Analysis of the Planar Intercept and Tracking Problem by Application of Optimal Control and Singular Perturbation Theory

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presented by

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1986
To My Parents
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Literature
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Introduction and Problem Statement

The purpose of designing a controller for a dynamical system is to achieve a certain desired behaviour of the output variables. In many practical applications this design goal may be expressed in terms of a performance index. If the performance index and additional restrictions (such as the differential equations of the dynamical system, constraints on states and controls) are "reasonably" formulated the optimal control input to the system may be determined by minimizing the performance index subject to the given constraints with respect to all feasible control inputs. Necessary and sufficient conditions for optimality of a control are treated within the framework of optimal control theory [1,2]. These conditions give a guideline for the design of nonlinear optimal controllers. Unfortunately the resulting equations cannot be solved analytically in most cases but iterative numerical methods must be applied to find a solution. Approximate suboptimal control laws may be obtained by appropriate linearization of the original problem. A very powerful approach to the derivation of nearly optimal control laws is the application of singular perturbation theory [32-35, 40-42]. It is applicable if the behaviour of the dynamic system is characterized by the simultaneous occurrence of very slow and very fast processes. This allows the identification and decoupling of different time scales and the decomposition of the original problem (which often is of high order) into a sequence of low order problems which are usually much easier to solve.

As a consequence of the principle of optimality [3] the optimal control at current time depends on the current system state and the future behaviour of the system. The latter property often requires the prediction of certain system parameters. Moreover, in many cases a number of system states cannot be measured. Therefore an estimator (filter) is required to estimate the current values of the unknown quantities. Thus, for the derivation of implementable (sub-)optimal control laws three subproblems have to be solved:

- the optimal control problem
- the filtering problem
- the prediction problem

In general these problems have to be solved simultaneously. In order to obtain simple approximate solutions they are usually treated separately. In the following they are investigated for the planar intercept problem. Singular perturbation theory is applied to analyze both the optimal control problem and the filtering problem.
Statement of the Problem:

Consider the planar intercept scenario depicted in figure 1. In the following investigation the pursuer M is a short-range air-to-air missile. The evader T is typically a fighter aircraft.

In an air combat scenario the missile is launched after target acquisition by the attacking aircraft. During the initial boost phase the missile then accelerates to maximum velocity and after fuel burn out is decelerated by the aerodynamic drag. (For cost reasons throttatable engines are usually not used for short range missiles.)

In order to close in on the target the missile uses onboard sensors producing information about the missile motion and the missile-target relative geometry. These signals are preprocessed and then fed into the guidance law. The guidance algorithm computes an acceleration command $a_c$ which is input to the autopilot acting on the flipper servos in order to produce the desired acceleration.

There are two important restrictions in this scenario:

a) Since thrust control is not possible as mentioned above the commanded acceleration is always directed perpendicular to the missile axis (see figure 2).

b) After launch the missile does not receive any information from a third observer such as the launching aircraft or a ground radar station. Information is available from the missile's own sensors only (two-point-guidance). The signal flow for this situation is depicted in figure 3.

Guidance law design is of course crucial to ensure intercept. The most widely used guidance law in the described scenario is proportional navigation (PN) [1,9,15,20,21]. The outstanding advantages of PN are its simplicity and the use of only little information about the missile-target relative geometry namely $\dot{R}$ and $\dot{\varphi}$ (figure 1). These measurements are readily available with a radar seeker. If the missile is equipped with a passive seeker (infrared) only the $\dot{\varphi}$-measurement is available. In many cases PN works even with this single information because $\dot{R}$ may often be assumed constant and can be estimated. Thus PN is very easy to implement and has proven to be very effective against non- or weakly-maneuvering targets. Modern air-combat scenarios are, however, dominated by high target accelerations that result in a substantial degradation of PN performance. Therefore great efforts have been
made since the 1960s to derive new guidance laws that are effective in the case of strongly evasive target maneuvers. Most of the investigations have been carried out within the framework of optimal control theory [1,2]. An overview on research topics of the USAF associated with guidance and control of tactical missiles is given in [21,23].

It is well known that the application of optimal control theory yields control laws requiring full state information at each point of time. In view of the restricted information about the target state that is available by radar or passive seekers it becomes clear that the implementation of guidance laws based on optimal control theory is not at all straightforward. The solution of the tracking problem, i.e. the estimation of unknown states associated with the target motion is a prerequisite to the implementability of modern guidance algorithms. Essentially the tracking problem is a nonlinear filtering problem. Several approaches to this problem ranging from extended Kalman filters (EKF) to multiple model estimators have been suggested. A survey is given in [69]. For maneuvering targets most of these estimation algorithms exhibit serious stability problems. These difficulties become even harder if the estimator is based on bearing or bearing-rate-only measurements which is unavoidable if a passive seeker is used. The stability problems are due to the lack of observability under certain geometric conditions and modeling errors that arise mainly because of the unknown target dynamics. While the observability and stability problems associated with bearings-only tracking analysis have been discussed for non-maneuvering targets [62-65,69] the analysis has not yet been extended to maneuvering targets. Moreover the interaction of estimator and guidance law has not been sufficiently addressed: The guidance law may be affected by the possible divergence of the estimator. On the other hand the performance of the estimator depends heavily on the guidance law.

This thesis tries to point out the possibilities and limits of optimal control theory for the derivation of implementable guidance laws and to enhance the closely related problems of target tracking mentioned above. It turns out that in fact the solution of the tracking problem is much more important than optimality of the guidance law. The main topic will therefore be the derivation of a new adaptive tracking filter for maneuvering targets based on bearing-rate-only measurements.
I
\[ \begin{align*}
M(t) & \quad R, \text{LOS} \\
T(t) & \quad T(t) \quad \text{initial LOS}
\end{align*} \]

M: missile \quad R: range
T: target \quad LOS: line of sight
I: point of intercept \quad \phi: bearing angle

*figure 1: geometry of the planar intercept problem*

L/m
\[ \begin{align*}
\alpha & \quad \text{angle of attack} \\
\alpha_c & \quad \text{commanded acceleration}
\end{align*} \]

L: lift \quad a: missile acceleration
D: drag \quad \alpha_c: commanded acceleration
CL: missile center line \quad \alpha: angle of attack
m: missile mass \quad v: missile velocity

*figure 2: aerodynamic forces*
figure 3: signal flow of two-point guidance
1. Derivation of Guidance Laws via Optimal Control Theory

In this chapter an overview is given on how optimal control theory has been applied for the derivation of missile guidance laws. The advantages and drawbacks of the resulting guidance algorithms with respect to their practical realization are discussed.

There are essentially three approaches for guidance law derivation within the framework of optimal control theory:

a) Pontryagin's minimum principle [1,2]
b) dynamic programming [1,3]
c) differential game theory [1,3]

Only the first approach will be discussed here in detail. Many of the conclusions, however, apply to the others as well.

1.1 Review of Pontryagin's Minimum Principle

Consider the nonlinear dynamic system given by the following set of first order differential equations:

\[
\dot{x} = f(x,u,t) \in \mathbb{R}^n \quad (1.1.1)
\]

subject to the initial conditions

\[
x(t_0) = x_0 \in \mathbb{R}^n \quad (1.1.2)
\]

and the final conditions

\[
r[x(t_f)] = 0 \in \mathbb{R}^k, \quad k \leq n \quad (1.1.3)
\]

with

- \(x \in \mathbb{R}^n\) state vector
- \(u \in \mathbb{R}^m\) control vector
- \(t \in I = [t_0, t_f] \subset \mathbb{R}\) time (independent variable)
- \(t_0, t_f\) initial, final time
- \(\dot{\cdot} = \frac{d}{dt}\) differentiation with respect to time
The control vector is restricted as follows: \( u \in G \subset \mathbb{R}^m \) for all \( t \) in \( I \)  

\[
\tag{1.1.4}
\]

The optimal control problem consists of determining the control \( u(t) \) in \( I \) such that the performance index

\[
J[x(.), u(.) :] = \int_{t_0}^{t_f} L(x(t), u(t), t) \, dt + \Phi(x(t_f)) \quad (1.1.5)
\]

is minimized subject to the constraints (1.1.1) to (1.1.4). \( L \) and \( \Phi \) denote scalar real valued penalty functions which are continuously differentiable with respect to all their arguments.

If \( u(t) \) is the optimal control satisfying (1.1.1) to (1.1.5) the following necessary conditions hold for \( t \in I \):

\[
\tag{1.1.6}
\]

with

\[
H = L(x,u,t) + \lambda^t f(x,u,t) \in \mathbb{R} \quad \text{Hamiltonian} \quad (1.1.7)
\]

\[
\lambda^t = - \frac{\partial H}{\partial x} = - \frac{\partial L}{\partial x}(x,u,t) - \lambda^t \frac{\partial f}{\partial x}(x,u,t) \in \mathbb{R}^n \quad (1.1.8)
\]

\[
\lambda^t(t_f) = \frac{\partial \Phi}{\partial x}(x(t_f)) + s^t \frac{\partial r}{\partial x}(x(t_f)) \quad \text{transversality condition} \quad (1.1.9)
\]

where \((\cdot)^t\) denotes the transposed.

\( \lambda \) is called the adjoint vector, and is an \( n \)-tuple of Lagrange multipliers. For a more general formulation of the problem and sufficient conditions for optimality see [1,2]. Aspects of implementation of (1.1.6) are discussed in the next section.
1.2 Application of Optimal Control to Missile Guidance: Basic Problems

Theoretically optimal guidance laws can be derived by stating the intercept problem as an optimal control problem. The guidance law is then given by (1.1.6). This equation can in general, however, only be evaluated by simultaneously solving (1.1.1), (1.1.6) and (1.1.8) subject to the boundary conditions (1.1.2), (1.1.3) and (1.1.9). Usually this two-point-boundary-value problem (TPBVP) has to be solved iteratively by numerical methods [1,4-8].

In principle equation (1.1.6) can be implemented as a feedback law by interpreting t as initial time, x(t) as associated initial condition and continuously solving the TPBVP on \([t, t_f]\) in order to determine the optimal control \(u(x(t))\). There are three severe drawbacks of this approach:

i) Real time solution of TPBVPs is not realistic due to the severe numerical difficulties that arise with this type of problems in most cases.

ii) The control law (1.1.6) requires the knowledge of all components of the state vector x. In many cases, however, only a part of the states can be measured. This is especially true for the intercept problem where little information about the target state is available.

iii) According to (1.1.6) \(u(t)\) depends on \(\lambda(t)\) which in turn depends via (1.1.9) on the final state \(x(t_f)\). This means that the optimal control strategy depends on the future behaviour of the system. As a consequence the determination of an optimal control law for an interceptor requires the knowledge of the future target maneuver. The latter is of course unknown. Therefore guidance law derivation via the minimum principle is always based on assumptions about the target behaviour and guidance law performance may be heavily degraded by deviations from the assumed nominal conditions.

In summary it can be stated that optimal guidance laws are not implementable because

- the associated TPBVP cannot be solved in real time
- the required full state information is not available
- the future target maneuver is unknown.
Remarks:
The difficulties ii) and iii) arise in the dynamic programming approach as well whereas i) is replaced by the need of solving the Hamilton-Jacobi-Bellman partial differential equation [1,2,3]. The application of differential game theory avoids iii) by determining an interceptor guidance law based on an optimal evader strategy [1,3,10,18,26,28]. This problem is even more difficult to solve than the TPBVP associated with the minimum principle.

The following possibilities exist to overcome the above mentioned difficulties:

- In order to avoid the on-line solution of a TPBVP, linearization methods are used that allow the derivation of guidance laws that can be represented in closed form or at least can be evaluated numerically in real time. Clearly these guidance laws are suboptimal.

- The problem of only partially measurable system states can be overcome by the application of estimators. As mentioned earlier the development of practicable estimators is crucial to the implementability of modern guidance laws and will be discussed in later sections.

- The determination of the future target maneuver is a prediction problem. It can be "solved" by assuming a target maneuver or by extrapolation of information gathered about the past maneuver. A good extrapolation is possible across several target time constants and therefore useful during the endgame. Predictions across long periods of time can only be based on assumptions.

In the following sections approximate methods to derive suboptimal implementable guidance laws are discussed.
1.3 **Guidance Laws Based on the LQ-Method**

In this section a class of guidance laws based on a linear approximation of the intercept dynamics and a quadratic cost criterion $J$ is presented.

1.3.1 **Review of the LQ-Method**

(see [1,2])

Consider the linear plant equation

\[
\begin{align*}
\dot{x} &= A(t) x(t) + B(t) u(t) & (1.3.1a) \\
x(t_0) &= x_0 & (1.3.1b) \\
x &\in \mathbb{R}^n, \, u &\in \mathbb{R}^m & (1.3.1c)
\end{align*}
\]

and the quadratic performance index

\[
J = \frac{1}{2} x^t(\tau_f)F x(\tau_f) + \frac{1}{2} \int_{\tau_0}^{\tau_f} [x^t(t)Q(t)x(t) + u^t(t)R(t)u(t)] \, dt & (1.3.2)
\]

with

\[
\begin{align*}
A(t) &\in \mathbb{R}^{n \times n} : \text{system matrix} \\
B(t) &\in \mathbb{R}^{n \times m} : \text{control matrix} \\
F, Q(t) &\in \mathbb{R}^{n \times n} : \text{symmetric positive semidefinite weighting matrices penalizing the states } x \\
R(t) &\in \mathbb{R}^{m \times m} : \text{symmetric positive definite weighting matrices penalizing the controls } u
\end{align*}
\]

For the ease of notation the time dependency will be dropped in the sequel.

Solving (1.1.6) to (1.1.9) for (1.3.1) and (1.3.2) yields a linear optimal control law of the form:

\[
u^* = -R^{-1}B^tKx & (1.3.3)
\]
where the (n×n)-matrix $K$ is positive definite and solves the Riccati-equation

$$\dot{K} = -A^tK - KA + KBR^{-1}B^tK - Q$$  \hspace{1cm} (1.3.4)

subject to the final condition

$$K(t_f) = F$$  \hspace{1cm} (1.3.5)

Discussion:

An important feature of this solution is that the Riccati equation (1.3.4) is decoupled from the state $x$. Hence $K(t), t \in [t_0, t_f]$, can be calculated and stored off-line by backward integration of (1.3.4/5). With $K(t)$ known $u(t)$ can easily be evaluated. The implementation of (1.3.3) requires the knowledge of all states $x$.

1.3.2 Application to the Planar Intercept Problem

In order to apply the LQ-method to the planar intercept problem the dynamic equations have to be cast in linear form and a quadratic performance index has to be chosen. For this purpose consider the situation depicted in figure 1.3.1.

In the cartesian coordinate system $(x,y)$ the kinematical equations of the intercept problem are linear:

$$\Delta \dot{x} = v_{Tx} - v_x = \Delta v_x$$  \hspace{1cm} (1.3.6a)

$$\Delta \dot{y} = v_{Ty} - v_y = \Delta v_y$$  \hspace{1cm} (1.3.6b)

$$\Delta \ddot{x} = a_{Tx} - a_x = \Delta \dot{v}_x$$  \hspace{1cm} (1.3.6c)

$$\Delta \ddot{y} = a_{Ty} - a_y = \Delta \dot{v}_y$$  \hspace{1cm} (1.3.6d)

A necessary and sufficient condition for intercept at final time is:

$$\Delta x(t_f) = 0$$  \hspace{1cm} (1.3.7a)

$$\Delta y(t_f) = 0$$  \hspace{1cm} (1.3.7b)

Problems arise in the formulation of the missile autopilot dynamics i.e. the transfer function relating the commanded acceleration $a_c$ to the actual missile acceleration $a$. It has already been mentioned that due to the absence of thrust control the
commanded acceleration is perpendicular to the missile center line (figure 2). For small angles of attack it may be assumed that

$$a_c \perp v$$

(1.3.8)

In other words only the acceleration component $a_L$ (see figure 1.3.1) can be controlled.

