Doctoral Thesis

A Study of several aero thermoelastic problems of aircraft structures in high-speed flight

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A Study of Several Aerothermoelastic Problems of Aircraft Structures in High-Speed Flight

THESIS

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BY

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Summary

Several aerothermoelastic problems pertaining to aircraft structures are treated, with emphasis being given to dynamic aeroelastic effects. The first part of the report deals with temperature, stress, and stiffness analysis. A review of the thermal aspects of high speed flight, with regard to both external and internal heat flow, and a résumé of the basic equations of elasticity, modified to include temperature effects, are first given to set down fundamentals and ingredients for the work which follows.

Energy and equilibrium equations applicable to platelike structures of variable thickness in the large deflection range are developed, and this is followed by a further treatment of the technique of stress function analysis, especially with regard to the treatment of boundary conditions and the analogy that may be drawn with plate bending. The solution of these problems by means of difference equations is then considered in detail, and then a new "summation-equation" method for solution is presented.

In the second part, the emphasis is on aeroelastic phenomenon. The "summation-equation" method is applied to a problem involving transient disturbances and is shown to yield remarkable accuracy in comparison to exact solution. The means for determining frequency response influence surfaces in one loading application without having to repeat for each load position is then given. A development of a flutter analysis through use of difference equations which considers the structures and the aerodynamics in a single combined operation follows.

A rather exhaustive treatment of the problem of panel flutter, both in the flat and buckled condition, is then made; special attention is focused on the behavior of the deflection and natural frequencies when air flow is present, on the negligible influence of aerodynamic damping at flutter, and on the characteristics merging together of two natural frequencies at flutter. Through application of the techniques of generalized harmonic analysis, a treatment then follows on the response of panels subject to an air flow which has random pressure disturbances. A special means is developed for considerably simplifying the solution phase of this problem by introducing the reciprocal flow system. Several new concepts are advanced, such as the vanishing of the generalized mass at flutter. The stress variation with airflow is shown to be rather similar to the deflection behavior as resonance in a simple mass oscillator is approached, or as the critical load in a column is approached.
Zusammenfassung


Im zweiten Teil liegt die Betonung auf aeroelastischen Vorgängen. Die Summengleichungsmethode wird auf ein System unter vorübergehender Störung angewendet und zeigt bemerkenswerte Genauigkeit im Vergleich mit der exakten Lösung. Hierauf wird erklärt, wie die Frequenzempfindlichkeitsflächen über sämtliche Lastangriffspunkte durch eine einzige Lastanwendung bestimmt werden können. Dann folgt noch die Entwicklung einer Methode zur Behandlung des Flatterproblems mittels Differenzgleichungen, wobei gezeigt wird, wie die elastischen und aerodynamischen Wirkungen sich gleichzeitig berücksichtigen lassen.

List of Symbols

\( a \)  
velocity of sound; plate length

\( a_n, b_n \)  
modal coefficients

\( b \)  
plate width

\( c \)  
column fixity coefficient

\( c, c_p, c_v \)  
specific heat

\( D \)  
flexural rigidity  
\( D = \frac{Eh^3}{12(1-\mu^2)} \)

\( e \)  
volume expansion; centroidal distance

\( E \)  
Young's modulus

\( F_x, F_y, F_{xy} \)  
edge forces

\( F(\omega) \)  
Fourier transform

\( g_a, g_s, g_r \)  
damping coefficients

\( G \)  
shearing modulus  
\( G = \frac{E}{2(1+\mu)} \)

\( h \)  
heat-transfer coefficient; plate thickness (thickness of stress carrying material only for built-up plates)

\( I \)  
moment of inertia; identity matrix

\( J \)  
Joule's equivalent (= 778 ft.-lb. per BTU)

\( k \)  
thermal conductivity; reduced frequency; panel frequency parameter; spring constant

\( L \)  
panel or beam length

\( \bar{L} \)  
turbulence scale

\( m \)  
mass per unit area for plates

\( M \)  
Mach number  
\( M = \frac{V}{a} \)

\( M_x, M_y, M_{xy} \)  
plate bending and twisting moments

\( N_x, N_y, N_{xy} \)  
resultant middle surface forces

\( p \)  
pressure, loading intensity, roots

\( Pr \)  
Prandtl Number \( \left( Pr = \frac{\mu cp}{k} \right) \), \( k \) = thermal conductivity

\( q \)  
heat flux per unit area

\( r \)  
recovery factor; ratio of axial load to Euler value for pin ends, also expressed in temperature ratio form; row matrix

\( R \)  
Gas constant; load ratio

\( s \)  
separation constant; load ratio

\( t, \tau \)  
time

10
\( T \)  
\( \Delta T \)  
\( u, v, w \)  
\( u \)  
\( U \)  
\( v \)  
\( V \)  
\( w_n, y_n \)  
\( W \)  
\( x, y, z \)  
\( X, Y, Z \)  
\( Z_n(\omega) \)  
\( \alpha \)  
\( \beta \)  
\( \beta_k \)  
\( \gamma \)  
\( \gamma_{xy}, \gamma_{xz}, \gamma_{yz} \)  
\( \delta \)  
\( \Delta \)  
\( \epsilon \)  
\( \epsilon_x, \epsilon_y, \epsilon_z \)  
\( \eta \)  
\( \eta_n \)  
\( \theta \)  
\( \lambda \)  
\( \mu \)  
\( \rho \)  
\( \sigma \)  
\( \sigma_x, \sigma_y, \sigma_z \)  
\( \tau_{xy}, \tau_{xz}, \tau_{yz} \)  
\( \nu \)  
\( \phi \)  
\( \phi_{r}, \phi_{t} \)  
\( \phi(\omega), \phi(0, \omega) \)  
\( \phi(x, \omega) \)  
\( \omega \)  

- temperature
- temperature change from ambient
- displacements
- normal velocity of plate surface
- strain energy
- flow velocity
- potential energy of applied and body forces
- modal functions
- total potential energy, \( W = U - V \)
- rectangular coordinates
- body forces
- impedance associated with \( n^{\text{th}} \) mode
- coefficient of thermal expression; absorptivity coefficient; exponential coefficient; velocity factor in random loading study
- damping coefficient; exponential coefficient
- support factor
- ratio of specific heat for gases
- components of shearing strain
- influence coefficient
- determinant; incremental value
- emissivity; distance interval in the \( x \)-direction
- components of strain
- non-dimensional distance, \( \eta = \frac{x}{L} \); matrix involving airfoil thickness derivative, see eq. (7.11); finite width correction factor in random loading study
- integrating factors
- radiation incidence angle
- Lame's constant; distance interval in the \( y \)-direction; panel velocity parameter
- Poisson’s ratio; viscosity
- density
- Stefan-Boltzman radiation constant; stress
- components of stress
- components of shearing stress
- kinematic viscosity \( \left( \nu = \frac{\mu}{\rho} \right) \)
- Airy’s stress function
- radiative heat flux per unit area
- power spectra
- cross spectra
- circular frequency
Subscripts

\( a, aw, w, s, e \) air or ambient, adiabatic wall, wall, stagnation and equilibrium, respectively (when used with temperature)

\( m, n, i, j \ldots \) integers designating the mode or deflection station

\( N, E, S, W, NE \ldots \) grid points

\( r \) reference, random

\( s \) structural

Note, for Fourier transforms and spectra, the quantity being referred to is often used as a subscript, also.

Notation

\( \dot{w} \) denotes \( \frac{\partial w}{\partial t} \)

\( \phi_x \) and \( \phi' \) both denote \( \frac{\partial \phi}{\partial x} \)

\( \nabla^2 \) the Laplace operator \( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \), \( \nabla^2 \nabla^2 = \nabla^4 \)

\( \bar{\cdot} \) a bar denotes modulus, conjugate, adjoint or time average

\( [ ] \) column matrix

\( [ ] \) row matrix

\( [ ] \) square matrix

\( [ ]' \) transpose of matrix

\( | | \) determinant

\( [ ] \) rectangular matrix
Introduction

This thesis aims to further the understanding of various problems in aero-thermoelasticity — the area of aeronautics in which the fields of aerodynamics, structures, and thermodynamics interact.

Aeroelasticity has become a dominant consideration in the design of high-speed aircraft during the past decade, and has grown in severity largely because of the conflicting necessity of building a structure strong enough to withstand the larger forces of supersonic flight, yet thin enough to slice through the atmosphere at these speeds. And there is clear evidence that the phenomenon will be aggravated by the high temperatures induced by aerodynamic heating at flight speeds several times the speed of sound [1–3]*. It is significant that the nineteenth Wright Brothers Lecture given in 1956 by Bisplinghoff was devoted exclusively to the aeroelastic problem. This excellent paper [1], has done a remarkable job in surveying the structural demands of supersonic flight and in assessing the possible influence on structural design.

The consideration of the supersonic flight regime opened a new era in aeronautics just a few years ago and the problems engendered are having a vital influence with respect to future aeronautical progress. As with most new problem areas that open up, progress is first marked with many contributions on specific items, as is attested by the many publications which have resulted, see, for example, the large bibliography on aeroelasticity alone given in [1]. Then as the problems begin to merge, or as existing analytical and experimental techniques become inadequate, it becomes increasingly difficult to isolate a clear-cut problem. For this reason, the choice was made in the present paper to consider several overlapping phases of the aeroelastic problem, rather than attempting to concentrate on a specific phase. A second reason is that it would thus be possible to set down under one cover a number of ideas that have occurred to the author during several years research work in the fields of aeroelasticity and structures.

Because treatment is made of a variety of subjects, the theme may not at first be evident; but let it be stated that the central theme aimed for is that of dynamic aeroelasticity. Thermal aspects and statics are considered first to set down fundamentals and because these are essential ingredients in dynamic.

* The numbers in [ ] refer to the numbered references at the end, page 107.
considerations. Although the sections of the present work are related, each may be read somewhat independently of the others, except perhaps for the final section on panels under random loading, which is closely linked with the previous section on panel flutter.

In a broad sense, thermal aspects are considered in the paper along generally accepted lines. As regards the influence of elevated temperatures on structural behavior, there are two principle effects: (1) a deterioration in strength properties associated with temperature level, and (2) a change in stiffness (usually adverse) as brought about by the thermal stresses that result from temperature distribution. The first of these is treated in a tacit sense throughout most of the first of the paper and then is considered specifically in later examples; the other is dealt with more explicitly throughout. As regards the treatment of the aerothermoelastic problem, it generally is the practice to separate the problem into two phases — the aerothermal problem which deals with the thermal equilibrium of heat transfer between the structure and boundary layer, including internal heat flow and radiation effects; and the aeroelastic problem, which deals with the equilibrium between aerodynamic, elastic, and inertial forces — and this practice has been followed herein. (For some applications, the determination of the thermal stresses is regarded as a separate stage between these two phases.) The coupling that exists between the two phases appears to be generally negligible.

From a specific point of view, this work has the following several aims: to review, to develop energy and equilibrium equations applicable to plate-like structures of variable thickness in the large deflection range, to add to the technique of stress function analysis, especially with regard to the treatment of boundary conditions and the analogy that may be drawn with plate bending, to develop newer methods of solving these problems through use of difference equations and a new "summation-equation" method suggested, to develop a simplified means for determining frequency response influence surfaces, to develop a flutter analysis through use of difference equations which considers the structures and the aerodynamics in a single combined operation, and to effect specific solutions of the problem of panel flutter and the response of panels subject to random loading where air flow is involved. The table of contents serves well to indicate the detailed coverage, and further introductory remarks are given at the beginning of most of the sections. Since each section contains concluding observations and can stand more or less alone, no overall conclusions of the work are presented.
Part I

Temperature, Stress and Stiffness Analysis

1. The Thermal Environment

A. Stagnation temperatures

The elementary appearing equation which has had a marked influence on aeronautical development is the equation for stagnation temperature

\[ T_s = T_a \left(1 + \frac{\gamma - 1}{2} M^2 \right). \] (1.1)

The frontispiece gives the temperatures indicated by this equation for atmospheric flight. A glance at this figure and fig. 1 on materials properties tells immediately why aircraft designers have been concerned about the high temperature problems of near and future flight.

Equation (1.1) follows directly from the expression for the change in enthalpy when an air stream is adiabatically brought to rest

\[ c_p (T_s - T_a) = \frac{1}{g J} \frac{v^2}{2}, \quad \text{for } c_p \text{ constant}, \] (1.2)

where \( T_a \) and \( T_s \) are respectively the free stream and stagnation temperatures (absolute), or from Bernoulli’s equation for steady flow

\[ \frac{1}{2} v^2 + \int \frac{d p}{p} = \text{const.}, \] (1.3)

when either expression is considered in conjunction with the fundamental gas laws for adiabatic processes

\[ a^2 = \frac{\gamma p}{\rho} = \gamma g R T; \quad c_p = \frac{\gamma}{\gamma - 1} \frac{R}{J}; \quad \gamma = \frac{c_p}{c_r}; \] (1.4)

\[ \frac{p}{p_0} = \left( \frac{\rho}{\rho_0} \right) ^\gamma = \left( \frac{T}{T_0} \right) ^{\gamma(\gamma - 1)}, \] (1.5)

where \( a \) is the velocity of sound. Thus, the substitution of relations (1.4) into equation (1.2) leads directly to equation (1.1); alternatively, the substitution of the pressure-density relation in equation (1.2) leads through integration to

\[ \frac{1}{2} v^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \text{const.}, \] (1.6)
Fig. 1. Property variations with temperature of possible alloys for air-frame construction (from ref. 1).
which then reduces to equation (1.1) when use is made of the pressure-temperature and the velocity of sound relations in application to a free stream and a stagnation point.

With $\gamma=1.4$ and $R=53.3$ for air, equation (1.1) may be written

\[
\begin{align*}
T_s - T_a &= 0.2 T_a M^2, \\
T_s - T_a &= 0.000179 v^2, \, ^\circ\text{F} \quad (v \text{ in mph}),
\end{align*}
\]

(1.7)

where the latter is useful since it does not depend on altitude. Written in this form, the temperatures on the left may be either absolute ($^\circ\text{R}$) or Fahrenheit.

B. Adiabatic wall temperatures

The process of velocity reduction in a boundary layer by viscous forces is nearly as much of a heat generator as is the actual arresting at a stagnation point. In fact, if no convective heat transfer to the surrounding air elements took place, the temperatures at the solid boundary would be equal to the stagnation temperatures. The convective heat transfer causes the temperatures to be slightly less and it is common to refer to the temperature that develops at the body boundary when no heat flow is allowed between the air and the body as the adiabatic wall temperature $T_{aw}$ (the maximum temperature possible at the surface). This adiabatic wall temperature is given by the equation

\[
T_{aw} = T_a \left(1 + r \frac{\gamma - 1}{2} M^2\right),
\]

(1.8)

where $r$ is called the temperature recovery factor, such that

\[
r = \frac{T_{aw} - T_a}{T_s - T_a}.
\]

(1.9)

This recovery factor has been found to be primarily a function of the Prandtl Number, $Pr$, and is also dependent on whether the flow is laminar or turbulent. The following relations have been found to apply with sufficient accuracy for practical purposes:

\[
r = (Pr)^{1/4} \quad \text{for laminar flow},
\]

\[
r = (Pr)^{1/4} \quad \text{for turbulent flow}.
\]

The Prandtl Number has been related to the temperature outside the boundary layer, but the variation is small and for most purposes a value of 0.72 may be used. This would give for turbulent flow, for example, a recovery factor of 0.9; thus, the adiabatic wall temperatures are seen to be only slightly less than the stagnation temperatures.
2. Determination of Temperatures Within the Structure

A. Heat transfer at body surface

1. Sources of heat flow. The total heat flux per unit area, $q$, at the surface of a body in high speed flight may result from at least four distinct causes, as depicted by the following sketch and equation:

$$q = h (T_{aw} - T_w) + \alpha \phi_r \cos \theta + \epsilon \sigma T_w^4 + \phi_i. \quad (2.1)$$

Strictly speaking, the first term in this equation implies an isothermal surface; it may be applied in fair approximation to non-isothermal surfaces, however, as will be indicated later. In the equation, $h$ is the boundary layer heat transfer coefficient, $T_w$ is the surface or wall temperature, $\alpha$ is the absorptivity coefficient, $\phi_r$ the radiative flux per unit area of surface normal to the line of sight from the airplane to the radiating object, $\theta$ is the angle of incidence between an outward directed normal to the surface and the incident ray, $\epsilon$ is the emissivity, $\sigma$ the Stefan-Boltzman constant, and $\phi_i$ is the heat flux from heat sources within the structure.

Of all the coefficients in equation (2.1), $h$, the boundary layer heat transfer coefficient, is perhaps the most difficult to establish. It is a function of many variables and has been the subject of much theoretical and empirical work in recent years, embodying flat plates, cones, cylinders, parabolic bodies of revolution and the like [4-11]. The following equations are given to show the nature of some of the results obtained. For steady flow over an isothermal semi-infinite flat plate, the following approximate formulas for heat-transfer coefficients have been suggested:

For laminar flow,

$$h = 0.332 k_a \left( \frac{v}{\nu} \right)^{1/4} (Pr)^{1/4}. \quad (2.2)$$

For turbulent flow,

$$h = 0.0296 k_a \left( \frac{v}{\nu} \right)^{1/4} (Pr)^{1/4} \left( \frac{T_a}{T_w} \right)^{0.44}. \quad (2.3)$$

In these formulas, $k_a$ is the thermal conductivity of air, $\nu$ is the coefficient of kinematic viscosity, and $v$ the flow velocity, all local static properties evaluated...
at the other edge of the boundary layer. The distance \( x \) in equation (2.2) is measured from the leading edge, while in equation (2.3), \( x \) is measured from the effective transition point.

In the case of non-isothermal flat plate, the work of Lighthill has led to the following result [7]

\[
q = -0.332 \frac{y}{r} \left( \frac{v}{\nu x} \right)^{1/3} (Pr)^{1/3} k_a \int_{\xi=0}^{\xi=x} \frac{d[T_w(\xi) - T_{aw}]}{\left[1 - \left(\frac{\xi}{x}\right)^{1/3}\right]^{1/3}}. \tag{2.4}
\]

For isothermal plate conditions, this equation yields the heat transfer coefficient given by equation (2.2).

In passing, it is remarked that among the many difficulties and approximations involved in applying such results as are indicated by equations (2.2) to (2.4) to airplane structures, one of the biggest difficulties lies in predicting the position of the transition point. The severity of the resulting structural problems is strongly dependent on the location of this point, since the heat transfer coefficients for laminar and turbulent flow differ considerably.

2. Relative magnitude of heat flow. The coefficients \( k_a, \nu, \) and \( Pr \) in equations (2.2) to (2.4) are all a function of the temperature at the other edge of the boundary layer. If this temperature is taken as the ambient air temperature, then it is possible to express the heat transfer coefficient in terms of altitude; equations (2.2) and (2.3) would appear, for example

\[
\dot{h} = c_H \left( \frac{M}{x} \right)^{1/5}, \quad \dot{h} = c_H \left( \frac{M}{x} \right)^{1/5} \left( \frac{T_a}{T_w} \right)^{0.44}, \tag{2.5}
\]

where the \( c_H \)'s have been computed as function of altitude and are given in fig. 2.
Fig. 3. Relative magnitude of heat fluxes.

Table 1. Conversion factors

Length, weight, and stress:

\[
\begin{align*}
1 \text{ in} & = 2.540 \text{ cm} \\
1000 \text{ ft} & = 0.3048 \text{ km} \\
1 \text{ lb} & = 453.6 \text{ g} \\
1 \text{ lb/ft}^3 & = 0.01602 \text{ g/cm}^3 \\
1000 \text{ psi} & = 0.703 \text{ kg/mm}^2
\end{align*}
\]

For energy units:

\[
\begin{align*}
1 \text{ Btu} & = 778 \text{ ft-lb (by definition)} = 1055 \text{ joules} = 252 \text{ cal.} \\
1 \text{ erg} & = 1 \text{ dyne-cm (by definition)} \\
1 \text{ joule} & = 10^7 \text{ ergs (by definition)} = 0.73756 \text{ ft-lb} \\
1 \text{ horsepower-hour} & = 2545 \text{ Btu} = 0.7457 \text{ kwhr} \\
1 \text{ kwhr} & = 860 \text{ international cal. (by definition of calorie)} = 1.341 \text{ hp-hr}
\end{align*}
\]

For power units:

\[
\begin{align*}
1 \text{ hp} & = 550 \text{ ft-lb per sec (by definition)} = 33.000 \text{ ft-lb/min} \\
& = 2545 \text{ Btu/hr} = 0.7457 \text{ kw} \\
1 \text{ watt} & = 1 \text{ joule/sec (by definition)} \\
1 \text{ kilowatt} & = 1000 \text{ watts (by definition)} = 1.341 \text{ hp} = 3413 \text{ Btu/hr}
\end{align*}
\]
From these equations, fig. 3 has been prepared to give an indication of the relative importance of the various terms that compose equation (2.1). The boundary layer heat rates shown in this figure were determined on the assumption that $T_w = T_o$; this implies that the airplane was instantaneously brought from rest to the Mach number indicated. Since $T_w$ will generally be higher than $T_o$ for finite acceleration flight, the rates shown are the maximum possible. In contrast, in computing the radiative heat flux $T_w$ was assumed equal to $T_{aw}$; the values shown are thus also a somewhat fictitious maximum rate possible since the radiative effect will never allow the surface to reach $T_{aw}$. Nevertheless, the figure allows an order of magnitude comparison to be made. The heat flux from solar radiation indicated is based on an absorptivity coefficient of 0.4. To link the results with a heat source of common everyday use, the heat flux as obtained from a 1000 watt, 9 in. diameter electric stove "burner" operating at 0.7 efficiency is also indicated on the figure. The enormous heating capabilities of the boundary layer during high speed flight is thus evident from the figure. (See energy and power conversion factors in table I.)

![Figure 4](image)

**Fig. 4.** Altitude and Mach number effects on equilibrium temperature of a flat plate, emissivity = 0.5.
Another indication of the relative importance of the various heat flux terms can be had by considering the actual equilibrium temperature that is reached when radiative effects are taken into account, see fig. 4, which is taken from [1]. The ordinate indicates the ratio of equilibrium temperature to adiabatic wall temperature, where the equilibrium temperature is defined as the temperature attained at the surface when the inward convective heat transfer from the boundary layer equals the outward heat transfer due to radiation.

From these figures, it may be seen that radiative effects are not very important for the lower speeds and low altitude, but that it has an important influence in reducing surface temperature at very high speeds and altitudes.

**B. Heat transfer and temperatures within the structure**

With the temperatures at the surface established, a next step in the thermal stress consideration of structures is to consider the transfer of heat within the structure. For solid bodies, this heat transfer is entirely conductive, while for built-up structures the transfer may involve both transfer forms—conduction and radiation; radiation, however, becomes important only at the higher temperatures. The heating by convection, involving air mass movement, is normally unimportant.

The theory of the transfer of heat by conduction, originated by Fourier, is based on the diffusion equation

\[
\frac{\partial}{\partial t} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) = c \rho \frac{\partial T}{\partial t},
\]

where \( T \) is the temperature at time \( t \) at any point whose location is given in terms of the orthogonal axes system \( x, y, z \), and \( k, c, \) and \( \rho \) are respectively the thermal conductivity, specific heat per unit mass, and density of the material. The boundary conditions may be mixed as follows:

- \( S_1 \) boundary where the temperature is prescribed,
- \( S_2 \) insulated boundary, \( k \frac{\partial T}{\partial n} = 0 \),
- \( S_3 \) boundary where heat is transferred, \( k \frac{\partial T}{\partial n} + q = 0 \).

where \( n \) is an inwardly drawn normal to the body surface. The interesting physical aspect about this equation is that the flow or diffusion phenomenon is fully damped and always tends to uniformity, never to overshoot and produce oscillation, as in the case of vibration or wave motion.

The presence of the damping or odd derivative term and mixed b.c., however, make the equation particularly difficult to solve. Elegant analytical solutions by Fourier series, Fourier integral, Green’s function, and the like,
Table II. Mechanical properties and physical constants of some metals and alloys. (Average room temperature values)

### Thermal conductivities of air, CO₂, and some metals

<table>
<thead>
<tr>
<th>Material</th>
<th>Density (lb cu in)</th>
<th>Melting point (°F)</th>
<th>Specific heat (Btu lb⁻⁰°F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum 75S</td>
<td>0.101</td>
<td>890–1180</td>
<td>0.21</td>
</tr>
<tr>
<td>Inconel “X”</td>
<td>0.300</td>
<td>2540–2600</td>
<td>0.105</td>
</tr>
<tr>
<td>Stainless Steel 347</td>
<td>0.286</td>
<td>2550–2600</td>
<td>0.12</td>
</tr>
<tr>
<td>Titanium</td>
<td>0.16</td>
<td>3130</td>
<td>0.13</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Material</th>
<th>Thermal expansion coef. (in⁻¹°F × 10⁻⁶)</th>
<th>Thermal conductivity (Btu Ft⁻HR⁻°F/in)</th>
<th>Tensile modulus (×10⁶ psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum 75S</td>
<td>12.9</td>
<td>840</td>
<td>10.4</td>
</tr>
<tr>
<td>Inconel “X”</td>
<td>7.6</td>
<td>102</td>
<td>31</td>
</tr>
<tr>
<td>Stainless Steel 347</td>
<td>9.3</td>
<td>110</td>
<td>28</td>
</tr>
<tr>
<td>Titanium</td>
<td>5.0</td>
<td>97</td>
<td>15</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Material</th>
<th>Yield strength (1000 psi)</th>
<th>Elongation in 2 in. (%)</th>
<th>Hardness (Brinell)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum 75S</td>
<td>80</td>
<td>10</td>
<td>150</td>
</tr>
<tr>
<td>Inconel “X”</td>
<td>120</td>
<td>25</td>
<td>380</td>
</tr>
<tr>
<td>Stainless Steel 347</td>
<td>120</td>
<td>5</td>
<td>300</td>
</tr>
<tr>
<td>Titanium</td>
<td>135</td>
<td>9</td>
<td>300</td>
</tr>
</tbody>
</table>
have only been made for very simple idealized shapes [12–15]. For intricate and built-up structures approximate methods must be employed for practical solutions, such as relaxation, difference equations, Galerkin, and variational methods [16–19]. It is remarked that perhaps more attention should be given to the inherent connection that this equation has with the wave equation of acoustical mechanics as regards method of solution.

