μ the measure on G which is the pushforward of the measure on $\mathbb{R} \times \Lambda \times \Theta$ which coincides with $w((\lambda, \theta))\mathcal{L}^1$ on each $\mathbb{R} \times \{(\lambda, \theta)\}$. Note that different choices of the weights in Weight will produce mutually singular measures on the asymptotic cone X, compare [23].

The next lemma provides estimates on the measures of balls and boxes.

Lemma 2.60. *Let S be a set of pairs of labels and $k \geq 1$; assume that whenever $(\lambda, \theta), (\lambda', \theta') \in S$ and $(\lambda, \theta) \neq (\lambda', \theta')$, then (λ', θ') cannot be obtained from (λ, θ) by modifying some of the first k-entries of λ and/or θ. For $i = 1, 2$ let $C_{\mathrm{gw},i} = \sum_{s \in \mathrm{Symb}_i} w_s$; then the measure of a box is given by:*

$$\mu\left(\mathrm{Box}\,(I, S, k)\right) = \mathcal{L}^1(I) \times C^k_{\mathrm{gw},1} C^k_{\mathrm{gw},2} \sum_{(\lambda,\theta)\in S} \prod_{n=k+1}^{\infty} w(\lambda(n), \theta(n)). \tag{2.61}$$

In particular, if $x = [(t, \lambda, \theta)]$:

$$\begin{aligned} \mu\left(\bar{B}(x, R)\right) &\approx R(C_{\mathrm{gw},1} C_{\mathrm{gw},2})^{\lg R} \sum_{\lambda \in S(x,R)} \prod_{n=\lg R+1}^{\infty} w(\lambda(n), \theta(n)) \\ &\approx R^{1+\log_8 C_{\mathrm{gw},1} + \log_8 C_{\mathrm{gw},2}} \sum_{\lambda \in S(x,R)} \prod_{n=\lg R+1}^{\infty} w(\lambda(n), \theta(n)). \end{aligned} \tag{2.62}$$

Proof. For each pair of labels (λ, θ) let $T^{(k)}_{\lambda,\theta}$ be the set of labels that can be obtained from (λ, θ) by making the first k entries of λ and/or θ arbitrary. We then compute as follows:

$$\begin{aligned} \mu\left(\mathrm{Box}\,(I, S, k)\right) &= \sum_{(\lambda,\theta)\in S} \sum_{(\tilde{\lambda},\tilde{\theta})\in T^{(k)}_{\lambda,\theta}} \mu\left(\mathrm{Box}\,(I, S, k) \cap Q(\mathbb{R} \times \{\tilde{\lambda}\} \times \{\tilde{\theta}\})\right) \\ &= \sum_{(\lambda,\theta)\in S} \sum_{(\tilde{\lambda},\tilde{\theta})\in T^{(k)}_{\lambda,\theta}} \mathcal{L}^1(I) w\left((\tilde{\lambda}, \tilde{\theta})\right) \\ &= \mathcal{L}^1(I) \sum_{(\lambda,\theta)\in S} \sum_{(\tilde{\lambda},\tilde{\theta})\in T^{(k)}_{\lambda,\theta}} \prod_{n=1}^{k} w\left((\tilde{\lambda}(n), \tilde{\theta}(n))\right) \cdot \prod_{n=k+1}^{\infty} w\left((\lambda(n), \theta(n))\right) \\ &= \mathcal{L}^1(I) \times C^k_{\mathrm{gw},1} C^k_{\mathrm{gw},2} \sum_{(\lambda,\theta)\in S} \prod_{n=k+1}^{\infty} w(\lambda(n), \theta(n)), \end{aligned} \tag{2.63}$$

which gives (2.61).

Now the first approximate equality in (2.62) follows from (2.61) and Lemma 2.49 by observing that for any C_0 there is a $C(C_0)$ such that:

$$\lg(C_0 R) \leq \lg R + C(C_0). \tag{2.64}$$

Finally, the last approximate equality in (2.62) follows observing that if $R > 1$ one has $\lg R \approx \log_8 R$. □

We can finally prove Lemma 2.6

Proof of Lemma 2.6. Using that $n \mapsto \nu_n(G)$ is lower semicontinuous if G is open and upper semicontinuous if G is compact, it suffices to show that one has, uniformly in p_n, λ_n:

$$\frac{\mu_G\left(B(p_n, \lambda_n t) \setminus B(p_n, \lambda_n(t-\varepsilon))\right)}{\mu_G\left(B(p_n, \lambda_n t)\right)} \leq O(\varepsilon^{1/2}). \tag{2.65}$$

For $s \in (0,1)$ let $L(s)$ denote the set of labels (λ, θ) of edges intersecting $\partial B(p_n, \lambda_n(1-s)t)$. Note that $s_1 < s_2$ implies $L(s_2) \supset L(s_1)$. However, as:

$$\frac{\lambda_n(1-s)t}{\lambda_n(1-\varepsilon^{1/2})t} \leq \frac{3}{2} \tag{2.66}$$

for ε sufficiently small and $s \geq \varepsilon$, any label $(\lambda, \theta) \in L(s) \setminus L(\varepsilon^{1/2})$ can differ from a label in $L(\varepsilon^{1/2})$ only at the j-th entry, where either:

$$j \in \left\{\lg(2\lambda_n(1-\varepsilon^{1/2})t), \lg(2\lambda_n(1-\varepsilon^{1/2})t) + 1\right\}, \tag{2.67}$$

or $j = j_0$, where j_0 is some fixed integer $> \lg(2\lambda_n(1-\varepsilon^{1/2})t) + 1$ (this can occur if the ball $B(p_n, \lambda_n t)$ contains a socket point of order greater than $\lg(2\lambda_n t)$). We thus obtain:

$$\frac{\mu_G\left(B(p_n, \lambda_n t) \setminus B(p_n, \lambda_n(t-\varepsilon))\right)}{\mu_G\left(B(p_n, \lambda_n t)\right)} \leq (C_{\mathrm{gw},1} C_{\mathrm{gw},2})^3 \frac{\lambda_n \varepsilon t}{\lambda_n \varepsilon^{1/2} t}, \tag{2.68}$$

from which (2.65) follows. □

3 Construction of good walks

In this section we prove the existence of **good walks** between points in G. These walks correspond to quasi-geodesics which are used to build the families of curves used to prove Poincaré inequalities.

Let $x, y \in G$; choose labels (λ_x, θ_x), (λ_y, θ_y) such that $x = \left[(\pi(x), \lambda_x, \theta_x)\right]$, $y = \left[(\pi(y), \lambda_y, \theta_y)\right]$ and the cardinality of the set:

$$\mathbb{N}(x, y) = \left\{k : (\lambda_x(k), \theta_x(k)) \neq (\lambda_y(k), \theta_y(k))\right\} \tag{3.1}$$

is minimal.

In the following C will denote a universal constant that can change from line to line and that can be explicitly estimated.

Definition 3.2. Given $x, y \in G$ with $d(x, y) > 1$ a **good walk** $W = \{w_0\, e_1\, w_1 \cdots e_L\, w_L\}$ from x to y is a walk having the following properties:

(GW1) $\operatorname{len} W \leq C d(x, y)$;
(GW2) $d(w_0, x), d(w_L, y) \in [0, 1)$;
(GW3) for $i > 0$ one has $d(w_i, x) \geq i/C$.

Remark 3.3. Intuitively condition **(GW1)** forces the path corresponding to W to be a quasigeodesic. Condition **(GW2)** forces W to start at a vertex adjacent to an edge containing x and end at a vertex adjacent to an edge containing y. Finally **(GW3)** forces W to move away from x at a linear rate.

In the following we will often use the following estimate.

Lemma 3.4. *If* $\lg d(x,y) < \max \mathbb{N}(x,y)$ *then for each* $k \in \mathbb{N}(x,y) \setminus \{\max \mathbb{N}(x,y)\}$ *one has* $\lg d(x,y) \geq k$.

Proof. Let w_0, w_1 be vertices of G with $\mathrm{ord}(w_0) \neq \mathrm{ord}(w_1)$, then:

$$d(w_0, w_1) \geq \left|\pi(w_0) - \pi(w_1)\right| \geq \sigma_{\min(\mathrm{ord}(w_0),\mathrm{ord}(w_1))}. \tag{3.5}$$

Take a geodesic walk W from x to y. Then there are $w_{j_0}, w_{j_1} \in W$ such that w_{j_0} is either a gluing or a socket point of order $\max \mathbb{N}(x,y)$ and w_{j_1} is either a gluing or a socket point of order k; let $\tilde{W}$ be a subwalk of W joining w_{j_0} and w_{j_1}, and observe that:

$$\mathrm{len}(W) \geq \mathrm{len}(\tilde{W}) \geq d(w_{j_0}, w_{j_1}) \geq \sigma_k. \tag{3.6}$$

□

The following Theorem is the first part of the construction of good walks under the additional assumption $\lg d(x,y) \geq k_{\max} = \max \mathbb{N}(x,y)$. Condition **(GWA2)** is just an estimate on the length of W. Condition **(GWA1)** is less transparent. It establishes what happens along W as we change the values of θ and λ to reach y. If $k \in \max \mathbb{N}(x,y)$ and we need only to change $\lambda(k)$, then after a "critical" vertex $w_{s(k)}$ we will always move through edges where either $\lambda(k) = \lambda_y(k)$ or $\lambda(k) = \{\spadesuit\}$ (this second option occurs when we need to get closer to a socket point of order $j > k$). If $k \in \max \mathbb{N}(x,y)$ and we need to change $\theta(k)$ (and possibly also $\lambda(k)$), then we need to pass through a "critical" vertex $w_{s(k)}$ which is a socket point and after passing through it $\theta(k)$ will remain equal to $\theta_y(k)$. An important constraint is that the map $k \mapsto s(k)$ is monotone increasing. Note that in the case $k \notin \mathbb{N}(x,y)$ and $k < k_{\max}$ we just define $w_{s(k)}$ to be a (gluing) point along the walk so that (3.8) holds.

Theorem 3.7. *If* $\lg d(x,y) \geq k_{\max} = \max \mathbb{N}(x,y)$ *there is a good walk W from x to y which has the following additional properties:*

(GWA1) *If $k \in \mathbb{N}(x,y)$ is such that $\theta_x(k) = \theta_y(k)$, there is a distinguished gluing or socket point $w_{s(k)}$ such that each edge e preceding $w_{s(k)}$ satisfies $\lambda_e(k) = \lambda_x(k)$, and each edge e following $w_{s(k)}$ satisfies either $\lambda_e(k) = \lambda_y(k)$ or $\lambda_e(k) = \{\spadesuit\}$. Moreover, in this case all edges e satisfy $\theta_e(k) = \theta_x(k)$. If $k \in \mathbb{N}(x,y)$ is such that $\theta_x(k) \neq \theta_y(k)$, there is a distinguished socket point $w_{s(k)}$ such that each edge e preceding $w_{s(k)}$ satisfies $\theta_e(k) = \theta_x(k)$ and $\lambda_e(k) = \lambda_x(k)$, and each edge e following $w_{s(k)}$ satisfies $\theta_e(k) = \theta_y(k)$ and either $\lambda_e(k) = \lambda_y(k)$ or $\lambda_e(k) = \{\spadesuit\}$. Moreover, the map $k \mapsto s(k)$ is monotone increasing and the subwalk W_k from $w_{s(k)}$ to $w_{s(k+1)}$ satisfies:*

$$\mathrm{len}\, W_k \approx \sigma_{k+1} \approx d(w_{s(k)}, w_{s(k+1)}); \tag{3.8}$$

(GWA2) *The walk W satisfies:*

$$\mathrm{len}\, W \approx \max\left\{\left|\pi(x) - \pi(y)\right|, \sigma_{k_{\max}}\right\}. \tag{3.9}$$

Proof. Without loss of generality we can assume $\pi(x) \leq \pi(y)$. If $\mathbb{N}(x,y) = \emptyset$ then x, y lie in some $Q(\mathbb{R} \times \{\lambda\} \times \{\theta\})$ and the construction of the walk is immediate. Let w_0 be the vertex of G satisfying $\pi(w_0) \in [\pi(x), \pi(x)+1)$, $(\lambda_{w_0}, \theta_{w_0}) = (\lambda_x, \theta_x)$ (if the labels for w_0 or x are not unique, one can choose them so that equality holds. Note that for a non-unique label (λ_p, θ_p) only one entry $(\lambda_p(m), \theta_p(m))$ is not uniquely determined). Order the elements of $\mathbb{N}(x,y)$ increasingly:

$$k_0 < k_1 < \cdots < k_q. \tag{3.10}$$

Now either $\theta_x(k_0) = \theta_y(k_0)$ or $\theta_x(k_0) \neq \theta_y(k_0)$. The goal is to construct a walk W_{k_0} of length comparable to σ_{k_0} which allows to change the k_0-th entries of the labels. We build W_{k_0} in two parts $W^{(-)}_{k_0}$ and $W^{(+)}_{k_0}$.

We now consider the first case $\theta_x(k_0) = \theta_y(k_0)$ which implies $\lambda_x(k_0) \neq \lambda_y(k_0)$; by Lemma 2.41 we can find a monotone increasing walk $W^{(-)}_{k_0}$ from w_0 to a gluing or a socket point $v^{(-)}_{k_0}$ of order k_0 such that:

1. $\pi(v^{(-)}_{k_0}) \in \left[\pi(w_0) + \sigma_{k_0}, \pi(w_0) + C\sigma_{k_0}\right]$;
2. all edges of $W^{(-)}_{k_0}$ have the same labels $(\lambda_{w_0}, \theta_{w_0})$;

3. len $W_{k_0}^{(-)} \in [\sigma_{k_0}, C\sigma_{k_0}]$.

$W_{k_0}^{(-)}$ is the first part of the walk W_{k_0} and we let $w_{s(k_0)} = v_{k_0}^{(-)}$. Let $\tilde{\lambda}_{w_0}$ be the label which agrees with λ_{w_0} except at the k_0-th entry $\tilde{\lambda}_{w_0}(k_0) = \lambda_y(k_0)$. The second part of the walk $W_{k_0}^{(+)}$ is a monotone walk of length len $W_{k_0}^{(+)} \in [1, \sigma_{k_0}]$ which terminates at a vertex of order 0 and whose edges have the same label $(\tilde{\lambda}_{w_0}, \theta_{w_0})$. We now consider the second case $\theta_x(k_0) \neq \theta_y(k_0)$ which is slightly more complicated. By Lemma 2.42 we can find a label-nonincreasing monotone walk $W_{k_0}^{(-)}$ from w_0 to a socket point $v_{k_0}^{(-)}$ such that:

1. $\pi(v_{k_0}^{(-)}) \in \left[\pi(w_0) + \sigma_{k_0}, \pi(w_0) + C\sigma_{k_0}\right]$.
2. $v_{k_0}^{(-)}$ has order k_0 and for $l > k_0$ one has $(\lambda(v_{k_0}^{(-)}; l), \theta(v_{k_0}^{(-)}; l)) = (\lambda(w_0; l), \theta(w_0; l))$.
3. len $W_{k_0}^{(-)} \in [\sigma_{k_0}, C\sigma_{k_0}]$.

