LOGICS AND GENERALIZED QUANTIFIERS

Abhandlung
zur Erlangung des Titels eines Doktors
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der
EIDGENÖSSISCHEN TECHNISCHEN HOCHSCHULE ZÜRICH

vorgelegt von
JOHANN ANDREAS MAKOWSKY
Dipl. Math. ETH-Zürich
geboren am 12. März 1948
von Zürich (Kt. Zürich)

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Prof. Dr. H. Läuchli, Referent
Prof. Dr. E. Specker, Korreferent

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5. The \(\Delta\)-Closure of Continuous Quantifiers.

Let \( L \supseteq L_{\omega_1}[Q',\ldots, Q^n] \) and \( \omega_1 \)-securable. By Corollary 2.2(iv) we know that \( L \) is not \(\Delta\)-closed. It is hence natural to ask for a description of \(\Delta(L)\). In this section we shall solve this problem completely for \( L \) \(\omega\)-continuous or \(\omega_1\)-securable and monadic (with only finitely many quantifiers).

Our main tool is the following lemma, which is proved in [MSS], cf. also [BGM]. Lemma 5.1 is independently, but much earlier, due to J.Barwise.\(^1\)

**Lemma 5.1.** Let \( L \) be a model theoretic language which is \(\Delta\)-closed and where the structure \( (\omega, +, \cdot, ' P) \), \( P \subseteq \omega \), is characterizable up to isomorphism (i.e. \( \{ (\omega, +, \cdot, ' P) \} \in PC_L \)), then \( L \) is an extension of \( L_A \) with \( A = P^+ \), the next admissible set containing \( P \).

From the classical procedure of coding finite structures (or formulas) within number theory, we have to collect some facts.

**Fact I.** There is an effective Gödel numbering for the formulas of \( L_{\omega_1}[Q',\ldots, Q^n] \) which we shall denote by \( \bar{\varphi} \) for \( \varphi \) a formula in \( L_{\omega_1}[Q',\ldots, Q^n] \).

**Fact II.** There is an effective coding of the finite structures of a given similarity type, denote it by \( c(\mathcal{U}) \) for \( \mathcal{U} \) such a structure.

**Fact III.** There is a formula in \( L_{\omega_1}[Q',\ldots, Q^n] \), \( T(x, y) \) which is true in the natural numbers iff \( x \) codes some finite structure \( \mathcal{U} \), \( y \) codes some formula \( \varphi \) and \( \mathcal{U} \models \varphi \). (cf.[Li 2]).

Having this in mind, we define

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\(^1\) Barwise had announced this to several people in a private communication.
Definition. Let a fixed coding of the finite structures of some similarity type in the natural numbers be given (recursively). If $Q$ is a \( \omega \)-securable quantifier of type \( \mu \), then \( G(Q) = \{ n \in \omega \mid n \text{ is a code for a finite structure in } K^Q \} \) is called the Gödelset of $Q$.

Theorem 5.2. Let $Q$ be an \( \omega \)-securable quantifier, $G(Q)$ its Gödelset; then $Q$ is definable in $L_A$ where $A = G(Q)^+$, the next admissible set containing $G(Q)$.

Proof: Since $A$ is closed under primitive recursive function, all the conjunctions constructed in the proof of Theorem 3.5 are in $A$. \( \square \)

To get a converse of Theorem 5.2, we prove a lemma.

Lemma 5.3. \( (\omega, +, \cdot, ', G(Q)) \) is characterizable in $L_{\omega^\omega}[Q]$, if $Q$ is \( \omega \)-securable and not definable in $L_{\omega^\omega}$.

Proof: By a lemma of [Li], \( (\omega, \cdot, +, ') \) is characterizable in $L_{\omega^\omega}[Q]$ since $L_{\omega^\omega}[Q]$ satisfies $LS(\omega)$ and is a proper extension of $L_{\omega^\omega}$. To define $G(Q)$ we observe that $n \in G(Q)$ iff
\[ \exists m T(n,m) \land m = R\chi[R_1, \ldots, R_n] \] holds in \( (\omega, \cdot, +, ') \) where \( (R_1, \ldots, R_n) \) is the similarity type of $Q$. Hence \( \exists m T(n,m) \land m = R\chi[R_1, \ldots, R_n] \) defines $G(Q)$ over \( (\omega, +, \cdot, ') \) and using the facts listed beforehand it is a formula of $L_{\omega^\omega}[Q]$. \( \square \)

Theorem 5.4. $\Delta(L_{\omega^\omega}[Q]) \sim L_A$, $A = G(Q)^+$.

