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APPLICATION OF A NEW SET OF CANONICAL ELEMENTS TO THE THREE-DIMENSIONAL RESTRICTED PROBLEM OF THREE BODIES

ABHANDLUNG

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3.3. Refinements

The goal of this part is to take into account one more term in the series (2,5). Since the order of the perturbation procedure is not to be improved, this refinement will not complicate to much the computational procedure, nevertheless the additional term will contribute considerably to the accuracy of the analytical solution, if we compare the improved analytical approximation with a numerical integration that takes into account the whole perturbation due to the third body. The additional term corresponds to the potential V_3 of the series (2,1) and is given by

$$V_3 = -\frac{k^2M'}{\varsigma^4} r^3 P_3(\cos\vartheta)$$

By means of (2,2) and (2,4) the polynomials $Q_{3j}(r,x_1,x_2,x_3)$ $R_{3j}(r,x_1,x_2,x_3)$ $(j=0,1,2,3)$ of (2,5) are found to be

$$Q_{30} = Q_{32} = R_{30} = R_{32} \equiv 0$$

$$Q_{31} = \frac{3}{8} r x_2 (r^2 - 5x_1^2) \qquad R_{31} = \frac{3}{8} r x_3 (r^2 - 5x_1^2)$$

$$Q_{33} = \frac{5}{8} r x_2 (x_2^2 - 3x_3^2) \qquad R_{33} = \frac{5}{8} r x_3 (3x_2^2 - x_3^2) \tag{98}$$

Thus

$$\frac{1}{4} r V_3 = -\varepsilon_3 \{ Q_{31}\cos[\nu(t-T)] + Q_{33}\cos[3\nu(t-T)] + R_{31}\sin[\nu(t-T)] + R_{33}\sin[3\nu(t-T)] \tag{99}$$

with $\varepsilon_3 = \frac{k^2M'}{4\varsigma^4}$.

Since we will not change the order of the perturbation procedure and because of the assumption (2,9), we have just to modify the perturbing part H_p of the Hamiltonian:

$$H_p = \varepsilon^2 H^{(2)} + \varepsilon^3 (H^{(3)} + \tilde{H}^{(3)}) \qquad (100)$$

where $H^{(2)}, H^{(3)}$ are given by (6), (7) and $\tilde{H}^{(3)}$ is to be expressed in terms of the PKS-elements (c.f. section 3.1) The result shows the same structure as formulae (7):

$$\tilde{H}^{(3)} = \sum_{\substack{j_0 \neq 0 \\ (j_0, j_2, j_3) \in \tilde{\mathcal{M}}^{(3)}}} \tilde{C}^{(3)}_{j_0 j_2 j_3}(\alpha_i) \cos(j_0 \tilde{\zeta} + j_2 \beta_2 + j_3 \beta_3)$$
$$+ \sum_{\substack{j_1 \neq 0 \\ (j_0, j_1, j_2, j_3) \in \tilde{\mathcal{N}}^{(3)}}} \tilde{D}^{(3)}_{j_0 j_1 j_2 j_3}(\alpha_i) \cos(j_0 \tilde{\zeta} + j_1 \beta_1 + j_2 \beta_2 + j_3 \beta_3) \qquad (101)$$

Both sums are finite. The variable $\tilde{\zeta}$ is defined by

$$\tilde{\zeta} = 2\nu\beta_0 - \nu T = \mu_3 z - \nu T \quad \text{where} \quad \mu_3 = 2\frac{\nu}{\varepsilon} = O(1)$$

and the coefficients $\tilde{C}^{(3)}, \tilde{D}^{(3)}$ are finite sums the terms of which have the following form

$$\text{const} \; \alpha_0^{-2} \alpha_1^{i_1} \alpha_2^{i_2} \alpha_3^{i_3} \sqrt{\alpha_1^2 - \alpha_2^2}$$

with

$$i_1, i_3 \in \{0, 1, 2, 3\} \qquad i_2 \in \{-3, -2, -1, 0, 1, 2\}$$
$$i_1 + i_2 + i_3 = 3$$

The Hamiltonian $\tilde{H}^{(3)}$ has to be added in the third von Zeipel equation (14d'). Because $\tilde{H}^{(3)}$ contains dominant short-periodic and short-periodic terms only it does not contribute to the long-periodic part $\overline{H}^{(3)}$. Thus we have to modify the generating function $G^{(2)}$ containing the dominant short-periodic terms only. This improvement of $G^{(2)}$ is of the form

$$\widetilde{G}^{(2)} = -\frac{1}{H^{(0)}_{\overline{\alpha}_0}\mu_3} \sum_{\substack{j_0 \neq 0 \\ (j_0, j_2, j_3) \in \widetilde{\mathcal{M}}^{(3)}}} \frac{\widetilde{C}^{(3)}_{j_0 j_2 j_3}(\overline{\alpha}_i)}{j_0} \sin(j_0 \widetilde{\zeta} + j_2\beta_2 + j_3\beta_3) \qquad (102)$$

