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Independence of the Zeros

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The Large Sieve, Monodromy, and Zeta Functions of Algebraic Curves, 2: Independence of the Zeros

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Using the sieve for Frobenius developed earlier by the author, we show that in a certain sense, the roots of the L-functions of most algebraic curves over finite fields do not satisfy any nontrivial (linear or multiplicative) dependency relations. This can be seen as an analogue of conjectures of $\mathbf{Q}$-linear independence among ordinates of zeros of L-functions over number fields. As a corollary of independent interest, we find for "most" pairs of distinct algebraic curves over a finite field the form of the distribution of the (suitably normalized) difference between the number of rational points over extensions of the ground field. The method of proof also emphasizes the relevance of random matrix models for this type of arithmetic questions. We also describe an alternate approach, suggested by Katz, which relies on Serre's theory of Frobenius tori.

1 Introduction

In a number of studies of the fine distribution of primes, there arises the issue of the existence of linear dependence relations, with rational coefficients, among zeros (or rather ordinates of zeros) of the Riemann zeta function, or more generally, of Dirichlet L-functions. This was important in disbelieving (then disproving, as done by Odlyzko

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and te Riele) Mertens's conjecture

$$\left|\sum_{n\leqslant x}\mu(n)\right| < \sqrt{x}, \qquad \text{for } x \geqslant 2 \tag{1.1}$$

as Ingham [7] showed how it implied that the zeros of $\zeta(s)$ are $\mathbf{Q}$-linearly dependent (in fact, that zeros "arbitrarily high" on the critical line are linearly dependent; for the most recent work in studying the left-hand side of (1.1) using the assumption of linear independence, see the work of Ng [21]).

More recently, this turned out to be crucial in understanding the "Chebychev bias" in the distribution of primes in arithmetic progressions (the apparent preponderance of primes $\equiv 3 \pmod 4$ as compared to those $\equiv 1 \pmod 4$, and generalizations of this), as discussed in depth by Rubinstein and Sarnak [24]. They introduce the "Grand Simplicity Conjecture" as the statement that the set of all ordinates $\gamma \geqslant 0$ of the nontrivial zeros ρ of Dirichlet L-functions $L(s,\chi)$ are $\mathbf{Q}$-linearly independent when χ runs over primitive Dirichlet characters and the zeros are counted with multiplicity (indeed, "simplicity" relates to the particular corollary of this conjecture that all zeros of Dirichlet L-functions are simple).

Building on the fact that our current knowledge of the behavior of zeros of zeta functions of (smooth, projective, geometrically connected) algebraic curves over finite fields is somewhat more extensive, we consider analogs of this type of independence questions in the context of finite fields. Let $C/\mathbf{F}_q$ be such an algebraic curve over a finite field with q elements and characteristic p, and let $g \geqslant 0$ be its genus. Its zeta function $Z(C,s)$ is defined (first for $s \in \mathbf{C}$ with $\mathrm{Re}(s)$ large enough) by either of the equivalent expressions

$$Z(C,s) = \exp\left(\sum_{n\geqslant 1}\frac{|C(\mathbf{F}_{q^n})|}{n}q^{-ns}\right) = \prod_{\substack{x \text{ closed}\\ \text{point in } C}}(1-N(x)^{-s})^{-1},$$

and it is well known (as proved by Schmidt) that it can be expressed as

$$Z(C,s) = \frac{L(C,s)}{(1-q^{-s})(1-q^{1-s})},$$

where $L(C,s) = P_C(q^{-s})$ for some polynomial $P_C(T) \in \mathbf{Z}[T]$ of degree $2g$. This polynomial (which is also called the L-function of $C/\mathbf{F}_q$) may be factored as

$$P_C(T) = \prod_{1 \leqslant j \leqslant 2g} (1 - \alpha_j T),$$

where the "roots" (or inverse roots, really) α_j, $1 \leqslant j \leqslant 2g$, satisfy $|\alpha_j| = \sqrt{q}$, as proved by Weil. This is well known to be the analog of the Riemann Hypothesis, as we recall: writing

$$\alpha_j = q^{w_j} e(\theta_j), \quad \text{where } w_j,\ \theta_j \in \mathbf{R}, \quad e(z) = e^{2i\pi z},$$

implies that the zeros ρ of $L(C,s)$ are given by

$$\rho = w_j + \frac{2\pi i \theta_j}{\log q} + \frac{2ik\pi}{\log q},$$

for $k \in \mathbf{Z}$, $1 \leqslant j \leqslant 2g$. So, Weil's result $|\alpha_j| = \sqrt{q}$ corresponds to $w_j = 1/2$, hence to $\mathrm{Re}(\rho) = 1/2$ for any zero ρ of $L(C,s)$.

It is clearly of interest to investigate the possible linear relations among those zeros as an analogue of the conjectures of linear independence for ordinates of zeros of Dirichlet L-functions. Note however that if we allow all imaginary parts, many "trivial" relations come from the fact that, e.g. the $\theta_j + k$, $k \in \mathbf{Z}$, are $\mathbf{Q}$-linearly dependent. One must therefore consider θ_j up to integers, and the simplest way to do this is to consider multiplicative relations

$$\prod_{1 \leqslant j \leqslant 2g} e(n_j \theta_j) = 1,$$

with $n_j \in \mathbf{Q}$ or, raising to a large power to eliminate the denominator, relations

$$\prod_{1 \leqslant j \leqslant 2g} \left(\frac{\alpha_j}{\sqrt{q}}\right)^{n_j} = 1,$$

with $n_j \in \mathbf{Z}$. This, in fact, also detects $\mathbf{Q}$-linear dependencies among the components of the vector

$$(1, \theta_1, \dots, \theta_{2g})$$

of size $2g+1$ (which is important for later applications).

We will indeed study this problem, but, at the same time, we will consider another independence question that seems fairly natural, even if no particular analog over number fields suggests itself: are the α_j, or the $1/\alpha_j$, linearly independent over $\mathbf{Q}$? (In fact, what we will prove about this will be helpful in one step of the study of the multiplicative case.)

In the multiplicative case, it is immediately clear that we have to take into account the functional equation

$$L(C, s) = q^{g(1-2s)} L(C, 1-s),$$

which may be interpreted as stating that for any j, q/α_j is also among the inverse roots. In particular, except if $\alpha_j = \pm\sqrt{q}$, there are identities $\alpha_j\alpha_k = q$ with $j \neq k$, leading to multiplicative relations of the form

$$\alpha_j\alpha_k = \alpha_{j'}\alpha_{k'},$$

(this is similar to the fact that a root $1/2 + i\gamma$ of $L(s, \chi)$, for a Dirichlet character χ, gives a root $1/2 - i\gamma$ of $L(s, \bar{\chi})$, which leads to the restriction of the Grand Simplicity Conjecture to nonnegative ordinates of zeros). Hence, the most natural question is whether those "trivial" relations are the only multiplicative relations.

Finally, since dealing with a single curve seems still far away of this Grand Simplicity Hypothesis, which involves all Dirichlet L-functions, an even more natural-looking analog would be to ask the following: given a family of curves, interpreted as an algebraic family $\mathcal{C} \to U$ of curves of genus g over some parameter variety $U/\mathbf{F}_q$, what (if any) multiplicative relations can exist among the $\alpha_j(t)/\sqrt{q}$ that are the inverse roots of the polynomials $P_{C_t}(T)$, for *all* $t \in U(\mathbf{F}_q)$?

We will prove in this paper some results that give evidence that this type of independence holds. Of course, for a fixed curve, it might well be that nontrivial relations do hold among the roots (see Section 6, for examples). However, looking at suitable algebraic families, we will show that for *most* curves C_t, $t \in U(\mathbf{F}_q)$, their zeros and inverse zeros are as independent as possible, both additively and multiplicatively.

The first idea that may come to mind (along the lines of [12]) is to use the fact that the set of matrices in a compact group such as $SU(N, \mathbf{C})$ or $USp(2g, \mathbf{C})$ for which the eigenvalues satisfy nontrivial relations is of measure zero (for the natural measure, induced from Haar measure), and hope to apply directly Deligne's Equidistribution Theorem, which states that after taking suitable limits, the zeros of families of polynomials

P_{C_t} become equidistributed with respect to this measure. However, the sets in question, though they are measure-theoretically insignificant, are also dense in the corresponding group, and this means equidistribution does not by itself guarantee the required result. So, instead of this approach, we will use more arithmetic information on the zeta functions (note, however, that for *multiplicative* relations, which are in a sense the most interesting, one can apply Deligne's Theorem, after some preliminary work involving the specific properties of the eigenvalues, see Section 7).

Here is now a sample statement, where we can easily give concrete examples. We use the following notation: given a finite family $\boldsymbol{\alpha} = (\alpha_j)$ of nonzero complex numbers, we write $\langle \boldsymbol{\alpha} \rangle_a$ for the $\mathbf{Q}$-vector subspace of $\mathbf{C}$ generated by the α_j, and $\langle \boldsymbol{\alpha} \rangle_m$ for the multiplicative subgroup of $\mathbf{C}^\times$ generated by the α_j. For an algebraic curve C over a finite field (respectively, for finitely many curves $\boldsymbol{C} = (C_1, \ldots, C_k)$ over a common base field), we denote by $\mathcal{Z}(C)$ the multiset of inverse zeros of $P_C(T)$ (respectively, by $\mathcal{Z}(\boldsymbol{C})$ the multiset of inverse zeros of the product $P_{C_1} \cdots P_{C_k}$), and similarly with $\tilde{\mathcal{Z}}(C)$ and $\tilde{\mathcal{Z}}(\boldsymbol{C})$ for the multisets of normalized inverse zeros $\alpha/\sqrt{q}$.

Proposition 1.1. Let $f \in \mathbf{Z}[X]$ be a squarefree monic polynomial of degree $2g$, where $g \geqslant 1$ is an integer. Let p be an odd prime such that p does not divide the discriminant of f, and let $U/\mathbf{F}_p$ be the open subset of the affine t-line, where $f(t) \neq 0$. Consider the algebraic family $\mathcal{C}_f \to U$ of smooth projective hyperelliptic curves of genus g given as the smooth projective models of the curves with affine equations

$$C_t \ : \ y^2 = f(x)(x-t), \quad \text{for } t \in U.$$

Then, for any extension $\mathbf{F}_q/\mathbf{F}_p$, we have

$$|\{t \in U(\mathbf{F}_q) \mid \text{there is a nontrivial linear relation among } \mathcal{Z}(C)\}| \ll q^{1-\gamma^{-1}}(\log q), \tag{1.2}$$

$$|\{t \in U(\mathbf{F}_q) \mid \text{there is a nontrivial multiplicative relation among } \tilde{\mathcal{Z}}(C)\}| \ll q^{1-\gamma^{-1}}(\log q), \tag{1.3}$$

where $\gamma = 4g^2 + 2g + 4 > 0$, the implied constants depending only on g. □

In order to explain precisely the meaning of the statements, and to state further generalizations more concisely, we introduce the following notation: for any finite set M

of complex numbers, we define

$$\mathrm{Rel}(M)_a = \left\{ (t_\alpha) \in \mathbf{Q}^M \,\middle|\, \sum_{\alpha\in M} t_\alpha \alpha = 0 \right\}, \tag{1.4}$$

$$\mathrm{Rel}(M)_m = \left\{ (n_\alpha) \in \mathbf{Z}^M \,\middle|\, \prod_{\alpha\in M} \alpha^{n_\alpha} = 1 \right\}, \tag{1.5}$$

the additive relation $\mathbf{Q}$-vector space and multiplicative relation group, respectively. Note that $\mathrm{Rel}(M)_m$ is a free abelian group.

Then, tautologically, the condition in (1.2) for a given curve may be phrased equivalently as

$$\mathrm{Rel}(\mathcal{Z}(C))_a = 0, \text{ or } \dim_{\mathbf{Q}} \langle\, \mathcal{Z}(C) \,\rangle_a < 2g, \text{ or } \langle\, \mathcal{Z}(C) \,\rangle_a \simeq \mathbf{Q}^{2g},$$

and the qualitative content of (1.2) is that this holds for most values of t.

The interpretation of (1.3) needs more care because of the "trivial" multiplicative relations among the $\tilde{\alpha} \in \tilde{\mathcal{Z}}(C_t)$. Precisely, from the functional equation, it follows that we can arrange the $2g$ normalized roots $\tilde{\alpha} = \alpha/\sqrt{q}$ in g pairs of inverses $(\tilde{\alpha}, \tilde{\alpha}^{-1})$, so that the multiplicative subgroup $\langle\, \tilde{\mathcal{Z}}(C_t) \,\rangle_m \subset \mathbf{C}^\times$ is of rank $\leqslant g$. For $M = \tilde{\mathcal{Z}}(C_t)$, this corresponds to the inclusion

$$\{(n_{\tilde{\alpha}}) \in \mathbf{Z}^M \mid n_{\tilde{\alpha}} - n_{\tilde{\alpha}^{-1}} = 0\} \subset \mathrm{Rel}(M)_m. \tag{1.6}$$

Denote by $\mathrm{Triv}(M)_m$ the left-hand abelian group (which makes sense for any $M \subset \mathbf{C}^\times$ stable under inverse), and let $\mathrm{Rel}_0(M)_m = \mathrm{Rel}(M)_m/\mathrm{Triv}(M)_m$ (the group of nontrivial relations). The interpretation of (1.3) is that most of the time, there is equality:

$$\mathrm{Rel}(\tilde{\mathcal{Z}}(C_t))_m = \mathrm{Triv}(\tilde{\mathcal{Z}}(C_t))_m, \quad \text{or} \quad \mathrm{Rel}_0(\tilde{\mathcal{Z}}(C_t))_m = 0, \tag{1.7}$$

(or, in fact, simply $\langle\, \tilde{\mathcal{Z}}(C_t) \,\rangle_m \simeq \mathbf{Z}^g$; this is because if $\langle\, \tilde{\mathcal{Z}}(C_t) \,\rangle_m$ is of rank g, comparing ranks implies that $\mathrm{Triv}(\tilde{\mathcal{Z}}(C_t))_m$ is of finite index in $\mathrm{Rel}(\tilde{\mathcal{Z}}(C_t))_m$, and the former is easily seen to be saturated in $\mathbf{Q}^{\tilde{\mathcal{Z}}(C_t)}$, so it is not a proper finite index subgroup of a subgroup of $\mathbf{Z}^{\tilde{\mathcal{Z}}(C_t)}$).

Moreover, yet another interpretation is the following. Assume still that $M \subset \mathbf{C}^\times$ is stable under inverse and of even cardinality $2g$; order its elements in some way so that

$$M = \{\tilde{\alpha}_1, \dots, \tilde{\alpha}_g, \tilde{\alpha}_1^{-1}, \dots, \tilde{\alpha}_g^{-1}\},$$

and write $\tilde{\alpha}_j = e(\theta_j)$, with $0 \leqslant \theta_j < 1$. Then, $\mathrm{Rel}_0(M)_m = 0$ if and only if the elements $(1, \theta_1, \dots, \theta_g)$ are $\mathbf{Q}$-linearly independent. Indeed, assuming the former, if we have a relation

$$t_0 + \sum_{1 \leqslant j \leqslant g} t_j \theta_j = 0,$$

with $(t_0, t_1, \dots, t_g) \in \mathbf{Q}^{g+1}$, multiplying by a common denominator Δ and exponentiating leads to

$$\prod_{1 \leqslant j \leqslant g} \tilde{\alpha}_j^{n_j} = 1,$$

where $n_j = \Delta t_j \in \mathbf{Z}$. This implies that $(n_1, \dots, n_g, 0, \dots, 0) \in \mathrm{Rel}_0(M)_m = \mathrm{Triv}(M)_m$, and, by definition (1.6), we deduce $n_j = 0$, $1 \leqslant j \leqslant g$, and then, $t_j = 0$ for all j. The converse is also easy.

Remark 1.2. In the spirit of the previous remark concerning Deligne's Equidistribution Theorem, note that the result may also be interpreted (though this is much weaker) as giving instances of the convergence of $\mu_n(A)$ to $\mu(A)$, where μ_n is the average of Dirac measures associated to the normalized geometric Frobenius conjugacy classes in $USp(2g, \mathbf{C})$ for $t \in \mathbf{F}_{q^n}$, $f(t) \neq 0$, while μ is the probability Haar measure on $USp(2g, \mathbf{C})$ and A is the set of unitary symplectic matrices with eigenangles that are nontrivially additively or multiplicatively dependent (so, in fact, $\mu(A) = 0$). □

Our second result encompasses the first one and is a first step toward independence for more than one curve. Again, we state it for the concrete families above.

Theorem 1.3. Let $f \in \mathbf{Z}[X]$ be a squarefree monic polynomial of degree $2g$, where $g \geqslant 1$ is an integer. Let p be an odd prime such that p does not divide the discriminant of f, and let $U/\mathbf{F}_p$ be the open subset of the affine line, where $f(t) \neq 0$. Let $\mathcal{C}_f \to U$ be the family of hyperelliptic curves defined in Proposition 1.1.