$W_{k_0}^{(-)}$ is the first part of the walk W_{k_0} and we let $w_{s(k_0)} = v_{k_0}^{(-)}$.

By Lemma 2.47 we find a label-nondecreasing monotone walk $W_{k_0}^{(+)}$ from $v_{k_0}^{(-)}$ to a vertex $v_{k_0}^{(+)}$ of order zero satisfying:

1. $\pi(v_{k_0}^{(+)}) \in \left[\pi(w_0) + \sigma_{k_0}, \pi(w_0) + C\sigma_{k_0}\right]$.
2. For $l \le k_0$ one has $(\lambda(v_{k_0}^{(+)}; l), \theta(v_{k_0}^{(+)}; l)) = (\lambda(y; l), \theta(y; l))$ and for $l > k_0$ $(\lambda(v_{k_0}^{(+)}; l), \theta(v_{k_0}^{(+)}; l)) = (\lambda(x; l), \theta(x; l))$.
3. len $W_{k_0}^{(+)} \in [\sigma_{k_0}, C\sigma_{k_0}]$.
4. All edges of $W_{k_0}^{(+)}$ satisfy $(\lambda_e(k_0), \theta_e(k_0)) = (\lambda_y(k_0), \theta_y(k_0))$.

The construction continues by induction on k_j, i.e. suppose we have constructed the subwalks $\left\{W_{k_0}, \cdots, W_{k_j}\right\}$ which form the first part of W. The first part $W_{k_{j+1}}^{(-)}$ of $W_{k_{j+1}}$ is a label-nonincreasing monotone walk $W_{k_{j+1}}^{(-)}$ from $v_{k_j}^{(+)}$ to a socket point $v_{k_{j+1}}^{(-)}$ of order k_{j+1} such that:

1. $\pi(v_{k_{j+1}}^{(-)}) \in \left[\pi(v_{k_j}^{(+)}) + \sigma_{k_{j+1}}, \pi(v_{k_j}^{(+)}) + C\sigma_{k_{j+1}}\right]$.
2. $v_{k_{j+1}}^{(-)}$ has order k_{j+1} and for $l > k_{j+1}$ one has $(\lambda(v_{k_{j+1}}^{(-)}; l), \theta(v_{k_{j+1}}^{(-)}; l)) = (\lambda(v_{k_j}^{(+)}; l), \theta(v_{k_j}^{(+)}; l))$.
3. len $W_{k_{j+1}}^{(-)} \in [\sigma_{k_{j+1}}, C\sigma_{k_{j+1}}]$.

We then let $w_{s(k_{j+1})} = v_{k_{j+1}}^{(-)}$.

By Lemma 2.47 we complete $W_{k_{j+1}}$ by finding a label-nondecreasing monotone walk $W_{k_{j+1}}^{(+)}$ from $v_{k_{j+1}}^{(-)}$ to a vertex $v_{k_{j+1}}^{(+)}$ such that:

1. $\pi(v_{k_{j+1}}^{(+)}) \in \left[\pi(v_{k_{j+1}}^{(-)}) + \sigma_{k_{j+1}}, \pi(v_{k_{j+1}}^{(-)}) + C\sigma_{k_{j+1}}\right]$.
2. For $l \le k_{j+1}$ one has $(\lambda(v_{k_{j+1}}^{(+)}; l), \theta(v_{k_{j+1}}^{(+)}; l)) = (\lambda(y; l), \theta(y; l))$ and for $l > k_{j+1}$ $(\lambda(v_{k_{j+1}}^{(+)}; l), \theta(v_{k_{j+1}}^{(+)}; l)) = (\lambda(x; l), \theta(x; l))$.
3. len $W_{k_{j+1}}^{(+)} \in [\sigma_{k_{j+1}}, C\sigma_{k_{j+1}}]$.
4. All edges of $W_{k_{j+1}}^{(+)}$ satisfy $(\lambda_e(k_{j+1}), \theta_e(k_{j+1})) = (\lambda_y(k_{j+1}), \theta_y(k_{j+1}))$.

When we reach $j = q$ we have constructed the first part $W^{(1)}$ of the walk W. Property **(GW3)** is satisfied because $W^{(1)}$ is monotone increasing and the part of **(GW2)** concerning w_0 is also satisfied; the additional condition **(GWA1)** is also satisfied on $W^{(1)}$, and needs only to be checked there because of the way in which we construct the second part $W^{(2)}$ of the walk.

There are two cases to consider to complete the proof.

(Case 1): $\pi(v_{k_q}^{(+)}) \le \pi(y)$; then $v_{k_q}^{(+)}$ and y belong to $Q(\mathbb{R} \times \{\lambda_y\} \times \{\theta_y\})$. Therefore, $W^{(2)}$ is constructed by taking a geodesic walk in $Q(\mathbb{R} \times \{\lambda_y\} \times \{\theta_y\})$ from $v_{k_q}^{(+)}$ to y. We need only to prove **(GW1)** which is a consequence of

(GWA2):

$$\begin{aligned} \operatorname{len} W &= \sum_{j=0}^{q} (\operatorname{len} W_{k_j}^{(-)} + \operatorname{len} W_{k_j}^{(+)}) + \pi(y) - \pi(v_{k_q}^{(+)}) \\ &\le C \sum_{j=0}^{q} \sigma_{k_j} + \pi(y) - \pi(v_{k_q}^{(+)}) \\ &\le C\sigma_{k_q} + \pi(y) - \pi(v_{k_q}^{(+)}); \end{aligned} \tag{3.11}$$

however, $\pi(x) \le \pi(v_{k_q}^{(-)}) \le \pi(y)$ and so $\sigma_{k_q} \le \pi(y) - \pi(x)$ which implies:

$$\operatorname{len} W \le C(\pi(y) - \pi(x)) \le Cd(x, y). \tag{3.12}$$

As π is 1-Lipschitz and as W is monotone increasing, we have len $W \ge \pi(y) - \pi(x)$ which completes the proof of **(GWA2)**.

(Case 2): $\pi(y) < \pi(v_{k_q}^{(+)})$; then $v_{k_q}^{(+)}$ and y belong to $Q(\mathbb{R} \times \{\lambda_y\} \times \{\theta_y\})$ and $W^{(2)}$ is constructed by taking a geodesic walk in $Q(\mathbb{R} \times \{\lambda_y\} \times \{\theta_y\})$ from $v_{k_q}^{(+)}$ to y; note that $W^{(2)}$ is monotone decreasing. Let:

$$W^{(2)} = \{z_0, \cdots, z_m = v_y\}, \tag{3.13}$$

where v_y is the unique vertex satisfying $(\lambda_y, \theta_y) = (\lambda_{v_y}, \theta_{v_y})$ and $\pi(v_y) \in [\pi(y), \pi(y) + 1)$. Note that:

$$\pi(x) \le \pi(y) \le \pi(v_y) \le C\sigma_{k_q} + \pi(x), \tag{3.14}$$

and so

$$\begin{aligned} \sigma_{k_q} = \operatorname{len} W &= \sum_{j=0}^{q} (\operatorname{len} W_{k_j}^{(-)} + \operatorname{len} W_{k_j}^{(+)}) + \pi(v_{k_q}^{(+)}) - \pi(y) \\ &\le C\sigma_{k_q} \le Cd(x, y), \end{aligned} \tag{3.15}$$

which establishes **(GW1)**, **(GWA2)** and the part of **(GW2)** concerning w_L.

If $\pi(z_m) \ge \pi(x) + \sigma_{k_q}/2$ then **(GW3)** holds for some universal constant C. Otherwise, let $m_0 \le m$ denote the first integer so that:

$$\pi(z_{m_0}) < \pi(x) + \sigma_{k_q}/2; \tag{3.16}$$

for $\tilde{m} \ge m_0$ we have $d(z_{\tilde{m}}, z_m) < \sigma_{k_q}/2$ as $W^{(2)}$ is a monotone decreasing geodesic walk; thus:

$$d(z_{\tilde{m}}, x) \ge d(z_m, x) - \sigma_{k_q}/2 \ge \sigma_{k_{\max}}/2 - 1, \tag{3.17}$$

and so **(GW3)** holds for some universal constant C (recall that $k_q = k_{\max}$). □

In the following theorem we complete the construction of good walks by analyzing the case $\lg d(x, y) < k_{\max} = \max \mathbb{N}(x, y)$; essentially this means that, as $d(x, y)$ is less than $\sigma_{k_{\max}}$, one is forced to choose a particular socket or gluing point to change $(\lambda_x(k_{\max}), \theta_x(k_{\max}))$ to $(\lambda_y(k_{\max}), \theta_y(k_{\max}))$. Specifically, one should think about the situation where $d(x, y)$ is insignificant next to $\sigma_{k_{\max}}$, which means that geodesics from x to y must pass near a given gluing or socket point. The following condition **(GWA3)** essentially says that we can find a gluing or socket point $u_{k_{\max}}$ (which must be a socket point if $\theta_x(k_{\max}) \neq \theta_y(k_{\max})$), then construct good walks W_x and W_y from x to $u_{k_{\max}}$ and $u_{k_{\max}}$ to y (respectively), which satisfy the conclusions of Theorem 3.7, and finally obtain W concatenating W_x and W_y.

Theorem 3.18. *If* $\lg d(x, y) < k_{\max} = \max \mathbb{N}(x, y)$ *then there is a good walk* W *from* x *to* y *which has the following additional property:*

(GWA3) *If $\theta(x, k_{\max}) = \theta(y, k_{\max})$ there is a distinguished gluing or socket point $u_{k_{\max}} \in W$ of order $k_{\max}$ such that each edge e preceding $u_{k_{\max}}$ satisfies $\lambda(e; k_{\max}) = \lambda(x; k_{\max})$ and each edge following $u_{k_{\max}}$ satisfies $\lambda(e; k_{\max}) = \lambda(y; k_{\max})$. Moreover, in this case all edges e of W satisfy $\theta(e; k_{\max}) = \theta(x; k_{\max})$. On the other hand, if $\theta(x, k_{\max}) \neq \theta(y, k_{\max})$ there is a distinguished socket point $u_{k_{\max}}$ such that each edge preceding $u_{k_{\max}}$ satisfies $(\lambda(e; k_{\max}), \theta(e; k_{\max})) = (\lambda(x; k_{\max}), \theta(e; k_{\max}))$ and each edge e following $u_{k_{\max}}$ satisfies $(\lambda(e; k_{\max}), \theta(e; k_{\max})) = (\lambda(y; k_{\max}), \theta(y; k_{\max}))$. Moreover, W can be decomposed into consecutive walks W_x and W_y where W_x is a good walk from x to $u_{k_{\max}}$ satisfying the conclusion of Theorem 3.7, and W_y is a good walk from $u_{k_{\max}}$ to y satisfying the conclusion of Theorem 3.7.*

Proof. The construction in the cases $\theta_x(k_{\max}) = \theta_y(k_{\max})$ and $\theta_x(k_{\max}) \neq \theta_y(k_{\max})$ is essentially the same, and we thus discuss only the latter case. The properties of the labels $(\lambda(e; k_{\max}), \theta(e; k_{\max}))$ follow from the construction and Theorem 3.7.

Take a geodesic walk W from x to y. Note that there must be a socket point $\tilde{u} \in W$ of order $k_{\max}$ so that:

$$d(x, \tilde{u}) + d(\tilde{u}, y) = d(x, y); \tag{3.19}$$

moreover, let $\mathcal{U}$ denote the set of socket points of order $k_{\max}$ and let $u_{k_{\max}}$ be an element of $\mathcal{U}$ at minimal distance from x so that $d(x, u_{k_{\max}}) \le d(x, \tilde{u}) \le d(x, y)$. Let $k \in \mathbb{N}(x, u_{k_{\max}})$; then if $k > k_{\max}$ a geodesic walk W from x to $u_{k_{\max}}$ would pass through either a gluing or a socket point of order k and by Lemma 3.4 we would have:

$$d(x, u_{k_{\max}}) = \operatorname{len} W \ge \sigma_{k_{\max}} > d(x, y), \tag{3.20}$$

yielding a contradiction. Hence $k \le k_{\max}$; note that $(\lambda(u_{k_{\max}}; k_{\max}), \theta(u_{k_{\max}}; k_{\max}))$ can take any value, and hence $k < k_{\max}$; we can then take a geodesic walk from x to $u_{k_{\max}}$ which must pass through either a gluing or a socket point of order k, and we apply Lemma 3.4 to conclude that:

$$d(x, u_{k_{\max}}) = \operatorname{len} W \ge \sigma_k. \tag{3.21}$$

Thus we can apply Theorem 3.7 to obtain a good walk W_x from x to $u_{k_{\max}}$. Note that (3.20) implies that $(\lambda(u_{k_{\max}}; l), \theta(u_{k_{\max}}; l)) = (\lambda(x; l), \theta(x; l))$ for $l > k_{\max}$; in particular, as $k_{\max} = \max \mathbb{N}(x, y)$, if $k \in \mathbb{N}(u_{k_{\max}}, y)$ we have $k < k_{\max}$. Let W be a geodesic walk from $u_{k_{\max}}$ to y; then it must pass through either a gluing or a socket point of order k and Lemma 3.4 implies:

$$d(y, u_{k_{\max}}) = \operatorname{len} W \ge \sigma_k; \tag{3.22}$$

therefore, we can apply Theorem 3.7 to obtain a good walk W_y from $u_{k_{\max}}$ to y. For later reference, we also note here that:

$$d(x, u_{k_{\max}}) + d(y, u_{k_{\max}}) \in [d(x, y), 3d(x, y)]. \tag{3.23}$$

The walk W is obtained by concatenating W_x and W_y so that it satisfies **(GWA3)**. Property **(GW1)** follows observing that:

$$\operatorname{len} W = \operatorname{len} W_x + \operatorname{len} W_y \le C\left(d(x, u_{k_{\max}}) + d(u_{k_{\max}}, y)\right), \tag{3.24}$$

and using (3.23) to conclude that:

$$\operatorname{len} W \le C d(x, y). \tag{3.25}$$

Property **(GW2)** holds because it holds for W_x and W_y. We discuss property **(GW3)** in some cases. We will denote by $C_1 \ge 2$ the constant in **(GW3)** provided by Theorem 3.7. In the following we use the notations $k^{(x)}_{\max} = \max \mathbb{N}(x, u_{k_{\max}})$ and $k^{(y)}_{\max} = \max \mathbb{N}(u_{k_{\max}}, y)$.

