Prom. Nr. 3139

HOMOTOPY THEORY IN GENERAL CATEGORIES

Von der

EIDGENÖSSISCHEN TECHNISCHEN HOCHSCHULE IN ZÜRICH

zur Erlangung
der Würde eines Doktors der Mathematik
genehmigte

PROMOTIONSARBEIT

Vorgelegt von

Peter Jost Huber
Dipl. Math. E. T. H.

von

Besenbüren (A. G.)

Referent: Herr Prof. Dr. B. Eckmann
Korreferent: Herr Prof. Dr. H. Hopf

Brühlsche Universitätsdruckerei Gießen
1961

Leer - Vide - Empty

HUBER, P. J.
Math. Annalen 144, 361—385 (1961)

Homotopy Theory in General Categories

By

PETER J. HUBER *) in Zürich and Geneva

Introduction

The homotopy groups of topological spaces have been generalized by ECKMANN-HILTON to two-space homotopy groups $\Pi_n(X, Y)$ that contain homotopy groups and cohomology groups (both with arbitrary coefficients) as well as their exact sequences, as special cases; the classical homotopy and cohomology are dual to each other in the sense of a simple (heuristic) duality that consists in interchanging X and Y. Furthermore, there is, in the category of modules over a ring — or more generally in any abelian category with sufficiently many injectives and projectives — an analogous homotopy theory and its dual (cf. ECKMANN [1]); here, the duality is not only heuristic, as in the category of spaces, but follows automatically. By the analogy in question, the imbedding of a topological space X into the cone CX, for instance, corresponds to the imbedding of a module X into an injective module $\overline{X}$; the topological suspension $\Sigma X = CX/X$ corresponds to the algebraic suspension $\Sigma X = \overline{X}/X$; and, in both categories, the homotopy groups may be defined with the aid of iterated suspensions.

These two heuristic principles

(a) the duality in the category of spaces,

(b) the analogy between spaces and modules,

have much stimulated the development of the Eckmann-Hilton homotopy theory. It will be shown in the present paper that these principles can be given a theoretical foundation.

For this purpose, we shall develop a semisimplicial homotopy theory in the framework of general categories, such that the homotopy theory of spaces and the one of modules are included as special cases, as well as the homotopy theory of maps of spaces, etc. Moreover, in all cases where this homotopy theory can be defined, full duality is obtained automatically, from the general duality principle in categories. Thereby, not only is a precise notion of the analogy between the homotopy theories for modules and spaces achieved, but it is also possible to simplify substantially some proofs. For instance, the exactness of the homotopy sequences in the categories of modules, of spaces, of pairs of

*) This research was partly supported by the U. S. Department of Army through its European Research Office.

modules, of pairs of spaces, of pairs of pairs, etc., may be established in one and the same proof; and it is to be noted that this single proof gives, by strict duality, both sides of the picture, e.g. the homotopy as well as the cohomology sequences in the category of spaces.

Our main tool will be the semisimplicial standard construction, originally devised by R. GODEMENT [4] to generate flabby resolutions in the category of sheaves. Since also the Hochschild homology theory of associative algebras (cf. [4]) and, moreover, the whole theory of derived functors in the categories of modules may be obtained with the aid of standard constructions, these turn out to be one of the most powerful tools of homological and homotopical algebra.

The standard constructions may be considered as being a generalization of the path space and cone constructions in topology. For instance, the triple $\{E, k, p\}$, consisting of the path functor E, of the natural fibre map $k(Y): EY \to Y$ and of a hitherto scarcely noticed natural map $p(Y): EY \to EEY$, constitutes a standard construction. Dually, the cone functor C, the natural imbedding $k(X): X \to CX$, and a certain map $p(X): CCX \to CX$ constitute a dual standard construction.

Section 1 introduces the terminology to be used in this paper, section 2 contains the definition of the standard construction, and in section 3 the semisimplicial complex associated with each standard construction is introduced, together with a preliminary discussion of its Kan homotopy groups. In section 5, the homotopy groups in the categories of modules will be treated. To simplify the pertinent proofs, large parts of them will be dealt with in the framework of general categories (parts of section 3, and section 4). Section 6 discusses the homotopy groups of topological spaces; section 7 contains, among other topics, an interesting generalization of the singular complex of a space. In section 8, the exactness of the homotopy sequence is proved for general categories. Section 9 contains a treatment of fibrations and cofibrations; it turns out that one may define the analogue, and establish the main properties, of the suspension ΣX (the cofibre of $k(X): X \to CX$) and of the loop space ΩX (the fibre of $k(Y): EY \to Y$) in the framework of general categories.

The author is greatly indebted to Professor ECKMANN for many hours of helpful and stimulating discussions.

1. Categories and Functors

A *category* $\mathfrak{K}$ consists of a non-empty class $\mathfrak{K}$ of *objects* $X, Y, \ldots,$ together with sets $\mathrm{Hom}(X, Y)$ of *morphisms* $u: X \to Y$ $(X, Y \in \mathfrak{K})$, and of an associative *composition* of morphisms $\circ: \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$, $(u, v) \to$ $\to v \circ u$, which has both-sided *identities* $1_X \in \mathrm{Hom}(X, X)$.

A morphism $u: X \to Y$ is an *isomorphism* if there exists a morphism $v: Y \to X$, such that $u \circ v$ and $v \circ u$ are the respective identities. X and Y then are said to be *isomorphic*.

An object $O \in \mathfrak{K}$ is called a *zero object* if, for all $X \in \mathfrak{K}$, the sets $\mathrm{Hom}(X, O)$ and $\mathrm{Hom}(O, X)$ consist of exactly one element 0_{XO} and 0_{OX} respectively. Two zero objects O, O' are always isomorphic, and we have $0_{OY} \circ 0_{XO} = 0_{O'Y} \circ 0_{XO'}$. If $\mathfrak{K}$ has a zero object, then in each set $\mathrm{Hom}(X, Y)$ we have a distinguished morphism $0 = 0_{XY} = 0_{OY} \circ 0_{XO}$, the *zero morphism.*

Let $\mathfrak{K}$ and $\mathfrak{L}$ be arbitrary categories. A *covariant functor* $F : \mathfrak{K} \to \mathfrak{L}$ assigns to each object $X \in \mathfrak{K}$ an object $F(X) \in \mathfrak{L}$, and to each morphism $u : X \to Y$ a morphism $F(u) : F(X) \to F(Y)$, such that $F(u \circ v) = F(u) \circ F(v)$, and that $F(1_X) = 1_{F(X)}$. If both categories $\mathfrak{K}$ and $\mathfrak{L}$ contain a zero object, then we shall mostly require that zero objects are preserved under F.

Let $F, G : \mathfrak{K} \to \mathfrak{L}$ be covariant functors. A *functor morphism* (or *natural transformation*) $\vartheta : F \to G$ consists of a family of morphisms $\vartheta(X) : F(X) \to G(X)$, such that the following diagram is commutative for all objects $X, Y \in \mathfrak{K}$ and all morphisms $u : X \to Y$.

$$\begin{array}{ccc} F(X) & \xrightarrow{F(u)} & F(Y) \\ {\scriptstyle \vartheta(X)} \downarrow & & \downarrow {\scriptstyle \vartheta(Y)} \\ G(X) & \xrightarrow{G(u)} & G(Y) \end{array}$$

The compositions $F \circ G$ of functors and $\vartheta' \circ \vartheta''$ of functor morphisms are defined in the obvious way. Mostly, we shall abbreviate $F \circ G$ to FG. Each functor morphism $\vartheta : F \to G$ induces a morphism of the compositions of F and G with covariant functors U, V:

$$U * \vartheta * V : UFV \to UGV,$$

which is defined by $(U * \vartheta * V)(X) = U\big(\vartheta(V(X))\big)$. If U (or V) is the identity functor I, then we abbreviate $U * \vartheta * V$ to $\vartheta * V$ (or $U * \vartheta$ respectively). The *identity morphism* $\iota : I \to I$ is defined by $\iota(X) = 1_X$; obviously, we have $F * \iota = \iota * F$ for any functor F.

For any category $\mathfrak{K}$, one defines the *dual category* $\mathfrak{K}'$, which consists of the same objects as $\mathfrak{K}$; the set $\mathrm{Hom}'(X, Y)$ of morphisms $X \to Y$ in $\mathfrak{K}'$ is identical with the set $\mathrm{Hom}(Y, X)$ of morphisms $Y \to X$ in $\mathfrak{K}$, and the composition of two morphisms u, v in $\mathfrak{K}'$ is defined as being the composition of v and u in $\mathfrak{K} : u \circ' v = v \circ u$. Evidently, $\mathfrak{K}'$ is a category, and we have $(\mathfrak{K}')' = \mathfrak{K}$.

Thus, one may say that $\mathfrak{K}$ and $\mathfrak{K}'$ are the same things, being described in two different languages; each statement S about the category $\mathfrak{K}$ may be translated into a statement about $\mathfrak{K}'$, and vice versa. This seems quite trivial. Now we consider a statement S about $\mathfrak{K}$, which belongs to the theory of categories (that means that S is composed only from logical terms and terms such as "object", "morphism", etc., and thus makes sense in arbitrary categories); then the same statement S makes sense in $\mathfrak{K}'$, since $\mathfrak{K}'$ is a category. Hence, S in $\mathfrak{K}'$ may be translated into a statement S' about $\mathfrak{K}$; S' again belongs to the theory of categories. These two statements S and S', both making sense in arbitrary categories, are called *dual* to each other. It follows that, if S can be proved from some axioms $A_1, A_2, \ldots$, these axioms as well as the proof belonging to the theory of categories, then the dual statement S' can be proved from the dual axioms $A_1', A_2', \ldots$. This *duality principle* will be used extensively in the sequel; mostly, however, it will be left to the reader to formulate the dual definitions and theorems.

Sometimes, the explicit introduction of the dual category has not only conceptual, but also technical advantages. For instance, a *contravariant functor* $F : \mathfrak{K} \to \mathfrak{L}$ is the same as a covariant functor $F : \mathfrak{K}' \to \mathfrak{L}$ or $F : \mathfrak{K} \to \mathfrak{L}'$, so that it suffices, at least in principle, to consider covariant functors only. This works for $\mathfrak{K} = \mathfrak{L}$, too; but if one tries to handle contravariant functors with the aid of the duality principle, without introducing explicitly the dual category, the case $\mathfrak{K} = \mathfrak{L}$ needs a rather awkward special treatment.

The following five formulas are valid for any covariant functors and any functor morphisms, as soon as they make sense (cf. GODEMENT [4], Appendice).

(I) $(U \circ V) * \vartheta = U * (V * \vartheta)$

(II) $\vartheta * (U \circ V) = (\vartheta * U) * V$

(III) $(U * \vartheta) * V = U * \vartheta * V = U * (\vartheta * V)$

(IV) $U * (\vartheta' \circ \vartheta'') * V = (U * \vartheta' * V) \circ (U * \vartheta'' * V)$

(V) $(\psi * G) \circ (U * \varphi) = (V * \varphi) \circ (\psi * F)$

for any two functor morphisms $\varphi : F \to G$, $\psi : U \to V$.