Proof: By Theorem 5.2, $L_{\omega^\omega}[Q] < L_A$, hence by the closure properties of $\Delta$ and the fact that $L_A$ is $\Delta$-closed, we have $\Delta(L_{\omega^\omega}[Q]) < L_A$. By Lemma 5.3 and Lemma 5.1 we have also $L_A \triangleleft \Delta(L_{\omega^\omega}[Q])$. \( \square \)
From Section 1 we know that there are $2^\aleph_0$ many $\omega$-securable quantifiers. Let us recall that one can arrange the coding of finite structures in such a way that the cardinality is a primitive recursive function, i.e. "$f(n) = m$ saying $n$ is the code of some structure of cardinality $m$" is primitive recursive. Using this, we get:

**Theorem 5.5.** For every subset $P$ of $\omega$ there is $\omega$-securable quantifier $Q^P$ such that $\Delta(L_{\omega_1}[Q^P]) = L_{P^+}$.

**Proof:** Let $P \subseteq \omega$, and take $Q^P$ as in Example 9 (Section 1). Clearly $Q^P$ is definable in $L_{P^+}$. Conversely define $P$ by $n \in P$ iff $n$ is the cardinality of a finite-connected structure in $K^P$, which, by the previous remarks, is expressible in $L_{\omega_1}[Q^P]$. 

**Corollary 5.6.** $L_{\omega_1 \omega}$ is the smallest $\Delta$-closed language containing all $\omega$-securable quantifiers.

**Proof:** This follows from Theorem 5.5 together with the remark that every $\phi \in L_{\omega_1 \omega}$ is in some $L_{P^+}$.

Corollary 5.6 says is that every formula $\phi$ of $L_{\omega_1 \omega}$ is equivalent, using additional predicates to a boolean combination of formulas, which are either (i) in $L_{\omega_1 \omega}$ (ii) of the form $Q\forall x[R_1, \ldots, R_n]$ with $Q$ $\omega$-securable and $R_1, \ldots, R_n$ prime formulas or (iii) of the form $\forall x(S(x) \iff Q\forall y[R_i(x, y), \ldots, R_n(x, y)])$ with $Q$ $\omega$-securable and $S, R_i$ $i \leq n$ prime formulas. Using more additional predicates, every formula of $L_{\omega_1 \omega}$ can also be written as in (iii). But models of formulas of type (iii) are complements of projections of $\omega$-securable classes. Hence, with the help of Theorem 2.13, we have
Theorem 5.6'. (i) Every elementary class of \( L_{\omega_1\omega} \) (of finite similarity type) is the projection of a finite boolean combination of projections of biinductive classes. (ii) Every class of structures closed under isomorphisms \( K \) such that \( K \) and its complement are projections of biinductive classes, is an elementary class in \( L_{\omega_1\omega} \) (iii) A monadic class of structures \( K \) closed under isomorphisms is biinductive iff \( K \) is elementary in \( L_{\omega_1\omega} \) and both in \( \forall \exists \) and \( \exists \forall \).

Proof: All these are just reformulations of Corollary 5.6, 2.12, 3.5, 3.6 and 4.2. \( \square \)

We now turn our attention to \( \omega_1 \)-continuous monadic quantifiers to get similar results.

Using Theorem 4.3 for monadic \( \omega_1 \)-continuous quantifiers and the previous remarks, we get easily:

Theorem 5.7: Let \( Q \) be \( \omega_1 \)-securable and monadic, \( Q \) not definable in \( L_{\omega_1\omega} \). Then \( \Delta(L_{\omega_1\omega}[Q]) \sim L_A \) with \( A = G(KQ)^+ \).

Proof: Exactly as the proof of Theorem 5.2. \( \square \)

Theorem 5.8. For every subset \( P \) of \( \omega \) there is a monadic \( \omega_1 \)-securable quantifier \( Q^P \) such that \( \Delta(L_{\omega_1\omega}[Q^P]) = L_{P^+} \).

Proof: Take \( K^P = \{(A,R)\mid \exists n \text{ for some } n \in P \} \). Then \( P \) is definable in \( L_{\omega_1\omega}[Q^P] \).

Corollary 5.9. \( L_{\omega_1\omega} \) is the smallest \( \Delta \)-closed logic containing all \( \omega_1 \)-securable monadic quantifiers.