For plate-like elements for which there is little variation in temperature across the thickness, the following equation for two dimensional heat flow applies

\[-\left(\frac{\partial}{\partial x}k h \frac{\partial T}{\partial x} + \frac{\partial}{\partial y}k h \frac{\partial T}{\partial y}\right) + c \rho h \frac{\partial T}{\partial t} = q,\]  

(2.7)

where \(h\) is the skin thickness, and \(q\) is the heat flux at the plate surface. If there is no heat flow in the \(y\)-direction, this equation yields the one-dimensional heat flow equation

\[-\frac{\partial}{\partial x}k h \frac{\partial T}{\partial x} + c \rho h \frac{\partial T}{\partial t} = q.\]  

(2.8)

This equation in conjunction with equation (2.1) has received extensive attention in thermal studies of built-up structures. As mentioned previously, the use of the convective term in equation (2.1) in the form shown gives rise to an inconsistency, since the form is based on an isothermal surface. However, some investigators have compared the results obtained by this application with the results obtained by use of the more appropriate heat flow relation given by equation (2.4), and found that the differences in the results were negligible for practical purposes [7].

Table II lists the materials properties \(k, c, \rho\), for several structural materials of interest. If \(k\) is constant in the foregoing heat flow equations, it is customary to combine the properties into a single quantity \(\frac{c \rho}{k}\), called the thermal diffusivity.

C. Some analogies

1. Torsional analogy. A point worthy of note is that equation (2.8) is similar to the equation for torsional deformation of a beam of variable cross-section, and that therefore, many of the methods of solution that have been devised for the torsional problem apply equally to this equation. In fact, the following mechanical torsional analogy for heat flow is suggested. If parabolic difference equations are used, equation (2.8) may be written

\[-\frac{1}{\epsilon}\{(kh)_{n-\nu_x} T_{n-1} - [(kh)_{n-\nu_x} + (kh)_{n+\nu_x}] T_n + (kh)_{n+\nu_x} T_{n+1}\}
+ \epsilon c \rho h \frac{\partial T_n}{\partial t} = \epsilon q_n,\]  

(2.9)
where \( \epsilon \) is the spacing between the stations designated by \( n - 1, n, n + 1, \) 
\((n - \frac{1}{2} \text{ and } n + \frac{1}{2} \) designate the half stations). It may be shown that this same equations applies to the following torsional system

Cross sections of built-up structures can be simulated by compounding a series of such torsional systems, where torsional rods simulating cross members such as webs are introduced through means of pulleys or gears.

2. **Electrical analogy.** The following electrical circuit [20] may be used as an analog to equation (2.8), for the case of adiabatic wall heating

![Electrical Circuit Diagram]

The analysis of a given section of the circuit leads to the following equation

\[
- \left( \frac{1}{R_{n-\frac{1}{2}}} \frac{E_{n-1}}{E_n} - \frac{1}{R_{n+\frac{1}{2}}} \frac{E_n}{E_{n+1}} + \frac{1}{R_{n+\frac{1}{2}}} \frac{E_{n+1}}{E_n} \right) + C_n \frac{\partial E_n}{\partial t} = \frac{1}{R} (E - E_n)
\]

which is seen to be exactly analogous to equation (2.9) when the convective heat transfer term of equation (2.1) is introduced. An extension of this system to take into account radiative effects is suggested as follows. If each of the condensers in the circuit are shunted by a diode whose current characteristics are proportional to the fourth power of the voltage across the diode, then an additional term will appear on the right-hand side of the equation which will exactly duplicate the radiation term in the equation.

3. **Air pressure analogy.** Because the diffusion of heat through a structure is similar to the diffusion of a gas through a porous substance, another analogy can be found as is suggested by the following sketch
With the use of a substance of small porosity, the diffusion can be made to be slow enough so that “quasi-static” measurements can be made. This system is applicable for 1-, 2-, or 3-dimensional systems and would appear especially useful for the study of intricate shapes.

3. Stress and Stiffness Analysis of Structures with Temperature Gradients

As noted in the introduction, structural behavior is adversely influenced by the elevated temperature environment because of a reduction in material strength properties (see fig. 1) and because of the thermal stresses which are brought about by a non-uniform temperature distribution. A non-uniformity in temperature causes unequal expansion throughout the structure, and, as redundancies are usually present, stresses develop; this thermal stress system, which is self-equilibrating, affects the deformation characteristics of the structure, causing usually an effective reduction in stiffness. Fig. 5 is an example illustration of this point, and shows the loss in torsional stiffness in a wing during the initial portion of an accelerated high speed flight; this case is a reflection of the well known fact that axial stresses may appreciably affect

Fig. 5. Influence of acceleration rate on the effective torsional stiffness of a solid steel double-wedge wing. (a) top. Flight histories. (b) bottom. Effective torsional stiffness histories (from ref. 3).
the torsional stiffness of thin beams. A further undesirable effect that may be brought about by high temperatures is that the material behavior may become inelastic and time dependent. This is a vastly complicated phenomenon, however, which lacks much in the way of fundamental knowledge and which seems to require extensive experimental study.

In the following sections on stress and stiffness analysis of elastic structures having temperature gradients, the use of an elevated temperature Young's modulus is implied.

A. Equations of elasticity

The direct approach to thermal stress analysis is to employ the ordinary equations of elasticity with modified stress-strain relations to take into account temperature gradients. The equations of elasticity have been treated extensively in textbooks (see for example [21] and [22]), and only a summary of the equations is given here.

1. 3-Dimensional. Let the reference axis system be orthogonal and designated by \( x, y, z \) with displacements represented by \( u, v, w \) and body forces by \( X, Y, Z \), respectively. The temperature above ambient is specified by \( \Delta T(x, y, z) \) and the direction cosines between a line normal to a surface and the axes are denoted by \( l, m, n \). The appropriate equations for a homogeneous isotropic body read as follows:

Equilibrium equations,

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0, \\
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0, \\
\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + Z = 0.
\]  

(3.1)

Boundary conditions,

\[
X_s = \sigma_x l + \tau_{xy} m + \tau_{xz} n, \\
Y_s = \sigma_y m + \tau_{yz} n + \tau_{xy} l, \\
Z_s = \sigma_z n + \tau_{xz} l + \tau_{yz} m.
\]  

(3.2)

Strain-displacement relations,

\[
\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \epsilon_z = \frac{\partial w}{\partial z}, \\
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.
\]  

(3.3)
Stress-strain relations,
\[ \varepsilon_x = \frac{1}{E} [\sigma_x - \mu (\sigma_y + \sigma_z)] + \alpha \Delta T, \]
\[ \varepsilon_y = \frac{1}{E} [\sigma_y - \mu (\sigma_x + \sigma_z)] + \alpha \Delta T, \]
\[ \varepsilon_z = \frac{1}{E} [\sigma_z - \mu (\sigma_x + \sigma_y)] + \alpha \Delta T. \]  
(3.4)
\[ \gamma_{xy} = \frac{1}{G} \tau_{xy}, \quad \gamma_{yz} = \frac{1}{G} \tau_{yz}, \quad \gamma_{zx} = \frac{1}{G} \tau_{zx}, \]
which inversely read
\[ \sigma_x = \lambda e + 2 G \varepsilon_x - \frac{\alpha E \Delta T}{1 - 2 \mu}, \]
\[ \sigma_y = \lambda e + 2 G \varepsilon_y - \frac{\alpha E \Delta T}{1 - 2 \mu}, \]
\[ \sigma_z = \lambda e + 2 G \varepsilon_z - \frac{\alpha E \Delta T}{1 - 2 \mu}, \]
(3.5)
\[ \tau_{xy} = G \gamma_{xy}, \quad \tau_{yz} = G \gamma_{yz}, \quad \tau_{zx} = G \gamma_{zx}, \]
where
\[ \alpha = \text{Coefficient of thermal expansion}, \]
\[ G = \frac{E}{2(1+\mu)}, \]
\[ e = \varepsilon_x + \varepsilon_y + \varepsilon_z, \]
\[ \lambda = \frac{\mu E}{(1+\mu)(1-2\mu)}. \]

The solution of any problem requires that all the foregoing equations be satisfied. However, if equations (3.5) and (3.3) are combined and substituted in equations (3.1), the following equations of equilibrium in terms of displacement are found
\[ (\lambda + G) \frac{\partial e}{\partial x} + G V^2 u + X - \frac{\alpha E}{1 - 2\mu} \frac{\partial \Delta T}{\partial x} = 0, \]
\[ (\lambda + G) \frac{\partial e}{\partial y} + G V^2 v + Y - \frac{\alpha E}{1 - 2\mu} \frac{\partial \Delta T}{\partial y} = 0, \]
\[ (\lambda + G) \frac{\partial e}{\partial z} + G V^2 w + Z - \frac{\alpha E}{1 - 2\mu} \frac{\partial \Delta T}{\partial z} = 0. \]  
(3.6)

The satisfaction of these equations, subject to the boundary conditions on \( u, v, \) and \( w, \) is now sufficient for unique solution. No general solution to the equations is known, but exact solutions for many special cases have been
found, and particularly when the equations are expressed in cylindrical and polar coordinate form.

2. 2-Dimensional (Plane stress). The rectangular axis system $x, y$ is used with $u, v$ the displacements and $X, Y$ the body forces. The temperature relative to ambient conditions is specified by $\Delta T(x, y)$ and the direction cosines between a line normal to an edge surface and the axes are designated by $l$ and $m$. In this case, the appropriate equations for a homogeneous elastic body are:

**Equilibrium equations,**

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0, \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y = 0. \tag{3.7}
\]

**Boundary conditions,**

\[
X_s = l\sigma_x + m\tau_{xy}, \quad Y_s = m\sigma_y + l\tau_{xy}. \tag{3.8}
\]

**Strain-displacement relations,**

\[
\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{3.9}
\]

**Stress-strain relations**

\[
\epsilon_x = \frac{1}{E}(\sigma_x - \mu \sigma_y) + \alpha \Delta T, \tag{3.10}
\]

\[
\epsilon_y = \frac{1}{E}(\sigma_y - \mu \sigma_x) + \alpha \Delta T, \tag{3.10}
\]

\[
\gamma_{xy} = \frac{1}{G}\tau_{xy},
\]

which inversely read

\[
\sigma_x = \frac{E}{1-\mu^2}(\epsilon_x + \mu \epsilon_y) - \frac{\alpha E \Delta T}{1-\mu^2},
\]

\[
\sigma_y = \frac{E}{1-\mu^2}(\epsilon_y + \mu \epsilon_x) - \frac{\alpha E \Delta T}{1-\mu^2},
\]

\[
\tau_{xy} = G \gamma_{xy}. \tag{3.11}
\]

Operations on equations (3.9) lead to the basic compatibility relation

\[
\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}, \tag{3.12}
\]

which, when expressed in terms of stresses, reads

\[
\nabla^2(\sigma_x + \sigma_y) = -(1+\mu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) - \alpha E \nabla^2 \Delta T. \tag{3.13}
\]
When no body forces are present, a significant simplification results by
the introduction of the Airy’s stress function $\phi$, such that

$$\begin{align*}
\sigma_x &= \frac{\partial^2 \phi}{\partial y^2}, \\
\sigma_y &= \frac{\partial^2 \phi}{\partial x^2}, \\
\tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y}.
\end{align*}$$

(3.14)

The introduction of these equations in equation (3.13) yields

$$\nabla^2 \nabla^2 \phi = \nabla^4 \phi = -\alpha E \nabla^2 \Delta T,$$

(3.15)

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

A complete solution to any 2-dimensional plane stress problem requires
the satisfaction of all of equations (3.7) through (3.10), or more simply, of
either equation (3.13) or (3.15), with due regard to boundary conditions.

It is perhaps worthy to mention the concept of equivalent body forces
that may be made with reference to the temperature terms, when the equa¬
tions of equilibrium are expressed in terms of the displacements, see equation
(3.6) in the case of 3-dimensions. In an analogous way, if equations (3.7) are
expressed in terms of the displacements through means of equations (3.11)
and (3.9), then it would be noted that for 2-dimensions the temperature terms
may be regarded as contributing to equivalent body forces defined as

$$X_1 = X - \frac{\alpha E}{1-\mu} \frac{\partial \Delta T}{\partial x}, \quad Y_1 = Y - \frac{\alpha E}{1-\mu} \frac{\partial \Delta T}{\partial y}.$$  

(3.16)

If these equivalent body forces are substituted in equation (3.13) with the
$\alpha E \nabla^2 \Delta T$ term suppressed, there results

$$\nabla^2 (\sigma_x + \sigma_y) = -(1+\mu) \left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} - \frac{\alpha E}{1-\mu} \nabla^2 \Delta T \right).$$

(3.17)

The stresses defined by this equation, however, are those which result from
the “applied” equivalent body forces only. On these stresses must be super¬
posed the initial stresses which come from the temperature but with initial
body forces added so that no strain results (these initial body forces exactly
cancel the part of equivalent body forces associated with temperature leaving
a net of only the given body forces). The initial stresses would be

$$\sigma_{xi} = -\frac{\alpha E \Delta T}{1-\mu}, \quad \sigma_{yi} = -\frac{\alpha E \Delta T}{1-\mu}, \quad \tau_{xyi} = 0$$

and hence the final stresses would be

$$\begin{align*}
\sigma_{x_i} &= \sigma_x + \sigma_{xi} = \sigma_x - \frac{\alpha E \Delta T}{1-\mu}, \\
\sigma_{y_i} &= \sigma_y + \sigma_{yi} = \sigma_y - \frac{\alpha E \Delta T}{1-\mu}, \\
\tau_{xyi} &= \tau_{xy},
\end{align*}$$

(3.18)
where $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ are the solutions of equation (3.17). If equations (3.18) are solved for $\sigma_x$ and $\sigma_y$ and the results are substituted in equation (3.17), there results

$$\nabla^2(\sigma_x + \sigma_y) = -(1-\mu) \left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) - \alpha E \nabla^2 \Delta T,$$

which agrees with equation (3.13).

A listing of the material properties that appear in the foregoing equations is given in table II for some materials of interest.

### B. Strains for large displacements

The foregoing equations of elasticity are linear as a consequence of the assumption of small displacements. In many applications, and particularly when thermal stresses are present, it is necessary to make explicit consideration of large deflection effects, which in general introduces non-linearities. The effect on strain of large deflection of plate elements, when stretching of the neutral surface is involved, is for example well known. It is of interest, however, to see what the resulting strains are when large displacements are present in all three of the orthogonal directions.

Thus, consider three fibers of length $dx$, $dy$, $dz$ emanating from a common point such that they are parallel respectively to the $x$, $y$, $z$ coordinate system. The displacement of the common point is designated by $u$, $v$, $w$, while the displacements of the ends of the fibers are equal to $u$, $v$, $w$ plus their incremental values, as shown in the following sketch.

If the final length of the $dx$ fiber is designated by $ds_x$, then the strain in this fiber is

$$\varepsilon_x = \frac{ds_x - dx}{dx} = \frac{ds_x}{dx} - 1.$$

(3.19)
The length $ds_x$ is given by

$$
\overline{ds_x}^2 = (dx + \frac{\partial u}{\partial x} \, dx)^2 + (dv/\partial x)^2 + (dw/\partial x)^2,
$$

which yields

$$
\frac{ds_x}{dx} = \left[ 1 + 2 \frac{\partial u}{\partial x} + \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right]^{1/2}.
$$

If this equation is expanded up to and including the second order term, and the results are substituted into equation (3.19), the following equation for the strain $\varepsilon_x$ is found

$$
\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial v}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2.
$$

(3.20a)

In a similar manner, the expressions for $\varepsilon_y$ and $\varepsilon_z$ are found to be

$$
\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial u}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2,
$$

(3.20b)

$$
\varepsilon_z = \frac{\partial w}{\partial z} + \frac{1}{2} \left( \frac{\partial u}{\partial z} \right)^2 + \frac{1}{2} \left( \frac{\partial v}{\partial z} \right)^2.
$$

(3.20c)

It is interesting to note that there is no second order term of $u$ in the first equation, or of $v$ in the second, or of $w$ in the third. The third term of the expansion causes the disappearance of these terms.

The shearing strain $\gamma_{xy}$ is found as follows. The length $ab$ in the above sketch is related to the fiber lengths $ds_x$ and $ds_y$ by the cosine law as follows

$$
\overline{ab}^2 = ds_x^2 + ds_y^2 - 2 ds_x ds_y \cos \psi.
$$

(3.21)

But $\psi = 90 - \gamma_{xy}$, and therefore $\cos \psi = \gamma_{xy}$. Equation (3.21), therefore yields the following equation for $\gamma_{xy}$

$$
\gamma_{xy} = \frac{ds_x^2 + ds_y^2 - \overline{ab}^2}{2 ds_x ds_y}.
$$

The substitution of the appropriate lengths in this equation (note, $\overline{ab}^2 = (x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2$) and expansion of the various terms, then yields for $\gamma_{xy}$ the equation

$$
\gamma_{xy} = \frac{\partial u}{\partial y} \left( 1 - \frac{\partial v}{\partial y} \right) + \frac{\partial v}{\partial y} \left( 1 - \frac{\partial u}{\partial y} \right) + \frac{\partial w}{\partial y} \frac{\partial w}{\partial x}.
$$

(3.22a)

By cyclic interchange of the subscripts the remaining two shearing strains are found as follows

$$
\gamma_{yz} = \frac{\partial v}{\partial z} \left( 1 - \frac{\partial w}{\partial z} \right) + \frac{\partial w}{\partial z} \left( 1 - \frac{\partial v}{\partial z} \right) + \frac{\partial w}{\partial z} \frac{\partial u}{\partial y},
$$

(3.22b)

$$
\gamma_{zx} = \frac{\partial w}{\partial x} \left( 1 - \frac{\partial v}{\partial x} \right) + \frac{\partial v}{\partial x} \left( 1 - \frac{\partial w}{\partial x} \right) + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y}.
$$

(3.22c)
The expressions for strains that have been conventionally used in the large deflection theory of plate are noted to be subcases of the foregoing general expressions.

C. Derivation of energy equation for plate-like structures with temperature gradients

The following is one means for deriving the energy equations applicable to the deformation of plate-like structures having temperature gradients. In the derivation, use will be made of the concept of “equivalent body forces” mentioned previously to take into account effects of temperature.

Consider the plate to have the coordinate system shown in the adjacent sketch

Let the temperature gradients $\Delta T(x, y)$ be applied first, but also apply supplemental body forces (1) and edge pressures to allow no strain. The stresses, body forces, and edge pressures would be (from equations (3.11) and (3.7))

\[
\begin{align*}
\sigma_x &= -\frac{\alpha E \Delta T}{1-\mu}, \\
X_1 &= \frac{\alpha E}{1-\mu} \frac{\partial \Delta T}{\partial x}, \\
\sigma_y &= -\frac{\alpha E \Delta T}{1-\mu}, \\
Y_1 &= \frac{\alpha E}{1-\mu} \frac{\partial \Delta T}{\partial y}, \\
\gamma_{xy} &= 0, \\
\sigma_{edge} &= \frac{\alpha E \Delta T}{1-\mu}.
\end{align*}
\]

Now apply the transverse loading $p_z$, the given body forces $X$, $Y$, and the given edge forces $F_x$, $F_y$, and $F_{xy}$, and also apply supplemental body forces (2), $X_2 = -\frac{\alpha E}{1-\mu} \frac{\partial \Delta T}{\partial x}$, $Y_2 = -\frac{\alpha E}{1-\mu} \frac{\partial \Delta T}{\partial y}$, and edge tensions of $\frac{\alpha E \Delta T}{1-\mu}$ and calculate the change in internal energy minus the work done by all the applied forces.

The stresses after displacement are

\[
\begin{align*}
\sigma_x &= \frac{E}{1-\mu^2} (\epsilon_x + \mu \epsilon_y) - \frac{\alpha E \Delta T}{1-\mu}, \\
\sigma_y &= \frac{E}{1-\mu^2} (\epsilon_y + \mu \epsilon_x) - \frac{\alpha E \Delta T}{1-\mu}, \\
\tau_{xy} &= G \gamma_{xy}.
\end{align*}
\]
Thus, the average stresses would be
\[
\begin{align*}
\sigma_x + \sigma_{x_0} &= \frac{1}{2} \frac{E}{1 - \mu^2} \left( \varepsilon_x + \mu \varepsilon_y \right) - \frac{\alpha E \Delta T}{1 - \mu}, \\
\sigma_y + \sigma_{y_0} &= \frac{1}{2} \frac{E}{1 - \mu^2} \left( \varepsilon_y + \mu \varepsilon_x \right) - \frac{\alpha E \Delta T}{1 - \mu}, \\
\tau_{xy} + \tau_{xy_0} &= \frac{1}{2} G \gamma_{xy}. 
\end{align*}
\]

The internal work per differential volume performed by these average stresses during displacement would then follow as
\[
dU = \left[ \frac{1}{2} \frac{E}{1 - \mu^2} (\varepsilon_x + \mu \varepsilon_y) \varepsilon_x - \frac{\alpha E \Delta T}{1 - \mu} \varepsilon_x + \frac{1}{2} \frac{E}{1 - \mu^2} (\varepsilon_y + \mu \varepsilon_x) \varepsilon_y - \frac{\alpha E \Delta T}{1 - \mu} \varepsilon_y + \frac{1}{2} G \gamma_{xy} \right] dx dy dz, \quad (3.26)
\]
\[
= \left[ \frac{1}{2} \frac{E}{1 - \mu^2} (\varepsilon_x^2 + \varepsilon_y^2 + 2 \mu \varepsilon_x \varepsilon_y) + \frac{1}{2} G \gamma_{xy}^2 - \frac{\alpha E \Delta T}{1 - \mu} (\varepsilon_x + \varepsilon_y) \right] dx dy dz.
\]

The work done by the supplemental body forces (2) cancels the work done by the existing supplemental body forces (1), since these forces are equal and opposite and since the (2) forces must be regarded as coming from a potential (that is, constant during displacement); likewise the work of the supplemental edge tensions cancels the work of the supplemental edge pressures. The only applied forces which yield a net work then are the given loads, and the work done by these loads is
\[
V = \iiint p_x w d x d y + \iiint (X u + Y v) d x d y d z + \int \left[ F_x u_0 + F_{xy} v_0 \right] \bigg|_{x_0}^{x} d y + \int \left[ F_y v_0 + F_{xy} u_0 \right] \bigg|_{y_0}^{y} d x, \quad (3.27)
\]

Now, if equation (3.26) is integrated over the plate volume and then combined with equation (3.27), the following energy equation appropriate to the deformation of plate-like structures with temperature gradients is found
\[
W = U - V = \iiint \left[ \frac{1}{2} \frac{E}{1 - \mu^2} (\varepsilon_x^2 + \varepsilon_y^2 + 2 \mu \varepsilon_x \varepsilon_y) + \frac{E}{4(1 + \mu)} \gamma_{xy}^2 - \frac{\alpha E \Delta T}{1 - \mu} (\varepsilon_x + \varepsilon_y) \right] d x d y d z - V. \quad (3.28)
\]

The energy theorem that applies to this expression is the theorem of minimum potential energy, which states that among all strain states satisfying the condition of compatibility and geometric boundary conditions, the actual state of equilibrium is the one which makes the potential energy a minimum. This minimum condition represents in reality the principle of virtual work, which
states that equilibrium exists when for any geometrically possible virtual
displacement, the work done by all the prescribed forces is equal to the change
in strain energy.

The application of this equation to determine the differential equation of
equilibrium applicable to thin wing sections having large deflections is made
in the next section.

D. Application of energy equation to derive differential equations of equilibrium
for large deflections

Consider that it is desired to determine the differential equations of equi¬
librium that are applicable to the large deflection of wings having cross sec¬
tions of the following type.

The first section is solid, of variable thickness and may have camber. The
second section may have variable thickness and camber also, but is built up
with a top and bottom cover plate, and a core material which is considered to
contribute virtually no stiffness against deformations. It is remarked that the
analysis of the type to follow can be made to apply to built-up thin wings by
adding the appropriate energy terms for whatever stringers, spar webs, or
ribs are present.

The analysis is based on the assumption that the deformations of the wing
can be expressed in terms of the deflections and in-the-plane distortions of a
surface contained originally in the \(xy\) plane. Thus consider that the deflection
of this surface (and all points in the thickness direction) is denoted by \(w\), and
for displacements \(u\) and \(v\) assume that

\[
\begin{align*}
\quad u &= u_0 - z \frac{\partial w}{\partial x}, \\
\quad v &= v_0 - z \frac{\partial w}{\partial y},
\end{align*}
\]

but regard \(u\) and \(v\) small relative to \(w\). From equations (3.20) and (3.22) the
strains follow then as

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2, \\
\varepsilon_y &= \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w}{\partial y^2} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2, \\
\gamma_{xy} &= \frac{\partial u_0}{\partial y} - z \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial v_0}{\partial x} - z \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}.
\end{align*}
\]

These expressions are substituted in equation (3.28), and the resulting
energy expression is then minimized by means of the calculus of variations,
where variations in $u_0$, $v_0$, and $w$ are allowed, to yield the appropriate differential equations of the problem. The equations found for variations in $u_0$ and $v_0$ are respectively

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + \bar{X} = 0, \quad \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} + \bar{Y} = 0,$$

(3.31a)

with boundary terms

$$(N_x - F_x) \delta u_0 \bigg|_{x_1}^{x_2} = 0, \quad (N_y - F_y) \delta v_0 \bigg|_{y_1}^{y_2} = 0,$$

$$\left(N_{xy} - F_{xy}\right) \delta u_0 \bigg|_{y_1}^{y_2} = 0, \quad \left(N_{xy} - F_{xy}\right) \delta v_0 \bigg|_{x_1}^{x_2} = 0,$$

(3.31b)

where

$$N_x = \int \sigma_x \, dz, \quad \bar{X} = \int \bar{X} \, dz,$$

$$N_y = \int \sigma_y \, dz, \quad \bar{Y} = \int \bar{Y} \, dz,$$

$$N_{xy} = \int \tau_{xy} \, dz.$$

Note, considerable algebraic manipulation may be avoided in performing the variational process by choosing the sequence of operations in a judicious manner. It is not advisable, for example, to substitute the expressions for strain in the energy expression, to expand the results, integrate in the thickness direction and then perform the variational process. It is better to first vary the energy expression in an implicit sense ($2 \epsilon_{xy} \delta \epsilon_x$ for the term $\epsilon_{xy}^2$, for example), then to substitute in the expressions for strain, to group together terms which can be recognized as a single quantity, like stress, and then to integrate in the thickness direction.