Rule (V) may be remembered with the aid of the commutative diagram:

$$\begin{array}{ccc} U \circ F & \xrightarrow{U * \varphi} & U \circ G \\ {\scriptstyle \psi * F}\downarrow & & \downarrow{\scriptstyle \psi * G} \\ V \circ F & \xrightarrow{V * \varphi} & V \circ G \end{array}$$

2. Definition of the Standard Construction

Let $\mathfrak{K}$ be an arbitrary category. A *standard construction* in $\mathfrak{K}$ is a triple $\{C, k, p\}$, consisting of a covariant functor $C : \mathfrak{K} \to \mathfrak{K}$, and of two functor morphisms $k : C \to I$, $p : C \to CC$, such that the following two axioms (GODEMENT [4], Appendice) are satisfied:

(SC 1) $$(k * C) \circ p = (C * k) \circ p = \iota * C$$

(SC 2) $$(p * C) \circ p = (C * p) \circ p$$

Example. In any category, there exists the trivial standard construction $\{I, \iota, \iota\}$, consisting of the identity functor and of the identity morphisms.

Each standard construction $\{C, k, p\}$ generates a semisimplicial functor

$$F_* = (F_n, d_n^i, s_n^i)_{n \geqq -1},$$

that is, a sequence of functors $F_n : \mathfrak{K} \to \mathfrak{K}$, together with face and degeneracy morphisms

$$d_n^i : F_n \to F_{n-1}$$
$$s_n^i : F_n \to F_{n+1} \qquad (0 \leqq i \leqq n),$$

F_* is defined as follows. Let $C^0 = I$ and $C^{n+1} = C \circ C^n$. Then

$$F_n = C^{n+1}$$
$$d_n^i = C^i * k * C^{n-i}$$
$$s_n^i = C^i * p * C^{n-i}.$$

Usually, we shall omit the lower indices of d_n^i and of s_n^i.

Sometimes it is convenient to treat $F_{-1} = I$ separately, and to consider F_* as a semisimplicial functor $F_*^+ = (F_n, d^i, s^i)_{n \geqq 0}$ with *augmentation* $d_0^0 = k : F_0 \to I$.

The face and degeneracy morphisms satisfy the usual semisimplicial commutation rules:

(a) $d^i d^j = d^{j-1} d^i \quad (i < j)$

(b) $s^i s^j = s^{j+1} s^i \quad (i \leqq j)$

(c) $d^i s^j = s^{j-1} d^i \quad (i < j)$

(d) $d^i s^i = d^{i+1} s^i = \text{identity}$

(e) $d^i s^j = s^j d^{i-1} \quad (i > j + 1)$.

This follows from (SC 1) and (SC 2) with the aid of the five rules at the end of the previous section; in fact, (b) is equivalent to axiom (SC 2), and (d) is equivalent to axiom (SC 1), whereas (a), (c) and (e) are valid in any case.

A *dual standard construction* is a triple $\{C, k, p\}$ consisting of a covariant (!) functor $C : \mathfrak{K} \to \mathfrak{K}$, and of functor morphisms $k : I \to C$, $p : CC \to C$, such that the axioms

(SC 1') $\quad p \circ (k * C) = p \circ (C * k) = \iota * C$

(SK 2') $\quad p \circ (p * C) = p \circ (C * p)$

are satisfied.

The duality principle implies that it suffices to consider only one kind of standard construction, and then to dualize the results, if necessary. For instance, it follows from duality that the functor $F^* = (F^n, d^i, s^i)$ belonging to a dual standard construction has a dual semisimplicial structure, i.e. the face and degeneracy morphisms

$$d_n^i : F^{n-1} \to F^n$$
$$s_n^i : F^{n+1} \to F^n$$

go into the opposite direction and satisfy the relations dual to (a)—(e).

3. Homotopy Groups. T-trivial Constructions

If we apply the semisimplicial functor F_* to an object $Y \in \mathfrak{K}$, we obtain a semisimplicial object $F_*(Y) = (F_n(Y), d^i(Y), s^i(Y))_{n \geqq -1}$. One would like to investigate the homotopy groups of $F_*(Y)$, but this is not possible in general, since the Kan homotopy groups (cf. [7]) are defined only for semisimplicial complexes, i.e. for semisimplicial objects in the category of sets. One could try to extend the definition of homotopy groups to more general categories — similarly as homological algebra has been extended to abelian categories — but here another way seems to be more promising: namely, to go over to an ordinary semisimplicial complex with the aid of a functor $T:\mathfrak{R}\to\mathfrak{M}$ with values in the category of sets.

For instance, we may choose the functor $\mathrm{Hom}(X,\ )$ of the category $\mathfrak{R}$, with a fixed first argument X, to obtain a semisimplicial complex

$$K_*(X, Y) = \mathrm{Hom}(X, F_*(Y)) = (\mathrm{Hom}(X, F_n(Y)), d^i, s^i)_{n\geqq -1}\,.$$

The Kan homotopy groups of $K_*(X, Y)$ will generalize the Eckmann-Hilton groups.

If the objects of the category $\mathfrak{R}$ are sets, and if the morphisms $u: X\to Y$ are mappings of the set X into the set Y, and are composed in the natural way, then we may choose, for instance, the functor T, which assigns to each object its underlying set.

If $\mathfrak{R}$ contains a zero object and C preserves zero objects, then it is convenient to choose for $\mathfrak{M}$ the category of sets with base element, and to require that T preserves zero objects. (The zero objects of $\mathfrak{M}$ are those sets which are reduced to the base element.) This assumption is verified for $T = \mathrm{Hom}(X,\ )$.

The homotopy groups of $TF_*(Y)$ now may be defined in any of the usual ways (cf. KAN [7]). First, we shall give a sufficient condition for them to be trivial.

Definition 3.1. Let $T:\mathfrak{R}\to\mathfrak{M}$ be a covariant functor with values in the category of sets. A standard construction $\{C, k, p\}$ in $\mathfrak{R}$ is called *T-trivial* if there exists a functor morphism $h: T\to TC$, such that

$$(T * k)\circ h = \iota * T\,.$$

Example. The trivial construction $\{I, \iota, \iota\}$ is T-trivial for all T, with $h = \iota * T$.

Theorem 3.2. *Let $\{C, k, p\}$ be a T-trivial standard construction. Then, the semisimplicial complex $TF_*^+(Y)$ has the component set*

$$\pi_0(TF_*^+(Y)) = T(Y)\,,$$

and each component has the homotopy groups

$$\pi_n(TF_*^+(Y)) = 0 \qquad (n > 0)\,.$$

Proof. We want to show that $TF_*^+(Y)$ is homotopy equivalent with the complex L_*^+ defined by $L_n = T(Y)$, $d^i = s^i =$ identity. Obviously, L_*^+ is a Kan complex, and the assertion of the theorem follows.

We recall the definition of the semisimplicial homotopy relation. Let P_*^+, Q_*^+ be arbitrary semisimplicial complexes. Two semisimplicial maps $f, g: P_*^+\to Q_*^+$ are homotopic (MOORE [10]) if and only if there exist functions

$$h_n^i: P_n\to Q_{n+1} \qquad (0\leqq i\leqq n),$$

such that

(1) $$d^0_{n+1} h^0_n = g_n$$

(2) $$d^{n+1}_{n+1} h^n_n = f_n$$

(3) $$d^i h^j = h^{j-1} d^i \qquad (i < j)$$

(4) $$d^{j+1} h^{j+1} = d^{j+1} h^j$$

(5) $$d^i h^j = h^j d^{i-1} \qquad (i > j+1)$$

(6) $$s^i h^j = h^{j+1} s^i \qquad (i \leqq j)$$

(7) $$s^i h^j = h^j s^{i-1} \qquad (i > j)\,.$$

Now, we shall define a homotopy h^i_n between the identity map $TF^{\ddagger}_{*}(Y) \to \to TF^{\ddagger}_{*}(Y)$ and a map f, which is a retraction map to a subcomplex isomorphic with $L^{\ddagger}_{*}$. To simplify the notation, we shall write d^i_n, s^i_n instead of $T(d^i_n(Y))$ and of $T(s^i_n(Y))$; the argument Y will be omitted in some other cases, too.

We define

$$h^0_n = h * C^{n+1}, \qquad (n \geqq -1)$$

$$h^i_n = (s^0)^i h^0_{n-i} (d^0)^i, \qquad (0 \leqq i \leqq n).$$

h^0_n verifies the following relations:

(A) $$d^0_{n+1} h^0_n = \text{identity} \qquad (n \geqq -1)$$

(B) $$d^{i+1}_{n+1} h^0_n = h^0_{n-1} d^i_n \qquad (n \geqq 0)$$

(C) $$s^{i+1}_{n+1} h^0_n = h^0_{n+1} s^i_n \qquad (n \geqq 0)\,.$$

Proof.

(A) $$d^0_{n+1} h^0_n = (T * k * C^{n+1}) \circ (h * C^{n+1}) = (T * k \circ h) * C^{n+1} = \text{identity}$$

(B) $$\begin{aligned} d^{i+1}_{n+1} h^0_n &= T * (C^{i+1} * k * C^{n-i}) \circ h * C^{n+1} \\ &= \big((TC * (C^i * k)) \circ (h * C^{i+1})\big) * C^{n-i} \\ &= \big((h * C^i) \circ (T * (C^i * k))\big) * C^{n-i} \\ &= h^0_{n-1} d^i_n \end{aligned}$$

(C) $$\begin{aligned} s^{i+1}_{n+1} h^0_n &= T * (C^{i+1} * p * C^{n-i}) \circ h * C^{n+1} \\ &= (TC * (C^i * p) \circ h * C^{i+1}) * C^{n-i} \\ &= \big((h * C^{i+2}) \circ (T * (C^i * p))\big) * C^{n-i} \\ &= h^0_{n+1} s^i_n\,. \end{aligned}$$

The relations (1)—(7) now follow from (A)—(C) by a straightforward calculation, which will be left to the reader. It turns out that f is the semisimplicial map defined by

$$f_n = (s^0)^n h (d^0)^{n+1},$$

and that we have

(8) $$ff = d^i s^j f = s^i d^j f = f\,.$$

In particular, we have $f_0 = hd^0$, and hence, $hd^0 f_0 = f_0$. Since $d^0 h =$ identity, it follows that d^0 induces a $1-1$ map of $f_0(TF_0(Y))$ onto $T(Y)$. It follows by induction that $f(TF_*^+(Y))$ is isomorphic with the complex L_*^+.

Furthermore, (8) implies that

$$f : TF_*^+(Y) \to L_*^+ \subset TF_*^+(Y)$$

is a retraction map. Since f is homotopic to the identity map, it follows that $TF_*^+(Y)$ and L_*^+ have the same homotopy type.

The above homotopy h_n^i need not be a deformation retraction (cf. MOORE [10]); but this is true if and only if we have

$$(T * p) \circ h = (h * C) \circ h .$$

Since we shall not need this result, the proof is omitted.