`(Case 1)`: $\pi(x) \le \pi(u_{k_{\max}}) \le \pi(y)$.

`(Case 1,1)`: W_x and W_y are both monotone. Then W is monotone and **(GW3)** holds.

`(Case 1,2)`: W_x is not monotone and W_y is monotone. As in Theorem 3.7 we decompose W_x is a first part $W_x^{(m)}$ which is monotone, and a second part $\{z_0, \cdots, z_m = u_{k_{\max}}\}$. Then $\operatorname{len} W_x \approx \sigma_{k^{(x)}_{\max}}$ and $d(z_i, x) \ge j_x(i)/C$ there

$j_x(i)$ is the index / position of z_i in the walk W_x, and C is a universal constant. Let $w \in W_y$ and $j_y(w)$ denote the position of w in W_y and $j(w)$ the position in W. If $j_y(w) < 2 \operatorname{len} W_x$, then $j(w) \le 3 \operatorname{len} W_x$ and so:

$$d(w,x) \ge d(w, z_m = u_{k_{\max}}) \ge \frac{\operatorname{len} W_x}{C_1} \ge \frac{j(w)}{3C_1}. \tag{3.26}$$

If $j_y(w) > 2 \operatorname{len} W_x$, then $d(w,x) \ge d(w, u_{k_{\max}}) - d(u_{k_{\max}}, x)$; as W_y is monotone we have $d(w, u_{k_{\max}}) \ge j_y(w)$ and so:

$$d(w,x) \ge j_y(w) - \operatorname{len} W_x \ge \frac{j_y(w)}{2}; \tag{3.27}$$

thus

$$j(w) = j_y(w) + \operatorname{len} W_x \le \frac{3}{2} j_y(w), \tag{3.28}$$

and so

$$d(w,x) > \frac{j(w)}{3}. \tag{3.29}$$

(`Case 1,3`): Suppose that W_x is monotone but W_y is not. As in Theorem 3.7 we decompose W_y in a first part $W_y^{(m)}$ which is monotone and a second part $\{z_0, \cdots, z_m = v_y\}$. On $W_y^{(m)}$ we obtain **(GW3)** as in (`Case 1,1`).

Note that:

$$\operatorname{len} W_x \approx \pi(u_{k_{\max}}) - \pi(x) \tag{3.30}$$

$$\operatorname{len} W_y \approx \operatorname{len} W_y^{(m)} + m \approx \sigma_{k_{\max}^{(y)}} \approx d(u_{k_{\max}}, y). \tag{3.31}$$

Note that for each i we have $d(z_i, u_{k_{\max}}) \ge \sigma_{k_{\max}^{(y)}}/C_1$. If $\pi(z_i) \ge \pi(u) + \sigma_{k_{\max}^{(y)}}/2$ we conclude that:

$$\begin{aligned} d(z_i, x) &\ge \pi(u_{k_{\max}}) - \pi(x) + \frac{\sigma_{k_{\max}^{(y)}}}{2} \gtrsim d(x, u_{k_{\max}}) + d(u_{k_{\max}}, y) \\ &\gtrsim_{(*)} d(x,y) \end{aligned} \tag{3.32}$$

where in (*) we used (3.23) and where the constant in the lower bound can be explicitly estimated in terms of C_1.

Suppose that $\pi(z_i) \in \left[\pi(u_{k_{\max}}), \pi(u_{k_{\max}}) + \sigma_{k_{\max}^{(y)}}/2\right)$. Then any geodesic walk from x to z_i must pass through some socket point $\tilde{u} \in \mathcal{U}$, and we would also have $k_{\max}^{(y)} \in \mathbb{N}(\tilde{u}, y)$ so that:

$$\begin{aligned} d(x, z_i) &\ge d(\tilde{u}, x) + \sigma_{k_{\max}^{(y)}} \ge d(x, u_{k_{\max}}) + \sigma_{k_{\max}^{(y)}} \\ &\gtrsim d(x, u_{k_{\max}}) + d(u_{k_{\max}}, y) \gtrsim d(x,y). \end{aligned} \tag{3.33}$$

The bounds (3.32), (3.33) imply that **(GW3)** holds on $\{z_0, \cdots, z_m\}$ with a constant that can be computed in terms of C_1.

(`Case 1,4`): W_x and W_y are both not monotone. The argument for (`Case 1,3`) can be adapted noting that $d(x, u_{k_{\max}}) \approx \sigma_{k_{\max}^{(x)}}$.

(`Case 2`): $\pi(u_{k_{\max}}) \le \pi(x) \le \pi(y)$. After reaching $u_{k_{\max}}$, the walk W starts to move in the direction of increasing values of π.

(`Case 2,1`): W_y is monotone. There is a $\theta > 0$ depending only on **(P2)** so that $\sigma_{l+\theta} \ge 3\sigma_l$ for each l, and there is a C_θ depending on **(P2)** so that $\sigma_{l+\theta} \le C_\theta \sigma_l$ for each l. Let $l = \lceil \lg d(x, u_{k_{\max}}) \rceil$ and fix $w \in W_y$. If $j(w) \le \sigma_{l+\theta}$ we have that any walk from x to w must pass through a socket point of order $k_{\max}$ and so:

$$\begin{aligned} d(w,x) &\ge d(x, u_{k_{\max}}) \gtrsim \sigma_l \gtrsim \sigma_{l+\theta} \\ &\ge j(w). \end{aligned} \tag{3.34}$$

Let $j(w) > \sigma_{l+\theta}$; then $d(w,x) \ge d(w, u_{k_{\max}}) - d(u_{k_{\max}}, x)$; as W_y is monotone, $d(w, u_{k_{\max}}) \ge j_y(w)$ and so:

$$d(w,x) \ge j_y(w) - \sigma_l \gtrsim j_y(w); \tag{3.35}$$

but:

$$\begin{aligned} j(w) = j_y(w) + \operatorname{len} W_x &\lesssim j_y(w) + \sigma_l \\ &\lesssim j_y(w), \end{aligned} \tag{3.36}$$

and so $d(w, x) \gtrsim j(w)$ where the constant in the lower bound can be estimated in terms of C_1, C_θ and θ.

(`Case 2,2`): W_y is not monotone. We decompose W_y as $W_y^{(m)} \cup \{z_0, \cdots, z_m = v_y\}$ and note that we can use (`Case 2,1`) on W_y. For $\{z_0, \cdots, z_m = v_y\}$ one can adapt the argument used in (`Case 1,3`).

(`Case 3`): $\pi(x) \leq \pi(y) \leq \pi(u_{k_{\max}})$. This case can be dealt with along the lines of (`Case 2`) except in the case in which W_y is not monotone, where a different estimate is required on the terminal part $\{z_0, \cdots, z_m = v_y\}$. Any walk from x to z_i must pass through socket points of orders $k_{\max}$ and $k_{\max}^{(y)}$ so that:

$$d(z_i, x) \geq d(x, u_{k_{\max}}) + \sigma_{k_{\max}^{(y)}}; \tag{3.37}$$

but W_y is not monotone, which implies $\sigma_{k_{\max}^{(y)}} \approx \operatorname{len} W_y$ which gives:

$$d(z_i, x) \gtrsim d(x, u_{k_{\max}}) + j_y(z_i); \tag{3.38}$$

but $d(x, u_{k_{\max}}) \gtrsim \operatorname{len} W_x$ and $j(z_i) = \operatorname{len} W_x + j_y(w_i)$ so that $d(z_i, x) \gtrsim j(z_i)$. □

4 The exponents for which the Poincaré inequality holds

4.1 Geometric characterizations of the Poincaré inequality

The proof of the Poincaré inequality will involve the construction of families of curves joining points in G. Overall, we have preferred to avoid using the language of pencils of curves employed by [10, 26], and preferred a probabilistic language. The rationale is that our construction is naturally modelled by Markov chains, a fact that also occurrs in the examples [4]. Specifically, we will deal with measurable functions defined on a probability space which take value in the set of (Lipschitz) curves on a metric space X; such maps will be called **random curves**. To a random curve Γ one can associate a measurable function defined on the same probability space and which takes values in the space of Radon measures on X by $\Gamma \mapsto \|\Gamma\|$ (the length measure); such a map will be called a **random measure**. Finally, the maps to the end and starting points of Γ, $\Gamma \mapsto \operatorname{end} \Gamma$ and $\Gamma \mapsto \operatorname{str} \Gamma$, produce **random points** in X. Here for a random point we just mean a measurable function defined on a probability space which takes values in the set of points of X; alternatively, one can think of a random point in terms of sampling points of X according to some probability measure P, which is the **law** of the random point. In particular, as a random curve Γ can be also thought in terms of choosing a curve according to some probability law, the extremes of Γ will be random points.

Finally, the support $\operatorname{spt} \Gamma$ of a random curve Γ is the set of edges that Γ crosses in positive measure with positive probability:

$$\operatorname{spt} \Gamma = \left\{ e : P_\Gamma(\|\Gamma\|(e) > 0) > 0 \right\}. \tag{4.1}$$

To disprove the Poincaré inequality we will use the notion of modulus of families of curves, which we now recall.

Definition 4.2. Let $P \geq 1$ and A be a family of locally rectifiable curves in the metric space X. We say that a Borel function $g : X \to [0, \infty]$ is admissible for A if for each $\gamma \in A$ one has:

$$\int g \, d\|\gamma\| \geq 1. \tag{4.3}$$

Having fixed a background measure ν on X, we define the P-modulus of A, $\operatorname{mod}_P(A)$, as the infimum of:

$$\int g^p \, d\nu \tag{4.4}$$

where g ranges over the set of functions admissible for A. We will be mainly interested in modulus when A is the family $A_{p,q}$ of locally rectifiable curves connecting two points p, q, and when ν is of the form:

$$\mu_{p,q}^{(C)} = \left(\frac{d(p,\cdot)}{\mu\left(B(p,d(p,\cdot))\right)} \chi_{B(p,Cd(p,q))} + \frac{d(q,\cdot)}{\mu\left(B(q,d(q,\cdot))\right)} \chi_{B(q,Cd(p,q))} \right) \mu, \tag{4.5}$$

where μ is a doubling measure on X and $C > 0$. In this case we will use the notation $\mathrm{mod}_P(p,q;\mu_{p,q}^{(C)})$ for the modulus of $A_{p,q}$ when the background measure is $\mu_{p,q}^{(C)}$.

We finally recall the definition of the **Riesz potential centred on** p:

$$\mu_p = \frac{d(p,\cdot)}{\mu\left(B(p,d(p,\cdot))\right)} \mu. \tag{4.6}$$

The following Theorem summarizes a geometric characterization of $(1,P)$-Poincaré inequalities. It combines results of Heinonen-Koskela [11], Hajłasz-Koskela [8], Keith [13], and Ambrosio, Di Marino and Savaré [1], and the proof is included just for the sake of completeness. Note that we will take Theorem 4.7 as the working definition of the Poincaré inequality, and so we will not need to recall the usual definition of the Poincaré inequality.

Theorem 4.7. *Let (X,μ) be a complete doubling metric measure space; then $P \in \mathrm{I}_{\mathrm{PI}}(X,\mu)$ if and only if one of the following equivalent conditions holds:*

1. *There is a universal constant C such that for each pair of points $p, q \in X$ one has:*

$$d(p,q)^{P-1} \mathrm{mod}_P(p,q;\mu_{p,q}^{(C)}) \geq C; \tag{4.8}$$

2. *There is a universal constant C such that any pair of points p, q can be joined by a random curve Γ satisfying:*

$$\left\| \frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu_{p,q}^{(C)}} \right\|_{\mathrm{L}^Q(\mu_{p,q}^{(C)})}^{Q} \leq C d(p,q). \tag{4.9}$$

Proof. The characterization of the Poincaré inequality in terms of (4.8) is due to Keith [13], who built on previous results of Heinonen-Koskela [10, 11], and Hajłasz-Koskela [8].

`Step 1: (1) implies (2).`

Consider the set A of locally rectifiable curves joining p to q; fix M large to be determined later and write $A = A_{\mathrm{exit}} \cup A_{\mathrm{long}} \cup A_{\mathrm{good}}$, where:

1. A_{exit} consists of the locally rectifiable curves in A which meet $X \setminus \bar{B}(\{p,q\}, Cd(p,q))$ in positive length;
2. A_{long} are the locally rectifiable curves in $A \setminus A_{\mathrm{exit}}$ which have length $\geq Md(x,y)$;
3. A_{good} are the rectifiable curves in $A \setminus (A_{\mathrm{exit}} \cup A_{\mathrm{long}})$.

We will now fix $\mu_{p,q}^{(C)}$ as the background measure with respect to which we compute moduli; using the test functions $g_{\mathrm{exit}} = 0$ on $\bar{B}(\{p,q\}, Cd(p,q))$ and $g_{\mathrm{exit}} = \infty$ elsewhere, and $g_{\mathrm{long}} = Md(p,q)$ on $\bar{B}(\{p,q\}, Cd(p,q))$ and 0 elsewhere, we see that:

$$\mathrm{mod}_P(A_{\mathrm{exit}}) = 0 \tag{4.10}$$

$$\mathrm{mod}_P(A_{\mathrm{long}}) \lesssim \frac{d(p,q)}{(Md(p,q))^P}; \tag{4.11}$$

thus for M sufficiently large,

$$d(p,q)^{P-1} \mathrm{mod}_P(A_{\mathrm{good}}) \geq C/2. \tag{4.12}$$

Instead of computing modulus on A_{good} we can compute it on the family of measures:

$$\Sigma_{\mathrm{good}} = \left\{ \mathcal{H}^1_\gamma : \gamma \in A_{\mathrm{good}} \right\} \tag{4.13}$$

Applying the main result of [1] we get a probability π on Σ_{good} such that, denoting by $\nu = \int_{\Sigma_{\text{good}}} \eta \, d\pi(\eta)$, we get:

$$\left\| \frac{d\nu}{d\mu^{(C)}_{p,q}} \right\|_{\mathrm{L}^Q(\mu^{(C)}_{p,q})} = \mathrm{mod}_P(\Sigma_{\text{good}})^{-1/P}; \tag{4.14}$$

using (4.12) we conclude that:

$$\left\| \frac{d\nu}{d\mu^{(C)}_{p,q}} \right\|_{\mathrm{L}^Q(\mu^{(C)}_{p,q})} \lesssim d(p,q)^{1/Q}. \tag{4.15}$$

Now, to each $\eta \in \Sigma_{\text{good}}$ we can associate a unique unit-speed curve $\gamma : [0, \operatorname{len}\gamma] \to X$ such that $\mathcal{H}^1_\gamma = \eta$. Thus π becomes the law of a random curve Γ with $\mathrm{E}[\|\Gamma\|] = \nu$ and then (4.9) follows from (4.15).