Let $k \geqslant 1$. For all finite fields $\mathbf{F}_q$ of characteristic p, and for all k-tuples $\boldsymbol{t} = (t_1, \dots, t_k) \in U(\mathbf{F}_q)^k$, denote $\boldsymbol{C}_{\boldsymbol{t}} = (C_{t_1}, \dots, C_{t_k})$. Then, we have

$$|\{\boldsymbol{t} \in U(\mathbf{F}_q)^k \mid \mathrm{Rel}(\mathcal{Z}(\boldsymbol{C}_{\boldsymbol{t}}))_a \neq 0\}| \ll c^k q^{k-\gamma^{-1}} (\log q),$$
$$|\{\boldsymbol{t} \in U(\mathbf{F}_q)^k \mid \mathrm{Rel}_0(\tilde{\mathcal{Z}}(\boldsymbol{C}_{\boldsymbol{t}}))_m \neq 0\}| \ll c^k q^{k-\gamma^{-1}} (\log q),$$

where $\gamma = 29kg^2 > 0$ and $c \geqslant 1$ is a constant depending only on g. In both estimates, the implied constant depends only on g. □

Remark 1.4. An important warning is not to read too much in this: the dependency of γ on k means the result is trivial for k unless we have

$$c^{-k} q^{1/\gamma} \to +\infty, \quad \text{i.e.} \quad \frac{\log q}{29kg^2} - k \log c \to +\infty,$$

which means essentially (for fixed g) that $k = o(\sqrt{\log q})$. However, it leads to nontrivial results for any fixed k, and even for k growing slowly as a function of $q \to +\infty$, and, in this respect, it is already quite interesting. Also, note that the "exceptional set" of k-tuples trivially contains those $\boldsymbol{t}$ where two coordinates coincide; there are $\gg q^{k-1}$ of them, and those "diagonals" would have to be excluded if one were to try to go beyond such a bound. □

Remark 1.5. We indicate the type of connections with distribution properties that arise. Those are of independent interest, and they show clearly the analogy with the discussion of the Chebychev Bias, in particular, why the independence issues appear naturally there (compare with the arguments in [24, §2,§3]).

Let $C/\mathbf{F}_q$ be any (smooth, projective, geometrically connected) algebraic curve of genus g, and choose g inverse roots α_j of the L-function of C, $1 \leqslant j \leqslant g$, so that

$$P_C(T) = \prod_{1 \leqslant j \leqslant g} (1 - \alpha_j T)\big(1 - \alpha_j^{-1} q T\big),$$

and write $\alpha_j = \sqrt{q} e(\theta_j)$ as before. For any $n \geqslant 1$, the number of points in $C(\mathbf{F}_{q^n})$ is given by

$$|C(\mathbf{F}_{q^n})| = q^n + 1 - \sum_{1 \leqslant j \leqslant g} \left(\alpha_j^n + \frac{q^n}{\alpha_j^n}\right) = q^n + 1 - 2q^{n/2} \sum_{1 \leqslant j \leqslant g} \cos 2\pi n\theta_j.$$

If $C/\mathbf{F}_q$ is such that $\mathrm{Rel}_0(\tilde{\mathcal{Z}}(C))_m = 0$, we know that the $g+1$ numbers 1 and $(\theta_j)_{i,j}$ are $\mathbf{Q}$-linearly independent. Hence, by Kronecker's theorem, the sequence

$$(2\pi n\theta_1, \dots, 2\pi n\theta_g) \in (\mathbf{R}/2\pi\mathbf{Z})^g$$

becomes equidistributed in $(\mathbf{R}/2\pi\mathbf{Z})^g$ as $n \to +\infty$, with respect to the Lebesgue measure on the torus. It follows that

$$\frac{|C(\mathbf{F}_{q^n})| - (q^n+1)}{2q^{n/2}}$$

becomes distributed like the image of Lebesgue measure under the map

$$\varphi : \begin{cases} (\mathbf{R}/2\pi\mathbf{Z})^g \to \mathbf{R} \\ (\theta_1, \dots, \theta_g) \mapsto \cos\theta_1 + \cdots + \cos\theta_g. \end{cases}$$

This distribution is, in fact, not unexpected: we have the well-known spectral interpretation

$$\frac{|C(\mathbf{F}_{q^n})| - (q^n+1)}{2q^{n/2}} = \mathrm{Tr}(F^n),$$

for $n \geqslant 1$, where $F \in USp(2g)$ is the unitarized Frobenius conjugacy class of C. A remarkable result due to Rains [23] states that for $n \geqslant 2g$, the eigenvalues of a Haar-distributed random matrix in $USp(2g, \mathbf{C})$ are distributed *exactly* like g independent points uniformly distributed on the unit circle, together with their conjugates. In particular, the limit distribution above is therefore the distribution law of the trace of such a random matrix.

Similarly, let now (C_1, C_2) be a pair of algebraic curves (smooth, projective, geometrically connected) of common genus $g \geqslant 1$ over $\mathbf{F}_q$, for which $\mathrm{Rel}_0(\tilde{\mathcal{Z}}(C_1, C_t))_m = 0$—for instance, any of the pairs given by Theorem 1.3 with $k = 2$. Write $\alpha_{i,j}$, $\theta_{i,j}$ for the inverse roots and arguments as above for C_i.

We compare the number of points on C_1 and C_2: we have

$$\frac{|C_1(\mathbf{F}_{q^n})| - |C_2(\mathbf{F}_{q^n})|}{q^{n/2}} = 2\sum_{1\leqslant j\leqslant g} (\cos 2\pi n\theta_{2,j} - \cos 2\pi n\theta_{1,j}). \tag{1.8}$$

The assumption that $\mathrm{Rel}_0(\tilde{\mathcal{Z}}(C_1,C_2))_m = 0$ gives now that the $2g+1$ numbers 1 and $(\theta_{i,j})_{i,j}$ are $\mathbf{Q}$-linearly independent, and thus, the sequence

$$(2\pi n\theta_{2,1},\dots,2\pi n\theta_{2,g},2\pi n\theta_{1,1},\dots,2\pi n\theta_{1,g})$$

becomes equidistributed in $(\mathbf{R}/2\pi\mathbf{Z})^{2g}$ as $n\to+\infty$ with respect to the Lebesgue measure on the $2g$-dimensional torus. So, the right-hand side of (1.8) becomes equidistributed as $n\to+\infty$ with respect to the image measure of the Lebesgue measure $d\theta$ by the same map as above, with $2g$ angles instead of g (since the cosine is an even function, it and its opposite have the same distribution).

Let μ_g be this measure, so that we have in particular, for any $a<b$, the limit

$$\frac{1}{N}\left|\left\{n\leqslant N\mid a<\frac{|C_2(\mathbf{F}_{q^n})|-|C_1(\mathbf{F}_{q^n})|}{q^{n/2}}<b\right\}\right|\to\int_a^b d\mu_g(t)$$

as $N\to+\infty$, and as a special case

$$\frac{1}{N}\left|\left\{n\leqslant N\mid |C_2(\mathbf{F}_{q^n})|<|C_1(\mathbf{F}_{q^n})|\right\}\right|\to\frac{1}{2}$$

as $N\to+\infty$. This (since the assumption on C_1 and C_2 is "almost always true") means that there is typically no "bias" that can lead to the number of points on C_1 being larger than that on C_2 when we look at extension fields of $\mathbf{F}_q$.

Furthermore, we can clearly interpret μ_g as the probability law of a sum

$$Y_g = 2\cos 2\pi X_1+\cdots+2\cos 2\pi X_{2g}$$

of $2g$ independent random variables $2\cos 2\pi X_j$, $1\leqslant j\leqslant 2g$, where each X_j is uniformly distributed on $[0,1]$ (there is no minus sign since the cosine and its opposite have the same distribution for uniform arguments). The characteristic function (in other words, Fourier transform) of such a random variable is given by

$$\varphi_g(t)=\boldsymbol{E}(e^{itY_g})=\left(\int_0^1 e^{2it\cos 2\pi\theta}d\theta\right)^{2g}=J_0(2t)^{2g},$$

where J_0 is the standard Bessel function. Furthermore, since

$$\boldsymbol{E}(Y_g)=0,\qquad \boldsymbol{E}\big(Y_g^2\big)=2g\boldsymbol{E}((2\cos 2\pi X_1)^2)=4g,$$

the Central Limit Theorem implies that $Y_g/2\sqrt{g}$ converges in law, as $g \to +\infty$, to a standard Gaussian random variable with variance 1. This means that, for curves C_1 and C_2 of large genus g, the further normalized difference

$$\frac{|C_2(\mathbf{F}_{q^n})| - |C_1(\mathbf{F}_{q^n})|}{2q^{n/2}\sqrt{g}}$$

will be distributed approximately like a standard Gaussian.

It would be interesting to know what other limiting distributions can occur for pairs of algebraic curves where there are nontrivial relations (such as those in Section 6). □

As far as relating a result like Theorem 1.3 to the Grand Simplicity Conjecture, even though the statement itself provides no direct evidence, the main point is in the method of proof, which can be interpreted as linking the problem with random matrix models for families of L-functions. The point is that the crucial input to apply the sieve for Frobenius, which is the main tool, is the fact that the families of curves considered have large (symplectic) monodromy, which in the Katz–Sarnak philosophy is the analog of the conjectured existence of "symmetry types" for families of L-functions such as Dirichlet characters (precisely, the latter are supposed to have unitary symmetry type, which is slightly different). We refer to [19] for a survey of recent developments in the area of random matrix models of L-functions and for discussion of the evidence available.

The idea of the proofs is, roughly, to first show that a certain maximality condition on the Galois group of the splitting field of an individual set of zeros implies the required independence (see Section 2, which uses methods developed by Girstmair to analyze relations between roots of algebraic equations). Then, we apply the sieve for Frobenius of the author (see [13] and [14, §8]) to check that most $\boldsymbol{C_t}$ satisfy this criterion (as can be guessed from the statement of Theorem 1.3, the main novel issue in applying the sieve is the need for some care in arguing uniformly with respect to k.) One can then see this type of argument as providing some kind of answer to the question asked by Katz (see [11, End of Section 1]) of what could be a number field analog of the irreducibility of zeta functions of curves, or of other (polynomial) L-functions over finite fields.

After the first version of this paper was completed, along the lines of the previous paragraph, Katz suggested to look at the implications of the theory of Frobenius tori of

Serre for this type of questions. It turns out that, indeed, one can use this theory (in the version described by Cheewhye Chin [3]) to get a different proof of the multiplicative independence of the zeros, for fixed k at least (in the setting of Theorem 1.3). The large monodromy assumption remains essential, but the analytic argument is a bit simpler since one can use a uniform effective version of the Chebotarev density theorem instead of the large sieve. This, however, does not significantly improve the final estimates. We sketch this approach in Section 7. We have chosen not to remove the earlier one because the sieve for Frobenius leads to added information that may be useful for other purposes (e.g. the linear independence of the roots is not controlled by Frobenius tori, see Remark 7.3), and because it is (in some sense) more elementary and accessible to analytic number theorists. For instance, if we look at elements of $Sp(2g, \mathbf{Z})$ obtained by random walks on such a discrete group, the approach based on Frobenius tori would not be available to show that the probability of existence of relations between eigenvalues of those matrices goes exponentially fast to 0, but it is an easy consequence of the large sieve of [14, §7] and the results of Section 2.

We provide general versions of independence statements for any family that has large (symplectic) monodromy. Analogs for other symmetry types are also easy to obtain; this is particularly clear from the point of view of Frobenius tori, but the sieve for Frobenius can also be adapted (see Jouve's thesis [9] for the case of "big" orthogonal monodromy).

Notation. As usual, $|X|$ denotes the cardinality of a set, $\mathfrak{S}_g$ is the symmetric group on g letters, $\mathbf{F}_q$ is a field with q elements. By $f \ll g$ for $x \in X$, or $f = O(g)$ for $x \in X$, where X is an arbitrary set on which f is defined, we mean synonymously that there exists a constant $C \geqslant 0$ such that $|f(x)| \leqslant Cg(x)$ for all $x \in X$. The "implied constant" is any admissible value of C. It may depend on the set X which is always specified or clear in context. On the other hand, $f = o(g)$ as $x \to x_0$ means that $f/g \to 0$ as $x \to x_0$.

An algebraic variety is meant to be a reduced, separated scheme of finite type, and most of those occurring will be affine. For $V/\mathbf{F}_q$, an algebraic variety over a finite field, $\nu \geqslant 1$ and $t \in V(\mathbf{F}_{q^\nu})$, we write $\mathrm{Fr}_{q^\nu,t}$ for the geometric Frobenius conjugacy class at t relative to the field $\mathbf{F}_{q^\nu}$; when ν is fixed, we simply write Fr_t. For a field k, we write $\bar{k}$ for an algebraic closure of k, and for an algebraic variety X over k, we write $\bar{X}$ for $X \times_k \bar{k}$, and we denote by η_X a geometric $\bar{k}$-valued point of X; whenever morphisms between fundamental groups are mentioned, the geometric points are assumed to be chosen in compatible fashion.

2 An Algebraic Criterion for Independence

Let $g \geqslant 1$ be a fixed integer, and let W_{2g} be the finite group of order $2^g g!$, which is described (up to isomorphism) by any of the following equivalent definitions:

- It is the group of permutations of a finite set M of order $2g$ that commute with a given involution c on M without fixed points:

$$\sigma(c(\alpha)) = c(\sigma(\alpha)), \qquad \text{for all } \alpha \in M;$$

  we write usually $c(\alpha) = \bar{\alpha}$, so that $\overline{\sigma(\alpha)} = \sigma(\bar{\alpha})$.
- Given a set M with $2g$ elements that is partitioned in a set N of g couples $\{x, y\}$, W_{2g} is the subgroup of the group of permutations of M, which permute the set of pairs N; as an example, we can take

$$M = \{-g, \dots, -1, 1, \dots, g\} \subset \mathbf{Z},$$

  with the pairs $\{-i, i\}$ for $1 \leqslant i \leqslant g$, and then, the condition for a permutation σ of M to be in W_{2g} is that

$$\sigma(-i) = -\sigma(i), \quad \text{for all } i,\ 1 \leqslant i \leqslant g.$$

- It is the semidirect product $\mathfrak{S}_g \ltimes \{\pm 1\}^g$, where $\mathfrak{S}_g$ acts on $\{\pm 1\}^g$ by permuting the coordinates.
- It is the subgroup of $GL(g, \mathbf{Q})$ of matrices with entries in $\{-1, 0, 1\}$, where one entry exactly in each row and column is nonzero (as explained in [1], except for seven values of g, this is, in fact, a finite subgroup of $GL(g, \mathbf{Q})$ with maximal order).
- Finally, it is the Weyl group of the symplectic group $Sp(2g)$, i.e. the quotient $N(T)/T$, where $T \subset Sp(2g)$ is a maximal torus (although this can be seen as the "real" reason this group occurs in our context, it is not at all necessary to know the details of this definition, or how it relates to the previous ones, to understand the rest of this paper).

Note that the second definition provides a short exact sequence

$$1 \to \{\pm 1\}^g \to W_{2g} \to \mathfrak{S}_g \to 1. \tag{2.1}$$

We will use mostly the first two definitions, the equivalence of which is particularly easy, indicating what is the set M and/or involution c under consideration. We let N be the quotient of M modulo the equivalence relation induced by c (with $\alpha \sim \bar{\alpha}$; this is the same as the set N of the second definition).

We now state some properties of the group W_{2g}, which we assume to be given with some set M and set N of couples on which W_{2g} acts, as in the second definition. For a given $\alpha \in M$, we write $\bar{\alpha}$ for the unique element such that $\{\alpha, \bar{\alpha}\} \in N$.

We let $F(M) = \mathbf{Q}^M$ be the $\mathbf{Q}$-vector space generated by M, with canonical basis $(f_\alpha)_{\alpha \in M}$, and we consider $F(M)$ as given with the associated permutation representation of W_{2g}.

Lemma 2.1. Let $g \geqslant 2$ be any integer, W_{2g}, M, N, and $F(M)$ as before. Then

(1) The group W_{2g} acts transitively on M, and acts on $M \times M$ with three orbits:

$$\Delta = \{(\alpha, \alpha) \mid \alpha \in M\}, \qquad \Delta_c = \{(\alpha, \bar{\alpha}) \mid \alpha \in M\},$$
$$O = \{(\alpha, \beta) \mid \alpha \neq \beta,\ \bar{\alpha} \neq \beta\}.$$

(2) The representation of W_{2g} on $F(M)$ decomposes as the direct sum

$$F(M) = \mathbf{1} \oplus G(M) \oplus H(M)$$

of the three subspaces defined by

$$\mathbf{1} = \mathbf{Q}\psi \subset F(M), \text{ where } \psi = \sum_{\alpha \in M} f_\alpha,$$
$$G(M) = \left\{ \sum_{\alpha \in M} t_\alpha f_\alpha \in F(M) \mid t_\alpha - t_{\bar{\alpha}} = 0,\ \alpha \in M, \text{ and } \sum_{\alpha \in M} t_\alpha = 0 \right\},$$
$$H(M) = \left\{ \sum_{\alpha \in M} t_\alpha f_\alpha \in F(M) \mid t_\alpha + t_{\bar{\alpha}} = 0,\ \alpha \in M \right\},$$

which are absolutely irreducible representations of W_{2g}. □

Proof. (1) The transitivity of W_{2g} on M is clear. Furthermore, it is obvious that the sets Δ, Δ_c, O form a partition of $M \times M$, and that Δ is the orbit of any fixed $(\alpha, \alpha) \in \Delta$ by transitivity.

To check that Δ_c is also an orbit, fix some $x_0 = (\alpha_0, \bar{\alpha}_0) \in \Delta_c$, and let $x = (\alpha, \bar{\alpha}) \in \Delta_c$ be arbitrary. If σ is any element of W_{2g} such that $\sigma(\alpha_0) = \alpha$, we have $\sigma(\bar{\alpha}) = \bar{\alpha}_0$; hence, $\sigma(x_0) = x$.

There remains to look at O. First, $O \neq \emptyset$ because $g \geqslant 2$ (so that there exists $(\alpha, \beta) \in M \times M$ with $\beta \notin \{\alpha, \bar{\alpha}\}$). Using the fact that for any $\gamma \neq \delta$ in M, there exists $\sigma \in W_{2g}$ such that $\sigma(\gamma) = \delta$ and σ acts as identity on $M - \{\gamma, \bar{\gamma}, \delta, \bar{\delta}\}$, it is clear that if $y = (\alpha, \beta) \in O$, then all elements of O of the form (α, γ) are in the orbit of y, and so are all elements of the form (γ, β).

