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Intersecting subvarieties of $\mathbf{G}_m^n$ with algebraic subgroups

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Abstract Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$. We bound the height of algebraic points on X that are in a certain sense close to the union of all algebraic subgroup of $\mathbf{G}_m^n$ of dimension $m < n/\dim X$. The bound we obtain is effective and will be expressed as a function of the height of X, the degree of X, and n. We then apply this bound to derive certain finiteness results if m is also strictly less than $n - \dim X$.

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1 Introduction

In this paper we study the intersection of a subvariety of the algebraic torus $\mathbf{G}_m^n$ with the union of all algebraic subgroups of fixed dimension. If not stated otherwise, all varieties are assumed to be defined over the field of algebraic numbers $\overline{\mathbf{Q}}$ and will be identified with their set of algebraic points.

We begin by describing a result of Bombieri, Masser, and Zannier which applies when the subvariety is a curve. Let m be an integer with $0 \leq m \leq n$, we define $\mathcal{H}_m$ to be the union of all algebraic subgroups of $\mathbf{G}_m^n$ with dimension less or equal to m. To avoid trivialities we set $\mathcal{H}_m = \emptyset$ for negative m. A *coset* is the translate of an algebraic subgroup of $\mathbf{G}_m^n$. Bombieri et al. [2] showed that if $X \subset \mathbf{G}_m^n$ is an irreducible algebraic curve not contained in a proper coset then the absolute logarithmic Weil height, to be defined in Sect. 2, is bounded on $X \cap \mathcal{H}_{n-1}$. Furthermore, using this height upper bound and, among other tools, a Lehmer-type height lower bound they showed that $X \cap \mathcal{H}_{n-2}$ is finite.

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Let H be any subset of $\mathbf{G}_m^n$ and let $\epsilon \in \mathbf{R}$, we define the "truncated cone" around H as

$$C(H,\epsilon) = \{ab;\ a \in H,\ b \in \mathbf{G}_m^n,\ h(b) \le \epsilon(1+h(a))\},$$

here $h(\cdot)$ denotes the absolute logarithmic Weil height, or just height for short. Kronecker's Theorem implies that the height vanishes precisely on the torsion points of $\mathbf{G}_m^n$, therefore $C(\mathcal{H}_m, 0) = \mathcal{H}_m$.

Let X be an irreducible closed subvariety of $\mathbf{G}_m^n$ and let X^{o} be the complement in X of the union of all positive dimensional cosets contained in X. If Γ is the division closure of a finitely generated subgroup of $\mathbf{G}_m^n$, Evertse [8] proved that $X^{\mathrm{o}} \cap C(\Gamma,\epsilon)$ is finite for $\epsilon > 0$ small enough. Earlier and Poonen [14] proved a related finiteness statement with $\mathbf{G}_m^n$ replaced by an algebraic group isogenous to the product of an abelian variety and an algebraic torus and with $C(\Gamma,\epsilon)$ replaced by the set of ab where $a \in \Gamma$ and where b has height at most ϵ. Rémond [15] then generalized Poonen's result by proving the finiteness of $X^{\mathrm{o}} \cap C(\Gamma,\epsilon)$ for small $\epsilon > 0$ if X is in an arbitrary semi-abelian variety.

From now on let X be an irreducible closed subvariety of $\mathbf{G}_m^n$ of unrestricted dimension. We pose the problem of bounding the height on the intersection $X^{\mathrm{oa}} \cap C(\mathcal{H}_m,\epsilon)$ for an appropriate m and $\epsilon > 0$ small enough. Here $X^{\mathrm{oa}} \subset X$ is Zariski open and will be defined further down. The study of finiteness results involving positive ϵ is reserved for a later article [10] which will use results from this article.

We mention the following simple proposition:

Proposition 1 *Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$ with $\dim X \ge 1$ and let $U \subset X$ be Zariski open and dense. Then $U \cap \mathcal{H}_{n-\dim X+1}$ has unbounded height and $U \cap \mathcal{H}_{n-\dim X}$ is infinite.*

Therefore to prove reasonable boundedness of height or finiteness results on the intersection of a Zariski open non-empty $U \subset X$ with $\mathcal{H}_m$ one must assume $m \le n - \dim X$, respectively, $m \le n - \dim X - 1$. A series of conjectures stated by Bombieri et al. [3] expect that these conditions on m are sufficient, at least when taking as U the subset X^{oa}, respectively, X^{ta} which we now define.

Following Bombieri et al. [3] we call an irreducible closed subvariety $Y \subset X$ *anomalous* if there exists a coset $H \subset \mathbf{G}_m^n$ such that $Y \subset H$ and

$$\dim Y \ge \max\{1, \dim X + \dim H - n + 1\}. \tag{1}$$

We define X^{oa} to be X deprived of all its anomalous subvarieties. If H is an algebraic subgroup we call Y *torsion anomalous*. We define X^{ta} to be X deprived of all its torsion anomalous subvarieties. Clearly, $X^{\mathrm{oa}} \subset X^{\mathrm{ta}}$. These definitions continue to make sense when all varieties in question are defined over $\mathbf{C}$.

Informally speaking, a positive dimensional $Y \subset X$ is anomalous if it is contained in the improper intersection of X with a coset.

In certain cases X^{oa} and X^{ta} can easily be determined: if X is a curve, then $X^{\mathrm{oa}} = X$ if X is not contained in a proper coset, and $X^{\mathrm{oa}} = \emptyset$ otherwise. Similarly, $X^{\mathrm{ta}} = X$ if

X is not contained in a proper algebraic subgroup, and $X^{\mathrm{ta}} = \emptyset$ otherwise. If X is a hypersurface, that is if $\dim X = n-1$, we will see below that $X^{\mathrm{oa}} = X^{\mathrm{o}}$.

Bombieri et al. [3] proved that X^{oa} is Zariski open in X, even if X is defined over $\mathbf{C}$. In the same article they conjectured that $X^{\mathrm{oa}} \cap \mathcal{H}_{n-\dim X}$ has bounded height. We generalize this conjecture by replacing $\mathcal{H}_{n-\dim X}$ by $C(\mathcal{H}_{n-\dim X}, \epsilon)$.

Conjecture 1 Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$. There exists $\epsilon > 0$ such that $X^{\mathrm{oa}} \cap C(\mathcal{H}_{n-\dim X}, \epsilon)$ is a set of bounded height.

Finally, they also stated the following conjecture [3]:

Conjecture 2 (Bombieri, Masser, and Zannier) Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety defined over $\mathbf{C}$. Then X^{ta} is Zariski open in X and $X^{\mathrm{ta}} \cap \mathcal{H}_{n-\dim X-1}$ is finite.

In Conjecture 2 one identifies X and $\mathcal{H}_m$ with their sets of complex points.

We also state a variant of Conjecture 2 with an ϵ.

Conjecture 3 Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$. There exists $\epsilon > 0$ such that $X^{\mathrm{oa}} \cap C(\mathcal{H}_{n-\dim X-1}, \epsilon)$ is finite.

See [10] for a variant of the previous conjecture with X^{ta} instead of X^{oa}.