The knowledge of the short-periodic terms of $\widetilde{H}^{(3)}$ is not necessary, therefore we need not to compute the coefficients $\widetilde{D}^{(3)}$. By means of (102) we are able to improve the transformations (33), (34). We list the result:

$$\begin{aligned}
\beta_i &= \overline{\beta}_i - \varepsilon G^{(1)}_{\overline{\alpha}_i} + \varepsilon^2\Big[-G^{(2)}_{\overline{\alpha}_i} - \widetilde{G}^{(2)}_{\overline{\alpha}_i} - S^{(2)}_{\overline{\alpha}_i} + \sum_{\sigma=2}^{3} G^{(1)}_{\overline{\alpha}_i\overline{\beta}_\sigma} G^{(1)}_{\overline{\alpha}_\sigma}\Big] \\
\alpha_\varsigma &= \overline{\alpha}_\varsigma + \varepsilon G^{(1)}_{\beta_\varsigma} + \varepsilon^2\Big[G^{(2)}_{\beta_\varsigma} + \widetilde{G}^{(2)}_{\beta_\varsigma} + S^{(2)}_{\beta_\varsigma} - \sum_{\sigma=2}^{3} G^{(1)}_{\beta_\varsigma\overline{\beta}_\sigma} G^{(1)}_{\overline{\alpha}_\sigma}\Big] \qquad (33') \\
\alpha_0 &= \overline{\alpha}_0 \quad + \varepsilon^2 G^{(1)}_{\overline{z}}
\end{aligned}$$

$$\begin{aligned}
\overline{\beta}_i &= \beta_i + \varepsilon G^{(1)}_{\alpha_i} + \varepsilon^2\Big[G^{(2)}_{\alpha_i} + \widetilde{G}^{(2)}_{\alpha_i} + S^{(2)}_{\alpha_i} - \sum_{\sigma=2}^{3} G^{(1)}_{\alpha_i\alpha_\sigma} G^{(1)}_{\beta_\sigma}\Big] \\
\overline{\alpha}_\varsigma &= \alpha_\varsigma - \varepsilon G^{(1)}_{\beta_\varsigma} + \varepsilon^2\Big[-G^{(2)}_{\beta_\varsigma} - \widetilde{G}^{(2)}_{\beta_\varsigma} - S^{(2)}_{\beta_\varsigma} + \sum_{\sigma=2}^{3} G^{(1)}_{\beta_\varsigma\alpha_\sigma} G^{(1)}_{\beta_\sigma}\Big] \qquad (34') \\
\overline{\alpha}_0 &= \alpha_0 - \varepsilon^2 G^{(1)}_{z} + \varepsilon^3\Big[-G^{(2)}_{z} - \widetilde{G}^{(2)}_{z} - S^{(2)}_{z} + \sum_{\sigma=2}^{3} G^{(1)}_{z\alpha_\sigma} G^{(1)}_{\beta_\sigma}\Big] \, . \qquad (34a')
\end{aligned}$$

3.4. Collection of formulae and numerical experiments

Next we list the computation procedure which we will apply in order to compute an approximate analytical solution $x_j(s), t(s)$ of the problem (1,1), which is specified in the sections 2.1 and 3.1 (see fig. 1).

1. Express the Hamiltonian (1,1) in terms of the PKS-elements of section 1.2:

$$H = 2\sqrt{\alpha_0}\,\alpha_1 + \varepsilon^2 \Big[A^{(2)}(\alpha_i) + B^{(2)}(\alpha_i)\cos 2\beta_2$$

$$+ \sum_{\substack{j_0 \neq 0 \\ (j_0, j_2, j_3) \in \mathcal{M}^{(2)}}} C^{(2)}_{j_0 j_2 j_3}(\alpha_i)\cos(j_0\zeta + j_2\beta_2 + j_3\beta_3) + \sum_{\substack{j_1 \neq 0 \\ (j_0, j_1, j_2, j_3) \in \mathcal{N}^{(2)}}} D^{(2)}_{j_0 j_1 j_2 j_3}(\alpha_i)\cos(j_0\zeta + j_1\beta_1 + j_2\beta_2 + j_3\beta_3)\Big]$$

$$+ \varepsilon^3 \Big[\sum_{\substack{j_0 \neq 0 \\ (j_0, j_2, j_3) \in \mathcal{M}^{(3)}}} C^{(3)}_{j_0 j_2 j_3}(\alpha_i)\cos(j_0\zeta + j_2\beta_2 + j_3\beta_3) + \text{ short-periodic terms}\Big]$$