Before considering the equation obtained from the variation of $w$, it is considered desirable to show how the two equations, (3.31a), may be reduced to a single equation by the introduction of a stress function $\phi$. If

$$N_x = h_r \frac{\partial^2 \phi}{\partial y^2}, \quad N_y = h_r \frac{\partial^2 \phi}{\partial x^2},$$

$$N_{xy} = -h_r \frac{\partial^2 \phi}{\partial x \partial y} - \bar{X} y - \bar{Y} x,$$

(3.33)

where $h_r$ is a convenient reference thickness and where $\bar{X}$ is assumed to be a function of $x$ only, $\bar{Y}$ a function of $y$ only, then equations (3.31a) are satisfied identically. Now integrate the expressions for strains (3.30) across the thickness of the plate, to yield
\[ \int_{h_i}^{h_s} \epsilon_x \, dz = \left[ \frac{\partial w_0}{\partial x} - \epsilon \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] h, \]

\[ \int_{h_i}^{h_s} \epsilon_y \, dz = \left[ \frac{\partial w_0}{\partial y} - \epsilon \frac{\partial^2 w}{\partial y^2} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right] h, \]

\[ \int_{h_i}^{h_s} \gamma_{xy} \, dz = \left( \frac{\partial w_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2\epsilon \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w \partial v}{\partial x \partial y} \right) h, \quad (3.34) \]

where \( h \) is to be regarded as the thickness of the stress carrying material only, not necessarily the total plate thickness, and \( \epsilon \) denotes the centroidal distance.

With appropriate differentiations of these expressions (after division by \( h \)), the following compatibility equation is found

\[ \frac{\partial^2}{\partial y^2} \frac{1}{h} \int \epsilon_x \, dz + \frac{\partial^2}{\partial x^2} \frac{1}{h} \int \epsilon_y \, dz - \frac{\partial^2}{\partial x \partial y} \frac{1}{h} \int \gamma_{xy} \, dz \]

\[ = \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2}{\partial x^2} \frac{\epsilon \partial^2 w}{\partial y^2} - 2 \frac{\partial^2}{\partial x \partial y} \frac{\epsilon \partial^2 w}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \frac{\epsilon \partial^2 w}{\partial x^2} \right). \quad (3.35) \]

To introduce the stress function in this compatibility equation, the expressions for strains as given by equations (3.10) are integrated in the thickness direction, to give, with the use of equations (3.32), the following expressions

\[ \int_{h_i}^{h_s} \epsilon_x \, dz = \frac{1}{E} (N_x - \mu N_y) + \alpha h_r \Phi, \quad \Phi = \frac{1}{h_r} \int_{h_i}^{h_s} \Delta T \, dz, \]

\[ \int_{h_i}^{h_s} \epsilon_y \, dz = \frac{1}{E} (N_y - \mu N_x) + \alpha h_r \Phi, \]

\[ \int_{h_i}^{h_s} \gamma_{xy} \, dz = \frac{1}{G} N_{xy}. \quad (3.36) \]

The substitution of these equations into equation (3.35), and the use of equations (3.33), gives the compatibility equation for large deflections of the plate in terms of the stress function finally in the form

\[ \frac{\partial^2}{\partial x^2} \frac{h_r}{E h} \frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial^2}{\partial x \partial y} \frac{h_r}{E h} \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \frac{h_r}{E h} \frac{\partial^2 \phi}{\partial y^2} \]

\[ - \frac{\mu}{E} \left( \frac{\partial^2}{\partial x^2} \frac{h_r}{E h} \frac{\partial^2 \phi}{\partial y^2} - 2 \frac{\partial^2}{\partial x \partial y} \frac{h_r}{E h} \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} \frac{h_r}{E h} \frac{\partial^2 \phi}{\partial x^2} \right) = - \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \frac{\alpha h_r \Phi}{h} \]

\[ + \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x^2} \epsilon \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 w}{\partial x \partial y} \epsilon \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} \epsilon \frac{\partial^2 w}{\partial x^2} \right), \quad (3.37) \]

where the body force terms have been dropped. For \( h \) constant and \( e = 0 \), equation (3.37) reduces simply to

\[ \nabla^2 \nabla^2 \phi = - \alpha E \nabla^2 \Delta T + E \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}. \quad (3.38) \]

37
The equation for \( w \) is now considered. It may be shown that the variation of the energy expression for small perturbations \( \delta w \) in the deflection, leads to the equation (again dropping body force terms)

\[
\frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p_z - \frac{\partial}{\partial x} N_x \frac{\partial w}{\partial x} - \frac{\partial}{\partial y} N_y \frac{\partial w}{\partial y} - \frac{\partial}{\partial x} N_{xy} \frac{\partial w}{\partial y} - \frac{\partial}{\partial y} N_{xy} \frac{\partial w}{\partial x},
\]

(3.39a)

with boundary conditions:

1. \( M_x = 0 \) \hspace{1cm} \text{or} \hspace{1cm} \delta \frac{\partial w}{\partial x} = 0 \hspace{1cm} x = x_1 \text{ and } x_2,

2. \( \frac{\partial M_x}{\partial x} - 2 \frac{\partial M_{xy}}{\partial y} + N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} = 0 \) \hspace{1cm} \text{or} \hspace{1cm} \delta w = 0 \hspace{1cm} \text{at} \hspace{1cm} x = x_1 \text{ and } x_2,

3. \( M_y = 0 \) \hspace{1cm} \text{or} \hspace{1cm} \delta \frac{\partial w}{\partial y} = 0 \hspace{1cm} y = y_1 \text{ and } y_2,

4. \( \frac{\partial M_y}{\partial y} - 2 \frac{\partial M_{xy}}{\partial x} + N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} = 0 \) \hspace{1cm} \text{or} \hspace{1cm} \delta w = 0 \hspace{1cm} \text{at} \hspace{1cm} y = y_1 \text{ and } y_2,

5. \( 2 M_{xy} = 0 \) \hspace{1cm} \text{or} \hspace{1cm} \delta w = 0 \hspace{1cm} \text{at} \hspace{1cm} y = y_1 \text{ and } y_2,

where

\[
M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right) + e N_x + \frac{\alpha E}{1 - \mu} (e h_r \bar{T} - h_r^2 \bar{T}_1),
\]

\[
M_y = -D \left( \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) + e N_y + \frac{\alpha E}{1 - \mu} (e h_r \bar{T} - h_r^2 \bar{T}_1),
\]

\[
M_{xy} = D (1 - \mu) \frac{\partial^2 w}{\partial x \partial y} - e N_{xy}
\]

and

\[
D = \frac{E h k_0^2}{1 - \mu^2}, \quad \bar{T} = \frac{1}{h_r} \int \Delta T \, dz, \quad \bar{T}_1 = \frac{1}{h_r^2} \int \Delta T \, dz.
\]

(3.39c)

(3.39d)

The fifth boundary condition indicates that a concentrated force develops at a corner when no deflection is permitted.

Substitution now of equations (3.39c) into equation (3.39a), and the use of the equilibrium equations (3.31a), gives finally the deflection equation

\[
\frac{\partial^2}{\partial x^2} D \left( \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right) + 2 \frac{\partial}{\partial x} \frac{\partial^2}{\partial x \partial y} D (1 - \mu) \frac{\partial w}{\partial y^2} + \frac{\partial^2}{\partial y^2} D \left( \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) = p_z + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y}
\]

\[
+ \frac{\partial^2 e N_x}{\partial x^2} + \frac{\partial^2 e N_y}{\partial y^2} + \frac{\partial^2 e N_{xy}}{\partial x \partial y} + \frac{\alpha E}{1 - \mu} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (e h_r \bar{T} - h_r^2 \bar{T}_1).
\]

(3.40)
This equation (with equation (3.33)), and equation (3.37) define completely the large deflection behavior of the platelike structures being considered under any combination of transverse loading, edge stresses, and "temperature". The appearance of terms which tend to deflect the plate even in the absence of applied loads is to be noted for the case of non-uniform temperature distribution in the thickness direction.

For \( t \) constant and \( e = 0 \), equation (3.40) reduces to

\[
P^2 \nabla^2 w = \frac{1}{D} \left( \mu + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} - \frac{\alpha E h^2}{1-\mu} \nabla^2 T_1 \right). \tag{3.41}
\]

This equation and equation (3.38), with temperature terms suppressed, correspond to the large deflection equations that were developed first by von Karman for plates of uniform thickness.

E. A simplified way of deriving the energy equation for plate bending when non-uniform in-the-plane stresses are present and no stretching of the middle surface is allowed

The derivation given by Timoshenko [21], for the energy equation applicable to combined bending and tension or compression of plates has often been referred to and used. This derivation, while leading to the correct answer, has never been quite satisfying to the author because of apparent inconsistencies in the manner by which the middle surface strains are handled (i.e., stresses are assumed to remain constant during bending, but to compute the energy of these middle surface forces, changes in strains are assumed to exist; later the assumption of no-stretching is introduced and this would seem to invalidate the earlier work, which appears somewhat inconsistent in itself).

The following is suggested as a simplified means for deriving the additional energy terms that take into account the presence of in-the-plane stresses during bending of plates when no stretching is allowed. The argument runs along the following lines. If the middle surface stresses are uniform during displacement and no stretching of the middle surface is allowed, then these stresses can of course do no work. But the only way in which uniform stresses can be induced is by edge forces. The consideration of the work done by these edge forces is then sufficient to obtain the correct energy expressions. (Note, when clamped edges are considered in this scheme, the edges, while not permitting rotation, must be considered free to move together, consistent with the assumption that the stresses remain constant.) In the case of non-uniform middle stresses, again no work can be performed by these stresses if no stretching is assumed. However, in this case, the only way in which non-uniform stresses can be brought about is by the application of body forces.
This is thus the key to the analysis; the consideration of the work done during displacement by these "substitute" body forces leads to the correct energy expression. Thus, for a plate (assuming for convenience constant thickness) under the action of transverse loading, edge and body forces, the complete potential energy expression would be

\[
U - V = \frac{D}{2} \int \int \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + 2 \mu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2(1-\mu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \, dx \, dy
\]

\[
- \int \left( \bar{X} u + \bar{Y} v \right) d x \, d y - \int p w \, d x \, d y - \int \left( u F_x + v F_y \right) \left. \right|_{x_0}^{x_1} \, dy
\]

(3.42)

\[
- \left. \int (v F_y + u F_{xy}) \right|_{y_0}^{y_1} \, dx.
\]

where, according to the concept mentioned above, the body forces are those which would produce the given middle surface stress distribution. From the equations of equilibrium, equations (3.31), these body forces would have to be

\[
\bar{X} = - \frac{\partial N_x}{\partial x} - \frac{\partial N_{xy}}{\partial y}, \quad \bar{Y} = - \frac{\partial N_y}{\partial y} - \frac{\partial N_{xy}}{\partial x},
\]

where in this case \( N_x = h \sigma_x, \) \( N_y = h \sigma_y, \) \( N_{xy} = h \tau_{xy}, \) and \( \bar{X} \) and \( \bar{Y} \) represent the summation of body forces across the thickness. The substitution of these equations into expression (3.42), gives, after integration by parts

\[
U - V = U - \int \int \left[ N_x \frac{\partial u}{\partial x} + N_{xy} \frac{\partial u}{\partial y} + N_y \frac{\partial v}{\partial y} + N_{xy} \frac{\partial v}{\partial x} + p w \right] \, d x \, d y. \quad (3.43)
\]

The assumption of no middle surface strains \( (\epsilon_x = \epsilon_y = \gamma_{xy} = 0) \) is now introduced; thus equations (3.30), with \( z = 0 \), yield

\[
\frac{\partial u}{\partial x} = - \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2, \quad \frac{\partial v}{\partial y} = - \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2.
\]

Then with substitution of these relations into equation (3.43), the following well known energy expression appears to complete the derivation

\[
U - V = \frac{D}{2} \int \int \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + 2 \mu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2(1-\mu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \, dx \, dy
\]

\[
+ \frac{1}{2} \int \int \left[ N_x \left( \frac{\partial w}{\partial x} \right)^2 + N_y \left( \frac{\partial w}{\partial y} \right)^2 + 2 N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \, dx \, dy - \int p w \, d x \, d y.
\]

F. The use of the stress function for obtaining thermal stresses

1. Derivation of differential equation and boundary conditions. While the concept of Airy’s stress function for stress analysis purposes is well known and extensively applied, little discussion appears to exist on the associated boundary conditions. The following treatment is therefore given to develop
these boundary conditions further, and to lead into the analogy that can be drawn between the stress function and plate bending.

The strain energy of a plate in terms of stresses can be shown to be

\[
U = \frac{h}{2E} \int \left[ \sigma_x^2 + \sigma_y^2 - 2\mu\sigma_x\sigma_y + 2(1+\mu)\tau_{xy}^2 + 2\alpha E\Delta T(\sigma_x + \sigma_y) \right] dxdy. \tag{3.45}
\]

This form of the energy is usually regarded as the complementary energy; with it there is associated a second principal energy theorem which states: among all possible states of stress which satisfy the equations of equilibrium and boundary conditions, the actual state is the one which makes the complementary energy a minimum. Now the equations of equilibrium (with body force terms suppressed) can be identically satisfied by the introduction of the stress function \( \phi \), such that \( \sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \). Thus, equation (3.45) in terms of the stress function would appear

\[
U = \frac{h}{2E} \int \left[ \left( \frac{\partial^2 \phi}{\partial y^2} \right)^2 + \left( \frac{\partial^2 \phi}{\partial x^2} \right)^2 - 2\mu \frac{\partial^2 \phi}{\partial y^2} \frac{\partial^2 \phi}{\partial x^2} + 2(1+\mu) \left( \frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \right] dxdy. \tag{3.46}
\]

Application now of the conventional minimization process by means of the calculus of variations leads to the differential equation for the stress function and the associated boundary conditions. The result is the differential equation (which agrees with equation (3.15))

\[
\nabla^2 \nabla^2 \phi + \alpha E \nabla^2 \Delta T = 0 \tag{3.47a}
\]

and the boundary conditions:

Along \( x = x_1 \), and \( x = x_2 \),

1. \( \phi_{xx} - \mu \phi_{yy} + \alpha E \Delta T = 0 \) or \( \delta \phi_x = 0 \),
2. \( \phi_{xxx} + (2 + \mu) \phi_{xyy} + \alpha E (\Delta T)_x = 0 \) or \( \delta \phi = 0 \).

Along \( y = y_1 \) and \( y = y_2 \),

3. \( \phi_{yy} - \mu \phi_{xx} + \alpha E \Delta T = 0 \) or \( \delta \phi_y = 0 \),
4. \( \phi_{yyy} + (2 + \mu) \phi_{xxy} + \alpha E (\Delta T)_y = 0 \) or \( \delta \phi = 0 \). \tag{3.47b}

At \( x = x_1 \) and \( x_2 \), and \( y = y_1 \) and \( y_2 \)

5. \( \frac{2(1+\mu)}{E} \phi_{xy} = 0 \) or \( \delta \phi = 0 \).

Since the strain energy was used in terms of stresses, and the equations of equilibrium were satisfied, it would be expected that the boundary conditions are conditions of compatibility, which indeed they are. It is of interest to
note that b. c. (5) leads to concentrated "corner forces" of magnitude 
\[ \frac{2(1+\mu)}{E} \frac{\partial^2 \phi}{\partial x \partial y} \]
whenever the stress function is restrained along the edges, just as in the case of plates, where, for example, the corners of a simply supported rectangular plate have to be "held down" when the plate is loaded.

Some physical insight into the four other b. c. follows, where an edge \( x = x_2 \) is chosen for illustration purposes; but first it is worthwhile to take cognizance of the following relations

\[ \varepsilon_y = \frac{\partial^2 u}{\partial y^2} = 0 \]
\[ \frac{\partial^2 u}{\partial y^2} = 0 \]
\[ \frac{\partial^2 u}{\partial x^2} = 0 \]
\[ \frac{\partial^2 u}{\partial y^2} = 0 \]

From b. c. (1), \( \delta \phi_x = 0 \)
From b. c. (2), \( \delta \phi = 0 \)

Note: \( \delta \phi = \delta \phi_x = 0 \) automatically insures that \( \sigma_x = \tau_{xy} = 0 \).

**Free edge**

\[ \varepsilon_y = \frac{\partial v}{\partial y} = 0 \]
\[ \frac{\partial^2 u}{\partial y^2} = 0 \]
\[ \frac{\partial^2 u}{\partial y^2} = 0 \]

From b. c. (1), \( \delta \phi_x = 0 \)
From b. c. (2), \( \delta \phi = 0 \)

**Partially clamped I**

\[ \varepsilon_y = \frac{\partial v}{\partial y} = 0 \]
\[ \frac{\partial^2 u}{\partial y^2} = 0 \]

From b. c. (1), \( \delta \phi_x = 0 \)
From b. c. (2), \( \delta \phi = 0 \)

**Partially clamped II**

\[ \varepsilon_y = \frac{\partial v}{\partial y} = 0 \]
\[ \frac{\partial u}{\partial y} = 0; \text{ thus } \frac{\partial^2 u}{\partial y^2} = 0 \]

From b. c. (1), \( \delta \phi_x = 0 \)
From b. c. (2), \( \delta \phi = 0 \)

**Fully clamped**

\[ \varepsilon_y = \frac{\partial v}{\partial y} = 0 \]
\[ \frac{\partial u}{\partial y} = 0; \text{ thus } \frac{\partial^2 u}{\partial y^2} = 0 \]

From b. c. (1), \( \delta \phi_x = 0 \)
From b. c. (2), \( \delta \phi = 0 \)
A listing of some of the properties of the surface defined by the Airy's stress function follows. The \( xy \) plane is taken as the plane of stress, and is horizontal; the ordinates to the surface \( \phi \) are plotted vertically. Consider two points, \( A \) and \( B \), on the surface, with their corresponding projections, \( a \) and \( b \), on the \( xy \) plane.

1. The planes tangent to \( \phi \) at \( A \) and \( B \) intersect along a line which projects on the \( xy \) plane into the line of action of the resultant force on any section between \( a \) and \( b \).

2. The change in slope in the direction of the straight line \( ab \), between the planes tangent to \( \phi \) at \( A \) and \( B \), is numerically equal to the component in the direction perpendicular to the line \( ab \), of the resultant force transmitted between \( a \) and \( b \).

3. The change in slope perpendicular to the line \( ab \), between the planes tangent to \( \phi \) at \( A \) and \( B \), is numerically equal to the component in the direction of \( ab \), of the resultant force transmitted between \( a \) and \( b \).

4. The distance to the surface \( \phi \) at point \( B \), measured perpendicular to the \( xy \) plane from the tangent plane at point \( A \), is numerically equal to the moment about an axis through \( b \) of the forces between \( a \) and \( b \) transmitted across any section through \( a \) and \( b \).

5. Consider a vertical section through \( a, b, A, \) and \( B \). The ordinates to \( \phi \), measured perpendicular to the \( xy \) plane from the chord through points \( A \) and \( B \), are numerically equal to the moments in a conjugate simple beam supported at \( a \) and \( b \), and loaded with an intensity of vertical loading equal at every point to the stress normal to \( ab \) at the corresponding point in the plane of stress.

2. The analogy with plate bending. An analogy of some interest is readily apparent between plates in bending and the stress function. The following formulation is made for the case of constant plate thickness, but may be extended without difficulty to variable thickness plates. Thus, the equations applicable to plate bending may be obtained from the stress function equations (and vice-versa) by replacing terms as follows:

<table>
<thead>
<tr>
<th>Replace</th>
<th>by</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi )</td>
<td>( w )</td>
</tr>
<tr>
<td>( \mu )</td>
<td>( -\mu )</td>
</tr>
<tr>
<td>( E )</td>
<td>( \frac{1}{D} )</td>
</tr>
<tr>
<td>( \varepsilon_y - \alpha \Delta T )</td>
<td>( -M_x )</td>
</tr>
<tr>
<td>( \varepsilon_x - \alpha \Delta T )</td>
<td>( -M_y )</td>
</tr>
<tr>
<td>( \frac{1}{2} \gamma_{xy} )</td>
<td>( -M_{xy} )</td>
</tr>
<tr>
<td>( \alpha \nabla^2 \Delta T )</td>
<td>( -\rho )</td>
</tr>
</tbody>
</table>
The following interesting pictorial comparison for the four types of edge condition mentioned previously may also be made.

\[
\begin{align*}
\phi &= 0, \quad \frac{\partial \phi}{\partial x} = 0; \quad w = 0, \quad \frac{\partial w}{\partial x} = 0; \\
\phi &= 0, \quad \epsilon_y = 0; \quad w = 0, \quad M_x = 0; \\
\frac{\partial \phi}{\partial x} &= 0, \quad \frac{\partial^2 u}{\partial y^2} = 0; \quad \frac{\partial w}{\partial x} = 0, \quad R_x = 0; \\
\epsilon_y &= 0, \quad \frac{\partial^2 u}{\partial y^2} = 0; \quad M_x = 0, \quad R_x = 0.
\end{align*}
\]

Edge condition in stress problem  Associated stress function b.c. or analogous plate b.c. in bending

As an example, consider a uniform rectangular free-free plate subject to a given temperature distribution; the stress function must then have a shape as depicted in the adjacent sketch.

By analogy, this shape can be exactly duplicated by considering the plate to be clamped along all four edges and loaded with a distribution \( \alpha E E^2 \Delta T \). The surface strains (curvatures) due to bending will then be proportional to the stresses that are present in the plate as a result of the applied temperature (i.e., the curvature in the \( y \)-direction gives the temperature stresses in the \( x \)-direction, etc.).

It should be recognized that while the two systems can be made mathematically analogous for non-homogeneous conditions on \( \phi \) and \( \frac{\partial \phi}{\partial x} \) at the boundaries, they will not be strictly so in a physical sense because the effect of Poisson’s ratio is just opposite in the two systems. When \( \phi \) and \( \frac{\partial \phi}{\partial x} \) at the boundaries are homogeneous, however, Poisson’s ratio does not appear and the systems become physically analogous as well. But whatever the situation, one of the chief merits of the analogy is that it provides a handy means for visualizing what shape the stress function should take, which is particularly desirable when the solution is made by energy or Galerkin procedures, since it is generally more easy to visualize how a plate bends under a given loading.
than it is to visualize the stress distribution that results when temperature is applied.

3. Examples. An assumed stress function used often for approximating the temperature stresses in a rectangular cantilever plate is given by the expression

$$\phi = K \alpha E \Delta T_1 (x^2 - a^2)(y^2 - b^2)^2,$$

(3.48)

where the coordinates are shown in fig. 6 ($b$ is used here to denote the half-width). By the energy approach, this equation is substituted into equation (3.46), and a minimization is made with respect to $K$. Reference [23] has shown, for example, that for a temperature distribution of the form

$$\Delta T = \Delta T_0 + \Delta T_1 \left(\frac{y}{b}\right)^n,$$

(3.49)

Fig. 6. Thermal stresses in square cantilever plate, $\Delta T = \Delta T_1 \left(\frac{y}{b}\right)^2$. 
the results for $K$ for a square plate ($a=2b$) would be

$$K = \frac{2.722}{a^6} \frac{n}{(n+3)(n+1)}.$$

Now it so happens that for this particular problem the temperature distribution that leads to the stress function given by equation (3.48) can be solved for exactly. Thus, it may be verified that the following temperature equation, together with equation (3.48) satisfy equation (3.47a) identically,

$$-\Delta T = KAT'J12(x^4 y^2 + x^2 y^4 - 4 y^4 (2 a^2 + b^2) - 4 x^4 (2 b^2 + a^2) + [6 (a^4 + b^4) + 8 a^2 b^2] [a_0 x^2 + (2 - a_0) y^2]) + a_1 x + b_1 y + c_1,$$

where $a_0$ is any value between 0 and 2, and $a_1$, $b_1$, and $c_1$ are arbitrary. The vast difference between this expression and equation (3.49), would make one wonder just how good the results of the foregoing example are. To obtain some measure of the accuracy of the results, the stresses that would develop due to a parabolic temperature distribution ($n=2$ in equation (3.49)) were computed by a difference-equation technique that is to be presented in the next section, and the results that were obtained are given in fig. 6, where a comparison is made with the approximate stresses that are obtained from equation (3.48). It is noted that in many areas of the plate, the approximate stresses are considerably in error.

The difference-equation solution also reveals that, even for this square plate, there is a tendency for the stress function to develop into a cylindrical surface at the root. The $\sigma_x$ stresses shown along $y=0$ and $y=b$ are of special significance, since the $\sigma_x$ stresses have a primary effect on the effective torsional stiffness of the wing. Better agreement would of course be obtained by adding additional "modal function" terms to equation (3.48), but then the simplicity of the expression would be lost, and it would be easier to make a difference-equations solution.

4. Numerical Solution of Plate Stress and Bending Equations

In practical considerations of the various problems that have been discussed so far in this paper (heat flow, temperature, stresses, plate bending), where exact solutions are not possible, two of the more popular and powerful methods for obtaining approximate solutions are the Rayleigh-Ritz and Galerkin procedures. The chief liabilities of these two approaches are the difficulty of choosing appropriate expansion functions when other than simple planforms are involved (particularly in the case of the Galerkin process, where, strictly speaking, each of the chosen functions should satisfy all the boundary conditions, not just the geometric conditions as in the case of the Rayleigh-
Ritz approach), and the excessive amount of set-up time involved, especially if a large number of functions are required to obtain converged results.

Another powerful means for solving these problems is by numerical methods, such as the use of difference equations. Although difference equations have been employed extensively in certain fields of study, such as heat flow, their application to aeronautical plate bending and temperature stress problems does not appear to be widespread. Mention should be made that difference equations have been developed extensively in oblique as well as rectangular coordinates in Civil Engineering studies of concrete slabs of skewed and rectangular planform [24], but most of these studies appear confined to slabs of uniform thickness. Mention should also be made of a recent excellent study pertaining to built-up wings [25], which makes use of difference equations along one coordinate, but employs chosen modal functions in the other coordinate direction.