*figure 1.3.1: planar intercept geometry*
From figure 1.3.1 follows:

\[ a_x = -a_L \sin \gamma - a_D \cos \gamma \quad (1.3.9) \]
\[ a_y = a_L \cos \gamma - a_D \sin \gamma \quad (1.3.10) \]
\[ \dot{\gamma} = \frac{a_L}{v} \quad (1.3.11) \]
\[ \dot{v} = -a_D \quad (1.3.12) \]

In case of a "perfect" autopilot \( a_L \) may be replaced by its commanded value \( a_c \). However, \( a_D \) is a highly nonlinear function of the missile velocity and the angle of attack (see appendix 1.1).

In order to obtain a simple linear model of the missile dynamics it is assumed that \( a_y \) can be controlled rather than \( a_L \). This assumption is true for small heading angles \( \gamma \). For the moment it is further assumed that the time-to-go

\[ t_{go} = t_f - t \quad (1.3.13) \]

is known. Equation (1.3.7b) is then a sufficient condition for intercept and the kinematic equations in \( x \)-direction (1.3.6a/c) become obsolete. Note that in principle (1.3.6a/c) and (1.3.7a) could be used to determine \( t_{go} \). The equations of the planar intercept problem can now be stated as follows:

\[ \Delta \dot{y} = v_T \gamma - v_y \quad (1.3.14a) \]
\[ \Delta \ddot{y} = a_T \gamma - a_y \quad (1.3.14b) \]
\[ \dot{a}_y = f_A(u) \quad (1.3.14c) \]
\[ \ddot{a}_y = f_T(t) \quad (1.3.14d) \]

where \( f_A \) designates a linear function approximating the dynamics of the missile autopilot and \( f_T \) describes the target maneuver. \( \Delta y \) is sometimes called missile-target separation.
By specifying $f_A$, a quadratic performance index $J$ and making assumptions about the unknown target maneuver $f_T$ various guidance laws of the form

$$a_c = u(\Delta y, \Delta \dot{y}, a_y, a_{Ty}, t_{go}) \quad (1.3.15)$$

can be derived.

Typical models for $f_A$ are:

$$a_y = u \quad \text{(perfect autopilot)} \quad (1.3.16a)$$

$$f_A = -\frac{1}{T_A}(a_y - u), \ T_A = \text{const.} \quad (1.3.16b)$$

For the target motion one usually chooses:

$$a_{Ty} = 0 \quad (1.3.17a)$$

or

$$f_T \equiv 0, \ a_{Ty} = \text{const.} \neq 0 \quad (1.3.17b)$$

The basic goal of the control (1.3.15) is to null or at least minimize the miss distance $\Delta y(t_f)$. At the same time one is interested in keeping low the control effort measured by

$$J = \int_{t_0}^{t_f} u^2(t) \, dt \quad (1.3.18)$$

in order to minimize the aerodynamic drag losses. Thus a typical optimal control formulation of the intercept problem is:

Find the control $u$ minimizing $J$ subject to the constraints (1.3.14) with

$$J = c_1 \Delta y^2(t_f) + c_2 \int_{t_0}^{t_f} u^2(t) \, dt \quad (1.3.19)$$

$$c_1, c_2 = \text{const.} \quad (1.3.20)$$

Guidance laws based on (1.3.19) have been derived in [1,12,17,21,24].
The Riccati equation associated with (1.3.14/19) can be solved analytically and the resulting control laws are of the form:

\[ u = k_1 \Delta y + k_2 \Delta \dot{y} + k_3 a_y + k_4 a_T y \]  
(1.3.21)

with

\[ k_i = k_i(t_{go}, T_A), \quad i=1, 2, 3, 4 \]  
(1.3.22)

Optimization with respect to (1.3.19) does not null the miss distance because the penalty term on \( u \) causes a trade-off between miss distance and control effort. The performance index

\[ J = \frac{1}{2} \int_{t}^{t_f} u^2(t) \, dt \]  
(1.3.23)

with the additional constraint

\[ \Delta y(t_f) = 0 \]  
(1.3.24)

ensures zero miss with minimum control effort. However, the gains \( k_i \) tend to infinity as \( t_{go} \) approaches zero [12,27].

Therefore the additional constraint

\[ |u| \leq u_{\text{max}} \]  
(1.3.25)

is added in [25,28]. Note that due to (1.3.25) the minimization is not an LQ-Problem any more. Other approaches differing from the LQ-method are given in [15,16] with

\[ J = |\Delta y(t_f)| \]  
(1.3.26)

and (1.3.25).

A more detailed literature overview is given in [23,29]. Comparisons of classical and modern guidance laws can also be found in [27,29,31].
1.3.3 Considerations on Guidance Law Implementation

The general form of the guidance laws discussed in the last subsection is

\[ u = u(\Delta y, \Delta \dot{y}, a_y, a_{Ty}, t_{go}) \]  \hspace{1cm} (1.3.27)

The implementation of (1.3.27) requires the knowledge of all state variables and \( t_{go} \). With the exception of \( a_y \) which can be measured by accelerometers there is no means of directly measuring the other variables. In the most favourable case measurements for \( R, \dot{R}, \phi \) (radar seeker), and \( \varphi \) (inertial or strapdown navigation) are available. In the worst case only \( \dot{\varphi} \) can be measured. With \( R, \dot{R}, \phi, \) and \( \varphi \) known \( \Delta y \) and \( \Delta \dot{y} \) can easily be determined. In the case of unknown \( \varphi \) (no inertial angular measurements) a Kalman filter was designed in [15] for the estimation of \( \Delta y \), \( \Delta \dot{y} \), and \( a_{Ty} \) based on measurements of \( \dot{\varphi} \) and accurate knowledge (noise free measurement) of \( R \) and \( \dot{R} \).

Other approaches will be discussed later. Most frequently the unknown variables are approximated by making the following assumptions:

i) \( |\Delta y| \ll R \)  \hspace{1cm} (1.3.28a)
ii) \( \dot{R} \approx \text{constant} \)  \hspace{1cm} (1.3.28b)

From (1.3.28) follows:

\[ \varphi = \frac{\Delta y}{R} \]  \hspace{1cm} (1.3.29a)

\[ t_{go} \approx -\frac{R}{\dot{R}} \]  \hspace{1cm} (1.3.29b)

With (1.3.29) \( u \) from (1.3.27) may be expressed in terms of polar coordinates measured by the target tracker:

\[ u = u(R, \dot{R}, \varphi, \dot{\varphi}, \hat{a}_{Ty}) \]  \hspace{1cm} (1.3.30)

Here \( \hat{a}_{Ty} \) has been replaced by \( \hat{a}_{Ty} \) to indicate that an estimated or assumed value of the target acceleration is used. In some cases (small LOS-rate, present LOS \( \approx \) reference LOS) further simplifications are possible allowing to drop the dependency on \( R \) and \( \varphi \) leaving:
\[ u = u(\dot{R}, \dot{\varphi}, \dot{a}_{Ty}) \]  \hspace{1cm} (1.3.31)

After replacing \( \dot{R} \) by an estimate \( \hat{R} \) (1.3.31) becomes a guidance law suitable for implementation with a passive seeker.

1.3.4 Some General Conclusions on Guidance Law Performance

The majority of guidance laws based on (1.3.14-26) have proven to be quite efficient although implementable versions are based on severe assumptions as discussed in the last subsection. Deviations from these assumptions affect optimality but not necessarily stability of the guidance loop. Heuristically this fact becomes clear by observing that a sufficient condition for intercept is:

\[ \dot{\varphi} = 0 \] \hspace{1cm} (1.3.32a)
\[ \dot{R} < 0 \] \hspace{1cm} (1.3.32b)

(1.3.32a) guarantees that the missile-target relative velocity is always directed along the LOS which ensures intercept.

For non-maneuvering targets (1.3.32) determines the time-optimal interceptor trajectory. In case of a target maneuver (1.3.32) still guarantees intercept but the associated interceptor trajectory is neither time-optimal nor does it minimize the control effort. Since most guidance laws try to at least approximately establish (1.3.32a) (simply by \( \dot{\varphi} \)-feedback) and differ in compensation terms for autopilot dynamics, target dynamics etc. their relative robustness is plausible. There are, however, some important experiences that should not be left unmentioned:

a) Guidance law performance can be significantly improved by taking into account the autopilot dynamics according to (1.3.16b) rather than using the simple model (1.3.16a). On the other hand these guidance laws are very sensitive to estimation errors in \( T_A, R \) and \( t_{go} \) [25,27]. Note that \( T_A \) is not a constant as assumed for guidance law derivation. Inspection of (1.3.10-12), (1.3.14c) and (1.3.16b) shows that \( T_A \) depends on the missile heading angle \( \gamma \), the missile velocity \( v \) and angle of attack \( \alpha \) since it is the component \( a_L \) that is controlled by the autopilot rather than \( a_T \). In a more sophisticated model one could replace the time constant \( T_A \) by a function
This function should be known in advance for all \( t \in [t_0, t_f] \) in order to solve the Riccati equation (1.3.4). In addition the \( t_{go} \)-estimate given by (1.3.29b) is far from accurate especially during the boost phase with large axial accelerations that lead to strong violations of assumption (1.3.28b). The effect of improved \( t_{go} \)-estimation is discussed in [24].

In view of these uncertainties the use of a model like (1.3.16b) seems questionable except for short \( t_{go} \), i.e. during the endgame.

b) Guidance laws derived by taking into account limited controls according to (1.3.25) result in much higher feedback gains \( k_i \) than guidance laws based on a quadratic control penalty term without saturation. This again leads to high sensitivity to errors in the design assumptions especially \( T_A \) and \( t_{go} \) [15,16]. Moreover high gains degrade guidance law performance in the presence of noise [25].

c) A substantial improvement of guidance law performance can be achieved by taking into account the target acceleration \( a_T \) [20,25] rather than assuming a zero acceleration nominal target trajectory.

Based on these experiences a guidance law that will be referred to as Extended Proportional Navigation (PE) will be derived in the next section.

### 1.3.5 Extended Proportional Navigation (PE)

In this section a guidance law designated as extended proportional navigation (PE) is derived by applying the previously discussed LQ-approach. The guidance law is compared with proportional navigation (PN) and in later sections with other nonlinear suboptimal guidance laws. It turns out that PE lends itself in some "optimal" way (to be specified later) for implementation in connection with an extended Kalman filter that solves the tracking problem.
1.3.5.1 Derivation of Guidance Law

Consider the scenario depicted in figure 1.3.2. At time $t_0$ the missile and the target are in their respective initial positions $M(t_0)$ and $T(t_0)$. The direction of the initial LOS $M(t_0)T(t_0)$ determines the orientation of a nonrotating missile-fixed cartesian coordinate system ($x,y$). Intercept occurs in the collision point $I$ at final time $t_f$. The target maneuvers along the path $T(t_0)I$.

*figure 1.3.2: intercept geometry associated with PE*
The optimal missile path with respect to (1.3.19) or (1.3.23/24) is a straight zero-maneuver trajectory (broken line in figure 1.3.2). The determination of the associated collision course $\gamma_{cs}$ requires a priori knowledge of the target maneuver which is usually unknown. Therefore (1.3.32) will be exploited to guarantee intercept. From figure 1.3.1 follows:

$$\dot{\varphi} = \frac{v_T \sin \gamma_{Ts} - v \sin \gamma_s}{R}$$  \hspace{1cm} (1.3.34)

with

$$\gamma_{Ts} = \gamma_T - \varphi$$  \hspace{1cm} (1.3.35)

$$\gamma_s = \gamma - \varphi$$  \hspace{1cm} (1.3.36)

(1.3.32) and (1.3.34) yield:

$$\sin \gamma_s = \frac{v_T}{v} \sin \gamma_{Ts} \quad \forall \ t \in [t_0, t_f]$$  \hspace{1cm} (1.3.37)

For a non-maneuvering target $\gamma_s$ is identical to the collision course $\gamma_{cs}$. In case of a maneuvering target (1.3.37) results to the missile trajectory $M(t_0)M(t_f)$ depicted in figure 1.3.2.

From (1.3.32) follows:

$$\ddot{\varphi} = 0$$  \hspace{1cm} (1.3.38)

Differentiation of (1.3.34) with respect to time renders:

$$\dddot{\varphi} = \frac{a_T y - a_y}{R} - \frac{2}{R} \frac{\dddot{R}}{R} \varphi$$  \hspace{1cm} (1.3.39)

where $a_T y, a_y$ denote the target and missile acceleration components perpendicular to the instantaneous LOS (figure 1.3.2).

(1.3.32/38/39) result in the nominal control

$$a_{y}^N = a_T y$$  \hspace{1cm} (1.3.40)

Obviously the steering law (1.3.40) is a target maneuver compensation associated with the nominal missile path defined by (1.3.32). In order to keep the missile on its nominal path in the presence of disturbances a guidance law of the form
\[ a_y = a_N^y + \Delta u \]  \hspace{1cm} (1.3.41)

is sought. \( \Delta u \) is determined in such a way that (1.3.32) is stabilized. Note that by (1.3.41) a perfect autopilot response according to (1.3.26a) is assumed.

Consider a deviation from the nominal path indicated by the missile position \( M'(t) \) in figure 1.3.2. This deviation produces a missile-target separation \( \Delta y(t) \). Note that deviations directed along the LOS don't affect the nominal condition (1.3.32) and therefore don't have to be considered. It is sufficient to control \( \Delta y \). For this purpose the state vector

\[ z^t = -(\Delta y, \Delta y) \]  \hspace{1cm} (1.3.42)

is formed. With (1.3.41) and (1.3.14b) \( z \) satisfies the simple equation

\begin{align*}
\dot{z}_1 &= z_2(t) \hspace{1cm} (1.3.43a) \\
\dot{z}_2 &= \Delta u(t) \hspace{1cm} (1.3.43b) \\
t \in [t_0, t_f] \hspace{1cm} (1.3.43c)
\end{align*}

In order to ensure collision with minimum additional control effort the following optimization problem is solved:

Minimize

\[ J = \frac{1}{2} \int_{t_0}^{t_f} [\Delta u(t)]^2 \, dt \]  \hspace{1cm} (1.3.44)

subject to

\[ z_1(t_f) = 0 \]  \hspace{1cm} (1.3.45)

and (1.3.43).

The problem is solved by application of the minimum principle (see section 1.1). The Hamiltonian is:

\[ H = \frac{1}{2} \Delta u^2 + \lambda_1 z_2 + \lambda_2 \Delta u \]  \hspace{1cm} (1.3.46)
with the adjoint variables satisfying

\[
\dot{\lambda}_1 = - \frac{\partial H}{\partial z_1} = 0 \Rightarrow \lambda_1 = \text{const.} \tag{1.3.47}
\]

\[
\dot{\lambda}_2 = - \frac{\partial H}{\partial z_2} = - \lambda_1 \tag{1.3.48}
\]

With (1.3.44/45) the transversality condition (1.1.9) reduces to:

\[
\lambda_2(t_f) = 0 \tag{1.3.49}
\]

The optimal control is \( H \)-minimizing and from (1.3.46) follows:

\[
\Delta u = -\lambda_2 \tag{1.3.50}
\]

The solution of (1.3.47 -49) is:

\[
\lambda_2(t) = (t_f - t) \lambda_1, \quad t \in [t_0, t_f] \tag{1.3.51}
\]

and with (1.3.50) the optimal control becomes:

\[
\Delta u(t) = -(t_f - t) \lambda_1 \tag{1.3.52}
\]

where the constant \( \lambda_1 \) is still unknown but can be determined by exploiting (1.3.45).