`Step 2: (2) implies (1).`

Take a random curve Γ satisfying (4.9) and let g be admissible for the curves joining p to q. Then:

$$\begin{aligned} 1 &\le \mathrm{E}\left[\int g \, d\|\Gamma\|\right] = \int g \, d\,\mathrm{E}[\|\Gamma\|] \\ &\le \|g\|_{\mathrm{L}^P(\mu^{(C)}_{p,q})} \left\| \frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu^{(C)}_{p,q}} \right\|_{\mathrm{L}^Q(\mu^{(C)}_{p,q})} \\ &\le C\|g\|_{\mathrm{L}^P(\mu^{(C)}_{p,q})} \cdot d(p,q)^{1/Q}, \end{aligned} \tag{4.16}$$

and (4.8) follows minimizing in g. □

4.2 Construction of Random curves

In this subsection we construct the ingredients to build the random curves used to verify the Poincaré inequality. This is the subsection where most of the technical work takes place. As we work with walks but need to produce random curves, we define the Lipschitz path associated to a walk as follows.

Definition 4.17. To a walk $W = \{w_0\, e_1\, w_1 \cdots e_l\, w_l\}$ we can canonically associate a 1-Lipschitz map Γ_W : $[0, \operatorname{len} W] \to G$ by letting $\Gamma_W|[l, l+1]$ be a unit speed parametrization of the edge e_l.

Our construction requires 3 building blocks, which are random curves that satisfy some constraints. These random curves will then be concatenated in the next subsection. As an overview we offer the following informal discussion:

- Theorem 4.23 associates to a monotone walk a random curve which gets "compressed" through a socket point. This situation arises when a random curve Γ joining x to y must pass through a given socket point ξ. In this case there will be a t_ξ such that $\Gamma(t_\xi) = \xi$ and so as $t \to t_\xi$ the random point $\Gamma(t)$ gets closer to ξ. As there is a constraint on the labels of ξ and as Γ is Lipschitz, the set of possible labels of the random point $\Gamma(t)$ will shrink as t approaches t_ξ. Intuitively, to prove a Poincaré inequality one must show that this shrinkage is not too fast, otherwise one cannot satisfy (4.9).
- Theorem 4.35 associates to a monotone walk W_0 a random curve which moves "parallel" to W_0. This situation arises when we have a random curve Γ which can take a finite set of values which are all *lifts* (compare Definition 4.18) of a given curve.
- Theorem 4.47 which explains how to "expand" a random curve so that as t increases the set of possible labels for $\Gamma(t)$ increases. Note that this situation is already familiar in the classical Poincaré inequality. For example, consider a random curve Γ joining x and y with $\operatorname{dom}\Gamma = [0, L]$ which is used to verify a Poincaré inequality by proving (4.9). One expects that as $t \to L/2$ the random point $\Gamma(t)$ can take a broader set of values, leading to a more diffused probability measure. On the other hand, as $t \to 0$ (resp. $t \to L$) one expects that the probability associated to $\Gamma(t)$ concentrates on x (resp. y).

We now define a notion of lift for walks used in the subsequent constructions. The idea is that given points w_0 and w_0' satisfying $\pi(w_0') = \pi(w_0)$ we can canonically lift a walk starting at w_0 to a walk starting at w_0'.

Definition 4.18. Let $W = \{w_0\, e_1\, w_1 \cdots e_l\, w_l\}$ and w_0' a point such that $\pi(w_0') = \pi(w_0)$. We construct a new walk $\{w_0'\, e_1'\, w_1' \cdots e_l'\, w_l'\}$ as follows. The vertex w_{i+1}' is adjacent to w_i' and is determined as follows. If w_i' is not a socket point the requirement $\pi(w_{i+1}') = \pi(w_{i+1})$ uniquely determines w_{i+1}'. Otherwise, assume that w_i' is a socket point of order k and let e_{i+1}' denote the edge between w_i' and w_{i+1}'. We require that $\lambda(e_{i+1}'; k) = \lambda(e_{i+1}; k)$ and $\theta(e_{i+1}'; k) = \theta(e_{i+1}; k)$ for all k. We say that W' is the **lift of W starting at** w_0' and we will denote it by $w_0' \cdot W$.

We now add some auxiliary definitions used in the constructions, e.g. when concatenating random curves. The idea is that when we need to concatenate a random curve Γ_0 to a random curve Γ_1 we need the probability measures associated to end Γ_0 and str Γ_1 to be compatible. We thus introduce canonical probabilities on subsets of $\pi^{-1}(s)$ (where $s \in \mathbb{Z}$) determined by constraints on λ and θ.

Definition 4.19. Let $p \in G$ a vertex with $\operatorname{ord}(p) = 0$ and $k \in \mathbb{N}$. Let $F(p, k)$ denote the set of those $p' \in G$ satisfying $\pi(p') = \pi(p)$ and $(\lambda_{p'}(l), \theta_{p'}(l)) = (\lambda_p(l), \theta_p(l))$ for $l > k$. For $k = 0$ we let $F(p, 0) = \{p\}$. To $F(p, k)$ we can associate a canonical probability measure P, which can be also thought of as the law of a random point in $F(p, k)$. The probability P satisfies:

$$\frac{P(p')}{P(p'')} = \frac{w((\lambda_{p'}, \theta_{p'}))}{w((\lambda_{p''}, \theta_{p''}))} \quad (\forall p, p' \in F(p, k)). \tag{4.20}$$

For $p' \in F(p, k)$ denote by $s(p')$ the finite string of pairs $\{(\lambda_p(j), \theta_p(j))\}_{j \le k}$; then:

$$P(p') = (C_{\mathrm{gw},1} C_{\mathrm{gw},2})^{-k} w(s(p')). \tag{4.21}$$

Given $F(p_0, k)$, $F(p_1, k)$ we define a canonical map $\tau : F(p_0, k) \to F(p_1, k)$ so that $\tau(p_0')$ is the unique point $p_1' \in F(p_1, k)$ such that $s(p_0') = s(p_1')$. Note that $\tau_\# P_0 = P_1$.

Let $p \in G$ a vertex and $k \in \mathbb{N}$. We denote by $F_\Theta(p, k)$ the set of those $p' \in G$ satisfying $\pi(p') = \pi(p)$, $\lambda_{p'} = \lambda_p$ and $\theta_{p'}(l) = \theta_p(l)$ for $l > k$. As above, to $F_\Theta(p, k)$ we associate a canonical probability P by requiring:

$$\frac{P(p')}{P(p'')} = \frac{w(\theta_{p'})}{w(\theta_{p''})} \quad (\forall p, p' \in F_\Theta(p, k)). \tag{4.22}$$

We now present the construction of a random curve which goes through a socket point ξ in G if one has a walk that passes through ξ. In the following, given a walk $W = \{w_0\, e_1\, w_1 \cdots e_l\, w_l\}$ we denote by W^{-1} the reversed walk $\{w_l, e_l, w_{l-1}, \cdots, e_1, w_0\}$.

Theorem 4.23. *Let W_0 be a monotone walk. Let $p_0 = \operatorname{str} W_0$, $\xi = \operatorname{end} W_0$. Assume that:*

(H1) $\operatorname{ord}(p_0) = 0$ *and ξ is a socket point of order $K \ge k$;*
(H2) $\operatorname{len} W_0 \in [\sigma_k, C_0\sigma_k]$ *and all edges of W_0 have the same Θ-label θ;*
(H3) *There are $(\tau_i)_{1 \le i \le k-1} \subset \mathbb{N} \cap [0, \operatorname{len} W_0]$ such that the map $i \mapsto \tau_i$ is strictly decreasing,* $\operatorname{len} W_0 - \tau_i \in [\sigma_i, C_0\sigma_i]$, *$w_{\tau_i}$ is either a gluing or a socket point of order i, and if $l \ge \tau_i + 1$ one has $\lambda(e_l; j) = \{\spadesuit\}$ for $i \le j \le k-1$;*
(H4) *If $w_s \in W$ satisfies* $\operatorname{ord}(w_s) \ge k$, *then $\lambda_{e_s} = \lambda_{e_{s+1}}$;*
(H5) *For an edge e_t of W_0 one has the following: if $t \in [1, \tau_{k-1}]$ then $\lambda_{e_t} = \lambda_{p_0}$; if $t \in (\tau_{i+1}, \tau_i]$ then $\lambda_{e_t}(l) = \lambda_{p_0}(l)$ for $l < i$ or $l \ge k$; if $t \in [\tau_1,$* $\operatorname{len} W_0]$ *then $\lambda_{e_t}(l) = \lambda_{p_0}(l)$ for $l \ge k$.*

Fix $J_{\mathrm{cut}} \in \mathbb{N} \cup \{0\}$ and let P_0 be the canonical probability on $F(p_0; k - J_{\mathrm{cut}})$. Construct a random curve Γ as follows: choose $p_0' \in F(p_0; k - J_{\mathrm{cut}})$ according to P_0 and let $\Gamma = \Gamma_{p_0' \cdot W_0}$. Then:

(C1) $\operatorname{end} \Gamma$ *has law P_1, where P_1 is the canonical probability on $F_\Theta(\xi; k - J_{\mathrm{cut}})$;*
(C2) $\operatorname{spt} \Gamma \subset B(\Gamma_W, C_1\sigma_{k-J_{\mathrm{cut}}})$*;*

(C3) *To each* $e \in \operatorname{spt}\Gamma$ *there is associated a unique* $\operatorname{in}(e)$ *such that* $\pi(e) = \pi(e_{\operatorname{in}(e)})$, *where* $e_{\operatorname{in}(e)}$ *is the* $\operatorname{in}(e)$*-th edge of* W_0, *and one has:*

$$\frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu}|e \approx_{C_1} C_{\mathrm{gw},1}^{-\lg(\operatorname{len} W_0 - \operatorname{in}(e))} C_{\mathrm{gw},2}^{-k+J_{\mathrm{cut}}} \times w_{\{\spadesuit\}}^{-k+\lg(\operatorname{len} W_0-\operatorname{in}(e))} \prod_{j=k}^{\infty} w(\lambda(e;j),\theta(e;j))^{-1}, \tag{4.24}$$

where C_1 *depends on* J_{cut}, C_0, **(P1)–(P3)** *and* Weight.

Remark 4.25. While the hypotheses **(H1)** and **(H2)** are clear, we offer more motivation for **(H3)–(H5)**. Condition **(H3)** is an assumption on how fast the labels of λ_e of the edges of W_0 approach λ_ξ. The point is that the entries of λ_e are switched to $\{\spadesuit\}$ in reverse order, from $k-1$ to 1, and that switching $\lambda_e(i)$ occurs at a distance from ξ comparable to σ_i. In condition **(H4)** we assume that if we pass through a gluing or socket point of order $> k$ we do not use it to change λ. Finally **(H5)** is a consistency condition for **(H3)**: we change as few labels as possible and after switching $\lambda(i)$ at w_{τ_i} we do not switch the value of $\lambda(i)$ again. Moreover, labels $\lambda(l)$ are never changed for $l \geq k$.

Concerning the conclusions, we point out that **(C3)** is the technical estimate quantifying that the "compression" of Γ is not too fast. This plays a crucial role in establishing the Poincaré inequality. Finally, note that J_{cut} is an integer parameter chosen for convenience, i.e. to create some "space" between the length of W and the maximum order of the entries of λ and θ that can differ from the corresponding values in λ_{p_0} and θ_{p_0}.

Proof. We prove **(C1)**. Let $\xi' = \operatorname{end}(p_0' \cdot W_0)$; we use the notation w_t, e_t for the vertices, respectively the edges of W_0; we use the notation w_t', e_t' for the corresponding edges and vertices of $p_0' \cdot W_0$. We note that if $t \geq \tau_i + 1$ **(H3)** implies that $\lambda(e_t'; l) = \{\spadesuit\}$ for $i \leq l \leq k-1$. We thus conclude that $\lambda(\xi'; l) = \{\spadesuit\}$ for $l \leq k-1$; for $l \geq k$ the label $\lambda_{e_t'}$ coincides with that of λ_{e_t} and so we conclude that $\lambda(\xi'; l) = \lambda(\xi; l)$ for $l \geq k$. Therefore, ξ' is a socket point of order K. By **(H2)** all edges of W_0 have the same label θ, and this implies that all edges of $p_0' \cdot W_0$ have the same label $\theta_{p_0'}$. As $\pi(\xi') = \pi(\xi)$, we conclude that ξ' is the point of $F_\Theta(\xi; k - J_{\mathrm{cut}})$ with label $\theta_{p_0'}$ and thus **(C1)** follows.

We now prove **(C2)**. Note that the i-th vertices w_i, w_i' of W_0 and $p_0' \cdot W_0$ have $\pi(w_i) = \pi(w_i')$, and the labels $(\lambda(w_i), \theta(w_i))$, $(\lambda(w_i'), \theta(w_i'))$ and can differ only in the first $k - J_{\mathrm{cut}}$ entries. Hence **(C2)** follows from Lemma 2.49.

We now prove **(C3)**. First let $e \in \operatorname{spt}\Gamma$ and assume that $e = e_l' \in p_0' \cdot W_0$, $e = e_{\tilde l}'' \in p_0'' \cdot W_0$. As the path W_0 is monotone, $l = \tilde l$ and there is a unique edge e_s of W_0 such that $\pi(e) = \pi(e_s)$. We can thus associate to e the unique integer $\operatorname{in}(e) = s$. We now turn to the proof of (4.24). For $p_0' \in \Lambda(p_0, k - J_{\mathrm{cut}})$ we will denote by $e(p_0'; l)$ the l-th edge of $p_0' \cdot W_0$.