So, given $y_1 = (\alpha, \beta)$ and $y_2 = (\gamma, \delta) \in O$, we can find σ_1 such that $\sigma(y_1) = (\alpha, \delta)$, then σ_2 such that

$$\sigma_1\sigma_2(\alpha, \beta) = \sigma_2(\alpha, \delta) = (\gamma, \delta) = y_2,$$

so, O is a single orbit as desired.

(2) Again, it is easily checked that $\mathbf{1}$, $G(M)$, and $H(M)$ are W_{2g}-invariant subspaces of $F(M)$, and it suffices to check that the representation $F(M) \otimes \mathbf{C}$ is a direct sum of three irreducible components. This means we must show that

$$\langle \chi, \chi \rangle = 3,$$

where χ is the character of the representation of W_{2g} on $F(M) \otimes \mathbf{C}$, as 3 can only be written as $1 + 1 + 1$ as sum of squares of positive integers. This is a well-known consequence of (1): since χ is real valued (as character of a permutation representation), we have $\langle \chi, \chi \rangle = \langle \chi^2, 1 \rangle$; further, χ^2 is the character of the permutation representation of W_{2g} on $M \times M$, and hence, as for any permutation character, the inner product $\langle \chi^2, 1 \rangle$ is the number of orbits of the action of W_{2g} on $M \times M$, which we saw is equal to 3 (for these facts, see, e.g. [25, Exercise 2.6]). ■

Remark 2.2. The first part of the lemma says that W_{2g} does not act doubly transitively on M, but is not so far from this, the orbit O being of much larger size than the diagonal orbit Δ and Δ_c (the graph of the involution c on M): we have $|\Delta| = |\Delta_c| = 2g$ and $|O| = 4g(g-1)$).

On the other hand, we have $\dim \mathbf{1} = 1$, $\dim G(M) = g - 1$, and $\dim H(M) = g$. If we select one element of each of the g pairs in N and number them as $(\alpha_i, \bar{\alpha}_i)$ for $1 \leqslant i \leqslant g$,

then bases of $\mathbf{1}$, $G(M)$, and $H(M)$ are given, respectively, by the vectors

$$\sum_{\alpha\in M} f_\alpha, \tag{2.2}$$

$$(f_{\alpha_i} + f_{\tilde{\alpha}_i}) - (f_{\alpha_{i+1}} + f_{\tilde{\alpha}_{i+1}}), \qquad 1 \leqslant i \leqslant g-1, \tag{2.3}$$

$$f_{\alpha_i} - f_{\tilde{\alpha}_i}, \qquad 1 \leqslant i \leqslant g. \tag{2.4}$$

Note that we also obtain from the definitions of $\mathbf{1}$ and $G(M)$ that

$$\mathbf{1} \oplus G(M) = \left\{ \sum_{\alpha\in M} t_\alpha f_\alpha \in F(M) \mid t_\alpha = t_{\tilde{\alpha}}, \quad \alpha \in M \right\}, \tag{2.5}$$

(which is none other than $\mathrm{Triv}(M)_m$, as defined in (1.6)).

In terms of "abstract" representation theory, the three subspaces are not hard to identify: notice first that both $\mathbf{1}$ and $G(M)$ are invariant under the subgroup $(\mathbf{Z}/2\mathbf{Z})^g$ in the exact sequence (2.1), hence are representations of the quotient $\mathfrak{S}_g$. It is clear that their direct sum is simply the standard permutation representation of the symmetric group. As for $H(M)$, looking at the action on the basis (2.4), one finds that it is isomorphic to the representation given by the embedding $W_{2g} \hookrightarrow GL(g, \mathbf{Q})$ of the last definition of W_{2g} (in particular, it is faithful). □

Corollary 2.3. Let $k \geqslant 1$ be an integer and $W = W_{2g} \times \cdots \times W_{2g}$, the product of k copies of W_{2g}, the jth copy acting on M_j. Consider the action of W on the disjoint union

$$M = \bigsqcup_{1\leqslant j\leqslant k} M_j,$$

where the jth factor acts trivially on M_i for $i \neq j$. Let $F(M)$ denotes the permutation representation of W on the $\mathbf{Q}$-vector space $\mathbf{Q}^M$ of dimension $2kg$. Then, $F(M)$ is $\mathbf{Q}$-isomorphic to the direct sum

$$F(M) \simeq k \cdot \mathbf{1} \oplus \bigoplus_{1\leqslant j\leqslant k} G_j \oplus \bigoplus_{1\leqslant j\leqslant k} H_j$$

of geometrically irreducible representations of W, where G_j is the representation $G(M_j)$ of the previous lemma, $(\sigma_1, \dots, \sigma_k)$ acting as σ_j, and similarly, H_j is $H(M_j)$ acting through the jth factor W_{2g}. □

Proof. This is clear from Lemma 2.1 and the definition of M. ■

Continuing with an integer $k \geqslant 1$, we now assume that we have polynomials P_1, ..., P_k with coefficients in a field $E \subset \mathbf{C}$ such that each of the splitting fields K_i/E of P_i has Galois group isomorphic to W_{2g}, acting by permutation on the set M_j of roots of P_j, and which are jointly linearly independent so that the splitting field K/E of the product

$$P = P_1 \cdots P_k \in E[X]$$

has Galois group naturally isomorphic to $W = W_{2g}^k$. Note that this implies, in particular, that the sets of roots of the polynomials P_j are disjoint. Then, the disjoint union M of Corollary 2.3 can be identified with the set of all roots of P.

We have the $\mathbf{Q}$-vector space $\langle M \rangle_a \subset \mathbf{C}$ generated by the set of roots of P, and the multiplicative abelian group $\langle M \rangle_m \subset \mathbf{C}^\times$, from which we may construct the $\mathbf{Q}$-vector space $\langle M \rangle_m \otimes_{\mathbf{Z}} \mathbf{Q}$. Using the Galois action by permutation of the roots, those two vector spaces are themselves representations of W, and moreover mapping each element of the canonical basis of $F(M) = \mathbf{Q}^M$ to the corresponding root, we have natural $\mathbf{Q}$-linear maps

$$F(M) = \mathbf{Q}^M \xrightarrow{r_a} \langle M \rangle_a, \qquad F(M) = \mathbf{Q}^M \xrightarrow{r_m} \langle M \rangle_m \otimes \mathbf{Q},$$

which are also maps of W-representations. By construction, we have

$$\ker(r_a) = \mathrm{Rel}(M)_a, \qquad \ker(r_m) = \mathrm{Rel}(M)_m \otimes \mathbf{Q},$$

where $\mathrm{Rel}(M)_a$ and $\mathrm{Rel}(M)_m$ are the relation groups defined in (1.4) and (1.5). Note that both $\mathrm{Rel}(M)_a$ and $\mathrm{Rel}(M)_m \otimes \mathbf{Q}$ are subrepresentations of the permutation representation $F(M)$.

Thus, we see that the problem of finding the possible relations among roots of a polynomial is transformed into a problem of representation theory (in the multiplicative case, one must also handle the possible loss of information in taking the tensor product with $\mathbf{Q}$: for instance, $\mathrm{Rel}(-1)_m = 2\mathbf{Z} \subset \mathbf{Z}$ and $1 \in \mathrm{Rel}(-1)_m \otimes \mathbf{Q}$ although $(-1)^1 \neq 1 \ldots$). This is in essence Girstmair's method, see, e.g. [4] (notice that there is nothing special in working with W-extensions in the above). Since Corollary 2.3 has described explicitly the decomposition of $F(M)$ as sum of irreducible representations of W, the theory of

linear representations of finite groups shows that there are very few possibilities for the subrepresentations $\mathrm{Rel}(M)_a$ and $\mathrm{Rel}(M)_m \otimes \mathbf{Q}$.

Proposition 2.4. Let $k \geqslant 1$ and $g \geqslant 2$ be integers. Let $P_1, \dots, P_k$ be polynomials satisfying the conditions above. With notation as above, in particular, $P = P_1 \cdots P_k$ and M the set of zeros of P, assume in addition that for any pair of roots $(\alpha, \bar{\alpha})$, we have $\alpha\bar{\alpha} \in \mathbf{Q}^\times$.

(1) We have

$$\mathrm{Rel}(M)_a = \bigoplus_{1 \leqslant j \leqslant k} \mathrm{Rel}(M_j)_a,$$

and for each j, we have either $\mathrm{Rel}(M_j)_a = 0$ or $\mathrm{Rel}(M_j)_a = \mathbf{1}$. The latter alternative holds if and only if

$$\sum_{\alpha \in M_j} \alpha = 0,$$

or equivalently, if $\mathrm{Tr}_{K/E}(\alpha) = 0$ for any $\alpha \in M_j$.

(2) We have

$$\mathrm{Rel}(M)_m \otimes \mathbf{Q} = \bigoplus_{1 \leqslant j \leqslant k} \mathrm{Rel}(M_j)_m \otimes \mathbf{Q}.$$

Moreover, assume that the rational number $\alpha\bar{\alpha} \in \mathbf{Q}$ is positive and independent of α, say, equal to m. Then, for $g \geqslant 5$ in the general case, and for $g \geqslant 2$ if $m = 1$, we have for each j that

$$\mathrm{Rel}(M_j)_m \otimes \mathbf{Q} = \begin{cases} \mathbf{1} \oplus G(M_j) & \text{if } m = 1, \\ G(M_j) & \text{otherwise.} \end{cases}$$

□

Proof. (1) From representation theory, we know that $\mathrm{Rel}(M)_a$ is the direct sum of some subset of the irreducible components of $F(M)$ corresponding to the decomposition in Corollary 2.3. This isomorphism shows that $F(M)$ decomposes as a direct sum over j of representations $F(M_j)$ depending on the jth factor of W, each of which is given by Lemma 2.1. Accordingly, $\mathrm{Rel}(M)_a$ is the direct sum over j of subrepresentations of $F(M_j)$. Those are representations of the jth factor W_{2g} extended by the identity to W, and tautologically, they correspond exactly to the relation space $\mathrm{Rel}(M_j)_a$ among zeros of P_j.

To finish the proof of (1), it suffices therefore to treat each P_j in turn, so we might as well assume $k = 1$ and remove the subscript j, using notation in Lemma 2.1 (in particular, writing now M instead of M_j). Noting that, for any $\alpha \in M$, the relation $\mathrm{Tr}_{K/E}(\alpha) = 0$ is equivalent with $\mathbf{1} \subset \mathrm{Rel}(M)_a$, the claim then amounts to saying that $G(M)$ and $H(M)$ cannot occur in $\mathrm{Rel}(M)_a$.

First, $G(M) \subset \mathrm{Rel}(M)_a$ means that

$$\sum_{\alpha} t_\alpha \alpha = 0, \tag{2.6}$$

whenever $(t_\alpha) \in \mathbf{Q}^M$ sum to zero and satisfy $t_\alpha - t_{\bar{\alpha}} = 0$ for $\alpha \in M$. In particular, fix a root α of P; we find that for any $\sigma \in W_{2g}$ with $\sigma(\alpha) \neq \alpha$, say $\sigma(\alpha) = \beta$, we have

$$(\alpha + \bar{\alpha}) - (\beta + \bar{\beta}) = (\alpha + \bar{\alpha}) - \sigma(\alpha + \bar{\alpha}) = 0,$$

for all $\sigma \in W_{2g} = \mathrm{Gal}(K/\mathbf{Q})$ not fixing α. Since the last relation is trivially valid for σ fixing α (hence $\bar{\alpha}$), it follows that $\alpha + \bar{\alpha} \in \mathbf{Q}$. From the assumption $\alpha\bar{\alpha} \in \mathbf{Q}^\times$, it follows that $\mathbf{Q}(\alpha)$ is a quadratic field. It must be the splitting field K of the polynomial P, and hence, this cannot occur under the conditions $g \geqslant 2$ and $\mathrm{Gal}(K/\mathbf{Q}) = W_{2g}$.

Similarly, $H(M) \subset \mathrm{Rel}(M)_a$ means that (2.6) holds whenever $(t_\alpha) \in \mathbf{Q}^M$ satisfy $t_\alpha + t_{\bar{\alpha}} = 0$. Using again a fixed root α of P, we obtain in particular

$$\alpha - \bar{\alpha} = 0, \tag{2.7}$$

which contradicts the fact that the elements α and $\bar{\alpha}$ are distinct.

(2) The proof of the direct sum decomposition for $\mathrm{Rel}(M)_m \otimes \mathbf{Q}$ is the same as that for additive relations, and hence, we are again reduced to the case $k = 1$ (and we write M instead of M_j). We first show that $G(M) \subset \mathrm{Rel}(M)_m \otimes \mathbf{Q}$ in all cases. Indeed, considering the generators (2.3) of $G(M)$, it suffices to show that

$$\frac{\alpha\bar{\alpha}}{\beta\bar{\beta}} = 1,$$

for all α and β, and this is correct from our assumption that $\alpha\bar{\alpha}$ is independent of α. (Note the tensor product with $\mathbf{Q}$ means this is not *equivalent* with $G(M) \subset \mathrm{Rel}(M)_m \otimes \mathbf{Q}$.)

Now, we consider the consequences of the possible inclusion of the subrepresentations $\mathbf{1}$, and $H(M)$ in $\mathrm{Rel}(M)_m \otimes \mathbf{Q}$. First, $\mathbf{1} \subset \mathrm{Rel}(M)_m \otimes \mathbf{Q}$ means exactly that for some

integer $n \geqslant 1$, we have

$$n\psi = \sum_{\alpha \in M} nf_\alpha \in \mathrm{Rel}(M)_m,$$

which is equivalent with

$$\prod_{\alpha \in M} \alpha^n = \left(\prod_{\alpha \in M} \alpha \right)^n = (N_{K/E}(\alpha))^n = 1,$$

or in other words, $N_{K/E}(\alpha)$ is a root of unity. But, the assumption that $\alpha\bar{\alpha} = m$ be a positive rational number independent of α implies that $N_{K/E}(\alpha) = m^g$, so $\mathbf{1} \subset \mathrm{Rel}(M)_m \otimes \mathbf{Q}$ if and only if $m = 1$.

It remains to exclude the possibility that $H(M) \subset \mathrm{Rel}(M)_m \otimes \mathbf{Q}$ to conclude the proof. But, instead of (2.7), this possibility implies now that, for some integer $n \geqslant 1$, we have

$$\alpha^{2n} = m^n \left(\frac{\alpha^{2n}}{m^n} \right) = m^n \left(\frac{\alpha}{\bar{\alpha}} \right)^n = m^n.$$

Hence, $K/\mathbf{Q}$ would be the Kummer extension $\mathbf{Q}(\sqrt{m}, \boldsymbol{\mu}_{2n})$, where $\boldsymbol{\mu}_{2n}$ is the group of $2n$th roots of unity. In particular, the Galois group of K/E would be solvable, which is false for W_{2g} if $g \geqslant 5$ (the nonsolvable group A_g occurs as one composition factor). For $m = 1$, the Galois group would be abelian, which is not the case of W_{2g} for $g \geqslant 2$. ■

Remark 2.5. Since there exist elements with trace zero generating a given number field, both cases of the alternative in (1) can occur. It should be clear, however, that $\mathrm{Rel}(M_j)_a = 0$ is the "most likely," and we will see this at work in Section 4. □

Remark 2.6. In [15, Proposition 2.1, Remark 2.2], we had proved for different purposes and using quite different methods a result that implied, as we remarked, that if the splitting field of the L-function of a curve $C/\mathbf{F}_q$ is W_{2g}, and if, in addition, the curve were ordinary (which can be interpreted as saying that the coefficient of T^g of P_C is not divisible by p), then the multiplicative group $\langle\, \mathcal{Z}(C) \,\rangle_m$ is free of rank $g+1$. This is almost the same as the case $k = 1$ of Proposition 2.4, but it would be very inconvenient below to have to assume ordinarity. As explained by Milne [20, 2.7], the freeness of the group generated by the inverse roots also has consequences for the Tate conjecture. □

Remark 2.7. Since this may be useful in other investigations, we quote the analog of Proposition 2.4 when W_{2g} is replaced by the symmetric group $\mathfrak{S}_n$, $n \geqslant 2$. The proof is easier than the previous one (because the natural action of $\mathfrak{S}_n$ on sets of order n is doubly transitive), and, in fact, is contained in the works of Girstmair. □

Proposition 2.8. Let $k \geqslant 1$ and $n \geqslant 2$ be integers. Let $P_1, \dots, P_k$ be polynomials with rational coefficients of degree n such that $P = P_1 \cdots P_k$ has splitting field K with Galois group $\mathfrak{S}_n^k$. Let M be the set of complex roots of P, M_j that of P_j.

(1) We have

$$\mathrm{Rel}(M)_a = \bigoplus_{1 \leqslant j \leqslant k} \mathrm{Rel}(M_j)_a,$$

and for each j, we have either $\mathrm{Rel}(M_j)_a = 0$, or

$$\mathrm{Rel}(M_j)_a = \mathbf{1} = \mathbf{Q} \cdot \sum_{\alpha \in M_j} \alpha,$$

and the latter alternative holds if and only if, for any $\alpha \in M_j$, we have $\mathrm{Tr}_{K/\mathbf{Q}}(\alpha) = 0$.

