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On the numerical solution of special second-order initial value problems

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ON THE NUMERICAL SOLUTION 
OF SPECIAL SECOND-ORDER 
INITIAL VALUE PROBLEMS.

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1964
ON THE NUMERICAL SOLUTION OF SPECIAL
SECOND-ORDER INITIAL VALUE PROBLEMS

William Perry Timlake

INTRODUCTION

Consider the special second-order initial value problem

\[ X^{(1)}(t) = f(1)(t, X(1)(t), \ldots, X(m)(t)) \]  \hspace{1cm} \left( \frac{d}{dt^2} \right) \]

\[ X(1)(a) = a_1, \quad X'(1)(a) = b(1), \quad i = 1, \ldots, m. \]

Since problems of this form frequently occur in mechanical problems (e.g., in orbit calculations), we limit ourselves to this particular form.

Although it is possible to write (0.1) as an equivalent first-order differential problem with the immediate applicability of the results given in [4],* it seems unnatural to introduce the computation of \( X(t) \) unless it is desired for some special use. For these reasons many methods have been devised, particularly by astronomers [6] and [8] to compute an approximation to the solution (if it exists) of (0.1).

*Numbers in brackets refer to the bibliography. A more complete list of literature pertinent to this general area is given in [3].
We shall study those finite difference schemes which, recursively from the equation
\[ A_k \mathbf{x}_{n+k} + \ldots + A_0 \mathbf{x}_n - h^2 \{ B_k f(t_{n+k}, \mathbf{x}_{n+k}) + \ldots + B_0 f(t_n, \mathbf{x}_n) \} = 0, \]
where \( A_i \) and \( B_i \) are diagonal matrices, generate a sequence of vectors \( \{ \mathbf{x}_n \} \) (\( n \geq k \)) whose \( i \)th component approximates \( X_i(t_n) \), the solution of (0.1). A more precise formulation of this class of methods will be given shortly. That diagonal matrices are permitted is a slight generalization over [4] in which only scalar multiples of the identity matrix are used. The basic intention is that if it is known a priori that certain of the components of (0.1) have a large derivative of a particular order but smaller values for subsequent orders, then a more refined technique may be used on this component. But at the same time a cruder (and hence less computation required) scheme may be used for the better behaved components.

Before giving a formal presentation, we offer a brief resume of the results. (For the moment it is assumed the reader will understand the terminology intuitively from its literal connotation.) In Theorem IF a necessary and sufficient condition on \( A_i, B_i \) (\( i = 1, \ldots, k \)) is given which guarantees that the sequence \( \{ \mathbf{x}_n \} \) converges to the solution of (0.1). This condition is the natural generalization of the analogous result of Dahlquist [9]. The method of proof
follows quite closely that of [4]. In Theorem IID we obtain an asymptotic formula which depicts the "rate of convergence" in terms of the truncation error and the starting error. In particular

\[ X(i)n = X(i)(t_n) + h^p e(i)(t_n) + h^q \sum_{j=1}^{d} d' j z^j X_{ij}(t_n) + O(h^r) \]

where \( X(i)n \) is the \( i \)th component of the vector generated by the recursion formula; \( X(i)(t_n) \) is the \( i \)th solution of (0.1); \( e_i(t_n), X_{ij}(t_n) \) are smooth functions; \( d' j \) is either zero or one; \( z^j \) is a complex number with absolute value 1; \( O(h^{p+2}) \) is the order of the truncation error; and \( O(h^{q+1}) \) is the order of the starting error. The method of proof follows in broad outlines that of a similar theorem in [4]. However, since we permit diagonal matrices, \( A_i \), which of course do not commute with arbitrary matrices, certain nontrivial complications arise.

In terms of this asymptotic expansion, statistical estimates of the round-off error are found. As might be expected, the mean of the accumulated round-off error is \( O(\mu/h^2) \) where \( \mu \) is a bound on the mean of the local round-off error. This dangerous growth property has been long recognized at least intuitively, and a summing technique was devised to combat this difficulty.