Since it is the opinion of the author that difference-equations techniques merit greater use, especially since high-speed large-capacity computers are now extensively used, the next section is devoted to the development of the difference equations that apply for the bending of plates of variable thickness. As indicated by the plate bending-stress function analogy discussed earlier, a similar development would apply for the temperature stress problem. Following this section, a new numerical summation-equation method is discussed.

A. By difference equations

1. Derivation of difference equations. The popular derivation proceeds by replacing the differentials in the governing differential equation by the approximating difference expressions. For points near or at a boundary, the differentials are expressed in terms of fictitious exterior points, and these fictitious points are then eliminated by means of the boundary conditions, which are also expressed in difference form. As a simple illustration, consider the difference equation applicable to point 4 in the following uniform cantilever example

\[ EI \frac{d^4 w}{dx^4} = p. \]

In terms of the well known difference approximation for \( \frac{d^4 w}{dx^4} \), the equation for point 4 would first read

\[ \frac{EI}{\epsilon^4} (w_2 - 4 w_3 + 6 w_4 - 4 w_5 + w_6) = p_4. \]
However, to insure zero moment at point 5, \( EI \frac{d^2 w}{dx^2} \bigg|_5 = 0 \), and therefore
\[
w_4 - 2w_5 + w_6 = 0.
\]
The elimination of \( w_6 \) between the two equations yields then the final equation for point 4
\[
w_2 - 4w_3 + 5w_4 - 2w_5 = \frac{\epsilon^4 p_4}{EI}.
\]

For simple situations, the foregoing procedure is straightforward and presents no difficulties. However, for the more complicated problems, such as would be involved in wings of complicated planform or variable thickness, the process can become tedious and may involve the introduction of uncertain steps. For these situations, another procedure may be used and this is to derive the difference equations through means of the energy equation. Essentially, the process consists of dividing the surface into a regular gridwork system, then by means of difference equations to write the total potential energy of the system in terms of the deflections at each grid point, and then to minimize this energy expression with respect to each deflection point. With this procedure, practically all guess work is removed.

To be specific, consider the determination of the difference equations of equilibrium that apply to plates of variable thickness when subject to lateral and neutral surface forces. The appropriate energy expression would be equation (3.44), with \( D \) brought under the integral sign
\[
U - V = \frac{1}{2} \int \int D \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + 2\mu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] dx dy - \int \int p w dx dy.
\]

The terms involving middle surface stresses have momentarily been dropped and will be treated separately in a later section. The grid point designation shown in the adjoining sketch will be employed.

For integration over the plate, it is convenient to use elements as indicated by area 1 for all \( \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2} \), and \( p \) terms, and elements designated by area 2 for the \( \frac{\partial^2 w}{\partial x \partial y} \) terms. The difference equations to be used are based on parabolic arcs, and typical expressions for the above grid pattern would be
\[
\frac{\partial^2 w_0}{\partial x^2} = \frac{1}{\epsilon^2} (w_W - 2w_0 + w_E),
\]
\[
\frac{\partial^2 w_0}{\partial y^2} = \frac{1}{\lambda^2} (w_S - 2w_0 + w_N),
\]
\[
\frac{\partial^2 w_{NE}}{\partial x \partial y} = \frac{1}{\epsilon \lambda} (w_{NE} - w_N - w_E + w_0).
\]

It is well to note also that the following moment equations apply:

For a general interior point 0,
\[
M_{x0} = -D \left( \frac{\partial^2 w_0}{\partial x^2} + \mu \frac{\partial^2 w_0}{\partial y^2} \right) = -\frac{D_0}{\epsilon^2} \left[ w_W - 2w_0 + w_E - \frac{\mu \epsilon^2}{\lambda^2} (w_S - 2w_0 + w_N) \right],
\]
\[
M_{y0} = -D \left( \frac{\partial^2 w_0}{\partial y^2} + \mu \frac{\partial^2 w_0}{\partial x^2} \right) = -\frac{D_0}{\lambda^2} \left[ w_S - 2w_0 + w_N + \frac{\mu \lambda^2}{\epsilon^2} (w_W - 2w_0 + w_E) \right],
\]
\[
M_{x,y,NE} = (1 - \mu) D \frac{\partial^2 w_{NE}}{\partial x \partial y} = D_{NE/E} \frac{1 - \mu}{\epsilon \lambda} (w_{NE} - w_N - w_E + w_0).
\]

For the point 0 located along a free edge, \( x = x_1 \) (where \( M_x = 0 \)),
\[
M_y = -D (1 - \mu^2) \frac{\partial^2 w}{\partial y^2} = -D_0 \frac{1 - \mu^2}{\lambda^2} (w_S - 2w_0 + w_N).
\]

For the point 0 located along a free edge, \( y = y_1 \) (where \( M_y = 0 \)),
\[
M_x = -D (1 - \mu^2) \frac{\partial^2 w}{\partial x^2} = -D_0 \frac{1 - \mu^2}{\epsilon^2} (w_W - 2w_0 + w_E).
\]

If trapezoidal type integration is considered, the minimization of equation (4.1) leads to the following general equation
\[
\frac{\partial (U - V)}{\partial w_i} = \sum_{n=1}^{N} \epsilon \lambda \eta_n D_n \left[ \frac{\partial^2 w_n}{\partial x^2} + \mu \frac{\partial^2 w_n}{\partial y^2} \right] \frac{\partial}{\partial w_i} \left( \frac{\partial^2 w_n}{\partial x^2} + \mu \frac{\partial^2 w_n}{\partial y^2} \right) + \frac{\partial^2 w_m}{\partial x \partial y} \frac{\partial}{\partial w_i} \frac{\partial^2 w_m}{\partial x \partial y} - \epsilon \lambda \eta_i p_i = 0,
\]
\[
\sum_{m=1}^{M} \lambda \eta_m 2 (1 - \mu) D_m \frac{\partial^2 w_m}{\partial x \partial y} \frac{\partial}{\partial w_i} \frac{\partial^2 w_m}{\partial x \partial y} - \epsilon \lambda \eta_i p_i = 0,
\]
\[
- \sum_{n=1}^{N} \epsilon \lambda \eta_n \left[ M_{x,n} \frac{\partial}{\partial w_i} \left( \frac{\partial^2 w_n}{\partial x^2} \right) + M_{y,n} \frac{\partial}{\partial w_i} \left( \frac{\partial^2 w_n}{\partial y^2} \right) \right] + 2 \epsilon \lambda \eta_m M_{x,y,m} \frac{\partial}{\partial w_i} \left( \frac{\partial^2 w_m}{\partial x \partial y} \right) - \epsilon \lambda \eta_i p_i = 0.
\]

where \( n \) is used to denote the grid station points such as 0, \( N, E, NE \), etc. (for which, type (1) areas apply), \( m \) denotes the half-station points \( \frac{NE}{2}, \frac{NW}{2} \), etc. (for type (2) areas), and the \( \eta_n \)'s are integration factors; these integration factors are unity for all interior points, but are something less than unity for
boundary points, depending on the extent to which the grid elements are truncated by the boundary. By application of either equation (4.4a) or (4.4b) to each type of point that may be present in a plate system, it is possible to obtain general and universally applicable difference expressions. The expression obtained from equation (4.4a) will be in terms of deflections, while those obtained from equation (4.4b) will be in terms of moments, which then must be converted to deflections by means of the moment-deflection relations; the use of these moment expressions has certain advantages in many cases, however, as will be seen. As examples, the following general equations result from the application of equation (4.4b) to several typical plate points:

General interior point,
\[ -\frac{1}{\varepsilon^2} \left( M_{\text{sw}} - 2 M_{x_0} + M_{x_E} \right) - \frac{1}{\lambda^2} \left( M_{y_0} - 2 M_{x_0} + M_{y_E} \right) \]
\[ + \frac{2}{\varepsilon \lambda} \left[ M_{\text{USE}/2} - M_{x_{\text{SW}/2}} - M_{x_{\text{SE}/2}} + M_{x_{\text{SW}/2}} \right] = p_0 \cdot (4.5a) \]

Point near a free edge,
\[ -\frac{1}{\varepsilon^2} \left( M_{\text{sw}} - 2 M_{x_0} + M_{x_E} \right) - \frac{1}{\lambda^2} \left( M_{y_0} - 2 M_{x_0} \right) \]
\[ + \frac{2}{\varepsilon \lambda} \left[ M_{\text{USE}/2} - M_{x_{\text{SW}/2}} - M_{x_{\text{SE}/2}} + M_{x_{\text{SW}/2}} \right] = p_0 \cdot (4.5b) \]

Point at a free edge,
\[ -\frac{1}{2 \varepsilon^2} \left( M_{\text{sw}} - 2 M_{x_0} + M_{x_E} \right) - \frac{1}{\lambda^2} M_{y_S} \]
\[ + \frac{2}{\varepsilon \lambda} \left[ -M_{x_{\text{SE}/2}} + M_{x_{\text{SW}/2}} \right] = \frac{1}{2} p_0 \cdot (4.5c) \]

Point near a free corner,
\[ -\frac{1}{\varepsilon^2} \left( M_{\text{sw}} - 2 M_{x_0} \right) - \frac{1}{\lambda^2} \left( M_{y_S} - 2 M_{y_0} \right) \]
\[ + \frac{2}{\varepsilon \lambda} \left[ M_{\text{USE}/2} - M_{x_{\text{SW}/2}} - M_{x_{\text{SE}/2}} + M_{x_{\text{SW}/2}} \right] = p_0 \cdot (4.5d) \]

Point on edge near free corner,
\[ -\frac{1}{2 \varepsilon^2} \left( M_{\text{sw}} - 2 M_{x_0} \right) - \frac{1}{\lambda^2} M_{y_S} \]
\[ + \frac{2}{\varepsilon \lambda} \left[ -M_{x_{\text{SE}/2}} + M_{x_{\text{SW}/2}} \right] = \frac{1}{2} p_0 \cdot (4.5e) \]

Point at free corner,
\[ -\frac{1}{2 \varepsilon^2} M_{\text{sw}} - \frac{1}{2} \frac{1}{\lambda^2} M_{y} \]
\[ + \frac{2}{\varepsilon \lambda} M_{x_{\text{SW}/2}} = \frac{1}{4} p_0 \cdot (4.5f) \]
Table III. General difference equations for a rectangular-cantilever wing of non-uniform thickness

| ε | \( M_{10} \) | \( M_{00} \) | \( M_{01} \) | \( M_{11} \) | \( M_{20} \) | \( M_{21} \) | \( M_{22} \) | \( M_{30} \) | \( M_{31} \) | \( M_{32} \) | \( M_{33} \) | \( M_{40} \) | \( M_{41} \) | \( M_{42} \) | \( M_{43} \) | \( M_{44} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

\[ \frac{\partial \mathbf{u}}{\partial \mathbf{z}} = \mathbf{u} \]
As regards boundary conditions, it is only necessary to insure that geometric conditions are satisfied if evaluation is made explicitly in terms of deflections, that is, by equation (4.4a). This is generally easy to do by means of "image" considerations. For points near or at free boundaries, however, evaluation proceeds in terms of fictitious grid points just off the boundaries (as though the deflection surface extended continuously). A minimization is then also made with respect to the deflection at these fictitious grid points to yield auxiliary difference equations. These auxiliary equations cause the automatic satisfaction of the boundary condition on shear and moment; they can be carried along with the other difference equations, or with them the fictitious deflection points may be eliminated.

In contrast, if the difference equations are established through means of equation (4.4b), which was done for equations (4.5a) through (4.5f), the boundary condition of zero edge moment is explicitly introduced. In this way, no exterior points become involved, and to obtain the final difference equations, it is only necessary to substitute in the difference equivalents for the moments, equations (4.3a) or (4.3c). In this substitution, due regard is given the geometric conditions.

2. Example applications. Consider the cantilever wing and grid system shown in Table III. The general difference equations for all such cantilever wings follow easily and directly from equations (4.5a) through (4.5f) and are given, also, in Table III. When applied to a double wedge airfoil section, these equations reduce to the single set of equations shown in Table IV. From these equations, the influence functions of the wing may readily be determined, or they may be used directly in aeroelastic studies as will be indicated in the section on flutter.

For a uniform plate equations (4.5a) and (4.3) would yield the deflection coefficients

\[
\begin{align*}
  c_1 &= \frac{e^4}{\lambda^4}, \\
  c_2 &= \frac{2}{\lambda^2} e^2, \\
  c_3 &= -4 \frac{e^2}{\lambda^2} \left(1 + \frac{e^2}{\lambda^2}\right), \\
  c_4 &= 1, \\
  c_5 &= -4 \left(1 + \frac{e^2}{\lambda^2}\right), \\
  c_6 &= 6 + 8 \frac{e^2}{\lambda^2} + 6 \frac{e^4}{\lambda^4},
\end{align*}
\]
Table IV. Difference equations for rectangular cantilever wing with double-wedge airfoil

\[
\epsilon = 1, \quad \mu = 0.3
\]

\[
D_0 = \frac{E h_0^3}{12 (1 - \mu^2)}
\]

\[
\begin{array}{cccccccccccccccccc}
5.4 & -0.7 & 0 & 0 & -11.8 & 1.9 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.7 & 5.4 & -0.7 & 0 & 1.9 & -11.8 & 1.9 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.7 & 5.4 & -0.7 & 0 & 1.9 & -11.8 & 1.9 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.7 & 2.52 & 0 & 0 & 0.7 & -4.34 & 0 & 0 & 0 & 1.82 & 0 & 0 & 0 & 0 & 0 \\
-11.8 & 1.9 & 0 & 0 & 124.8 & -40.4 & 4 & 0 & -131.4 & 29.7 & 0 & 0 & 32 & 0 & 0 & 0 & 0 \\
1.9 & -11.8 & 1.9 & 0 & -40.4 & 120.8 & -40.4 & 4 & 29.7 & -134.4 & 29.7 & 0 & 0 & 32 & 0 & 0 & 0 & 0 \\
0 & 1.9 & -11.8 & 0.7 & 4 & -40.4 & 116.8 & -30 & 0 & 29.7 & -131.4 & 28.5 & 0 & 0 & 32 & 0 & 0 & 0 \\
0 & 0 & 1.9 & -4.34 & 0 & 4 & -30 & 45.44 & 0 & 0 & 20.1 & -51.66 & 0 & 0 & 14.56 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & -131.4 & 29.7 & 0 & 0 & 512.4 & -204.2 & 32 & 0 & -131.4 & 29.7 & 0 & 0 & 4 & 0 \\
0 & 4 & 0 & 0 & 29.7 & -131.4 & 29.7 & 0 & -204.2 & 480.4 & -204.2 & 32 & 29.7 & -131.4 & 29.7 & 0 & 0 & 4 & 0 \\
0 & 0 & 4 & 0 & 0 & 29.7 & -131.4 & 20.1 & 32 & -204.2 & 448.4 & -121 & 0 & 29.7 & -131.4 & 20.1 & 0 & 0 & 4 \\
0 & 0 & 0 & 1.82 & 0 & 0 & 28.5 & -51.66 & 0 & 32 & -121 & 131.68 & 0 & 0 & 28.5 & -51.66 & 0 & 0 & 1.82 \\
0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & -131.4 & 29.7 & 0 & 0 & 124.8 & -40.4 & 4 & 0 & -11.8 & 1.9 & 0 \\
0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 29.7 & -131.4 & 29.7 & 0 & -40.4 & 120.8 & -40.4 & 4 & 1.9 & -11.8 & 1.9 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 29.7 & -131.4 & 28.5 & 4 & -40.4 & 116.8 & -30 & 0 & 1.9 & -11.8 & 0.7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14.56 & 0 & 0 & 20.1 & -51.66 & 0 & 4 & -30 & 45.44 & 0 & 0 & 1.9 & -4.34 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & -11.8 & 1.9 & 0 & 0 & 5.4 & -0.7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1.9 & -11.8 & 1.9 & 0 & -0.7 & 5.4 & -0.7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1.9 & -11.8 & 1.9 & 0 & -0.7 & 5.4 & -0.7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.82 & 0 & 0 & 0.7 & -4.34 & 0 & 0 & -0.7 & 2.52 \\
\end{array}
\]

\begin{array}{cccccccccccccccccc}
w_2 & \frac{1}{2} p_2 \\
w_3 & \frac{1}{2} p_3 \\
w_4 & \frac{1}{2} p_4 \\
w_5 & \frac{1}{2} p_5 \\
w_6 & p_7 \\
w_7 & p_8 \\
w_8 & p_9 \\
w_9 & p_{10} \\
w_{10} & p_{11} \\
w_{11} & p_{12} \\
w_{12} & \frac{32 \epsilon^4}{D_0} \\
w_{13} & \frac{1}{2} p_{13} \\
w_{14} & \frac{1}{2} p_{14} \\
w_{15} & \frac{1}{2} p_{15} \\
w_{16} & \frac{1}{2} p_{16} \\
w_{17} & \frac{1}{2} p_{17} \\
w_{18} & \frac{1}{2} p_{18} \\
w_{19} & \frac{1}{2} p_{19} \\
w_{20} & \frac{1}{2} p_{20} \\
w_{21} & \frac{1}{2} p_{21} \\
w_{22} & \frac{1}{2} p_{22} \\
w_{23} & \frac{1}{2} p_{23} \\
w_{24} & \frac{1}{2} p_{24} \\
w_{25} & \frac{1}{2} p_{25}
\end{array}
Table V. Difference equations for clamped rectangular plate

\[ \nabla^4 \phi = -\alpha E \nabla^2 \Delta T \]

with \( \Delta T = \Delta T_1 \left( \frac{y}{b} \right)^2 \)

and \( a = 2b; \quad \epsilon = \frac{a}{4} = 2\lambda. \)

\[
\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = \frac{64}{a^2} (\phi_{j-1} - 2\phi_j + \phi_{j+1})_{x=m\epsilon}
\]

\[
\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = \frac{16}{a^2} (\phi_{i-1} - 2\phi_i + \phi_{i+1})_{y=n\lambda}
\]
such that

\[ c_6 w_0 + c_5 (w_W + w_E) + c_3 (w_S + w_N) + c_2 (w_{NE} + w_{NW} + w_{SE} + w_{SW}) \]
\[ + c_4 (w_{WE} + w_{EE}) + c_1 (w_{SS} + w_{NN}) = \frac{\varepsilon^4 p_0}{D}. \]  

When \( \varepsilon = \lambda \), these coefficients become the familiar values that apply in the case of uniform plates with a square grid pattern.

As a second example, consider the application of equation (4.6) to a uniform rectangular plate clamped on all four edges. The grid system used and the equations found for this case are shown in table V; by use of the analogy between plate bending and the stress function mentioned previously, it was through this set of equations that the stress results shown in fig. 6 were obtained.

The third example is intended to illustrate the use of exterior points in the case of more complicated planforms. Thus consider the delta wing

where only a few grid points have been used purely for convenience of illustration. In difference form the potential energy expression for this case would appear

\[ U - V = \frac{D}{2\varepsilon^2} \left\{ \frac{1}{8} (2 w_4)^2 + \frac{1}{2} (2 w_1)^2 + \frac{1}{4} (2 w_2)^2 + \frac{1}{8} (-2 w_1 + w_3)^2 + \frac{1}{8} (-2 w_2 + w_3)^2 + \frac{1}{8} (-2 w_2 + w_3)^2 + \frac{1}{8} (w_2 - 2 w_3 + w_6)^2 \right\} \]

where the exterior points are employed as though the deflection surface extended beyond the boundaries of the plate. Note, that the integration factors \( \frac{1}{4}, \frac{1}{1}, \frac{1}{8} \) are introduced, in accordance with the amount of material that is present in each grid element. The minimization of equation (4.7) yields the equations shown in table VI. These equations may be solved simultaneously as they stand, or the last five equations (which come from the minimization of \( w_4 \) through \( w_8 \)) may be solved for \( w_4 \) through \( w_8 \), and the results substituted in the first three equations to yield the three basic equilibrium equations of the system in terms of \( w_1, w_2, \) and \( w_3 \) only.

Thus no particular difficulty arises in the treatment of other than simple
### Table VI. Difference equations for Delta Wing

<table>
<thead>
<tr>
<th></th>
<th>(21)</th>
<th>(-10 + 2\mu)</th>
<th>(2 - \mu)</th>
<th>(-4)</th>
<th>(-4)</th>
<th>(0)</th>
<th>(0)</th>
<th>(0)</th>
<th>(1)</th>
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<td>(-10 + 2\mu)</td>
<td>(17\frac{1}{2} + 2\mu)</td>
<td>(-4\frac{1}{2} - \frac{1}{2}\mu)</td>
<td>(1)</td>
<td>(2 - \frac{3}{2}\mu)</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{4}\mu)</td>
<td>(-2 - 2\mu)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2 - \mu)</td>
<td>(-4\frac{1}{2} - \frac{1}{2}\mu)</td>
<td>(5)</td>
<td>(0)</td>
<td>(-2\frac{1}{2} + 1\frac{1}{2}\mu)</td>
<td>(-\frac{1}{2} - \frac{1}{2}\mu)</td>
<td>(-\frac{1}{2} - \frac{1}{2}\mu)</td>
<td>(\mu)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-4)</td>
<td>(1)</td>
<td>(0)</td>
<td>(4 - 2\mu)</td>
<td>(\mu)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-4)</td>
<td>(2 - \frac{3}{2}\mu)</td>
<td>(-2\frac{1}{2} + 1\frac{1}{2}\mu)</td>
<td>(\mu)</td>
<td>(3\frac{1}{2} - 2\mu)</td>
<td>(\frac{1}{4}\mu)</td>
<td>(\frac{1}{4}\mu)</td>
<td>(0)</td>
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<td></td>
</tr>
<tr>
<td>(0)</td>
<td>(\frac{1}{4})</td>
<td>(-\frac{1}{2} - \frac{1}{2}\mu)</td>
<td>(0)</td>
<td>(\frac{1}{4}\mu)</td>
<td>(\frac{1}{4}\mu)</td>
<td>(\frac{1}{4}\mu)</td>
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<tr>
<td>(0)</td>
<td>(\frac{1}{4}\mu)</td>
<td>(-\frac{1}{2} - \frac{1}{2}\mu)</td>
<td>(0)</td>
<td>(\frac{1}{4})</td>
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<tr>
<td>(1)</td>
<td>(-2 - 2\mu)</td>
<td>(\mu)</td>
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<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(1)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\frac{p_1 e^t}{D} = \sum w_i
\]

\[
\begin{align*}
\frac{p_2 e^t}{D} &= 0 \\
\frac{p_3 e^t}{4D} &= 0 \\
\frac{p_4 e^t}{4D} &= 0
\end{align*}
\]
planform. The following three sketches are given to suggest the grid system that may be used in other sample cases

The main point to note is that in the treatment of the grid elements near boundaries, care should be taken to introduce appropriate integration factors to take into account that the element may be only partially intersected by material.

3. Inclusion of middle-surface forces. The amount contributed to the total energy of the system by middle surface forces is given by

\[ \frac{1}{2} \iint \left[ N_x \left( \frac{\partial w}{\partial x} \right)^2 + N_y \left( \frac{\partial w}{\partial y} \right)^2 + 2 N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dx \, dy. \]

With finite difference application in mind, the minimization of this expression can be written symbolically as

\[ \iint \left\{ N_x \frac{\partial w}{\partial x} \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial x} \right) + N_y \frac{\partial w}{\partial y} \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial y} \right) + N_{xy} \frac{\partial w}{\partial x} \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial y} \right) + \frac{\partial w}{\partial y} \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial x} \right) \right\} dx \, dy. \] (4.8)

The summation over the plate surface can be made in several ways, but a suggested scheme employs the elements indicated in the adjoining grid pattern.

Type a elements are used to sum the work done by the $N_x$ forces, type $b$ for $N_y$ and type $c$ for $N_{xy}$; where the stress that is considered to act in each element is taken as the given stress that is present at the center of each element, respectively. With this choice of elements and the grid notation used earlier, expression (4.8) would then yield for the minimization with respect to $w_0$, for example, the expression.
\[
\left\{ w_0 \left[ \frac{1}{\epsilon^2} (N_{xN}\epsilon_x + N_{xw}\epsilon_x) + \frac{1}{\lambda^2} (N_{yN}\epsilon_y + N_{yw}\epsilon_y) + \frac{1}{2\epsilon^2} (N_{xNS}\epsilon_x - N_{xSW}\epsilon_x + N_{xSE}\epsilon_x + N_{xSW}\epsilon_x) \right] \\
- \frac{1}{\epsilon^2} (w_W N_{xw} + w_E N_{xw}) - \frac{1}{\lambda^2} (w_S N_{yS} + w_N N_{yS}) \\
+ \frac{1}{2\epsilon^2} \left( -w_{NE} N_{xNS} + w_{NW} N_{xNW} + w_{SE} N_{xSE} - w_{SW} N_{xSW} \right) \right\} \epsilon \lambda.
\]

The inclusion of terms of this type in the difference equations then yields equations which take into account directly the influence middle surface forces have on the deflection characteristics of the plate. For future reference, let the complete difference equations be represented by the matrix equation

\[
[B] [w] + [N] [w] = [p], \tag{4.9}
\]

where \(B\) represents the matrix of deflection coefficients for bending, as derived in the previous section, and \(N\) is the matrix of deflection coefficients that are associated with the middle surface forces, as derived in this section. Finally, equation (4.9) may be written simply

\[
[S] [w] = [p], \tag{4.10}
\]

where

\[
S = B + N.
\]

4. Reduction to fewer degrees of freedom. One of the inherent beauties of difference equations (and ironically one of the features that leads to their biggest drawback) is the ease by which they may be set up. Another nice feature is that the physical significance of the various terms remains intact in their use. But while they are relatively easy to set up, the drawback that appears is the fact that generally a large number of grid points and hence equations must be employed to retain a reasonable degree of accuracy. Thus, most of the work comes in their solution. This drawback has, of course, been considerably offset by the use of high-speed large-capacity computing machines.

In the following, two means are suggested for reducing a set of difference equations to a system having fewer degrees of freedom, so as to simplify the solution phase of the problem when desired.