(2) We have

$$\mathrm{Rel}(M)_m = \bigoplus_{1 \leqslant j \leqslant k} \mathrm{Rel}(M_j)_m,$$

and for each j, $\mathrm{Rel}(M_j)_m$ is one of the following:

$$0\,, \qquad m_j\mathbf{Z} \cdot \sum_{\alpha \in M_j} \alpha \qquad n_j\mathbf{Z}^{M_j}\,, \qquad m'_j \cdot \left\{ (n_\alpha) \mid \sum_\alpha n_\alpha = 0 \right\},$$

where $m_j \in \{1, 2\}$, $n_j \in \{3, 4, 6\}$, $m'_j \in \{2, 3\}$. The third case holds when M_j is the set of roots of unity of order n_j. The second case holds when the third one does not and $N_{K/\mathbf{Q}}(\alpha) = (-1)^{m_j - 1}$, i.e. when the $\alpha \in M_j$ are units, not roots of unity. The fourth case occurs when the two previous do not, and α satisfies a Kummer equation $\alpha^{m'_j} = \beta \in \mathbf{Q}^\times$, β not an m'_jth power of an integer. □

3 The Simplest Case: Proof of Proposition 1.1

We start with a proof of Proposition 1.1, although it is subsumed in Theorem 1.3, because we can quote directly from earlier results of the author on Galois groups of splitting fields of numerators of the zeta functions in those families of curves (we recall also that the first qualitative result on this topic is due to Chavdarov [2]). This means that we can avoid setting up anew the general sieve for Frobenius, and, in particular, we do not need to refer explicitly to the fairly sophisticated algebraic geometry that is involved.

Consider then a squarefree monic polynomial $f \in \mathbf{Z}[X]$ of degree $2g$ and an odd prime p not dividing the discriminant of f. Let $q \neq 1$ be a power of p. For each $t \in \mathbf{F}_q$ with $f(t) \neq 0$, we consider the (smooth projective model of the) hyperelliptic curve

$$C_t \ : \ y^2 = f(x)(x-t),$$

which is of genus g, so that the L-function $P_t \in \mathbf{Z}[T]$ of C_t, as defined in the introduction, has degree $2g$.

For a fixed q, we say that $t \in \mathbf{F}_q$ is *special* if any one of the following condition holds:

- We have $f(t) = 0$.
- The Galois group of the splitting field of P_t is not isomorphic to W_{2g} (which is the largest it can be because of the functional equation of the zeta function).
- The sum of the inverse roots $\alpha \in \mathcal{Z}(C_t)$ is 0.

Then, under the assumptions stated, it follows from Theorem 8.1 in [14] (see also [13, Theorem 6.2]) that

$$|\{t \in \mathbf{F}_q \mid t \text{ is special}\}| \ll q^{1-\gamma^{-1}}(\log q),$$

where $\gamma = 4g^2 + 2g + 4$ and the implied constant depends only on g. More precisely, those results only deal with the first two conditions (of which the second is, of course, the one that is significant), but the simplest type of sieve (or rather uniform Chebotarev density theorem) shows that

$$|\{t \in \mathbf{F}_q \mid f(t) \neq 0 \text{ and the sum of inverse roots of } P_t \text{ is zero}\}| \ll q^{1-\gamma^{-1}},$$

simply because it is an algebraic condition on the coefficients of the polynomial (see the proof of Theorem 1.3 for details in the general case $k \geqslant 2$).

Consider now any $t \in \mathbf{F}_q$ that is not special. We will show that the roots of the zeta function of C_t satisfy the two independence conditions in Proposition 1.1, and this will finish the proof, in view of the bound on the number of special parameters t.

Because it is fixed, we drop the dependency on t from the notation from now on, unless this creates ambiguity. The additive case is clear from the first part of Proposition 2.4 applied with $k = 1$, $m = q$, and

$$P = T^{2g} P_t(T^{-1}) \in \mathbf{Z}[T]$$

(which has the $\alpha \in \mathcal{Z}(C_t)$ as roots) since the splitting field K of this polynomial is the same as that of P_t; hence, its Galois group is indeed W_{2g}, and the sum of the roots of P is nonzero for t not special, by the very definition.

Now, we come to the multiplicative independence of the normalized inverse roots. Recall first that with $M = \tilde{\mathcal{Z}}(C_t)$, and involution given by

$$\bar{\alpha} = c(\alpha) = \frac{1}{\alpha},$$

the desired conclusion (1.7) can be rephrased as

$$\mathrm{Rel}(\tilde{\mathcal{Z}}(C_t))_m = \{(n_{\tilde{\alpha}}) \in \mathbf{Z}^M \mid n_{\tilde{\alpha}} - n_{\tilde{\alpha}^{-1}} = 0\},$$

and the left-hand side does contain the right-hand side, so only the reverse inclusion is required.

The elements of M are roots of the polynomial

$$Q_t = T^{2g} P_t(q^{-1/2} T^{-1}) \in \mathbf{Q}(\sqrt{q})[T],$$

which creates a slight complication: if (as seems natural) we extend scalars to $E = \mathbf{Q}(\sqrt{q})$ to have $Q_t \in E[T]$, there is a possibility that the Galois group of its splitting field (over E) is not W_{2g} anymore (e.g. when $\sqrt{q}$ is in the splitting field of P_t). We deal with this by looking at the squares of the inverse roots.

Let

$$M' = \{\tilde{\alpha}^2 \mid \tilde{\alpha} \in M = \tilde{\mathcal{Z}}(C_t)\} = \{\alpha^2/q \mid \alpha \in \mathcal{Z}(C_t)\};$$

the second expression shows that $M' \subset K = \mathbf{Q}(\mathcal{Z}(C_t))$, so the field $F = \mathbf{Q}(M')$ is a subfield of K. Its Galois group is the group of those $\sigma \in \mathrm{Gal}(K/\mathbf{Q})$ that fix all α^2 for $\alpha \in \mathcal{Z}(C_t)$, i.e.

such that $\sigma(\alpha) \in \{\alpha, -\alpha\}$ for all α. If $\sigma \in \mathrm{Gal}(K/F)$ is not the identity, there exists some $\alpha \in \mathcal{Z}(C_t)$ such that $\beta = \sigma(\alpha)$ is equal to $-\alpha$, and this leads to $\alpha + \beta = 0$, in particular, to $\mathrm{Rel}(\mathcal{Z}(C_t))_a \neq 0$. Since this contradicts the previous observation that the elements of $\mathcal{Z}(C_t)$ are $\mathbf{Q}$-linearly independent when t is not special, we have, in fact, $\mathrm{Gal}(K/F) = 1$, and so $F = K$.

We can now apply (2) of Proposition 2.4, with $k = m = 1$ and P taken to be the polynomial with zeros M', namely

$$\prod_{\gamma \in M'} (T - \gamma) = \prod_{\tilde{\alpha} \in M} (T - \tilde{\alpha}^2) \in \mathbf{Q}[T],$$

with $F = K$ such that $\mathrm{Gal}(F/\mathbf{Q}) = W_{2g}$, acting by permutation of the set M' with the involution

$$c(\gamma) = \gamma^{-1}, \quad \text{i.e.} \quad c(\tilde{\alpha}^2) = \tilde{\alpha}^{-2}.$$

Since $\gamma c(\gamma) = 1$ for all $\gamma \in M'$, we obtain

$$\mathrm{Rel}(M')_m \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{1} \oplus G(M') = \{(n_\gamma) \in \mathbf{Q}^{M'} \mid n_\gamma - n_{c(\gamma)} = 0, \quad \gamma \in M'\}$$

(see (2.5)).

Now note that since $\mathrm{Rel}(M')_m$ is free, the natural map $\mathrm{Rel}(M')_m \to \mathrm{Rel}(M')_m \otimes \mathbf{Q}$ is injective. Note also the tautological embedding $\mathrm{Rel}(M)_m \xrightarrow{i} \mathrm{Rel}(M')_m$ induced by the map $\mathbf{Z}^M \to \mathbf{Z}^{M'}$ that maps any basis vector $f_{\tilde{\alpha}}$ of $\mathbf{Z}^M$ to $f_{\tilde{\alpha}^2} \in \mathbf{Z}^{M'}$. If $m \in \mathrm{Rel}(M)_m$, we have

$$i(m) \in \{(n_\gamma) \in \mathbf{Q}^{M'} \mid n_\gamma - n_{c(\gamma)} = 0, \quad \gamma \in M'\}$$

and this means that $\mathrm{Rel}(M)_m = \mathrm{Triv}(M)_m$, as desired.

4 Application of the Sieve for Frobenius

We are now going to apply the sieve for Frobenius to produce extensions with Galois groups W_{2g}^k to which we can apply the results of Section 2 to prove Theorem 1.3 and related results.

For this, we need to generalize the estimate for nonmaximality of the Galois group used in the proof of Proposition 1.1 to situations involving W_{2g}^k. For this purpose,

we will again use sieve, and we first recall the main statement for completeness. We use the version from [14, Chapter 8] (the version in [13] would also suffice for our purposes), in the situation of a general higher-dimensional parameter space. However, we extend it slightly to allow tame ramification instead of prime-to-p monodromy (see the comments following the statement for a quick explanation if this is unfamiliar).

We will mention later on the (very small) improvements that can sometimes be derived when the parameter space is a product of curves.

Theorem 4.1. Let p be a prime number, $q \neq 1$ a power of p. Let $V/\mathbf{F}_q$ be a smooth affine geometrically connected algebraic variety of dimension $d \geqslant 1$. Assume that V can be embedded in $\mathbf{A}^N$ using r equations of degree $\leqslant \delta$, and assume also that $\bar{V}$ has a compactification for which it is the complement of a divisor with normal crossing so that the tame (geometric) fundamental group $\pi_1^t(\bar{V}, \eta_{\bar{v}})$ is defined. Let Λ be a set of primes $\ell \neq p$. For each $\ell \in \Lambda$, assume given a lisse sheaf $\mathcal{F}_\ell$ of $\mathbf{F}_\ell$-vector spaces, corresponding to an homomorphism

$$\rho_\ell \, : \, \pi_1(V, \eta_v) \to GL(r, \mathbf{F}_\ell),$$

which is tamely ramified, so that ρ_ℓ restricted to the geometric fundamental group factors through the tame quotient:

$$\pi_1(\bar{V}, \eta_{\bar{v}}) \to \pi_1^t(\bar{V}, \eta_{\bar{v}}) \to GL(r, \mathbf{F}_\ell).$$

Let G_ℓ, G_ℓ^g be the corresponding arithmetic and geometric monodromy groups, i.e.

$$G_\ell = \rho_\ell(\pi_1(V, \eta_v)), \qquad G_\ell^g = \rho_\ell(\pi_1(\bar{V}, \eta_{\bar{v}})) = \rho_\ell(\pi_1^t(\bar{V}, \eta_{\bar{v}})),$$

and assume that for any distinct primes ℓ, $\ell' \in \Lambda$, the map

$$\pi_1(\bar{V}, \eta_{\bar{v}}) \to G_\ell^g \times G_{\ell'}^g \tag{4.1}$$

is onto.

Let $\gamma_0 \in G_\ell/G_\ell^g$ be the element such that all the geometric conjugacy classes Fr_t map to γ_0 for $t \in V(\mathbf{F}_q)$, as in the short exact sequence

$$1 \to G_\ell^g \to G_\ell \to G_\ell/G_\ell^g \to 1.$$

Then, for any choices of subsets $\Omega_\ell \subset G_\ell$ such that the image of Ω_ℓ in G_ℓ/G_ℓ^g is $\{\gamma_0\}$, and for any $L \geqslant 2$, we have

$$|\{t \in V(\mathbf{F}_q) \mid \rho_\ell(\mathrm{Fr}_t) \notin \Omega_\ell \text{ for all } \ell \leqslant L\}| \leqslant (q^d + CL^A q^{d-1/2})H^{-1}, \tag{4.2}$$

where, π running below over irreducible representations of G_ℓ, we have

$$H = \sum_{\substack{\ell \leqslant L \\ \ell \in \Lambda}} \frac{|\Omega_\ell|}{|G_\ell^g| - |\Omega_\ell|}, \tag{4.3}$$

$$A \leq 1 + \max_{\ell \leq L}\left\{2\frac{\log|G_\ell|}{\log \ell} + \max_\pi \frac{\log \dim \pi}{\log \ell} + \sum_\pi \frac{\log \dim \pi}{\log \ell}\right\} \leq 1 + \frac{7}{2}\max_{\ell \leq L}\frac{\log|G_\ell|}{\log \ell}, \tag{4.4}$$

$$C = 12N2^r(3 + r\delta)^{N+1}. \tag{4.5}$$

□

Proof. The pieces are collected from [14, (8.11), Proposition 8.7], or the corresponding results in [13] (where there is an extraneous factor κ that can be removed as explained in [14]). The only difference is the assumption that the sheaves are tamely ramified instead of the geometric monodromy groups being of order prime to p. However, the proof goes through with this weaker assumption, because the only place this was used was in applying the multiplicativity of the Euler–Poincaré characteristic in a finite Galois étale cover of degree prime to the characteristic. This result of Deligne and Lusztig holds for tamely ramified covers more generally (see [6, 2.6, Corollaire 2.8]). ■

Remark 4.2. The generalization to tamely ramified sheaves is useful to avoid assuming that $p > 2g + 1$ when looking at families of curves to ensure that $Sp(2g, \mathbf{F}_\ell)$ has order prime to p (for instance, in Theorem 1.3). The difference between the two is that tame ramification of an homomorphism $\pi_1(\bar{V}, \eta_{\bar{v}}) \to GL(n, \mathbf{F}_\ell)$ (with $\ell \neq p$) only requires that the p-Sylow subgroups of the ramification groups at infinity act trivially on $\mathbf{F}_\ell^n$, whereas having geometric monodromy group of order prime to p means that the whole p-Sylow subgroup of the fundamental group acts trivially.

Note however that in Remark 5.4, we explain how one could also prove Theorem 1.3 using only ramification theory for curves. □

We derive from Theorem 4.1 a theorem generalizing the maximality of splitting fields to $k \geqslant 2$. Recall first that a family $(\mathcal{F}_\ell)$ of lisse sheaves of free $\mathbf{Z}_\ell$-modules on an algebraic variety $V/\mathbf{F}_q$ is a compatible system if, for any finite extension $\mathbf{F}_{q^v}/\mathbf{F}_q$, any $t \in V(\mathbf{F}_{q^v})$, the characteristic polynomial

$$\det(1 - \mathrm{Fr}_{q^v,t}\, T \mid \mathcal{F}_\ell) \in \mathbf{Z}_\ell[T]$$

is, in fact, in $\mathbf{Z}[T]$ and is independent of ℓ.

Theorem 4.3. Let p be a prime number, $q \neq 1$ a power of p, $g \geqslant 2$ and $k \geqslant 1$ integers. Let $V/\mathbf{F}_q$ be a smooth affine geometrically connected algebraic variety of dimension $d \geqslant 1$. Assume that V can be embedded in $\mathbf{A}^N$ using r equations of degree $\leqslant \delta$, and define the constant $C(N, r, \delta)$ as in equation (4.3). Assume also that $\bar{V}$ has a compactification for which it is the complement of a divisor with normal crossing, so that the tame geometric fundamental group $\pi_1^t(\bar{V}, \eta_{\bar{v}})$ is defined.

Let Λ be a set of primes $\ell \neq p$ with positive density, i.e. such that

$$\pi_\Lambda(L) = \sum_{\substack{\ell \leqslant L \\ \ell \in \Lambda}} 1 \gg \pi(L), \tag{4.6}$$

for $L \geqslant L_0$, the smallest element of Λ, the implied constant depending on Λ. For each $\ell \in \Lambda$, assume given on V a tamely ramified lisse sheaf $\tilde{\mathcal{F}}_\ell$ of free $\mathbf{Z}_\ell$-modules of rank $2kg$ with $Sp(2g)^k$ symmetry, i.e. given by representations

$$\tilde{\rho}_\ell \,:\, \pi_1(V, \eta_v) \to C\,Sp(2g, \mathbf{Z}_\ell)^k.$$

Let $\tilde{\mathcal{F}}_{j,\ell}$ be the lisse sheaves given by composition

$$\pi_1(V, \eta_v) \to C\,Sp(2g, \mathbf{Z}_\ell)^k \to C\,Sp(2g, \mathbf{Z}_\ell),$$

and assume that for each j, $1 \leqslant j \leqslant k$, the family $(\tilde{\mathcal{F}}_{j,\ell})_{\ell \in \Lambda}$ is a compatible system.

Then, $(\tilde{\mathcal{F}}_\ell)$ is also a compatible system; for $t \in V(\mathbf{F}_q)$, let

$$P_t = \det(1 - \tilde{\rho}_\ell(\mathrm{Fr}_t)T) \in \mathbf{Z}[T].$$

Assume that this system has maximal geometric monodromy modulo ℓ, in the sense that the geometric monodromy group G_ℓ^g of $\tilde{\mathcal{F}}_\ell/\ell\tilde{\mathcal{F}}_\ell$ is equal to $G_\ell^g = Sp(2g, \mathbf{F}_\ell)^k$ for all $\ell \in \Lambda$.

Then, we have

$$|\{t \in V(\mathbf{F}_q) \mid \text{ the splitting field of } P_t \text{ is not maximal}\}| \ll gc^k C^{2\gamma^{-1}} q^{d-\gamma^{-1}} (\log q) \tag{4.7}$$

where $\gamma = 29kg^2$, for some constant $c \geqslant 1$ depending only on g, where the implied constant depends only on Λ. Here, maximality for P_t means that the Galois group is isomorphic to W_{2g}^k.