Integration of (1.3.43) with \( \Delta u \) from (1.3.52) yields:

\[
z_2(t) = z_2(t_0) - \frac{1}{2} \lambda_1 (t_f - t_0)^2 + \frac{1}{2} \lambda_1 (t_f - t_0)^2 \tag{1.3.53}
\]

\[
z_1(t) = z_1(t_0) + \frac{1}{6} \lambda_1 (t_f - t_0)^3 + z_2(t_0) (t - t_0) + \frac{1}{2} \lambda_1 (t_f - t_0)^2 (t - t_0) - \frac{1}{6} \lambda_1 (t_f - t_0)^3 \tag{1.3.54}
\]

With (1.3.45) one obtains:

\[
z_1(t_f) = 0 = z_1(t_0) + z_2(t_0) (t_f - t_0) - \frac{1}{3} \lambda_1 (t_f - t_0)^3 \tag{1.3.55}
\]
After replacing $t_0$ by the current time $t$ and solving for $\lambda_1$ one finally arrives at:

$$
\lambda_1 = 3 \frac{z_1(t) + z_2(t) t_{go}}{t_{go}^3}
$$

(1.3.56)

with

$$
t_{go} = t_f - t \quad \text{time-to-go}
$$

(1.3.57)

(1.3.56) in (1.3.52) yields the optimal control

$$
\Delta u(z_1, z_2, t_{go}) = -3 \frac{z_1(t) + z_2(t) t_{go}}{t_{go}^2}
$$

(1.3.58)

Using the assumptions (1.3.28) the following approximations hold (figure 1.3.2):

$$
\varphi \approx -\frac{z_1}{R}
$$

(1.3.59)

$$
\dot{\varphi} \approx -\frac{z_2}{R} + \frac{\dot{R} z_1}{R^2}
$$

(1.3.60)

$$
R \approx -\dot{R} t_{go}
$$

(1.3.61)

Substituting (1.3.61) in (1.3.60) yields:

$$
\dot{\varphi} = \frac{z_1(t) + z_2(t) t_{go}}{\dot{R} t_{go}^2}
$$

(1.3.62)

and with (1.3.58) one ends up with

$$
\Delta u = -3 \dot{R} \dot{\varphi}
$$

(1.3.63)

Remark: (1.3.63) is the well known proportional navigation (PN) the general form of which is

$$
\text{PN:} \quad \Delta u = -\lambda_0 \dot{R} \dot{\varphi}
$$

(1.3.64)

$\lambda_0$ : navigation constant
The optimal value of $\lambda_0$ according to (1.3.63) was first derived in [1]. After substituting (1.3.64) in (1.3.41) the complete guidance law becomes:

$$\text{PE: } u = a_{Ty} - \lambda_0 \dot{R}\dot{\varphi} \tag{1.3.65}$$

Discussion:

a) PE is a combination of target maneuver compensation and PN. For non-maneuvering targets it is identical to PN. This is desirable since PN has proven to be very effective in this case. The only but severe difficulty in implementing (1.3.65) is the need to estimate $a_{Ty}$.

b) An important feature of PE is that this guidance law is not based on any assumptions about the future target maneuver but uses \textit{instantaneous} information only. Therefore the sensitivity problems associated with the extrapolation of wrong system parameters into the future in order to solve the TPBVP (1.1.1-1.1.9) are avoided (see discussion in section 1.3.4).

c) By substituting (1.3.65) into (1.3.29) one obtains:

$$\ddot{\varphi} = (\lambda_0 - 2)\frac{\dot{R}\dot{\varphi}}{R} \tag{1.3.66}$$

It follows immediately that (1.3.32) is stable only for $\lambda_0 \geq 2$ (note that $\dot{R} < 0$). This is a well known result from PN [9].

d) A derivation of (1.3.65) can also be found in [20]. Other versions based on an assumed constant target acceleration are given in [12,25,27] but here the sensitivity problems mentioned in b) occur.

In the next subsection some simulation results comparing PN and PE are presented.
1.3.5.2 Comparison: PE versus PN

In this section simulation results comparing the performance of PE and PN are discussed. First some remarks on the implementation of (1.3.65) are necessary.

Implemented version of PE/PN

According to (1.3.41) \( u \) is the commanded acceleration perpendicular to the LOS. As mentioned earlier the only acceleration component that can actually be controlled is \( a_L \) (small angles of attack assumed, see figures 2 and 1.3.1). Therefore a commanded value for \( a_L \) denoted by \( a_c \) will be calculated from \( u \). Neglecting the aerodynamic drag \( a_D \) (figure 1.3.1) \( a_c \) can be determined from figure 1.3.3. It follows:

\[
a_c = \frac{u}{\cos \gamma} = \frac{1}{\cos \gamma} \left( a_{Ty} - \lambda_0 \dot{R} \dot{\varphi} \right)
\]  

(1.3.67)

In the absence of an inertial measurement unit \( \gamma \) is unknown however. A good approximation for \( \gamma \) is the lead angle \( \gamma_L \) (figure 1.3.2) which is usually calculated prior to launch. A preferable approach is replacing \( \gamma \) by a design parameter \( \gamma_L \) that accounts for the intercept geometry given by the collision course \( \gamma_c \) (figure 1.3.2) and drag losses due to \( a_D \). The final version of PE as implemented in the simulation program then becomes:

PE:

\[
a_c = \frac{1}{\cos \gamma_L} \left( a_{Ty} - \lambda_0 \dot{R} \dot{\varphi} \right)
\]  

(1.3.68)

\[ \text{figure 1.3.3: commanded acceleration} \]
By omitting $a_{T y}$ in (1.3.68) one obtains the implemented version of PN. Experience from many simulation runs shows that the value

$$\cos \gamma_L = 0.8 \quad (1.3.69)$$

produces good results for all investigated intercept scenarios. This value is therefore used for all subsequent simulations.

**Simulations**

The purpose of the following simulations is to illustrate the effect of target maneuver compensation on guidance law performance. The target maneuvers considered here are constant speed maneuvers i.e. the target acceleration $a_T$ is directed perpendicular to the velocity vector $v_T$ (figure 1.3.4). During the initial phase $t \in [t_0,t_e]$ the target moves unaccelerated. For $t > t_e$ the target acceleration changes from zero to a constant value $a_{T0}$ according to equation (1.3.70) and figure 1.3.5:

$$a_T = 0 \quad t_0 \leq t \leq t_e \quad (1.3.70a)$$
$$a_T = a_{T0} \left[ 1 - \exp\left(\frac{t - t_e}{t_v}\right) \right] \quad t > t_e \quad (1.3.70b)$$

$t_f$ is the time of closest approach of missile and target. The associated distance is called miss distance and will be denoted by $R_f$. Due to the missile’s warhead effectiveness the target is considered to be hit if

$$R_f < R_{\text{max}} = 5\text{m} \quad (1.3.71)$$

For $R_f > R_{\text{max}}$ the missile missed the target. $R_f$ is a measure of guidance law performance and will be investigated as a function of

$$\Delta t = t_f - t_e \quad (1.3.72)$$

$\Delta t$ is the time that is left for the missile to react to the target’s evasive maneuver. The function $R_f(\Delta t)$ is therefore suitable to reveal the effect of target maneuver compensation for maneuvers of type (1.3.70).
Two engagement scenarios will be investigated. For both scenarios the following parameters are fixed:

**target maneuver:**

\[ a_{T0} = 6g \]

\[ t_v = 0.1s \]
intercept geometry: \[ R_0 = 3.5 \text{ km}, \quad h = 10 \text{ km} \]
\[ v_0 = 0.9 \text{ Ma}, \quad v_T = 0.9 \text{ Ma} \]

The intercept scenarios are defined by the remaining parameters \( y_0 \) and \( y_{T0} \):

- scenario 1: \( y_{T0} = 90^\circ, y_0 = 20^\circ \)
- scenario 2: \( y_{T0} = 0^\circ, y_0 = 0^\circ \)

The functions \( R_f(\Delta t) \) obtained for the guidance laws PN and PE are depicted in figures 1.3.6 and 1.3.7 for scenario 1 and 2, respectively.

Discussion of results:

Figures 1.3.6 and 1.3.7 show that \( R_f \) is nearly independent of \( \Delta t \) for the guidance law PE and \( \Delta t > 0.8 \text{ s} \). A sharp rise of \( R_f \) occurs for target evasive maneuvers shortly before intercept (\( \Delta t < 0.8 \text{ s} \)). Here the missile has no time to correct its path due to its dynamic lags.

In contrast to PE the results of PN show a strong dependence on \( \Delta t \). For small \( \Delta t \) PN produces the same miss distance as PE because the missile does not maneuver any more in both cases. For scenario 1 all results lie well within the hit range \( R_{\text{max}} \). However, in scenario 2 PN produces large miss distances (see table 1.3.1) when used with the "optimal" navigation constant \( \lambda_0 = 3 \) (see equation 1.3.63) that was used for all simulations of scenario 1. Only after choosing \( \lambda_0 = 6 \) PN yields acceptable miss distances with a sharp rise in \( R_f \) for small \( \Delta t \).

The guidance law behaviour for both scenarios indicates that PN is very sensitive to both intercept scenario and target maneuver whereas PE behaves rather indifferently. The sensitivity of PN is easily understood by substituting

\[ \text{PN:} \quad a_y = \Delta u = -\lambda_0 \dot{R} \dot{\varphi} \quad (1.3.73) \]

into (1.3.39) resulting in:
From (1.3.74) it becomes clear that any target maneuver $a_{Ty}$ perpendicular to the LOS destabilizes the nominal condition (1.3.32a) resulting in large miss distances. This can be prevented by choosing a high navigation constant $\lambda_0$ in order to keep $\dot{\phi}$ small as was done in scenario 2. In scenario 1 PN with $\lambda_0 = 3$ produces low miss distances despite the target maneuver. Here the target acceleration is mainly directed along the LOS. Hence, $a_{Ty}$ is small and the performance of PN is not seriously affected (figure 1.3.8). On the other hand the target performs a strong maneuver perpendicular to the LOS in scenario 2 (figure 1.3.9) and therefore degrades PN performance. This is why in practice $\lambda_0$ is selected within the range of $\sim 2.5$ to $\sim 8$ according to the intercept scenario. Target maneuver compensation eliminates these sensitivity problems and allows for the use of a navigation constant that is independent of the engagement. Moreover the feedforward of $a_{Ty}$ results in fast reaction to target evasive maneuvers reducing the miss distance especially for small $\Delta t$.

$$\ddot{\phi} = \frac{a_{Ty}}{R} + (\lambda_0 - 2) \dot{R} \dot{\phi}$$

(1.3.74)

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<th>$\Delta t$ [s]</th>
<th>$R_f$ [m]</th>
</tr>
</thead>
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<tr>
<td>0.14</td>
<td>0.23</td>
</tr>
</tbody>
</table>

*table 1.3.1: miss distances of PN with $\lambda_0 = 3$ in scenario 2*
figure 1.3.6: miss distances in scenario 1

figure 1.3.7: miss distances in scenario 2
Figure 1.3.8: Typical missile and target flight path in scenario 1

Figure 1.3.9: Typical missile and target flight path in scenario 2
1.3.6 Summary

In this section an overview about the derivation of guidance laws based on LQ-design and related approaches was given. The approximation of the intercept dynamics by linear time invariant equations is common to all these approaches enabling one to solve the resulting optimal control problem analytically in most cases. Thus guidance laws are obtained in closed form. Most of them may be viewed as extensions of PN.

One version of extended proportional navigation taking into account the target maneuver by a simple compensation term was derived. A simulation study revealed that target maneuver compensation leads to substantial improvements of guidance law performance and makes adaption of the navigation constant $\lambda_0$ to the engagement scenario obsolete. The main obstacle to the implementation of PE is the problem of target maneuver estimation. This topic will be addressed in chapter 2.

In the following sections guidance laws derived by application of singular perturbation theory will be investigated. These guidance laws are a useful reference to judge the performance of PN and PE. Moreover singular perturbation theory gives a deep insight into the structure of modern guidance laws and reveals the advantages and limits of optimal control theory in this field.
1.4 Guidance Laws based on Singular Perturbation Theory

Perturbation methods are an important tool for the analysis of ordinary and partial differential equations. These methods are applicable if the problem under consideration exhibits "small" perturbation terms which, after adequate scaling, manifest themselves by the occurrence of one or several perturbation parameters $\varepsilon \ll 1$. In many cases it is possible to construct expansions in terms of $\varepsilon$ that approximate the exact solution with reasonable accuracy. The importance of perturbation methods lies in the fact that the calculation of these approximations is often much easier than the determination (if possible at all) of an exact solution and leads to a substantial reduction of the computational effort if the problem is solved numerically.

The question of existence and properties of expansions associated with perturbed problems are treated in perturbation theory. An introduction to perturbation methods is given in [32–34]. Applications to control theory are summarized in [35] and [43].

A special class of perturbed problems are singularly perturbed problems. They are of special interest here. A well known approach to solve this type of problems is the method of matched asymptotic expansions (MAE) [32,33,34,36]. The MAE-method will be applied here to obtain approximate solutions to the optimal control formulation of the planar intercept problem. First the basic definitions and ideas of this method will be introduced.

1.4.1 Basic Definitions of Perturbation Theory

The following discussion will be restricted to the investigation of ordinary differential equations of the type

$$\dot{z} = h(z, t, \varepsilon) \quad (1.4.1)$$

with

- $z \in \mathbb{R}^n$ dependent variables
- $t \in \mathbb{R}$ independent variable (time)
- $\varepsilon \in \mathbb{R}$ "small" perturbation parameter

$$\left(\begin{array}{c}
\dot{z} \\
\dot{t}
\end{array}\right) = \left(\begin{array}{c}
h(z, t, \varepsilon) \\
1
\end{array}\right)$$
With the assumption that (1.4.1) has a unique solution for a given initial condition

\[ z(t_0) = z_0 \]  

(1.4.2)

the exact solution of (1.4.1/2) on the interval \( t \in I = [t_0, t_f] \) will be denoted as

\[ z(t, z_0, \varepsilon) \]  

(1.4.3)

Assume for the following that \( z \) is scalar. Extension to the vector case is straightforward. \( z(t, z_0, \varepsilon) \) is usually not available in closed form. In many cases, however, it is possible to solve (1.4.1) analytically for \( \varepsilon = 0 \). This motivates an expansion of (1.4.3) in terms of known functions \( \psi_i \) of the following form:

\[
z(t, z_0, \varepsilon) \approx \tilde{z}(t, z_0, \varepsilon) = \sum_{i=0}^{N-1} \psi_i(t, \varepsilon) \quad (1.4.4)
\]

Since in general (1.4.3) cannot be represented exactly by a finite expansion there will be a truncation error

\[
R_N(t, \varepsilon) = z(t, z_0, \varepsilon) - \tilde{z}(t, z_0, \varepsilon) \quad (1.4.5)
\]

due to the neglect of terms of order \( \geq N \). The behaviour of \( R_N(t, \varepsilon) \) as \( \varepsilon \) approaches 0 determines whether (1.4.4) is a meaningful expansion or not i.e. whether it can be used to approximate \( z(t, z_0, \varepsilon) \). In the following definitions the Landau symbols [32,34] are introduced. They will be used to describe the properties of (1.4.4). Consider the two scalar functions \( f(t, \varepsilon), g(t, \varepsilon) \).