(c.f. (6) - (11))

2. Solve the modified von Zeipel eqations (2,29):

$$\overline{H}^{(0)} = 2\sqrt{\overline{\alpha}_0}\,\overline{\alpha}_1$$

$$\overline{H}^{(1)} = 0 \qquad S^{(1)} = 0$$

$$\overline{H}^{(2)} = A^{(2)}(\overline{\alpha}_i) + B^{(2)}(\overline{\alpha}_i)\cos 2\overline{\beta}_2$$

$$G^{(1)} = -\frac{1}{\mu_2 H^{(0)}_{\overline{\alpha}_0}} \sum_{\substack{j_0 \neq 0 \\ (j_0, j_2, j_3) \in \mathcal{M}^{(2)}}} \frac{C^{(2)}_{j_0 j_2 j_3}(\overline{\alpha}_i)}{j_0}\sin(j_0\zeta + j_2\beta_2 + j_3\beta_3)$$

$$S^{(2)} = -\frac{1}{H^{(0)}_{\overline{\alpha}_1}} \sum_{\substack{j_1 \neq 0 \\ (j_0, j_1, j_2, j_3) \in \mathcal{N}^{(2)}}} \frac{D^{(2)}_{j_0 j_1 j_2 j_3}(\overline{\alpha}_i)}{j_1}\sin(j_0\zeta + j_1\beta_1 + j_2\beta_2 + j_3\beta_3)$$

$$\overline{H}^{(3)} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\!\!\int_0^{2\pi} \sum_{\varsigma=2}^{3} H^{(2)}_{\overline{\alpha}_\varsigma} G^{(1)}_{\beta_\varsigma}\, d\zeta\, d\beta_1 = A^{(3)}(\overline{\alpha}_i) + B^{(3)}(\overline{\alpha}_i)\cos 2\overline{\beta}_2$$

$$G^{(2)} = -\frac{1}{\mu_2 H^{(0)}_{\overline{\alpha}_0}} \Big\{ \sum_{\substack{j_0 \neq 0 \\ (j_0, j_2, j_3) \in \mathcal{M}^{(3)}}} \frac{C^{(3)}_{j_0 j_2 j_3}(\overline{\alpha}_i)}{j_0}\sin(j_0\zeta + j_2\beta_2 + j_3\beta_3)$$

$$+ \int \Big[\frac{1}{2\pi}\int_0^{2\pi} \sum_{\varsigma=2}^{3} H^{(2)}_{\overline{\alpha}_\varsigma} G^{(1)}_{\beta_\varsigma}\, d\beta_1 - \overline{H}^{(3)} - \overline{H}^{(2)}_{\beta_2} G^{(1)}_{\overline{\alpha}_2} \Big] d\zeta \Big\}$$

(c.f. (15) -(23))

3. Compute the initial conditions $\alpha_i^{(0)} = \alpha_i(s=0)$, $\beta_i^{(0)} = \beta_i(s=0)$, $(i=0,1,2,3)$ of the PKS-elements from the given physical initial data $x_j^{(0)} = x_j(t=0)$, $p_j^{(0)} = p_j(t=0)$, $(j=1,2,3)$ according to (1,26),(1,27). Calculate the initial conditions $\gamma_i^{(0)}, \delta_i^{(0)}$ of the longperiodic system (29) by means of the transformations $(\alpha_i, \beta_i) \longrightarrow (\bar{\alpha}_i, \bar{\beta}_i) \longrightarrow (\gamma_i, \delta_i)$:

$$\bar{\alpha}_0^{(0)} = \alpha_0^{(0)} - \varepsilon^2 G_z^{(1)} + \varepsilon^3 \Big[- G_z^{(2)} - S_z^{(2)} + \sum_{\sigma=2}^{3} G_{z\alpha_\sigma}^{(1)} G_{\beta_\sigma}^{(1)} \Big]$$

$$\bar{\alpha}_\varsigma^{(0)} = \alpha_\varsigma^{(0)} - \varepsilon G_{\beta_\varsigma}^{(1)} + \varepsilon^2 \Big[- G_{\beta_\varsigma}^{(2)} - S_{\beta_\varsigma}^{(2)} + \sum_{\sigma=2}^{3} G_{\beta_\varsigma\alpha_\sigma}^{(1)} G_{\beta_\sigma}^{(1)} \Big] \qquad (\varsigma = 1,2,3)$$

