-LOGICS AND GENERALIZED QUANTIFIERS

Abhandlung
zur Erlangung des Titels eines Doktors
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der
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5. The A-Closure of Continuous Quantifiers.

Let \( L \not\supset L_{\omega}^{\omega}[Q', \ldots, Q^n] \) and \( \omega_1 \)-seurable. By Corollary 2.2(iv) we know that \( L \) is not A-closed. It is hence natural to ask for a description of \( \Delta(L) \). In this section we shall solve this problem completely for \( L \) \( \omega_\)-continuous or \( \omega_1 \)-seurable and monadic (with only finitely many quantifiers).

Our main tool is the following lemma, which is proved in [MSS], cf. also [BGM]. Lemma 5.1 is independently, but much earlier, due to J. Barwise.¹

Lemma 5.1. Let \( L \) be a model theoretic language which is A-closed and where the structure \( \langle \omega, +, \cdot, ' \rangle, P \rangle \subseteq \omega \) is characterizable up to isomorphism (i.e. \( \langle \omega, +, \cdot, ' \rangle, P \rangle \in PC_L \)), then \( L \) is an extension of \( L_A \) with \( A = P^+ \), the next admissible set containing \( P \).

From the classical procedure of coding finite structures (or formulas) within number theory, we have to collect some facts.

Fact I. There is an effective Gödel numbering for the formulas of \( L_{\omega}^{\omega}[Q', \ldots, Q^n] \) which we shall denote by \( \Gamma \phi \) for \( \phi \) a formula in \( L_{\omega}^{\omega}[Q', \ldots, Q^n] \).

Fact II. There is an effective coding of the finite structures of a given similarity type, denote it by \( c(\mathcal{U}) \) for \( \mathcal{U} \) such a structure.

Fact III. There is a formula in \( L_{\omega}^{\omega}[Q', \ldots, Q^n] \), \( T(x, y) \) which is true in the natural numbers iff \( x \) codes some finite structure \( \mathcal{U} \), \( y \) codes some formula \( \phi \) and \( \mathcal{U} \models T(x, y) \). (cf. [Li 2]).

Having this in mind, we define

¹ Barwise had announced this to several people in a private communication.
Definition. Let a fixed coding of the finite structures of some similarity type in the natural numbers be given (recursively). If \( Q \) is a \( \omega \)-securable quantifier of type \( \mu \), then \( G(Q) = \{ n \in \omega \mid n \text{ is a code for a finite structure in } K^Q \} \) is called the Gödelset of \( Q \).

Theorem 5.2. Let \( Q \) be an \( \omega \)-securable quantifier, \( G(Q) \) its Gödelset; then \( Q \) is definable in \( L_A \) where \( A = G(Q)^+ \), the next admissible set containing \( G(Q) \).

Proof: Since \( A \) is closed under primitive recursive function, all the conjunctions constructed in the proof of Theorem 3.5 are in \( A \).

To get a converse of Theorem 5.2, we prove a lemma.

Lemma 5.3. \( (\omega, +, \cdot, ', G(Q)) \) is characterizable in \( L_\omega Q \), if \( Q \) is \( \omega \)-securable and not definable in \( L_\omega \).

Proof: By a lemma of [Li], \( (\omega, \cdot, +, ') \) is characterizable in \( L_\omega [Q] \) since \( L_\omega [Q] \) satisfies \( LS(\omega) \) and is a proper extension of \( L_\omega \). To define \( G(Q) \) we observe that \( n \in G(Q) \) iff \( \exists m (n, m) A m = \Gamma Qx[R_1, \ldots, R_n] \) holds in \( (\omega, \cdot, +, ') \) where \( (R_1, \ldots, R_n) \) is the similarity type of \( Q \). Hence \( \exists m (n, m) A m = \Gamma Qx[R_1, \ldots, R_n] \) defines \( G(Q) \) over \( (\omega, +, \cdot, ') \) and using the facts listed beforehand it is a formula of \( L_\omega [Q] \).

Theorem 5.4. \( \Delta(L_\omega Q) \sim L_A, A = G(Q)^+ \).

Proof: By Theorem 5.2, \( L_\omega Q \) \( \prec \) \( L_A \), hence by the closure properties of \( \Delta \) and the fact that \( L_A \) is \( \Delta \)-closed, we have \( \Delta(L_\omega Q) \) \( \prec \) \( L_A \).