Consider that the potential energy of the plate system is given formally by

\[
U - V = \int F(w) dA = \text{Min}.
\]

In using a grid system with deflections \(w_i\) at the grid points, this would read

\[
\sum_{n=1}^{N} F_n(w_1, w_2, w_3, \ldots) \epsilon \lambda = \text{Min}. \tag{4.11}
\]

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The minimization yields

\[ \sum_{n=1}^{N} \frac{\partial F_n}{\partial w_i} = 0, \quad i = 1, 2, 3, \ldots \]  

(4.12)

and these are the difference equations of the system.

Now suppose that the deflection is expanded in a series of chosen modal functions, such that

\[ w = a_1z_1 + a_2z_2 + a_3z_3 + \cdots, \]

which for point \( i \) would be

\[ w_i = a_1z_{i1} + a_2z_{i2} + a_3z_{i3} + \cdots, \]  

(4.13)

where the \( a_n \)'s are unknown coefficients. If equation (4.13) had been substituted into equation (4.11), the minimization would have given (in finite summation form)

\[ \sum_{n=1}^{N} \frac{\partial F_n}{\partial a_j} = 0, \quad j = 1, 2, 3, \ldots \]  

(4.14)

But

\[ \frac{\partial F_n}{\partial a_j} = \frac{\partial F_n}{\partial w_1} \frac{\partial w_1}{\partial a_j} + \frac{\partial F_n}{\partial w_2} \frac{\partial w_2}{\partial a_j} + \frac{\partial F_n}{\partial w_3} \frac{\partial w_3}{\partial a_j} + \cdots = \frac{\partial F_n}{\partial w_1} z_{j1} + \frac{\partial F_n}{\partial w_2} z_{j2} + \frac{\partial F_n}{\partial w_3} z_{j3} + \cdots \]

Thus equation (4.14) becomes

\[ \sum_{n=1}^{N} \left( \frac{\partial F_n}{\partial w_1} z_{j1} + \frac{\partial F_n}{\partial w_2} z_{j2} + \frac{\partial F_n}{\partial w_3} z_{j3} + \cdots \right) = 0, \quad j = 1, 2, 3, \ldots \]  

(4.15)

Now with reference to equation (4.12), this equation has a very simple interpretation. It states that, in terms of the \( a_j \)'s, the minimum condition is given by adding together the first difference equation multiplied by \( z_{j1} \), the second by \( z_{j2} \), the third by \( z_{j3} \), and so forth, and to express the \( w_i \)'s in terms of the \( a_j \)'s. A reduced system having degrees of freedom equal to the number of terms chosen in the deflection expansion is thus obtained. It is noted that the reduction process resembles a Galerkin Procedure.

The other reduction scheme involves a least squares process. Substitute equation (4.13) into the difference equations and consider that the equations are not satisfied by an amount \( \epsilon_n \). Make

\[ \sum_{n=1}^{N} \epsilon_n^2 \]

a minimum. Thus

\[ \sum_{n=1}^{N} \epsilon_n \frac{\partial \epsilon_n}{\partial a_j} = 0, \quad j = 1, 2, 3, \ldots \]  

(4.16)

will be another set of equations for determining the \( a_j \)'s.
B. A new numerical "summation-equation" method for solving partial differential equations

There follows a numerical "summation-equation" method for solving certain partial differential equations. It differs from a difference-equations approach in that only integrations are involved. Its chief asset is that, with some sacrifice in labor, it gives remarkable accuracy with only a few grid points. The key to the approach is as follows: a surface is assumed to be present over the region being considered for each term in the differential equation, and unknown values of the terms are assigned at grid points. Integration is then performed along the grid lines for each term, giving due regard to the boundary conditions, until a like quantity is reached. The equating of this common quantity then yields the relations that must exist between the various differential terms, and these when substituted in the given differential equation allow solution to be made by ordinary simultaneous equations.

As an illustration, consider the membrane problem

\[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{p}{N}. \]

Divide membrane into a grid system and integrate along grid lines. For \( \frac{\partial^2 w}{\partial x^2} \),

\[ \frac{\partial w}{\partial x} = \left. \frac{\partial w}{\partial x} \right|_{x=0} + \int_0^x \frac{\partial^2 w}{\partial x^2}, \]

\[ w = w|_{x=0} + \left. \frac{\partial w}{\partial x} \right|_{x=0} x + \int_0^x \int_0^x \frac{\partial^2 w}{\partial x^2}, \]

\[ = -\frac{x}{a} \int \int \frac{\partial^2 w}{\partial x^2} + \int \int \frac{\partial^2 w}{\partial x^2} \quad \text{since} \quad w|_{x=0} = w|_{x=a} = 0. \]

Similarly, for \( \frac{\partial^2 w}{\partial y^2} \), it is found that

\[ w = -\frac{y}{b} \int \int \frac{\partial^2 w}{\partial y^2} + \int \int \frac{\partial^2 w}{\partial y^2} \quad \text{since} \quad w|_{y=0} = w|_{y=b} = 0, \]

where the integral sign symbolically indicates a numerical integration of some sort involving the unknowns \( \frac{\partial^2 w}{\partial x^2} \) and \( \frac{\partial^2 w}{\partial y^2} \), \( n \) being the \( n^{th} \) grid point. But the deflections as given by equations (4.17) and (4.18) must be equal, thus
which thus establishes the relation that must exist between $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial^2 w}{\partial y^2}$ since they apply to a common deflection surface. In matrix notation equation (4.19) would appear

$$A \frac{\partial^2 w}{\partial x^2} = B \frac{\partial^2 w}{\partial y^2},$$

(4.20)

where here $A$ and $B$ are to be interpreted as square matrices, and $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial^2 w}{\partial y^2}$ are column matrices. Write the differential equation in matrix form also, and multiply through by either $A$ or $B$, say $A$

$$A \frac{\partial^2 w}{\partial x^2} + A \frac{\partial^2 w}{\partial y^2} = -A \frac{p}{N}.$$

But from equation (4.20), this becomes simply

$$(B + A) \frac{\partial^2 w}{\partial y^2} = -A \frac{p}{N},$$

which is solved by simultaneous means.

For this particular problem, the work is little greater than if difference equations had been used. Note, that solution has been made for the highest order differential in the differential equation, and that lower order differentials then follow by integration, a feature which is usually preferred in numerical approaches, because of the inherent accuracy of such processes as compared to differentiating.

As a specific example of the foregoing, consider a membrane with sinusoidal loading with the seemingly absurd assumption of one grid point at the center only

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{p_0 \sin \pi x \sin \pi y}{N}.$$

For this problem, it is convenient to write

$$\frac{\partial^2 w}{\partial x^2} = -\left(\frac{\partial w}{\partial x}\right)_x \frac{x}{\epsilon} - 2,$$

$$\frac{\partial^2 w}{\partial y^2} = -\left(\frac{\partial w}{\partial y}\right)_y \frac{y}{\lambda} - 2.$$
By integration, with due regard to the boundary conditions these equations yield

\[ w_1 = -\frac{5\varepsilon^2}{12} \frac{(\partial^2 w)}{\partial x^2} = -\frac{5\lambda^2}{12} \frac{(\partial^2 w)}{\partial y^2} \]

or

\[ \left( \frac{\partial^2 w}{\partial y^2} \right)_1 = \frac{\varepsilon^2}{\lambda^2} \left( \frac{\partial^2 w}{\partial x^2} \right)_1. \]

But the differential equation also states that

\[ \left( \frac{\partial^2 w}{\partial x^2} \right)_1 + \left( \frac{\partial^2 w}{\partial y^2} \right)_1 = -\frac{p_0}{N}. \]

The solution of these two equations thus yields

\[ w_1 = \frac{5}{48} \frac{p_0}{1} \frac{1}{a^2 + \frac{1}{b^2}}. \]

For this problem, the exact solution is

\[ w_1 = \frac{1}{\pi^2} \frac{p_0}{1} \frac{1}{a^2 + \frac{1}{b^2}}, \]

and it would follow that

\[ \frac{w_{\text{approx.}}}{w_{\text{exact}}} = 1.028. \]

An error of only 2.8% is noted. The solution of this problem by difference equations gives a deflection which is in error by 23.3%; the marked increase in accuracy by the summation process is thus illustrated.

This "summation" process may also be applied to plates, of variable or constant thickness. Because the procedure depends a lot on the specific configuration being considered, a detailed discussion will not be given. It is mentioned, however, that for plates of constant thickness, one starts with unknown distributions of \( \frac{\partial^4 w}{\partial x^4}, \frac{\partial^4 w}{\partial y^4}, \) and \( \frac{\partial^4 w}{\partial x^2 \partial y^2}. \) The \( \frac{\partial^4 w}{\partial x^4} \) is integrated twice in the \( x \)-direction, and \( \frac{\partial^4 w}{\partial x^2 \partial y^2} \) is integrated twice in the \( y \)-direction, to give two relations for \( \frac{\partial^4 w}{\partial x^2 \partial y^2} \), which are then equated to give the relation that must exist between \( \frac{\partial^4 w}{\partial x^4} \) and \( \frac{\partial^4 w}{\partial x^2 \partial y^2} \); likewise, \( \frac{\partial^4 w}{\partial y^4} \) is integrated twice in the \( y \)-direction, and \( \frac{\partial^4 w}{\partial x^2 \partial y^2} \) twice in the \( x \)-direction, to give two relations for \( \frac{\partial^4 w}{\partial x^2 \partial y^2} \), which also are equated to yield the relation between \( \frac{\partial^4 w}{\partial y^4} \) and \( \frac{\partial^4 w}{\partial x^2 \partial y^2} \). Enough relations are thus found which together with the differential equation allows solution to be made. For variable thickness, one starts with the differential equation in terms of the moments \( M_x, M_y \) and \( M_{xy} \), and then later makes use of the moment deflection relations.
To illustrate the results that are obtained in application of this "summation" method to plates, a rectangular plate subject to sinusoidal loading was treated, using only a center grid point as in the case of the membrane example given before. The result for center deflection was found to be only 3% greater than the exact value. A corresponding difference solution gave a 52% error.

Difference solutions can of course be made to yield similar accuracies, but more grid points have to be used. But since it must be admitted that the set-up work is more tedious in the summation method, one should not try to state which is to be preferred. Each has its merits, and for the moment, let it be sufficient to say that the "summation" procedure provides another solution scheme which not only may be useful in itself, but which appears to provide a method by which other approximate methods may be evaluated when exact solutions are not possible.
Part II

Aeroelastic Phenomena

5. Application of "Summation-Equation" Method to Transient or Initial Value Problems

The following indicates the application of the "summation-equation" method discussed in the preceding section to initial value problems by application to a beam with time dependent loading.

Consider the differential equation
\[ \frac{\partial^2}{\partial x^2} E I \frac{\partial^2 w}{\partial x^2} = -m \ddot{w} + p(x, t). \] (5.1)

With the use of the influence coefficients for the beam, the solution for \( w \) on the left-hand side may be given by the matrix equation
\[ w = -\delta m \ddot{w} + \delta p, \] (5.2)

where \( \delta \) is to be regarded as the matrix of influence coefficients, \( m \) a diagonal matrix of masses, and \( w, \ddot{w}, \) and \( p \) column matrices. It should be noted that the use of influence coefficients infers integrations along the span.

Next, consider the following equation which represents the deflection that is obtained by a numerical integration of successive equally-spaced acceleration values of spacing \( \epsilon \) (see development in ref. 26).

\[
\begin{bmatrix}
    w_0 \\
    w_2 \epsilon \\
    w_3 \epsilon \\
    \vdots
\end{bmatrix}
= \begin{bmatrix}
    1 & 1 \\
    1 & 2 \\
    1 & 3 \\
    \vdots & \vdots
\end{bmatrix}
+ \epsilon^2
\begin{bmatrix}
    2 & 1 & 0 & 0 & 0 \\
    5 & 6 & 1 & 0 & 0 \\
    8 & 6 & 2 & 1 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
    \ddot{w}_0 \\
    \ddot{w}_2 \epsilon \\
    \ddot{w}_3 \epsilon \\
    \ddots
\end{bmatrix},
\] (5.3)

where the subscripts denote the value of time. The \( w \) in this equation may denote a single function, or it may denote a matrix column of deflections, such as the deflections at given spanwise stations along a beam, as it is chosen to represent in the present application.
Now consider equation (5.2) at time $t = \epsilon$, but replace $w$ by $w_\epsilon$ as given by equation (5.3), thus

$$w_0 + \epsilon \dot{w}_0 + \epsilon^2 \left( \frac{2}{6} \ddot{w}_0 + \frac{1}{6} \dddot{w}_0 \right) = -\delta m \ddot{w}_\epsilon + \delta p_\epsilon. \tag{5.4}$$

Since the initial values $w_0$, $\dot{w}_0$, and $\ddot{w}_0$ are known, or presumed calculable, solution may be made for $\ddot{w}_\epsilon$, and is

$$\ddot{w}_\epsilon = \left[ \delta m + \frac{\epsilon^2}{6} I \right]^{-1} \left\{ -\frac{2}{6} \epsilon^2 \ddot{w}_0 - \epsilon \dot{w}_0 - w_0 + \delta p_\epsilon \right\}. \tag{5.5}$$

With $\ddot{w}_\epsilon$ established, solution for $\dddot{w}_\epsilon$ follows next in a similar manner

$$\dddot{w}_2 \epsilon = \left[ \delta m + \frac{\epsilon^2}{6} I \right]^{-1} \left\{ -\frac{5}{6} \epsilon^2 \dddot{w}_0 - \ddot{w}_\epsilon - 2 \epsilon \dot{w}_0 - w_0 + \delta p_2 \epsilon \right\}, \tag{5.6}$$

and likewise

$$\dddot{w}_3 \epsilon = \left[ \delta m + \frac{\epsilon^2}{6} I \right]^{-1} \left\{ -\frac{8}{6} \epsilon^2 \dddot{w}_0 - 2 \ddot{w}_\epsilon - \dddot{w}_2 \epsilon - 3 \epsilon \dot{w}_0 - w_0 + \delta p_3 \epsilon \right\}. \tag{5.7}$$

This type solution may be carried on as far as is desired. However, a rather remarkable simplification may be made. If equation (5.5) through (5.7) are multiplied by 1, $-2$, 1, respectively, and the results are added, there results

$$\dddot{w}_3 \epsilon = \left[ 2 I - \left( \delta m + \frac{\epsilon^2}{6} I \right)^{-1} \right] \dddot{w}_2 \epsilon - \dddot{w}_\epsilon + \left[ \delta m + \frac{\epsilon^2}{6} I \right]^{-1} \delta \left( p_\epsilon - 2 p_2 \epsilon + p_3 \epsilon \right), \tag{5.8}$$

and indeed it would follow in a general way that, for $n \geq 3$

$$\dddot{w}_n \epsilon = \left[ 2 I - \left( \delta m + \frac{\epsilon^2}{6} I \right)^{-1} \right] \dddot{w}_{(n-1)} \epsilon - \dddot{w}_{(n-2)} \epsilon + \left[ \delta m + \frac{\epsilon^2}{6} I \right]^{-1} \delta \left( p_{(n-2)} \epsilon - 2 p_{(n-1)} \epsilon + p_n \epsilon \right). \tag{5.9}$$

Thus, solution of the problem may be made by first solving equations (5.5) and (5.6), and then applying the general recurrence equation (5.9) thereafter. It is remarked that if equation (5.1) had contained a damping term, or a nonsteady lift integral term as would be obtained in the treatment of transient aeroelastic effects, the same process would apply without difficulty [26].

As a specific illustration and critical test of the above process, the following cantilever beam was treated.

\[ \text{with } \delta = \frac{L^3}{48EI} \left[ \begin{array}{cc} 2 & 5 \\ 5 & 16 \end{array} \right]. \]
The time interval $\epsilon$ was based on the second mode frequency and is given by $\omega_2 \epsilon = \frac{2 \pi}{N}$, where $N$ was taken as 11.97. The equations corresponding to equations (5.5), (5.6), and (5.9) are

\[
\begin{align*}
\mathbf{m} \ddot{\mathbf{w}}_1 &= \begin{bmatrix} 0.218620 & -0.0681767 \\ -0.136353 & 0.0549959 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{w} \end{bmatrix}, \\
\mathbf{m} \ddot{\mathbf{w}}_2 &= \begin{bmatrix} 0.218620 & -0.0681767 \\ -0.136353 & 0.0549959 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{w} \end{bmatrix}, \\
\mathbf{m} \ddot{\mathbf{w}}_3 &= \begin{bmatrix} 1.78138 & 0.0681767 \\ 0.136353 & 1.94500 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{w} \end{bmatrix} - \begin{bmatrix} \mathbf{w} \\ \mathbf{w} \end{bmatrix},
\end{align*}
\]

where the subscripts denoting time have been brought outside the matrices, and the subscripts on $w$ here represent beam stations. No load terms appear in the last equation because a unit step input was assumed. The solution as obtained by these equations is given in fig. 7, together with the exact solution of the problem, which is given by the equations

\[
m \ddot{w}_1 = \frac{0.3274}{0.6072} (\cos \omega_1 t - \cos \omega_2 t); \quad \omega_2 = 5.151 \omega_1,
\]

\[
m \ddot{w}_2 = \frac{1}{0.6072} (\cos \omega_1 t + 0.21436 \cos \omega_2 t); \quad \omega_2 t = \frac{2 \pi}{11.97} n.
\]
Very satisfactory agreement is noted, and it may be remarked that if these acceleration results are integrated to give deflections, virtually no difference in the results would be found.

In the use of a recurrence equation such as equation (5.9), the question of its inherent stability characteristics should be raised. Therefore, the stability was investigated by substituting in the expression \( z = \bar{z} e^{\lambda t} \), where

\[
\bar{z} = \begin{bmatrix}
\bar{w}_1 \\
\bar{w}_2
\end{bmatrix}.
\]

A fourth order characteristic equation resulted, the roots of which were all neutrally stable (\( \alpha = 0 \) in \( \lambda = \alpha + i \beta \)); there is therefore no tendency for computations to diverge or converge. This finding lends credit to the approach and conforms appropriately to the physics of the system; that is, the homogeneous motion should be sinusoidal since no damping was introduced, and further the numerical process does not appear to introduce any apparent damping, either negative or positive.

6. The Rapid Calculation of Frequency Response Influence Surfaces

With reference to a plate, suppose that it is desired to determine a moment at a point \( A \) due to the successive application of unit oscillatory loads at points 1, 2, 3, etc. It so happens that a very useful and interesting theorem can be developed which indicates how this may be done in one operation without having to repeat for each load position. This development, which is an extension of the static consideration given for difference equations in [27], and which is intimately associated with the reciprocal laws of structures, follows.

A. By difference equations

Let

\[
Sw + \beta \ddot{w} + m \dot{w} = p,
\]

(6.1)

represent the matrix equation of difference equations which applies to the system under consideration. The matrix \( S \), for example, is the square stiffness matrix that is derivable by the method given in the section on difference equations, and which takes into account the bending stiffness of the plate and the influence of middle surface forces, if present (see equation (4.10). For oscillating conditions, with

\[
p = \bar{p} e^{i \omega t}, \quad w = \bar{w} e^{i \omega t},
\]

(6.2)

equation (6.1) becomes

\[
[S + i \omega \beta - \omega^2 m] \bar{w} = \bar{p},
\]

(6.3)

and hence

\[
\bar{w} = Z^{-1} \bar{p}; \quad Z = [S + i \omega \beta - \omega^2 m],
\]

(6.4)
where it is noted that $Z^{-1}$ is symmetric and is in the nature of an influence coefficient matrix. It follows, then, that the deflection due to a unit oscillatory load at $n$ would be

$$\bar{w}_{(n)} = Z^{-1} \bar{p}_{(n)},$$  \hfill (6.5)

where it should be noted carefully here that the $w_{(n)}$ represent the deflection surface due to the unit oscillatory load at $n$, and $\bar{p}_{(n)}$ represents a column loading matrix with unity at the $n$th element and all other elements zero. The moment at $A$ is given by $M_A = M_A e^{i\omega t}$; then $M_{A_{(n)}}$ follows as

$$\bar{M}_{A_{(n)}} = [r] [\bar{w}_{(n)}] = [r] Z^{-1} [\bar{p}_{(n)}],$$  \hfill (6.6)

where $r$ is a row matrix composed of the difference coefficients necessary to convert deflection values into moment, see equation (4.3a). With this equation, a row matrix may be built up which denotes the moments that result from the successive application of the unit load at all the stations, thus

$$[\bar{M}_{A(1)} \bar{M}_{A(2)} \bar{M}_{A(3)} \cdots] = r [\bar{w}_{(1)} \bar{w}_{(2)} \bar{w}_{(3)} \cdots] = r Z^{-1} [\bar{p}_{(1)} \bar{p}_{(2)} \bar{p}_{(3)} \cdots],$$  \hfill (6.7)

where $[\bar{w}_{(1)} \bar{w}_{(2)} \bar{w}_{(3)} \cdots]$ is a square matrix composed of the deflection columns, and $[\bar{p}_{(1)} \bar{p}_{(2)} \bar{p}_{(3)} \cdots]$ is absorbed since it is the identity matrix. The transpose of equation (6.7) is

$$\begin{bmatrix}
\bar{M}_{A(1)} \\
\bar{M}_{A(2)} \\
\bar{M}_{A(3)} \\
\vdots
\end{bmatrix} = [Z^{-1}]' [r'],$$

$$= Z^{-1} r',$$  \hfill (6.8)

where the last result follows because $Z^{-1}$ is symmetrical.

Return now to equation (6.4) and ask the following significant question: is it possible to find an oscillatory loading distribution which will produce deflections which are the same as the moments given by equation (6.8)? If equations (6.4) and (6.8) are set equal, then

$$Z^{-1} \bar{p} = Z^{-1} r'$$  \hfill (6.9)

and it is seen that if the loading $p$ is made equal to $r'$, then the desired goal will be accomplished.

Thus, the following theorem may be stated: If the coefficients of the difference expression that converts deflection to a moment at a point, say $A$, are applied simultaneously as loads at their respective positions, then the resulting deflection will be an influence surface, such that the deflection at any point $B$ represents the moment at $A$ that would result from a unit load application at $B$.  

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B. By modal function method

A conclusion similar to that given in the preceding section can also be derived when the modal function approach is used. To avoid notation conflict, let stress, instead of moment, be the frequency response quantity of interest in this derivation. Thus, let

\[ w = a_1 w_1 + a_2 w_2 + a_3 w_3 + \cdots \]  

(6.10)

represent the deflection surface, where the \( w_n \)'s are the natural mode deflection surfaces, and the \( a_n \)'s are functions of time only, and are established through means of the well known equation

\[ M_n \ddot{a}_n + M_n \beta_n \dot{a}_n + M_n \omega_n^2 a_n = \int p w_n \, dA, \]  

(6.11)

where \( M_n \) and \( \omega_n \) are the generalized mass and frequency of the \( n \)th mode, and \( p \) is the loading. For sinusoidal conditions

\[ p = \bar{p} e^{i \omega t}, \quad a_n = \bar{a}_n e^{i \omega t}, \]

equation (6.11) would yield

\[ \bar{a}_n = \frac{\int \bar{p} w_n \, dA}{M_n \omega_n^2 - M_n \omega^2 + i \omega M_n \beta_n} = \frac{\int \bar{p} w_n \, dA}{Z_n}. \]  

(6.12)

Now consider matrix notation to apply. With equation (6.12), equation (6.10) would appear, after writing \( w = \bar{w} e^{i \omega t} \),

\[ \bar{w} = \epsilon \lambda \left( \frac{w_1 \bar{p}}{Z_1} w_1 + \frac{w_2 \bar{p}}{Z_2} w_2 + \cdots \right), \]  

(6.13)

where \( \bar{w}, w_1, w_2, \ldots \), and \( \bar{p} \) are column matrices and \( w_n' \) is the transpose of \( w_n \). Since the stress, \( \sigma = \bar{\sigma} e^{i \omega t} \), is obtained from the deflection by a differential operation, equation (6.13) also applies when written in terms of stresses, thus

\[ \bar{\sigma} = \epsilon \lambda \left( \frac{\sigma_1 \bar{p}}{Z_1} \sigma_1 + \frac{\sigma_2 \bar{p}}{Z_2} \sigma_2 + \cdots \right), \]  

(6.14)

where \( \bar{\sigma}, \sigma_1, \sigma_2, \ldots \) are column matrices of stresses. From this equation, the stress at a point \( A \) is given by

\[ \bar{\sigma}_A = \epsilon \lambda \left( \frac{\sigma_1 \bar{p}}{Z_1} w_1 + \frac{\sigma_2 \bar{p}}{Z_2} w_2 + \cdots \right), \]

and if unit oscillatory loads are successively applied at the various station points, the following equation applies

\[ [\bar{\sigma}_{A(1)} \bar{\sigma}_{A(2)} \bar{\sigma}_{A(3)} \cdots] = \epsilon \lambda \left( \frac{\sigma_1 \bar{p}}{Z_1} [w_1 \bar{p} \bar{p} \cdots] + \frac{\sigma_2 \bar{p}}{Z_2} [w_2 \bar{p} \bar{p} \cdots] + \cdots \right) \]

\[ = \epsilon \lambda \left( \frac{\sigma_1}{Z_1} w_1 + \frac{\sigma_2}{Z_2} w_2 + \cdots \right), \]  

(6.15)

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where $\bar{p}_{(1)}, \bar{p}_{(2)} \ldots$ are unit load columns which form the identity matrix, which can then be absorbed to lead to the last line shown. The transpose of equation (6.15) is

$$\begin{bmatrix} \bar{\sigma}_{A(1)} \\ \bar{\sigma}_{A(2)} \\ \bar{\sigma}_{A(3)} \\ \vdots \end{bmatrix} = \varepsilon \lambda \left( \frac{\sigma_{1A}}{Z_1} w_1 + \frac{\sigma_{2A}}{Z_2} w_2 + \cdots \right). \tag{6.16}$$

It is now possible to compare equations (6.13) and (6.16) to see if a loading distribution can be introduced into the first of these which will yield a deflection which has the same values as the stresses given by equation (6.16). Thus, if equations (6.13) and (6.16) are set equal

$$\frac{w_1'}{Z_1} + \frac{w_2'}{Z_2} w_2 + \cdots = \frac{\sigma_{1A}}{Z_1} w_1 + \frac{\sigma_{2A}}{Z_2} w_2 + \cdots \tag{6.17}$$

From this equation, it is seen that the desired goal can be achieved if the following equation is satisfied

$$\begin{bmatrix} w_1' \\ w_2' \\ w_3' \\ \vdots \end{bmatrix} \bar{p} = \begin{bmatrix} \sigma_{1A} \\ \sigma_{2A} \\ \sigma_{3A} \\ \vdots \end{bmatrix}. \tag{6.18}$$

The loading values obtained from this equation, when applied to the structure will lead to the desired influence surface. When the number of load points exceeds the number of $a$-coefficients chosen in equations (6.10), as is usually the case, all those loads in excess may arbitrarily be set equal to zero, so that solution of equation (6.18) can be made. It is noted that from a calculation point of view it is not even necessary to consider equation (6.18). All that has to be done is to replace $w_1' \bar{p}$ by $\sigma_1$, $w_2' \bar{p}$ by $\sigma_2$, etc., in equation (6.13); the deflection that is obtained will indicate the stress that would develop at point $A$ if a unit oscillatory load were applied at the position where the deflection is being considered.