$$\bar{\beta}_i^{(0)} = \beta_i^{(0)} + \varepsilon G_{\alpha_i}^{(1)} + \varepsilon^2 \Big[G_{\alpha_i}^{(2)} + S_{\alpha_i}^{(2)} - \sum_{\sigma=2}^{3} G_{\alpha_i\alpha_\sigma}^{(1)} G_{\beta_\sigma}^{(1)} \Big]$$

$$\bar{\alpha}_1^{(0)} = \frac{1}{2\sqrt{\bar{\alpha}_0^{(0)}}} \Big[\frac{k^2 M}{4} - \varepsilon^2 \bar{H}^{(2)}(\bar{\alpha}_i^{(0)}, \bar{\beta}_2^{(0)}) - \varepsilon^3 \bar{H}^{(3)}(\bar{\alpha}_i^{(0)}, \bar{\beta}_2^{(0)}) \Big]$$

$$\gamma_i^{(0)} = \gamma_i(\tau=0) = \bar{\alpha}_i^{(0)}, \quad \delta_i^{(0)} = \delta_i(\tau=0) = \bar{\beta}_i^{(0)}$$

(c.f. (33) - (35),(25),(27))

4. Solve the long-periodic system (29) attached to the Hamiltonian (28) (c.f. section 3.2):

$$\tau = \varepsilon^2 s$$

$$\gamma_\varsigma(\tau) = \gamma_\varsigma^{(0)} \qquad (\varsigma = 0,1,3)$$

$$\gamma_2(\tau) = \gamma_1 \sqrt{c + (b-c)\,\mathrm{sn}^2\Big[\frac{\ell}{g}(\tau-\tau_0); k\Big]}$$

$$\delta_0(\tau) = \tilde{\delta}_0 + \mathcal{A}_0 \tau + \mathcal{B}_0 \tilde{E}(u_1)$$

$$\delta_1(\tau) = \tilde{\delta}_1 + \mathcal{A}_1 \tau + \mathcal{C}_1 \tilde{\Pi}(u_1, \alpha_1^2)$$

$$\delta_3(\tau) = \tilde{\delta}_3 + \mathcal{A}_3 \tau + \mathcal{B}_3 \tilde{E}(u_1) + \mathcal{C}_3 \tilde{\Pi}(u_1, \alpha_3^2)$$

$$\delta_2(\tau) = \begin{cases} \frac{1}{2}\sigma(\tau) \arccos \dfrac{\mathcal{R} - B - 12x^2\eta}{B}, & \text{if } (y_1^2 \mathcal{R}') \in G_1 \\ \frac{1}{2}\sigma(\tau) \arccos \dfrac{\mathcal{R} - B - 12x^2\eta}{B} + \frac{\pi}{2}\Big[n + \frac{1}{2}(1-\sigma(\tau))\Big], & \text{if } (y_1^2 \mathcal{R}') \in G_2 \end{cases}$$

(c.f. (36),(55),(65) - (68),(81),(82))

5. Compute $\alpha_i(s), \beta_i(s)$ by means of the transformations

$$(\gamma_i'', \delta_i) \longrightarrow (\bar{\alpha}_i, \bar{\beta}_i) \longrightarrow (\alpha_i, \beta_i):$$

$$\bar{\alpha}_i(s) = \gamma_i''(\tau = \varepsilon^2 s)$$

$$\bar{\beta}_0(s) = \delta_0(\tau = \varepsilon^2 s) + \frac{\gamma_1''}{\sqrt{\gamma_0''}} s \qquad \bar{\beta}_1(s) = \delta_1(\tau = \varepsilon^2 s) + 2\sqrt{\gamma_0''}\, s$$

$$\bar{\beta}_2(s) = \delta_2(\tau = \varepsilon^2 s) \qquad \bar{\beta}_3(s) = \delta_3(\tau = \varepsilon^2 s)$$

$$\alpha_\varsigma(s) = \bar{\alpha}_\varsigma + \varepsilon G^{(1)}_{\bar{\beta}_\varsigma} + \varepsilon^2 \Big[G^{(2)}_{\bar{\beta}_\varsigma} + S^{(2)}_{\bar{\beta}_\varsigma} - \sum_{\sigma=2}^{3} G^{(1)}_{\bar{\beta}_\varsigma \bar{\beta}_\sigma} G^{(1)}_{\bar{\alpha}_\sigma} \Big] \quad (\varsigma = 1,2,3)$$

$$\alpha_0(s) = \bar{\alpha}_0 + \varepsilon^2 G^{(1)}_{\tau}$$

$$\beta_i(s) = \bar{\beta}_i - \varepsilon G^{(1)}_{\bar{\alpha}_i} + \varepsilon^2 \Big[- G^{(2)}_{\bar{\alpha}_i} - S^{(2)}_{\bar{\alpha}_i} + \sum_{\sigma=2}^{3} G^{(1)}_{\bar{\alpha}_i \bar{\beta}_\sigma} G^{(1)}_{\bar{\alpha}_\sigma} \Big]$$