We now fix $e \in \operatorname{spt}\Gamma$ and assume that $\operatorname{in}(e) = s$. We first consider the case $s \in [1, \tau_{k-1}]$. Then by **(H5)** there is a unique $p_0' \in F(p_0; k - J_{\mathrm{cut}})$ such that e is the s-th edge of $p_0' \cdot W_0$. In this case by **(H2)–(H3)** $\lg(\operatorname{len} W_0 - \operatorname{in}(e))$ is comparable to k up to a multiplicative constant depending on C_0. Assume now that $s \in (\tau_i, \tau_{i+1}]$; then e is the s-the edge of $p_0' \cdot W_0$ if and only if:

$$\theta_{p_0'} = \theta_{e_s} \tag{4.26}$$

$$\lambda(p_0'; j) = \lambda(e; j) \quad (1 \leq j < i); \tag{4.27}$$

note also that in this case $\lg(\operatorname{len} W_0 - \operatorname{in}(e))$ is comparable to i. Finally by **(H3)** if $s \in [\tau_1, \operatorname{len} W_0]$ e is the s-th egdge of $p_0' \cdot W_0$ whenever $p_0' \in F(p_0; k - J_{\mathrm{cut}})$ satisfies $\theta_{p_0'} = \theta_e$. Note that in this case $\lg(\operatorname{len} W_0 - \operatorname{in}(e))$ is comparable to 1. We can now put all this information together:

$$\begin{aligned}\operatorname{weight}(\mathrm{E}[\|\Gamma\|]; e) &= \sum \left\{P_0(p_0') : e(p_0'; \operatorname{in}(e)) = e,\, p_0' \in F(p_0'; k - J_{\mathrm{cut}})\right\}\\ &\approx \sum \left\{P_0(p_0') : \theta_{p_0'} = \theta_e,\, \lambda\left(e(p_0'; \operatorname{in}(e)); j\right) = \lambda(e; j) \ \text{ for } j < \lg(\operatorname{len} W - \operatorname{in}(e))\right\}\\ &\approx C_{\mathrm{gw},2}^{-k+J_{\mathrm{cut}}} \prod_{j=1}^{k-J_{\mathrm{cut}}} w\left(\theta_e(j)\right) C_{\mathrm{gw},1}^{-\lg(\operatorname{len} W_0 - \operatorname{in}(e))} \times \prod_{j=1}^{\lg(\operatorname{len} W_0 - \operatorname{in}(e))} w(\lambda(e; j)).\end{aligned} \tag{4.28}$$

On the other hand,

$$\text{weight}(\mu; e) = \prod_{j=1}^{\infty} w\left(\lambda(e;j), \theta(e;j)\right) \tag{4.29}$$

and so (4.24) follows by taking the quotient of (4.28) and (4.29). □

Corollary 4.30. *Suppose that W_0 satisfies the assumptions of Theorem 4.23 and let $p \in G$. Assume that for some $C_0 > 0$ one has:*

$$\operatorname{dist}(p, \operatorname{spt}\Gamma) \approx_{C_0} \sigma_k. \tag{4.31}$$

Then there is a $C_1 = C_1(C_0, J_{\text{cut}})$ such that:

$$\left\| \frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu_p} \right\|_{\mathrm{L}^Q(\mu_p)}^{Q} \approx_{C_1} \sum_{l=1}^{k} (w_{\{\spadesuit\}}^{-1} C_{\text{gw},1})^{l(Q-1)} \sigma_{k-l}. \tag{4.32}$$

Proof. By assumption (4.31) we have that on the edges of $\operatorname{spt}\Gamma$:

$$\frac{d\mu_p}{d\mu} \approx_{C(C_0)} (C_{\text{gw},1} C_{\text{gw},2})^{-k} \prod_{n=k+1}^{\infty} w\left((\lambda(p;n), \theta(p;n))\right)^{-1}. \tag{4.33}$$

We now obtain the following estimate using that $W_0|[\tau_{i+1}, \operatorname{len} W_0]$ has a number of edges $\lesssim \sigma_i$:

$$\begin{aligned}
\left\| \frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu_p} \right\|_{\mathrm{L}^Q(\mu_p)}^{Q} &= \left(\sum_{\substack{e:\\ \operatorname{in}(e)\in[0,\tau_{k-1})}} + \sum_{i=k-1}^{1} \sum_{\substack{e:\\ \operatorname{in}(e)\in[\tau_{i+1},\tau_i)}} + \sum_{\substack{e:\\ \operatorname{in}(e)\in[\tau_1,\operatorname{len} W]}} \right) \left(\left(\frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu_p} |_e \right)^{Q} \frac{d\mu_p}{d\mu} |_e \times \text{weight}(\mu; e) \right) \\
&\approx \sum_{\substack{e:\\ \operatorname{in}(e)\in[0,\tau_{k-1})}} (C_{\text{gw},1} C_{\text{gw},2})^{-k} \prod_{n=k+1}^{\infty} w\left((\lambda(p;n), \theta(p;n)\right)^{-1} \text{weight}(\mu; e) \\
&\quad + \sum_{i=1}^{k-1} \sum_{\substack{e:\\ \operatorname{in}(e)\in[\tau_i,\tau_{i+1})}} (w_{\{\spadesuit\}}^{-1} C_{\text{gw},1})^{(k-i)Q} (C_{\text{gw},1} C_{\text{gw},2})^{-k} \times \prod_{n=k+1}^{\infty} w\left((\lambda(p;n), \theta(p;n)\right)^{-1} \text{weight}(\mu; e) \\
&\quad + \sum_{\substack{e:\\ \operatorname{in}(e)\in[\tau_k,\operatorname{len} W]}} (w_{\{\spadesuit\}}^{-1} C_{\text{gw},1})^{kQ} (C_{\text{gw},1} C_{\text{gw},2})^{-k} \times \prod_{n=k+1}^{\infty} w\left((\lambda(p;n), \theta(p;n)\right)^{-1} \text{weight}(\mu; e) \\
&\approx \sum_{l=1}^{k} (w_{\{\spadesuit\}}^{-1} C_{\text{gw}})^{l(Q-1)} \sigma_{k-l}.
\end{aligned} \tag{4.34}$$

□

In the following theorem we construct a random curve which moves "parallel" to a given walk W.

Theorem 4.35. *Let $W = \{w_0\, e_1\, w_1 \cdots e_l\, w_l\}$ be a monotone walk joining p_0 to p_1 where $\operatorname{ord}(p_i) = 0$. Let P_i denote the canonical probability measure on $F(p_i; k)$.*

To each $p_0' \in F(p_0; k)$ we associate a walk $W_{p_0'}$ as follows. We let $w_0' = p_0'$. Then, e_i' and (hence) w_{i+1}' are determined by w_i' and e_{i-1}' as follows. First $\pi(e_i') = \pi(e_i)$. If $\operatorname{ord}(w_i') = 0$ or w_i' is not a gluing or a socket point the previous requirement uniquely determines e_i'. If w_i' is either a gluing or a socket point of order $> k$ we take the edge e_i' satisfying the additional requirement $\left(\lambda(e_i'; \operatorname{ord}(w_i')), \theta(e_i'; \operatorname{ord}(w_i'))\right) = \left(\lambda(e_i; \operatorname{ord}(w_i)), \theta(e_i; \operatorname{ord}(w_i))\right)$. If w_i' is a socket point of order $\le k$ then e_i' is determined by the additional requirement that $(\lambda_{e_i'}, \theta_{e_i'}) = (\lambda_{e_{i-1}'}, \theta_{e_{i-1}'})$.

Let Γ be the random curve determined by choosing p_0' according to P_0 and letting $\Gamma = W_{p_0'}$. Then the following holds:

(C1) $\operatorname{end}\Gamma$ *has law P_1;*
(C2) $\operatorname{spt}\Gamma \subset B(\Gamma_W, C\sigma_k)$*;*

(C3) *For $e \in \operatorname{spt}\Gamma$ one has:*

$$\frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu}|e \approx_{C_1} = (C_{\mathrm{gw},1}C_{\mathrm{gw},2})^{-k}\prod_{j=k}^{\infty} w\left((\lambda_e(j),\theta_e(j))\right)^{-1}, \tag{4.36}$$

where C_1 depends on **(P1)–(P3)** *and* Weight.

Proof. Fix $p_0' \in F(p_0;k)$ and let e_t denote the t-th edge of W and e_t' the t-th edge of $W_{p_0'}$. One has $\pi(e_t) = \pi(e_t')$; moreover, the choice of behaviour at gluing and socket points implies that:

$$(\lambda(e_t';j),\theta(e_t';j)) = \begin{cases}(\lambda(p_0';j),\theta(p_0';j)) & \text{if } j \le k\\ (\lambda(e_t;j),\theta(e_t;j)) & \text{if } j > k.\end{cases} \tag{4.37}$$

Thus, for $e \in \operatorname{spt}\Gamma$ there are a unique $t \in \mathbb{N}$ and a unique $p_0' \in F(p_0;k)$ such that e is the t-th edge of $W_{p_0'}$. We now prove **(C1)**. Observe that the end point p_1' of $W_{p_0'}$ satisfies:

$$\pi(p_1') = \pi(p_1) \tag{4.38}$$

$$(\lambda(p_1';j),\theta(p_1';j)) = \begin{cases}(\lambda(p_0';j),\theta(p_0';j)) & \text{if } j \le k\\ (\lambda(p_1;j),\theta(p_1;j)) & \text{otherwise.}\end{cases} \tag{4.39}$$

Then, using the definition of the map τ in Definition 4.19, we get $p_1' = \tau(p_0')$ and so **(C1)** follows.

Statement **(C2)** is proven like in Theorem 4.23.

We now show statement **(C3)**. Let $e \in \operatorname{spt}\Gamma$ and let (t,p_0') be the unique pair such that e is the t-th edge of $W_{p_0'}$. Then:

$$\text{weight}\left(\mathrm{E}[\|\Gamma\|];e\right) = P(p_0') = (C_{\mathrm{gw},1}C_{\mathrm{gw},2})^{-k}\prod_{j=1}^{k} w\left((\lambda(p_0';j),\theta(p_0';j))\right), \tag{4.40}$$

and the result follows dividing (4.40) by weight$(\mu;e)$. □

Corollary 4.41. *Let W satisfy the assumptions of Theorem 4.35 and let $p \in G$. Assume that for some $C_0 > 0$ one has:*

$$\operatorname{dist}(p,\operatorname{spt}\Gamma) \approx_{C_0} \sigma_k, \tag{4.42}$$

and that len $W \le C_0\sigma_k$. *Then there is a $C_1 = C_1(C_0)$ such that:*

$$\left\|\frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu_p}\right\|_{\mathrm{L}^Q(\mu_p)}^Q \lesssim_{C_1} \sigma_k. \tag{4.43}$$

Proof. By assumption (4.42) we have

$$\frac{d\mu_p}{d\mu} \approx_{C(C_0)} (C_{\mathrm{gw},1}C_{\mathrm{gw},2})^{-k}\prod_{n=k+1}^{\infty}\left(w((\lambda(p;n),\theta(p;n))\right)^{-1}. \tag{4.44}$$

on the edges of $\operatorname{spt}\Gamma$. Then for $e \in \operatorname{spt}\Gamma$ one has:

$$\frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu_p} \approx 1. \tag{4.45}$$

On the other hand, len $W \lesssim \sigma_k$ and so:

$$\begin{aligned}\left\|\frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu_p}\right\|_{\mathrm{L}^Q(\mu_p)}^Q &= \sum_{t=1}^{\text{len } W}\sum_{p_0'\in F(p_0;k)}\sum_{\substack{e\text{ is the}\\ t\text{-th edge of}\\ W_{p_0'}}}\left(\frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu_p}\right)^Q \times \frac{d\mu_p}{d\mu}\,\text{weight}(\mu;e)\\ &\approx_{C(C_0)} \sum_{t=1}^{\text{len } W}\sum_{p_0'\in F(p_0;k)}(C_{\mathrm{gw},1}C_{\mathrm{gw},2})^{-k}\prod_{j=1}^{k} w\left((\lambda(p_0';j),\theta(p_0';j)\right)\\ &\lesssim \sigma_k.\end{aligned} \tag{4.46}$$

□

In the following theorem we assume that the walk is monotone increasing for concreteness; the same result holds if the walk is monotone decreasing. The goal is to build a random curve which "expands" gaining access to new labels. This is needed to get the estimate (4.9).

Theorem 4.47. *Let W be a monotone increasing walk joining p_0 to p_1 where $\operatorname{ord}(p_i) = 0$ and $\operatorname{len} W \in [\sigma_k/2, \sigma_k]$. Assume that all edges in W have the same label. Then there is a C_0 which depends only on* **(P1)**–**(P3)** *such that the following holds whenever $J_{\text{cut}} \geq C_0$. Let:*

$$\left(\lambda(p_0; k - J_{\text{cut}} + 1), \theta(p_0; k - J_{\text{cut}} + 1)\right) = (s_0, t_0), \tag{4.48}$$

and choose $(s_1, t_1) \in \operatorname{Symb}_1 \times \operatorname{Symb}_2 \setminus \{(s_0, t_0)\}$.