Moreover, write $P_{j,t} = \det(1 - T\,\mathrm{Fr}_t \mid \tilde{\mathcal{F}}_{j,\ell})$; then, we also have

$$|\{t \in V(\mathbf{F}_q) \mid \text{the sum of inverse roots of some } P_{j,t} \text{ is zero}\}| \ll kC^{2\gamma^{-1}} q^{d-\gamma^{-1}}, \tag{4.8}$$

where the implied constant depends only on Λ. □

Proof. First, notice that we have immediately the factorization

$$\det(1 - \tilde{\rho}_\ell(\mathrm{Fr}_{q^\nu,t})T) = \prod_{1 \leqslant j \leqslant k} \det(1 - \tilde{\rho}_{j,\ell}(\mathrm{Fr}_{q^\nu,t})T),$$

for any $t \in \mathbf{F}_{q^\nu}$, $\nu \geqslant 1$, so that the compatibility of the systems $(\tilde{\mathcal{F}}_{j,\ell})_\ell$ implies that of $(\tilde{\mathcal{F}}_\ell)_\ell$, as stated. In particular, for $t \in V(\mathbf{F}_q)$, we write

$$P_t(T) = \prod_{1 \leqslant j \leqslant k} P_{j,t}(T), \quad \text{with} \quad P_{j,t} = \det(1 - \tilde{\rho}_{j,\ell}(\mathrm{Fr}_t)T).$$

Each $\tilde{\mathcal{F}}_{j,\ell}$ has maximal symplectic geometric monodromy modulo ℓ, since those monodromy groups are the images of the composite

$$\pi_1(\tilde{V}, \eta_{\tilde{v}}) \xrightarrow{\rho_\ell} Sp(2g, \mathbf{F}_\ell)^k \to Sp(2g, \mathbf{F}_\ell),$$

which are surjective (the first one by the maximal monodromy assumption on $\mathcal{F}_\ell$). In particular, the splitting field of $P_{j,t}$ over $\mathbf{Q}$ has Galois group isomorphic to a subgroup of W_{2g} (by the customary functional equation), and the splitting field of P_t over $\mathbf{Q}$ has Galois group isomorphic to a subgroup of W_{2g}^k. This justifies the interpretation of the maximality adjective in the statement of the theorem.

We now recall the basic facts that allow sieve methods to detect this type of maximality:

- For any $\ell \in \Lambda$, the reduction of P_t modulo ℓ is the characteristic polynomial of $\rho_\ell(\mathrm{Fr}_t)$.
- If a polynomial $Q \in \mathbf{Z}[T]$ of degree r is such that Q reduces modulo a prime ℓ to a squarefree polynomial (of degree r), which is the product of n_1 irreducible factors of degree $1, \dots, n_r$ irreducible factors of degree r, then as a subgroup of permutations of the roots of Q, the Galois group of the splitting field Q contains an element with cycle structure consisting of n_1 fixed points, n_2 disjoint 2-cycles, ...
- If a subgroup H of a finite group G has the property that $H \cap c \neq \emptyset$ for all conjugacy classes $c \subset G$, then, $H = G$.

Implementing this, let us first define a q-symplectic polynomial R (with coefficient in a ring B) to be a polynomial in $B[T]$ of even degree such that $R(0) = 1$ and

$$q^{(\deg P)/2} T^{(\deg P)} R\left(\frac{1}{qT}\right) = R(T),$$

which is, of course, the "functional equation" for $\det(1 - Tg)$ for any symplectic similitude with multiplicator q. In particular, and this is why we need the notion, the characteristic polynomials $\det(1 - \rho_{j,\ell}(\mathrm{Fr}_t)T)$ are q-symplectic.

In [14, Proof of Theorem 8.13], as in [13], we explicitly described four subsets $\tilde{\Omega}_{1,\ell}, \dots, \tilde{\Omega}_{4,\ell}$ of q-symplectic polynomials of degree $2g$ in $\mathbf{F}_\ell[T]$ such that a q-symplectic polynomial in $\mathbf{Z}[T]$ of degree $2g$ with nonmaximal splitting field satisfies $P \pmod{\ell} \notin \tilde{\Omega}_{i,\ell}$ for some i and all ℓ. From this, we construct the 4^k subsets

$$\tilde{\Omega}_{i,\ell} = \prod_{1 \leqslant j \leqslant k} \tilde{\Omega}_{i_j,\ell}, \qquad i = (i_1, \dots, i_k) \text{ with } i_j \in \{1, 2, 3, 4\},$$

of the set of q-symplectic polynomials of degree $2kg$ in $\mathbf{F}_\ell[T]$.

It may be the case (we do not know if this happens or not) that a $P \in \mathbf{Z}[T]$, which is q-symplectic of degree $2kg$ and splits as

$$P = P_1 \cdots P_k, \qquad P_j \in \mathbf{Z}[T], \ q\text{-symplectic of degree } 2g \tag{4.9}$$

(so that the Galois group of its splitting field is a subgroup of W_{2g}^k) has nonmaximal splitting field, but is not detected by those subsets (i.e. for all i, the factors P_j reduce modulo some ℓ to elements of $\tilde{\Omega}_{i,\ell}$): the only obvious consequence here of the case $k = 1$ is that the Galois group of the splitting field, as a subgroup of W_{2g}^k, surjects to each of the k-components W_{2g}.

We bypass this problem by adding a fifth subset $\tilde{\Omega}_{0,\ell}$ defined as

$$\tilde{\Omega}_{0,\ell} = \{ f \in \mathbf{F}_\ell[T] \mid \ f \text{ is } q\text{-symplectic and is a product of } 2g \text{ distinct linear factors} \}$$

(which, therefore, corresponds to the trivial element of a Galois group), and (re)define now $\tilde{\Omega}_{i,\ell}$ in the obvious way for i a k-tuple with entries in $\{0, 1, 2, 3, 4\}$. The point is that if a q-symplectic polynomial $P \in \mathbf{Z}[T]$ of degree $2kg$ factoring as above (4.9) has splitting field strictly smaller than W_{2g}^k, then, for some $i \in \{0, 1, 2, 3, 4\}^k$, we have

$$(P_j \pmod{\ell})_j \notin \tilde{\Omega}_{i,\ell}$$

for all primes ℓ. Indeed, arguing by contraposition, it would follow otherwise by using

$$i = (0, \dots, 0, i, 0, \dots, 0), \qquad 1 \leqslant i \leqslant 4,$$

(where the nonzero coordinate is the jth one, $1 \leqslant j \leqslant k$), and the case $k = 1$, that the Galois group, as a subgroup of W_{2g}^k, contains

$$1 \times \cdots \times 1 \times W_{2g} \times 1 \cdots \times 1,$$

where the W_{2g} occurs at the jth position. Consequently, the Galois group must be the whole of W_{2g}. In particular, we only need to use the $4k$ tuples described in this argument.

Now if we denote (with obvious notation for the multiplicator)

$$\Omega_{i,\ell} = \{ g \in CSp(2g, \mathbf{F}_\ell)^k, \ m(g) = (q, \dots, q), \ \det(1 - Tg) \in \tilde{\Omega}_{i,\ell} \}$$

for $\ell \in \Lambda$, then we see that the left-hand side, say, $N(L)$, of (4.7) is at most

$$N(L) \leqslant \sum_i |\{t \in V(\mathbf{F}_q) \mid \det(1 - T\rho_\ell(\mathrm{Fr}_t)) \notin \Omega_{i,\ell}, \text{ for } \ell \in \Lambda\}|$$

(where the sum ranges over the $4k$ tuples used before).

Each of the terms in this sum may be estimated by the sieve for Frobenius as in Theorem 4.1, provided the last assumption (4.1) is checked. Here, it means showing that

$$\pi_1(\bar{V}, \eta_{\bar{v}}) \to Sp(2g, \mathbf{F}_\ell)^k \times Sp(2g, \mathbf{F}_{\ell'})^k$$

is onto, for $\ell \neq \ell'$ in Λ, and this follows from Lemma 4.4 below, which is a variant of Goursat's lemma.

The outcome of the sieve for Frobenius is the upper bound

$$N(L) \leqslant 4k(q^d + C\,L^A q^{d-1/2})H^{-1}$$

for C given by (4.5) and

$$A \leqslant 29kg^2, \qquad H = \min_i \sum_{\substack{\ell \leqslant L \\ \ell \in \Lambda}} \frac{|\Omega_{i,\ell}|}{|Sp(2g, \mathbf{F}_\ell)|^k}.$$

The former, which is quite rough but good enough for our purpose, follows from the right-hand inequality in equation (4.4), together with the easy bound

$$|C\,Sp(2g, \mathbf{F}_\ell)| \leqslant (\ell + 1)^{2g^2+g+1},$$

(note that the better bounds for the dimension and sum of dimension of irreducible representations of G_ℓ, which are described in [14, Example 5.8(2)] could also be used, if one tried to optimize the value of A, e.g. for small values of g).

To obtain a lower bound for H, we recall from [14, Proof of Theorem 8.13] again that there exists a constant $c_g > 0$ (which could also be specified more precisely) such that, for $\ell \geqslant 3$ and $1 \leqslant i \leqslant 4$, we have

$$\frac{|\tilde{\Omega}_{i,\ell}|}{|Sp(2g, \mathbf{F}_\ell)|} \geqslant c_g,$$

while the same counting arguments lead also to

$$\frac{|\tilde{\Omega}_{0,\ell}|}{|Sp(2g,\mathbf{F}_\ell)|} \geqslant c'_g,$$

(see also [2, §3], [14, Appendix B]) for $\ell \geqslant 2g+1$, for some other constant c'_g (now extremely small, of the order of $|W_{2g}|^{-1}$). Replacing c_g by $\min(c_g, c'_g)$, we have

$$H \geqslant c_g^{-k}\pi_\Lambda(L) \gg c_g^{-k}\frac{L}{\log L},$$

by equation (4.6); this bound holds for $L > L_0$ and the implied constant depending only on Λ (L_0 can be taken as $\max(2g+1, \text{smallest element of } \Lambda)$).

The outcome is therefore that we have

$$N(L) \ll 4kc_g^{-k}(q^d + CL^A q^{d-1/2})(\log L)L^{-1},$$

for $L > L_0$, the implied constant depending only on Λ.

As usual, we select L so that

$$CL^A = q^{1/2}, \quad \text{i.e.} \quad L = (qC^{-2})^{1/(2A)},$$

if this is $> L_0$. This leads to

$$N(L) \ll 4kc_g^{-k}q^{d-1/(2A)}(\log q)C^{1/A},$$

where the implied constant depends only on Λ. This last inequality is trivial if $L \leqslant L_0$ if we take the implied constant large enough (indeed, if the implied constant is $\geqslant L_0 \geqslant 2g+1$), and so by doing so if necessary, we finish the proof of (4.7).

As for the proof of (4.8), it follows the same idea, but is much easier since we only need to "sieve" by a single well-chosen prime $\ell \in \Lambda$ (what is called "individual equidistribution" in [14], and is really the uniform explicit Chebotarev density theorem here, as in [16]). Indeed, the sum of inverse roots of some $P_{j,t}$ is zero if and only if the coefficient of T in $P_{j,t}$ is zero.

So, let $\tilde{\Upsilon}_\ell$ be the set of q-symplectic polynomials of degree $2g$ in $\mathbf{F}_\ell[T]$ where the coefficient of T is nonzero, and Υ_ℓ the set of matrices g in $CSp(2g, \mathbf{F}_\ell)$ with multiplicator q with $\det(1 - Tg) \in \Upsilon_\ell$. Then, the left-hand side of equation (4.8) is bounded by

$$\begin{aligned} M(\ell) &= |\{t \in V(\mathbf{F}_q) \mid P_{j,t} \pmod{\ell} \notin \tilde{\Upsilon}_\ell, \text{ for } 1 \leqslant j \leqslant k\}| \\ &= |\{t \in V(\mathbf{F}_q) \mid \rho_{j,\ell}(\mathrm{Fr}_t) \notin \Upsilon_\ell, \text{ for } 1 \leqslant j \leqslant k\}| \\ &\leqslant |\{t \in V(\mathbf{F}_q) \mid \rho_\ell(\mathrm{Fr}_t) \notin \Upsilon_\ell^k\}|, \end{aligned}$$

for any prime ℓ. It is clear from the counting results in [14, Appendix B] that we have

$$\frac{|\Upsilon_\ell|}{|Sp(2g, \mathbf{F}_\ell)|} = 1 + O(\ell^{-1}), \text{ and therefore, } \frac{|\Upsilon_\ell|^k}{|Sp(2g, \mathbf{F}_\ell)|^k} = 1 + O(k\ell^{-1})$$

for all $\ell \geqslant 3$, $\ell \geqslant k$, the implied constant depending only on g. Applying Theorem 4.1 with Λ replaced by $\{\ell\}$ for any fixed $\ell \in \Lambda$, we find

$$M(\ell) \leqslant (q^d + C\ell^A q^{d-1/2}) \left(1 - \frac{|\Upsilon_\ell|^k}{|Sp(2g, \mathbf{F}_\ell)|^k}\right) \ll k(q^d + C\ell^A q^{d-1/2})\ell^{-1},$$

for $\ell \geqslant k$, the implied constant depending only on g from which the proof of equation (4.8) finishes as before by choosing a value of ℓ in a dyadic interval around the value $(C^{-2}q)^{1/(2A)}$. ∎

Here is the group theoretic lemma we used in the proof.

Lemma 4.4. Let $k \geqslant 1$ be an integer, ℓ_1, ℓ_2 distinct odd primes. Let $G_1 = Sp(2g, \mathbf{F}_{\ell_1})$ and $G_2 = Sp(2g, \mathbf{F}_{\ell_2})$. If H is a subgroup of $G_1^k \times G_2^k$ that surjects to G_1^k and G_2^k under the two projection maps, then, in fact, $H = G_1^k \times G_2^k$. □

Proof. We can write $G_1^k \times G_2^k$ as a product of $2k$ factors, say B_j, $1 \leqslant j \leqslant 2k$. Moreover, for any i, j, $1 \leqslant i < j \leqslant 2k$, the projection $H \to B_i \times B_j$ is onto: this follows from the assumption if B_i and B_j are isomorphic (to G_1 or G_2), and from the usual Goursat lemma (as in [2, Proposition 5.1]) if B_i and B_j are not. Since moreover G_1 and G_2 are both equal to their commutator subgroups, the conclusion follows from [2, Lemma 5.2]. ∎

Remark 4.5. One can show that, for *any* compatible system $(\mathcal{F}_\ell)$ of lisse sheaves with $Sp(2g)^k$ monodromy, on a smooth curve over a finite field at least, there exist some compatible systems of lisse sheaves $(\mathcal{F}_{j,\ell})$, $1 \leqslant j \leqslant k$, such that the monodromy of $\mathcal{F}_{j,\ell}$ is $Sp(2g)$ and the representation ρ_ℓ associated with $\mathcal{F}_\ell$ is given, up to isomorphism, by

$$\rho_\ell(x) = (\rho_{j,\ell}(x))_{1\leqslant j\leqslant k}, \tag{4.10}$$

in terms of those associated with $\mathcal{F}_{j,\ell}$ (this amounts to a choice of orderings of the projections

$$p_j \,:\, Sp(2g)^k \to Sp(2g),$$

as ℓ varies, so that the sheaves $p_j(\mathcal{F}_\ell)$ are compatible, for $1 \leqslant j \leqslant k$). This is a consequence of Lafforgue's proof of the global Langlands correspondence over function fields: fix some $\ell_0 \neq p$ and define ρ_{j,ℓ_0}, so that the above formula is valid for ℓ_0; then, Lafforgue shows that there exist compatible systems $(\tilde{\rho}_{j,\ell})$ for which $\tilde{\rho}_{j,\ell} = \rho_{j,\ell_0}$ (see [17, Theorem VII.6, (v)], using the fact that the geometric monodromy of $\tilde{\rho}_{j,\ell}$ is $Sp(2g)$, hence this sheaf is irreducible). Define $\tilde{\rho}_\ell$ by the analog of (4.10); then, this compatible system (or its semisimplification) must be isomorphic to ρ_ℓ because they have same characteristic polynomials of Frobenius at all closed points.

After twisting to reduce the $CSp(2g)^k$ case to $Sp(2g)^k$, this means that the compatible systems considered in the theorem are very likely the most general ones with the given monodromy for smooth parameter spaces. It would be interesting to prove this directly and in general, but this structure is obvious in our applications, so we did not try to do this. □

Remark 4.6. This theorem is interesting in itself as a complement to the earlier results of [14, §8] and [13]: not only do most curves (in a family with large monodromy) have large Galois group, but their polynomial L-functions tend to be independent of each other. Note also that there are families of number fields that are pairwise linearly disjoint, but not globally disjoint (for instance, take $\mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt{3})$, $\mathbf{Q}(\sqrt{6})$, where the compositum is biquadratic, and not of degree 8), although if the Galois groups are perfect groups, pairwise disjointness does imply global disjointness (again by [2, Lemma 5.2]). Because W_{2g} is not perfect, Theorem 4.3 cannot be deduced directly from the cases $k = 1$, $k = 2$, and playing with intersections and inclusion/exclusion. □

Corollary 4.7. Let the data $(p, q, g, k, V/\mathbf{F}_q, N, r, \delta, d, \Lambda, (\tilde{\mathcal{F}}_\ell))$ be as in Theorem 4.3 above. For $t \in V(\mathbf{F}_q)$, let $\mathcal{Z}_t$ be the set of α such that

$$\det(1 - T\,\mathrm{Fr}_t \mid \tilde{\mathcal{F}}_\ell) = \prod_{\alpha \in \mathcal{Z}_t} (1 - \alpha T),$$

and let $\tilde{\mathcal{Z}}_t$ be the set of $\alpha/\sqrt{q}$ for $\alpha \in \mathcal{Z}_t$. Let C be the constant defined in *(4.5)*.