Conjecture 1 was showed with $\epsilon = 0$ for curves [2] and planes [4] by Bombieri, Masser, and Zannier and for hypersurfaces [19] by Bombieri and Zannier. Although no explicit bounds for the height were given. Recently, Maurin [11] gave a proof of Conjecture 2 for curves defined over $\overline{\mathbf{Q}}$. Finally, Conjecture 2 is also known [3] for varieties of dimension $n-2$ defined over $\overline{\mathbf{Q}}$.

Pink [13] and Zilber [21] independently stated conjectures related to Conjecture 2. More precisely, let for the moment X be an irreducible closed subvariety of a semi-abelian variety both defined over $\mathbf{C}$. Zilber's Conjecture states that the intersection of X with the union of all algebraic subgroup of codimension at least $\dim X + 1$ is contained in a finite union of proper algebraic subgroups. In particular, if we assume that X is not contained in a proper algebraic subgroup then this intersection is not Zariski dense in X. Pink's Conjecture 1.3 [13] on mixed Shimura varieties also implies this non-density statement (cf. Theorem 6.3 [13]).

Theorem 1, the main result of this paper, goes in the direction of Conjecture 1. We obtain boundedness of height when the algebraic subgroups involved have dimension strictly less than $n/\dim X$. This strong restriction on the subgroup dimension allows us to prove Conjecture 1 only in some special cases. On the other hand we obtain explicit bounds for the height in terms of the degree of X, the height of X, and n. Already in the case $\epsilon = 0$ and when X is an arbitrary curve no explicit height bounds have appeared in literature.

We use Philippon's definition [12] of $h(X)$, the height of X. In simple terms, the height of X controls the heights of the coefficients of a certain set of defining equations for X whereas the degree of X controls their degrees.

To formulate Theorem 1 we shall refine the definition of X^{oa}: let t be an integer with $0 \le t \le n$, we define $X^{\mathrm{oa},t}$ to be X deprived of all its irreducible closed subvarieties Y that are contained in a coset H satisfying (1) and $\dim H \le t$. Clearly we have

$$X = X^{\mathrm{oa},0} \supset X^{\mathrm{oa},1} \supset \cdots \supset X^{\mathrm{oa},n} = X^{\mathrm{oa}}. \tag{2}$$

In Lemma 4 we will prove $X^{\mathrm{oa},n-\dim X} = X^{\mathrm{oa}}$. If $\dim X < n$, it is not difficult to show

$$X^{\mathrm{o}} = X^{\mathrm{oa},1}, \tag{3}$$

so the $X^{\mathrm{oa},t}$ interpolate between X^{o} and X^{oa} as $1 \leq t \leq n - \dim X$. As promised above we can now show $X^{\mathrm{o}} = X^{\mathrm{oa}}$ if $\dim X = n - 1$, indeed just apply (3) to the conclusion of Lemma 4.

Theorem 1 *Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$. Let s be an integer with* $\dim X \leq s \leq n$ *and let m be an integer with $m \cdot s < n$. If*

$$\epsilon \leq (c(n)\deg(X)^{\frac{n}{n-ms}})^{-1} \tag{4}$$

and $p \in X^{\mathrm{oa},n-s} \cap C(\mathcal{H}_m, \epsilon)$ then

$$h(p) \leq c(n)\deg(X)^{\frac{ms}{n-ms}}(h(X) + \deg(X)) \tag{5}$$

where we can take $c(n) = (2n)^{30n^2}$.

We do not claim that the height bound is best-possible in terms of $h(X)$ or $\deg(X)$. Let us consider for a moment a family of varieties of fixed dimension and fixed degree such that $X^{\mathrm{oa}} \neq \emptyset$ for all X in this family. Then $X^{\mathrm{oa}} \subset X$ is Zariski open and dense [3]. Proposition 1 implies that $X^{\mathrm{oa}} \cap \mathcal{H}_{n-\dim X}$ is Zariski dense in X. By Zhang's inequality (17) which compares height and essential minimum of a variety, one easily deduces that an upper bound for the height on $X^{\mathrm{oa}} \cap \mathcal{H}_{n-\dim X}$ must be at least linear in $h(X)$. Of course $m = n - \dim X$ and $s = \dim X$ are not always admissible in Theorem 1, but in the cases where they are, it would be interesting to know the best-possible dependency on the degree of the factor in front of $h(X)$ in (5). Zhang's inequality shows that this factor is of order at least $\deg(X)^{-1}$.

Here is a short sketch of the proof of Theorem 1: given p as in the theorem, we use geometry of numbers to construct an irreducible algebraic subgroup H of dimension $n - s$ such that the coset pH has small height compared with the height of p and controlled degree. We provide two methods to bound the height of pH: the first one involves the Arithmetic Bézout Theorem and the second one Zhang's inequality. Up to a factor depending only on n, both approaches lead to the same bound. Because $p \in X^{\mathrm{oa},n-s}$, the intersection $X \cap pH$ will have $\{p\}$ as an irreducible component. Finally, we apply the Arithmetic Bézout Theorem again and obtain a bound for the height of p.

The proof of Theorem 1 does not use the fact that $X^{\mathrm{oa}} \subset X$ is Zariski open. For m and s as in the theorem our proof shows that the height is bounded on $U \cap C(\mathcal{H}_m, \epsilon)$ for some Zariski open non-empty $U \subset X$ which contains $X^{\mathrm{oa},n-s}$ if the following hypothesis on X is satisfied:

if $\varphi : \mathbf{G}_m^n \to \mathbf{G}_m^s$ is a surjective homomorphism of algebraic groups then $\dim \varphi(X) = \dim X$. (6)

In the proof of Theorem 1 we will see that (6) follows from $X^{\mathrm{oa},n-s} \neq \emptyset$, an assumption we may certainly make in the proof.

It is tempting to take $s = \dim X$ in Theorem 1, since then one can take m as large as possible in function of $\dim X$ and since $X^{\mathrm{oa},n-\dim X} = X^{\mathrm{oa}}$. If X is a curve or a hypersurface, then $m = n - 1$, respectively, $m = 1$ is admissible. In these two cases the theorem implies Conjecture 1 with an explicit height bound.

If $X \neq \mathbf{G}_m^n$ then taking $m = 1$ and $s = n - 1$ also has interesting consequences. Indeed, $X^{\mathrm{o}} = X^{\mathrm{oa},1}$ from the discussion above, so Theorem 1 gives an explicit height bound for the points in $X^{\mathrm{o}} \cap C(\mathcal{H}_1, \epsilon)$. We have recovered an explicit version of Bombieri and Zannier's Theorem (Theorem 1, p. 523 [19]).

By Theorem 2 below, a subset of $X^{\mathrm{ta}} \cap \mathcal{H}_m$ of bounded height is finite if $m = n - \dim X - 1$. The proof of this result uses a relative Lehmer-type height lower bound by Amoroso and David [1].

Theorem 2 (Bombieri, Masser, and Zannier, Lemma 9 [4]) *Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$ of dimension r. If $B \in \mathbf{R}$, then*

$$\{p \in X^{\mathrm{ta}} \cap \mathcal{H}_{n-r-1};\ \ h(p) \leq B\}$$

is finite.

If we combine the Theorem of Bombieri, Masser, and Zannier with our Theorem 1 we get a finiteness result with $\epsilon = 0$.

Corollary 1 *Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$ of dimension $r \geq 1$ and let m be an integer with $m < \min\{n/r, n - r\}$. Then $X^{\mathrm{oa}} \cap \mathcal{H}_m$ is finite.*

This corollary is an immediate consequence of Theorem 1 in the case $s = \dim X$, Theorem 2, and the fact that $X^{\mathrm{oa}} \subset X^{\mathrm{ta}}$.