In the case of a T-trivial construction, the augmentation $d^0 : TF_0(Y) \to$ $\to T(Y)$ is epimorphic, since $d^0 h =$ identity. Since the diagram

$$\begin{array}{ccc} TF_0(Y) & \xrightarrow{d^0} & T(Y) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle \text{id.}} \\ TF_0(Y) & \xrightarrow{d^0} & T(Y) \end{array}$$

is commutative, theorem 3.2 implies that $TF_*(Y)$ is the disjoint union of components $K_*^{(y)}$ defined by

$$K_n^{(y)} = (d^0)^{-(n+1)}(y), \qquad y \in T(Y) .$$

Usually, the homotopy sets are defined only for non-augmented semisimplicial complexes. However, sometimes it is useful to extend the definitions to the augmented case, especially if semisimplicial groups are involved. More generally, let K_* be a semisimplicial object in a category of sets with base element, satisfying the Kan condition. Then, according to KAN [7], one defines $\pi_n(K_*)$, $n \geqq 0$ as being the set of equivalence classes of

$$\Gamma_n = \{\sigma \in K_n \mid d^i \sigma = 0 \quad \text{for all} \quad i\} ,$$

modulo the homotopy relation $\sim$.

This definition carries over to the augmented case without any change; the above remarks and theorem 3.2 then imply that, in the case of a T-trivial construction, the homotopy of $TF_*(Y)$ is trivial in *all* dimensions $n \geqq 0$. We do not define (-1)-dimensional homotopy.

4. Induced Standard Constructions

Let $\mathfrak{K}$ and $\mathfrak{L}$ be arbitrary categories which are connected by a pair of covariant functors F and G:

$$F : \mathfrak{K} \to \mathfrak{L} , \quad G : \mathfrak{L} \to \mathfrak{K} .$$

Then, each functor $C : \mathfrak{K} \to \mathfrak{K}$ induces a functor $\bar{C} = FCG : \mathfrak{L} \to \mathfrak{L}$.

Now, we want to show that a standard construction $\{C, k, p\}$ in the category $\mathfrak{K}$ induces a standard construction $\{\bar{C}, \bar{k}, \bar{p}\}$ in $\mathfrak{L}$, if F and G are adjoint functors. First, we need a theorem on adjoint functors.

Theorem 4.1. *Let $F : \mathfrak{K} \to \mathfrak{L}$, $G : \mathfrak{L} \to \mathfrak{K}$ be covariant functors. The following two assertions are equivalent:*

(a) *There exists an isomorphism of functors*

$$\gamma : \mathrm{Hom}(F,\ \ ) \to \mathrm{Hom}(\ \ , G)\,,$$

i.e., F and G are adjoint functors in the sense of Kan [8]. *Usually, we shall identify* $\mathrm{Hom}(FX, Y) = \mathrm{Hom}(X, GY)$ *by γ.*

(b) *There exist two functor morphisms*

$$\zeta : I \to GF$$
$$\eta : FG \to I$$

which satisfy the relations

$$(\eta * F) \circ (F * \zeta) = \iota * F : F \to FGF \to F$$
$$(G * \eta) \circ (\zeta * G) = \iota * G : G \to GFG \to G\,.$$

Proof. (b) $\to$ (a). We define

$$\gamma : \mathrm{Hom}(FX, Y) \to \mathrm{Hom}(X, GY)$$
$$\beta : \mathrm{Hom}(X, GY) \to \mathrm{Hom}(FX, Y)$$

by $\gamma(u) = G(u) \circ \zeta(X)$, $\beta(v) = \eta(Y) \circ F(v)$.

The diagram

$$\begin{array}{ccccc} FX & \xrightarrow{(F*\zeta)(X)} & FGFX & \xrightarrow{FG(u)} & FGY \\ & \searrow^{(\iota*F)(X)} & \downarrow{\scriptstyle (\eta*F)(X)} & & \downarrow{\scriptstyle \eta(Y)} \\ & & FX & \xrightarrow{\quad u \quad} & Y \end{array}$$

is commutative. Thus

$$\eta(Y) \circ FG(u) \circ (F * \zeta)(X) = \eta(Y) \circ F(G(u) \circ \zeta(X)) = u\,;$$

therefore, we have $\beta \circ \gamma =$ identity. In the same way, one obtains $\gamma \circ \beta =$ identity. We have further to show that γ is natural with respect to X and Y, i.e. that the diagram

$$\begin{array}{ccc} \mathrm{Hom}(FX, Y) & \xrightarrow{(v, w)_*} & \mathrm{Hom}(FX', Y') \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\ \mathrm{Hom}(X, GY) & \xrightarrow{(v, w)_*} & \mathrm{Hom}(X', GY') \end{array}$$

is commutative for all morphisms $v : X' \to X$, $w : Y \to Y'$, or, equivalently, that we have

$$G(w) \circ \gamma(u) \circ v = \gamma(w \circ u \circ F(v))$$

for all morphisms $u : FX \to Y$. But this follows immediately from the definition of γ.

(a) → (b). We define ζ and η by

$$\zeta(X) = \gamma(1_{FX}), \quad \eta(Y) = \gamma^{-1}(1_{GY}).$$

First, we show that ζ and η are functor morphisms, i.e. that we have

$$\zeta(X) \circ v = GF(v) \circ \zeta(X')$$

and

$$\eta(X) \circ FG(v) = v \circ \eta(X'),$$

for all morphisms $v: X' \to X$. We have

$$\zeta(X) \circ v = \gamma(1_{FX}) \circ v = \gamma(F(v)),$$

and

$$GF(v) \circ \zeta(X') = GF(v) \circ \gamma(1_{FX'}) = \gamma(F(v)),$$

which proves the first assertion; the proof for η is dual.

Furthermore, we have

$$G(\eta(Y)) \circ \zeta(G(Y)) = G(\gamma^{-1}(1_{GY})) \circ \gamma(1_{FGY}) = \gamma(\gamma^{-1}(1_{GY}) \circ 1_{FGY}) = 1_{GY}$$

and

$$\eta(F(X)) \circ F(\zeta(X)) = \gamma^{-1}(1_{GFX}) \circ F(\gamma(1_{FX})) = \gamma^{-1}\big(\gamma(\gamma^{-1}(1_{GFX})) \circ \circ \gamma(1_{FX})\big) = 1_{FX},$$

which proves (b).

Theorem 4.2. *Let $F: \mathfrak{K} \to \mathfrak{L}$, $G: \mathfrak{L} \to \mathfrak{K}$ be covariant adjoint functors, such that we may identify* $\mathrm{Hom}(FX, Y) = \mathrm{Hom}(X, GY)$. *Then, each standard construction $\{C, k, p\}$ in $\mathfrak{K}$ induces a standard construction $\{\overline{C}, \overline{k}, \overline{p}\}$ in $\mathfrak{L}$, namely (using the notations of theorem* 4.1*):*

$$\overline{C} = FCG$$
$$\overline{k} = \eta \circ (F * k * G)$$
$$\overline{p} = F * ((C * \zeta * C) \circ p) * G.$$

Proof. We have to verify the axioms (SC 1) and (SC 2). With the aid of the rules (I)—(V) of section 1, and of the fact that (SC 1) is satisfied by the construction $\{C, k, p\}$, we obtain

$$\begin{aligned}(\overline{k} * \overline{C}) \circ \overline{p} &= (\eta \circ F * k * G) * FCG \circ F * (C * \zeta * C \circ p) * G \\ &= \eta * FCG \circ F * (k * GF \circ C * \zeta) * CG \circ F * p * G \\ &= \eta * FCG \circ F * (\zeta \circ k) * CG \circ F * p * G \\ &= \eta * FCG \circ F * (\zeta * C \circ k * C \circ p) * G \\ &= (\eta * F \circ F * \zeta) * CG\end{aligned}$$

$$\begin{aligned}(\overline{C} * \overline{k}) \circ \overline{p} &= FCG * (\eta \circ F * k * G) \circ F * (C * \zeta * C \circ p) * G \\ &= FCG * \eta \circ FC * (GF * k \circ \zeta * C) * G \circ F * p * G \\ &= FCG * \eta \circ FC * (\zeta \circ k) * G \circ F * p * G \\ &= FC * (G * \eta \circ \zeta * G) \circ F * (C * k \circ p) * G \\ &= FC * (G * \eta \circ \zeta * G).\end{aligned}$$

If, in addition, the assumptions of theorem 4.1 (b) are satisfied, then both expressions are equal to $\iota * FCG$; hence, (SC 1) is valid. Since (SC 2) is satisfied by the construction $\{C, k, p\}$, we obtain

$$\begin{aligned}(\bar{p} * \bar{C}) \circ \bar{p} &= F * (C * \zeta * C \circ p) * G * FCG \circ F * (C * \zeta * C \circ p) * G \\ &= F * (C * \zeta * CGFC \circ p * GFC \circ C * \zeta * C \circ p) * G \\ &= F * (C * \zeta * CGFC \circ (p * GF \circ C * \zeta) * C \circ p) * G \\ &= F * (C * \zeta * CGFC \circ (CC * \zeta \circ p) * C \circ p) * G \\ &= F * (C * (\zeta * CGF \circ C * \zeta) * C \circ p * C \circ p) * G \\ &= FC * (GFC * \zeta \circ \zeta * C) * CG \circ F * (p * C \circ p) * G \\ (\bar{C} * \bar{p}) \circ \bar{p} &= FCG * F * (C * \zeta * C \circ p) * G \circ F * (C * \zeta * C \circ p) * G \\ &= F * (CGFC * \zeta * C \circ C * (GF * p \circ \zeta * C) \circ p) * G \\ &= F * (CGFC * \zeta * C \circ C * (\zeta * CC \circ p) \circ p) * G \\ &= F * (CGFC * \zeta * C \circ C * \zeta * CC \circ C * p \circ p) * G \\ &= FC * (GFC * \zeta \circ \zeta * C) * CG \circ F * (p * C \circ p) * G\end{aligned}$$

Hence, (SC 2) is satisfied by the construction $\{\bar{C}, \bar{k}, \bar{p}\}$, too.

By dualizing either in $\mathfrak{K}$ or in $\mathfrak{L}$, or in both categories, we obtain three additional types of induced constructions; an example will occur in section 5.2.