**Definition 1:** If there exist \( A > 0 \) and \( \varepsilon_0 > 0 \) such that

\[
|f(t, \varepsilon)| \leq A |g(t, \varepsilon)| \quad \forall |\varepsilon| \leq \varepsilon_0
\]

(1.4.6)

then one writes

\[
f(t, \varepsilon) = O[g(t, \varepsilon)] \quad \text{as} \quad \varepsilon \to 0
\]

(1.4.7)

**Definition 2:** If \( A \) and \( \varepsilon_0 \) are independent of \( t \in I \) then (1.4.7) is *uniformly* valid on I.
Remark: (1.4.6) may be replaced by

\[ \lim_{\varepsilon \to 0} \left| \frac{f(t, \varepsilon)}{g(t, \varepsilon)} \right| < \infty \tag{1.4.8} \]

Consequently (1.4.7) holds uniformly on \( I \) if (1.4.8) is valid for all \( t \in I \).

**Definition 3:** If there exists \( \varepsilon_0 > 0 \) such that

\[ |f(t, \varepsilon)| \leq \delta |g(t, \varepsilon)| \quad \forall \delta > 0 \quad \text{and} \quad \forall |\varepsilon| \leq \varepsilon_0 \tag{1.4.9} \]

then one writes

\[ f(t, \varepsilon) = o[g(t, \varepsilon)] \quad \text{as} \quad \varepsilon \to 0 \tag{1.4.10} \]

**Definition 4:** If \( \varepsilon_0 \) is independent of \( t \in I \) then (1.4.10) holds uniformly on \( I \).

Remark: (1.4.9) may be replaced by

\[ \lim_{\varepsilon \to 0} \left| \frac{f(t, \varepsilon)}{g(t, \varepsilon)} \right| = 0 \tag{1.4.11} \]

Consequently (1.4.10) holds uniformly on \( I \) if (1.4.11) is valid for all \( t \in I \).

The notation introduced above is now used to state the properties of meaningful expansions (1.4.4).

**Definition 5:** A sequence of functions \( \delta_N(\varepsilon) \) is called *asymptotic sequence* if

\[ \delta_N(\varepsilon) = o[\delta_{N-1}(\varepsilon)] \quad \text{as} \quad \varepsilon \to 0 \tag{1.4.12} \]

**Definition 6:** \( z(t, \varepsilon) \) is called an *asymptotic expansion* if and only if

\[ z(t, \varepsilon) = \sum_{i=0}^{N-1} \varphi_i(t, \varepsilon) + O[\delta_N(\varepsilon)] \quad \text{as} \quad \varepsilon \to 0 \tag{1.4.13} \]
Definition 7: (1.4.13) is called a Poincaré expansion if
\[
\psi_1(t, \epsilon) = a_1(t) \delta_1(\epsilon)
\] (1.4.14)

Remarks:

i) Asymptotic expansions may be divergent, i.e.
\[
R_N(t, \epsilon) \overset{N \to \infty}{\longrightarrow} \infty \quad \text{for any value of } \epsilon
\] (1.4.15)

Nevertheless these expansions can be used to approximate \( z(t, \epsilon) \) if
\[
\lim_{\epsilon \to 0} R_N(t, \epsilon) = 0 \quad \text{for any finite } N
\] (1.4.16)
i.e. if the truncation error can be made arbitrarily small by choosing \( \epsilon \) sufficiently small (for examples see [32]). Note that (1.4.16) is a property of Poincaré expansions.

ii) The representation (1.4.13) is not unique. However, the coefficients \( a_i(t) \) of the Poincaré expansion (1.4.14) are uniquely determined once the functions \( \delta_i(\epsilon) \) have been specified.

Definition 8: The expansion
\[
z(t, \epsilon) = \sum_{i=0}^{N-1} \psi_i(t, \epsilon) + R_N(t, \epsilon)
\] (1.4.17)
is said to be uniformly valid on \( I \) if
\[
R_N(t, \epsilon) = O[\delta_N(t, \epsilon)] \quad \text{uniformly for all } t \in I
\] (1.4.18)
Otherwise (1.4.17) is nonuniformly valid on \( I \) and is called a singular perturbation expansion.

Remarks:

i) In most cases asymptotic expansions are nonuniform. The reasons of nonuniformity and techniques to circumvent them are discussed in detail in [32,33]. Here only one type of nonuniformity is of interest that occurs if the highest
derivatives in (1.4.1) are multiplied by \( \varepsilon \). This problem and the method of MAE that restores uniformity will be discussed in the next section.

ii) The following abbreviated notation of (1.4.17) will be used in the sequel:

\[
z(t, \varepsilon) \sim \sum_{i} \psi_i(t, \varepsilon) \quad (1.4.19)
\]

### 1.4.2 The Method of Matched Asymptotic Expansions (MAE)

Consider (1.4.1/2) with

\[
z^t = (x^t, y^t) \in \mathbb{R}^n \quad (1.4.20)
\]

and

\[
h^t(z, t, \varepsilon) = [f(z, t), \frac{1}{\varepsilon} g(z, t)] \in \mathbb{R}^n \quad (1.4.21)
\]

(1.4.1) can then be rewritten in the following form:

\[
\begin{align*}
\dot{x} &= f(x, y, t) \in \mathbb{R}^{n_1} \quad (1.4.22a) \\
\varepsilon \dot{y} &= g(x, y, t) \in \mathbb{R}^{n_2} \quad (1.4.22b) \\
n_1 + n_2 &= n \quad (1.4.22c)
\end{align*}
\]

with

\[
\begin{align*}
x(t_0) &= x_0 \quad (1.4.23a) \\
y(t_0) &= y_0 \quad (1.4.23b)
\end{align*}
\]

The goal is now to develop asymptotic expansions

\[
\begin{align*}
x &\sim x^o = \sum_{i} \psi_i(t, \varepsilon) \quad (1.4.24a) \\
y &\sim y^o = \sum_{i} \xi_i(t, \varepsilon) \quad (1.4.24b)
\end{align*}
\]

that approximately solve (1.4.22/23).
A common choice for $\psi_i$ and $\xi_i$ are the powers of $\epsilon$, i.e.

$$\psi_i = x_i^0(t) \epsilon^i$$  \hspace{1cm} (1.4.25a)

$$\xi_i = y_i^0(t) \epsilon^i$$  \hspace{1cm} (1.4.25b)

An important advantage of asymptotic expansions in terms of powers of $\epsilon$ is that besides addition and subtraction also multiplication of power series is well defined, i.e. yields again an asymptotic expansion. In general this is not true. Operations like exponentiation and differentiation with respect to $t$ or $\epsilon$ are usually not well defined, not even for power series. They result in nonuniformities. For details see [32]. Note also that (1.4.24/25) is a Poincaré-type asymptotic expansion.

By (1.4.25) the problem of determining $\psi_i$ and $\xi_i$ is reduced to calculating the $\epsilon$-independent coefficients $x_i^0(t)$ and $y_i^0(t)$. This can be done by inserting (1.4.24/25) into (1.4.22) yielding:

$$\dot{x}_0^0 + \epsilon \dot{x}_1^0 + \epsilon^2 \dot{x}_2^0 + ... = f(x_0^0 + \epsilon x_1^0 + \epsilon^2 x_2^0 + ..., y_0^0 + \epsilon y_1^0 + \epsilon^2 y_2^0 + ..., t)$$  \hspace{1cm} (1.4.26a)

$$\epsilon \dot{y}_0^0 + \epsilon^2 \dot{y}_1^0 + \epsilon^3 \dot{y}_2^0 + ... = g(x_0^0 + \epsilon x_1^0 + \epsilon^2 x_2^0 + ..., y_0^0 + \epsilon y_1^0 + \epsilon^2 y_2^0 + ..., t)$$  \hspace{1cm} (1.4.26b)

Expansion of the right hand sides in terms of powers of $\epsilon$ up to first order yields:

$$\dot{x}_0^0 + \epsilon \dot{x}_1^0 + ... = f_0^0 + \epsilon f_1^0 + ...$$  \hspace{1cm} (1.4.27a)

$$\epsilon \dot{y}_0^0 + \epsilon^2 \dot{y}_1^0 + ... = g_0^0 + \epsilon g_1^0 + ...$$  \hspace{1cm} (1.4.27b)

with

$$f_0^0 = f(x_0^0, y_0^0, t)$$  \hspace{1cm} (1.4.28a)

$$f_1^0 = \frac{\partial f}{\partial x}(x_0^0, y_0^0, t) x_1^0 + \frac{\partial f}{\partial y}(x_0^0, y_0^0, t) y_1^0$$  \hspace{1cm} (1.4.28b)

and $g_0^0$, $g_1^0$ defined analogously.
Since (1.4.27) must be valid for arbitrary values of \( \epsilon \) equality holds only if coefficients of equal powers of \( \epsilon \) are identical. Thus one obtains the following set of equations for \( x_i^0 \) and \( y_i^0 \):

\[
\begin{align*}
  x_0^0 &= f_0^0 \\
  x_1^0 &= f_1^0 \\
  &\quad \vdots \\
  0 &= g_0^0 \\
  y_0^0 &= g_1^0 \\
  &\quad \vdots
\end{align*}
\]  

(1.4.29a)  

(1.4.29b)  

(1.4.30a)  

(1.4.30b)

Substitution of (1.4.24/25) into (1.4.23) renders:

\[
\begin{align*}
  x_0^0(t_0) + \epsilon x_1^0(t_0) + \ldots &= x_0 \quad \text{for all } \epsilon \\
  y_0^0(t_0) + \epsilon y_1^0(t_0) + \ldots &= y_0 \quad \text{for all } \epsilon
\end{align*}
\]  

(1.4.31a)  

(1.4.31b)

which results in the following initial conditions for (1.4.29/30):

\[
\begin{align*}
  x_0^0(t_0) &= x_0 \\
  x_i^0(t_0) &= 0 \quad i=1,2,\ldots \\
  y_0^0(t_0) &= y_0 \\
  y_i^0(t_0) &= 0 \quad i=1,2,\ldots
\end{align*}
\]  

(1.4.32a)  

(1.4.32b)  

(1.4.33a)  

(1.4.33b)

In order to solve (1.4.29/30) for \( x_0^0 \) and \( y_0^0 \) equation (1.4.30a) is written more explicitly using (1.4.28a):

\[
g_0^0 = g(x_0^0, y_0^0, t) = 0
\]  

(1.4.34)

Assume for the moment that \( x_0^0 \) is known. (1.4.34) then is a set of \( n_2 \) nonlinear algebraic equations with \( n_2 \) unknowns \( y_0^0(t) \). It is further assumed that (1.4.34) has a unique root.
\[ y_0^\circ(t) = \Phi[x_0^\circ(t), t] \]  \hfill (1.4.35)

(1.4.35) can now be substituted into (1.4.29a) yielding:

\[ \dot{x}_0^\circ = \mathbf{f}[x_0^\circ, \Phi[x_0^\circ(t), t], t] \]  \hfill (1.4.36)

After integration of (1.4.36) with (1.4.32a) the zero-order approximation \( x_0^\circ(t) \) of \( x \) is obtained and \( y_0^\circ(t) \) is determined via (1.4.35). Evaluation of (1.4.35) for \( t = t_0 \) and comparison with (1.4.33a) reveals, however, a contradiction in the initial conditions since in general

\[ \Phi[x_0^\circ(t_0), t_0] \neq y_0 \]  \hfill (1.4.37)

Moreover from (1.4.32b/33b) follows that the initial conditions cannot be matched by higher order terms. Therefore the expansion (1.4.24/25) is nonuniform on \([t_0, t_f]\) with the nonuniformity occurring in the neighbourhood of \( t_0 \). The reason for this behaviour becomes clear by observing that (1.4.29a/30a) is a system of \( n_1 \) differential equations and \( n_2 \) algebraic equations. Thus the system order has been reduced from originally \( n \) in (1.4.22) to \( n_1 \). Consequently only \( n_1 \) of the \( n \) initial conditions (1.4.23) can be satisfied. (1.4.32/33) will therefore be replaced by the following still unknown initial conditions:

\[ x_0^\circ(t_0) = x_{i0}^\circ ; \quad i = 0,1,2,... \]  \hfill (1.4.38)

\[ y_0^\circ(t_0) = \Phi[x_{00}^\circ, t_0] \]  \hfill (1.4.39)

\[ y_1^\circ(t_0) = y_{i0}^\circ ; \quad i = 1,2,... \]  \hfill (1.4.40)

The first order approximations of \( x \) and \( y \) can be calculated by substituting (1.4.35) into (1.4.30b). With (1.4.28b) one obtains:

\[ \dot{\Phi}[x_0^\circ(t), t] = g_x x_1^\circ(t) + g_y y_1^\circ(t) \]  \hfill (1.4.41)

with

\[ (.)_r = \frac{\partial(\.)}{\partial r} [x_0^\circ(t), y_0^\circ(t), t] ; \quad r = x, y \]  \hfill (1.4.42)
Provided that \( g_y \) is nonsingular (1.4.41) can be solved for \( y_1^0(t) \). With \( y_1^0(t) \) known \( x_1^0(t) \) is obtained by forward integration of (1.4.29b) subject to the initial condition (1.4.38). In this manner all higher order coefficients of \( x_i^0, y_i^0, i > 0 \) can be found successively by first solving an \( n_2 \) dimensional algebraic equation for \( y_i^0 \) and then integrating a \( n_1 \) dimensional system of differential equations rendering \( x_i^0 \). Note that both (1.4.29) and (1.4.30) are linear for \( i > 0 \) [36]. Note also that the \( y_i^0 \) are determined by \( x_{i0}^0 \) similarly as \( y_0^0(t_0) \) and therefore \( x_{i0}^0 \) are the only unknowns of this problem.

The equations (1.4.35/36) that define the zero-order approximations \( x_0^0(t), y_0^0(t) \) can also be obtained by setting \( \varepsilon = 0 \) in (1.4.22). Therefore one has

\[
\lim_{\varepsilon \to 0} x(t) = x_0^0(t) \quad (1.4.43a)
\]

\[
\lim_{\varepsilon \to 0} y(t) = y_0^0(t) \quad (1.4.43b)
\]

With (1.4.25) it follows from (1.4.43) that condition (1.4.16) is satisfied, i.e. (1.4.24) is a meaningful approximation for \( z(t) \). On the other hand the system order in (1.4.22) reduces from \( n \) to \( n_1 \) and therefore only \( n_1 \) initial conditions can be satisfied as mentioned earlier. In general it is therefore valid:

\[
\lim_{t \to t_0} z_0^0(t) = \lim_{t \to t_0} \lim_{\varepsilon \to 0} z(t) = \lim_{\varepsilon \to 0} \lim_{t \to t_0} z(t) \quad (1.4.44)
\]

(1.4.44) expresses the nonuniformity of the asymptotic expansion in \( t_0 \), i.e. the expansion is valid for all \( t \in I \) except a neighbourhood of \( t_0 \).