Then, we have

$$|\{t \in V(\mathbf{F}_q) \mid \mathrm{Rel}(\mathcal{Z}_t)_a \neq 0\}| \ll gc^k C^{2\gamma^{-1}} q^{d-\gamma^{-1}} (\log q),$$

and

$$|\{t \in V(\mathbf{F}_q) \mid \mathrm{Rel}_0(\tilde{\mathcal{Z}}_t)_m \neq 0\}| \ll gc^k C^{2\gamma^{-1}} q^{d-\gamma^{-1}} (\log q)$$

for some constant $c \geqslant 1$ depending only on g, where $\gamma = 29gk^2$ and the implied constant depends only on Λ. □

Proof. As in the proof of Proposition 1.1, and with notation as in the statement of Theorem 4.3, let us call *special* any $t \in V(\mathbf{F}_q)$ such that:

- The splitting field of $\det(1 - T\,\mathrm{Fr}_t \mid \tilde{\mathcal{F}}_\ell) \in \mathbf{Z}[T]$ (which is independent of ℓ) has Galois group W_{2g}^k.
- For some j, $1 \leqslant j \leqslant k$, the sum of the inverse roots of $P_{j,t}$ is zero.

  By (4.7) and (4.8), we have

  $$|\{t \in V(\mathbf{F}_q) \mid t \text{ is special}\}| \ll gc^k C^{2\gamma^{-1}} q^{d-\gamma^{-1}} (\log q),$$

  where $\gamma = 29kg^2$, for some constant $c \geqslant 1$ depending only on g where the implied constant depends only on Λ.

Now, arguing exactly as in the proof of Proposition 1.1 in Section 3, using Proposition 2.4 (the first part of which reduces the general case of arbitrary k to that of $k = 1$ by excluding "cross-relations"), we find that if t is not special, then there is no $\mathbf{Q}$-linear dependency relation among the $\alpha \in \mathcal{Z}_t$, and also that the only multiplicative relations among the $\tilde{\alpha} \in \tilde{\mathcal{Z}}_t$ are the obvious ones, which concludes the proof. ■

5 Proof of Theorem 1.3

We can now prove Theorem 1.3 by direct applications of the results of the previous section. First, we state a lemma concerning fundamental groups that seems to be well known, but for which we did not find a reference in the literature. (It also holds in much greater generality certainly, but we simply state what we need.) The argument of the proof was suggested by Liu.

Lemma 5.1. Let U, V be smooth affine-connected schemes of finite type over the algebraic closure k of a finite field. Fix a geometric point η of $U \times V$, and let η', η'' be its images in U and V, respectively. Then, the natural map

$$\pi_1(U \times_k V, \eta) \xrightarrow{\varphi} \pi_1(U, \eta') \times \pi_1(V, \eta'')$$

is surjective. □

Proof. We suppress the base points, which are fixed, for simplicity. It suffices to show that the image Π of the map is dense in $\pi_1(U) \times \pi_1(V)$, since Π is closed (φ is continuous and the fundamental groups are compact). This means that for any open set $W \subset \pi_1(U) \times \pi_1(V)$, we must show that $\Pi \cap W \neq \emptyset$. Since we have the product topology on the target, we may assume that $W = W_1 \times W_2$, where $W_1 \subset \pi_1(U)$, $W_2 \subset \pi_1(V)$, are open. The profinite topology of the fundamental groups is also such that a basis of open sets are those of the form $W_i = x_i G_i$, where x_i is arbitrary and G_i is a normal subgroup of finite index. Thus, we must show that there exists $\sigma \in \Pi$, which is congruent to x_1 modulo G_1 and to x_2 modulo G_2, i.e. $p_i(\sigma) = x_i \pmod{G_i}$, where

$$p_1 : \pi_1(U) \to \pi_1(U)/G_1 = H_1, \qquad p_2 : \pi_1(V) \to \pi_1(V)/G_2 = H_2$$

are the two projections. If we let E_1 (respectively, E_2) denote the connected étale cover of U (respectively, V) associated with G_1 (respectively, G_2), this means that we must find $\sigma \in H$ that acts like x_1 on $E_1 \to U$ and like x_2 on $E_2 \to V$.

However, let $E = E_1 \times_k E_2$. Because k is algebraically closed, E is a connected Galois covering of $U \times_k V$ with Galois group $H_1 \times H_2$, hence there is a surjective homomorphism

$$\pi_1(U \times V) \to H_1 \times H_2,$$

and $\sigma = \varphi(\sigma')$ will work for any $\sigma' \in \pi_1(U \times V)$, which maps to $(x_1 \pmod{G_1}, x_2 \pmod{G_2})$ under this homomorphism. ■

Remark 5.2. It is not the case that the map in Lemma 5.1 is injective in general. There are issues of wild ramification in positive characteristic that prevent this, see [27, Exposé X, Remarques 1.10], for examples, (even for $U = V$ the affine line). However, the prime-to-p parts of $\pi_1(U \times V)$ and $\pi_1(U) \times \pi_1(V)$ are isomorphic (see [27, Exposé XIII, Proposition 4.6], under assumptions of existence of resolution of singularity, and [22] in general). More generally, the latter paper shows that there is isomorphism for the tame fundamental group (when this is defined). □

Proof of Theorem 1.3. We will apply Theorem 4.3 with $V = U^k$, where U is the complement of the set of zeros of the squarefree polynomial f defining the family of hyperelliptic curves. The geometric parameters for V are given by $N = 2k$, $r = k$, and $\delta = 2g + 1$, since we can embed U^k in $\mathbf{A}^{2k}$ (with coordinates (x_j, y_j)) using the k equations

$$x_j f(y_j) = 1, \qquad 1 \leqslant j \leqslant k.$$

Thus, the constant C in (4.5) satisfies

$$C \leqslant 24(2g+1)2^k(3 + (2g+2)k)^{2k+1}$$

(notice that this constant grows superexponentially in terms of k, but it will be raised to a very small power later on; going back to the original proof of the large sieve inequality in this particular case, one can replace this constant by one that grows "only" exponentially, see Remark 5.4; the improvements on the final results are barely visible).

Since $\bar{U}$ is the complement of $2g+1$ points in the projective line $\mathbf{P}^1/\bar{\mathbf{F}}_q$, $\bar{V}$ is the complement of $(2g+1)^k$ coordinate hyperplanes in $\mathbf{P}^k/\bar{\mathbf{F}}_q$, which forms a divisor with normal crossings, so that the tame fundamental group is well defined for $\bar{V}$.

Let $f : \mathcal{C} \to U$ be the morphism defining the (compactified) family of curves, which, we recall, are given by the affine equations

$$C_t \ : \ y^2 = f(x)(x - t),$$

and let

$$p_j \ : \ V \to U, \qquad 1 \leqslant j \leqslant k,$$

denote the coordinate projections. We use the family of sheaves

$$\tilde{\mathcal{F}}_\ell = \bigoplus_{1\leqslant j\leqslant k} p_j^* R^1 f_! \mathbf{Z}_\ell,$$

for $\ell \in \Lambda$, the set of odd primes $\neq p$. By construction, the associated sheaves $\tilde{\mathcal{F}}_{j,\ell}$ are each copies of $R^1 f_! \mathbf{Z}_\ell$, and hence, they form compatible systems of lisse sheaves of free $\mathbf{Z}_\ell$-modules of rank $2g$, in fact, with

$$\det(1 - T \operatorname{Fr}_{q^\nu,t} \mid R^1 f_! \mathbf{Z}_\ell) = P_{C_t}(T) \in \mathbf{Z}[T], \qquad \text{for } \nu \geqslant 1,\ t \in U(\mathbf{F}_{q^\nu}).$$

Each $R^1 f_! \mathbf{Z}_\ell$, for $\ell \geqslant 3$, $\ell \neq p$, corresponds to a homomorphism

$$\rho'_\ell : \pi_1(U, \eta_U) \to CSp(2g, \mathbf{Z}_\ell),$$

which is tamely ramified (see [12, Lemma 10.1.12]) the symplectic structure coming from Poincaré duality for curves. In turn, $\mathcal{F}_\ell$ is also tamely ramified. Indeed, the corresponding homomorphism, restricted to the geometric fundamental group, factors as follows:

$$\pi_1(\bar{V}, \eta_{\bar{V}}) \to \pi_1(\bar{U}, \eta_{\bar{U}})^k \to \pi_1^t(\bar{U}, \eta_{\bar{U}})^k \to Sp(2g, \mathbf{Z}_\ell)^k,$$

and it is essentially tautological (this amounts to saying that $V \xrightarrow{p_j} U$ induces an homomorphism on the respective tame fundamental groups) that for all j, the jth component homomorphism

$$\pi_1(\bar{V}, \eta_{\bar{V}}) \to \pi_1^t(\bar{U}, \eta_{\bar{U}})$$

also factors through $\pi_1^t(\bar{V}, \eta_{\bar{V}})$; consequently, the original homomorphism also factors through the tame fundamental group of $\bar{V}$.

From all this, it follows that $(\tilde{\mathcal{F}}_\ell)_{\ell\in\Lambda}$ is a compatible system of free $\mathbf{Z}_\ell$-modules of rank $2kg$, which is tamely ramified, and such that we have

$$\det(1 - T \operatorname{Fr}_{\boldsymbol{t}} \mid \tilde{\mathcal{F}}_\ell) = \prod_{1\leqslant j\leqslant k} P_{C_{t_j}}(T)$$

for any $\boldsymbol{t} = (t_1, \dots, t_k) \in V(\mathbf{F}_q)$.

To compute the geometric monodromy group of $\mathcal{F}_\ell = \tilde{\mathcal{F}}_\ell/\ell\tilde{\mathcal{F}}_\ell$, we appeal to Lemma 5.1 and induction to ensure that we have a surjective homomorphism

$$\pi_1(\bar{V}, \eta_{\bar{v}}) \xrightarrow{\prod p_{j*}} \pi_1(\bar{U}, \eta_{\bar{v}})^k \tag{5.1}$$

and we observe that the representation ρ_ℓ corresponding to $\mathcal{F}_\ell$ factors as

$$\pi_1(V, \eta_v) \xrightarrow{\prod p_{j*}} \pi_1(U, \eta_v)^k \to CSp(2g, \mathbf{F}_\ell)^k, \tag{5.2}$$

the last homomorphism being $(\rho'_\ell, \dots, \rho'_\ell)$, where ρ'_ℓ corresponds to the sheaf $R^1 f_! \mathbf{F}_\ell$ on U.

Then, we invoke (as in [2], [13], and [14] for $k = 1$) the remarkable theorem of Yu according to which the image of ρ'_ℓ restricted to $\pi_1(\bar{U}, \eta_{\bar{v}})$ (i.e. the geometric monodromy group modulo ℓ) is equal to $Sp(2g, \mathbf{F}_\ell)$ for all odd primes (Hall [5] has given another proof, whereas Yu's proof is unpublished). This together with equations (5.2) and (5.1) immediately implies that the geometric monodromy group G_ℓ^g of $\mathcal{F}_\ell$ is $Sp(2g, \mathbf{F}_\ell)^k$, as needed to apply Theorem 4.3.

We note also that the value of C above, and $\gamma = 29kg^2$, leads by trivial bounds to

$$C^{2\gamma^{-1}} \ll k^{(4g^2)^{-1}},$$

for $g \geqslant 1$, $k \geqslant 1$, with an absolute implied constant. Applying Corollary 4.7, we find that the number of $\boldsymbol{t} \in V(\mathbf{F}_q)$ for which either $\mathrm{Rel}(\mathcal{Z}(\boldsymbol{C}_{\boldsymbol{t}}))_a \neq 0$ or $\mathrm{Rel}_0(\tilde{\mathcal{Z}}(\boldsymbol{C}_{\boldsymbol{t}}))_m \neq 0$ is at most

$$\ll gc^k k^{(4g^2)^{-1}} q^{k-\gamma^{-1}} (\log q) \ll c_1^k q^{k-\gamma^{-1}} (\log q)$$

for any $c_1 > c$, where the implied constants depends only on g. This concludes the proof of Theorem 1.3. ∎

It is clear that, *mutatis mutandis*, we have proved the following more general statement instead of Theorem 1.3.

Proposition 5.3. Let p be a prime number, $q \neq 1$ a power of p, and $k \geqslant 1$ integers. Let $U_1, \dots, U_k$ be smooth affine curves over $\mathbf{F}_q$ and

$$\mathcal{C}_j \xrightarrow{f_j} U_j$$

families of smooth projective curves of genus $g_j \geqslant 1$ such that, for some set Λ of primes of positive density, the geometric monodromy of $R^1 f_{j,!}\mathbf{F}_\ell$ is $Sp(2g_j, \mathbf{F}_\ell)$ for $\ell \in \Lambda$. Let $U = U_1 \times \cdots \times U_k$.

Then, with obvious notation, we have

$$|\{\boldsymbol{t} \in U(\mathbf{F}_q) \mid \mathrm{Rel}(\mathcal{Z}(\boldsymbol{C}_{\boldsymbol{t}}))_a \neq 0\}| \ll c^k q^{k-\gamma^{-1}}(\log q),$$
$$|\{\boldsymbol{t} \in U(\mathbf{F}_q) \mid \mathrm{Rel}_0(\tilde{\mathcal{Z}}(\boldsymbol{C}_{\boldsymbol{t}}))_m \neq 0\}| \ll c^k q^{k-\gamma^{-1}}(\log q),$$

where $\gamma = 29(g_1^2 + \cdots + g_k^2) > 0$ for some constant $c \geqslant 1$ depending only on $(g_1, \dots, g_k)$. In both estimates, the implied constant depends only on Λ, $(g_1, \dots, g_k)$ and the Euler–Poincaré characteristic of the curves $\bar{U}_i$. □

Remark 5.4. We explain now how to replace the constant C in (4.2) by a smaller one in the case above where $V = U^k$ with U a smooth affine curve, complement of the zeros of a polynomial f of degree m in the affine line.

More precisely, in Theorem 4.1, suppose that V is of this type. Let p_i, $1 \leqslant i \leqslant k$, denote the ith coordinate map $V \to U$. Assume then that we have sheaves $(\mathcal{G}_\ell)$ on the curve U that arise by reduction modulo ℓ from a compatible system $(\tilde{\mathcal{G}}_\ell)_\ell$ such that the sheaves $\mathcal{F}_\ell$ are given by

$$\mathcal{F}_\ell = \bigoplus_{1 \leqslant j \leqslant k} p_j^* \mathcal{G}_\ell$$

(note that it is not necessary here to assume that the sheaves are tamely ramified, but they must form a compatible system, which is not assumed in Theorem 4.1). Let ρ_ℓ (respectively, τ_ℓ) be the representations of $\pi_1(V, \eta_v)$ associated to $\mathcal{F}_\ell$ (respectively, $\mathcal{G}_\ell$).

From the proof of the large sieve inequality and the setting of the sieve for Frobenius, a bound for C derives from a uniform estimate for the "exponential sums"

$$S(\pi, \pi') = \sum_{\boldsymbol{t} \in V(\mathbf{F}_q)^k} \mathrm{Tr}(\pi(\rho_\ell(\mathrm{Fr}_{\boldsymbol{t}})))\overline{\mathrm{Tr}(\pi'(\rho_{\ell'}(\mathrm{Fr}_{\boldsymbol{t}})))}$$

for primes $\ell, \ell' \in \Lambda$ and irreducible representations π (respectively, π') of G_ℓ (respectively, $G_{\ell'}$); see [14, §2.2; Proposition 2.9; §8.3].

For sheaves of the above type, the monodromy group G_ℓ of $\mathcal{F}_\ell$ is clearly isomorphic to H_ℓ^k, where H_ℓ is the monodromy group of $\mathcal{G}_\ell$. Correspondingly, the representations π

and π' factor as external tensor products

$$\pi = \boxtimes_{1\leqslant j\leqslant k}\pi_j, \qquad \pi' = \boxtimes_{1\leqslant j\leqslant k}\pi'_j,$$

where π_j (respectively, π'_j) are uniquely defined irreducible representations of G_ℓ (respectively, $G_{\ell'}$), and since $\mathrm{Fr}_{\boldsymbol{t}} = (\mathrm{Fr}_{t_1}, \dots, \mathrm{Fr}_{t_k})$, the exponential sum itself factors (cohomologically speaking, this reflects the Künneth formula for the groups $H^i_c(\bar{V}, \pi(\mathcal{F}_\ell) \otimes \pi'(\mathcal{F}_{\ell'}))$, where the tensor product is the external one if $\ell \neq \ell'$, which occur after applying the Grothendieck–Lefschetz trace formula directly to $S(\pi, \pi')$) as

$$S(\pi, \pi') = \prod_{1\leqslant j\leqslant k} \sum_{t\in U(\mathbf{F}_q)} \mathrm{Tr}(\pi_j(\tau_\ell(\mathrm{Fr}_t)))\overline{\mathrm{Tr}(\pi'(\tau_{\ell'}(\mathrm{Fr}_t)))},$$

where each term is now a 1-variable sum of the type discussed for the large sieve on a parameter curve. Using the bounds in [14, Propositions 8.6(2) and 8.7], it is easy to deduce that the constant C in equation (4.3) may be replaced with

$$C' = (1 - \chi_c(\bar{U}) + mw)^k,$$

where w is the sum of Swan conductors of $\mathcal{F}$ at the points at infinity (see [13, §4] for the definition; it vanishes in the case of tame ramification). Thus, we obtain a bound that "only" grows exponentially in k. However, this turns out to be a fairly inconsequential gain in the applications in this paper at least. □

6 Examples of Relations among Zeros

In this section, we wish to give explicit examples of L-functions over finite fields where the (inverse) roots satisfy some multiplicative relations (for additive relations, see Remark 7.3). Numerically, we tried to find such relations by looking (using GP's function `lindep`) for "small" dependency relations between the components of the vectors $(\pi, \theta_1, \dots, \theta_g)$, where $\pm\theta_j \in [0, 2\pi[$ are the arguments of the $2g$ inverse roots considered. It is easy to confirm rigorously a relation obtained this way, since all numbers involved are algebraic (but, on the other hand, if some of the large relations found by `lindep` are genuine, we have missed them . . .).