If $\{C, k, p\}$ is the trivial standard construction, then the induced construction $\{\bar{C}, \bar{k}, \bar{p}\}$ is not necessarily trivial (cf. section 5.1). T-triviality, however, is hereditary in some sense:

Theorem 4.3. *Let $\{C, k, p\}$ be a T-trivial standard construction in $\mathfrak{K}$, and let $F: \mathfrak{K} \to \mathfrak{L}$, $G: \mathfrak{L} \to \mathfrak{K}$ be adjoint functors. Then, the induced construction $\{\bar{C}, \bar{k}, \bar{p}\}$ in $\mathfrak{L}$ is $\bar{T}$-trivial, with $\bar{T} = TG$ and $\bar{h} = (T * \zeta * CG) \circ (h * G)$.*

Proof.
$$\begin{aligned}(\bar{T} * \bar{k}) \circ \bar{h} &= TG * (\eta \circ F * k * G) \circ T * \zeta * CG \circ h * G \\ &= TG * \eta \circ (T * (GF * k \circ \zeta * G) \circ h) * G \\ &= TG * \eta \circ (T * (\zeta \circ k) \circ h) * G \\ &= T * (G * \eta \circ \zeta * G) = \iota * TG = \iota * \bar{T} .\end{aligned}$$

5. Examples of Induced Constructions

5.1. *The projective homotopy groups of modules.* Let $\mathfrak{M}$ be the category of sets with base element, and let $\mathfrak{L}$ be the category of unitary left Λ-modules over a ring Λ with unit element. We define two functors $F: \mathfrak{M} \to \mathfrak{L}$, $G: \mathfrak{L} \to \mathfrak{M}$ as follows. F assigns to a set X the free module over X, with the only relation: base element $= 0$; G assigns to each module its underlying set, with the zero element as base element. We have a natural identification

$$\mathrm{Hom}(FX, Y) = \mathrm{Hom}(X, GY), \ (X \in \mathfrak{M}, \ Y \in \mathfrak{L});$$

hence, these two functors are adjoint. The trivial standard construction $\{I, \iota, \iota\}$ in $\mathfrak{M}$ induces a standard construction $\{C, k, p\}$ in $\mathfrak{L}$:

$$\begin{aligned}C &= FG \\ k &= \eta: FG \to I \\ p &= F * \zeta * G: FG \to FGFG .\end{aligned}$$

Obviously, the semisimplicial module complex $F^{\ddagger}_{*}(Y)$ belonging to this construction consists of free modules. Theorem 4.3 implies that $\{C, k, p\}$ is G-trivial, and theorem 3.2 and the remarks at the end of section 3 imply that $GF_{*}(Y)$ and $F_{*}(Y)$ have trivial homotopy groups.

J. C. MOORE has proved that the homotopy groups of a semisimplicial abelian group complex (K_n, d^i, s^i) are canonically isomorphic with the homology groups of the chain complex $(K_n, \partial = \Sigma(-1)^i d^i)$ (MOORE [10]; obviously, the theorem remains true for augmented complexes).

Therefore, the chain complex $(F^{\ddagger}_{*}(Y), \partial)$ is a free, a fortiori projective, resolution of Y.

Theorem 5.1. *The Kan homotopy groups of the semisimplicial abelian group complex*

$$K_{*}(X, Y) = \mathrm{Hom}(X, F_{*}(Y)), \ (X, Y \in \mathfrak{L})$$

are canonically isomorphic with the Eckmann-Hilton projective homotopy groups

$$\pi_n(K_{*}(X, Y)) = \underline{\Pi}_{n+1}(X, Y) \qquad (n \geq 0)\,.$$

Proof. An application of Moore's theorem yields that the homotopy groups $\pi_n(K_{*}(X, Y))$ are isomorphic with the homology groups of $K_{*}(X, Y)$, this latter being considered as a chain complex with differentiation $\partial = \Sigma\,(-1)^i d^i$. These homology groups are, by definition, the Eckmann-Hilton homotopy groups — with a trivial shift of dimensions. (Since the (-1)-dimensional Kan homotopy groups are not defined, we may obtain $\underline{\Pi}_0(X, Y)$ as (-1)-dimensional homology group of $K_{*}(X, Y)$, but not as homotopy group).

Of course, one may use $F_{*}(Y)$ to obtain a "semisimplicial" definition of the functors Ext and Tor, and of other derived functors; GODEMENT has given a similar definition of sheaf cohomology.

Now, we proceed to an explicit description of the construction $\{C, k, p\}$ and of the associated semisimplicial module complex $F_{*}(Y)$.

$F_{-1}(Y) = Y$ is the module to be resolved. $F_0(Y) = C(Y)$ is the free left Λ-module generated by the elements $\langle y\rangle$, $y \in Y$, with the single relation $\langle 0\rangle = 0$.

$F_1(Y) = C(C(Y))$ is therefore generated by the elements

$$\langle \lambda_1 \langle y_1\rangle + \cdots + \lambda_m \langle y_m\rangle\rangle,\ \lambda_i \in \Lambda,\ y_i \in Y, \text{etc.}$$

It suffices to define $k(Y)$ and $p(Y)$ on the generators of $C(Y)$:

$$k(Y): \langle y\rangle \to y$$
$$p(Y): \langle y\rangle \to \langle\langle y\rangle\rangle\,.$$

Hence, $d^i_n(Y): F_n(Y) \to F_{n-1}(Y)$ acts on the elements of $F_n(Y)$ by cancelling the bracket number $(i+1)$, $s^i_n(Y): F_n(Y) \to F_{n+1}(Y)$ by doubling the bracket number $(i+1)$, if the numbering of the brackets starts from outside.

Remark. The above procedure may be generalized to any arbitrary Abelian category admitting a projective generator and infinite direct sums. (For the notion of generator, see [5].)

5.2. *The injective homotopy groups of modules.* Let $\mathfrak{L}$ be the category of left Λ-modules, and $\mathfrak{R}$ the category of right Λ-modules. We define two functors $U: \mathfrak{R} \to \mathfrak{L}$, $V: \mathfrak{L} \to \mathfrak{R}$ as follows.

Let Q_1 be the additive group of the rational numbers modulo the integers (or, more generally, any divisible group containing this group). For each right Λ-module X, we define the left Λ-module

$$U(X) = X' = \mathrm{Hom}_Z(X, Q_1),$$

the module structure being induced by

$$(\lambda \varphi)(x) = \varphi(x\lambda).$$

Similarly, we define for each left Λ-module Y a right Λ-module

$$V(Y) = Y' = \mathrm{Hom}_Z(Y, Q_1).$$

Since we have a natural identification

$$\mathrm{Hom}_\Lambda(X, Y') = \mathrm{Hom}\ \ (Y, X'), \quad \text{by} \quad \varphi(x)(y) = \psi(y)(x),$$

these two contravariant functors U, V are adjoint.

The homomorphisms $\zeta: X \to X''$ and $\eta: Y \to Y''$ are the natural imbeddings into the "bidual" modules. The standard construction $\{C, k, p\}$ of 5.1 induces a dual standard construction $\{\overline{C}, \overline{k}, \overline{p}\}$ in $\mathfrak{R}$, which we shall investigate now. First, we need some propositions concerning the functors U and V. Since they are valid for both functors, we prefer the notation with primes: X', Y'.

Proposition 5.2. *If X is Λ-projective, the X' is Λ-injective.*

Proof. We shall treat only the case where X is a left Λ-module. We have to show that for each exact sequence of right Λ-modules

$$0 \to A \to B,$$

the induced sequence

$$\mathrm{Hom}_\Lambda(B, X') \to \mathrm{Hom}_\Lambda(A, X') \to 0$$

is exact.

Since X is Λ-projective, the sequence

$$0 \to A \underset{\Lambda}{\otimes} X \to B \underset{\Lambda}{\otimes} X$$

is exact; since Q_1 is Z-injective, it follows that the sequence

$$\mathrm{Hom}_Z(B \underset{\Lambda}{\otimes} X, Q_1) \to \mathrm{Hom}_Z(A \underset{\Lambda}{\otimes} X, Q_1) \to 0$$

is exact. The assertion now follows by an application of the associativity formulas:

$$\mathrm{Hom}_Z(B \underset{\Lambda}{\otimes} X, Q_1) = \mathrm{Hom}_\Lambda(B, \mathrm{Hom}_Z(X, Q_1)), \text{ etc.}$$

Proposition 5.3. *If one of the sequences*

$$A \xrightarrow{u} B \xrightarrow{v} C$$

$$C' \xrightarrow{v^*} B' \xrightarrow{u^*} A'$$

is exact, the other is exact, too.

Proof. If the first sequence is exact, then the second is exact, since $\mathrm{Hom}_Z(\ \ , Q_1)$ is an exact functor. Therefore, we assume that the first sequence is not exact; this means either that there is an element $x \in A$, such that $vux \neq 0$, or that there is a $y \in \mathrm{Ker}\, v$, such that $y \notin \mathrm{Im}\, u$. In the first case, let G_{vux} be the cyclic subgroup of C, generated by vux (C being considered as a Z-module). Since Q_1 contains cyclic subgroups of any finite order, there exists a Z-homomorphism $\varphi' : G_{vux} \to Q_1$, which does not map vux onto zero. Since Q_1 is Z-injective, φ' may be extended to a Z-homomorphism $\varphi : C \to Q_1$. But then we have $(u^* v^* \varphi)(x) = \varphi(vux) \neq 0$; thus, $u^* v^* \varphi \neq 0$, and the second sequence is not exact. In the second case, let H_y be the cyclic subgroup of $B/(\mathrm{Im}\, u)$, which is generated by $y/(\mathrm{Im}\, u)$. Here, too, there exists a Z-homomorphism $\psi' : H_y \to Q_1$, which does not map y onto zero. ψ' may be extended to a Z-homomorphism $\psi : B \to Q_1$. ψ maps $\mathrm{Im}\, u$ onto 0, and is therefore in the kernel of u^*. On the other hand, $\psi(y) \neq 0$; thus, ψ is not in the image of v^*, and hence the second sequence is not exact.

Now, let $\overline{F}^*$ be the semisimplicial functor generated by $\{\overline{C}, \overline{k}, \overline{p}\}$. We shall show that the cochain complex

$$\{\overline{F}^*_{\ddagger}(X), \partial\} = \{\overline{F}^n(X), \partial = \Sigma\, (-1)^i d^i\}_{n \geqq 0}$$

is an injective resolution of X.

Proposition 5.2 implies that $\overline{C} = VCU$ assigns to each module X an injective module $(C(X'))'$; hence, the modules $\overline{F}^n(X)$ are injective for $n \geqq 0$. Theorem 4.3 implies that the construction $\{\overline{C}, \overline{k}, \overline{p}\}$ is GU-trivial, since $\{C, k, p\}$ is G-trivial. Therefore, the semisimplicial complexes $G\big(U(\overline{F}^*(X))\big)$ and $U(\overline{F}^*(X))$ have trivial homotopy groups, and the chain complex $\big(U(\overline{F}^*(X)), \partial\big)$ has trivial homology groups. Proposition 5.3 now implies that $(\overline{F}^*(X), \partial)$ has trivial cohomology groups (here, use is made of the fact that U is an additive functor). As in 5.1, we have

Theorem 5.4. *The Kan homotopy groups of the semisimplicial abelian group complex*

$$\overline{K}_*(X, Y) = \mathrm{Hom}(\overline{F}^*(X), Y)$$

are canonically isomorphic with the Eckmann-Hilton injective homotopy groups

$$\pi_n(\overline{K}_*(X, Y)) = \overline{\Pi}_{n+1}(X, Y) \qquad (n \geqq 0).$$

As in 5.1, $\overline{\Pi}_0(X, Y)$ may be obtained as (-1)-dimensional homology group of the chain complex, but not as Kan homotopy group.

6. The Topological Cone Construction

Let $\mathfrak{K}$ be the category of topological spaces with basepoint, $\mathrm{Hom}(X, Y)$ being the set of basepoint preserving continuous maps $X \to Y$, with the natural rule of composition. We shall define a dual standard construction $\{C, k, p\}$ in $\mathfrak{K}$.

The functor C is the cone construction:

$$CX = [0, 1] \times X/\{0\} \times X \cup [0, 1] \times \{0\},$$

$[0, 1]$ denoting the real interval $0 \leqq t \leqq 1$, with the point 0 as basepoint.