\( y_0^0(t) \) from (1.4.35) is the quasistationary solution of (1.4.22b), i.e. the solution obtained by neglecting the dynamics of this subsystem. For small \( \varepsilon \) and

\[
f(z, t) = O[g(z, t)] \quad (1.4.45)
\]

the states \( y \) change much faster than the states \( x \). In the sequel \( x \) will therefore be referred to as slow and \( y \) as fast variables, respectively. Under certain stability
conditions mentioned later \( y \) reaches its quasistationary solution almost immediately as \( \varepsilon \to 0 \). This justifies neglection of the \( y \)-dynamics for \( t > t_0 \). In the neighbourhood of \( t_0 \) the behaviour of \( y \) is, however, determined by the transition \( y^i \) from the initial conditions to the quasistationary solution \( y^0 \) (figure 1.4.1). The associated time interval is called boundary layer. The expansion (1.4.24/25) approximates \( x, y \) outside the boundary layer and is called outer solution. A new expansion called inner solution and denoted by \( x^i, y^i \) is required to represent \( x, y \) inside the boundary layer where the \( y \)-dynamics must not be neglected.

\[
\begin{align*}
\frac{d}{dt} &= \frac{1}{\varepsilon} \frac{d}{d\tau} \\
\tau &= \frac{t - t_0}{\varepsilon} \quad (1.4.46)
\end{align*}
\]

is introduced (for a motivation of this transformation and more general transformations see [33, chapt.12]). By this transformation the neighbourhood of \( t_0 \) is mapped to the interval \([0, \infty]\) as \( \varepsilon \to 0 \). \( \tau \) is called the time scale of the fast variables. From (1.4.46) follows

\[
\frac{d}{dt} = \frac{1}{\varepsilon} \frac{d}{d\tau} \quad (1.4.47)
\]

\textbf{figure 1.4.1: behaviour of the fast variables in the boundary layer}
Substitution of (1.4.47) into (1.4.22) yields:

\[
\begin{align*}
x' &= \epsilon f(x, y, \tau \epsilon + t_0) \\
y' &= g(x, y, \tau \epsilon + t_0)
\end{align*}
\]  
(1.4.48a, b)

with

\[
(\gamma)' = \frac{d(\gamma)}{d\tau}
\]  
(1.4.48c)

subject to the initial conditions:

\[
\begin{align*}
x(\tau = 0) &= x_0 \\
y(\tau = 0) &= y_0
\end{align*}
\]  
(1.4.49a, b)

\(x^i\) and \(y^i\) will be represented in the same manner as \(x^0\) and \(y^0\). In analogy to (1.4.24/25) one obtains:

\[
\begin{align*}
x &\sim x^i = \sum_k x^i_k(t) \epsilon^k \\
y &\sim y^i = \sum_k y^i_k(t) \epsilon^k
\end{align*}
\]  
(1.4.50a, b)

Substitution of (1.4.50) into (1.4.48) and expanding \(f\) and \(g\) in powers of \(\epsilon\) results in:

\[
\begin{align*}
x^i_0 + \epsilon x^i_1 + \epsilon^2 x^i_2 + \ldots &= \epsilon f^i_0 + \epsilon^2 f^i_1 + \ldots \\
y^i_0 + \epsilon y^i_1 + \ldots &= g^i_0 + \epsilon g^i_1 + \ldots
\end{align*}
\]  
(1.4.51a, b)

with

\[
\begin{align*}
f^i_0 &= f(x^i_0, y^i_0, \tau) \\
f^i_1 &= \frac{\partial f}{\partial x}(x^i_0, y^i_0, \tau) x^i_1 + \frac{\partial f}{\partial y}(x^i_0, y^i_0, \tau) y^i_1
\end{align*}
\]  
(1.4.52a, b)

and likewise for \(g^i_0\) and \(g^i_1\).

Since the inner solution represents the transient behaviour in the boundary layer one demands that
lim \left[ \lim_{\epsilon \to 0} \lim_{\tau \to 0} \begin{pmatrix} x_i(\tau) \\ y_i(\tau) \end{pmatrix} \right] = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \quad (1.4.53)

Equating equal powers of \( \epsilon \) yields:

\begin{align*}
x_0^1 &= 0 \\
x_1^1 &= \epsilon_0 \\
y_0^1 &= \varepsilon_0 \\
y_1^1 &= \varepsilon_1 \\
\end{align*}

(1.4.54a, 1.4.54b, 1.4.55a, 1.4.55b)

The initial conditions for (1.4.54/55) are obtained by inserting (1.4.50) into (1.4.53):

\begin{align*}
x_0^1(0) &= x_0 \\
x_k^1(0) &= 0 \quad \forall \; k > 0 \\
y_0^1(0) &= y_0 \\
y_k^1(0) &= 0 \quad \forall \; k > 0 \\
\end{align*}

(1.4.56a, 1.4.56b, 1.4.56c, 1.4.56d)

From (1.4.54a/56a) follows:

\begin{equation}
\begin{align*}
x_0^1(\tau) &= \text{constant} = x_0 \\
y_0^1(\tau) &= \text{known} = y_0 \\
\end{align*}
\end{equation}

(1.4.57)

\( y_0^1(\tau) \) can be calculated by inserting (1.4.57) into (1.4.55a) and integrating forward with (1.4.56c). Once \( y_0^1(\tau) \) is known (1.4.54b) can be integrated subject to (1.4.56b) rendering \( x_1^1(\tau) \). This in turn can be used to solve (1.4.55b) with (1.4.56d). In this manner all coefficients \( x_k^1 \) and \( y_k^1 \) can be determined successively. As was the case for the outer solution all equations associated with \( k > 0 \) are linear. Equations (1.4.54a) and (1.4.55a) can be obtained by setting \( \epsilon = 0 \) in (1.4.48) expressing the fact that

\begin{equation}
\lim_{\epsilon \to 0} z(\tau) = z_0^1(\tau) \quad (1.4.58)
\end{equation}

\( \tau \) fixed.
From (1.4.58) follows with (1.4.56)

\[
\lim_{\tau \to 0} \lim_{\epsilon \to 0} z(\tau) = z_0(0) = z_0 = \lim_{\epsilon \to 0} \lim_{\tau \to 0} z(\tau)
\]  

(1.4.59)

Hence the expansion \( z^1 \) represents \( z \) uniformly on \( \tau \in [0, \infty] \).

Up to this point two asymptotic expansions for \( z \) namely the outer solution \( z^0 \) and the inner solution \( z^1 \) have been constructed. \( z^0 \) represents \( z \) outside the boundary layer while \( z^1 \) is uniformly valid within the boundary layer that has been blown up by the stretching transformation (1.4.46). In the following the two solutions will be combined to form a composite solution that is uniformly valid on \( t \in [t_0, t_f] \). This procedure is called **boundary layer matching**. Since the inner solution describes the transient behaviour from the initial conditions to the quasistationary solution it is evident that \( z^1 \) should satisfy the initial conditions at \( t=0 \). This matching condition has already been established in (1.4.53) and was used to construct \( z^1 \). In addition \( z^1 \) should asymptotically approach the outer solution as \( \tau \to \infty \) as the transient dies out and the system becomes quasistationary. This condition can be stated in the following way [36]:

\[
\lim_{\epsilon \to 0} \lim_{t \to t_0} \lim_{\tau \to \infty} [x^0(t, \epsilon) - x^1(\tau, \epsilon)] = 0
\]  

(1.4.60a)

\[
\lim_{\epsilon \to 0} \lim_{t \to t_0} \lim_{\tau \to \infty} [y^0(t, \epsilon) - y^1(\tau, \epsilon)] = 0
\]  

(1.4.60b)

The matching procedure essentially serves the determination of the unknown initial conditions (1.4.38). It depends on the number of terms considered in the expansions (1.4.24), (1.4.50). Simple expansions are usually only obtained if \( z^0 \) and \( z^1 \) are truncated after the zero-order term. In view of later applications to the intercept problem the matching will be demonstrated only for this case here. For matching of expansions including the first order terms the reader is referred to [32, 34, 36, 37].
The zero-order expansions of $x$ and $y$ are:

- $x^0(t) = x^0(t)$ outersolution  \hspace{1cm} (1.4.61a)
- $x^i(t) = x^i(\varepsilon t) = x_0$ inner solution  \hspace{1cm} (1.4.61b)
- $y^0(t) = y^0(t)$ outersolution  \hspace{1cm} (1.4.61c)
- $y^i(t) = y^i(\varepsilon t)$ inner solution  \hspace{1cm} (1.4.61d)

With (1.4.61) one obtains from (1.4.60):

$$x_0^0(t_0) = x_0 \hspace{1cm} (1.4.62)$$

Note that (1.4.62) is valid regardless of the number of terms considered in the expansions $x^0$ and $x^i$ [36].

**Theorem 1.4:**

The matching condition (1.4.60b) is always satisfied if $y^0_0(t)$ is a stable root of (1.4.34) and the initial condition $(x_0, y_0)$ is in the domain of attraction of this root [39-42].

Under the conditions of this theorem the expansions $z^0$ and $z^i$ exist and with (1.4.56/57) and (1.4.62) their zero-order approximations are determined. For a short discussion of the theorem see appendix 1.2. In a last step the two solutions will be combined to render a composite solution $z^c$ which is uniformly valid on $[t_0, t_f]$. The simplest form of $z^c$ is the additive composition

$$z^c(t) = z^0(t) + z^i\left(\frac{t - t_0}{\varepsilon}\right) - z^{cp} \hspace{1cm} (1.4.63)$$

where $\tau$ was replaced by (1.4.46) in $z^i$.

Other forms of $z^c$ are discussed in [32,34,40]. $z^{cp}$ denotes the "common part" of $z^0$ and $z^i$ in the overlapping zone of inner and outer solution marked with $t^*$ in figure 1.4.1. It is defined by the matching conditions (1.4.60). With (1.4.61) it follows immediately that

$$x^{cp} = x^0(t_0) = x_0 \hspace{1cm} (1.4.64a)$$
$$y^{cp} = y^0(t_0) \hspace{1cm} (1.4.64b)$$
Hence the composite expansions for $x$ and $y$ become:

\begin{align*}
x(t) & \sim x_0^0(t) + x_0^1(t) - \frac{t-t_0}{\varepsilon} - x_0 + O(\varepsilon) = x_0^0(t) + O(\varepsilon) \quad (1.4.65a) \\
y(t) & \sim y_0^0(t) + y_0^1(t) - \frac{t-t_0}{\varepsilon} - y_0^0(t_0) + O(\varepsilon) \quad (1.4.65b)
\end{align*}

**Discussion:**

The zero-order composite solution for $x$ is identical to the zero-order outer solution $x_0^0(t)$. This is easily understood by observing that $x_0^0(t)$ satisfies the initial conditions according to (1.4.62). Therefore no boundary layer correction of the slow variables is necessary for the zero-order approximation. Inspection of (1.4.65b) shows that the terms $y_0^0$ cancel out as $t \to t_0$. Thus $y(t)$ is represented correctly by the inner solution $y_0^1$ in the boundary layer which by (1.4.56) satisfies the initial conditions. Outside the boundary layer, i.e. for $t > t^*$ the terms $y_0^1$ and $y_0^0(t_0)$ cancel out under the conditions of theorem 1.4. $y(t)$ is then represented by the quasistationary solution $y_0^0(t)$.

Note that by construction of the composite solution (1.4.65) determination of $t^*$ is avoided. For later applications the calculation of the zero-order approximations is summarized:

1. **outer solution:**
   Set $\varepsilon = 0$ in (1.4.22). One obtains a system of $n_1$ differential equations for the slow variables $x$ and $n_2$ algebraic equations (1.4.34) for the fast variables $y$. Since the system order has been reduced from $n$ to $n_1$ only $n_1$ initial conditions (1.4.23a) can be satisfied. The outer solution is obtained by solving the algebraic equations for $y$ resulting in the outer solution

$$y_0^0(t) = \Phi[x_0^0(t), t] \quad (1.4.66)$$

Note: In general this solution is not unique. By theorem 1.4 it must be stable and the initial conditions $x_0$ must lie in its domain of attraction. Otherwise a boundary layer correction for $y$ does not exist.
\( x^0(t) \) is calculated by forward integration of the reduced system

\[
\dot{x}^0 = f\{ x^0(t), \Phi[ x^0(t), t], t \} \tag{1.4.67}
\]

subject to the initial conditions

\[
x^0(t_0) = x(t_0) \tag{1.4.68}
\]

While \( x^0(t) \) is the uniformly valid zero-order approximation of \( x \) on \( t \in [t_0, t_f] \), \( y^0(t) \) converges nonuniformly in \( t_0 \) and must be corrected in the boundary layer.

2. Inner solution:

The expansions of \( x, y \) in the boundary layer are called inner solution \( x^i, y^i \). They exist only under the stability conditions of theorem 1.4. Their calculation is done by performing the time-scale transformation (1.4.46) yielding the system equations (1.4.48) in the stretched \( \tau \) - time-scale. For \( x^i_0 \) one obtains:

\[
x^i_0(\tau) = x(t_0) \tag{1.4.69}
\]

\( y^i_0(\tau) \) is calculated by integrating

\[
y^i_0(\tau) = g(x(t_0), y^i_0(\tau), \tau) \tag{1.4.70}
\]

with

\[
y^i_0(0) = y(t_0) \tag{1.4.71}
\]

3. Composite solution:

The zero-order composite solution is given by

\[
x(t) \sim x^0(t) + O(\varepsilon) \tag{1.4.72}
\]

\[
y(t) \sim y^0(t) + y^i_0(\frac{t - t_0}{\varepsilon}) - y^0(t_0) + O(\varepsilon) \tag{1.4.73}
\]

Concluding remarks:

While the calculation of a zero-order uniformly valid expansion for the singularly perturbed system (1.4.22/23) is relatively simple the determination of higher order terms is considerably more complicated. The difficulties arise mainly in the boundary layer matching. It turns out that in contrast to \( x^0 \) the higher order terms \( x^k \) with
k > 0 have a boundary layer correction $x_k^i$ with initial conditions that involve limits of integrals that in general cannot be solved analytically [36]. In view of (1.4.15) it is not even guaranteed that accuracy will actually improve by taking into account higher order terms. This is why in most cases only the zero-order approximations are used.

The MAE-method has been illustrated here for initial value problems. The application to TPBVPs is discussed in [34,35,36,37,39,43]. In the next subsection a review of the MAE-method as applied to multiple-time-scale problems is given.

1.43 Multiple Time Scale Problems

In this section the application of the MAE-method to multiple-time-scale problems will be discussed shortly. The procedure is a straightforward generalization of the method described in the preceding section. The theoretical foundations can be found in [41,42]. Only zero-order approximations will be discussed here.

Consider the following system of $n + m$ differential equations:

$$\dot{x} = f(x, y, t) \in \mathbb{R}^n \quad (1.4.74)$$
$$\varepsilon_1 \dot{y}_1 = g_1(x, y, t) \in \mathbb{R}^{m_1} \quad (1.4.75a)$$
$$\varepsilon_2 \dot{y}_2 = g_2(x, y, t) \in \mathbb{R}^{m_2} \quad (1.4.75b)$$
$$\ldots$$
$$\varepsilon_k \dot{y}_k = g_k(x, y, t) \in \mathbb{R}^{m_k} \quad (1.4.75c)$$

with

$$t \in [t_0, t_f]$$
$$y^t = [y_1^t, y_2^t, \ldots, y_k^t] \in \mathbb{R}^m \quad (1.4.76)$$

$$m = \sum_{j=1}^{k} m_j \quad (1.4.77)$$
\[
\lim_{\epsilon_j+1 \to 0} \frac{\epsilon_j+1}{\epsilon_j} = 0 \quad (1.4.78)
\]

initial conditions:

\[
x(t_0) = x_0 \quad (1.4.79a)
\]
\[
y_j(t_0) = y_{j0} ; \ j = 1,2,...,k \quad (1.4.79b)
\]

Remark:
Since only the zero-order approximation is considered here the subscript ( )_0 denoting the zero order term has been omitted.

Equation (1.4.78) ensures that \( y_{j+1} \) is fast with respect to \( y_j \). In this problem one therefore has a slow time scale of the \( x \)-variables and \( k \) fast time scales decoupled by (1.4.78). For the outer solution one obtains, as in section (1.4.2), an expansion that holds nonuniformly on \([t_0, t_f]\). Uniformity of the expansion is restored by constructing an inner solution in the boundary layer of each time scale.