(c.f. (25),(33))

Calculate $x_j(s)$, $t(s)$, $(j = 1,2,3)$ from $\alpha_i(s), \beta_i(s)$ by applying the transformation (1,25).

Numerical examples

We have applied this computation procedure to the problem of section 2.1 (see fig. 1) using the following data:

$k^2 M = 398601\ km^3 s^{-2}$	(M = earth)
$k^2 M' = 4902.66\ km^3 s^{-2}$	(M' = moon)
$\varsigma = 384400\ km$	(= mean distance earth-moon)
$\nu = \sqrt{\dfrac{k^2 M + k^2 M'}{\varsigma^3}}$	(= angular velocity of the moon)

Examples 1 to 6

All these examples deal with the special case of Hill's potential, i.e. only the first term V_2 of the series (2,1)

has been taken into account. The examples differ only in the choice of the initial conditions.
We have computed the approximate solution $x_i^H(s), t^H(s)$ for different values of the independent variable s. These values of s are chosen such that they correspond approximately to n revolutions of the mobile. In order to test the accuracy of this analytical solution, we have compared it with the results $\overline{x}_i^H(t^H(s))$ of a numerical integration of the same problem. As measure of error we use the residual $|\Delta x^H|$ defined by

$$|\Delta x^H| = \sqrt{\sum_{i=1}^{3} [\overline{x}_i^H(t^H) - x_i^H(t^H)]^2}$$

For comparative reasons we have computed the quantity

$$|\Delta x^0| = \sqrt{\sum_{i=1}^{3} [\overline{x}_i^H(t^H) - x_i^0(t^H)]^2}$$

where $x_i^0(t^H)$ is the position of the mobile in the <u>unperturbed</u> motion.
Each line of the tables Ex1 to Ex6 contains the following information:

$$n \;,\; t^H(s) \;,\; x_i^H(s) \;,\; |\Delta x^H| \;,\; |\Delta x^0|$$

<u>Examples 7 to 10</u>: Test of the refinements of section 3.3

In these examples two approximate analytical solutions are confronted. The first one is the same as in the examples 1 to 6. The second one which we denote by $x_i^R(s), t^R(s)$ has been computed according to the refinements of section 3.3, i.e. the second term V_3 of the expansion (2,1) of the perturbing potential is considered additionally. Both approximate solutions are compared with the numerically integrated solution $\overline{x}_i(t)$ which takes the <u>whole</u> perturbation due to the third body into account. We allow for this perturbation by adopting the perturbing force

$$P = k^2 M' \left[\frac{1}{\Delta^3}(\vec{x} - \vec{a}) + \frac{1}{\rho^3}\vec{a} \right]$$

(Δ = distance mobile - third body).
The initial conditions of the three solutions satisfy the following relations

$$t^H(0) = t^R(0) = \bar{t}(0) = 0 \;, \quad x_i^H(0) = x_i^R(0) = \bar{x}_i(0) = x_i(0)$$

Two residuals are computed:

$$|\Delta x^H| = \sqrt{\sum_{i=1}^{3} \left[\bar{x}_i(t^H) - x_i^H(t^H) \right]^2}$$

$$|\Delta x^R| = \sqrt{\sum_{i=1}^{3} \left[\bar{x}_i(t^R) - x_i^R(t^R) \right]^2}$$

The definition of the comparative value $|\Delta x^0|$ is the same as above.
The tables Ex7 to Ex10 contain the following information:

first line: $n,\ t^H(s)\ ,\ x_i^H(s),\ |\Delta x^H|\ ,\ |\Delta x^0|$

second line: $n,\ t^R(s)\ ,\ x_i^R(s),\ |\Delta x^R|,\ |\Delta x^0|$

The tables Ex7 and Ex8 differ in the choice of the initial data and the tables Ex9 and Ex10 show the position of the same mobile as Ex7, Ex8 respectively, but after a larger number of revolutions.