$k(X) : X \to CX$ is the natural imbedding of X into the base of the cone, defined by $k(X)(x) = (1, x)$.

$p(X): CCX \to CX$ is defined by $p(X)(t_0, t_1, x) = (t_0 t_1, x)$.

First, we have to verify some points:

(1) p is compatible with the identifications being made in $[0, 1] \times [0, 1] \times X$ to get the space CCX:

$$p(X)(0, t_1, x) = p(X)(t_0, 0, x) = p(X)(t_0, t_1, 0) = 0 .$$

(2) k and p are functor morphisms; this is obvious.

(3) Axiom (SC 1′) is valid, since

$$p(X) \circ k(CX): (t, x) \to (1, t, x) \to (t, x)$$
$$p(X) \circ C(k(X)): (t, x) \to (t, 1, x) \to (t, x) .$$

(4) Axiom (SC 2′) is valid, since

$$p(X) \circ p(CX): (t_0, t_1, t_2, x) \to (t_0 t_1, t_2, x) \to (t_0 t_1 t_2, x)$$
$$p(X) \circ C(p(X)): (t_0, t_1, t_2, x) \to (t_0, t_1 t_2, x) \to (t_0 t_1 t_2, x) .$$

Thus, the axioms (SC 1′) and (SC 2′) essentially express the fact that the real interval $[0, 1]$ is a multiplicative monoid.

The dual semisimplicial functor F^* assigns to each topological space X a sequence of topological spaces

$$F^0(X) = CX, F^1(X) = CCX, \ldots$$

together with continuous face and degeneracy operators. $K_*(X, Y) = \mathrm{Hom}(F^*(X), Y)$, then, is an ordinary semisimplicial complex.

Theorem 6.1. *The Kan homotopy groups of the semisimplicial complex $K_*(X, Y)$ are canonically isomorphic with the Eckmann-Hilton homotopy groups*

$$\pi_n(K_*(X, Y)) = \Pi_{n+1}(X, Y) \qquad (n \geqq 0) .$$

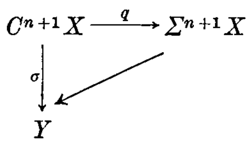
(For $n = 0$, only the right hand side carries a group structure.)

Proof. (1) First, one shows that $K_*(X, Y)$ satisfies the Kan condition. This is an almost immediate consequence of the fact that the union of all faces, except one, of an n-cube is a retract of this cube.

(2) In each set $K_n(X, Y)$, the zero map $0: C^{n+1}X \to Y$ is distinguished. The sets

$$\Gamma_n = \{\sigma \in K_n \mid d^i \sigma = 0 \quad \text{for all } i\} , \qquad (n \geqq 0)$$

consist of those and only those maps $\sigma: C^{n+1}X \to Y$ which may be factored through the $(n+1)$-fold suspension $\Sigma^{n+1}X$, q being the canonical map of $C^{n+1}X$ onto $\Sigma^{n+1}X$:



Thus, we may identify $\Gamma_n = \mathrm{Hom}(\Sigma^{n+1}X, Y)$, $(n \geqq 0)$.

If we take the non-augmented complex $K_*^+(X, Y)$, this description is valid only for $n > 0$. For $n = 0$, we have $\Gamma_0 = K_0(X, Y)$, since then in K_0 no face operator is defined.

(3) The Kan homotopy sets are defined as the sets of equivalence classes of Γ_n modulo the semisimplicial homotopy relation:

$$\pi_n(K_*(X, Y)) = \Gamma_n/\sim .$$

First, we shall investigate $\Gamma_0/\sim$ for the non-augmented case. Let σ, τ be 0-simplexes. σ and τ are semisimplicially homotopic if and only if there exists a 1-simplex ϱ, such that $d^0\varrho = \sigma$, $d^1\varrho = \tau$. Therefore, we have maps

$$\sigma, \tau : [0, 1] \times X \to Y$$

$$\varrho : [0, 1] \times [0, 1] \times X \to Y$$

satisfying

$$\sigma(0, x) = \sigma(t, 0) = \tau(0, x) = \tau(t, 0) = 0$$

$$\varrho(0, t_1, x) = \varrho(t_0, 0, x) = \varrho(t_0, t_1, 0) = 0$$

$$\varrho(1, t, x) = \sigma(t, x)$$

$$\varrho(t, 1, x) = \tau(t, x) .$$

The last two equations imply that σ and τ coincide on the base of the cone: $\sigma(1, x) = \tau(1, x)$.

ϱ determines an ordinary homotopy Φ_t of the maps σ and τ by

$$\Phi_t(s, x) = \Phi(t, s, x) = \varrho\big(s + t(1 - s), s/(s + t(1 - s)), x\big), (t, s) \neq (0, 0)$$

$$\Phi(0, 0, x) = 0 .$$

Then, we have

$$\Phi_0(s, x) = \varrho(s, 1, x) = \tau(s, x)$$

$$\Phi_1(s, x) = \varrho(1, s, x) = \sigma(s, x)$$

$$\Phi_t(0, x) = \Phi_t(s, 0) = 0$$

$$\Phi_t(1, x) = \varrho(1, 1, x) = \sigma(1, x) = \tau(1, x) .$$

We shall see presently that Φ is continuous; thus, Φ_t is a basepoint-preserving homotopy, which, in addition, leaves the base of the cone pointwise fixed. Continuity of Φ at $(0, 0, x)$ may be proved as follows. Let U be an open neighborhood of $0 \in Y$. For each $s \in [0, 1]$, we choose a cubic open neighborhood $V(s) = V_0(s) \times V_1(s) \times V_2(s)$ of the point $(0, s, x)$, such that $\varrho(V(s)) \subset U$. The sets $V_1(s)$ constitute an open cover of the compact interval $[0, 1]$; we may choose a finite subcover $V_1(s_1), \ldots, V_1(s_m)$. We put

$$V_0 = \bigcap_i V_0(s_i), \quad V_2 = \bigcap_i V_2(s_i) .$$

Then, we have $\varrho(V_0 \times [0, 1] \times V_2) \subset U$, and continuity of Φ follows.

Conversely, if Φ_t is a homotopy of two maps $\sigma, \tau : CX \to Y$ satisfying the above relations, then we may construct a semisimplicial homotopy by putting

$$\varrho(u, v, x) = \Phi(u(1 - v)/(1 - uv), uv, x), (u, v) \neq (1, 1),$$

$$\varrho(1, 1, x) = \Phi(t, 1, x) = \sigma(1, x) = \tau(1, x) .$$

The 1-simplex ϱ then gives the desired homotopy. The continuity of ϱ may be shown by a proof similar to that above for the continuity of Φ.

Hence, two 0-simplexes $\sigma, \tau : CX \to Y$ are semisimplicially homotopic if and only if the corresponding maps

(a) agree on the base of the cone, and

(b) are homotopic in the ordinary sense relative to the base of the cone.

In the case of the augmented complex, this may be simplified, since then all simplexes of Γ_0 map the base of the cone onto 0. Therefore, we have then $\pi_0(K_*(X, Y)) = \Pi(\Sigma X, Y)$.

$\Gamma_n/\sim$, $n > 0$, may be treated similarly. Two simplexes $\sigma, \tau \in \Gamma_n$ are homotopic if there is a $(n+1)$-simplex ϱ, such that $d^n \varrho = \sigma$, $d^{n+1}\varrho = \tau$, and $d^i \varrho = 0$, $i < n$.

But now the bases of the various cones will be mapped onto the basepoint of Y in any case; hence, we obtain for both the augmented and the non-augmented complex:

Two n-simplexes $\sigma, \tau \in \Gamma_n$, $n > 0$, are semisimplicially homotopic if and only if the corresponding maps

$$\sigma, \tau : \Sigma^{n+1} X \to Y$$

are homotopic in the ordinary sense. Therefore, the *sets*

$$\pi_n(K_*(X, Y)) \quad \text{and} \quad \Pi_{n+1}(X, Y), \qquad n > 0,$$

may be identified canonically.

(4) It remains to show that this canonical identification induces an isomorphism of the group structures. This may be proved easiest by using the fact that the group structure in $\pi_n(K_*(X, Y))$, $n \geqq 1$, is natural with respect to X and Y. If this group structure is carried over to $\Pi_{n+1}(X, Y)$ with the aid of the canonical identification, we obtain a natural group structure in this latter set; but, according to Hilton [6], the natural group structure of $\Pi_{n+1}(X, Y)$ is uniquely determined for $n > 0$. For $n = 0$, we have no group structure in $\pi_0(K_*(X, Y))$.

Remark. The above semisimplicial definition of homotopy groups of two spaces X, Y has a small flaw: it does not give us the set $\Pi(X, Y)$; and $\Pi_1(X, Y)$ is obtained only without its group structure. This may be remedied as follows.

We replace the dual semisimplicial object $F^*(X)$ by the subobject $\tilde{F}^*(X)$ consisting of the subspaces

$$\tilde{F}^n(X) = \left\{(t_0, \ldots, t_n, x) \in F^n(X) \mid t_0 t_1 \ldots t_n = \frac{1}{2}\right\}$$

with the induced face and degeneracy operators.

$\tilde{F}^n(X)$ may be identified with the space

$$\Delta_n \times X / \Delta_n \times \{0\},$$

where Δ_n is the Euclidean n-simplex, the face and degeneracy operators being the usual ones. By taking logarithms, $s_i = -\log(t_i)/\log(2)$, we obtain the usual parametrization of Δ_n, too: $\Sigma s_i = 1$, $s_i \geqq 0$.

We put

$$\tilde{K}_*(X, Y) = \operatorname{Hom}(\tilde{F}^*(X), Y) = \{\operatorname{Hom}(\tilde{F}^n(X), Y), d^i, s^i\}_{n \geqq 0}.$$

Theorem 6.2. *The* Kan *homotopy groups of* $\tilde{K}_*(X, Y)$ *are canonically isomorphic with the Eckmann-Hilton homotopy groups:*

$$\pi_n(\tilde{K}_*(X, Y)) = \Pi_n(X, Y), \qquad (n \geqq 0),$$

and, if $X = S_0$ *is the* 0*-sphere, then* $\tilde{K}_*(X, Y)$ *may be identified with the singular complex of* Y.

Proof. Let $\tilde{\Gamma}_n$ be the set of n-simplexes of $\tilde{K}_*(X, Y)$ having faces $d^i\sigma = 0$ for all i. $\tilde{\Gamma}_n$ may be identified with the set of continuous maps

$$\sigma : \Delta_n \times X \to Y$$

having the property

$$\sigma(\dot{\Delta}_n \times X \cup \Delta_n \times \{0\}) = 0 .$$

Therefore, after choosing suitable homeomorphisms, we may identify Γ_n with the set of basepoint-preserving maps

$$\Sigma^n X \to Y .$$

If two simplexes $\sigma, \tau \in \tilde{\Gamma}_n$ are semisimplicially homotopic, then there exists a $(n+1)$-simplex ϱ, such that

$$d^n\varrho = \sigma, \; d^{n+1}\varrho = \tau, \; d^i\varrho = 0 \qquad (i < n) .$$

As above, one defines an ordinary homotopy Φ_t between the maps σ and τ by putting

$$\Phi_t(t_0, \ldots, t_{n-1}, s, x) = \varrho\big(t_0, \ldots, t_{n-1}, s + t(1-s), s/(s + t(1-s)), x\big) .$$

$\left(\text{Of course, } \Phi_t \text{ is defined only on the surface } t_0 t_1 \ldots t_{n-1}\, s = \frac{1}{2}\right)$.