The previous corollary implies the finiteness statement of Conjecture 2 for curves, hypersurfaces, and also if $\dim X = n - 2$ but always with X^{ta} replaced by the possibly smaller X^{oa} and if X is defined over $\overline{\mathbf{Q}}$. By taking $n = 5$, $r = 2$, and $m = 2$ we observe that Corollary 1 implies finiteness for surfaces in $\mathbf{G}_m^5$ with the correct subgroup dimension:

Corollary 2 *Let $X \subset \mathbf{G}_m^5$ be an irreducible closed algebraic surface defined over $\overline{\mathbf{Q}}$, then $X^{\mathrm{oa}} \cap \mathcal{H}_2$ is finite.*

It does not seem possible to adapt the proof of Theorem 2 to give finiteness results with $X^{\mathrm{ta}} \cap \mathcal{H}_{n-r-1}$ replaced by $X^{\mathrm{oa}} \cap C(\mathcal{H}_{n-r-1}, \epsilon)$ for some small $\epsilon > 0$. In a further paper [10] we will prove a variant of Theorem 2 for curves with $X^{\mathrm{oa}} \cap C(\mathcal{H}_{n-2}, \epsilon)$ instead of $X^{\mathrm{ta}} \cap \mathcal{H}_{n-2}$. Together with Theorem 1 we will then be able to conclude Conjecture 3 for curves and even obtain a bound for the cardinality.

It is probably possible to adapt the methods presented in this article to treat the case where $\mathbf{G}_m^n$ is replaced by a power of an elliptic curve or a more general abelian variety. Such an adaptation could provide an explicit height bound in Viada's Theorem [18] on the intersection of a curve contained in a power of an elliptic curve with the union of all proper algebraic subgroups.

The article is organized in the following way: in Sect. 2 we define the height functions which we use throughout the paper. In Sect. 3 we recall some results on algebraic subgroups of $\mathbf{G}_m^n$ and bound the height of a coset. In Sect. 4 we do some geometry of numbers which will be used in the proof of Theorem 1 in Sect. 5.

2 Heights

Throughout the paper $c_1, \dots, c_9$ will denote positive, effectively computable constants which depend only on n.

Let K be a number field. By a place of K we mean an absolute value normalized such that its restriction to $\mathbf{Q}$ is either the standard complex absolute value or a p-adic absolute value for some prime p. The former places are called *infinite* and the latter *finite*. It is well-known that the infinite places correspond one-to-one to embeddings of K into $\mathbf{C}$ defined up-to complex conjugation. By abuse of notation we also use the symbol v to denote its restriction to a subfield of K. Let K_v be the completion of K with respect to v. With our choice of normalization, the product formula (Proposition 1.4.4 [5]) $\prod_v |x|_v^{[K_v:\mathbf{Q}_v]} = 1$ holds for all x in K^*, the non-zero elements of K.

Say $p = [p_0 : \cdots : p_n] \in \mathbf{P}^n$ with $p_i \in K$. We define the *absolute logarithmic Weil height* as

$$h(p) = \sum_v \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} \log \max\{|p_0|_v, \dots, |p_n|_v\} \tag{7}$$

where the sum runs over all places of K. We also define an alternative height using the 2-norm at the infinite places:

$$\begin{aligned} h_2(p) = & \sum_{v \text{ finite}} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} \log \max\{|p_0|_v, \dots, |p_n|_v\} \\ & + \sum_{v \text{ infinite}} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} \log((|p_0|_v^2 + \cdots + |p_n|_v^2)^{1/2}). \end{aligned}$$

The product formula implies that both heights are independent of the chosen projective coordinates, in particular they are non-negative. Furthermore, both heights are independent of the number field containing the coordinates (Lemma 1.5.2 [5]) and can be compared using

$$h(p) \le h_2(p) \le h(p) + \frac{1}{2}\log(n+1). \tag{8}$$

These inequalities follow by comparing corresponding local terms in the definition of both heights.

Let X be an irreducible closed subvariety of $\mathbf{P}^n$. The *degree* of X, denoted by $\deg(X)$, equals the cardinality of the intersection of X with an $(n - \dim X)$-dimensional linear subvariety of $\mathbf{P}^n$ in general position. With this characterization we extend the degree to possibly reducible Zariski closed subsets of $\mathbf{P}^n$. The degree of a Zariski closed subset of $\mathbf{P}^n$ is then the sum of the degrees of its irreducible components of maximal dimension.

Recall that X is defined over $\overline{\mathbf{Q}}$ by our convention. The *height* $h(X)$ of X, a non-negative real number, is defined according to Philippon [12]. The general idea behind this definition is to take a Chow form of X, a multihomogeneous polynomial in algebraic coefficients defined uniquely up to multiplication by a non-zero scalar, and then to define the height of X as an appropriate height of this form. The precise choice of metrics used in the height of the Chow form is motivated by Arakelov theory and Bost, Gillet, and Soulé's height, called Faltings height, defined in Sect. 3.1 [6].

At this point we must warn the reader about a possible ambiguity: a point $p \in \mathbf{P}^n$ already has a well-defined height given by (7) which does not necessarily equal the height $h(\{p\})$ of the corresponding variety. In fact by (3.1.6) [6] we have $h(\{p\}) = h_2(p)$. To avoid confusion the height of a point considered as a variety will always be written as $h(\{p\})$ or $h_2(p)$. If the brackets are omitted in the first case we mean the height defined in (7).

Let us assume for the moment that X is the irreducible component of the zero set of a single non-zero homogeneous polynomial f. Let $\tilde{f}$ be an irreducible factor of f which defines X. We may consider a polynomial as a point in some projective space by taking its coefficients as the projective coordinates. Therefore, the heights $h(f)$ and $h(\tilde{f})$ are well-defined. It follows from [12, p. 347] and standard estimates that $h(X) \le h(\tilde{f}) + c_1\deg(\tilde{f})$. Surely, $\deg(\tilde{f}) \le \deg(f)$ and by Theorem 1.6.13 [5] we also have $h(\tilde{f}) \le h(f) + (\log 2)(n+1)\deg(f)$. Hence

$$h(X) \le h(f) + c_2\deg(f). \tag{9}$$

There is a embedding $\iota : \mathbf{G}_m^n \hookrightarrow \mathbf{P}^n$ given by taking a point $(p_1, \dots, p_n)$ to $[1 : p_1 : \cdots : p_n]$. If $p \in \mathbf{G}_m^n$ we set $h(p) = h(\iota(p))$ and $h_2(p) = h_2(\iota(p))$. Having defined the degree and height of a subvariety of $\mathbf{P}^n$ it makes sense to speak of the degree and height of a subvariety of $\mathbf{G}_m^n$ using this embedding. Concretely, if $Z \subset \mathbf{G}_m^n$ is any subset and Z' is the Zariski closure of $\iota(Z)$ we define $\deg(Z) = \deg(Z')$ and if Z' is irreducible we define $h(Z) = h(Z')$.