It is interesting to remark here that in the case of linear relations between roots of unrestricted rational polynomials, Berry, Dubickas, Elkies, Poonen, and Smyth [1] have

found for any integer $n \geqslant 1$ what is the largest degree $d = d(n)$ for which there exists an algebraic number α of degree d over $\mathbf{Q}$ such that its conjugates span a $\mathbf{Q}$-vector space of dimension n; in fact, they show that $d(n)$ is the same as the maximal order of a finite subgroup of $GL(n, \mathbf{Q})$, and then, invoke results of Feit, Weisfeiler—which depend on the classification of finite simple groups—that give this value. As we already recalled at the beginning of Section 2, except for seven exceptional cases, such a group is isomorphic to W_{2n}, so that $d(n) = 2^n n!$. Among the remaining cases, for instance, we have $d(4) = 1152$. There are also similar (less complete) results for multiplicative relations.

Example 6.1. We started by looking at purely numerical examples using previous computations of roughly $160,000$ zeta functions of hyperelliptic curves of genus 3 in two particular families of the type occurring in Theorem 1.3 (computed using Magma [18], see [14, End of §8.6]), over fields $\mathbf{F}_{5^k}$, $k \leqslant 8$. We had found only about 50 nonirreducible L-functions, and among these, only three curves over $\mathbf{F}_{5^8}$ in the family

$$y^2 = (x^2 + 6x - 1)(x - t),$$

which have irreducible polynomial L-functions (of degree 6) having Galois groups the dihedral group D_{12}. However, upon examination of the roots, it turns out that there are no nontrivial relations (although there certainly exist self-reciprocal polynomials with this Galois group and some interesting multiplicative relations).

This confirms, of course, the "genericity" of the independence of the roots, and suggests that the upper bounds in Proposition 1.1 are far from the truth (however, we only did very spotty checks for relations involving multiple zeta functions, i.e. corresponding to $k \geqslant 2$). □

Example 6.2. In view of the lack of success of the previous item, a natural way to try to construct examples without looking at curves directly is to use the fact that for (most) choice of polynomial P satisfying the functional equation (with respect to a power of prime $q \neq 1$) and Riemann Hypothesis, there exists, if not an algebraic curve C, at least an abelian variety $A/\mathbf{F}_q$ where the L-function (more precisely, the reversed characteristic polynomial of the geometric Frobenius acting on $H^1(\bar{A}, \mathbf{Z}_\ell)$, which we call the L-function to simplify) is exactly given by this polynomial. This is due to Honda and Tate (see [28]) and allows us to simply look for polynomials with roots satisfying nontrivial relations.

One simple way to do this is to consider $q = p$ and take a polynomial that splits as a product

$$\prod_{1 \leqslant j \leqslant g} (1 - a_j T + pT^2),$$

where $a_j \in \mathbf{Z} - \{0\}$ (to avoid ordinarity issues) satisfies $|a_j| < 2\sqrt{p}$. Honda–Tate theory then implies that this polynomial *is* the L-function for some abelian variety $A/\mathbf{F}_p$ of dimension g, which is, in fact, isogenous to the product of the elliptic curves corresponding to the factors $1 - a_j T + pT^2$. Since the inverse roots α_j, β_j with

$$\prod_{1 \leqslant j \leqslant g} (1 - a_j T + pT^2) = \prod_{1 \leqslant j \leqslant g} (1 - \alpha_j T)(1 - \beta_j T)$$

are given by

$$\alpha_j = \frac{a_j + i\sqrt{4p - a_j^2}}{2}, \qquad \beta_j = \frac{a_j - i\sqrt{4p - a_j^2}}{2},$$

one can try to select p and a_j, so that the quadratic fields $\mathbf{Q}(i\sqrt{4p - a_j^2})$ are identical for all j; this locates all $2g$ roots in the same imaginary quadratic field, and one may hope for nontrivial relations. Of course, we can take $a_j = a$ for all j, but this is cheating, and similarly, using signs $a_j = \pm a$ leads to factors that are all geometrically isomorphic elliptic curves. More interestingly, one should look for a_j's with distinct absolute values, so that A becomes a product of g pairwise nonisogenous elliptic curves.

This can happen, but this type of behavior is actually pretty restricted: we need to find distinct a_j's, and integers f_j, such that

$$4p = a_j^2 + d f_j^2, \qquad 1 \leqslant j \leqslant g,$$

for a common squarefree value of d. This means that

$$p = N_{\mathbf{Q}(\sqrt{-d})/\mathbf{Q}}\left(\frac{a_j}{2} + \frac{f_j\sqrt{-d}}{2}\right),$$

and by standard properties of quadratic fields, the ideal $\mathfrak{a}$ generated by $w_j = \frac{a_j}{2} + \frac{f_j\sqrt{-d}}{2}$ in the ring of integers of $\mathbf{Q}(\sqrt{-d})$ is unique up to conjugation (this w_j is necessarily an integer because its norm (p) and its trace (a_j) are). The only way to obtain distinct values

is therefore to replace w_j by some other generator of $\mathfrak{a}$, i.e. by εw_j, where $\varepsilon \in \mathbf{Q}(\sqrt{-d})$ is a unit. If $\mathbf{Q}(\sqrt{-d})$ is of discriminant $\neq -4, -3$, only $-w_j$ is permitted, which simply amounts to replacing a_j by $-a_j$. So, the interesting possibilities are when $d = 1$ or $d = 3$.

In the first case, the units are $\pm 1, \pm i$, and if we write $p = N_{\mathbf{Q}(i)/\mathbf{Q}}(a/2 + ib/2)$, then besides $a_1 = |a|$, we can take $a_2 = |b|$ to obtain the two distinct positive solutions. Note, moreover, that this is possible if and only if $p \equiv 1 \pmod 4$ by Fermat's theorem on primes that are sums of two squares.

In the second case where $d = 3$, which can occur if and only if $4p$ is of the form $a^2 + 3b^2$, i.e. if and only if $p \equiv 1 \pmod 3$, there are six units, equal to $\pm 1, \pm j, \pm j^2$, where $j = (-1 + i\sqrt{3})/2$. Writing

$$p = N_{\mathbf{Q}(\sqrt{-3})/\mathbf{Q}}\left(\frac{a}{2} + \frac{b\sqrt{-3}}{2}\right),$$

with $a \geqslant 1$, $b \geqslant 1$ integers, and multiplying by j and j^2, we find that there are three possible (positive) values for a, namely

$$a, \quad \frac{a+3b}{2}, \quad \frac{|a-3b|}{2}.$$

Note that in passing the following amusing property: if those three values (say x, y, z) are ordered, so that $x < y < z$, then we have $z = x + y$. Indeed, this amounts to the identities

$$\begin{aligned} \frac{a+3b}{2} + \frac{a-3b}{2} &= a, \quad \text{if } a > 3b, \\ \frac{3b-a}{2} + a &= \frac{a+3b}{2}, \quad \text{if } a < 3b. \end{aligned}$$

Here is a simple example for $d = 3$, with $g = 3$, $p = 541$ (the 100th prime); we find that the three values of a are $a_1 = 17$, $a_2 = 29$, $a_3 = 46$, and, indeed, we have

$$4p - a_1^2 = 1875 = 3 \times 5^4, \quad 4p - a_2^2 = 1323 = 3^3 \times 7^2, \quad 4p - a_3^2 = 48 = 3 \times 2^4,$$

so the corresponding inverse roots are

$$\alpha_1 = \frac{17 + 25i\sqrt{3}}{2}, \quad \alpha_2 = \frac{29 + 21i\sqrt{3}}{2}, \quad \alpha_3 = \frac{46 + 4i\sqrt{3}}{2},$$

in $\mathbf{Q}(\sqrt{-3})$. If we let $\tilde{\alpha}_j = \alpha_j/\sqrt{p}$, then the reader will easily check that we have the relation

$$\tilde{\alpha}_1^2\tilde{\alpha}_2^{-4}\tilde{\alpha}_3^2 = 1.$$

□

Example 6.3. Another type of examples can be obtained from the work of Katz [10] on G_2-equidistribution for some families of exponential sums. Precisely, for $p \neq 2, 7$, consider the exponential sums defined by

$$S_m(t) = \sum_{x \in \mathbf{F}_{q^m}^\times} \chi_2(N_{\mathbf{F}_{q^m}/\mathbf{F}_q}(x)) e\left(\frac{\mathrm{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_p}(x^7 + tx)}{p}\right), \qquad m \geqslant 1, \quad t \in \mathbf{F}_q, \quad q = p^\nu,$$

where χ_2 is the quadratic character of $\mathbf{F}_q$. Katz shows that it has the property that, for $t \in \mathbf{F}_q$, the zeta function

$$\exp\left(\sum_{m \geqslant 1} S_m(t)\frac{T^m}{m}\right)$$

is a polynomial of degree 7 in $\mathbf{Z}[\zeta_7][T]$, where ζ_7 is a primitive mth root of unity, and that when properly normalized by dividing by $(-G)^m$, where G is the Gauss sum given by

$$G = \sum_{x \in \mathbf{F}_q^\times} \chi_2(x) e\left(-7\frac{\mathrm{Tr}_{\mathbf{F}_q/\mathbf{F}_p}(x)}{p}\right),$$

it is the characteristic polynomial of a semisimple matrix in $SO(7, \mathbf{C})$, which lies in a conjugate of the exceptional group G_2. By the known structure of a maximal torus in such a group (as explained in [10, (5.5)]), its inverse roots are of the form

$$(1, \tilde{\alpha}, \tilde{\beta}, \tilde{\alpha}\tilde{\beta}, \tilde{\alpha}^{-1}, \tilde{\beta}^{-1}, (\tilde{\alpha}\tilde{\beta})^{-1}), \tag{6.1}$$

and we see clearly some interesting relations.

Performing the computations (with Magma) for $p = 5$, $t = 1$, we obtain that the inverse roots are $\sqrt{5}$, and numbers given approximately by

$$\alpha = 1.80901699437494742410229341 - i \cdot 1.31432778029783401506417271, \quad 5/\alpha = \bar{\alpha},$$
$$\beta = -1.22569983594963888407429447 5 + i \cdot 1.87020317403030527715765010 5, \quad 5/\beta = \bar{\beta},$$
$$\gamma = 0.10766584719974403586970764 07 - i \cdot 2.23347443803298572010538348 3, \quad 5/\gamma = \bar{\gamma},$$

the first two of which are roots of

$$P_1 = X^4 - 5X^3 + 15X^2 - 25X + 25$$

(which is Galois over $\mathbf{Q}$ with cyclic group $\mathbf{Z}/4\mathbf{Z}$), while the other four are roots of

$$P_2 = X^8 + 5X^6 - 20X^5 + 5X^4 - 100X^3 + 125X^2 + 625,$$

which has (non-abelian) splitting field of degree 16 over $\mathbf{Q}$ (the Galois group is generated by the permutations (1 2 6 5)(3 7 4 8) and (2 4)(3 5), for some ordering of the roots). Corresponding to the pattern (6.1), one finds that

$$\frac{\alpha}{\sqrt{5}} \cdot \frac{\beta}{\sqrt{5}} \cdot \frac{\gamma}{\sqrt{5}} = 1.$$

Note that one finds that there are four roots (not related by inversion), say, x, y, z, t, of P_2 that satisfy a relation $x^{-1}y^3zt^{-3} = 1$. So, it would be interesting to know if this polynomial P_2 corresponds to an algebraic curve of genus 4 over $\mathbf{F}_5$ (experimentally, what would be the number of points of this curve over $\mathbf{F}_{5^n}$, i.e.

$$5^n + 1 - (x^n + y^n + z^n + t^n + x^{-n} + y^{-n} + z^{-n} + t^{-n})$$

are nonnegative integers, as they should; the sequence starts 6, 36, 66, 596, 3126,..., and only the first two terms could have been negative). □

Example 6.4. Other systematic investigations can be done in cases where the zeta functions of families of curves are explicitly known, or computable with easily available tools. We first discuss briefly some examples related to modular curves (see the next example for the case of Fermat curves).

Let $N \geqslant 1$ be an integer, and consider the modular curve $X_0(N)$ over the finite field $\mathbf{F}_p$ for some $p \nmid N$. From Eichler–Shimura theory and Atkin–Lehner theory, the polynomial L-function of $X_0(N)/\mathbf{F}_p$ is given by

$$P_N(T) = \prod_f (1 - a_f(p)T + pT^2)^{m(f)},$$

where f runs over the finite set of primitive forms of weight 2 for any $\Gamma_0(M)$, where $M \mid N$ ("newforms" in Atkin–Lehner terminology), with $a_f(p)$ being the pth Hecke eigenvalue

of f. If f is of conductor M, then the multiplicity $m(f)$ of f is $d(N/M)$, the number of divisors of N/M. This is often $\geqslant 2$, showing the existence of multiple roots of P_N, hence of some multiplicative relations. It is natural to restrict to the "new" part, which means taking instead of $X_0(N)$ the new part $J_0(N)^{\mathrm{new}}$ of its Jacobian variety. The L-function of this abelian variety is

$$P_N^*(T) = \prod_{f \text{ level } N} (1 - a_f(p)T + pT^2) \in \mathbf{Z}[T]$$

(note that $P_N^* = P_N$ if N is a prime, for instance).

The $a_f(p)$ are totally real algebraic integers, with usually distinct degrees. We used Magma to compute some of the polynomials P_N^*, taking levels N prime roughly up to 300 and primes p in $\{5, 7, 11, 13\}$ (coprime with N); this amounts to about $1,000$ cases.

What happens experimentally is that a large majority (roughly 85%) of the splitting fields of $1 - a_f(p)T + pT^2$ (over $\mathbf{Q}$) have Galois group $W_{2\deg(a_f(p))}$. This does not exclude cross-relations for different f of the same level, but small-scale tests only found a few of those in remaining multiple factors (e.g. $(1 + T + 5T^2)^2$ divides P_{167}^* for the prime $p = 5$).

Even when the Galois group of a factor is smaller than $W_{2\deg(a_f(p))}$, most of the time there is no extra relation. The few exceptions correspond to factors of degree 4 of the type

$$1 - aT^2 + p^2T^4$$

(e.g. $1 + 17T^2 + 121T^4$ divides P_{67}^* and P_{313}^* for the prime 11, $1 + 6T^2 + 49T^4$ divides P_{29}^* for the prime 7), where there are relations of the type $\alpha^2 = \beta^2$. Similar even polynomials could probably occur also for other values. □

Example 6.5. Let F_m be the Fermat curve defined by

$$F_m \,:\, x^m + y^m + z^m = 0$$

in the projective plane (more general diagonal hypersurfaces could also be considered). The zeta functions of these curves over all finite fields are well known, going back to Weil at least. We assume that $q \neq 1$ is a power of a prime for which $q \equiv 1 \pmod m$, and we consider $F_m/\mathbf{F}_q$. Let then X_m be the set of $m - 1$ nontrivial characters in the cyclic

group (of order m) of characters of order m of $\mathbf{F}_q^\times$. Let

$$g(\chi) = \sum_{x \in \mathbf{F}_q^\times} \chi(x) e(\mathrm{Tr}(x)/p)$$

for $\chi \in X_m$ be the associated Gauss sum, and let, moreover, A_m be the set of 3-tuples $(\chi_0, \chi_1, \chi_2) \in X_m^3$ such that $\chi_0\chi_1\chi_2$ is trivial. Then (see, e.g. [8, §11.3, Theorem 2]), the L-function of F_m is the polynomial

$$\prod_{(\chi_0,\chi_1,\chi_2)\in A_m} (1 - q^{-1} g(\chi_0)g(\chi_1)g(\chi_2)T) \in \mathbf{Z}[T],$$

so that, in particular, the distinct normalized inverse roots are the numbers

$$\frac{g(\chi_0)g(\chi_1)g(\chi_2)}{q^{3/2}}, \qquad (\chi_0, \chi_1, \chi_2) \in A_m. \tag{6.2}$$

We can see here many multiplicative relations: first of all, in A_m, permutations are permitted, and since the inverse roots only depend on the set $\{\chi_0, \chi_1, \chi_2\}$, there will typically be multiplicities among the numbers (6.2), which is, of course, a well-known fact (it is interesting to note that Ulmer [29] has recently used properties of zeta functions of Fermat curves to construct examples of abelian varieties $A/\mathbf{F}_q(t)$, which have bounded ranks in towers of extensions of the form $\bar{\mathbf{F}}_q(t^{1/d})$, d ranging over powers of suitable primes or integers not divisible by p; the crucial properties for him are, however, the prime factorizations of the inverse roots).

But, even among roots taken without the obvious multiplicities arising from permutations, and with only one of each pair $(\alpha, q/\alpha)$ preserved, nontrivial relations will arise because the order of A_m modulo those restrictions grows quicker than m. Indeed, let B_m be the set of different triplets (χ_0, χ_1, χ_2) modulo permutations, and modulo the inversions. Since we have

$$|A_m| = \frac{(m-1)^3 - (m-1)}{m},$$

there are at least $|A_m|/12$ elements in B_m, which is roughly $m^2/12$ as m gets large. A product restricted to representatives of B_m, with exponents $\boldsymbol{n} = (n_b)$, leads to an expression

of the type

$$q^{-3U(\boldsymbol{n})/2}\prod_{\chi\in X_m} g(\chi)^{u_\chi(\boldsymbol{n})},$$

where the u_χ are linear forms with integral coefficients and $U(\boldsymbol{n})$ is the sum of the n_b. So, to produce a relation, it *suffices* to find $\boldsymbol{n}$ such that

$$U(\boldsymbol{n})=0 \quad \text{and} \quad u_\chi(\boldsymbol{n})=0, \quad \text{for } \chi\in X_m.$$

These are m linear relations with integral coefficients, so quite quickly there will less of them than there are coefficients available, guaranteeing the existence of nonzero solutions.