Conversely, each ordinary homotopy Φ_t defines a semisimplicial homotopy, as above.

The last part of the theorem is obvious.

7. Adjoint Constructions

Let us assume that the functor C in a dual standard construction $\{C, k, p\}$ admits a right adjoint E. In other words, we have a natural equivalence

$$\gamma : \mathrm{Hom}(CX, Y) \to \mathrm{Hom}(X, EY) .$$

Then, the functor morphisms k and p admit adjoint morphisms k' and p' respectively, satisfying the axioms (SC 1) and (SC 2); thus, $\{E, k', p'\}$ is a standard construction. k' and p' are defined by

$$k'(X) = \gamma^{-1}(1_{EX}) \circ k(EX)$$
$$p'(X) = \gamma\big(\gamma(\gamma^{-1}(1_{EX}) \circ p(EX))\big) .$$

The somewhat lengthy verification of the axioms will be omitted. It follows that E generates a semisimplicial functor F_*, which is adjoint to the functor F^* belonging to C; in fact, γ induces an isomorphism of the semisimplicial complexes

$$\mathrm{Hom}(F^*(X), Y) \to \mathrm{Hom}(X, F_*(Y)) .$$

For instance, the topological cone construction C admits a right adjoint E: the path functor (EY is the space of paths beginning in the basepoint of Y, topologized by the compact-open topology). It follows that we obtain the same homotopy groups by "resolving" either X by cone constructions, or Y by path space constructions.

By the way, the above use of the word "resolve" is not quite correct and should better be avoided. If, for instance, the space Y is a topological abelian group, then the complex F_*^+, generated by path space constructions, is a semisimplicial topological abelian group complex. The associated chain complex (obtained by putting $\partial = \Sigma(-1)^i d^i$), however, is by no means a resolution of the group Y; its homology groups are, essentially, the homotopy groups of the space Y.

The modified semisimplicial complex $\tilde{K}_*(X, Y)$ admits a very interesting interpretation using adjointness. The right adjoint functor $\tilde{F}_*$ of $\tilde{F}^*$ assigns to each space its singular complex, topologized by the compact-open topology. More precisely, $\tilde{F}_*(Y)$ consists of the spaces

$$\tilde{F}_n(Y) = \mathrm{Map}(\Delta_n, Y)\,,$$

where $\mathrm{Map}(\Delta_n, Y)$ denotes the set of (not necessarily basepoint-preserving) continuous maps $\Delta_n \to Y$, topologized by the compact-open topology, and having the constant map $\Delta_n \to 0$ as basepoint.

Then, the natural identification

$$\gamma : \mathrm{Hom}(\Delta_n \times X/\Delta_n \times \{0\}, Y) = \mathrm{Hom}(X, \mathrm{Map}(\Delta_n, Y))$$

induces an isomorphism of the semisimplicial complexes $\mathrm{Hom}(\tilde{F}^*(X), Y)$ and $\mathrm{Hom}(X, \tilde{F}_*(Y))$, which proves the adjointness of $\tilde{F}^*$ and $\tilde{F}_*$.

Thus, $\tilde{K}_*(X, Y)$ may be interpreted as being the set of basepoint-preserving continuous maps of X into the topologized singular complex of Y.

If X is a Hausdorff k-space (Kelley [9], p. 230), then we may still go further. Let $\mathrm{Map}'(X, Y)$ be the subset of $\mathrm{Map}(X, Y)$, consisting of the basepoint-preserving maps. Then, the identification

$$\mathrm{Map}'(X, \mathrm{Map}(\Delta_n, Y)) = \mathrm{Map}(\Delta_n, \mathrm{Map}'(X, Y))$$

(Eckmann and Huber [3]) implies that $\tilde{K}_*(X, Y)$ may be identified with the singular complex of $\mathrm{Map}'(X, Y)$. This applies to a rather large class of spaces, since all Hausdorff spaces, which are either CW-complexes or locally compact, or satisfy the first axiom of countability, are k-spaces.

8. The Category of Pairs. Exact Sequences

Let $\mathfrak{K}$ be any category. Then, the *category of pairs (or category of morphisms)* $\mathfrak{P}(\mathfrak{K})$, or simply $\mathfrak{P}$, is defined as follows. The objects u, v of $\mathfrak{P}$ are the morphisms of $\mathfrak{K}$; the morphisms $\Phi : u \to v$ of $\mathfrak{P}$ are those pairs (φ_1, φ_2) of morphisms of $\mathfrak{K}$ which make the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi_1} & Y_2 \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X_2 & \xrightarrow{\varphi_2} & Y_1 \end{array}$$

commutative. The composition of morphisms in $\mathfrak{P}$ is defined in the obvious way.

Each standard construction $\{C, k, p\}$ in $\mathfrak{K}$ defines a standard construction $\{C', k', p'\}$ in $\mathfrak{P}$, by

$$C'(u) = C(u)$$
$$C'(\varphi_1, \varphi_2) = (C(\varphi_1), C(\varphi_2))$$
$$k'(u) = (k(X_1), k(X_2))$$
$$p'(u) = (p(X_1), p(X_2)) .$$

It is easy to verify the axioms (SC 1) and (SC 2); since no confusion is possible, we shall omit the strokes at C', k' and p' from now on.

These constructions allow the introduction of the semisimplicial complexes $K_*(X, Y)$ and $K_*(u, v)$ in $\mathfrak{K}$ and $\mathfrak{P}$ respectively, and the definition of homotopy groups

$$\pi_n(X, Y) = \pi_n(K_*(X, Y)), \text{ and } \pi_n(u, v) = \pi_n(K_*(u, v)) .$$

It should be noted that the dimensional notation is practically forced on us by the semisimplicial structure and does not quite agree with that introduced by ECKMANN and HILTON.

From now on, we shall suppose that the following two conditions are satisfied:

(A) The category $\mathfrak{K}$ contains a zero object O, and C preserves zero objects. Obviously, $\mathfrak{P}$ then contains a zero object, too.

(B) The complexes $K_*(X, Y)$ and $K_*(u, v)$ satisfy the Kan condition for all X, Y, u, v.

It is not known to the author whether (B) is really necessary; perhaps one might avoid it by using Kan's functor Ex^∞.

Now, we want to relate pair homotopy groups $\pi_n(u, v)$ with absolute groups $\pi_n(X, Y)$. Obviously, we have for any $v: Y_1 \to Y_2$

$$K_*(0_{OX}, v) = K_*(X, Y_2)$$
$$K_*(0_{XO}, v) = K_*(X, Y_1) .$$

Therefore, we may identify the corresponding homotopy groups.

Let $u: X_1 \to X_2$ be an object of $\mathfrak{P}$. The morphisms $\alpha = (0_{OX_1}, 1_{X_2})$ and $\beta = (0_{OO}, u)$ induce semisimplicial maps

$$K_*(u, v) \xrightarrow{\alpha^*} K_*(0_{OX_2}, v) \xrightarrow{\beta^*} K_*(0_{OX_1}, v) .$$

By using the above identifications and choosing $v = k(Y)$, we obtain the following sequence of homotopy groups and natural group homomorphisms

$$\pi_n(u, k(Y)) \xrightarrow{\alpha^*} \pi_n(X_2, Y) \xrightarrow{\beta^*} \pi_n(X_1, Y) .$$

Theorem 8.1. *There exists a natural boundary homomorphism ∂, which turns this sequence into an exact sequence*

$$\cdots \xrightarrow{\partial} \pi_n(u, k(Y)) \xrightarrow{\alpha^*} \pi_n(X_2, Y) \xrightarrow{\beta^*} \pi_n(X_1, Y) \xrightarrow{\partial} \pi_{n-1}(u, k(Y)) \xrightarrow{\alpha^*} \cdots$$

Proof. ∂ may be defined as follows. Let the morphism

$$\varphi : X_1 \to C^{n+1} Y$$

be a representative of the class $[\varphi] \in \pi_n(X_1, Y)$; then, $d^i \varphi = 0$ for all i. $\partial [\varphi]$ now is defined as being the class of $(\varphi, 0)$:

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi} & C^n(C\,Y) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle C^n(k(Y)) = d^n} \\ X_2 & \xrightarrow{0} & C^n\,Y \end{array}$$

We have to show

(1) ∂ depends only on the class $[\varphi]$.

Let $\varphi \sim \varphi'$, then there exists a $(n+1)$-simplex η, such that $d^0\eta = \varphi$, $d^1\eta = \varphi'$, $d^i\eta = 0$ $(i > 1)$; $(\eta, 0)$ then defines a homotopy between $(\varphi, 0)$ and $(\varphi', 0)$. (Here we use the fact that we obtain the same homotopy relation by taking the first or the last face operators.)

(2) ∂ is natural; this is obvious.

(3) ∂ is a group homomorphism. Let $n \geqq 2$, and let $[\chi] = [\varphi] + [\psi]$, the sum being defined by a $(n+1)$-simplex η; $d^0\eta = \psi$, $d^1\eta = \chi$, $d^2\eta = \varphi$, $d^i\eta = 0$ $(i > 2)$. The simplex $(\eta, 0)$ then yields $[\chi, 0] = [\varphi, 0] + [\psi, 0]$.

(4) $\beta^* \alpha^* = 0$.

The semisimplicial map $\beta^* \alpha^* : K_*(u, k(Y)) \to K_*(X_1, Y)$ may be factored through $K_*(1_{X_1}, k(Y))$. This complex has trivial homotopy: let $(\varphi, \psi) \in K_n(1_{X_1}, k(Y))$ be a representative of some homotopy class. Then, the $(n+1)$-simplex $(s^{n+1}\varphi, \varphi)$ defines a homotopy $(\varphi, \psi) \sim (0, 0)$.

(5) $\partial \beta^* = 0$.

Let φ be a representative of a homotopy class of $\pi_n(X_2, Y)$. Then, $\partial \beta^* [\varphi] = [\varphi u, 0]$, and the desired homotopy $(\varphi u, 0) = (0, 0)$ is furnished by $(s^n \varphi u, \varphi)$.

(6) $\alpha^* \partial = 0$; this is obvious.

(7) Let $\beta^* [\varphi] = 0$. Since $\beta^* [\varphi] = [\varphi u]$, there exists a η, with $d^{n+1}\eta = \varphi u$, $d^i\eta = 0$ $(i < n+1)$. Then, $\alpha^* [\eta, \varphi] = [\varphi]$.

(8) Let $\partial [\varphi] = 0$. Then we have a n-simplex (η, ϑ), such that $d^{n+1}\eta = \vartheta u$, $d^n(\eta, \vartheta) = (\varphi, 0)$, $d^i(\eta, \vartheta) = (0, 0)$ for $i < n$. It follows that $\vartheta u \sim \varphi$; hence, $\beta [\vartheta] = [\varphi]$.