From the point of view of Arakelov theory the height of a variety should be understood as an arithmetic version of the degree. The arithmetic analog of Bézout's Theorem is called the Arithmetic Bézout Theorem. We use a version due to Philippon. If Y is another irreducible closed subvariety of $\mathbf{P}^n$, the Arithmetic Bézout Theorem bounds the height of the irreducible components in $X \cap Y$ from above in terms of heights and degrees of X and Y:

Theorem 3 *Let X and Y be irreducible closed subvariety of $\mathbf{P}^n$ defined over $\overline{\mathbf{Q}}$. If $Z_1, \dots, Z_g$ are the irreducible components of $X \cap Y$, then*

$$\sum_{i=1}^{g} h(Z_i) \le \deg(X)h(Y) + \deg(Y)h(X) + c_3\deg(X)\deg(Y),$$

where we can take $c_3 = 3n^2$.

Proof This follows from Philippon's Theorem 3 [12]; the value of the constant c_3 follows from standard estimates. Bost et al. (Theorem 5.5.1 [6]) gave a version of the Arithmetic Bézout Theorem for the projective height on cycles of projective space. □

This theorem also holds if X and Y are irreducible closed subvarieties of $\mathbf{G}_m^n$.

3 Algebraic subgroups of $\mathbf{G}_m^n$

We introduce some notation which eases calculations in $\mathbf{G}_m^n$. Let $p = (p_1, \ldots, p_n) \in \mathbf{G}_m^n$ and $u = (\alpha_1, \ldots, \alpha_n)^{\mathrm{t}} \in \mathbf{Z}^n$ where ${}^{\mathrm{t}}$ means transpose. We set $p^u = p_1^{\alpha_1} \ldots p_n^{\alpha_n}$. Let U be the $n \times s$ matrix with columns $u_1, \ldots, u_s \in \mathbf{Z}^n$, we set $p^U = (p^{u_1}, \ldots, p^{u_s}) \in \mathbf{G}_m^s$. For $q \in \mathbf{G}_m^n$ we have $(pq)^U = p^U q^U$. If V is a matrix with s rows and integer coefficients, then $(p^U)^V = p^{UV}$. Finally, we define the morphism of algebraic groups $\varphi_{(u_1,\ldots,u_s)} : \mathbf{G}_m^n \to \mathbf{G}_m^s$ by sending p to $(p^{u_1}, \ldots, p^{u_s})$.

Let $p, q \in \mathbf{G}_m^n$, then by bounding local terms in (7) from above we obtain the useful inequalities $h(pq) \le h(p) + h(q)$ and $h(p) \le h(p_1) + \cdots + h(p_n) \le nh(p)$. The definition (7) also implies that the height function is invariant under multiplication by torsion points of $\mathbf{G}_m^n$. Furthermore, using the product formula one can show $h(p^{-1}) \le nh(p)$. In the special case $n = 1$ we have $h(p^m) = |m|h(p)$ for $m \in \mathbf{Z}$. If U is as in the previous paragraph we conclude

$$h(p^U) \le s \sum_{k=1}^{n} x_k h(p_k), \tag{10}$$

where x_k is the maximum of the absolute values of the elements of the kth row of U.

For $u \in \mathbf{R}^n$ let $|u|$ be the Euclidean norm of u. Furthermore, if $L \in \mathbf{R}[X_1, \ldots, X_n]$ is a linear form then $|L|$ will denote the Euclidean norm of the coefficient vector of L.

Let $\Lambda \subset \mathbf{Z}^n$ be a subgroup, we define

$$\mathcal{H}(\Lambda) = \{p \in \mathbf{G}_m^n;\;\; p^u = 1 \text{ for all } u \in \Lambda\}$$

this is an algebraic subgroup of $\mathbf{G}_m^n$. We set $\det \Lambda$ to be the determinant of Λ considered as a lattice in $\mathbf{R}^n$, that is a discrete subgroup.

Algebraic subgroups of $\mathbf{G}_m^n$ and subgroups of $\mathbf{Z}^n$ are closely related:

Proposition 2 *Let $\Lambda \subset \mathbf{Z}^n$ be a subgroup of rank s, then* $\dim \mathcal{H}(\Lambda) = n - s$ *and*

$$\deg(\mathcal{H}(\Lambda)) \le c_4 \det \Lambda \tag{11}$$

where we can take $c_4 = n^n$. Furthermore, for any algebraic subgroup $H \subset \mathbf{G}_m^n$ there exists a subgroup $\Lambda \subset \mathbf{Z}^n$ with $H = \mathcal{H}(\Lambda)$.

Proof The equality $\dim \mathcal{H}(\Lambda) = n - s$ follows from Proposition 3.2.7 [5]. The last statement in the assertion is Corollary 3.2.15 [5].

Inequality (11) can be proved as follows. Let ω_s be the Lebesgue measure of the Euclidean unit ball in $\mathbf{R}^s$, i.e. $\omega_s = \pi^{s/2}/\Gamma(s/2+1)$ where Γ is the gamma function.

By Minkowski's Second Theorem there exist linearly independent $u_1, \ldots, u_s \in \Lambda$ with $|u_1| \cdots |u_s| \omega_s \leq 2^s \det \Lambda$. Say the u_i generate the subgroup $\Lambda' \subset \Lambda$. Then by the first part of the proposition $\mathcal{H}(\Lambda)$ and $\mathcal{H}(\Lambda')$ have equal dimension. We recall that any two irreducible components of an algebraic group have the same dimension. Since $\mathcal{H}(\Lambda) \subset \mathcal{H}(\Lambda')$ the irreducible components of $\mathcal{H}(\Lambda)$ are irreducible components of $\mathcal{H}(\Lambda')$ and so $\deg(\mathcal{H}(\Lambda)) \leq \deg(\mathcal{H}(\Lambda'))$. Now $x^{u_1} = \cdots = x^{u_s} = 1$ are defining equations for $\mathcal{H}(\Lambda')$. By multiplying these with suitable monomials we get s polynomials of degrees bounded by $\sqrt{n}|u_1|, \ldots, \sqrt{n}|u_s|$, respectively, which are defining equations for $\mathcal{H}(\Lambda')$. By Bézout's Theorem (Example 8.4.6, p. 148 [9]) we conclude $\deg(\mathcal{H}(\Lambda)) \leq \deg(\mathcal{H}(\Lambda')) \leq n^{s/2} 2^s \deg(\Lambda)/\omega_s$ The inequality $n^{s/2} 2^s/\omega_s \leq n^n$ can be derived elementarily, and so the lemma follows. □

Proposition 3 *Say $s \geq 1$, let $u_1, \ldots, u_s \in \mathbf{Z}^n$ be linearly independent, and let $H \subset \mathcal{H}(u_1\mathbf{Z} + \cdots + u_s\mathbf{Z})$ be an irreducible component. If $p \in \mathbf{G}_m^n$, then*

$$h(pH) \leq c_6 |u_1| \cdots |u_s| \left(1 + \sum_{k=1}^{s} \frac{h(p^{u_k})}{|u_k|} \right)$$

where we can take $c_6 = 2^n n^{n+3}$.