Choose by Lemma 2.42 a monotone increasing walk $W_0^{(\text{new})}$ from p_0 to a socket point ξ of order $k - J_{\text{cut}} + 1$, and which satisfies $\operatorname{len} W_0^{(\text{new})} \leq \operatorname{len} W$. Let $\hat{p}_1 \in F(p_1; k - J_{\text{cut}} + 1)$ be the point satisfying:

$$\left(\lambda(\hat{p}_1; j), \theta(\hat{p}_1; j)\right) = \begin{cases} \left(\lambda(p_1; j), \theta(p_1; j)\right) & \text{for } j \neq k - J_{\text{cut}} + 1 \\ (s_1, t_1) & \text{for } j = k - J_{\text{cut}} + 1. \end{cases} \tag{4.49}$$

Using Lemma 2.47 obtain a monotone increasing walk $W_{1/2}^{(\text{new})}$ from ξ to a point $\hat{p}_{1/2}$ such that $\lambda_{\hat{p}_{1/2}} = \lambda_{\hat{p}_1}$, $\theta_{\hat{p}_{1/2}} = \theta_{\hat{p}_1}$ and:

$$\operatorname{len} W_{1/2}^{(\text{new})} \leq \operatorname{len} W - \operatorname{len} W_0^{(\text{new})}. \tag{4.50}$$

Finally concatenate $W_{1/2}^{(\text{new})}$ with a monotone increasing walk whose edges have constant label $(\lambda_{\hat{p}_1}, \theta_{\hat{p}_1})$ to obtain a walk $W_1^{(\text{new})}$ joining ξ to $\hat{p}_1$ and satisfying:

$$\operatorname{len} W_0^{(\text{new})} + \operatorname{len} W_1^{(\text{new})} = \operatorname{len} W. \tag{4.51}$$

Construct a random curve as follows. Choose $p_0' \in F(p_0; k - J_{\text{cut}})$ using the probability P_0. Then with probability:

$$(C_{\text{gw},1} C_{\text{gw},2})^{-1} w((s_0, t_0)) \quad (\text{event } E^{(\text{old})}) \tag{4.52}$$

let Γ be the canonical path $\Gamma_{p_0' \cdot W}$ associated to $p_0' \cdot W$. For $(s, t) \neq (s_0, t_0)$ let $\hat{p}_{1,s,t}'$ be the point in $F(p_1; k - J_{\text{cut}} + 1)$ such that:

$$\left(\lambda(\hat{p}_{1,s,t}'; j), \theta(\hat{p}_{1,s,t}'; j)\right) = \begin{cases} \left(\lambda(p_0'; j), \theta(p_0'; j)\right) & \text{if } j \neq k - J_{\text{cut}} + 1 \\ (s, t) & \text{if } j = k - J_{\text{cut}} + 1. \end{cases} \tag{4.53}$$

Then with probability:

$$(C_{\text{gw},1} C_{\text{gw},2})^{-1} w\left((s, t)\right) \quad (\text{event } E_{s,t}^{(\text{new})}), \tag{4.54}$$

let Γ be the canonical path associated to the walk:

$$p_0' \cdot W_0^{(\text{new})} \star \left(\hat{p}_{1,s,t}' \cdot (W_1^{(\text{new})})^{-1}\right)^{-1}. \tag{4.55}$$

Then the following hold:

(C1) $\operatorname{end} \Gamma$ *has law P_1 on $F(p_1; k - J_{\text{cut}} + 1)$;*
(C2) $\operatorname{spt} \Gamma \subset B(\Gamma_W, C\sigma_{k - J_{\text{cut}} + 1})$*;*
(C3) *Let*

$$E^{(\text{new})} = \bigcup_{(s,t) \neq (s_0, t_0)} E_{(s,t)}^{(\text{new})}; \tag{4.56}$$

let $\Gamma^{(\text{old})}$ denote Γ conditioned on $E^{(\text{old})}$ and $\Gamma^{(\text{new})}$ denote Γ conditioned on $E^{(\text{new})}$. Then for each $e \in \operatorname{spt} \Gamma$ there is a unique $\operatorname{in}(e) \in \mathbb{N}$ such that $\pi(e) = \pi(e_{\operatorname{in}(e)})$ where $e_{\operatorname{in}(e)}$ is the $\operatorname{in}(e)$-th edge of W. If $e \in \operatorname{spt} \Gamma^{(\text{new})}$ one has:

$$\frac{d\,\mathrm{E}[\|\Gamma^{(\text{new})}\|]}{d\mu}\Big|e \approx_{C_1} C_{\text{gw},1}^{-T(e)} w_{\{\spadesuit\}}^{-k+T(e)} C_{\text{gw},2}^{-k} \prod_{j=k}^{\infty} w\left(\left(\lambda(e; j), \theta(e; j)\right)\right)^{-1}, \tag{4.57}$$

where

$$T(e) = \begin{cases} \lg(\operatorname{len} W_0^{(\mathrm{new})} - \operatorname{in}(e)) & \text{if } \max(\pi(e)) \le \pi(\xi) \\ \lg(\operatorname{in}(e) - \operatorname{len} W_0^{(\mathrm{new})}) & \text{otherwise;} \end{cases} \tag{4.58}$$

and if $e \in \operatorname{spt} \Gamma^{(\mathrm{old})}$ *then:*

$$\frac{d\,\mathrm{E}[\|\Gamma^{(\mathrm{old})}\|]}{d\mu}|e \approx_{C_1} (C_{\mathrm{gw},1}C_{\mathrm{gw},2})^{-k} \prod_{j=k}^{\infty} w\left((\lambda(e;j), \theta(e;j))\right)^{-1}, \tag{4.59}$$

where C_1 *depends on* J_{cut}, **(P1)**–**(P3)** *and* Weight.

Remark 4.60. Theorem 4.47 corresponds to the notion of “expanding” pencils of curves as discussed by Heinonen and Semmes [10, 26]. However, here there is a substantial difference with previously known examples of PI-spaces, as we need to pass through a socket point in order to expand the random curve (or the pencil). This process entails some degree of “compression” in the expansion, and this compression must be controlled as it obstructs the Poincaré inequality.

Concretely, we want Γ to start in $F(p_0; k - J_{\mathrm{cut}})$ and end in $F(p_1; k - J_{\mathrm{cut}} + 1)$, where J_{cut} is an integer parameter chosen for convenience, i.e. to create some “space” between the length of W and the maximum order of entries of λ or θ which differ from the corresponding ones in λ_{p_0} and θ_{p_0}. While to reach $F(p_1; k - J_{\mathrm{cut}})$ we can just use a “parallel lift” (compare the definition of $\Gamma_{p_0'\cdot W}$ using $p_0' \cdot W$), to access points $\tilde{p}_1 \in F(p_1; k - J_{\mathrm{cut}} + 1)$ with $(\lambda_{\tilde{p}_1}(k - J_{\mathrm{cut}} + 1), \theta_{\tilde{p}_1}(k - J_{\mathrm{cut}} + 1)) \neq (\lambda_{p_0}(k - J_{\mathrm{cut}} + 1), \theta_{p_0}(k - J_{\mathrm{cut}} + 1))$ we will use the socket point ξ.

Specifically, we build a path $W_0^{(\mathrm{new})} \star W_1^{(\mathrm{new})}$ so that we reach from p_0 the point $\hat{p}_1$ whose label is defined in (4.49). In this way we can modify the $(k - J_{\mathrm{cut}} + 1)$-th entry of labels. This construction is then generalized to an arbitrary starting point $p_0' \in F(p_0; k - J_{\mathrm{cut}})$ by using (4.55).

Heuristically, the event $E^{(\mathrm{old})}$ means that we just follow a path ending in $F(p_1; k - J_{\mathrm{cut}})$ while the event $E^{(\mathrm{new})}$ means that we pass through ξ. Then the technical part of the argument boils down in showing that if the probability of $E^{(\mathrm{old})}$ is chosen correctly one gets the estimates (4.57) and (4.59) which will be needed in verifying the Poincaré inequality.

Proof. We first explain why the construction of the walks $W_0^{(\mathrm{new})}$, $W_{1/2}^{(\mathrm{new})}$ and $W_1^{(\mathrm{new})}$ can be carried out. If C_0 is sufficiently large, one can ensure that whenever $J_{\mathrm{cut}} \ge C_0$, and if C is the constant appearing in Lemmas 2.42, 2.47, one has:

$$2C\sigma_{k-J_{\mathrm{cut}}} \le \operatorname{len} W, \tag{4.61}$$

and thus one can construct $W_0^{(\mathrm{new})}$ and $W_{1/2}^{(\mathrm{new})}$ satisfying:

$$\operatorname{len} W_0^{(\mathrm{new})} + \operatorname{len} W_{1/2}^{(\mathrm{new})} \le \operatorname{len} W. \tag{4.62}$$

We now explain why the concatenation in (4.55) is well-defined. Note that $W_0^{(\mathrm{new})}$ and $(W_1^{(\mathrm{new})})^{-1}$ satisfy the assumptions of Theorem 4.23; referring to the notation of Theorem 4.23, we have to set $K = k$ where k is now given by the integer $k - J_{\mathrm{cut}} + 1$ used in this Theorem; for $W_0^{(\mathrm{new})}$ the value of J_{cut} now used in Theorem 4.23 is 0, while for $(W_1^{(\mathrm{new})})^{-1}$ the value of J_{cut} now used in Theorem 4.23 is 1. Now, Theorem 4.23 ensures that both $p_0' \cdot W_0^{(\mathrm{new})}$ and $(\hat{p}_{1,s,t}' \cdot W_1^{(\mathrm{new})})^{-1}$ end at the point $\xi' \in F_\Theta(\xi; k - J_{\mathrm{cut}} + 1)$ such that $\theta(p_0'; l) = \theta(\xi'; l)$ for $l \neq k - J_{\mathrm{cut}} + 1$. Therefore, the concatenation in (4.55) is well-defined.

We now turn to the proof of **(C1)**. Let $p_0' = \operatorname{str} \Gamma$; conditional on the event $E^{(\mathrm{old})}$ one has that $\operatorname{end} \Gamma = p_1'$ where p_1' is the point of $F(p_1; k - J_{\mathrm{cut}} + 1)$ satisying $(\lambda_{p_0'}, \theta_{p_0'}) = (\lambda_{p_1'}, \theta_{p_1'})$. The probability of the event:

$$\left\{\operatorname{str} \Gamma = p_0'\right\} \cap E^{(\mathrm{old})} \tag{4.63}$$

is:

$$
\begin{aligned}
&(C_{\mathrm{gw},1}C_{\mathrm{gw},2})^{-k+J_{\mathrm{cut}}}\prod_{n=1}^{k-J_{\mathrm{cut}}} w\left((\lambda(p_0';n),\theta(p_0';n))\right)\cdot(C_{\mathrm{gw},1}C_{\mathrm{gw},2})^{-1}w\left((s_0,t_0)\right)\\
&\qquad\qquad = (C_{\mathrm{gw},1}C_{\mathrm{gw},2})^{-k+J_{\mathrm{cut}}-1}\prod_{n=1}^{k-J_{\mathrm{cut}}+1} w\left((\lambda(p_0';n),\theta(p_0';n))\right). \qquad (4.64)
\end{aligned}
$$

Conditional on the event $E^{(\mathrm{new})}_{s,t}$ one has end $\Gamma = \hat{p}'_{1,s,t}$, and the probability of the event

$$\left\{\operatorname{str}\Gamma = p_0'\right\}\cap E^{(\mathrm{new})}_{s,t} \qquad (4.65)$$

is:

$$
\begin{aligned}
&(C_{\mathrm{gw},1}C_{\mathrm{gw},2})^{-k+J_{\mathrm{cut}}}\prod_{n=1}^{k-J_{\mathrm{cut}}} w\left((\lambda(p_0';n),\theta(p_0';n))\right)\cdot(C_{\mathrm{gw},1}C_{\mathrm{gw},2})^{-1}w\left((s,t)\right)\\
&\qquad\qquad = (C_{\mathrm{gw},1}C_{\mathrm{gw},2})^{-k+J_{\mathrm{cut}}-1}\prod_{n=1}^{k-J_{\mathrm{cut}}+1} w\left((\lambda(\hat{p}'_{1,s,t};n),\theta(\hat{p}'_{1,s,t};n))\right). \qquad (4.66)
\end{aligned}
$$

We thus conclude that **(C1)** holds

For **(C2)** we can apply the same argument as in Theorem 4.23.

We now prove **(C3)**. The fact that in(e) is well-defined follows from the monotonicity of the walks W, $W_0^{(\mathrm{new})}$ and $W_1^{(\mathrm{new})}$. As all edges of W have the same label, for $p_0' \in F(p_0;k-J_{\mathrm{cut}})$ one has that $p_0'\cdot W = W_{p_0'}$, where $W_{p_0'}$ is defined as in Theorem 4.35. Therefore, the estimate (4.59) on the Radon-Nikodym derivative of $\mathrm{E}\left[\|\Gamma^{(\mathrm{old})}\|\right]$ can be obtained from (4.36). Let now $t_\xi = \operatorname{in}(e_\xi)$ where e_ξ is the last edge of $W_0^{(\mathrm{new})}$. As remarked above, the walk $W_0^{(\mathrm{new})}$ satisfies the assumptions of Theorem 4.23. Thus, if $e\in\operatorname{spt}\Gamma^{(\mathrm{new})}$ and $\operatorname{in}(e)\le t_\xi$ we can apply (4.24) to get (4.57) with $T(e) = \lg(\operatorname{len} W_0 - \operatorname{in}(e))$. On the other hand, also the path $(W_1^{(\mathrm{new})})^{-1}$ satisfies the assumptions of Theorem 4.23. In this case the point end $\Gamma^{(\mathrm{new})}$ avoids the sets of points $p_1'\in F(p_1;k-J_{\mathrm{cut}}+1)$ such that:

$$(\lambda(p_1';k-J_{\mathrm{cut}}+1),\theta(p_1';k-J_{\mathrm{cut}}+1)) = (s_0,t_0); \qquad (4.67)$$

in applying Theorem 4.23 this can only introduce a multiplicative error lying in $[(C_{\mathrm{gw},1}C_{\mathrm{gw},2})^{-1}, C_{\mathrm{gw},1}C_{\mathrm{gw},2}]$ in the estimate (4.24). Note also that if $\operatorname{in}(e)\ge t_\xi$, considering the reverse walk $(W_1^{(\mathrm{new})})^{-1}$, the integer in$(e)$ in (4.24) must be replaced with len W − in(e) and thus the proof of (4.57) is complete. □

Corollary 4.68. *Let W be as in Theorem 4.47 and let $p\in G$. Assume that for some $C_1>0$ one has:*

$$\operatorname{dist}(p,\operatorname{spt}\Gamma)\approx_{C_1}\sigma_k. \qquad (4.69)$$

Then there is a $C_2 = C_2(C_1, J_{\mathrm{cut}})$ such that:

$$\left\|\frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu_p}\right\|^Q_{\mathrm{L}^Q(\mu_p)} \approx_{C_2} \sum_{l=1}^{k}(w^{-1}_{\{\spadesuit\}}C_{\mathrm{gw}})^{l(Q-1)}\sigma_{k-l}. \qquad (4.70)$$

Proof. We first apply convexity of the Q-th power of the $\mathrm{L}^Q(\mu_p)$ norm to get:

$$
\begin{aligned}
\left\|\frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu_p}\right\|^Q_{\mathrm{L}^Q(\mu_p)} &= \left\|P(E^{(\mathrm{new})})\frac{d\,\mathrm{E}[\|\Gamma^{(\mathrm{new})}\|]}{d\mu_p} + P(E^{(\mathrm{old})})\frac{d\,\mathrm{E}[\|\Gamma^{(\mathrm{old})}\|]}{d\mu_p}\right\|^Q_{\mathrm{L}^Q(\mu_p)}\\
&\le P(E^{(\mathrm{new})})\left\|\frac{d\,\mathrm{E}[\|\Gamma^{(\mathrm{new})}\|]}{d\mu_p}\right\|^Q_{\mathrm{L}^Q(\mu_p)} + P(E^{(\mathrm{old})})\left\|\frac{d\,\mathrm{E}[\|\Gamma^{(\mathrm{old})}\|]}{d\mu_p}\right\|^Q_{\mathrm{L}^Q(\mu_p)};
\end{aligned}
\qquad (4.71)
$$

let $t_\xi = \operatorname{in}(e_\xi)$ where e_ξ is the last edge of $W_0^{(\mathrm{new})}$. By assumption (4.69) we can apply Corollary 4.30 to $\Gamma^{(\mathrm{new})}|[0,t_\xi]$ and $\Gamma^{(\mathrm{new})}|[t_\xi,\operatorname{len}W]$. Similarly, by assumption (4.69) we can apply Corollary 4.41 to $\Gamma^{(\mathrm{old})}$. Thus, (4.70) follows substituting (4.32), and (4.43) in (4.71). □

4.3 Proof of the Poincaré inequality

In this subsection we join the random curves constructed in Subsection 4.2 to prove the Poincaré inequality.