(9) Let $\alpha^* [\varphi, \psi] = 0$. Then we have a η, such that $d^{n+1}\eta = \psi$, $d^i\eta = 0$ $(i < n+1)$. Now, consider the $n+2$ $(n+1)$-simplexes $\sigma_i = 0$ $(0 \leqq i < n)$, $\sigma_{n+1} = \varphi$, $\sigma_{n+2} = \eta u$. By the Kan condition, we may find a $(n+2)$-simplex σ, such that $d^i\sigma = \sigma_i$ $(i \neq n)$. Then, the simplex (σ, η) defines a homotopy $(\varphi, \psi) \sim (d^n\sigma, 0)$; thus, $\partial [d^n\sigma] = [\varphi, \psi]$.

This rather simple proof of the exactness of the homotopy sequence has several great advantages, as compared with the proofs given previously by Eckmann and Hilton:

(1) It is dualizable.

(2) It is valid in the category of modules as well as in the category of topological spaces.

(3) It is valid in the categories of pairs, of pairs of pairs, etc., of modules and spaces respectively, since it is easy to verify that the respective semisimplicial complexes satisfy the Kan condition.

9. Fibrations and Cofibrations in General Categories

It is even possible to introduce the notion of fibration and cofibration in general categories with the aid of a standard construction.

First, we define the *kernel* and the *cokernel* of a morphism. Let $\mathfrak{K}$ be a category containing a zero object, and let $u : X \to Y$ be a morphism. A pair (U, j) consisting of an object U and of a morphism $j : U \to X$ is called a *kernel of u*, if

(1) $u \circ j = 0$, and

(2) for each $Z \in \mathfrak{K}$ and each $v : Z \to X$ with $u \circ v = 0$ there exists one and only one morphism $w : Z \to U$, so that $j \circ w = v$

$$\begin{array}{ccccc} U & \xrightarrow{j} & X & \xrightarrow{u} & Y \\ {\scriptstyle w}\uparrow & \nearrow{\scriptstyle v} & & & \\ Z & & & & \end{array}$$

Obviously, (U, j) is uniquely determined up to equivalence, and j is a monomorphism (i.e., $j \circ w = j \circ w'$ implies $w = w'$). By abuse of language, we shall denote the object U as kernel and omit j.

The definition of the *cokernel of u* is dual.

Let $\mathfrak{K}$ be a category containing a zero object, and let $\{C, k, p\}$ be a standard construction in $\mathfrak{K}$, generating a semisimplicial functor F_* and semisimplicial complexes $K_*(X, Y) = \mathrm{Hom}(X, F_*(Y))$. As usual, we assume that zero objects are preserved under C.

Definition 9.1. A morphism $u : X_1 \to X_2$ is called a *cofibration* if the induced semisimplicial map

$$u^* : K_*(X_2, Y) \to K_*(X_1, Y)$$

is a semisimplicial fibration for all $Y \in \mathfrak{K}$. The cokernel X_3 of u then will be called *cofibre of u* (provided that this cokernel exists).

One may show that, in the category of topological spaces, each cofibration in the ordinary sense (homotopy extension property for arbitrary range spaces) is a cofibration in the above sense. It is not known to the author whether the converse is true. A similar statement is valid for the fibrations, to be defined later on.

The semisimplicial fibre of u^* consists of the set of all morphisms $\sigma : X_2 \to C^{n+1} Y$, $(n \geqq -1)$, satisfying $\sigma \circ u = 0$. Thus, it may be identified with $K_*(X_3, Y)$.

Then, one may infer from the general theory of semisimplicial complexes that we have exact homotopy sequences

$$\cdots \to \pi_n(X_3, Y) \to \pi_n(X_2, Y) \to \pi_n(X_1, Y) \to \pi_{n-1}(X_3, Y) \to \cdots$$

This result is valid even if $K_*(X, Y)$ does not satisfy the Kan condition, but from now on we shall again consider only Kan complexes.

A comparison with theorem 8.1 suggests the following excision theorem:

Theorem 9.2. *Let $u : X_1 \to X_2$ be a cofibration with cofibre (X_3, v). Then, $(0_{X_1 O},\ v)$ induces an isomorphism*

$$J : \pi_n(X_3,\ Y) \to \pi_n(u,\ k(Y))\,.$$

Proof. First, we show that J is epimorphic. Let (φ, ψ)

$$\begin{array}{ccc} X_1 & \xrightarrow{\ \varphi\ } & C^{n+1}(C\,Y) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle d^{n+1}} \\ X_2 & \xrightarrow{\ \psi\ } & C^{n+1}\,Y \end{array}$$

be a representative of a homotopy class of $\pi_n(u, k(Y))$. Since u^* is a semisimplicial fibre map, we may find a $(n+1)$-simplex $\tau : X_2 \to C^{n+2}\,Y$, such that $\tau \circ u = \varphi$, $d^{n+1}\tau = \psi$, $d^i\tau = 0$ $(i < n)$.

Then, $(s^{n+1}\,\varphi, \tau)$ defines a homotopy $(\varphi, \psi) \sim (0, d^n\tau)$. We have $d^n\tau \circ u = 0$; thus, there exists a $\omega : X_3 \to C^{n+1}\,Y$, satisfying $d^n\tau = \omega \circ v$. Hence, $J[\omega] = [\varphi, \psi]$; thus J is epimorphic.

Now, we want to show that J is monomorphic, which is equivalent to showing that $(0, \omega v) \sim (0, \omega' v)$ implies $\omega \sim \omega'$. Let (η, ϑ) be a homotopy between $(0, \omega v)$ and $(0, \omega' v)$; i.e. $d^{n+2}\eta = \vartheta \circ u$, $d^{n+1}(\eta, \vartheta) = (0, \omega' v)$, $d^n(\eta, \vartheta) = (0, \omega v)$, $d^i(\eta, \vartheta) = (0, 0)$, $i < n$. Since u^* is a semisimplicial fibre map, there is a $\sigma : X_2 \to C^{n+3}\,Y$, such that $\sigma \circ u = \eta$, $d^{n+2}\sigma = \vartheta$, $d^n\sigma = s^n\,\omega v$, $d^i\sigma = 0$, $i < n$. We have $(d^{n+1}\sigma) \circ u = d^{n+1}\eta = 0$; thus, there exists a $\chi : X_3 \to \to C^{n+2}\,Y$, satisfying $\chi v = d^{n+1}\sigma$. Since v is an epimorphism, the relations $d^{n+1}\chi v = \omega' v$, $d^n\chi v = \omega v$, $d^i\chi v = 0$, $i < n$, imply $d^{n+1}\chi = \omega'$, $d^n\chi = \omega$, $d^i\chi = 0$, $i < n$. Hence, $\omega \sim \omega'$.

The dualizations of definition 9.1 and of theorem 9.2 show some interesting features. The ordinary dualization procedure in categories leads to

Definition 9.3. Let $\mathfrak{K}$ be a category containing a zero object, and let $K_*(X, Y)$ be the semisimplicial complex induced by a *dual* standard construction in $\mathfrak{K}$. Then, a morphism $u : Y_2 \to Y_1$ is called a *fibration* if the induced semisimplicial map

$$u^* : K_*(X,\ Y_2) \to K_*(X,\ Y_1)$$

is a semisimplicial fibration for all $X \in \mathfrak{K}$. The kernel Y_3 of u then will be called *fibre* of u (provided that this kernel exists).

It follows, as above, that the semisimplicial fibre of u^* may be identified with $K_*(X, Y_3)$, and that we have an exact sequence

$$\cdots \to \pi_n(X,\ Y_3) \to \pi_n(X,\ Y_2) \to \pi_n(X,\ Y_1) \to \pi_{n-1}(X,\ Y_3) \to \cdots.$$

Theorem 9.4. *Let $u : Y_2 \to Y_1$ be a fibration with fibre (Y_3, v). Then, $(v, 0_{O Y_1})$ induces an isomorphism*

$$J : \pi_n(X,\ Y_3) \to \pi_n(k(X),\ u)\,.$$

However, it is possible to introduce fibrations without dualizing completely, that is, without replacing the standard construction by a dual one:

Definition 9.5. Let $\mathfrak{K}$ be a category containing a zero object, and let $K_*(X, Y)$ be the semisimplicial complex induced by an (ordinary) standard construction in $\mathfrak{K}$. Then, a morphism $u: Y_2 \to Y_1$ is called a *fibration* if the induced semisimplicial map

$$u_*: K_*(X, Y_2) \to K_*(X, Y_1)$$

is a semisimplicial fibration for all $X \in \mathfrak{K}$. *If C commutes with kernels* (i.e., if $C(\operatorname{Ker} u) = \operatorname{Ker} C(u)$), then the kernel Y_3 of u will be called *fibre* of u (provided that this kernel exists).

The assumption that C commutes with kernels is needed to prove that the semisimplicial fibre of u_* may be identified with $K_*(X, Y_3)$. The exact fibre sequence then follows as above. Since the concept of fibration is defined only relative to a specific standard construction, the definitions 9.3 and 9.5 cannot conflict; moreover, if a pair of adjoint constructions is used, then the two definitions are equivalent. Of course, definition 9.5 may be dualized too ...

Theorem 9.6. *Let $\mathfrak{K}$ be a category containing a zero object, and let $\{C, k, p\}$ be a standard construction in $\mathfrak{K}$, such that the complexes $K_*(X, Y)$ are Kan complexes. Then, the morphism $k(Y): CY \to Y$ is a fibration.*

Proof. Let $f: K_*(X, CY) \to K_*(X, Y)$ be the semisimplicial map induced by $k(Y)$. We have to show that for every n $(n-1)$-simplexes $\sigma_0, \ldots, \sigma_{k-1}, \sigma_{k+1}, \ldots, \sigma_n \in K_*(X, CY)$ satisfying $d^i\sigma_j = d^{j-1}\sigma_i$ for $i < j$, and $i, j \neq k$, and every n-simplex $\tau \in K_*(X, Y)$ satisfying $d^i\tau = f\sigma_i$, $i \neq k$, there exists a n-simplex $\sigma \in K_*(X, CY)$, such that $d^i\sigma = \sigma_i$ for $i \neq k$, and $f\sigma = \tau$. If we consider the σ_i as n-simplexes of $K_*(X, Y)$, then, by the Kan condition, we may find a $(n+1)$-simplex $\sigma \in K_*(X, Y)$, such that $d^i\sigma = \sigma_i$ for $i \neq k$, $i \leqq n$ and $d^{n+1}\sigma = \tau$. This σ, interpreted as n-simplex of $K_*(X, CY)$, has the desired properties.

Theorem 9.7. *Under the assumptions of theorem* 9.6, *the object CY has trivial homotopy:*

$$\pi_n(X, CY) = 0\,, \qquad n \geqq 0\,.$$

Proof. Let σ be a n-simplex of $K_*(X, CY)$, such that $d^i\sigma = 0$, $(0 \leqq i \leqq n)$. We interpret σ as $(n+1)$-simplex of $K_*(X, Y)$, form $\varrho = s^{n+1}\sigma$, and reinterpret ϱ as $(n+1)$-simplex of $K_*(X, CY)$. We have $d^{n+1}\varrho = \sigma$, $d^i\varrho = 0$ $(i \leqq n)$, and therefore $\sigma \sim 0$. (Here we consider, as usual, the augmented complex; for the non-augmented one, the argument would have failed in dimension 0, since there no face operator is defined.)