Proof Say $u_k = (u_{1k}, \ldots, u_{nk})^{\mathrm{t}}$. Let Y_k be an irreducible component of the zero set of the Laurent polynomial $X_1^{u_{1k}} \cdots X_n^{u_{nk}} - p^{u_k}$ in $\mathbf{G}_m^n$. Then Y_k is a coset and by Proposition 2 we have

$$\deg(Y_k) \leq c_4 |u_k|. \tag{12}$$

Since Y_k is a hypersurface we may us (9) to bound $h(Y_k)$ as follows

$$h(Y_k) \leq h(p^{u_k}) + c_2 \deg(Y_k) \leq h(p^{u_k}) + c_2 c_4 |u_k|. \tag{13}$$

Let Z_k be an irreducible component of the intersection $Y_1 \cap \cdots \cap Y_k$. The Arithmetic Bézout Theorem gives

$$h(Z_k) \leq \deg(Z_{k-1}) h(Y_k) + \deg(Y_k) h(Z_{k-1}) + c_3 \deg(Z_{k-1}) \deg(Y_k) \tag{14}$$

where Z_{k-1} is an irreducible component of $Y_1 \cap \cdots \cap Y_{k-1}$ such that Z_k is an irreducible component of $Z_{k-1} \cap Y_k$. Then Z_{k-1} is the translate of an irreducible component of the algebraic subgroup associated to the subgroup of $\mathbf{Z}^n$ generated by $u_1, \ldots, u_{k-1}$. By Hadamard's inequality this subgroup has determinant at most $|u_1| \cdots |u_{k-1}|$, so by (11) we may bound

$$\deg(Z_{k-1}) \leq c_4 |u_1| \cdots |u_{k-1}|. \tag{15}$$

We apply (12) and (13) to bound the degree and height, respectively, of Y_k in (14):

$$h(Z_k) \leq (h(p^{u_k}) + c_2 c_4 |u_k| + c_3 c_4 |u_k|) \deg(Z_{k-1}) + c_4 |u_k| h(Z_{k-1}),$$

and then apply (15) to bound the degree of Z_{k-1}:

$$h(Z_k) \le c_4|u_1|\cdots|u_{k-1}|h(p^{u_k}) + c_5|u_1|\cdots|u_k| + c_4|u_k|h(Z_{k-1}). \qquad (16)$$

The proposition follows by applying induction to (16) since pH is of the form Z_s. □

Let $X \subset \mathbf{G}_m^n$ be an irreducible closed subvariety. The *essential minimum* of X associated to the height h_2 is

$$\mu^{\mathrm{ess}}(X) = \inf\{\theta \in \mathbf{R};\ \{p \in X;\ h_2(p) \le \theta\} \text{ is Zariski dense in } X\}.$$

A well-known result of Zhang [20] compares the quotient $h(X)/\deg(X)$ with $\mu^{\mathrm{ess}}(X)$:

$$\mu^{\mathrm{ess}}(X) \le \frac{h(X)}{\deg(X)} \le (1 + \dim X)\mu^{\mathrm{ess}}(X). \qquad (17)$$

We provide an alternative way to deduce the height upper bound in the previous proposition, which also leads us to the value of c_6, by applying the upper bound in Zhang's inequality.

Alternative proof of Proposition 3 Let U be the $n \times s$ matrix with columns $u_1, \ldots, u_s$ and let Λ be the rank s subgroup of $\mathbf{Z}^n$ generated by the u_k. We write $u_k = (u_{1k}, \ldots, u_{nk})^{\mathrm{t}}$ and after possibly permuting coordinates we may assume that

$$U' = \begin{bmatrix} u_{11} & \cdots & u_{1s} \\ \vdots & & \vdots \\ u_{s1} & \cdots & u_{ss} \end{bmatrix}$$

is an $s \times s$ minor of U with maximal absolute determinant. We set $\Delta = \det U'$, by the Cauchy–Binet formula $(\det \Lambda)^2$ is equal to the sum over the squares of the determinants of all $s \times s$-minors of U. Hence we may bound

$$\det \Lambda \le \binom{n}{s}^{\frac{1}{2}} |\Delta| \le 2^{\frac{n}{2}}|\Delta| \quad \text{and} \quad |\Delta| \le |u_1|\cdots|u_s|, \qquad (18)$$

the second bound follows from Hadamard's inequality. The subgroup Λ defines an algebraic subgroup $\mathcal{H}(\Lambda) \subset \mathbf{G}_m^n$ of dimension $n - s$ by Proposition 2. Let H be the irreducible component of $\mathcal{H}(\Lambda)$ that contains the unit element. If $\pi : \mathbf{G}_m^n \to \mathbf{G}_m^{n-s}$ is the projection onto the last $n - s$ coordinates, then $\pi(pH) = \mathbf{G}_m^{n-s}$ because $\Delta \ne 0$. Since the torsion points of $\mathbf{G}_m^{n-s}$ lie Zariski dense we conclude that the set $V = \pi|_{pH}^{-1}(\text{torsion points of } \mathbf{G}_m^{n-s})$ lies Zariski dense in pH.

Let $w \in V \subset pH$, then $w = ph$ for some $h \in H$. We define the $(n-s) \times s$-matrix U'' by

$$U = \begin{bmatrix} U' \\ U'' \end{bmatrix}.$$

Furthermore, let us write $w = (w', w'')$ with $w' \in \mathbf{G}_m^s$ and $w'' \in \mathbf{G}_m^{n-s}$. Then $w'^{U'} w''^{U''} = w^U = p^U h^U = p^U$. So

$$w'^{\Delta} = w'^{U'(^\# U')} = p^{U(^\# U')} w''^{-U''(^\# U')}$$

where $^\# U'$ is the adjugate matrix of U'. Elementary height properties provide

$$|\Delta| h(w) = |\Delta| h(w') \leq s h(w'^{\Delta}) = s h(p^{U(^\# U')} w''^{-U''(^\# U')}).$$

Note that w'' is a torsion point, hence we may omit it above. We apply (10) to get

$$|\Delta| h(w) \leq s h(p^{U(^\# U')}) \leq s^2 \sum_{k=1}^{s} x_k h(p^{u_k}).$$

Here x_k denotes the maximum of the absolute values of the elements of the kth row of $^\# U'$. Linear algebra and Hadamard's inequality imply $x_k \leq \frac{|u_1| \cdots |u_s|}{|u_k|}$ and so

$$h(w) \leq s^2 \frac{|u_1| \cdots |u_s|}{|\Delta|} \sum_{k=1}^{s} \frac{h(p^{u_k})}{|u_k|}.$$

We compare the heights $h(w)$ and $h_2(w)$ with (8) and use $s \leq n$ to conclude

$$\begin{aligned} h_2(w) &\leq \frac{1}{2} \log(n+1) + n^2 \frac{|u_1| \cdots |u_s|}{|\Delta|} \sum_{k=1}^{s} \frac{h(p^{u_k})}{|u_k|} \\ &\leq n^2 \frac{|u_1| \cdots |u_s|}{|\Delta|} \left(1 + \sum_{k=1}^{s} \frac{h(p^{u_k})}{|u_k|} \right), \end{aligned} \tag{19}$$

the second inequality follows from the second inequality in (18). Inequality (19) holds for all $w \in V$.