Definition 4.72. Given $P \geq 1$ we denote by Q the conjugate exponent $P/(P-1)$. Let $\mathrm{I}_{\mathrm{neck}}$ denote the range of exponents $P \geq 1$ such that there is a $C = C(P)$ such that for each $k \in \mathbb{N}$ one has:

$$\sum_{l=1}^{k} (w_{\{\spadesuit\}}^{-1} C_{\mathrm{gw},1})^{l(Q-1)} \frac{\sigma_{k-l}}{\sigma_k} \leq C. \tag{4.73}$$

Minding that $\sigma_k = 8^k$, we obtain:

$$\mathrm{I}_{\mathrm{neck}} = \left(\log_8(w_{\{\spadesuit\}}^{-1} C_{\mathrm{gw},1}) + 1, \infty\right). \tag{4.74}$$

Note that as $P \searrow \log_8(w_{\{\spadesuit\}}^{-1} C_{\mathrm{gw},1}) + 1$ the constant $C(P) \nearrow \infty$. Now $w_{\{\spadesuit\}}^{-1} C_{\mathrm{gw},1} = 1 + w_{\{\spadesuit\}}^{-1}$ and thus varying $w_{\{\spadesuit\}}^{-1}$ we can prescribe $\mathrm{I}_{\mathrm{neck}}$ to be any open ray (a, ∞) where $a > 1$.

Theorem 4.75. *For $P \in \mathrm{I}_{\mathrm{neck}}$ the metric measure space (G, μ) satisfies a $(1, P)$-Poincaré inequality, i.e. $\mathrm{I}_{\mathrm{neck}} \subset \mathrm{I}_{\mathrm{PI}}(G, \mu)$.*

Proof. We apply Theorem 4.7, i.e. for any pair of points (x, y) we show the existence of a random curve Γ satisfying:

$$\operatorname{spt} \Gamma \subset B(\{x, y\}, Cd(x, y)), \tag{4.76}$$

$$\left\| \frac{d\,\mathrm{E}[\|\Gamma\|]}{d(\mu_x + \mu_y)} \right\|_{\mathrm{L}^Q(\mu_x+\mu_y)}^Q \lesssim_{C_Q} d(x, y), \tag{4.77}$$

where C does not depend on x, y, and C_Q does not depend on x, y but depends on Q. Γ is built by concatenating curves obtained by using Theorems 4.23, 4.35, 4.47. We observe that if $\operatorname{end} \Gamma_0 = \operatorname{str} \Gamma_1$ the random curves Γ_0, Γ_1, up to translating their domains, can be concatenated to obtain a random curve $\Gamma_0 \star \Gamma_1$.

`Step 1: First part of building "half" of a random curve joining` x `to` y.

Fix points x, y and assume that $\max \mathbb{N}(x, y) \leq \lg d(x, y)$. This assumption will be removed in `Step 2`. Using Theorem 3.7 we can choose a good walk from x to y satisfying **(GWA1)** and **(GWA2)**. We let $K = \lg d(x, y)$. We thus have a uniform constant C_0 such that:

$$C_0 \sigma_K \geq \operatorname{len} W \tag{4.78}$$

$$d(x, w_i) \geq C_0^{-1} i \quad (w_i \in W \text{ is the } i\text{-th vertex}). \tag{4.79}$$

For the moment let C be the maximum of the constants occurring at points **(C2)** of Theorems 4.23, 4.35, 4.47. We can find $C_1 = C(C_0)$, $J_1 = J(C_0)$ such that, if $J \geq J_1$ and $\tilde{w}$ satisfies:

$$d(\tilde{w}, w_i) \leq C \sigma_{\lg i - J}, \tag{4.80}$$

then one has:

$$d(\tilde{w}, x) \geq C_1^{-1} i. \tag{4.81}$$

We now subdivide W into subwalks $\{W_\alpha\}_{\alpha \in I}$ (I is a finite set of integers), the idea being that W can be thought of as a concatenation of the $\{W_\alpha\}$. More precisely, this can be formalized by using a strictly increasing map $\alpha \mapsto m_\alpha$, and letting W_α denote the part of W starting at the m_α-th vertex w_{m_α} and ending at the $m_{\alpha+1}$-th vertex $w_{m_{\alpha+1}}$. Note that we obtain an order relation $<$ on $\{W_\alpha\}_{\alpha \in I}$ where $W_\alpha < W_{\alpha+1}$.

Using the properties of the good walk constructed in Theorem 3.7 we obtain a J_2 such that there is a decomposition of W into monotone subwalks $\{W_\alpha\}_{\alpha \in I}$ having the following properties:

(Dec1) For each $k \in \{J_2, \cdots, K\}$ there is a $W_\alpha = W_k^{(\mathrm{exp})}$ satisfying the assumptions of Theorem 4.47 and:

$$\operatorname{dist}(W_\alpha, x) \approx_C \sigma_k; \tag{4.82}$$

(Dec2) For each $k \in \mathbb{N}(x,y)$ such that $\theta_x(k) \neq \theta_y(k)$, there is a $W_\alpha = W_k^{(\mathrm{neck})}$ which can be decomposed into subwalks $\tilde{W}_0$, $\tilde{W}_1$ which satisfy the following: one has end $\tilde{W}_0 = w_{s(k)} = \operatorname{str} \tilde{W}_1$; moreover, for $J_{\mathrm{cut}} \geq J_2$ the walks $\tilde{W}_0$ and $\tilde{W}_1^{-1}$ satisfy the assumptions of Theorem 4.23 where $\xi = w_{s(k)}$;

(Dec3) For each of the remaining walks W_α there is a k such that:

$$\operatorname{len} W_\alpha \leq C\sigma_k \tag{4.83}$$

$$\operatorname{dist}(W_\alpha, x) \geq C^{-1}\sigma_k. \tag{4.84}$$

Γ is constructed by concatenating curves Γ_α for each $\alpha \in I$. This is done inductively, and one starts by letting $\Gamma_1 = \Gamma_{W_1}$ with probability 1. The next step depends on which of the conditions **(Dec)** is satisfied by $W_{\alpha+1}$:

- Case of **(Dec1)**. We have $W_{\alpha+1} = W_k^{(\mathrm{exp})}$ and we know that end Γ_α is a random point in $F(w_{m_{\alpha+1}}; k - J_{\mathrm{cut}})$ whose law is the canonical probability. We obtain $\Gamma_{\alpha+1}$ applying Theorem 4.47, so that end $\Gamma_{\alpha+1}$ is a random point in $F(w_{m_{\alpha+1}}; k - J_{\mathrm{cut}} + 1)$ whose law is the canonical probability. Moreover, by (4.82) we can apply Corollary 4.68 to conclude that:

$$\left\| \frac{d\,\mathrm{E}[\|\Gamma_{\alpha+1}\|]}{d\mu_x} \right\|_{\mathrm{L}^Q(\mu_x)}^Q \approx_{C_2} \sum_{l=1}^{k} (w_{\{\spadesuit\}}^{-1} C_{\mathrm{gw}})^{l(Q-1)} \sigma_l, \tag{4.85}$$

where C_2 is a uniform constant depending on the constants $C_0, C_1, C, J_0, J_1, J_{\mathrm{cut}}$. Moreover, by the assumption on P we have that there is a uniform constant C_3 depending on C_2 and Q such that:

$$\left\| \frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu_x} \right\|_{\mathrm{L}^Q(\mu_x)}^Q \lesssim_{C_3} \sigma_k. \tag{4.86}$$

- Case of **(Dec2)**. We have $W_{\alpha+1} = W_k^{(\mathrm{neck})}$ and we know that end Γ_α is a random point in $F_\Theta(w_{m_{\alpha+1}}; k - J_{\mathrm{cut}})$ whose law is the canonical probability. We apply Theorem 4.23 to build $\tilde{\Gamma}_0$ from $\tilde{W}_0$. We then take the canonical probability on $F_\Theta(w_{m_{\alpha+2}}; k - J_{\mathrm{cut}})$ and use again Theorem 4.23 to build $\tilde{\Gamma}_1$ from $\tilde{W}_1^{-1}$. We obtain $\Gamma_{\alpha+1}$ by concatenating $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1^{-1}$ subject to the following additional prescription; suppose that str $\tilde{\Gamma}_0 = p_0'$; then one takes str $\tilde{\Gamma}_1 = \tau(p_0')$ where $\tau : F(w_{m_{\alpha+1}}; k - J_{\mathrm{cut}}) \to F(w_{m_{\alpha+2}}; k - J_{\mathrm{cut}})$ is the canonical map of Definition 4.19. Note that:

$$\operatorname{spt} \tilde{\Gamma}_0 \cap \operatorname{spt} \tilde{\Gamma}_1 = \{w_{s(k)}\}, \tag{4.87}$$

as the labels of the edges in spt $\tilde{\Gamma}_0$ and spt $\tilde{\Gamma}_1$ have different k-th entries. Moreover, as $\xi = w_{s(k)}$ and $d(x, w_{s(k)}) \approx_C \sigma_k$, we can apply Corollary 4.30 to obtain the estimate:

$$\left\| \frac{d\,\mathrm{E}[\|\Gamma_{\alpha+1}\|]}{d\mu_x} \right\|_{\mathrm{L}^Q(\mu_x)}^Q \approx_{C_2} \sum_{l=1}^{k} (w_{\{\spadesuit\}}^{-1} C_{\mathrm{gw}})^{l(Q-1)} \sigma_{k-l}, \tag{4.88}$$

where C_2 is a uniform constant depending on the constants $C_0, C_1, C, J_0, J_1, J_{\mathrm{cut}}$. Moreover, by the assumption on P we have that there is a uniform constant C_3 depending on C_2 and Q such that:

$$\left\| \frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu_x} \right\|_{\mathrm{L}^Q(\mu_x)}^Q \lesssim_{C_3} \sigma_k. \tag{4.89}$$

- Case of **(Dec3)**. We know that end Γ_α is a random point in $F(w_{m_{\alpha+1}}; k - J_{\mathrm{cut}})$ and that len $W_{\alpha+1} \leq C\sigma_k$. We build $\Gamma_{\alpha+1}$ by applying Theorem 4.35. In particular, the assumptions of Corollary 4.41 are also met an so we have:

$$\left\| \frac{d\,\mathrm{E}[\|\Gamma_{\alpha+1}\|]}{d\mu_x} \right\|_{\mathrm{L}^Q(\mu_x)}^Q \lesssim_{C_2} \sigma_k, \tag{4.90}$$

where C_2 is a uniform constant depending on the constants C_0, C_1, C, J_0 and J_1.

Note that by the choice of C_1, if $\operatorname{spt}\Gamma_\alpha \cap \operatorname{spt}\Gamma_\beta \neq \emptyset$, then $|\alpha - \beta| \le C_4$, where C_4 is a uniform constant. We thus obtain that:

$$\left\| \frac{d\,\mathrm{E}[\|\Gamma\|]}{d\mu_x} \right\|^Q_{\mathrm{L}^Q(\mu_x)} \lesssim_{C_Q} d(x,y) \tag{4.91}$$

and that for some uniform C:

$$\operatorname{spt}\Gamma \subset B(x, Cd(x,y)). \tag{4.92}$$

`Step 2: Modifying Step 1 if` $\max \mathbb{N}(x,y) > \lg d(x,y)$.

In this case W is given by Theorem 3.18. If $\theta_x(k_{\max}) = \theta_y(k_{\max})$ the construction can proceed as in `Step 1` because at $u_{k_{\max}}$ there is no change of the θ-label.

We now discuss the modifications for the case $\theta_x(k_{\max}) \neq \theta_y(k_{\max})$. We first enlarge W at $w_i = u_{k_{\max}}$ by inserting 4 subwalks $\{\tilde W_i\}_{i=0}^3$ between w_i and w_{i+1}. Let $M = \lg d(x, u_{k_{\max}})$, and let e denote the edge of W before $u_{k_{\max}}$. We take $\tilde W_0$ to be a monotone geodesic walk whose edges have all the same label (λ_e, θ_e), with $\operatorname{len} \tilde W_0 = \sigma_M$ and $d(\tilde W_0, x) \ge C_1^{-1}\sigma_M$. For $\tilde W_1$ we take $\tilde W_0^{-1}$. Let now e denote the edge of W after $u_{k_{\max}}$. Then $\tilde W_2$ is a monotone geodesic walk whose edges have all the same label (λ_e, θ_e), with $\operatorname{len} \tilde W_2 = \sigma_M$ and $d(\tilde W_2, x) \ge C_1^{-1}\sigma_M$. For $\tilde W_3$ we take $\tilde W_2^{-1}$.