7. Flutter Analysis at High $M$

A. Piston theory

Aerodynamic "piston" theory has received much attention lately and appears to be a useful tool for approximating the aerodynamic forces that develop on moving two-dimensional lifting surfaces traveling at high Mach numbers [28—30]. It is based on a fundamental relation in acoustics which
gives the pressure that develops on a plane moving so as to produce plane sound waves (or a piston moving in a one-dimensional channel, from whence the name of the theory is derived). A point function relation between perturbation velocity and pressure thus results, and this provides for a great simplification in the treatment of the nonsteady motion of lifting surfaces, allowing closed form solutions to be derived in many cases. A brief derivation of the fundamental relations follows.

The consideration of the differential relation that exists at the plane surface

\[ dp = \rho a du, \quad (7.1) \]

in which \( u \) is the normal velocity of the plane surface and \( a \) is the local velocity of sound given by

\[ a^2 = \frac{dp}{\rho} = \gamma \frac{p}{\rho}, \]

together with the adiabatic gas law, equation (1.5), leads directly to the long established acoustical relation

\[ p = p_0 \left( 1 + \frac{\gamma - 1}{2} \frac{u}{a_0} \right)^{\frac{2\gamma}{\gamma - 1}}, \quad (7.2) \]

where \( p_0, a_0 \) are for undisturbed or "free stream" conditions.

If equation (7.2) is expanded and use is made of the expression for speed of sound in the undisturbed medium, there results (with terms up through \( u^2 \), for example)

\[ p - p_0 = \rho_0 a_0^2 \left( \frac{u}{a_0} + \frac{\gamma + 1}{4} \frac{u^2}{a_0^2} \right), \quad (7.3) \]

where \( p - p_0 \) represents the perturbation pressure.

Consider the application of this equation to an airfoil, see following sketch

\[ \begin{array}{c}
\text{Consider the application of this equation to an airfoil, see following sketch} \\
\text{The vertical velocity for a point on the surface having a deflection } w_s \text{ is} \\
\text{Let } w_s \text{ for the upper surface be}
\end{array} \]

\[ u = \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) w_s. \quad (7.4) \]

Let \( w_s \) for the upper surface be

\[ w_1 = w(x, t) + Z_1(x). \quad (7.5) \]
Then
\[ u_1 = \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) w + v \frac{\partial Z_1}{\partial x}. \] (7.6)

The substitution of this equation into equation (7.3) gives the disturbance pressure on the upper surface
\[ p_1 - p_0 = \rho_0 a_0^2 \left[ \left( 1 + \frac{\gamma + 1}{2} \frac{v}{a_0} \frac{\partial Z_1}{\partial x} \right) \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) \frac{w}{a_0} + v \frac{\partial Z_1}{\partial x} \right] \]
\[ + \frac{\gamma + 1}{4} \frac{v^2}{a_0^2} \left( \frac{\partial Z_1}{\partial x} \right)^2 + \frac{\gamma + 1}{4} \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right)^2 \frac{w^2}{a_0^2}. \] (7.7)

This equation, with \( Z_2 = 0 \), gives the pressure that develops on skin panels and will be used in a later section. For the airfoil, the lower surface \( w_x = w(x, t) - Z_2(x) \) \((Z_2 \text{ positive downward})\) is treated likewise, and the difference in pressures between upper and lower surface is then taken to yield the loading across the airfoil with the result
\[ \Delta p = p_1 - p_0 - (p_2 - p_0) = \rho_0 a_0^2 \left[ \left( 2 + \frac{\gamma + 1}{2} \frac{v}{a_0} \frac{\partial (Z_1 + Z_2)}{\partial x} \right) \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) \frac{w}{a_0} \right] \]
\[ + \frac{v}{a_0} \frac{\partial (Z_1 - Z_2)}{\partial x} + \gamma + 1 \frac{v^2}{4} \frac{a_0^2}{a_0^2} \left[ \left( \frac{\partial Z_1}{\partial x} \right)^2 - \left( \frac{\partial Z_2}{\partial x} \right)^2 \right]. \] (7.8)

For \( Z_1 = Z_2 \) and \( h = 2Z_1 \), this loading becomes
\[ \Delta p = 2 \rho_0 a_0^2 \left( 1 + \frac{\gamma + 1}{4} \frac{v}{a_0} \frac{\partial h}{\partial x} \right) \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) \frac{w}{a_0}. \] (7.9)

These are thus the two-dimensional air forces that are given by "piston" theory. The theory has yielded flutter results which compare favorably with the results indicated by more involved theories, and because of the relative ease of application, it appears especially useful for making trend studies. It also is attractive as a convenient means for defining reference flutter velocities in the supersonic range, and may thus serve a role much the same as incompressible unsteady flow theory has in the subsonic range. As a check on the domain of applicability of the theory, equation (7.9) was used to derive the lift and moment on a thin airfoil oscillating both in vertical motion and pitch, and the results were compared with the indications of linearized 2-dimensional oscillatory flow theory. Regions of very good agreement were found as follows: for \( M > 3 \) regardless of the value of reduced frequency \( k \), and for \( 1 < M < 3 \), as long as \( k > 3 - M \).

B. Destabilizing effects

It has been recalled in the introduction and elsewhere in the paper that a structure in a high temperature environment loses stiffness due to thermal stresses and further because of a reduction in materials properties. In aero-
elastic considerations, these effects are generally detrimental, or destabilizing. Another destabilizing effect can come from the effect of body shape on the aerodynamics. It is noted that airfoil thickness terms appears in equation (7.8), and these terms indicate a pronounced influence on aeroelastic phenomena at high Mach number. In fact, it was considered at one time, that if a wing were safe from flutter in the transonic range, that it would be entirely free from flutter for speeds beyond this, due to the fact that the center of pressure moved rearward to about the midchord position. Studies made with thickness effects included, however, have indicated that wings may have critical supersonic flutter regions as well.

Without presenting specific case history flutter studies, the following treatment is given to indicate the nature and extent of the destabilizing effects of thickness. Consider the two airfoils in the following sketch, where a thickness of 3\% and an M of 5 are assumed.

![Airfoil Sketch](image)

Application of equation (7.9) indicates that thickness effects would modify the pressures on the airfoil according to the diagrams given below the airfoils. There is no net change in total force, but there is a marked influence on moment; the added force that develops in the front half due to thickness effects (or the deficiency in force in the rear half) is the same for both examples; however, the destabilizing effect is greater for the second case because of a longer moment arm.

It may be remarked that it is unfortunate that an airfoil of shape

![Airfoil Shape](image)
is not a good aerodynamic shape; from an aeroelastic point of view a stabilizing moment would develop and as regards heat flow, it would appear desirable because maximum thickness is present where the heat influx is the most severe.

C. Flutter analysis with difference equations

A flutter analysis by difference equations and piston theory proceeds simply by combining equation (4.10) and equation (7.8) or (7.9). For an airfoil in motion, the total loading that acts at a point is the sum of inertia and aerodynamic loadings, thus (for symmetrical airfoils)
\[ p = -m\ddot{w} - 2\rho a \left( 1 + \frac{\gamma + 1}{4} M \frac{\partial h}{\partial x} \right) \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) w. \] (7.10)

where the subscript zero denoting free-stream conditions has been dropped for convenience; this practice will be followed throughout the remainder of the report.

In matrix notation this equation would read
\[ p = -mw - 2\rho a \eta \dot{w} - 2\rho a v \eta C w, \] (7.11)
where \( m \) is a diagonal mass matrix, \( \eta = 1 + \frac{\gamma + 1}{4} M \frac{\partial h}{\partial x} \) a diagonal matrix which takes into account thickness, \( C \) a matrix which produces \( \frac{\partial}{\partial x} \), and \( \dot{w}, \ddot{w} \) and \( w \) are column matrices.

With the adjacent coordinate system inferred

the substitution of equation (7.11) into equation (4.10), gives
\[ Sw + 2\rho a v \eta C w + 2\rho a \eta \dot{w} + mw = 0, \] (7.12)
where \( S \) is the complete stiffness matrix of the difference equations. For flutter, let
\[ w = \bar{w} e^{i\omega t}, \] (7.13)
and so equation (7.12) becomes
\[ [S + 2\rho a v \eta C + i2\rho a \omega \eta - \omega^2 m] \bar{w} = 0. \] (7.14)

The vanishing of the determinant of these equations then defines flutter. This difference equation set-up for flutter is noted to consider both the aerodynamics and structures in a single combined operation, a feature which has been often regarded as a desired goal of the aeroelastician.

Another important point to note about the foregoing difference equation set-up for flutter is that complex numbers appear only in the main diagonal elements of the flutter determinant, and this should considerably ease the solution problem of the determinant. There are indications that for some applications an even greater simplification may be made as regards the solution for flutter. The idea is embodied in the following argument.

If equation (7.14) is considered without the damping term \( i2\rho a \omega \eta \), then an equation for a conservative system having natural vibration frequencies results. The presence of the flow term \( 2\rho a v \eta C \), however, has a critical influence on these natural frequencies, and because the term is associated with
the odd derivative term $\frac{\partial w}{\partial x}$, its effect will be destabilizing (mathematically, a non-self-adjoint system is involved); that is, it causes certain of the natural frequencies to approach one another as the velocity is increased as illustrated in the accompanying sketch.

At a certain critical velocity, two of these natural frequencies coalesce, and for higher velocities, these two frequencies lose their real significance and become complex conjugates, $\omega_R + i\omega_I$ and $\omega_R - i\omega_I$. The second of these indicates an unstable condition; i.e., substitution in equation (7.13) gives $w = \tilde{w} e^{\omega t + i\omega x}$ which represents a divergent oscillation. Thus, it is seen that for sufficiently high velocities, the conservative stable system can become unstable.

This critical velocity may be regarded as a conservative estimate of the flutter speed, since the actual flutter speed will be slightly higher because of the damping term in equation (7.14) (the amount that the flutter speed is greater is only that amount which will cause a degree of instability which is just sufficient to offset the damping term). An idea of how much greater the flutter speed may be from the critical velocity can be obtained by considering the application of equation (7.14) to a wing of constant thickness. For this case, the equation may be written in the form (since $\gamma = I$, the identity matrix)

$$[S + 2\rho a v C] \bar{w} = \omega^2 \left(1 - i \frac{2\rho a}{m\omega}\right) m \bar{w}. \quad (7.15)$$

Now regard $\omega^2 \left(1 - i \frac{2\rho a}{m\omega}\right)$ as the complex frequency that is found in a solution which makes use of a plot of $g$ vs. $v$, and regard the quantity $\frac{2\rho a}{m\omega}$ as in the nature of a $g$; then a curve of the adjoining type would be found.

The critical speed as obtained from the conservative system is that indicated for $g = 0$, while the actual flutter speed is the speed indicated for a $g = \frac{2\rho a}{m\omega}$. For most practical cases, this value of $g$ is a very small quantity, and hence, there should be little difference between $v_f$ and $v_{cr}$. Note, $\frac{2\rho a}{m\omega}$ may be shown to be like the damping ratio $\frac{B}{B_{cr}}$. It also may be expressed as $\frac{1}{Mk} \frac{\rho_l}{h/b}$, where $k$ is reduced frequency in terms of half chord $\frac{b}{2}$, and $\rho_m$ is density of the wing material; from this expression, the very small magnitude of the term is readily apparent.
A similar type conclusion as above would be expected to apply to wings of variable thickness. The argument is not pure conjecture, for it will be shown later in the section on panels that the influence of the damping term is indeed negligible. Thus, it appears that the damping term may be omitted in the application of equation (7.14) to plate-like wings. Any critical speed that is obtained should be a close approximation to the flutter speed, especially if appreciable chordwise bending is involved, since a chordwise bending type of flutter is not too unlike the flutter of panels, where the damping effect is known to be unimportant. It is remarked, however, that the neglect of the damping term may by-pass a flutter mode that might possibly have a lower flutter speed, if this mode involves primarily overall bending and torsion of the wing, with little chordwise deformation.

D. Reduction to real determinant

The following is a means for reducing the complex flutter determinant as obtained from equation (7.14) to a real flutter determinant. For convenience in writing let

\[ A_1 = S + 2 \rho a v \eta C - \omega^2 m, \]
\[ A_2 = 2 \rho a \omega \eta, \]  \hspace{1cm} (7.16)

and then let \( \bar{w} = \bar{w}_1 + i \bar{w}_2 \). Equation (7.14) would then appear

\[ [A_1 + i A_2] (\bar{w}_1 + i \bar{w}_2) = 0. \] \hspace{1cm} (7.17)

Expand, giving

\[ A_1 \bar{w}_1 - A_2 \bar{w}_2 = 0, \]
\[ A_2 \bar{w}_1 + A_1 \bar{w}_2 = 0. \]

Solve the first of these for \( \bar{w}_2 \), and substitute in the second giving

\[ ([A_2^{-1} A_1]^2 + I) \bar{w}_1 = 0, \] \hspace{1cm} (7.18)

where \( A_2^{-1} \) is a simple diagonal matrix. The determinant of this equation is thus the desired all real flutter determinant. With equations (7.16) this determinant reads

\[ A = \left[ \eta^{-1} (S + 2 \rho a v \eta C - \omega^2 m) \right]^2 + \omega^4 4 \rho^2 a^2 I, \]
\[ = 0 \quad \text{for flutter.} \] \hspace{1cm} (7.19)

The value of this determinant is always real and positive, and may be proved as follows. If equation (7.17) is multiplied by \( A_2^{-1} \), the determinant obtained is

\[ |A_2^{-1} A_1 + i I| = A_1 + i A_2. \]

Multiply this by its conjugate

\[ |A_2^{-1} A_1 + i I| |A_2^{-1} A_1 - i I| = A_1^2 + A_2^2. \]
But the product of the determinants of two matrices is equal to the determinant of the product of the two matrices, hence

$$A_1^2 + A_2^2 = \det[A_1^{-1} A_1 + iI] \det[A_2^{-1} A_1 - iI] = [\det[A_2^{-1} A_1]^2 + I].$$

Since the left-hand side is always positive, so too must the right-hand side be. But the determinant on the right-hand side is the same as the determinant of the all real flutter equation (7.18); thus, the proof is completed.

8. Flutter of Panels

A. Exact solution for flutter of flat panels

Through the use of aerodynamic piston theory, an exact solution may be made for the flutter of certain flat panels. Application is made here to semi-infinite flat panels with axial load, for both hinged and clamped edge conditions. The coordinate system is as follows

The differential equation for this system is

$$\frac{\partial^4 w}{\partial x^4} + \sigma h \frac{\partial^2 w}{\partial x^2} = p = -m \ddot{w} - 2 \rho a \dot{w} - \rho a v \frac{\partial w}{\partial x},$$

where the aerodynamic loading as given by the two terms on the far right comes from equation (7.7), with $Z_1 = 0$. It is noted that the damping term has been doubled on the assumption that the quiescent air beneath the panel provides a damping equal to the amount obtained from the upper surface.

Also, it may be remarked that the introduction of the factor $\frac{M}{\sqrt{M^2 - 1}}$ in the aerodynamic terms would bring them into accord with terms indicated by certain other aerodynamic theories; these terms would then reduce, for example, to the well known Ackeret result for steady-state supersonic flow. This term, however, differs only slightly from unity at the higher Mach numbers, where piston theory is supposed to apply, and is therefore omitted. The solution to follow is an extension of the solution given by Hedgepeth [31], in which he neglects the damping term and treats a simply supported plate.

With

$$w = \bar{w}(x) e^{\alpha t},$$

equation (8.1) becomes

$$D \frac{d^4 \bar{w}}{d x^4} + \sigma h \frac{d^2 \bar{w}}{d x^2} + \rho a v \frac{d \bar{w}}{d x} + 2 \rho a \alpha \bar{w} + m \alpha^2 \bar{w} = 0,$$
where in general \( \alpha \) is a complex number, \( \alpha = \beta + i \omega \). The problem is thus one of determining the value of \( \nu \) which will cause \( \beta \) to vanish, or pass from a negative to a positive value. It is convenient to introduce the notation \( \eta = \frac{x}{L} \), and to express equation (8.3) in the non-dimensional form

\[
\frac{d^4 \bar{w}}{d \eta^4} + \frac{\alpha h L^2}{D} \frac{d^2 \bar{w}}{d \eta^2} + \frac{\rho a v L^3}{D} \frac{d \bar{w}}{d \eta} + \left( \frac{2 \rho a L^4}{D} \alpha + \frac{m L^4}{D} \alpha^2 \right) \bar{w} = 0, \quad (8.4a)
\]

or simply

\[
\bar{w}^{IV} + \rho \bar{w}' + \lambda \bar{w}'' + k \bar{w} = 0, \quad (8.4b)
\]

where

\[
R = \frac{\alpha h L^2}{D} = \pi^2 \frac{\alpha h}{(\sigma h)_r} = \pi^2 r,
\]

\[
\lambda = \frac{\rho a v L^3}{D}, \quad (8.5)
\]

\[
k = \frac{2 \rho a L^4}{D} \alpha + \frac{m L^4}{D} \alpha^2 = \pi^4 g_a \frac{\alpha}{\omega_r} + \pi^4 \frac{\alpha^2}{\omega_r^2},
\]

in which the following reference quantities have been introduced for later convenience

\[
(\sigma h)_r = \pi^2 \frac{D}{L^2} \quad \text{(Euler buckling load for pin ends),}
\]

\[
\omega_r = \pi^2 \sqrt{\frac{D}{m L^4}} \quad \text{(first natural freq. for pin ends),}
\]

\[
g_a = \frac{2 \rho a}{m \omega_r} = 0.702 \sqrt{1 - \frac{\mu^2}{\rho_m a_m h^2}} \quad \text{(damping coefficient based on } \omega_r).\]

Now let

\[
\bar{w} = e^{\nu \eta}
\]

giving the characteristic equation

\[
p^4 + R p^3 + \lambda p + k = 0. \quad (8.7)
\]

Examination of this equation indicates that the four roots may be given by

\[
p_1 = -\epsilon + b, \\
p_2 = -\epsilon - b, \\
p_3 = \epsilon + i c, \\
p_4 = \epsilon - i c. \quad (8.8)
\]

But by the theorem in algebra which relates the coefficients of the characteristic equation to the sum of the product of the roots (taken one, two, etc. at a time), the values of \( b, c, \) and \( \epsilon \) must be given by
In terms of the four roots the deflection is

$$w = A_1 e^{\rho_1 \eta} + A_2 e^{\rho_2 \eta} + A_3 e^{\rho_3 \eta} + A_4 e^{\rho_4 \eta},$$

where the $A_i$'s are coefficients to be determined by the boundary conditions. The satisfaction of these boundary conditions leads to the following stability equation for flutter:

For simply supported ends, $w(0) = w''(0) = w(1) = w''(1) = 0$

$$i 16 \epsilon^2 b c \cosh 2 \epsilon - (b + i c)^2 \left[4 \epsilon^2 - (b - i c)^2\right] \cosh (b + i c)$$
$$+ (b - i c)^2 \left[4 \epsilon^2 - (b + i c)^2\right] \cosh (b - i c) = 0.$$ (8.11)

For fixed ends, $w(0) = w'(0) = w(1) = w'(1) = 0$

$$i 4 b c \cosh 2 \epsilon + \left[-4 \epsilon^2 + (b - i c)^2\right] \cosh (b + i c)$$
$$- \left[-4 \epsilon^2 + (b + i c)^2\right] \cosh (b - i c) = 0.$$ (8.12)

To solve these stability equations, the following procedure is used. For a given $R$, a value of $\lambda$ is chosen; then $\epsilon = \epsilon_1 + i \epsilon_2$ is varied until a solution of the stability equation is found. With a root $\epsilon$ found, $k = k_1 + ik_2$ is determined from the last of equations (8.9). Then a solution is made for $\alpha$ by the last of equations (8.5) in terms of this $k$

$$\alpha = \beta + i \omega = -\frac{\omega_r g_a^2}{2} \pm \sqrt{\left(\frac{\omega_r g_a^2}{2}\right)^2 + \frac{\omega_r^2}{\pi^4} (k_1 + i k_2)},$$ (8.13a)

and this result is examined to see what conditions on $g_a$ are necessary to make $\beta$ vanish.

It is found that for flutter ($\beta \geq 0$), the following condition on $g_a$ must exist

$$\pi^4 g_a^2 > \frac{k_2^2}{-k_1},$$ (8.13b)

and that the flutter frequency is given by

$$\omega_f = \frac{\sqrt{-k_1}}{\pi^2 \omega_r}.$$ (8.13c)

Note, $k_1$ always turns out to be a negative number.

Some of the results obtained by the above solution for $r = 0$ are shown in fig. 8. From the curves shown, a significant observation can be made. The critical values of $\frac{\lambda}{\pi^4} = 3.52$ and (6.52) indicated by the horizontal lines are the
results that would be obtained if the damping term in equation (8.1) were omitted, and the value for pin-ends corresponds to the critical value obtained by Hedgepeth [31]. In this reference, use is made of the argument presented in section 7 that when two frequencies of the conservative system coalesce, flutter must be imminent. Now, the values of $g_a$ usually found in practice (see equation (8.6) and fig. 9) will be located at the far left in the figure, where there is little difference in the actual flutter boundary and the horizontal line. This thus bears out the contention made previously in the paper that for practical purposes, the damping term may be disregarded; in other words, it is sufficient to consider only the conservative system — at least this is true for panels.

As a matter of interest, the flutter frequency and reduced frequency of flutter found for the pin-ended case are shown in fig. 10, and in fig. 11 typical modes shapes of flutter are shown for both cases (note how the deflection is squashed to the rear of the panel). These figures indicate that there is a very pronounced second mode content at flutter, and suggest that a Rayleigh-Ritz
Fig. 10. Frequency and reduced frequency at flutter for panels with hinged edges, $r = 0$.

Fig. 11. Typical flutter mode shapes for panels.

a) Hinged ends.

b) Clamped ends

Fig. 12. Flutter boundaries for flat panels with axial load.
or Galerkin solution using 3 modes might be adequate for approximate solution. Later results will bear further on this point.

Further solutions of the flutter stability equations are shown in fig. 12, where the critical value of \( \lambda \) is plotted against \( r \), which denotes the ratio of the applied axial load to the Euler buckling load for pin ends. At first thought it may seem that the results are trivial for loads greater than the respective Euler loads (\( r > 1 \) for pin ends, \( r > 4 \) for fixed ends) but the study given in the next section on the large deflection behavior of the plates will show that these results have a very real significance.

The flutter velocity follows readily from fig. 12 and the expression for \( \lambda \) given by the second of equations (8.5). Thus, with \( a_m^2 = \frac{E}{\rho_m} \), which defines the speed of sound in an elastic body, the expression for \( \lambda \) yields

\[
M = \frac{v}{a} = \frac{\lambda}{12 (1 - \mu^2)} \frac{\rho_m a_m^2 h^3}{\rho a^2 L^3} = 0.0917 \frac{\rho_m a_m^2 h^3}{\rho a^2 L^3} = 8.93 \left( \frac{\lambda}{\pi^4} \right) \frac{\rho_m a_m^2 h^3}{\rho a^2 L^3} \text{ for } \mu = 0.3. \tag{8.14}
\]

Fig. 13. Variation of panel flutter Mach number with altitude, material, and slenderness.

To facilitate the use of this equation, figs. 13 and 14 have been prepared. The ordinate in fig. 13 is based on room temperature values for the speed of sound in the materials; when high temperatures are involved, these ordinates may be adjusted downward through means of fig. 14, which indicates the variation in \( a_m^2 \) with temperature (note, \( a_m^2 \) is proportional to the modulus \( E \)).

**B. Flutter of panels with large deflections**

1. **Static considerations.** To start this section, it is fitting to see how much temperature change is required to cause buckling of the semi-infinite panels being treated. If the axial load that develops due to a temperature rise is
equated to the Euler buckling load, the following equations for critical buckling temperatures are obtained

\[
\Delta T_{cr} = \frac{c \pi^2 h^2}{12(1-\mu^2) \alpha L^2} \quad \text{for } \sigma_y = 0,
\]

\[
\Delta T_{cr} = \frac{c \pi^2 h^2}{12(1+\mu) \alpha L^2} \quad \text{for } \epsilon_y = 0,
\]

where \( c \), the column fixity coefficient, is 1 for pin ends and 4 for clamped ends.

Critical temperature values as obtained from the first of these equations, which is for the condition of unrestrained expansion in the semi-infinite direction, are shown in fig. 15. The values obtained from the second equation, which considers completely restrained expansion in the semi-infinite direction, would be 0.7 of the values shown, assuming a Poisson’s ratio \( \mu \) of 0.3. The critical temperatures are thus noted to be surprisingly low.

With the realization that the temperatures possible in high speed flight are far in excess of these values, so that large deflections result, it appears mandatory that any treatment of panels should also give due consideration to their behavior in the large deflection region.

The following extends the work given in [2] and [32], and will yield essential ingredients for the random loading study which follows in a later section.