Since $V \subset pH$ lies Zariski dense, the right-hand side of (19) is an upper bound for $\mu^{\text{ess}}(pH)$. Zhang's inequality (17) and $1 + \dim pH = 1 + n - s \leq n$ imply

$$\begin{aligned} h(pH) &\leq \deg(pH)(1 + \dim pH)\mu^{\text{ess}}(pH) \\ &\leq n^3 |u_1| \cdots |u_s| \frac{\deg(pH)}{|\Delta|} \left(1 + \sum_{k=1}^{s} \frac{h(p^{u_k})}{|u_k|} \right) \end{aligned} \tag{20}$$

Using (11) we may bound the degree of pH as follows

$$\deg(pH) = \deg(H) \leq \deg(\mathcal{H}(\Lambda)) \leq n^n \det \Lambda \leq (2n)^n |\Delta| \tag{21}$$

here the last inequality follows from the first inequality in (18). We insert (21) into (20) and conclude the proof. □

4 Geometry of numbers

We begin with a lemma which will later enable us to apply geometry of numbers to construct a coset of small height and controlled degree.

Lemma 1 *Let* $1 \leq m \leq n$ *and let* $a \in \mathcal{H}_m$. *There exist linear forms* $L_1, \ldots, L_m \in \mathbf{R}[X_1, \ldots, X_n]$ *such that* $|L_j| \leq 1$ *and*

$$h(a^u) \leq c_7 \max_{1 \leq j \leq m} \{|L_j(u)|\} h(a)$$

for all $u \in \mathbf{Z}^n$ *where we can take* $c_7 = n^2 4^n$.

Proof Let $a = (a_1, \ldots, a_n) \in \mathcal{H}_m$, by Proposition 2 the a_i lie in a finitely generated subgroup of $\overline{\mathbf{Q}}^*$ of rank at most m. By enlarging this subgroup if necessary we may assume that the rank equals m. By a result of Schlickewei (Theorem 1.1, [17]) there exist multiplicatively independent $g_1, \ldots, g_m \in \overline{\mathbf{Q}}^*$ and roots of unity $\zeta_1, \ldots, \zeta_n$ such that

$$a_i = \zeta_i g_1^{v_{i1}} \cdots g_m^{v_{im}} \quad \text{for some} \quad v_{ij} \in \mathbf{Z} \tag{22}$$

and

$$h(g_1^{b_1} \cdots g_m^{b_m}) \geq 4^{-m} \max\{|b_1| h(g_1), \ldots, |b_m| h(g_m)\} \tag{23}$$

for all $(b_1, \ldots, b_m) \in \mathbf{Z}^m$.

We define $A = \max_{i,j}\{|v_{ij}| h(g_j)\}$. The assertion of the lemma obviously holds if all a_i are roots of unity. So let us assume that at least one v_{ij} is non-zero, then $A > 0$. For $1 \leq j \leq m$ we define the linear forms

$$\tilde{L}_j = v_{1j} X_1 + \cdots + v_{nj} X_n \quad \text{and} \quad L_j = \frac{h(g_j)}{nA} \tilde{L}_j; \tag{24}$$

we have $|L_j| \leq 1$.

Let $u \in \mathbf{Z}^n$, then (22) and (24) imply

$$a^u = \xi g_1^{\tilde{L}_1(u)} \cdots g_m^{\tilde{L}_m(u)}$$

for some root of unity ξ. We apply standard height properties and (24) to conclude

$$\begin{aligned} h(a^u) &= h\left(g_1^{\tilde{L}_1(u)} \cdots g_m^{\tilde{L}_m(u)}\right) \\ &\leq |\tilde{L}_1(u)| h(g_1) + \cdots + |\tilde{L}_m(u)| h(g_m) \\ &= nA(|L_1(u)| + \cdots + |L_m(u)|) \\ &\leq mnA \max\{|L_1(u)|, \ldots, |L_m(u)|\}. \end{aligned} \tag{25}$$

We choose i_0 and j_0 such that $A = |v_{i_0 j_0}| h(g_{j_0})$. Then (22) and (23) imply

$$h(a) \geq h(a_{i_0}) \geq 4^{-m} |v_{i_0 j_0}| h(g_{j_0}) = 4^{-m} A.$$

We insert this inequality into (25) and use $m \leq n$ to complete the proof. □

In the next lemma we approximate zeros of linear forms with real coefficients by integral vectors with controlled norm. The proof uses Minkowski's Second Theorem.

Lemma 2 *Let $1 \leq m \leq n$ and let $L_1, \ldots, L_m \in \mathbf{R}[X_1, \ldots, X_n]$ be linear forms with $|L_j| \leq 1$. If $\rho \geq 1$, there exist $\lambda_1, \ldots, \lambda_n$ with $0 < \lambda_1 \leq \cdots \leq \lambda_n$ and linearly independent $u_1, \ldots, u_n \in \mathbf{Z}^n$ such that*

$$|u_k| \leq \lambda_k, \quad |L_j(u_k)| \leq \rho^{-1}\lambda_k, \text{ and } \lambda_1 \cdots \lambda_n \leq c_8 \rho^m, \tag{26}$$

where we can take $c_8 = 2n^n$.

Proof Let $\Lambda \subset \mathbf{R}^{m+n}$ be the rank n lattice generated by the columns of the $(m+n) \times n$-matrix

$$A = \begin{bmatrix} \rho L_1 \\ \vdots \\ \rho L_m \\ E \end{bmatrix}$$

where E is the $n \times n$ unit matrix and the L_i are identified with their coefficient vectors in $\mathbf{R}^n$. By the Cauchy–Binet formula we have $\det \Lambda = (\det A^{\mathrm{t}} A)^{1/2} = (\sum_{A'} (\det A')^2)^{1/2}$ where the sum ranges over all $n \times n$ minors A' of A. By Hadamard's inequality we deduce $|\det A'| \leq \rho^m$ and so

$$\det \Lambda \leq \binom{n+m}{n}^{\frac{1}{2}} \rho^m \leq 2^{\frac{n+m}{2}} \rho^m \leq 2^n \rho^m. \tag{27}$$

Let Q be the Euclidean unit ball in $\mathbf{R}^{m+n}$ and V the n-dimensional $\mathbf{R}$-vector space generated by Λ. Recall that the n-dimensional Lebesgue measure of $Q \cap V$ is ω_n, the measure of the Euclidean unit ball in $\mathbf{R}^n$.

Let $0 < \lambda_1 \leq \cdots \leq \lambda_n$ be the successive minima of Λ with respect to the convex, symmetric, and compact set $Q \cap V$. Minkowski's Second Theorem and (27) imply

$$\lambda_1 \cdots \lambda_n \leq \frac{2^n}{\omega_n} \det \Lambda \leq \frac{4^n}{\omega_n} \rho^m \leq 2n^n \rho^m, \tag{28}$$

the last inequality can be derived elementarily. For $1 \leq k \leq n$ there exist

$$v_k \in (\lambda_k Q) \cap \Lambda \text{ with } v_1, \ldots, v_n \text{ linearly independent.} \tag{29}$$

Hence there are $u_k \in \mathbf{Z}^n$ with

$$v_k = (\rho L_1(u_k), \ldots, \rho L_m(u_k), u_k). \tag{30}$$

Clearly the $u_1, \ldots, u_n$ are also linearly independent. The first two inequalities in (26) follow from (29) and (30). The last one is just (28). □

Lemma 3 *Let $1 \le m \le n$ and let $L_1, \ldots, L_m \in \mathbf{R}[X_1, \ldots, X_n]$ be linear forms with $|L_j| \le 1$. If $T \ge 1$, then for any integer s with $1 \le s \le n$ there exist linearly independent $u_1, \ldots, u_s \in \mathbf{Z}^n$ such that $|u_1| \cdots |u_s| \le T$ and*

$$|u_1| \cdots |u_s| \frac{|L_j(u_k)|}{|u_k|} \le c_8 T^{1-\frac{n}{ms}} \quad \text{for} \quad 1 \le j \le m \text{ and } 1 \le k \le s$$

with c_8 from Lemma 2.