By Lemma 5.3 and Lemma 5.1 we have also \( L_A \) \( \prec \) \( \Delta(L_\omega Q) \).
From Section 1 we know that there are \(2^\aleph_0\) many \(\omega\)-securable quantifiers. Let us recall that one can arrange the coding of finite structures in such a way that the cardinality is a primitive recursive function, i.e. "\(f(n) = m\) saying \(n\) is the code of some structure of cardinality \(m\)" is primitive recursive. Using this, we get:

**Theorem 5.5.** For every subset \(P\) of \(\omega\) there is \(\omega\)-securable quantifier \(Q^P\) such that \(\Delta(L^{\omega\omega}[Q^P]) = L_{p^+}\).

**Proof:** Let \(P \subseteq \omega\), and take \(Q^P\) as in Example 9 (Section 1). Clearly \(Q^P\) is definable in \(L_{p^+}\). Conversely define \(P\) by \(n \in P\) iff \(n\) is the cardinality of a finite-connected structure in \(K^P\), which, by the previous remarks, is expressible in \(L^{\omega\omega}[Q^P]\). \(\square\)

**Corollary 5.6.** \(L_{\omega^1\omega}\) is the smallest \(\Delta\)-closed language containing all \(\omega\)-securable quantifiers.

**Proof:** This follows from Theorem 5.5 together with the remark that every \(\phi \in L_{\omega^1\omega}\) is in some \(L_{p^+}\). \(\square\)

Corollary 5.6 says is that every formula \(\phi\) of \(L_{\omega^1\omega}\) is equivalent, using additional predicates to a boolean combination of formulas, which are either (i) in \(L^{\omega\omega}\) (ii) of the form \(Q\overline{x}[R_1, \ldots, R_n]\) with \(Q\) \(\omega\)-securable and \(R_1, \ldots, R_n\) prime formulas or (iii) of the form \(\forall \overline{x}(S(\overline{x}) \iff QY[R_1(\overline{x}, \overline{y}), \ldots, R_n(\overline{x}, \overline{y})])\) with \(Q\) \(\omega\)-securable and \(S, R_i\) \(i \leq n\) prime formulas. Using more additional predicates, every formula of \(L^{\omega\omega}\) can also be written as in (iii). But models of formulas of type (iii) are complements of projections of \(\omega\)-securable classes. Hence, with the help of Theorem 2.13, we have
Theorem 5.6'. (i) Every elementary class of $L_{\omega_1^\omega}$ (of finite similarity type) is the projection of a finite boolean combination of projections of biinductive classes. (ii) Every class of structures closed under isomorphisms $K$ such that $K$ and its complement are projections of biinductive classes, is an elementary class in $L_{\omega_1^\omega}$ (iii) A monadic class of structures $K$ closed under isomorphisms is biinductive iff $K$ is elementary in $L_{\omega_1^\omega}$ and both in $\forall \exists$ and $\exists \forall$.

Proof: All these are just reformulations of Corollary 5.6, 2.12, 3.5, 3.6 and 4.2. □

We now turn our attention to $\omega_1$-continuous monadic quantifiers to get similar results.

Using Theorem 4.3 for monadic $\omega_1$-continuous quantifiers and the previous remarks, we get easily:

**Theorem 5.7:** Let $Q$ be $\omega_1$-securable and monadic, $Q$ not definable in $L_{\omega_1^\omega}$. Then $\Delta(L_{\omega_1^\omega}[Q]) \sim L_{\delta}$ with $\delta = G(K^Q)^+$.  

Proof: Exactly as the proof of Theorem 5.2. □

**Theorem 5.8.** For every subset $P$ of $\omega$ there is a monadic $\omega_1$-securable quantifier $Q^P$ such that $\Delta(L_{\omega_1^\omega}[Q^P]) = L_{P^+}$.  

Proof: Take $K^P = \{(A,R)|R^A = n \text{ for some } n \in P\}$. Then $P$ is definable in $L_{\omega_1^\omega}[Q^P]$.  

**Corollary 5.9.** $L_{\omega_1^\omega}$ is the smallest $\Delta$-closed logic containing all $\omega_1$-securable monadic quantifiers.