The outer solution \( x^0, y^0 \) is calculated exactly as described in section 1.4.2. Setting

\[
\epsilon_j = 0 ; \ j = 1,2,...,k \quad (1.4.80)
\]

in (1.4.75) results in (1.4.34/36) with

\[
g^t = [g_1^t, g_2^t, ..., g_k^t] \quad (1.4.81)
\]

The solution is given by (1.4.66-68). The boundary layers are determined by sequentially applying the time-scale transformation

\[
\tau_j = \frac{t - t_0}{\epsilon_j} ; \ j = 1,2,...,k \quad (1.4.82)
\]

In this way one obtains \( k \) decoupled boundary layer equations of order \( m_1, m_2,..., m_k \).

The original problem of order \( n+m \) is therefore broken up into a
sequence of $k+1$ subproblems of orders $n, m_1, m_2, ..., m_k$. This feature is especially important for the approximate solution of optimal control problems since a suitable time-scale separation (which should be physically meaningful of course) allows for the construction of subproblems that can be solved easily. The boundary layer calculation proceeds as follows:

Applying the $j$-th time-scale stretching (1.4.82) and setting $\varepsilon_j = 0$ for all $j$ results with (1.4.78) in

\[ \begin{align*}
  j_x^i' &= 0 \\
  j_y^i' &= 0 \\
  j_{y1}^i &= 0 \\
  j_{yj-1}^i &= 0
\end{align*} \]

slow variables

of $j$-th boundary-layer

(1.4.83)

\[ \begin{align*}
  j_{yj}^i &= g_j(x, y, \varepsilon_j) \text{ transient of } y_j \\
  0 &= g_{j+1}
\end{align*} \]

quasistationary solution of fast

variables of $j$-th boundary-layer

(1.4.84)

(1.4.85)

where ( )' is defined in (1.4.48c).

Since the inner solutions must satisfy the initial conditions it follows from (1.4.83):

\[ \begin{align*}
  j_x^i(\tau_j) &= x_0 \\
  j_{y1}^i(\tau_j) &= y_{10} : 1 = 1, 2, ..., j
\end{align*} \]

(1.4.86a)

(1.4.86b)

After solving (1.4.85) for $j_{yj+1}, ..., j_{yk}$ (1.4.84) can be integrated forward with (1.4.86). Under analogous conditions as mentioned in section (1.4.2) the inner solution $j_{y_j}$ will approach the quasistationary solution associated with
\[ g_j = 0 \quad (1.4.87) \]

that is

\[ \lim_{\varepsilon_j \to 0} y_j^\varepsilon (\tau_j) = \Phi_j[x, y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_k, \varepsilon_j \tau_j] \quad (1.4.88) \]

where \( \Phi_j \) is a stable root of (1.4.87)

Composite solutions can be constructed according to (1.4.72/73).

**Remarks:**

- A very common choice for \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \) is:

  \[ \varepsilon_j = \varepsilon^j; \quad j = 1, 2, \ldots, k \quad (1.4.89) \]

  Obviously (1.4.89) satisfies (1.4.78). Note that the zero-order expansions are independent of \( \varepsilon_j \). The scaling of differential equations is therefore done in a heuristic manner rather than actually figuring out numbers for \( \varepsilon_j \).

- A special case leading to substantial simplifications is:

  \[ n = m_1 = m_2 = \ldots = m_k = 1 \quad (1.4.90) \]

  Here a sequence of scalar problems has to be solved. However it is emphasized that the scaling of differential equations is not arbitrary but should reproduce the physical properties of the system \((x, y)\). Otherwise the resulting expansions will not approximate the actual system behaviour. This fact must be kept in mind especially if the scaling is done by heuristic methods as mentioned above. An approach to estimate \( \varepsilon_j \) is discussed in the next section where guidance laws will be derived by applying the MAE-method.
1.4.4 Scaling of the Intercept Problem

The purpose of this section is to find a scaled representation of the nonlinear equations of the planar intercept problem. The analysis will first be restricted to a horizontal plane. Extensions to a vertical plane are delivered in section 1.4.8.

The scaling procedure will reveal several time scales in the system dynamics and lead to estimates of the associated perturbation parameters \( \varepsilon \) introduced in (1.4.75). First consider the unscaled equations of the planar intercept problem in a cartesian coordinate system \((x,y)\) according to figure 1.4.2 and appendix 1.1:

**missile-target relative motion:**

\[
\Delta \dot{x} = v_{T} \cos \gamma_{T} - v \cos \gamma
\]

(1.4.91a)

\[
\Delta \dot{y} = v_{T} \sin \gamma_{T} - v \sin \gamma
\]

(1.4.91b)

**missile dynamics:**

\[
\dot{v} = \frac{T - D}{m}
\]

(1.4.92a)

\[
\dot{\gamma} = \frac{L}{mv}
\]

(1.4.92b)

\[
\dot{q} = \frac{M}{I}
\]

(1.4.92c)

\[
\dot{\theta} = q
\]

(1.4.92d)

**target trajectory:**

\[
v_{T} = v_{T}(t)
\]

(1.4.93a)

\[
\gamma_{T} = \gamma_{T}(t)
\]

(1.4.93b)

with

\[
\Delta x, \Delta y : \text{missile-target relative position}
\]

\[
v, v_{T} : \text{missile velocity, target velocity}
\]

\[
\gamma, \gamma_{T} : \text{missile heading, target heading in } (x,y)
\]

\[
T : \text{missile thrust vector}
\]

\[
D : \text{drag}, \quad D = D(h, v, \alpha, \delta)
\]

(1.4.94a)

\[
L : \text{lift}, \quad L = L(h, v, \alpha, \delta)
\]

(1.4.94b)

\[
M : \text{aerodynamic torque}, \quad M = M(h, v, \alpha, \delta, q)
\]

(1.4.94c)

CL: missile center line
\( \alpha \) : angle of attack  
\( \delta \) : flipper deflection  
\( q \) : pitch rate  
\( \theta \) : pitch angle  

\( m \): missile mass  
\( I \): missile moment of inertia  

**Simplification:**  
For the type of missile under investigation here the guidance loop is activated after fuel burn out. Therefore the missile thrust need not be considered for guidance law derivation and one has:

\[
T = 0 \quad (1.4.95)
\]

It follows that the missile mass \( m \) and the moment of inertia \( I \) remain constant.
In the following, dimensionless states denoted by superscript * are defined:

\[ \Delta x = \Delta x^* R_R, \quad \Delta y = \Delta y^* R_R, \quad v = v^* v_R \]  
\[ \gamma = \gamma^* R_R, \quad \theta = \theta^* R_R, \quad q = q^* q_R \]  
\[ L = L^* L_R, \quad D = D^* D_R, \quad M = M^* M_R \]  
\[ t = t^* R_R \]  
\[ v_T = v_T^* v_R, \quad \gamma_T = \gamma_T^* R_R \]  

(1.4.96a,b,c)

(1.4.97a,b,c)

(1.4.98a,b,c)

(1.4.99a,b)

The reference values denoted by subscript R are chosen in such a way that the maximum values of the scaled quantities \((\cdot)^*\) are \(O(1)\). They are summarized in the following table:

<table>
<thead>
<tr>
<th>Reference</th>
<th>Typical Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R_R): initial range</td>
<td>2 km .... 10 km</td>
</tr>
<tr>
<td>(v_R): average missile velocity</td>
<td>1.5 Ma .... 2.5 Ma</td>
</tr>
<tr>
<td>(\gamma_R): total missile heading angle increment</td>
<td>0 deg .... 90 deg</td>
</tr>
<tr>
<td>(\theta_R): total missile pitch angle increment</td>
<td>0 deg .... 110 deg</td>
</tr>
<tr>
<td>(q_R): maximum pitch rate</td>
<td>(\sim 0.5 \text{ rad/s})</td>
</tr>
<tr>
<td>(L_R): maximum lift</td>
<td>(\sim 2000 \text{ kp})</td>
</tr>
<tr>
<td>(D_R): maximum drag</td>
<td>(\sim 900 \text{ kp})</td>
</tr>
<tr>
<td>(M_R): maximum aerodynamic torque</td>
<td>(\sim 200 \text{ kpm})</td>
</tr>
</tbody>
</table>

*Table 1.4.1: Reference Values*

After substitution of (1.4.95-99) into (1.4.91/92) and applying the time scale transformation

\[ \frac{d}{dt} = \frac{v_R}{R_R} \frac{d}{dt^*} =: \frac{v_R}{R_R} (\cdot)' \]  

(1.4.100)
one obtains:

\[ x^* = v^*_T \cos(y_R y^*_R) - v^* \cos(y_R y^*_R) \] (1.4.101a)
\[ y^* = v^*_T \sin(y_R y^*_R) - v^* \sin(y_R y^*_R) \] (1.4.101b)
\[ \varepsilon_1 v^* = -D^* \] (1.4.102a)
\[ \varepsilon_2 y^* = L^* \] (1.4.102b)
\[ \varepsilon_3 q^* = M^* \] (1.4.102c)
\[ \varepsilon_4 \theta^* = q^* \] (1.4.102d)

with

\[ \varepsilon_1 = \frac{v_R}{R_R} \frac{m_R}{D_R} \] (1.4.103a)
\[ \varepsilon_2 = \frac{v_R}{R_R} \frac{y_R}{L_R} \frac{m}{L_R} \] (1.4.103b)
\[ \varepsilon_3 = \frac{v_R}{R_R} \frac{q_R}{M_R} \] (1.4.103c)
\[ \varepsilon_4 = \frac{v_R}{R_R} \frac{\theta_R}{q_R} \] (1.4.103d)

Definitions:

\[ t_{\text{max}} = \frac{R_R}{v_R} \text{ characteristic time (~ problem duration)} \] (1.4.104a)

\[ \bar{a}_a = \frac{v_R}{t_{\text{max}}} \text{ average axial acceleration} \] (1.4.104b)

\[ \bar{a}_l = \frac{v_R y_R}{t_{\text{max}}} \text{ average lateral acceleration} \] (1.4.104c)
\[
\bar{q} := \frac{\dot{q}_R}{t_{\text{max}}} \quad \text{average pitch acceleration} \quad (1.4.104d)
\]
\[
\bar{q} := \frac{\theta_R}{t_{\text{max}}} \quad \text{average pitch rate} \quad (1.4.104e)
\]
\[
a_{\text{amax}} := \frac{D_R}{m} \quad \text{maximum axial acceleration} \quad (1.4.105a)
\]
\[
a_{\text{imax}} := \frac{L_R}{m} \quad \text{maximum lateral acceleration} \quad (1.4.105b)
\]
\[
\dot{q}_{\text{max}} := \frac{M_R}{I} \quad \text{maximum pitch acceleration} \quad (1.4.105c)
\]
\[
q_{\text{max}} := q_R \quad \text{maximum pitch rate} \quad (1.4.105d)
\]

(1.4.104/105) in (1.4.103) yields:

\[
\varepsilon_1 = \frac{\bar{a}_a}{a_{\text{amax}}} \quad (1.4.106a)
\]
\[
\varepsilon_2 = \frac{\bar{a}_l}{a_{\text{imax}}} \quad (1.4.106b)
\]
\[
\varepsilon_3 = \frac{\bar{q}}{q_{\text{max}}} \quad (1.4.106c)
\]
\[
\varepsilon_4 = \frac{\bar{q}}{q_{\text{max}}} \quad (1.4.106d)
\]
Discussion:

The scaling parameters $\varepsilon$ are of the form:

$$\varepsilon = \frac{\bar{s}}{s_{\text{max}}}$$  \hspace{1cm} (1.4.107)

$\bar{s}$ is the average value of $s$ and $s_{\text{max}}$ is its maximum value. Due to the scaling, the right hand sides of (1.4.101/102) are all $O(1)$. Fast dynamic behaviour with respect to (1.4.101) the scaling factor of which is 1 is therefore characterized by:

$$\varepsilon \ll 1$$  \hspace{1cm} (1.4.108)

This means with (1.4.107) that the fast states produce trajectories with changes that are low in the average but can be large locally. Typical values for $\varepsilon_i$ calculated from the reference values in table 1.4.1 are given below:

<table>
<thead>
<tr>
<th></th>
<th>$R_R = 2$ km</th>
<th>$R_R = 10$ km</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_1$</td>
<td>1.5</td>
<td>0.3</td>
</tr>
<tr>
<td>$\varepsilon_2$</td>
<td>0.1 ($\gamma_R = 10^\circ$)</td>
<td>0.02 ($\gamma_R = 10^\circ$)</td>
</tr>
<tr>
<td></td>
<td>0.9 ($\gamma_R = 90^\circ$)</td>
<td>0.2 ($\gamma_R = 90^\circ$)</td>
</tr>
<tr>
<td>$\varepsilon_3$</td>
<td>0.008</td>
<td>0.002</td>
</tr>
<tr>
<td>$\varepsilon_4$</td>
<td>0.05 ($\theta_R = 10^\circ$)</td>
<td>0.01 ($\theta_R = 10^\circ$)</td>
</tr>
<tr>
<td></td>
<td>0.4 ($\theta_R = 90^\circ$)</td>
<td>0.09 ($\theta_R = 90^\circ$)</td>
</tr>
</tbody>
</table>

*table 1.4.2: scaling parameters*
The estimates obtained in table 1.4.2 show that (1.4.108) is best satisfied by $\epsilon_3$ and $\epsilon_4$ suggesting that the missile pitch motion is well decoupled from the missile-target relative motion. Comparison of $\epsilon_4$ with the other scaling parameters reveals the decoupling of the pitch rate dynamics from all other states which justifies the usual autopilot-design procedure based on a linearization of (1.4.92c) with constant $v$ and $\gamma$ [44]. The $v$- and $\gamma$-dynamics depend strongly on the intercept scenario. Clearly, $v$ has to be considered a slow variable here sharing the time scale of $\Delta x$ and $\Delta y$. If thrust control is possible $v$ may be chosen as a fast variable decoupled from (1.4.101) as was done in [38]. In many cases $\gamma$ may be considered decoupled from $\Delta x$, $\Delta y$ and $v$. Only in the case of very short-range high-manuever scenarios it shares the time scale of (1.4.101) and (1.4.102a). Therefore the following scaling is suggested:

- slow variables : $\Delta x$, $\Delta y$, $v$
- fast variable : $\gamma$
- very fast variables : $q$, $\theta$

*table 1.4.3: time scales*

The subsequent application of the method of MAE will be restricted to the construction of zero-order approximations. Since these approximations are independent of the values of $\epsilon_i$ it is feasible to introduce an artificial scaling of the equations (1.4.91/92). The scaling will reflect the time scale selection given in table 1.4.3 but uses the original variables rather than their scaled counterparts. The scaling must satisfy (1.4.78) which is guaranteed by the use of (1.4.89). Thus one obtains the following perturbed equations:

\[
\begin{align*}
\dot{\Delta x} &= v_T \cos \gamma_T - v \cos \gamma \\
\dot{\Delta y} &= v_T \sin \gamma_T - v \sin \gamma \\
\dot{v} &= \frac{-D}{m} \\
\epsilon \dot{\gamma} &= \frac{L}{mv} \\
\epsilon^2 \dot{q} &= \frac{M}{I} \\
\epsilon^2 \dot{\theta} &= q
\end{align*}
\]
initial conditions:
\[ \Delta x(t_0) = Ax_0, \quad v(t_0) = v_0, \quad q(t_0) = q_0 \]  
\[ \Delta y(t_0) = Ay_0, \quad y(t_0) = y_0, \quad \theta(t_0) = \theta_0 \]  
(1.4.112a,b,c,d,e,f)

Remark: The artificial scaling of the original problem is sometimes called forced singular perturbation [38]. It can easily be proved that the original scaling and the artificial scaling render equivalent zero-order solutions [55].