![Figure 15: Critical temperature for buckling of panels.](image-url)
Consider the following system

\[ \lambda_i = \frac{1}{2} \int_0^L w_i' d x, \quad \lambda_f = \frac{1}{2} \int_0^L w_f' d x, \quad (8.15) \]

where a prime denotes \( \frac{\partial}{\partial x} \). With these quantities, the following compatibility equation must exist

\[ \alpha \Delta T L - (\lambda_f - \lambda_i) \frac{\sigma}{E} L = \frac{\sigma h}{k} + \alpha \Delta T_s L, \quad (8.16) \]

where \( \sigma \) is the axial stress; this condition applies when there is no restraint against expansion in the semi-infinite direction \((\sigma_y = 0)\). This equation yields the axial compressive stress

\[ \sigma = \frac{E \beta_k}{L} \left[ \alpha L (\Delta T - \Delta T_s) - \frac{1}{2} \int_0^L w_f'^2 d x + \frac{1}{2} \int_0^L w_i'^2 d x \right], \quad (8.17a) \]

where \( \beta_k \) is a support factor given by

\[ \beta_k = \frac{1}{1 + \frac{E h}{L k}}. \quad (8.17b) \]

If piston theory is assumed, the basic differential equation is the same as that given by equation (8.1); with equation (8.17), this equation becomes

\[ D \frac{\partial^4 (w_f - w_i)}{\partial x^4} + \frac{E \beta_k h}{L} \left[ \alpha L (\Delta T - \Delta T_s) - \frac{1}{2} \int_0^L w_f'^2 d x + \frac{1}{2} \int_0^L w_i'^2 d x \right] \frac{\partial^2 w_f}{\partial x^2} \]

\[ = -m \ddot{w}_f - 2 \rho a \dot{w}_f - \rho a v \frac{\partial}{\partial x} w_f. \quad (8.18) \]

Let \( w_f - w_i = w_s + \omega \), where \( w_s \) and \( \omega \) represent static and time dependent deflections, respectively. With this consideration on \( w_f \), equation (8.18) may be separated into the two equations (for the case of \( w_i = 0 \)
\[
D \frac{d^4 w_s}{dx^4} + E \beta_k h \left[ \alpha (\Delta T - \Delta T_s) - \frac{1}{2L} \int_0^L w_s'^2 \, dx + \frac{1}{2L} \int_0^L w_i'^2 \, dx \right] \frac{d^2 w_s}{dx^2} = -\rho a v \frac{dw_i}{dx},
\]
(8.19)

\[
D \frac{\partial^4 w}{\partial x^4} + E \beta_k h \left[ \alpha (\Delta T - \Delta T_s) - \frac{1}{2L} \int_0^L (w_s' + w')^2 \, dx + \frac{1}{2L} \int_0^L w_i'^2 \, dx \right] \frac{\partial^2 w}{\partial x^2} = -\rho \bar{w} - 2\rho a \dot{w} - \rho a v \frac{\partial w}{\partial x}.
\]
(8.20)

It is noted that these equations are strongly non-linear. However, instructive approximate solutions of them can be made, and in a few cases, exact solutions even exist. The first of these will be treated now and the second will be handled in the next section.

In equation (8.19), let
\[
w_s = a_1 w_1 + a_2 w_2,
\]
(8.21)

where the \( w_n \)'s satisfy the equation
\[
D \frac{d^4 w_n}{dx^4} = \omega_n^2 m w_n.
\]
(8.22)

A Galerkin type solution leads then to the following nonlinear simultaneous equations, for the case of \( w_i = 0 \)
\[
\frac{\omega_1^2}{\omega_r^2} A_1 - \frac{B_1}{\pi^2} \left[ r - \frac{E \beta_k h}{2D} \left( a_1^2 B_1 + a_2^2 B_2 \right) \right] a_1 - \frac{\lambda}{\pi^4} A_{21} a_2 = 0,
\]
(8.23)

\[
\frac{\lambda}{\pi^4} A_{21} a_1 + \frac{\omega_r^2}{\omega_r^2} A_2 - \frac{B_2}{\pi^2} \left[ r - \frac{E \beta_k h}{2D} \left( a_1^2 B_1 + a_2^2 B_2 \right) \right] a_2 = 0,
\]

where \( \lambda \) and \( \omega_r \) are given by equations (8.5) and (8.6) and
\[
A_n = \frac{1}{\pi} \int_0^L w_n^2 \, d\eta, \quad \left( \eta = \frac{x}{L} \right), \quad A_{mn} = \frac{1}{\pi} \int_0^L w_m w_n' \, d\eta = -A_{nm},
\]
\[
B_n = \frac{1}{\pi} \int_0^L w_n'^2 \, d\eta, \quad r = \frac{(\Delta T - \Delta T_s)_r}{(\Delta T - \Delta T_s)_r},
\]
(8.24)

Application of equation (8.23) to the cases of pin and clamped ends was made; the following gives the modes shapes and frequencies used, and indicates the coefficients and equations that result:
For pin ends,

\[ w_1 = \sin \frac{\pi x}{L}, \quad \omega_1^2 = \frac{\pi^4 D}{m L^4}, \]
\[ w_2 = \sin \frac{2\pi x}{L}, \quad \omega_2^2 = 16 \pi^4 \frac{D}{m L^4}, \]
\[ A_1 = A_2 = \frac{1}{2}, \quad A_{21} = \frac{4}{3}, \quad B_1 = \frac{\pi^2}{2}, \quad B_2 = 2 \pi^2, \]
\[ 1 - r + \frac{E \beta_k h}{4 D} \left( a_1^2 + 4 a_2^2 \right) a_1 - \frac{8 \lambda}{3 \pi^4} a_2 = 0, \]
\[ \frac{8 \lambda}{3 \pi^4} a_1 + \left[ 16 - 4 r + \frac{E \beta_k h}{D} (a_1^2 + 4 a_2^2) \right] a_2 = 0. \] (8.25)

For clamped ends,

\[ w_1 = 1 - \cos \frac{2\pi x}{L}, \quad \omega_1^2 = 502 \frac{D}{m L^4}, \]
\[ w_2 = \sin \frac{\pi x}{L} \sin \frac{2\pi x}{L}, \quad \omega_2^2 = 3810 \frac{D}{m L^4}, \]
\[ A_1 = 1.566, \quad A_2 = 0.26, \quad A_{21} = 2.14, \quad B_1 = 2 \pi^2, \quad B_2 = \frac{5}{4} \pi^2, \]

Fig. 16. Magnitude of 1st mode component of buckled panel shape as function of velocity and temperature.
\[
\left[ 8 - 2 r + \frac{E \beta_k h}{D} \left( 2 a_1^2 + \frac{\pi^2}{4} a_2^2 \right) \right] a_1 - 2.14 \frac{\lambda}{\pi^4} a_2 = 0,
\]
\[
2.14 \frac{\lambda}{\pi^4} a_1 + \left[ 10.16 - \frac{5}{4} r + \frac{5 E \beta_k h}{8 D} \left( 2 a_1^2 + \frac{\pi^2}{4} a_2^2 \right) \right] a_2 = 0.
\]

(8.26)

(Note, the trigonometric approximations listed here for \( w_1 \) and \( w_2 \) for clamped ends were introduced simply as a convenient means for evaluating the \( B_n \)'s and a quantity \( Q_n \) which appears later. Otherwise, the exact functions satisfying equation (8.22) for clamped ends were used, where the maximum deflections were made to correspond to those of the trigonometric approximations.

Note also, that with the above numbers, the term \( \frac{\omega_1^2}{\omega^2} A_1 \) would evaluate to 8.07; this number was made 8 in equation (8.26) so that \( 8 - 2 r \) would be equal to zero, as it should, when \( r = 4 \), since \( r = 4 \) corresponds to Euler buckling for clamped ends. The variation in natural mode shape with axial load is being reflected by this slight discrepancy.)

The solution obtained from these equations is shown in fig. 16 and in the following table

<table>
<thead>
<tr>
<th>Hinged ends</th>
<th>Fixed ends</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\lambda}{\pi^4} )</td>
<td>( \frac{E \beta_k h}{4 D} a_1^2 )</td>
</tr>
<tr>
<td>0</td>
<td>( r - 1 )</td>
</tr>
<tr>
<td>( \frac{3}{8} \sqrt{5} )</td>
<td>( \frac{5}{6} \left( r - \frac{3}{2} \right) )</td>
</tr>
<tr>
<td>( \frac{3}{8} \sqrt{8} )</td>
<td>( \frac{2}{3} \left( r - 2 \right) )</td>
</tr>
<tr>
<td>( \frac{9}{8} )</td>
<td>( \frac{1}{2} \left( r - \frac{5}{2} \right) )</td>
</tr>
</tbody>
</table>

Two important conclusions can be drawn from this figure. For a certain range in \( r \) above the critical buckling value, the application of an increasing stream velocity to a buckled panel will gradually decrease the deflection and will eventually cause the panel to be blown flat; for values of \( r \) greater than 2.5

![Fig. 17. Static stability boundaries.](image-url)
and 6.06 for the hinged and clamped panels, respectively, an increasing velocity will tend to decrease the buckle size, but before the panels reach flatness a static instability develops, and the velocity necessary to attain this unstable condition is independent of \( r \). Fig. 17 indicates these two stability boundaries.

2. Dynamic considerations—frequencies and flutter. The flutter boundaries for the panels in a flat condition have been studied, and it remains to investigate the flutter characteristics for the panels in a buckled state. This can be done by use of equation (8.20), which can now be treated because equation (8.19) has been solved. A Galerkin type solution of this equation, with

\[
w = b_1 w_1 + b_2 w_2,
\]

and \( w_s \) given by the previous section, leads, for example, to the following determinant for the pin-ended case (again for \( w_i = 0 \))

\[
\begin{vmatrix}
1 - r + \frac{E \beta_k h}{4D} (3a_1^2 + 4a_2^2) - \omega_r^2 + ig_{r, \omega} \omega_r - \frac{8 \lambda}{3\pi^4} + \frac{E \beta_k h}{4D} 8a_1a_2 \\
\frac{8 \lambda}{3\pi^4} + \frac{E \beta_k h}{4D} 8a_1a_2 - 16 - 4r + \frac{E \beta_k h}{4D} (4a_1^2 + 48a_2^2) - \omega_r^2 + ig_{r, \omega} \omega_r \\
0 - \frac{24 \lambda}{5\pi^4} - 81 - 9r - \omega_r^2 + ig_{r, \omega} \omega_r \\
\end{vmatrix} = 0, \quad (8.27)
\]

where solution has been made on the assumption that \( w \) represents a small perturbation and hence is negligibly small in comparison to \( w_s \) in the integrals in equation (8.20); a linearization thus results. Investigation of this equation, and the corresponding equation for clamped ends, indicates that no flutter condition is possible in the stable buckled range. If flutter is to occur for \( r \) less than the critical values of 2.5 and 6.06 for the two plate cases, it must occur while the panel is flat or has been unbuckled by the stream. The significance of extending the results to \( r \) greater than 1 and 4 in figure 12 is thus evident.

The complete stability boundaries of the plates are then the combination of figs. 12 and 17, and this is shown in fig. 18, where other information has also been added as will be discussed; for flat plates, the value of \( r \) has the same significance, whether expressed in terms of a stress ratio, equation (8.5), or in terms of a temperature ratio, equation (8.24).

It is of interest to see how exact and modal solutions compare for flutter of flat plates. A Galerkin type solution of equation (8.1) would lead to the following equations:

For pin-ends (with three modes)

\[
\begin{vmatrix}
1 - r - \frac{\omega^2}{\omega_r^2} + ig_{a, \omega} \omega_r - \frac{8 \lambda}{3\pi^4} & 0 & b_1 \\
\frac{8 \lambda}{3\pi^4} - 16 - 4r + \frac{E \beta_k h}{4D} (4a_1^2 + 48a_2^2) - \frac{24 \lambda}{5\pi^4} & 0 & b_2 \\
0 & \frac{24 \lambda}{5\pi^4} & 81 - 9r - \frac{\omega^2}{\omega_r^2} + ig_{a, \omega} \omega_r & b_3 \\
\end{vmatrix} = 0. \quad (8.28)
\]
Fig. 18. Complete stability boundaries for hinged and clamped plates.
For clamped ends

\[
\begin{vmatrix}
8 - 2r - 1.566 \frac{\omega^2}{\omega_r^2} + i 1.566 g_a \frac{\omega}{\omega_r} & -2.14 \frac{\lambda}{\pi^4} \\
2.14 \frac{\lambda}{\pi^4} & 10.16 - \frac{5}{4} r - 0.26 \frac{\omega^2}{\omega_r^2} + i 0.26 g_a \frac{\omega}{\omega_r}
\end{vmatrix}
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} = 0. \quad (8.29)
\]

Expansion of the determinant of equation (8.28) leads to the following two equations for the real part and imaginary part, respectively,

\[
\left( \frac{\lambda}{\pi^4} \right)^2 = -\left(1-r-x\right) (16-4r-x) (81-9r-x) + g_a^2 x (98-14r-3x); \quad x = \frac{\omega^2}{\omega_r^2}, \quad (8.30a)
\]

\[
\left( \frac{\lambda}{\pi^4} \right)^2 = -\left[\left(1-r-x\right) (97-13r-2x) + (16-4r-x) (81-9r-x)\right] + g_a^2 x. \quad (8.30b)
\]

Simultaneous solution of these equations to yield the flutter condition can be made in such a way as to give another indication of the importance of damping and also to illustrate how the "natural" frequencies of the conservative system vary with velocity. As an example, fig. 19 represents plots of equations (8.30) for \( r = 2 \). The intersections on this figure designate flutter; these intersections thus yield both the values of \( \lambda \) and frequency at flutter and indicate the influence of \( g_a \). It is seen that the flutter points for various \( g_a \) are very

Fig. 19. Velocity-frequency diagram showing negligible influence of aerodynamic damping at flutter (vertical distance between solid and dashed curves exaggerated twice for clarity).
close, both in $\lambda$ and in frequency, to the peak of the $g=0$ curve, where two of the "natural" frequencies have merged into one. Thus the argument presented earlier that the merging of two natural frequencies indicates a proximity to flutter is further substantiated.

The modal solution flutter results are indicated also in fig. 18. For hinged ends, it is seen that the 3-mode solution agrees fairly well with the exact solution. The two mode solution, which agrees with the results given in [2] and [32], yields, however, substantially lower values of $\lambda$. For clamped ends, the two mode solution also leads to lower values of $\lambda$ in contrast to the exact solution. This two mode solution resembles somewhat the results given in [33], but differs considerably in numbers. No three mode solution was made for clamped ends, but presumably the results would be fairly close to the exact solution.

The experimental points shown for the clamped-end case were taken from

![Diagram](image1.png)

**Fig. 20.** Panel flutter trends including elevated temperature effects; for buckled panels, $\frac{\lambda}{\pi^4} = 1.52$, and with $T_w = T_{aw}$ (recovery factor of 0.9).

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[33], and are seen to conform well to the theoretical findings; there is still a need, however, for additional experimental work for $\lambda$ closer to the stability boundaries.

The flutter results discussed are for a zero initial deflection. With the aid of fig. 16, it can be reasoned, however, that the consideration of finite initial deflection would lead to boundary curves of the type labeled $w_i$ finite in fig. 18. Thus, little if any change would be expected in the stability boundaries if initial deflections are present.

From fig. 18, fig. 20 has been prepared. This figure is more in the nature of a design figure, and indicates the influence that various parameters, such

![Graph showing natural frequencies for hinged and clamped ends.](image)

Fig. 21. "Natural" frequencies.
as altitude, $\frac{L}{h}$ ratio, material, and elevated temperature have on the flutter speed of panels (see eq. (8.14) and figs. 13 and 14).

Since flutter is strongly linked with the natural frequency of a structure, it is of interest to examine the “natural” frequencies of the panels. The frequencies as obtained by setting $g=0$ in equation (8.28) (using 2 modes only), equation (8.29), equation (8.27), and the corresponding equation for buckled clamped plates, are given in fig. 21. The curves for $\lambda = 0$ in the case of pin ends agree with the results presented in [34], and the experimental points shown along these curves are the values given in this reference; remarkable agreement between theory and experiment is noted. In the flat plate range of the figure, the frequencies gradually merge together as the velocity, or $\lambda$, is increased. In the buckled plate range (large values of $r$), the frequencies become more separated. There would be a dynamic instability or flutter at very high values of $\lambda$, but these would be far in excess of the value which causes static instability; large deflection dynamic instabilities would probably also be present for higher $\lambda$, as some tests have indicated. For the pin-end case, the lower frequency for $\frac{\lambda}{x^2} = \frac{9}{8}$ is of special interest, since this frequency is zero for all values of $r$ greater than 2.5, and indicates complete lack of stiffness — hence, static instability. For the clamped plate, only the frequencies for $\lambda = 0$ are shown, but the general trend in the frequencies with increasing $\lambda$ would be the same as for the pin-ended case, and for this case, the lower frequency would be zero for all values of $r > 6.06$ when $\frac{\lambda}{x^2}$ reached 1.52.

C. Treatment of finite width

In this section, the flutter of rectangular panels is treated. It will be shown that the solution given in section 8. A for semi-infinite panels may be applied also to finite width panels by a simple change in the definition of some of the basic parameters. The discussion will be limited to flat panels.

Consider that a temperature rise $\Delta T$ in the panel has brought about neutral surface forces, $-h \sigma_x$ and $-h \sigma_y$ (see equations (3.11), with $\epsilon_x = \epsilon_y = \gamma_{xy} = 0$).

From equations (3.41) and (7.7), the appropriate differential equation of the system would be

$$\nabla^4 w = \frac{1}{D}\left(-m \ddot{w} - 2 \rho a \dot{w} - \rho a v \frac{\partial w}{\partial x} - \sigma_x h \frac{\partial^2 w}{\partial x^2} - \sigma_y h \frac{\partial^2 w}{\partial y^2}\right).$$

This equation may be reduced to a differential equation in $x$ alone, and written in a form analogous to equation (8.4), through application of the following successive steps. Let

$$w = f(x) g(y) e^{\alpha t}$$
and substitute this expression in equation (8.31); then multiply through by \( g(x) \) and integrate across the width; finally, introduce the notation \( \eta = \frac{x}{L} \).

The result is the equation

\[
f^\prime + R f'' + \lambda f + k f = 0,
\]

where a prime denotes \( \frac{\partial}{\partial \eta} \), and

\[
R = \frac{\sigma_x h L^2}{D} - 2 L^2 c_0 \frac{c_1}{c_0} = \pi^2 r, \quad \lambda = \frac{\rho a v L^2}{D},
\]

\[
k = \frac{2 \rho a L^4}{D} \alpha + \frac{m L^4}{D} \alpha^2 - \frac{\sigma_y h L^4}{D} c_1 + L^4 c_2 c_0,
\]

in which

\[
c_0 = \int_0^b g^2 \, dy, \quad c_1 = \int_0^b \left( \frac{\partial g}{\partial y} \right)^2 \, dy, \quad c_2 = \int_0^b \left( \frac{\partial^2 g}{\partial y^2} \right)^2 \, dy.
\]

Thus, if the form of the deflection shape in the \( y \)-direction is assumed, it is possible to reduce the two-dimensional flutter equation to the one-dimensional equation given by equation (8.32). This equation is identical to equation (8.4), and therefore the solutions given in that section should also apply here. The solutions of the stability equations, however, have a slightly different interpretation because the parameters \( R \) and \( k \) are defined differently.

Before specific results are considered, let attention be focused on the parameter \( k \). It was shown in section 8. A that the damping term has only a minute influence on the flutter speed, and hence, may be dropped. With no damping, the exponential coefficient \( \alpha \) may be then written simply \( \alpha = i \omega \). The two expressions for \( k \) given by equations (8.5) and (8.33a) would thus appear (dropping the \( \alpha \) term which comes from damping)

\[
k_{1D} = -\pi^4 \frac{\omega_{1D}^2}{\omega_r^2}, \quad k_{2D} = -\pi^4 \frac{\omega_{2D}^2}{\omega_r^2} - \frac{\sigma_y h L^4}{D} c_1 + L^4 c_2 c_0.
\]

These \( k \)'s are equal, however, since they are associated with a common stability (or frequency) equation, and hence

\[
\frac{\omega_{2D}^2}{\omega_r^2} = \frac{\omega_{1D}^2}{\omega_r^2} - \frac{\sigma_y h L^4}{\pi^4 D} c_1 + \frac{L^4 c_2}{\pi^4 c_0}.
\]

Thus, the rectangular panel frequencies may be obtained directly from the frequencies for the one-dimensional case through means of this equation. When the restraint against expansion is similar in both the \( x \)- and \( y \)-directions, equation (8.35) may be reduced further, since \( \sigma_x = \sigma_y \). With this equality in stress, equations (8.33a) would yield
\[
\frac{\sigma_x h L^2}{D} = \frac{\sigma_y h L^2}{D} = \pi^2 r + 2 \frac{L^2 c_1}{c_0}.
\]

Equation (8.35) may therefore be written
\[
\frac{\omega_{2D}^2}{\omega_r^2} = \frac{\omega_{1D}^2}{\omega_r^2} - \frac{L^2}{\pi^2} \frac{c_1}{c_0} r + \frac{L^4}{\pi^4} \left( \frac{c_2}{c_0} - \frac{2}{c_0} \right) \left( \frac{c_2}{c_0} \right).
\]

Now return to the consideration of the results obtained for specific cases. Suppose \( g(y) \) is chosen as follows
\[
g(y) = \sin \frac{\pi y}{b} \quad \text{for simple support},
\]
\[
g(y) = \frac{1}{2} \left( 1 - \cos \frac{2\pi y}{b} \right) \quad \text{for clamped edges}.
\]

These functions would lead to
\[
c_0 = \frac{b}{2}, \quad c_1 = \frac{\pi^2}{2 b}, \quad c_2 = \frac{\pi^4}{2 b^3},
\]
\[
c_0 = \frac{3}{8} b, \quad c_1 = \frac{\pi^2}{2 b}, \quad c_2 = \frac{2\pi^4}{b^3},
\]

where the top row applies to the simply supported case. The substitution of these equations into the first of equations (8.33a) and into (8.36) gives the following results for \( r \) and frequency:

For simply supported edges,
\[
r = \frac{\sigma_x h}{(\sigma_x h)_r} - \frac{2}{3} \frac{L^2}{b^2}, \quad \frac{\omega_{2D}^2}{\omega_r^2} = \frac{\omega_{1D}^2}{\omega_r^2} - \frac{L^2}{b^2} r - \frac{L^4}{b^4}.
\]

For clamped edges,
\[
r = \frac{\sigma_x h}{(\sigma_x h)_r} - \frac{8}{3} \frac{L^2}{b^2}, \quad \frac{\omega_{2D}^2}{\omega_r^2} = \frac{\omega_{1D}^2}{\omega_r^2} - \frac{4}{3} \frac{L^2}{b^2} r + \frac{16}{9} \frac{L^4}{b^4}.
\]

The solution is thus completed. The flutter boundaries given in fig. 12 apply also to rectangular panels; there is no change in the interpretation of \( \lambda \), but \( r \) is now defined according to either equations (8.38) or (8.39). Effectively, \( r \) is shifted to the left of what the value would be if no width effects were present, thereby leading to an increase in \( \lambda \), as would be expected. The frequency results shown in fig. 21 can be also made to apply through use of the frequency expressions, as well as the expressions for \( r \), given by equations (8.38) and (8.39). It is noted that both \( r \) and frequency for the rectangular panels reduce to their former interpretation when \( b \to \infty \).
9. Panel Behavior with Flow Field Having Random Disturbances

The problem of panel excitation due to random loading, such as is produced by jet and boundary layer "noise", has received much attention because of attendant vibration and structural fatigue difficulties. A useful tool in the treatment of this problem has been the power spectral techniques of generalized harmonic analysis. Most of the applications of these techniques have been made on the assumption that the loading is in phase over the complete panel surface so that effectively a single random input is involved, and experimental investigations made to simulate this assumption have shown very encouraging agreement between measured and theoretically predicted stresses. Generally, however, the loading is not in phase over the surface since a high velocity flow over the panel may be involved, or since the average eddy size, or turbulent "scale", of the intense noise generators is usually small, and hence, it becomes necessary to treat the system as though multiple random inputs are involved. The following is intended to contribute to the problem by considering a panel having a turbulence flow over the surface, where the assumption is made that the velocity is sufficiently high so that the change in the intensity of the noise pulses is negligible during the time required to traverse the panel.

It is remarked that rather complete discussions of power spectral techniques, which are to be employed, are given in [35] and [36], in which references to various dynamic problems in aeronautics are also indicated. Attention is also directed to [37] through [39], where consideration is given the condition of multiple random inputs in connection with the gust loading problem on aircraft.

A. The input spectra

Consider the following system

\[ p_1(t) \]

where a flow field having random pressure fluctuations is considered to be passing with velocity \( v \) over the panel. Let \( p_1(t) \) represent the time history of random pressures at point 1, and assume that the velocity is sufficiently high so that these pressures move over the panel with negligible change; the pressure at point 2 would then be \( p_1(t-\tau) \), where \( \tau = \frac{x}{v} \). To establish the spectra of the pressures, write the Fourier transforms,
\[ F_{p_1}(\omega) = \int_{-T}^{T} p_1(t) e^{-i\omega t} dt, \]  
\[ F_{p_2}(x, \omega) = \int_{-T}^{T} p_1(t-\tau) e^{-i\omega \tau} dt = e^{-i\omega \tau} F_{p_1} = e^{-i\omega \frac{x}{\tau}} F_{p_1}, \]  
(9.1)

where the result for \( F_{p_2} \) follows by a change in variable. The power spectra of the pressure at point 1, and the cross spectra of the pressures at points 1 and 2 then follow as

\[ \phi_1(\omega) = \lim_{T \to \infty} \frac{1}{2\pi T} F_{p_1}(\omega) \overline{F}_{p_1}(\omega), \]
\[ \phi_{12}(x, \omega) = \lim_{T \to \infty} \frac{1}{2\pi T} F_{p_1}(\omega) \overline{F}_{p_2}(x, \omega) = e^{i\omega \frac{x}{\tau}} \phi_1(\omega), \]  
(9.2)

where a bar designates the complex conjugate. These relations denote the appropriate input spectra to the system.