Theorem 9.8. *Under the assumptions of theorem* 9.6, *together with the additional assumptions that $k(Y)$ has a kernel ΩY and that C commutes with kernels, we have*

$$\pi_n(X, \Omega Y) = \pi_{n+1}(X, Y)\,, \qquad (n > 0)\,.$$

Proof. This follows immediately from the exact fibre sequence, and from theorems 9.6 and 9.7.

Thus, the fibre ΩY of $k(Y): CY \to Y$ has properties approximately corresponding to those of a loop space. It will be left to the reader to dualize theorems 9.6, 9.7 and 9.8; the theorem dual to 9.8 states that the cofibre ΣX of the morphism $k(X): X \to CX$ of a dual standard construction has the formal properties of a suspension.

References

[1] ECKMANN, B.: Homotopie et dualité, Colloque de Topologie Algébrique, p. 41. Louvain 1956.

[2] ECKMANN, B., et P. J. HILTON: Groupes d'homotopie et dualité, Compt. rend. **246**, 2444, 2555, 2991 (1958); **247**, 620 (1958); **248**, 2054 (1959).

[3] ECKMANN, B., and P. J. HUBER: Spectral sequences for homology and homotopy groups of maps. Mimeographed Note (Battelle Memorial Institute, Geneva 1960).

[4] GODEMENT, R.: Théorie des faisceaux. Paris: Hermann 1958.

[5] GROTHENDIECK, A.: Sur quelques points d'algèbre homologique. Tôhoku Math. J. **9**, 119 (1957).

[6] HILTON, P. J.: Homotopy theory and duality. Cornell University 1959 (Mimeographed lecture notes).

[7] KAN, D. M.: A combinatorial definition of homotopy groups. Ann. Math. **67**, 282 (1958).

[8] KAN, D. M.: Adjoint functors. Trans. Am. Math. Soc. **87**, 294 (1958).

[9] KELLEY, J. L.: General Topology. New York: Van Nostrand 1955.

[10] MOORE, J. C.: Seminar on Algebraic Homotopy Theory, Lecture notes. Princeton University, 1955—56.

(Received March 28, 1961)

Druck: Brühlsche Universitätsdruckerei Gießen

Zusammenfassung in deutscher Sprache

Die Homotopiegruppen der topologischen Räume sind von ECKMANN-HILTON zu Gruppen $\Pi_n(X, Y)$ verallgemeinert worden, die von zwei Räumen X, Y abhängen, und welche die Homotopie- und Cohomologiegruppen (für beliebige Koeffizientenbereiche), nebst den entsprechenden exakten Sequenzen, als Spezialfälle enthalten. Die klassische Homotopie und Cohomologie sind zueinander dual im Sinne einer einfachen (heuristischen) Dualität, welche im Vertauschen von X und Y besteht.

Eine analoge Homotopietheorie, samt einer dualen, existiert in der Kategorie der Moduln über einem Ring — oder allgemeiner, in jeder abelschen Kategorie mit genügend vielen injektiven und projektiven Objekten (vgl. ECKMANN [1]); hier ist die Dualität nicht nur heuristisch, wie in der Kategorie der Räume, sondern gilt streng: für jeden beweisbaren Satz ist automatisch auch der duale Satz beweisbar. In der fraglichen Analogie entspricht z. B. die Einbettung eines topologischen Raumes X in den Kegel CX der Einbettung eines Moduls X in einen injektiven Modul $\overline{X}$; die topologische Einhängung $\Sigma X = CX/X$ entspricht der algebraischen $\Sigma X = \overline{X}/X$; und in beiden Fällen können die Homotopiegruppen mit Hilfe iterierter Einhängungen definiert werden.

Diese beiden heuristischen Prinzipien

a) die Dualität in der Kategorie der Räume,

b) die Analogie zwischen Räumen und Moduln,

haben entscheidenden Einfluß auf die Entwicklung der Eckmann-Hiltonschen Homotopietheorie gehabt. In dieser Arbeit wird nun gezeigt, daß die beiden Prinzipien theoretisch begründet werden können.

Zu diesem Zweck wird eine Homotopietheorie im Rahmen allgemeiner Kategorien entwickelt, welche die Homotopietheorien der Räume und der Moduln als Spezialfälle enthält, ebenso wie die Homotopietheorie der Abbildungen von Räumen, usw. Außerdem liefert ein allgemeines Dualitätsprinzip in allen Fällen, wo diese Homotopietheorie definiert werden kann, eine Dualität im strengen Sinn. Dadurch wird nicht nur ein präziser Begriff der Analogie zwischen den Homotopietheorien der Räume und der Moduln gewonnen, sondern es ist auch möglich, gewisse Beweise wesentlich zu vereinfachen. Zum Beispiel folgt die Exaktheit der Homotopiesequenzen in den Kategorien der Moduln, der Räume, der Paare von Moduln, der Paare von Räumen, der Paare von Paaren, usw., aus ein- und demselben Beweis, und es ist hervorzuheben, daß dieser eine Beweis, vermöge der Dualität, beide Seiten des Bildes liefert, also z. B. in der Kategorie der Räume die Exaktheit sowohl der Homotopie- als auch der Cohomologiesequenz.

Unser wichtigstes Werkzeug ist dabei die semisimpliziale Standardkonstruktion, welche ursprünglich von R. Godement [4] erfunden und zur Erzeugung der Garbencohomologie verwendet worden ist. Da auch die Hochschildsche Homologietheorie der assoziativen Algebren (vgl. [4]) und die Theorie der derivierten Funktoren in den Kategorien der Moduln mit Hilfe von Standardkonstruktionen erhalten werden können, erweisen sich diese als eines der mächtigsten Hilfsmittel der homologischen und homotopischen Algebra.

Die Standardkonstruktionen können als eine Verallgemeinerung der topologischen Wegeraum- und Kegelkonstruktionen aufgefaßt werden. Zum Beispiel ist das Tripel $\{E, k, p\}$, bestehend aus dem Wegeraumfunktor E (der jedem Raum Y den Raum EY der im Basispunkt von Y beginnenden Wege zuordnet), aus der natürlichen Faserabbildung $k(Y){:}EY \to Y$ (die jedem Weg seinen Endpunkt zuordnet) und aus einer sonst kaum beachteten natürlichen Abbildung $p(Y){:}EY \to EEY$, eine Standardkonstruktion in der Kategorie der topologischen Räume mit Basispunkt. Dual bilden der Kegelfunktor C (der jedem Raum X den Kegel CX über X zuordnet), die natürliche Einbettung $k(X){:}X \to CX$ von X in die Grundfläche des Kegels und eine gewisse Abbildung $p(X){:}CCX \to CX$ eine duale Standardkonstruktion in derselben Kategorie.

Abschnitt 1 dient zur Einführung der Terminologie, Abschnitt 2 enthält die Definition der Standardkonstruktion, und in Abschnitt 3 wird der zu einer Standardkonstruktion gehörende semisimpliziale Komplex eingeführt, dessen Kansche Homotopiegruppen die Eckmann-Hiltonschen Gruppen verallgemeinern werden. In Abschnitt 5 werden die Homotopiegruppen in der Kategorie der Moduln behandelt. Die sogenannten projektiven, resp. injektiven Homotopiegruppen werden mit Hilfe von zwei verschiedenen Standardkonstruktionen erzeugt.

Die erste, $\{C, k, p\}$, besteht aus dem Funktor C, der jedem Modul Y den freien Modul CY über der Menge Y zuordnet; $k(Y)$ ist die natürliche Projektion von CY auf Y, die jedem Basiselement von CY das gleichbezeichnete Element von Y zuordnet, während $p(Y)$ jedem Basiselement von CY das gleichbezeichnete Basiselement von CCY zuordnet. Die zweite, $\{\bar{C}, \bar{k}, \bar{p}\}$, hängt eng mit der ersten zusammen; der Funktor $\bar{C}$ ordnet jedem Modul X einen injektiven Modul $\bar{C}X$ zu, während $\bar{k}(X){:}X \to \bar{C}X$ eine natürliche Einbettung ist. Es wird gezeigt, daß die Kanschen Homotopiegruppen der entsprechenden semisimplizialen Komplexe gerade die Eckmann-Hiltonschen projektiven, resp. injektiven Homotopiegruppen sind. Zur Vereinfachung der entsprechenden Beweise wurden Teile davon in den allgemein-kategorietheoretischen Rahmen vorverlegt (Abschnitte 3 und 4). Abschnitt 6 leistet das entsprechende für die Homotopiegruppen der Räume; als Standardkonstruktionen werden die Kegel- und Wegeraumkonstruktionen benützt. Abschnitt 7 enthält unter anderem eine interessante Verallgemeinerung des singulären Komplexes: ebenso, wie man die Hurewiczschen Homotopiegruppen $\pi_n(Y)$ eines Raumes Y auch als Kansche Homotopiegruppen des singulären Komplexes von Y deuten kann, können die Eckmann-Hiltonschen Gruppen

$\Pi_n(X, Y)$ auch als Kansche Homotopiegruppen eines verallgemeinerten singulären Komplexes von Y aufgefaßt werden, der in den gewöhnlichen singulären Komplex übergeht, falls X die Nullsphäre ist. In Abschnitt 8 wird die Exaktheit der Homotopiesequenz für allgemeine Kategorien bewiesen. Abschnitt 9 enthält eine Untersuchung der Faserungen und Cofaserungen; es wird gezeigt, daß man bereits im Rahmen der allgemeinen Kategorien die Analoga der Einhängung ΣX (d. h. der Cofaser von $k(X): X \to CX$) und des Schleifenraumes ΩY (d. h. der Faser von $k(Y): EY \to Y$) definieren und ihre Haupteigenschaften beweisen kann.

Lebenslauf

Ich wurde am 25. März 1934 in Wohlen (Aargau) geboren. Ich habe in Wohlen die Primar- und Bezirksschule und anschließend in Aarau die Aargauische Kantonsschule besucht, wo ich im Frühjahr 1954 die Maturitätsprüfung (Typus B) bestand. Nach einem Semester an der Universität Zürich habe ich in den Jahren 1954—1958 an der Abteilung für Mathematik und Physik der Eidgenössischen Technischen Hochschule studiert und dort im Herbst 1958 das Diplom als Mathematiker erworben. Von 1958 bis 1960 war ich Assistent für Geometrie bei Herrn Prof. Dr. B. Eckmann; im Jahr 1959/60 war ich außerdem Mitarbeiter von Herrn Prof. Dr. B. L. van der Waerden. Zur Zeit bin ich am Battelle Memorial Institute in Genf angestellt.

Herrn Prof. Dr. B. Eckmann, unter dessen Leitung ich diese Arbeit ausführte, möchte ich an dieser Stelle für sein ständiges Interesse und für viele wertvolle Ratschläge herzlich danken.