1.4.5 Derivation of Guidance Law

In this section a guidance law is sought that solves the following minimum-time problem:
\[ J = \int_{t_0}^{t_f} dt \]  
(1.4.113)
subject to the final constraints:
\[ \Delta x(t_f) = 0 \]  
(1.4.114a)
\[ \Delta y(t_f) = 0 \]  
(1.4.114b)
and the dynamic constraints (1.4.109-111).

The choice of (1.4.113) is motivated by the desire to minimize the time for target evasive maneuvers while (1.4.114) guarantees intercept. The exact problem can only be solved numerically as discussed in sections 1.1 and 1.2. In the following the procedure described in sections 1.4.2/3 will be applied to derive a zero-order solution of the perturbed problem formulated above.

1.4.5.1 Outer Solution

Setting \( \varepsilon = 0 \) in (1.4.109-111) yields the reduced system:
\[ \Delta x^0 = v_T \cos y_T - v_0 \cos y^0 \]  
(1.4.115a)
\[ \Delta y^0 = v_T \sin y_T - v_0 \sin y^0 \]  
(1.4.115b)
\[ v^0 = \frac{-D^0}{m} \]  
(1.4.115c)
\[ 0 = \frac{L^0}{m v^0} \quad \text{(1.4.116a)} \]

\[ 0 = \frac{M^0}{I} \quad \text{(1.4.116b)} \]

\[ 0 = q^0 \quad \text{(1.4.116c)} \]

The matching condition (1.4.62) yields:

\[ \Delta x^0(t_0) = \Delta x_0 \quad \text{(1.4.117a)} \]
\[ \Delta y^0(t_0) = \Delta y_0 \quad \text{(1.4.117b)} \]
\[ v^0(t_0) = v_0 \quad \text{(1.4.117c)} \]

An analogous matching at final time \( t_f \) renders with (1.4.114):

\[ \Delta x^0(t_f) = 0 \quad \text{(1.4.118a)} \]
\[ \Delta y^0(t_f) = 0 \quad \text{(1.4.118b)} \]

From (1.4.116) follows:

\[ L^0 = M^0 = q^0 = 0 \quad \text{(1.4.119)} \]

(1.4.119) describes a straight-line missile trajectory. Clearly, on this path one has:

\[ \alpha^0 = 0 \quad \text{(1.4.120a)} \]
\[ \delta^0 = 0 \quad \text{(1.4.120b)} \]

Substitution of (1.4.120) into (1.4.94) yields:

\[ D^0 = D(h, v^0, \alpha^0, \delta^0) \quad \text{(1.4.121)} \]

The Hamiltonian associated with the reduced equations (1.4.115) with (1.4.113) is:

\[ H^0 = 1 + \lambda_1^0 (v_T \cos \gamma_T - v^0 \cos \gamma^0) + \lambda_2^0 (v_T \sin \gamma_T - v^0 \sin \gamma^0) - \frac{\lambda_3^0 D^0}{m} \quad \text{(1.4.122)} \]

with

\[ H^0 = 0 \quad \text{(1.4.123)} \]

since the final time is free.
Note that the fast variables $\gamma^0$, $q^0$ and $\phi^0$ may be regarded as control variables of the reduced problem \[36,38,51,54\]. $q^0$ is already determined by (1.4.119) and from (1.4.120a) follows (see figure 1.4.2):
\[
\phi^0 = \gamma^0
\]  
(1.4.124)

This leaves $\gamma^0$ as the only control variable. According to section 1.1 the necessary conditions of optimality are:
\[
\frac{\partial H}{\partial x^0} = 0 \rightarrow \lambda_1^0 \text{ constant} 
\]  
(1.4.125a)

\[
\frac{\partial H}{\partial y^0} = 0 \rightarrow \lambda_2^0 \text{ constant} 
\]  
(1.4.125b)

\[
\frac{\partial H}{\partial v^0} = \lambda_1^0 \cos \gamma^0 + \lambda_2^0 \sin \gamma^0 + \lambda_3^0 \frac{1}{m} \frac{\partial D^0}{\partial v^0} 
\]  
(1.4.126)

subject to the transversality condition:
\[
\lambda_3^0(t_f) = 0 \quad \text{since } v(t_f) \text{ is free.} 
\]  
(1.4.127)

The optimal heading $\gamma_{opt}^0$ minimizes $H^0$, hence:
\[
\gamma_{opt}^0 = \arg \min_{\gamma^0} H^0 
\]  
(1.4.128)

and is determined by
\[
\frac{\partial H^0}{\partial \gamma_{opt}^0} = 0 = \lambda_1^0 \sin \gamma_{opt}^0 - \lambda_2^0 \cos \gamma_{opt}^0 
\]  
(1.4.129)

resulting in
\[
tg \gamma_{opt}^0 = \frac{\lambda_2^0}{\lambda_1^0} = \text{constant} 
\]  
(1.4.130)
Since \( \gamma_1^0, \gamma_2^0 \) are unknown, \( \gamma_{\text{opt}}^0 \) will be determined by exploiting (1.4.118). Observing that \( \gamma_{\text{opt}}^0 \) is constant it is easily seen that the optimal missile path is a straight line connecting the initial position and the point of intercept. \( \gamma_{\text{opt}}^0 \) is the collision course (figure 1.4.3).

The calculation of \( \gamma_{\text{opt}}^0 \) has to be done iteratively by choosing a value \( \gamma_i^0 \), integrating (1.4.115), checking the end conditions (1.4.118), and determining a corrected value \( \gamma_i^{0+1} \) based on the error in (1.4.118). This procedure can only be carried out if the target maneuver is known. It amounts to a prediction of the collision point. In order to develop a prediction algorithm the following target maneuver is assumed:

\[
\begin{align*}
\beta_T & \perp v_T \\
v_T &= \text{constant}
\end{align*}
\]

Substitution of (1.4.131) into (1.4.115) yields with (1.4.117):

\[
\begin{align*}
\Delta x^0(t_f) &= 0 = v_T \int_{t_0}^{t_f} \cos \gamma_T(t) \, dt - \cos \gamma_{\text{opt}}^0 \int_{t_0}^{t_f} v^0 \, dt + \Delta x_0 & (1.4.132a) \\
\Delta y^0(t_f) &= 0 = v_T \int_{t_0}^{t_f} \sin \gamma_T(t) \, dt - \sin \gamma_{\text{opt}}^0 \int_{t_0}^{t_f} v^0 \, dt + \Delta y_0 & (1.4.132b)
\end{align*}
\]
(1.4.132) is a set of two equations for the two unknowns \( y^o \) and \( t_f \) or the time-to-go:

\[
t_{go} = t_f - t_0 \tag{1.4.133}
\]

They can be solved for \( y^o \) yielding:

\[
tg y^o = \frac{\Delta y_0 + v_T \int_{t_0}^{t_f} \sin \gamma_T(t) \, dt}{\Delta x_0 + v_T \int_{t_0}^{t_f} \cos \gamma_T(t) \, dt} \tag{1.4.134}
\]

With (1.4.134) only one of the equations (1.4.132) needs to be iterated for \( t_{go} \). The prediction algorithm can now be stated as follows:

1. choose starting value for \( t_{go} \)
2. integrate (1.4.115c) and \( v^o \), then evaluate (1.4.132a) with \( y^o \) from (1.4.134)
3. check \( \Delta x^o(t_f) \): if \( \left| \Delta x^o(t_f) \right| > \) tolerance correct \( t_{go} \) and go to 2), otherwise store obtained values for \( y^o \), \( t_{go} \) and \( v^o(t_f) \)

\textit{table 1.4.4: prediction algorithm}

A realization of this scheme is given in section 1.4.6.2. Note that the only integrations involved concern the determination of the missile's straight flight path. This can be calculated off-line and stored for several altitudes \( h \) so that on-line integrations may be replaced by table look-up and interpolation.

For later usage the adjoint variables \( \lambda^o_1 \), \( \lambda^o_2 \) and \( \lambda^o_3 \) are now determined. (1.4.122/123) can be solved for \( \lambda^o_3 \). Using (1.4.130) one obtains:

\[
\lambda^o_3 = \frac{m}{D^o} \{ 1 + \lambda^o_1 [(v_T \cos \gamma_T - v^o \cos y^o_{opt}) + (v_T \sin \gamma_T - v^o \sin y^o_{opt}) \, tg y^o_{opt}] \}
\]

(1.4.135)
At final time (1.4.127) can be exploited to solve (1.4.135) for the constant Lagrange multiplier $\lambda_1^0$:

$$\lambda_1^0 = \frac{1}{v_T \cos \gamma_T - v_f^0 \cos \gamma_{opt}^0 + \tan \gamma_{opt} \left[ v_T \sin \gamma_T - v_f^0 \sin \gamma_{opt}^0 \right]} = \cos \gamma_{opt}^0 \left[ v_T \cos (\gamma_T - \gamma_{opt}) - v_f^0 \right]$$  \hspace{1cm} (1.4.136)

with

$$v_f^0 = v^0(t_f)$$  \hspace{1cm} (1.4.137)

Substitution into (1.4.135) yields:

$$\lambda_3^0 = \frac{m}{D^0} \left[ \frac{v^0 - v_f^0}{v_T \cos (\gamma_T - \gamma_{opt}) - v_f^0} \right]$$  \hspace{1cm} (1.4.138)

From (1.4.130) one obtains:

$$\lambda_2^0 = \lambda_1^0 \tan \gamma_{opt}^o = - \frac{\sin \gamma_{opt}^0}{v_T \cos (\gamma_T - \gamma_{opt}) - v_f^0}$$  \hspace{1cm} (1.4.139)

The outer solution is now completely determined. In the next step the first boundary layer, i.e. the $\gamma$-transient will be derived.

1.4.5.2 First Boundary Layer

This boundary layer describes the $\gamma$-transient from the initial condition $\gamma_0$ to the outer solution $\gamma_{opt}^0$. Since no end conditions on $\gamma$ have been imposed there is no terminal boundary layer for this state. Following again section (1.4.3) the time scale transformation

$$\tau = \frac{t - t_0}{\varepsilon}, \quad (\cdot)' = \frac{d(\cdot)}{d\tau} = \varepsilon \frac{d(\cdot)}{dt}$$  \hspace{1cm} (1.4.140)
is applied to (1.4.109-111). For $\epsilon = 0$ the transformed equations become:

\begin{align*}
1_x^i &= x_0 \quad \text{(1.4.141a)} \\
1_y^i &= y_0 \quad \text{(1.4.141b)} \\
1_v^i &= v_0 \quad \text{(1.4.141c)} \\
(1_y^i)' &= \frac{1_L^i}{mv_0}, \quad 1_y^i(0) = y_0 \quad \text{(1.4.142)} \\
0 &= \frac{1_M^i}{1} \quad \text{(1.4.143a)} \\
0 &= 1_q^i \quad \text{(1.4.143b)}
\end{align*}

where the notation of section (1.4.3) has been adopted. Note that (1.4.141) is a matching condition (see section 1.4.2). The Hamiltonian associated with (1.4.113) and (1.4.141/142) is:

\begin{equation}
1_H^i = 1 + \lambda_{10} (v_T \cos \gamma_T - v_0 \cos \gamma) + \lambda_{20} (v_T \sin \gamma_T - v_0 \sin \gamma) \\
- \lambda_{30} \frac{1_D^i}{m} + \lambda_4^i \frac{1_L^i}{mv_0}
\end{equation}

with

\begin{align*}
1_H^i &= 0 \quad \text{(1.4.145)} \\
\lambda_{k0} = \lambda_k (t_0); \quad k = 1,2,3 \quad \text{(1.4.146)}
\end{align*}

The fast variables of this time scale are $1_q^i$ and $1_\theta^i$ which play the role of control variables. Since $1_q^i$ is determined by (1.4.143b) $1_\theta^i$ remains as the only control. Its optimal value is determined by:

\begin{equation}
1_\theta^i_{\text{opt}} = \arg \min_{1_\theta^i} 1_H^i
\end{equation}

From figure 1.4.2 follows:

\begin{equation}
1_\alpha^i = 1_\theta^i - 1_y^i
\end{equation}
(1.4.147) can therefore be rewritten as:

\[
1_{\alpha_{\text{opt}}} = \arg \min_{1\alpha^i} 1_{H^i} = \arg \min_{1\alpha^i} \left[ -\lambda_{30} \frac{1_{D^i}}{m} + \frac{1_{L^i}}{m v_0} \right]
\]  

(1.4.149)

with

\[
1_{L^i} = L(h, v_0, 1_{\alpha^i}, 1_{\delta^i})
\]

(1.4.150a)

\[
1_{D^i} = D(h, v_0, 1_{\alpha^i}, 1_{\delta^i})
\]

(1.4.150b)

Note that \(1_{\delta^i}\) is determined by (1.4.143a) with (1.4.94c).

**Definition:**

\[
1_{H^0}(1_{\alpha^i}, 1_{\gamma^i}) = 1 + \lambda_{10} (v_T \cos y_T - v_0 \cos 1_{\gamma^i}) + \lambda_{20} (v_T \sin y_T - v_0 \sin 1_{\gamma^i}) - \lambda_{30} \frac{1_{D^i}}{m}
\]

(1.4.151)

i.e. \(1_{H^0}\) is the value of the outer Hamiltonian in the first boundary layer. Substitution of (1.4.151) into (1.4.144) and solving (1.4.145) for \(1_{\lambda_4^i}\) yields:

\[
1_{\lambda_4^i} = - \frac{1_{H^0}(1_{\alpha_{\text{opt}}^i}, 1_{\gamma^i})}{1_{L^i}(1_{\alpha_{\text{opt}}^i})} \frac{1_{L^i}}{m v_0}
\]

(1.4.152)

(1.4.149) with (1.4.152) is an implicit equation for \(1_{\alpha_{\text{opt}}^i}\) since \(1_{\lambda_4^i}\) depends on \(1_{\alpha_{\text{opt}}^i}\).

The minimization (1.4.149) must therefore be carried out iteratively. This result has not been obtained in the literature because simplified missile models using the lift as a control variable and expressing \(D\) as a function of \(L\) rather than \(\alpha\) have been used [38,46,47,49]. As a consequence the structure of the boundary layer control law (1.4.149) namely the minimization of a weighted sum of drag and lift has not been revealed.

It can be shown (see appendix 1.3) that \(1_{\lambda_4^i}\) tends to zero as \(1_{\gamma^i}\) approaches the outer solution \(1_{\gamma_0^i}\). For \(1_{\lambda_4^i} = 0\) the control law (1.4.149) minimizes the drag which results in \(1_{\alpha_{\text{opt}}^i} = 0\). This result is in accordance with the fact that the outer solution is a time-optimal trajectory and hence must be drag-minimizing. Clearly (1.4.149) controls the course error.
\[ e_\gamma = l^i_{\gamma} - \gamma_{\text{opt}} \]  

(1.4.153)

The choice of \( l^i_{\alpha_{\text{opt}}} \) is a trade-off between time-optimal correction of \( e_\gamma \), i.e. lift maximization (for \( \lambda_{30} = 0 \)) and minimization of induced drag losses (\( l^i_{\lambda_4} = 0 \)) that increase time-to-go.