**B. The response and output spectra**

1. **General relations.** The differential equation for this system is obtained by adding a structural damping term and an external loading term to equation (8.1), thus

\[ D \frac{\partial^4 w}{\partial x^4} + \sigma_s \frac{\partial^2 w}{\partial x^2} = -m \ddot{w} - 2 \rho a \dot{w} - \rho a v \frac{\partial w}{\partial x} - \beta_s \dot{w} + p_r, \]  
(9.3)

where \( \beta_s \) is the structural damping coefficient and \( p_r \) is the external loading, which in this case is the random loading. Solution of this equation can be made by a modified Galerkin type solution, and in the treatment to follow, special consideration is given to the modal functions that are used in the deflection expansion, which is

\[ w = a_1 y_1 + a_2 y_2 + \cdots \]  
(9.4)

where the \( a_i \)'s are functions of time.

Now consider that stress is the primary response quantity of interest; then in accordance with equation (9.4), the stress may be written

\[ \sigma = a_1 \sigma_1 + a_2 \sigma_2 + \cdots \]  
(9.5)

where \( \sigma_i \) is the stress at a given point that is produced by the deflection \( y_i \). With the output spectra in mind, the Fourier transform of equation (9.5) is

\[ F_\sigma(\omega) = \sigma_1 F_{\sigma_1}(\omega) + \sigma_2 F_{\sigma_2}(\omega) + \cdots \]  
(9.6)
The output spectrum for stress then follows directly, and for a two mode expansion would be

\[
\phi_\sigma = \lim_{T \to \infty} \frac{1}{2\pi T} F_\sigma \overline{F}_\sigma = \lim_{T \to \infty} \frac{1}{2\pi T} (\sigma_1 F_{a_1} + \sigma_2 F_{a_2}) (\sigma_1 \overline{F}_{a_1} + \sigma_2 \overline{F}_{a_2})
\]

\[
= \sigma_1^2 \phi_{a_1} + \sigma_2^2 \phi_{a_2} + 2 \sigma_1 \sigma_2 \text{Re} \phi_{a_1} \phi_{a_2}, \tag{9.7}
\]

where \(\text{Re}\) denotes the real part. The problem has thus become one of determining the spectra and cross-spectra of the \(a_i\)’s.

2. Use of the adjoint or reciprocal flow system to effect modal solution. It is convenient to use the natural modes of the following differential equation to solve equation (9.3)

\[
D \frac{d^4 y_m}{dx^4} + \sigma_x \frac{d^2 y_m}{dx^2} + \rho a v \frac{dy_m}{dx} - \omega_m^2 m y_m = 0. \tag{9.8}
\]

However, the solution of equation (9.3) becomes rather involved if a normal type Galerkin process is used, because the modes \(y_m\) are not orthogonal due to the presence of the odd derivative term \(\frac{dy_m}{dx}\), coupling between the modes will thus result. To avoid this difficulty, it is convenient to introduce also the following differential equation, which will be referred to as the “adjoint” of equation (9.8)

\[
D \frac{d^4 \overline{y}_n}{dx^4} + \sigma_x \frac{d^2 \overline{y}_n}{dx^2} - \rho a v \frac{d \overline{y}_n}{dx} - \omega_n^2 m \overline{y}_n = 0. \tag{9.9}
\]

Equations (9.8) and (9.9) differ only in the sign of the odd derivative term. Physically, a reversed flow system is implied, and as such, the natural frequencies of the two systems will be the same, with deflections being related by \(\overline{y}_n(x) = y_n(L-x)\). It may be recalled that equation (9.8) corresponds to the “conservative” system that was considered in the section on flutter, and that this equation led to flutter velocities which were very close to the values obtained when the aerodynamic damping term was included. The treatment here is thus expected to be linked rather strongly with the flutter results. As a matter of fact, the frequencies of equation (9.8) are those given in the flat plate region in fig. 21 (these frequencies, it will be recalled, are two mode approximations).

Equations (9.8) and (9.9), when considered together, exhibit some very interesting and useful properties. Suppose these equations are multiplied by \(y_n\) and \(y_m\), respectively, and the results are integrated over the panel length; there results, after some integrations by parts

\[
D \int y_n \overline{y}_n' ' \ dx - \sigma_x h \int y_n \overline{y}_n' \ dx + \rho a v \int y_n' \overline{y}_n \ dx - \omega_n^2 m \int y_n \overline{y}_n \ dx = 0,
\]

\[
D \int \overline{y}_n \overline{y}_n' ' \ dx - \sigma_x h \int \overline{y}_n \overline{y}_n' \ dx + \rho a v \int \overline{y}_n' \overline{y}_n \ dx - \omega_n^2 m \int \overline{y}_n \overline{y}_n \ dx = 0.
\]
By subtraction, the following result is found

\[(\omega_m^2 - \omega_n^2) m \int y_m y_n \, dx = 0,\]

from which it follows that

\[m \int_0^L y_m \bar{y}_n \, dx = 0 \quad m \neq n,
\]

\[= M_n \quad m = n. \quad (9.10)\]

This is the basic orthogonality relation for the two systems, and physically the relation states that no net work is produced if the inertia forces of a mode in one system move through the displacements of any non-corresponding mode in the other system.

The solution of equation (9.3) can now be made in an elegant manner as follows. Substitute equation (9.4) (the modes of which satisfy equation (9.8)) into equation (9.3), multiply by the adjoint mode \(y_n\), integrate over the length, and apply relations (9.10); the result is

\[M_n \ddot{a}_n + (2 \rho a + \beta_s) \frac{M_n}{m} \dot{a}_n + \omega_n^2 M_n a_n = \int_0^L \bar{y}_n P_r \, dx. \quad (9.11)\]

This equation is noted to have a remarkable similarity to the well known equation that is obtained in the modal expansion treatment of structures in conjunction with Lagrange’s dynamical equation. The generalized mass of the present system can become negative, however, which represents a radical contrast to the usual concept that the generalized mass is a positive finite quantity. If the Fourier transform of this equation is taken (where the Fourier transform of \(\ddot{a}_n\) and \(\dot{a}_n\) are \(-\omega^2 F_{an}(\omega)\) and \(i \omega F_{an}(\omega)\), respectively), the following solution for \(F_{an}\) is obtained

\[F_{an} = \frac{1}{M_n \omega_n^2} \int \bar{y}_n P_r \, dx \frac{1}{1 - \omega^2 + \frac{\beta_s + 2 \rho a}{m \omega_n^2} \frac{\omega}{\omega_n}} = \frac{1}{M_n \omega_n^2} \int \bar{y}_n P_r \, dx \frac{1}{Z_n}. \quad (9.12)\]

where

\[F_{pr}(x, \omega) = \int_{-\infty}^{\infty} p_r(x, t) e^{-i \omega t} dt.\]

However, \(F_{pr}\) is that given by the second of equations (9.1); thus, equation (9.12) may be written

\[F_{an} = \frac{1}{M_n \omega_n^2} \frac{F_{pr} \int_0^L \bar{y}_n e^{-i \omega x} \, dx}{Z_n} = \frac{F_{pr} P_n}{M_n \omega_n^2 Z_n}, \quad (9.13a)\]

where

\[P_n = \int_0^L \bar{y}_n e^{-i \omega x} \, dx. \quad (9.13b)\]
Substitution of equation (9.13a) into equation (9.7) yields the final result for output spectra of stress

\[ \phi_\sigma = \sigma_1^2 \frac{P_1 \overline{P}_1 \phi_1}{M_1^2 \omega_1^4 Z_1 Z_1} + \sigma_2^2 \frac{P_2 \overline{P}_2 \phi_1}{M_2^2 \omega_2^2 Z_2 Z_2} + 2 \sigma_1 \sigma_2 \phi_1 \frac{Re}{M_1 M_2 \omega_1^2 \omega_2^2 Z_1 Z_2}. \]  

(9.14)

A few comments on the damping terms may now be in order. If structural damping had been introduced as \( (1 + \beta \frac{\partial}{\partial t}) \frac{\partial^2 w}{\partial x^2} \), as it has sometimes been done in theoretical studies, then the damping term in the general response equation would appear as \( (\beta \omega_n^2 M_n + \frac{2 \rho a M_n}{m}) \dot{a}_n \); \( \beta_s \) in this case would be \( \beta m \omega_n^2 \). As compared with experimental findings, this \( \beta_s \) is known to increase too rapidly with the higher modes. On the other hand, if \( \beta_s \) is chosen as a constant, the damping is probably not sufficient for the higher modes. Thus, not only as a compromise, but because a structural damping analogous to the viscous damping in a simple damped oscillator would result, \( \beta_s \) was chosen equal to \( g_s m \omega_n \); this choice causes the ratio of structural damping to critical damping to be the same for all modes (critical damping = \( 2 \omega_n M_n \)). Actually, the choice is not too important because the aerodynamic damping usually will overshadow the structural damping — if external damping devices are not provided. With the above choice on structural damping, the impedance will hereinafter be used as follows

\[ Z_n = 1 - \frac{\omega^2}{\omega_n^2} + i \left( g_s + g_a \frac{\omega_r}{\omega_n} \right) \frac{\omega}{\omega_n}, \]  

(9.15)

where \( \omega_r \) (see equation (8.6)) has been introduced for convenience.

The term \( g_a \frac{\omega_r}{\omega_n} \) represents the ratio of aerodynamic damping to critical damping, and it is seen that this ratio decreases with the higher modes. This may have a tie-in with the fact that appreciable higher mode content has been observed in experimental panel response studies involving flow across the surface.

Return now to equation (9.14); with this equation, the complete output spectra for stress may be evaluated, from which certain statistical characteristics of the time history of stress may be estimated, such as mean square stress, number of peaks in excess of a given stress, number of zero crossings, etc. (see refs. [35] and [36] for applicable formulas and limitations). In many cases, the equation can be simplified considerably; studies made with the equation have indicated that when the frequencies \( \omega_1 \) and \( \omega_2 \) are widely separated, the last term contributes only a negligible amount and hence may be neglected. This is usually the case, except for certain combinations of low velocity and high \( r \) (see fig. 21), and except for nearness to flutter.
With the neglect of the third term in equation (9.14), a rather simple expression can be derived for the mean square stress, a quantity which is usually of prime interest. Each of the first two terms yield spectral components which are similar to that obtained from a simple damped oscillator system, as is illustrated in the following sketch

\[ \phi_\sigma = \frac{c_n(\omega)}{(1 - \frac{\omega^2}{\omega_n^2})^2 + g_n^2 \frac{\omega^2}{\omega_n^2}}, \]

\[ g_n = g_s + g_u \frac{\omega_r}{\omega_n}, \]

\[ s_n \approx \frac{\pi}{2} \frac{\omega_n c_n(\omega_n)}{g_n}, \]

\[ c_n(\omega_n) = \frac{\sigma_n^2}{M_n^2 \omega_n} \left[ \phi_1 P_n \bar{P}_n \right]_{\omega=\omega_n}. \]

Then if the concept used by Liepmann [40] for determining the area under the output spectra of a simple damped oscillator is applied to each component, the following equation for mean square stress is obtained

\[ \bar{\sigma}^2 = \frac{\pi}{2} \frac{\sigma_1^2}{M_1^2 \omega_1^3} \frac{[\phi_1 P_1 \bar{P}_1]_{\omega=\omega_1}}{g_s + g_u \frac{\omega_r}{\omega_1}} + \frac{\pi}{2} \frac{\sigma_2^2}{M_2^2 \omega_2^3} \frac{[\phi_1 P_2 \bar{P}_2]_{\omega=\omega_2}}{g_s + g_u \frac{\omega_r}{\omega_2}}. \] (9.16)

This relation is exact if \( \phi_1 P_n \bar{P}_n \) is flat (it will be seen that \( \bar{P}_1 P_1 \) is nearly so in the practical range of interest), and for a varying \( \phi_1 P_n \bar{P}_n \), is a good approximation if the spectral components are sharply peaked as they usually will be.

C. Example applications

It is first necessary to establish the modes \( y_n \) and \( \bar{y}_n \), and the quantities \( M_n, \omega_n, P_n \bar{P}_n \) and \( \sigma_n \), all of which are a function of the velocity. To do this it is convenient to make use of the Galerkin solution of equation (9.8), which is given by equations (8.28) and (8.29) with the imaginary terms suppressed. The modal functions used in this solution, it may be recalled, are those given by equations (8.22). The deflections in equations (9.8) and (9.9) may therefore be written

\[ y_n = a_{1n} w_1 + a_{2n} w_2, \]

\[ \bar{y}_n = a_{1n} w_1 - a_{2n} w_2. \] (9.17)

Where \( a_{1n} \) and \( a_{2n} \) replace \( b_1 \) and \( b_2 \), respectively, in equations (8.28) and (8.29), and the result for \( \bar{y}_n \) follows by virtue of the fact that \( \bar{y}_n(x) = y_n(L-x) \), which implies simply that all antisymmetrical modal functions in the two expansions should have opposite signs. The quantities \( M_n, P_n \bar{P}_n \), and \( \sigma_n \), when evaluated in terms of equation (9.17), become

100
\[ M_1 = mL(a_{11}A_1 - a_{21}A_2), \]
\[ M_2 = mL(-a_{12}A_1 + a_{22}A_2), \]
\[ P_1 \overline{P}_1 = L^2[a_{11}Q_1 \overline{Q}_1 - a_{11}a_{21}(Q_1 \overline{Q}_2 + \overline{Q}_1 Q_2) + a_{21}Q_2 \overline{Q}_2], \]
\[ P_2 \overline{P}_2 = L^2[a_{12}Q_1 \overline{Q}_1 - a_{12}a_{22}(Q_1 \overline{Q}_2 + \overline{Q}_1 Q_2) + a_{22}Q_2 \overline{Q}_2], \]
\[ \sigma_1 = a_{11}s_1 + a_{21}s_2, \]
\[ \sigma_2 = a_{12}s_1 + a_{22}s_2, \]

where
\[ A_n = \int_0^1 w_n^2 d\eta, \]
\[ Q_n = \int_0^1 w_ne^{-ik\eta}d\eta, \hspace{1cm} k = \frac{\omega L}{v}, \]

and \( s_1 \) and \( s_2 \) are the stresses associated with \( w_1 \) and \( w_2 \), respectively.

The remaining discussion is now restricted to plates with clamped ends, since this condition is more representative of practical situations. The solution of equation (8.29) yields the following two natural frequencies and the two corresponding relations between the modal coefficients
\[ \frac{\omega_1^2}{\omega_r^2} = 0.638(4-r)(1+\alpha) + 2.405(8.17-r)(1-\alpha), \]
\[ \frac{\omega_2^2}{\omega_r^2} = 0.638(4-r)(1-\alpha) + 2.405(8.17-r)(1+\alpha), \]
\[ a_{21} = -2.45\alpha_1a_{11}, \]
\[ a_{12} = -0.4075\alpha_1a_{22}, \]

where \( \alpha = \sqrt{1 - \frac{v^2}{\nu^2}}, \hspace{1cm} \alpha_1 = \sqrt{\frac{1 - \alpha}{1 + \alpha}}. \)

For this case the modal approximations listed in conjunction with equation (8.26), leads to the following expression for \( Q_n \overline{Q}_n \):
\[ Q_1 \overline{Q}_1 = \frac{32\pi^4(1 - \cos k)}{k^2(4\pi^2 - k^2)^2}, \]
\[ Q_2 \overline{Q}_2 = \frac{32\pi^4k^2(1 + \cos k)}{(\pi^2 - k^2)^2(9\pi^2 - k^2)^2}, \]
\[ Q_1 \overline{Q}_2 = -i\frac{32\pi^4\sin k}{(\pi^2 - k^2)(4\pi^2 - k^2)(9\pi^2 - k^2)}. \]

The applicable \( A_n \)'s are \( A_1 = 1.566, A_2 = 0.26 \), and if the stress point is taken on the surface at an edge (position of maximum stress), then \( s_1 = 56\frac{Eh}{2L^2} \) and \( s_2 = \pm 63.1\frac{Eh}{2L^2} \) apply, where the plus sign is used for the leading edge, the minus sign for the trailing edge. (Also see note following equation (8.26)).
The substitution of the above coefficients and expressions into equations (9.18) gives then, after arbitrarily setting \( a_1 = 1 \) and \( M_1 = M_2 \) (which yields \( a_2 = 2.45 \)), the following coefficients for the flat-clamped plate:

\[
M_1 = M_2 = 1.566 m L (1 - a_1^2),
\]

\[
P_1 \bar{P}_1 = L^2 (Q_1 \bar{Q}_1 + 6.02 \alpha_1^2 \bar{Q}_2 \bar{Q}_2),
\]

\[
P_2 \bar{P}_2 = L^2 (a_1^2 \bar{Q}_1 \bar{Q}_1 + 6.02 \bar{Q}_2 \bar{Q}_2),
\]

\[
\sigma_1 = 56 \frac{E h}{2 L^2} (1 \pm 2.76 \alpha_1) \quad + \text{for trailing edge,}
\]

\[
\sigma_2 = 56 \frac{E h}{2 L^2} (\alpha_1 \pm 2.76) \quad - \text{for leading edge,}
\]

\[
\omega_1, \omega_2 \text{ (see eqs. (9.19)),}
\]

\[
k_n = \frac{\omega_n L}{v} = g_a \frac{\pi^4}{2 \lambda_{cr}} \frac{\omega_n}{v_f}.
\]

It is of interest to note that the generalized masses and stresses are functions only of the ratio \( \frac{v}{v_f} \). To facilitate the use of these expressions, fig. 22 has been prepared. The use of this figure in conjunction with equation (9.16), provides for a quick method for estimating the mean square stress.

To illustrate how the stresses vary with velocity, the results obtained for a panel having \( g_a = 0.227 \) and \( r = 0 \) are shown in fig. 23. Except for very low velocities or velocities in the neighborhood of flutter, the first mode is seen to be the primary source of stress. It is well to recall here, however, that the first mode referred to here is for the panel system with air flow; this mode is composed of both the first and second natural modes of the system without air flow, and the amount of second natural mode content becomes rather sizable as the flutter speed is approached (see eqs. (9.17) and (9.19) and \( \alpha_1 \) in fig. 22). The stress variation with velocity for the first mode is nearly linear at first, but as the flutter velocity is approached there is a very marked rise in the stress. This would indicate that even a small amount of noise input would give rise to high stresses in the neighborhood of flutter, as might be expected. The action is very similar to the amplitude behavior of a mass oscillator as resonance is approached, or of a column with initial imperfections as the critical load is approached.

Other sample results for \( r = 0 \) are shown in fig. 24; in this case only the contribution of the first air-flow mode is given. Again the linear rising portion and the abrupt rise is noted in the curves. The figure also indicates the influence of structural damping; it is seen that the structural damping values usually found in practice (\( g_s \) generally less than 0.01) are only mildly beneficial.
Fig. 22. Effect of velocity on output stress coefficients.

Fig. 23. Typical variation in components of mean-square stress.
The foregoing applications were restricted to the case of flat plates. It should be evident, however, that a similar treatment can be given buckled plates. It is only necessary to obtain the relations which correspond to equations (9.19) from the vibration equations for buckled plates (see equation (8.27)) and then make use of equation (9.18) and (9.16) as was done here.

To close this section, attention is drawn to a new concept that appears to arise out of the analytical treatment made. From fig. 22, the generalized mass is noted to vanish as the flutter velocity is reached. It is this vanishing that leads to the abrupt stress rise noted in the examples. Thus, in contrast to the usual concept that flutter is associated with a vanishing aerodynamic damping, here appears a concept of a vanishing generalized mass.

![Fig. 24. Examples showing mean-square stress due to random loading (Aluminum panels, Alt. = 40,000 ft., r=0, no temperature effects included).](image)

D. Modification for finite width panel

In principle, panels having finite width can be treated in a manner similar to that given the semi-infinite panels in the previous sections. The work is somewhat more complicated however, and involves the introduction of cross-correlation and cross-spectra functions. In the following treatment of a rectangular panel of width $b$, certain plausible assumptions about the input are introduced to lead to an expression similar to equation (9.16), but which contains a simple modification factor that approximately takes into account finite width effects. The derivation is given with little discussion.

The equation for $F_{a_n}$ (see equation (9.12)) for a rectangular panel would read

$$F_{a_n} = \frac{\int \int w_n(x, y) F_p(x, y, \omega) \, dy \, dx}{M_n \omega_n^2 Z_n(\omega)} = \frac{F_{Q_n}}{M_n \omega_n^2 Z_n},$$
where \( Q_n \) is used in this section to denote the generalized force

\[
Q_n = \int_0^{Lb} \int_0^b \omega_n(x,y) p(x,y,t) \, dy \, dx,
\]

and where \( \omega_n \), \( M_n \), \( \omega_n \), and \( Z_n \) now apply to the two-dimensional modes. On a parallel with equations (9.1), it is reasonable that

\[
F_p = G_p(y, \omega) e^{-i \omega x / v}
\]

and assume \( \omega_n \) \( /^{(x)} \) (\( x \)) \( g_n(y) \). Then

\[
F_{Q_n} = \int_0^{Lb} \int_0^b \omega_n F_p \, dy \, dx = \int_0^b \int_0^b \omega_n G_p(y, \omega) \, dy \, dx = P_n \int_0^b g_n(y) G_p(y, \omega) \, dy,
\]

from which the spectrum follows

\[
\phi_{Q_n} = \lim_{T \to \infty} \frac{1}{2 \pi T} \int_0^{b b} \int_0^b \omega_n(y) g_n(\xi) G_p(y, \omega) \, dy \, d \xi.
\]

Now consider that the cross spectrum between the pressures at any two points along the leading edge of the plate depends only on the separation distance \( s \) between the two points. The spectral function \( \phi_{Q_n} \) then can be written

\[
\phi_{Q_n} = P_n \int_0^{b b} \int_0^b \omega_n(y) g_n(\xi) \phi_p(y - \xi, \omega) \, dy \, d \xi,
\]

\[
= P_n \int_0^{b b} \int_0^b \omega_n(y) g_n(y - s) \phi_p(s, \omega) \, dy \, ds,
\]

\[
= 2 P_n \int_0^{b b} \omega_n(y) g_n(y - s) \phi_p(s, \omega) \, dy \, ds,
\]

where the last two results follow in succession by a change in variable and by an interchange in the order of integration, respectively.

Suppose that \( g_n(y) \) is taken as \( \frac{1}{2} \left( 1 - \cos \frac{2 \pi y}{b} \right) \); then it may be shown that \( 0.391 b e^{-7.7 b^*} \) is a fair approximation of the \( y \)-integral in \( \phi_{Q_n} \). The spectrum \( \phi_{Q_n} \) may thus be written

\[
\phi_{Q_n} = 2 P_n \int_0^b 0.391 b e^{-7.7 b^*} \phi_p(s, \omega) \, ds.
\]

Now assume that the input pressures at a point on the leading edge of the panel lead to a spatially dependent correlation function which can be approximated by the function

\[
R(r) = \frac{r_0}{r} e^{-2.3 \frac{r}{L^*}},
\]

where the factor 2.3 has been introduced on the basis that there is a 10\% correlation when the shift distance \( x \) equals the scale of turbulence \( \bar{L} \). Assume
further that the turbulence is axisymmetric; then the cross-correlation function for pressure between any two points separated by distances $x=vt$ in the $x$-direction and $s$ in the $y$-direction would read

$$R(t) = \rho^2 e^{-2.3 s^t \frac{v^t}{L^t}} e^{-2.3 \frac{v^t}{L^t}},$$

where $l = \frac{x}{v}$,

from which the cross-spectra follows as

$$\phi_p (s, \omega) = \rho^2 e^{-2.3 \frac{s^t}{L^t} \frac{1}{\pi}} \int_{-\infty}^{\infty} e^{-2.3 \frac{v^t}{L^t} e^{-i \omega t} dt},$$

$$= e^{-2.3 \frac{s^t}{L^t} \phi_p (0, \omega)}.$$

The cross spectrum is thus given as the product of the point spectrum and a simple function which depends only on the separation distance $s$. This cross-spectrum is now substituted into the above approximation for $\phi_{Q_n}$, and leads to an integrand which can be integrated exactly if the upper limit is considered to be infinity. This assumption on the limit is quite in order since the contribution to the integral beyond $b$ is negligible. The final result is thus simply

$$\phi_{Q_n} \approx \phi_p (0, \omega) P_n \bar{P}_n \frac{0.25 b^2}{\sqrt{1 + 0.3 \frac{b^2}{L^2}}}.$$

This expression replaces the quantity $\phi_p P_n \bar{P}_n$ in equation (9.14). The factor $0.25 b^2$ will be cancelled by a similar factor that results from the evaluation of the generalized mass term $M_n$ in terms of the rectangular panel modes. The resulting expression is thus similar to equation (9.14), except that the reduction factor $\frac{1}{\sqrt{1 + 0.3 \frac{b^2}{L^2}}}$ is present at each term. This factor approximately takes into account the influence the turbulence scale has on the output response. If for example the scale is $\frac{1}{5}$ the width of the panel, then the output response is but 0.34 the value that would be obtained if no "scale" effect were present.
References


24. Several University of Illinois Engineering Bulletins during the late “30’s and early “40’s.
Biography

The following is a resume of the educational background and work experience of John C. Houbolt. He was born in Altoona, Iowa, on April 10, 1919, but spent the major portion of his youth in Illinois, near Joliet. His first two college years were at Joliet Junior College, and then, through a high scholastic award and scholarship, he finished his remaining two years of undergraduate study at the University of Illinois, receiving his Bachelor of Science Degree in June 1940. Through a special research graduate assistantship he joined the University staff, and in June 1942 was awarded the Master of Science degree.

During the school years he spent a short while as a Junior Engineer for the City of Waukegan, and as a Bridge Engineer for the Illinois Central Railroad. In 1942 he joined the Structures Division of the National Advisory Committee for Aeronautics at Langley Field, Va., and in 1949 was made Assistant Chief of the newly formed Dynamic Loads Division, a capacity he has held since. Just prior to this appointment he spent a six months' tour of duty in the Flutter and Vibration Department of the Royal Aircraft Establishment, Farnborough, Hants., England, as an exchange research scientist.

During his employment with the NACA, he has been a staff member on the University of Virginia graduate extension division, and has taught various courses such as Elastic Stability, Structural Vibrations, and Aircraft Vibration and Flutter. In 1956, through a program administered by Princeton University, he was awarded a Rockefeller Public Service Award, which in conjunction with a leave of absence granted by the NACA, made possible his study at the Eidgenössische Technische Hochschule, Zürich, and the dissertation herein presented.

John C. Houbolt.