One then proceeds as in `Step 1`, by subdividing W. The subdivision must satisfy the additional requirement that the $\{\tilde W_i\}_{i=0}^3$ are subwalks of the subdivision, and we have only to specify how to construct the corresponding $\{\tilde\Gamma_i\}_{i=0}^3$. On $\tilde W_0$ we apply Theorem 4.35 and Corollary 4.41 and obtain the estimate:

$$\left\| \frac{d\,\mathrm{E}[\|\tilde\Gamma_0\|]}{d\mu_x} \right\|^Q_{\mathrm{L}^Q(\mu_x)} \lesssim_{C_2} \sigma_M. \tag{4.93}$$

Then $\tilde\Gamma_1$ and $\tilde\Gamma_2$ are built by applying Theorem 4.23 and Corollary 4.30 to $\tilde W_1$ and $\tilde W_2^{-1}$ respectively. Note that $\operatorname{str}\tilde\Gamma_2$ is taken to be a random point in $F(\operatorname{str}\tilde W_2^{-1}; M - J_{\mathrm{cut}})$ whose law is the canonical probability. We build $\tilde\Gamma_{12}$ by concatenating $\tilde\Gamma_1$ and $\tilde\Gamma_2^{-1}$ with the additional prescription that if $\operatorname{str}\tilde\Gamma_1 = p_1'$ then $\operatorname{str}\tilde\Gamma_2 = \tau(p_1')$ where

$$\tau : F(\operatorname{str}\tilde W_1; M - J_{\mathrm{cut}}) \to F(\operatorname{str}\tilde W_2^{-1}; M - J_{\mathrm{cut}}) \tag{4.94}$$

is the canonical map of Definition 4.19. We thus obtain the estimate:

$$\left\| \frac{d\,\mathrm{E}[\|\tilde\Gamma_{12}\|]}{d\mu_x} \right\|^Q_{\mathrm{L}^Q(\mu_x)} \lesssim_{C_3} \sigma_M. \tag{4.95}$$

Finally, $\tilde\Gamma_3$ is obtained by applying Theorem 4.35 and Corollary 4.41 to $\tilde W_3$. We then have the estimate:

$$\left\| \frac{d\,\mathrm{E}[\|\tilde\Gamma_3\|]}{d\mu_x} \right\|^Q_{\mathrm{L}^Q(\mu_x)} \lesssim_{C_2} \sigma_M. \tag{4.96}$$

With these modifications, one obtains (4.91), (4.92) where the constants have possibily worsened compared to `Step 1`.

`Step 3: building a random curve satisfying (4.76), (4.77).`

Fix $x, y \in G$ at distance > 1. We choose a vertex z of order 0 satisfying:

$$\left| d(z,x) - \frac{d(x,y)}{2} \right| \le 1 \tag{4.97}$$

$$\left| d(z,y) - \frac{d(x,y)}{2} \right| \le 1; \tag{4.98}$$

we then choose $J_{\mathrm{cut},x}$ and $J_{\mathrm{cut},y}$ larger than J_2 of `Step(s) 1, 2` such that:

$$|J_{\mathrm{cut},x} - J_2| \le 3 \tag{4.99}$$

$$|J_{\mathrm{cut},y} - J_2| \le 3 \tag{4.100}$$

$$\lg d(z,x) - J_{\mathrm{cut},x} = \lg d(z,y) - J_{\mathrm{cut},y}. \tag{4.101}$$

We then construct random curves Γ_x connecting x to $F(z; \lg d(z,x) - J_{\mathrm{cut},x})$, and Γ_y connecting y to $F(z; \lg d(z,y) - J_{\mathrm{cut},y})$ using `Steps 1,2`. Note that (4.101) implies that end Γ_x and end Γ_y have the same law. We can thus obtain Γ by concatenating Γ_x and Γ_y^{-1}. Now (4.76) follows from (4.92) and (4.97), (4.98). On the other hand, (4.77) follows from (4.91) and:

$$\begin{aligned} \left\| \frac{d\,\mathrm{E}[\|\Gamma\|]}{d(\mu_x+\mu_y)} \right\|^Q_{\mathrm{L}^Q(\mu_x+\mu_y)} &\le \left\| \frac{d\,\mathrm{E}[\|\Gamma_x\|]}{d\mu_x} \frac{d\mu_x}{d(\mu_x+\mu_y)} + \frac{d\,\mathrm{E}[\|\Gamma_y\|]}{d\mu_y} \frac{d\mu_y}{d(\mu_x+\mu_y)} \right\|^Q_{\mathrm{L}^Q(\mu_x+\mu_y)} \\ &\le 2^{Q-1} \left(\left\| \frac{d\,\mathrm{E}[\|\Gamma_x\|]}{d\mu_x} \right\|^Q_{\mathrm{L}^Q(\mu_x)} + \left\| \frac{d\,\mathrm{E}[\|\Gamma_y\|]}{d\mu_y} \right\|^Q_{\mathrm{L}^Q(\mu_y)} \right) \\ &\lesssim_{C_Q} d(x,y). \end{aligned} \tag{4.102}$$

□

4.4 Lack of the Poincaré inequality

To show that a $(1,P)$ Poincaré inequality does not hold if P is sufficiently small, we produce pairs of points such that the modulus estimate (4.8) does not hold.

Lemma 4.103. *Fix a constant $C_0 \ge 1$; then there are constants $M = M(C_0)$, $l = l(C_0)$ such that the following holds. Let (λ, θ) be labels such that $\lambda(j) = \{\spadesuit\}$ for $j \le k + M$. Let $m \in \mathbb{Z}$ have order $M + k$ and let $R = 3C_0\sigma_k$. In the box*

$$B_{\mathrm{bad}} = \mathrm{Box}\left([m-R, m+R], (\lambda,\theta), k+l\right) \tag{4.104}$$

select two points p_0, p_1 such that:

1. *$\pi(p_0) = m - \sigma_k$ and $\pi(p_1) = m + \sigma_k$;*
2. *$\lambda_{p_0} = \lambda_{p_1}$;*
3. *$\theta(p_0; j) = \theta(p_1; j)$ if $j \neq k + M$ and $\theta(p_0; k+M) \neq \theta(p_1; k+M)$.*

Then there is a constant $C_1(C_0, P)$ such that:

$$d(p_0,p_1)^{P-1}\mathrm{mod}_P(p_0,p_1;\mu^{(C_0)}_{p_0,p_1}) \le \frac{C_1}{(k-1)^P} \sum_{i=1}^{k-1} \left(\frac{\sigma_k}{\sigma_i}\right)^{P-1} (w_{\{\spadesuit\}} C^{-1}_{\mathrm{gw},1})^{k-1-i}. \tag{4.105}$$

Proof. Let $\xi \in B_{\mathrm{bad}}$ denote the socket point of label (λ,θ) such that $\pi(\xi) = m$. Let γ be a continuous curve joining p_0 to p_1. Note that by possibly enlarging C_0 we have $d(p_0,p_1) \approx_{C_0} \sigma_k$ and so for $l(C_0)$ sufficienlty large, by Lemma 2.49 we have:

$$B(\{p_0,p_1\}, C_0 d(p_0,p_1)) \subset B_{\mathrm{bad}}. \tag{4.106}$$

If $M(C_0)$ is sufficiently large, the only integer of order $k + M$ contained in $\pi(B_{\mathrm{bad}})$ is m. To estimate $\mathrm{mod}_P(p_0; p_1, \mu^{(C_0)}_{p_0,p_1})$ we need to produce an appropriate Borel function g. For the moment we let $g = \infty$ on B^c_{bad} and then the case of interest becomes when γ stays in B_{bad}; in particular, γ must pass through a socket point $\xi' \in F_\Theta(\xi; k+l)$.

Let $s \in \mathrm{dom}\,\gamma$ be the first time when $\gamma(s) \in F_\Theta(\xi; k+l)$ and let $\gamma_1 = \gamma|[0,s]$. Note that:

$$[m - \sigma_{k-1}, m] \subset \pi \circ \gamma([0,s]); \tag{4.107}$$

for $i < k$ let $t_i = m - \sigma_i$ and let ϱ_i be the last time such that $\pi \circ \gamma_1(\varrho_i) = t_i$. Let $E(i)$ denote the set of edges $e \in B_{\mathrm{bad}}$ such that $\pi(e) \subset [t_i, t_{i-1}]$. As there are no integers of order i in $[t_i, m]$ we conclude that the curve $\gamma_1|[\varrho_i, \varrho_{i-1}]$ passes through edges $\{e_1, \cdots, e_l\} \subset E(i)$ such that:

$(E(i),$**1**$)$ e_a and e_{a+1} are adjacent, $t_i \in \pi(e_1)$ and $t_{i-1} \in \pi(e_l)$;
$(E(i),$**2**$)$ $l \ge \sigma_i - \sigma_{i-1}$;

$(E(i),\mathbf{3})$ $\lambda(e_a;j) = \{\spadesuit\}$ for $j \geq i$;
$(E(i),\mathbf{4})$ $\theta(e_a;j) = \theta(p_0;j)$ for $j > k + l$.

We now complete the definition of g by defining $g|B_{\mathrm{bad}}$ as follows: if $e \in E(i)$ for some i and $(E(i),\mathbf{3})$ and $(E(i),\mathbf{4})$ hold, we let $g = (k-1)^{-1}(\sigma_i - \sigma_{i-1})^{-1}$; otherwise, we let $g = 0$. We now obtain the following lower bound:

$$\begin{aligned}\int g\, d\mathcal{H}^1_\gamma &\geq \sum_{i=1}^{k-1} \int \chi_{E(i)} g\, d\mathcal{H}^1_\gamma \\ &\geq \sum_{i=1}^{k-1} \int_{\varrho_i}^{\varrho_{i-1}} \chi_{E(i)}(\gamma(\tau)) g(\gamma(\tau))\, d\tau \\ &\geq \sum_{i=1}^{k-1} \frac{\sigma_i - \sigma_{i-1}}{(k-1)(\sigma_i - \sigma_{i-1})} = 1,\end{aligned} \tag{4.108}$$

where we let $\sigma_0 = 0$.

Note now that $\gamma_1|[\varrho_{k-1}, s]$ is at distance $\approx_{C_2} \sigma_k$ from p_0, p_1, where C_2 is a uniform constant. Therefore, we have:

$$\frac{d\mu^{(C_0)}_{p_0,p_1}}{d\mu} \mid (\gamma_1[\varrho_{k-1}, s]) \approx_{C(C_2)} (C_{\mathrm{gw},1} C_{\mathrm{gw},2})^{-k-1} \prod_{j>k-1} w\left(\lambda(p_0;j), \theta(p_0;j)\right)^{-1}; \tag{4.109}$$

note that $g \neq 0$ in B_{bad} only on $\bigcup_{i=1}^{k-1} E(i)$, and let $\tilde{E}(i)$ denote the set of edges of $E(i)$ satisfying $(E(i),\mathbf{3})$ and $(E(i),\mathbf{4})$; as g vanishes on $E(i) \setminus \tilde{E}(i)$, we have for some $C_1(C_0, C_2, P, M, l)$:

$$\begin{aligned}\int g^P\, d\mu^{(C_0)}_{p_0,p_1} &\lesssim_{C_1} \sum_{i=1}^{k-1} \frac{1}{(k-1)^P \sigma_i^P} (C_{\mathrm{gw},1} C_{\mathrm{gw},2})^{-k+1} \times \sum_{e \in \tilde{E}(i)} \prod_{j \leq k-1} w(\lambda_e(j), \theta_e(j)) \\ &\lesssim_{C_1} \sum_{i=1}^{k-1} \frac{\sigma_i}{(k-1)^P \sigma_i^P} (C_{\mathrm{gw},1} C_{\mathrm{gw},2})^{-k+1} \times w_{\{\spadesuit\}}^{k-1-i} C^i_{\mathrm{gw},1} C^{k+l}_{\mathrm{gw},2} \\ &\lesssim_{C_1} \sum_{i=1}^{k-1} \frac{\sigma_i}{(k-1)^P \sigma_i^P} (w_{\{\spadesuit\}} C^{-1}_{\mathrm{gw},1})^{k-1-i},\end{aligned} \tag{4.110}$$

from which (4.105) follows. □

Theorem 4.111. *If $P \leq 1 + \log_8(w^{-1}_{\{\spadesuit\}} C_{\mathrm{gw},1})$ then $P \notin \mathrm{I_{PI}}(G, \mu)$. Thus, $\mathrm{I_{neck}} = \mathrm{I_{PI}}(G, \mu)$.*

Proof. We show that for any value of C, (1) in Theorem 4.7 fails. For any $k \geq 1$ we can find a *bad box* B_{bad} satisfying the assumptions of Lemma 4.103. Hence we find sequences of pairs of points $(p_0^{(k)}, p_1^{(k)}) \in G^2$ such that:

$$\begin{aligned}d\left(p_0^{(k)}, p_1^{(k)}\right)^{P-1} \mathrm{mod}_P(p_0^{(k)}, p_1^{(k)}; \mu^{(C)}_{p_0^{(k)},p_1^{(k)}}) &\leq \frac{C}{(k-1)^P} \sum_{i=1}^{k-1} \left(\frac{\sigma_k}{\sigma_i}\right)^{P-1} \times (w_{\{\spadesuit\}} C^{-1}_{\mathrm{gw},1})^{k-1-i} \\ &= \frac{C}{w_{\{\spadesuit\}} C^{-1}_{\mathrm{gw},1} (k-1)^P} \times \sum_{i=1}^{k-1} (8^{P-1} w_{\{\spadesuit\}} C^{-1}_{\mathrm{gw},1})^{k-i}.\end{aligned} \tag{4.112}$$

As $P \leq 1 + \log_8(w^{-1}_{\{\spadesuit\}} C_{\mathrm{gw},1})$, the rhs. of (4.112) goes to 0 as $k \nearrow \infty$. □

Remark 4.113. Note that as $w^{-1}_{\{\spadesuit\}} C_{\mathrm{gw},1} \nearrow \infty$ one has $\min \mathrm{I_{PI}}(G, \mu) \to \infty$, i.e. the range of exponents for which a Poincaré inequality holds gets narrower and narrower. On the other hand, as $w^{-1}_{\{\spadesuit\}} C_{\mathrm{gw},1} \searrow 1$, $\min \mathrm{I_{PI}}(G, \mu) \to 1$ and thus the range of exponents for which the Poincaré inequality holds can be arbitrarily prescribed. However, as either $w^{-1}_{\{\spadesuit\}} C_{\mathrm{gw},1}$ goes to 1 or ∞, the doubling constant of μ_G blows up.

5 Putting all together

In this section we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. The existence of the measures $\{\mu_{P_c}\}_{P_c}$ follows combining Theorems 2.15, 4.75, 4.111 and Remark 4.113.

The projection map $\pi : G \to \mathbb{R}$ passes to the limit giving a 1-Lipschitz map $\pi : X \to \mathbb{R}$. The geodesic lines of the form $\mathbb{R} \times \{\lambda\} \times \{\theta\}$ pass to the limit and give a Fubini-like representation of the measure μ_{P_c}. To this Fubini representation one can associate a Weaver derivation D, i.e. a horizontal vector field as in [24].

The verification that (X, π) is a chart is standard and can be carried out in two ways. The first way uses a Sobolev-space argument like Sec. 9 in [4]. The second uses D and the Stone-Weierstrass Theorem for Lipschitz Algebras as in Example [27, Example 5E].

The claim about the Assouad-Nagata dimension follows because the graph G has Assouad-Nagata dimension 1 and the Assouad-Nagata dimension is stable in passing to asymptotic cones. □

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