Proof If $T < c_8^{\frac{s}{n}}$, then

$$T^{1-\frac{n}{sm}} \ge T^{1-\frac{n}{s}} \ge T^{-\frac{n}{s}} > c_8^{-1}.$$

In this case it suffices to take for $u_1, \ldots, u_s$ any distinct standard basis elements of $\mathbf{R}^n$.

So let us assume that $T \ge c_8^{\frac{s}{n}}$. We set $\rho = c_8^{-\frac{1}{m}} T^{\frac{n}{ms}} \ge 1$. Applying Lemma 2, we get λ_k and u_k as in (26) with

$$|u_1| \cdots |u_s| \le \lambda_1 \cdots \lambda_s \le (\lambda_1 \cdots \lambda_n)^{\frac{s}{n}} \le c_8^{\frac{s}{n}} \rho^{\frac{ms}{n}} = T.$$

Furthermore, by (26) and the inequality above we may estimate

$$|u_1| \cdots |u_s| \frac{|L_j(u_k)|}{|u_k|} \le \rho^{-1} \lambda_1 \cdots \lambda_s \le \rho^{-1} T = c_8^{\frac{1}{m}} T^{1-\frac{n}{ms}} \le c_8 T^{1-\frac{n}{ms}}$$

for all $1 \le j \le m$ and $1 \le k \le s$ since $c_8 \ge 1$. □

5 Proof of Theorem 1

We first prove Proposition 1: let $x_1, \ldots, x_n$ denote the coordinate functions on $X \subset \mathbf{G}_m^n$. Set $r = \dim X$, after possibly permuting coordinates we may assume that $x_1, \ldots, x_r$ are algebraically independent over $\overline{\mathbf{Q}}$.

Let $\pi : \mathbf{G}_m^n \to \mathbf{G}_m^r$ denote the projection onto the first r coordinates. Then $\pi(X)$ is Zariski dense in $\mathbf{G}_m^r$, and so is $\pi(U)$. The image $\pi(U)$ contains a Zariski open dense subset $V \subset \mathbf{G}_m^r$ (Theorem, p. 219 [7]). The torsion points of $\mathbf{G}_m^r$ lie Zariski dense and hence their intersection with V is infinite. Any point $p \in U$ such that $\pi(p)$ is a torsion point already lies in $\mathcal{H}_{n-r}$, therefore $U \cap \mathcal{H}_{n-r}$ is infinite.

Let $\pi' : \mathbf{G}_m^n \to \mathbf{G}_m^{r-1}$ denote the projection onto the first $r-1$ coordinates (if $r = 1$ we take $\mathbf{G}_m^{r-1}$ a point). Again $\pi'(U)$ lies Zariski dense and thus contains $V' \neq \emptyset$ which is Zariski open in $\mathbf{G}_m^{r-1}$. Now V' must contain a torsion point of $\mathbf{G}_m^{r-1}$. By the Theorem on the Dimension of the Fibres (first Theorem, p. 228 [7]), $\pi'|_U$ has positive dimensional fibres. By taking the fibre above a torsion point in V' we see that $U \cap \mathcal{H}_{n-r+1}$ has unbounded height. □

Lemma 4 *We have* $X^{\mathrm{oa},n-\dim X} = X^{\mathrm{oa}}$.

Proof Indeed by (2) it suffices to prove that if $p \in Y \subset pH$ where $Y \subset X$ is an irreducible subvariety and H is an algebraic subgroup satisfying (1), then $p \notin X^{\mathrm{oa},n-\dim X}$. If $\dim H \le n - \dim X$, then we are done; hence say $\dim H > n - \dim X$. By Proposition 2 there exist linearly independent $u_1, \dots, u_{n-\dim H} \in \mathbf{Z}^n$ such that $H = \mathcal{H}(u_1\mathbf{Z} + \cdots + u_{n-\dim H}\mathbf{Z})$. We may find $v_1, \dots, v_{\dim H + \dim X - n} \in \mathbf{Z}^n$ such that the u_i, v_i are linearly independent. We apply Proposition 2 again to deduce that the v_i define an algebraic subgroup $H' \subset \mathbf{G}_m^n$ with $\dim H' = 2n - \dim H - \dim X$ and $\dim H \cap H' = n - \dim X$. Let Y' be an irreducible component of $Y \cap pH'$ containing p. We have $\dim Y' \ge \dim Y + \dim H' - n$ (§8.2, p. 137 [9]) and use (1) to conclude $\dim Y' \ge 1$. Furthermore, $Y' \subset p(H \cap H')$ where the right side is a coset of dimension $n - \dim X$. Therefore $p \notin X^{\mathrm{oa},n-\dim X}$ and the lemma is established. □

Before giving the main argument for the proof of Theorem 1 we make some preliminary observations.

Assume for the moment that m is an integer with $1 \le m \le n$. Say $p \in C(\mathcal{H}_m, \epsilon)$ and $\epsilon \le (2n)^{-1}$. Then $p = ab$ with $a \in \mathcal{H}_m$ and $h(b) \le \epsilon(1 + h(a))$. By elementary height properties described in Sect. 3 we have $h(a) = h(pb^{-1}) \le h(p) + h(b^{-1}) \le h(p) + nh(b)$. So $h(a) \le h(p) + n\epsilon(1 + h(a)) \le h(p) + \frac{1}{2}(1 + h(a))$. We conclude that

$$h(a) \le 1 + 2h(p) \quad \text{and} \quad h(b) \le 2\epsilon(1 + h(p)). \tag{31}$$

For $T \in \mathbf{R}$ and an integer s with $1 \le s \le n$ we define the finite set

$$\Phi_s(T) = \{\varphi_{(u_1,\dots,u_s)} : \mathbf{G}_m^n \to \mathbf{G}_m^s;\ \ u_1, \dots, u_s \in \mathbf{Z}^n \text{ linearly independent and } |u_1| \cdots |u_s| \le T\}.$$

If $\varphi \in \Phi_s(T)$, then $\ker \varphi$, the kernel of φ, is an algebraic subgroup of $\mathbf{G}_m^n$ of dimension $n - s$ by Proposition 2. Hence φ is surjective by the Theorem on the Dimension of the Fibres.

Lemma 5 *Let* $1 \le m \le n$ *be an integer, let* $p \in C(\mathcal{H}_m, \epsilon)$ *with* $\epsilon \le (2n)^{-1}$, *and let* $T \ge 1$. *Say* s *is an integer with* $1 \le s \le n$, *there exists* $\varphi \in \Phi_s(T)$ *such that if* H *is the irreducible component of* $\ker \varphi$ *containing* 1, *then*

$$h(pH) \le c_9((T^{1-\frac{n}{ms}} + T\epsilon)(h(p) + 1) + T) \quad \textit{and} \quad \deg(pH) \le c_4 T \tag{32}$$

where we can take $c_9 = (2n)^{10n}$.