Here is the example of $m=7$: denoting the characters in X_m by ω^j, $1\leqslant j\leqslant 6$, for some generator ω, there are 30 elements in A_7, and 8 basic triplets up to permutation (listed as exponents of ω), namely,

$$(1,1,5)\quad (1,2,4)\quad (1,3,3)\quad (2,2,3)\quad (2,6,6)\quad (3,5,6)\quad (4,4,6)\quad (4,5,5).$$

Among those, it is easy to check that the inverse roots (6.2) corresponding to the last four ones are inverses of those corresponding to the first four ones, leaving four elements in B_7. Then, one finds that the matrix of equations, with columns indexed by the remaining four triplets in order, is

$$\begin{pmatrix} 1 & 1 & 1 & 1\\ 2 & 1 & 1 & 0\\ -1 & 1 & 0 & 2\\ 0 & -1 & 2 & 1\\ 0 & 1 & -2 & -1\\ 1 & -1 & 0 & -2\\ -2 & -1 & -1 & 0 \end{pmatrix}$$

and even though we still have more relations than parameters in this particular case, one checks that the integral kernel of this matrix is nonzero, being of rank 1 and generated by the row vector

$$(1,-1,-1,1)$$

(in fact, the first equation $U(\boldsymbol{n}) = 0$ is redundant here, since the sum of coefficients in each column is constant). This means that for any Fermat curve F_7 over $\mathbf{F}_q$ with $q \equiv 1 \pmod 7$, there will be four roots $\tilde{\alpha}_1, \dots, \tilde{\alpha}_4$, such that

$$\tilde{\alpha}_1\tilde{\alpha}_2^{-1}\tilde{\alpha}_3^{-1}\tilde{\alpha}_4 = 1. \qquad \square$$

7 Frobenius Tori and Multiplicative Independence

In this section, we review briefly the theory of Frobenius tori of Serre, in the version of Chin [3, §5], and explains how it leads to more direct proofs of statements of multiplicative independence of normalized zeros of L-functions in the case of families with large symplectic monodromy. In Remark 7.3, we give examples showing that, on the other hand, this technique does not lead (at least directly) to results concernant linear independence.

Consider a finite field $\mathbf{F}_q$ and a continuous representation

$$\mathrm{Gal}(\bar{\mathbf{F}}_q/\mathbf{F}_q) \xrightarrow{\rho} GL(r, \mathbf{Q}_\ell)$$

for some $\ell \neq p$. Serre defines the Frobenius torus $\mathbf{T}_\rho$ associated to ρ to be the connected component of the identity of the diagonalizable algebraic group $\mathbf{H}_\rho/\mathbf{Q}_\ell$, which is the Zariski closure in $GL(r)/\mathbf{Q}_\ell$ of the subgroup generated by the semisimple part of $\rho(\mathrm{Fr}_{\mathbf{F}_q})$. The character group $\mathrm{Hom}(\mathbf{H}_\rho, \mathbf{G}_m)$ of $\mathbf{H}_\rho$ is canonically isomorphic to the multiplicative group $\langle M_\rho \rangle_m$ generated by the set

$$M_\rho = \{\lambda_1, \dots, \lambda_r\}$$

of eigenvalues of $\rho(\mathrm{Fr}_{\mathbf{F}_q})$. Since a diagonalizable group is determined by its character group (see, e.g. [26, §3.2] for the basic theory), and since there is an exact sequence

$$0 \to \mathrm{Rel}(M_\rho)_m \to \mathbf{Z}^r \to \langle M_\rho \rangle_m \to 0,$$

we see that to know the group H_ρ is equivalent to knowing the group of multiplicative relations among the eigenvalues of $\rho(\mathrm{Fr}_{\mathbf{F}_q})$, showing the relevance of this theory to questions of multiplicative independence of Frobenius eigenvalues.

It follows, in particular, that if the image of ρ lies in a group (isomorphic to) $Sp(2g, \mathbf{Q}_\ell)^k$ for some nondegenerate alternating pairing and $k \geqslant 1$, then we have:

$$\mathrm{Rel}_0(M_\rho)_m = 0 \text{ if and only if } \mathbf{T}_\rho = \mathbf{H}_\rho \text{ is a maximal torus in } Sp(2g)^k/\mathbf{Q}_\ell.$$

Serre proved the first statement of the following type in the case of abelian varieties over number fields; the precise statement is a very special case of [3, Theorem 5.7].

Theorem 7.1 (J.-P. Serre; C. Chin)**.** Let $V/\mathbf{F}_q$ be a smooth affine algebraic variety of dimension $d \geqslant 1$. Let $k \geqslant 1$, and let

$$\rho : \pi_1(V, \eta_v) \to Sp(2g, \mathbf{Q}_\ell)^k$$

be a continuous representation such that the image of $\pi_1(\bar{V}, \eta_{\bar{v}})$ under ρ is Zariski dense in the algebraic group $Sp(2g)^k/\mathbf{Q}_\ell$. Assume that the following conditions hold.

(1) The representation ρ is pointwise pure of weight 0.

(2) There exists $C \geqslant 0$ such that, for every closed point x of V, with residue field of degree $n \geqslant 1$ over $\mathbf{F}_q$, every eigenvalue α of $\rho(\mathrm{Fr}_x)$ and every p-adic valuation v of $\mathbf{Q}(\alpha)$, we have

$$|v(\alpha)| \leqslant C\,|v(q^n)|.$$

(3) There exists $D \geqslant 0$ such that, for every closed point x of V, with residue field of degree $n \geqslant 1$ over $\mathbf{F}_q$, every eigenvalue α of $\rho(\mathrm{Fr}_x)$ and every p-adic valuation v of $\mathbf{Q}(\alpha)$, we have

$$D\,\frac{v(\alpha)}{v(q^n)} \in \mathbf{Z}.$$

Then, there exists a nonempty conjugacy-invariant Zariski open subset $W_k \subset Sp(2g)^k/\mathbf{Q}_\ell$ such that, for any $x \in V(\mathbf{F}_q)$, the Frobenius torus $\mathbf{T}_x$ associated to the local representation ρ_x defined by the composite

$$\mathrm{Spec}(\mathbf{F}_q) \xrightarrow{x} \pi_1(V, \eta_v) \xrightarrow{\rho} Sp(2g, \mathbf{Q}_\ell)^k$$

is a maximal torus in $Sp(2g)^k/\mathbf{Q}_\ell$ if $\rho(\mathrm{Fr}_{x,\mathbf{F}_q}) \in W_k$. □

Consider now the situation of Theorem 1.3, for a fixed value of $k \geqslant 1$: $f \in \mathbf{Z}[X]$ is a squarefree monic polynomial of degree $2g$, where $g \geqslant 1$ is an integer, p is an odd prime such that p does not divide the discriminant of f. Let $U/\mathbf{F}_p$ be the open subset of the affine line where $f(t) \neq 0$, and denote again by $\mathcal{C}_f \to U$ the family of hyperelliptic curves defined in Proposition 1.1. Fix an odd prime $\ell \neq p$ such that q is a square in $\mathbf{Q}_\ell$, and consider the lisse $\mathbf{Q}_\ell$-sheaf ρ corresponding to

$$\bigoplus_{1 \leqslant j \leqslant k} R^1 p_{j,!}\mathbf{Q}_\ell.$$

Fixing a square root $\alpha = \sqrt{p} \in \mathbf{Q}_\ell$, we can form the rank 1 sheaf $\alpha^{-\deg(\cdot)} = \mathbf{Q}_\ell(1/2)$ on U^k (see the discussion in [12, 9.1.9]), and the twist

$$\left(\bigoplus_{1 \leqslant j \leqslant k} R^1 p_{j,!}\mathbf{Q}_\ell\right)(1/2),$$

which has the property that the corresponding representation $\rho' = \rho \otimes \alpha^{\deg(\cdot)}$ takes value in the group $Sp(2g, \mathbf{Q}_\ell)^k$ (instead of $CSp(2g, \mathbf{Q}_\ell)^k$), and which is pointwise pure of weight 0 by the Riemann Hypothesis for curves over finite fields. Other well-known properties of curves over finite fields imply that conditions (2) and (3) of Theorem 7.1 hold for ρ'. The last condition, in particular, has to do with p-adic divisibility properties of the zeros of the L-functions of the curves in the family, and can be obtained, for instance, from Honda–Tate theory (see, e.g. [28]) (for more complicated sheaves, checking this assumption typically involves crystalline cohomology; see, e.g [3, Theorem 3.2], where it is proved to hold, using the techniques of Lafforgue's proof of the global Langlands correspondence over function fields, for any lisse sheaf, which is irreducible with determinant of finite order (e.g. trivial) on a smooth curve).

Thus, we deduce from Theorem 7.1 and the remark before the statement of this theorem that there exists a nonempty conjugacy-invariant Zariski dense subset $W_k \subset Sp(2g)^k$ such that, for any power $q = p^n \neq 1$, and for $\boldsymbol{t} \in U(\mathbf{F}_q)^k$, we have

$$\mathrm{Rel}_0(\tilde{\mathcal{Z}}(\boldsymbol{C}_{\boldsymbol{t}}))_m = 0$$

unless $\rho'(\mathrm{Fr}_{\boldsymbol{t},\mathbf{F}_q}) \in W_k$. Hence, defining C_k to be the closed complement of W_k in $Sp(2g)^k$, we have

$$|\{\boldsymbol{t} \in U(\mathbf{F}_q)^k \mid \mathrm{Rel}_0(\tilde{\mathcal{Z}}(\boldsymbol{C}_{\boldsymbol{t}}))_m \neq 0\}| \leqslant |\{\boldsymbol{t} \in U(\mathbf{F}_q)^k \mid \rho'(\mathrm{Fr}_{\boldsymbol{t},\mathbf{F}_q}) \in C_k\}|.$$

Since C_k is closed of dimension $< \dim Sp(2g)^k$, we can apply Deligne's Equidistribution Theorem to deduce directly

$$\lim_{q\to+\infty} \frac{1}{q^k} |\{\boldsymbol{t} \in U(\mathbf{F}_q)^k \mid \mathrm{Rel}_0(\tilde{\mathcal{Z}}(\boldsymbol{C}_{\boldsymbol{t}}))_m \neq 0\}| = 0.$$

This is a qualitative statement, but it can be made quantitative, for fixed k, by appealing to an explicit uniform Chebotarev density theorem, as in [16], and by reduction modulo ℓ^m for some well-chosen $m \geqslant 1$. Precisely, ρ' has a natural $\mathbf{Z}_\ell$-structure, and by the monodromy result of Yu (already used in the proof of Theorem 1.3) and some fairly standard group theory, the homomorphisms

$$\pi_1(\tilde{U}^k, \bar{\eta}) \to Sp(2g, \mathbf{Z}/\ell^m\mathbf{Z})^k$$

are surjective for all $m \geqslant 1$. Since C_k is a proper closed subset of $Sp(2g)^k$, the order of the image $C_{k,m}$ of C_k modulo ℓ^m satisfies

$$\frac{|C_{k,m}|}{|Sp(2g, \mathbf{Z}/\ell^m\mathbf{Z})^k|} \ll \frac{1}{\ell^m} \tag{7.1}$$

for $m \geqslant 1$, where the implied constant depends only on k. We have

$$|\{\boldsymbol{t} \in U(\mathbf{F}_q)^k \mid \rho'(\mathrm{Fr}_{\boldsymbol{t},\mathbf{F}_q}) \in C_k\}| \leqslant |\{\boldsymbol{t} \in U(\mathbf{F}_q)^k \mid \rho'(\mathrm{Fr}_{\boldsymbol{t},\mathbf{F}_q}) \in C_{k,m}\}|.$$

By the Chebotarev density theorem, we obtain

$$\begin{aligned}|\{\boldsymbol{t} \in U(\mathbf{F}_q)^k \mid \rho'(\mathrm{Fr}_{\boldsymbol{t},\mathbf{F}_q}) \in C_{k,m}\}| = {} & \frac{|C_{k,m}|}{|Sp(2g, \mathbf{Z}/\ell^m\mathbf{Z})^k|} q^k \\ & + O(q^{k-1/2} |Sp(2g, \mathbf{Z}/\ell^m\mathbf{Z})^k| |C_{k,m}|^{1/2}),\end{aligned}$$

where the implied constant depends only on g and k (as in [16, Theorem 1.1], but arguing as in the beginning of Theorem 1.3 to estimate the relevant sum of Betti numbers in such a way that the dependency only involves g and k). Using (7.1) and rough estimates, this

gives

$$|\{\boldsymbol{t}\in U(\mathbf{F}_q)^k \mid \rho'(\mathrm{Fr}_{\boldsymbol{t},\mathbf{F}_q})\in C_{k,m}\}| \ll \ell^{-m}q^k + O(q^{k-1/2}\ell^{6mg^2k}),$$

and the choice of m as the integer $m\geqslant 1$ such that

$$\frac{1}{2(6g^2k+1)}\frac{\log q}{\log \ell}-1\leqslant m<\frac{1}{2(6g^2k+1)}\frac{\log q}{\log \ell}$$

(if m exists, but otherwise the result becomes trivial) leads to

$$|\{\boldsymbol{t}\in U(\mathbf{F}_q)^k \mid \rho'(\mathrm{Fr}_{\boldsymbol{t},\mathbf{F}_q})\in C_k\}| \ll \ell q^{k-\gamma^{-1}}$$

with $\gamma=2(6g^2k+1)$, where the implied constant depends only on g and on k.

Compared to Theorem 1.3, two issues arise. The first is the apparent dependency on the choice of ℓ such that q is a square in $\mathbf{Q}_\ell$, but this is mostly cosmetic. It is clear that one can take $\ell\leqslant p$; so, the "vertical" direction $q=p^n$ with $n\to+\infty$ is dealt in this manner. As is, the "horizontal" direction $q=p\to+\infty$ requires something close to the Generalized Riemann Hypothesis (over $\mathbf{Q}$) (which implies $\ell\ll(\log p)^2$), but it is also certainly possible to prove directly a variant of Theorem 7.1 for sheaves of weight 1 to avoid the twist by $\alpha^{\deg(\cdot)}$ required to obtain a sheaf of weight 0.

The second issue is the uniformity in terms of k, which is more delicate, but would be necessary to obtain a result as strong as Theorem 1.3. To deal with it, one needs to write down more explicitly the closed subset C_k that occurs in the proof, and more precisely (by the standard point-counting estimates for varieties over finite fields), one needs to have an estimate for the number of (geometric) irreducible components of C_k. Since C_k is described quite concretely in [3, p. 37], obtaining a bound seems feasible (C_k is the union of Zariski closures of conjugates of a finite set of subgroups $\mathbf{H}$ of a maximal torus such that the connected component of $\mathbf{H}$ is among a finite set of subtori, and has index $\leqslant N$, where N is some integer depending only on g), but it is not obvious (to the author) how to do it efficiently. Certainly, counting only the number of subgroups $\mathbf{H}$ leads to a bound worse than exponential in terms of k, which would give a worse dependency on k than Theorem 1.3, but one may hope that not all subgroups lead to different irreducible components after conjugation and taking the Zariski closure.

Remark 7.2. Another interesting contrast between this proof of Theorem 1.3 (for fixed k) and the previous one is that this one depends crucially on using p-adic information

about the eigenvalues (via Condition (3) of Theorem 7.1), whereas the first one does not require any p-adic input (if one uses Remark 5.4, at least, because otherwise there is, hidden in the proof of the necessary estimates for sums of Betti numbers, some p-adic arguments of Bombieri and Adolphson–Sperber). □

Remark 7.3. We now show that the Frobenius torus does not control the linear relations between the Frobenius eigenvalues. Let $A = E_1 \times \cdots \times E_g$ be the product of g pairwise nonisogenous ordinary elliptic curves over a finite field $\mathbf{F}_q$. Then, for a fixed prime $\ell \neq p$, the Frobenius torus of A (which corresponds to $\mathbf{T}_\rho$ for the representation ρ on $H^1(A, \mathbf{Q}_\ell)$, twisted as before so that the eigenvalues are of modulus 1) is a maximal torus of $Sp(2g)/\mathbf{Q}_\ell$.

Let $(\lambda_i, q/\lambda_i)$ denote the eigenvalues of the Frobenius automorphism for E_i, and let $a_i = \lambda_i + q/\lambda_i$ be the trace of Frobenius, which is an integer. The set

$$M_A = \{\lambda_1, q/\lambda_1, \dots, \lambda_g, q/\lambda_g\},$$

is the set of all Frobenius eigenvalues of A. So, we see that any nontrivial linear relation between the a_i's (which exist in abundance) gives a nontrivial linear relation between the elements of M_A: defining $T_A = \{a_1, \dots, a_g\}$, there is an injection

$$\begin{cases} \mathrm{Rel}(T_A)_a \hookrightarrow \mathrm{Rel}(M_A)_a \\ (n_i) \quad \mapsto \quad (m_\lambda), \quad \text{where } m_{\lambda_i} = m_{q/\lambda_i} = n_i, \end{cases}'$$

and since $\mathrm{Rel}(T_A)_a$ is a $\mathbf{Z}$-module of rank $g-1$ (the a_i being nonzero), the rank of $\mathrm{Rel}(M_A)_a$ is $\geqslant g-1$.

(Note that, conversely, we can fix arbitrarily a g-tuple $(n_i)_{1\leqslant i\leqslant g}$, and then find, for all prime powers $q = p^k$ large enough, some $(a_i)_{1\leqslant i\leqslant g}$ with $p \nmid a_i$ and $|a_i| \leqslant 2\sqrt{q}$, such that

$$\sum_{1\leqslant i\leqslant g} n_i a_i = 0,$$

and building the corresponding elliptic curves, this shows that any arbitrarily fixed linear relation of this type can be obtained from an abelian variety with maximal Frobenius torus.) □

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