Proof We have $p = ab$ with $a \in \mathcal{H}_m$ and $h(b) \leq \epsilon(1+h(a))$. Let $1 \leq k \leq s$. By Lemmas 1 and 3 there exist linearly independent $u_1, \ldots, u_s \in \mathbf{Z}^n$ such that $|u_1|\cdots|u_s| \leq T$ and

$$|u_1|\cdots|u_s|\frac{h(a^{u_k})}{|u_k|} \leq 2^{2n+1}n^{n+2}T^{1-\frac{n}{ms}}h(a) \leq 2^{2n+2}n^{n+2}T^{1-\frac{n}{ms}}(1+h(p)); \quad (33)$$

the second inequality used the first part of (31). We set $\varphi = \varphi_{(u_1,\ldots,u_s)} \in \Phi_s(T)$. By elementary height inequalities, cf. Sect. 3, we have $h(b^{u_k}) \leq n|u_k|h(b)$, hence using the second part of (31) we get

$$|u_1|\cdots|u_s|\frac{h(b^{u_k})}{|u_k|} \leq n|u_1|\cdots|u_s|h(b) \leq nTh(b) \leq 2n\epsilon T(1+h(p)). \quad (34)$$

By (33) and (34), and since $h(p^{u_k}) \leq h(a^{u_k}) + h(b^{u_k})$ we conclude

$$|u_1|\cdots|u_s|\frac{h(p^{u_k})}{|u_k|} \leq 2^{2n+2}n^{n+2}(T^{1-\frac{n}{ms}} + \epsilon T)(1+h(p)). \quad (35)$$

The degree bound in (32) follows from Proposition 2 since $u_1, \ldots, u_s$ generate a subgroup of $\mathbf{Z}^n$ of determinant at most $|u_1|\cdots|u_s| \leq T$. To bound $h(pH)$ we apply (35) to Proposition 3. □

We now prove Theorem 1.

To avoid trivialities say $\dim X \geq 1$, $m \geq 1$, and $\epsilon \geq 0$. Clearly we may assume $X^{\mathrm{oa},n-s} \neq \emptyset$.

We will assume that $T \geq 1$ is fixed and depends only on $\deg(X)$, n. At the end of the proof we will see how to choose this constant appropriately.

Say $\varphi : \mathbf{G}_m^n \to \mathbf{G}_m^s$ is a surjective homomorphism of algebraic groups. We set

$$Z_\varphi = \{p \in X;\ \ p \text{ is not an isolated point of } \varphi|_X^{-1}(\varphi|_X(p))\}$$

The Semi-continuity Theorem of Chevalley (second Theorem on p. 228 [7]) implies that $Z_\varphi \subset X$ is Zariski closed.

We claim $Z_\varphi \subset X\backslash X^{\mathrm{oa},n-s}$. Indeed say $p \in Z_\varphi$, then p is contained in an irreducible subvariety $Y \subset X$ of positive dimension with $\varphi(Y) = \varphi(p)$. So Y is contained in the coset $p\ker\varphi$ of dimension $n-s$. Since $\dim X \leq s$ we conclude $p \notin X^{\mathrm{oa},n-s}$, thus $Z_\varphi \subset X\backslash X^{\mathrm{oa},n-s}$. In particular $Z_\varphi \neq X$ by our assumption $X^{\mathrm{oa},n-s} \neq \emptyset$. Therefore $\varphi|_X$ has generically finite fibres and the Theorem on the Dimension of the Fibres tells us that $\dim \varphi(X) = \dim X$. This establishes the claim made just below (6).

We define Z as the union

$$Z = \bigcup_{\varphi \in \Phi_s(T)} Z_\varphi.$$

Clearly $Z \subsetneq X$ is Zariski closed because the union is taken over a finite set. Furthermore, $X^{\mathrm{oa},n-s} \subset X\backslash Z$ by the arguments in the previous paragraph.

To complete the proof it suffices to show that we have bounded height on $(X\backslash Z)\cap C(\mathcal{H}_m,\epsilon)$, hence $X\backslash Z$ is U mentioned just above (6). Let us assume $p\in(X\backslash Z)\cap C(\mathcal{H}_m,\epsilon)$ with $m\cdot s<n$. Clearly, we have $\epsilon\leq(2n)^{-1}$. By Lemma 5 there exists $\varphi\in\Phi_s(T)$ such that if H is the irreducible component of $\ker\varphi$ containing 1, then (32) holds. Since $p\notin Z_\varphi$ we conclude that $\{p\}$ is an irreducible component of the intersection $X\cap pH$. The Arithmetic Bézout Theorem implies

$$h(p)\leq h(\{p\})\leq\deg(X)h(pH)+\deg(pH)h(X)+3n^2\deg(X)\deg(pH),\quad(36)$$

in the first inequality we used $h(\{p\})=h_2(p)$ and (8). We apply the bound $\deg(pH)\leq n^nT$ from Lemma 5 and inequality (32) to (36) to conclude that $h(p)$ is at most

$$c_9\deg(X)((T^{1-\frac{n}{ms}}+\epsilon T)(h(p)+1)+T)+n^nTh(X)+3n^{n+2}T\deg(X),$$

we recall $c_9=(2n)^{10n}$, so

$$h(p)\leq c_9\deg(X)(T^{1-\frac{n}{ms}}+\epsilon T)(h(p)+1)+c_9Th(X)+2c_9T\deg(X).\quad(37)$$

Since $c_9\geq 1$ and as $n>ms$ we may fix $T=(3c_9\deg(X))^{\frac{ms}{n-ms}}\geq 1$. With this choice we have

$$c_9\deg(X)T^{1-\frac{n}{ms}}=\frac{1}{3}\quad\text{and}\quad T\leq(3c_9)^n\deg(X)^{\frac{ms}{n-ms}}.\quad(38)$$

Furthermore, (4) implies

$$\begin{aligned}\epsilon c_9\deg(X)T&=\epsilon c_9\deg(X)(3c_9\deg(X))^{\frac{ms}{n-ms}}=\epsilon c_9(3c_9)^{\frac{ms}{n-ms}}\deg(X)^{\frac{n}{n-ms}}\\&\leq c(n)^{-1}c_9(3c_9)^{\frac{ms}{n-ms}}.\end{aligned}$$

Since $c(n)=(2n)^{30n^2}\geq(3(2n)^{10n})^n\geq(3c_9)^{\frac{n}{n-ms}}=3c_9(3c_9)^{\frac{ms}{n-ms}}$ we have

$$\epsilon c_9\deg(X)T\leq\frac{1}{3}.$$

Inequality (37) and the bound for T in (38) give us

$$\begin{aligned}h(p)&\leq 2+(3c_9)^{n+1}\deg(X)^{\frac{ms}{n-ms}}h(X)+2(3c_9)^{n+1}\deg(X)^{\frac{n}{n-ms}}\\&\leq(3c_9)^{2n}\deg(X)^{\frac{ms}{n-ms}}h(X)+3(3c_9)^{2n}\deg(X)^{\frac{n}{n-ms}}\\&\leq 3(3c_9)^{2n}\deg(X)^{\frac{ms}{n-ms}}(h(X)+\deg(X)).\end{aligned}$$

The theorem follows from $3(3c_9)^{2n}=3(3(2n)^{10n})^{2n}\leq 2^{2+4n}(2n)^{20n^2}\leq c(n)$. □

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