\( l^i_{\alpha_{\text{opt}}} \) may be transformed to an acceleration command. Substituting \( l^i_{\alpha_{\text{opt}}} \) into (1.4.150) yields:

\[
\begin{align*}
1^i_{L_{\text{opt}}} &= L(h, v_0, l^i_{\alpha_{\text{opt}}}, l^i_\delta) \\
1^i_{D_{\text{opt}}} &= D(h, v_0, l^i_{\alpha_{\text{opt}}}, l^i_\delta)
\end{align*}
\]

(1.4.154a)  

(1.4.154b)

According to figure 2 the commanded acceleration becomes:

\[
 a_c = \frac{1}{m} (1^i_{L_{\text{opt}}} \cos l^i_{\alpha_{\text{opt}}} + 1^i_{D_{\text{opt}}} \sin l^i_{\alpha_{\text{opt}}})
\]

(1.4.155)

Note that the sign of \( L \) is determined by the sign of \( e_\gamma \). (1.4.155) is the zero-order guidance law associated with the perturbed problem (1.4.113/114) with (1.4.109-111).

1.4.5.3 Second Boundary Layer

According to section (1.4.3) one could proceed now by applying the time scale transformation

\[
\tau_2 = \frac{t}{\epsilon^2}, \quad \frac{d}{dt} = \frac{1}{\epsilon^2} \frac{d}{d\tau_2}
\]

(1.4.156)

to (1.4.109-111), set \( \epsilon = 0 \), and solve the resulting optimal control problem in the time scale of the fast variables \( q, \theta \). This procedure amounts to the design of an autopilot as indicated in section 1.4.4. The task of the autopilot is to establish the commanded acceleration (1.4.155). Since the autopilot is not of interest here the analysis of this boundary layer will not be carried out.
1.4.5.4 **Summary**

The singular perturbation analysis of the planar intercept problem shows that the optimal guidance scheme consists of three elements:

a) a predictor that calculates the collision course based on a predicted collision point

b) the guidance law (1.4.149/155) that corrects the course error while keeping induced drag losses low

c) the autopilot

It is evident that the main drawback of this guidance scheme is the use of a predictor requiring knowledge of the usually unpredictable target maneuver. The sensitivity problems associated with inaccurate extrapolation of system parameters (here \( a_T \) and \( \gamma_T \)) have already been addressed in the context of LQ-design based guidance laws in section 1.3. They are especially dramatic if the guidance scheme is used in conjunction with estimators for \( a_T \) and \( \gamma_T \) (see chapter 2).

1.4.6 **Aspects of Implementation**

In this section the basic difficulties in the implementation of the guidance scheme developed in section 1.4.5 are discussed. Attention is focussed on the problem of deviation from the scaling assumptions, on algorithms to solve the prediction problem as well as the minimization (1.4.149), and on information requirements.

1.4.6.1 **Deviation from the Scaling Assumptions**

The guidance law (1.4.149) is based on the assumption that the time scale of the missile-heading-angle dynamics is decoupled from \( \Delta x, \Delta y \) and \( v \). These states are assumed "frozen" in the \( \gamma \)-time scale. It is, however, easy to take into account changes in the slow variables by *updating* the outer solution. \( t_0 \) can be interpreted as current time. Consequently the initial conditions (1.4.117) become the current values of the slow variables. After replacing all initial values by the respective current values in (1.4.134/141/142/144) equation (1.4.149) becomes a feedback law of the form:

\[
^1 \sigma^i_{\text{opt}} = ^1 \sigma^i_{\text{opt}}(\gamma_c, v_T, a_T, h, v, \gamma, \delta)
\]  

(1.4.157)
The continuous update of the outer solution improves accuracy of the zero-order approximation given by (1.4.149) and extends its validity to the case of only mildly decoupled time scales [38,48]. This is an important condition for the applicability of SP-theory to the short-range intercept problem where the scaling assumptions may be violated for certain scenarios (see table 1.4.2). In practical applications the update will be carried out with a finite update interval \( T_S \).

1.4.6.2 The Prediction Algorithm

The structure of the prediction algorithm has already been given in table 1.4.3. The Newton method will be applied to carry out the correction of \( t_{go} \) in step 3. If \( t_0 \) is the current time (1.4.132a) can be written as

\[
\Delta x(t_p) = F(t_{go})
\]  

(1.4.158)

In the \( i+1 \)-st iteration step the corrected value of \( t_{go} \) is:

\[
t_{go}^{i+1} = t_{go}^i - \frac{F(t_{go}^i)}{F'(t_{go}^i)}
\]  

(1.4.159)

with

\[
F'(t_{go}) = \frac{dF(t_{go})}{dt_{go}}
\]  

(1.4.160)

In order to evaluate \( F \) and \( F' \) a coordinate frame must be chosen. Since no inertial angular measurements are available a straightforward choice is the cartesian frame defined by the LOS at time \( t_0 = t \). This frame will be called seeker frame and is denoted by

\[
Z^s = (x_s, y_s)
\]  

(1.4.161)

\( Z^s \) will be kept fixed for all calculations of the prediction algorithm. From figure 1.4.4 follows that in \( Z^s \) one has:

\[
\Delta x_0 = R(t) \quad \text{(1.4.162a)}
\]

\[
\Delta y_0 = 0 \quad \text{(1.4.162b)}
\]
with (1.4.134) one obtains:

$$
\text{tg } \gamma_s^o = \frac{\int_0^{t_{go}} v_T \sin \gamma_{Ts}(\tau) \, d\tau}{R(t) + \int_0^{t_{go}} v_T \cos \gamma_{Ts}(\tau) \, d\tau}
$$

(1.4.163)

where the subscript $s$ denotes values with respect to $Z^S$. With (1.4.162) and figure (1.4.4) $F$ becomes:

$$
F(t_{go}) = v_T \int_0^{t_{go}} \cos \gamma_{Ts}(\tau) \, d\tau - \cos \gamma_s^o \int_0^{t_{go}} v_T^o(\tau) \, d\tau + R(t)
$$

(1.4.164)
Substitution into (1.4.160) yields:

\[ F(t_{go}) = \frac{aF}{\partial t_{go}}(t_{go}) + \frac{\partial F}{\partial \gamma_s^o} \frac{\partial \gamma_s^o}{\partial t_{go}}(t_{go}) \]  

(1.4.165)

with (see appendix 1.4)

\[ \frac{\partial \gamma_s^o}{\partial t_{go}}(t_{go}) = v_T \frac{x_{Ts}(t_{go}) \sin \gamma_{Ts}(t_{go}) - y_{Ts}(t_{go}) \cos \gamma_{Ts}(t_{go})}{R^2(t_{go})} \]  

(1.4.166)

\[ x_{Ts}(t_{go}) = R(t) + v_T \int_0^{t_{go}} \cos \gamma_{Ts}(\tau) \, d\tau \]  

(1.4.167a)

\[ y_{Ts}(t_{go}) = v_T \int_0^{t_{go}} \sin \gamma_{Ts}(\tau) \, d\tau \]  

(1.4.167b)

\[ R(t_{go}) = \sqrt{x_{Ts}^2(t_{go}) + y_{Ts}^2(t_{go})} \]  

(1.4.168)

Mechanization of the prediction algorithm is done by choosing the state vector

\[ \xi^t = (v, \int v(\tau) \, d\tau, \gamma_{Ts}, \int \sin \gamma_{Ts}(\tau) \, d\tau, \int \cos \gamma_{Ts}(\tau) \, d\tau)^t \]  

(1.4.169)

with

\[ \dot{\xi}_1 = - \frac{D^0(\tau)}{m}, \quad \xi_1(0) = v(t) \]  

(1.4.170a)

\[ \dot{\xi}_2 = \xi_1(\tau), \quad \xi_2(0) = 0 \]  

(1.4.170b)

\[ \dot{\xi}_3 = \frac{a_T(\tau)}{v_T}, \quad \xi_3(0) = \gamma_{Ts}(t) \]  

(1.4.170c)

\[ \dot{\xi}_4 = \sin \xi_3(\tau), \quad \xi_4(0) = \sin \gamma_{Ts}(t) \]  

(1.4.170d)

\[ \dot{\xi}_5 = \cos \xi_3(\tau), \quad \xi_5(0) = \cos \gamma_{Ts}(t) \]  

(1.4.170e)

\[ r \in [0, t_{go}], \quad \dot{()} = \frac{d}{dr}(()) \]  

(1.4.171)
The prediction algorithm now proceeds as follows:

step 1: \( i = 0 \), choose initial value \( t^i_{go} \)

step 2: solve (1.4.170) for \( r \in [0, t^i_{go}] \to \xi^i \)

step 3: compute predicted target position:

\[
\begin{align*}
    x^i_{Ts} &= R(t) + v_T \xi^i_3, \\
    y^i_{Ts} &= v_T \xi^i_4
\end{align*}
\]

step 4: compute range-to-go:

\[
R^i = \sqrt{\left( x^i_{Ts} \right)^2 + \left( y^i_{Ts} \right)^2}
\]

step 5: compute collision course (see figure 1.4.4):

\[
\begin{align*}
    \sin \gamma^o_1 &= \frac{y^i_{Ts}}{R^i}, \\
    \cos \gamma^o_1 &= \frac{x^i_{Ts}}{R^i} \\
    \gamma^o_1 &= \arctg \frac{\sin \gamma^o_1}{\cos \gamma^o_1}
\end{align*}
\]

step 6: compute

\[
\frac{\partial \gamma^o_1}{\partial t^i_{go}} = \gamma^o_1'
\]

\[
\gamma^o_1' = v_T \frac{x^i_{Ts} \sin \xi^i_3 - y^i_{Ts} \cos \xi^i_3}{R^i}
\]

step 7: compute \( F'(t^i_{go}) \):

\[
F'(t^i_{go}) = v_T \cos \xi^i_3 - \xi^i_1 \cos \gamma^o_1 + \gamma^o_1' \xi^i_1 \sin \gamma^o_1
\]

step 8: calculate \( F(t^i_{go}) \):

\[
F(t^i_{go}) = x^i_{Ts} - \xi^i_2 \cos \gamma^o_1
\]
step 9: compute improved estimate $t_{go}^{i+1}$

$$
\Delta t_{go}^i = - \frac{F(t_{go}^i)}{F'(t_{go}^i)}
$$

$$
t_{go}^{i+1} = t_{go}^i + \Delta t_{go}^i
$$

step 10: if $|\Delta t_{go}^i| < \text{tolerance}$ then $t_{go} = t_{go}^{i+1}$
otherwise: $i = i + 1$, go to 2

*table 1.4.5: prediction algorithm*

Remarks:
1) In order to execute the algorithm the following information must be available:

- current missile velocity: $v(t)$
- straight-flight drag coefficient for $D^0$-calculation: $c_D^0$
- missile mass: $m$
- current range: $R(t)$
- current target heading: $\gamma_{Ts}(t)$
- target velocity: $v_T$
- target acceleration profile: $a_T(\tau), \tau \in [0, t_{go}]$

*table 1.4.6: information requirements for prediction algorithm*

Knowledge of $v(t)$, $R(t)$, $m$, and $c_D^0$ is necessary to predict the missile flight path. $v(t)$ and $R(t)$ can be measured or estimated (Kalman filter), $m$ is a known constant and $c_D^0$ is usually available in the form of tabulated data as a function of altitude and Ma-number. In order to save on-line computing time it is possible to precompute and store the straight line missile-trajectories. Note that for the constant-altitude case only one trajectory for each altitude has to be stored. On-line integration of (1.4.170a/b) may therefore be replaced by interpolation in appropriate tables.
The current target state $\gamma_{{T_s}}(t)$, $v_T$, and $a_T(t)$ cannot be measured but may be available from an estimator like a Kalman-filter. However, no information is available about $a_T(\tau)$ for $\tau > 0$. Therefore assumptions about $a_T(\tau)$ must be made to solve the prediction problem. It is evident that deviations of the assumed target acceleration profile from the actual one will cause large prediction errors especially in the case of large $t_{go}$ and will therefore heavily degrade guidance law performance.

ii) The prediction has to be carried out at every update of the outer solution. Let $t_k$ and $t_{k+1}$ be two consecutive update times, i.e.

$$t_{k+1} = t_k + T_S$$  \hspace{1cm} (1.4.172)

If $t_{go}(k)$ is the value of $t_{go}$ found at $t_k$ a good estimate for $t_{go}(k+1)$ is:

$$t_{go}(k+1) = t_{go}(k) - T_S$$  \hspace{1cm} (1.4.173)

With this start value the prediction algorithm achieves sufficient accuracy with at most two iteration steps.

1.4.6.3 $\frac{1}{H^1}$ - minimization

In order to develop an algorithm to carry out the minimization (1.4.149) the equation is cast into a more suitable form. With (1.4.136/139/151/152) $\lambda_4^1$ becomes:

$$i_{\lambda_4^1} = \frac{m v_0}{\lambda_{30} \frac{1}{L_i(\alpha_{\text{opt}}^i)}} \left[ \lambda_{30} \frac{1}{D_i(\alpha_{\text{opt}}^i)} - \frac{v_0 \cos e_\gamma - v_f^o}{v_T \cos (\gamma_T - \gamma_{\text{opt}}^o) - v_f^o} \right] \hspace{1cm} (1.4.174)$$

Inserting (1.4.138) with (1.4.146) renders:

$$i_{\lambda_4^1} = \lambda_{30} \left[ \frac{1}{D_i(\alpha_{\text{opt}}^i)} - \frac{v_0 \cos e_\gamma - v_f^o}{\Delta v} \frac{D_0}{1} \right] \frac{v_0}{\frac{1}{L_i(\alpha_{\text{opt}}^i)}} \hspace{1cm} (1.4.175)$$

where

$$e_\gamma = 1_\gamma - \gamma_{\text{opt}}^o \quad \text{course error}$$  \hspace{1cm} (1.4.176)

$$\Delta v = v_0 - v_f^o$$  \hspace{1cm} (1.4.177)

$$D_0 = D^o(v_0)$$  \hspace{1cm} (1.4.178)
Substitution of (1.4.175) into (1.4.144) yields:

\[ 1_H^i(1\alpha^i) = -\frac{\lambda_{30}}{m} \left[ 1_D^i(1\alpha^i_{\text{opt}}) - \lambda_L(1\alpha^i_{\text{opt}}) 1_L^i(1\alpha^i) \right] \quad (1.4.179) \]

with

\[ \lambda_L(1\alpha^i_{\text{opt}}) = \frac{D_0}{1_L^i(1\alpha^i_{\text{opt}})} \frac{1_D^i(1\alpha^i_{\text{opt}})}{D_0} - \frac{v_0 \cos e_{\gamma} - v_f^o}{\Delta v} \quad (1.4.180) \]

It can be shown (appendix 1.5) that:

\[ \text{sign}(\lambda_{30}) = -1 \quad (1.4.181) \]

With (1.4.179/181) the minimization (1.4.149) may therefore be replaced by:

\[ 1\alpha^i_{\text{opt}} = \arg \min_\alpha [1_D^i(\alpha) - \lambda_L(1\alpha^i_{\text{opt}}) 1_L^i(\alpha)] \quad (1.4.182) \]

Let \( \lambda^*_L \) and \( \lambda_L(\alpha_k) \) denote the optimal value of \( \lambda_L \) according to (1.4.180) and the value obtained after replacing \( 1\alpha^i_{\text{opt}} \) by an estimate \( \alpha_k \), respectively. Using \( \lambda_L(\alpha_k) \) in (1.4.182) instead of \( \lambda^*_L \) yields:

\[ \tilde{\alpha}(\alpha_k) = \arg \min_\alpha [1_D^i(\alpha) - \lambda_L(\alpha_k) 1_L^i(\alpha)] \quad (1.4.183) \]

hence \( 1\alpha^i_{\text{opt}} \) is the solution of:

\[ \tilde{\alpha}(\alpha_k) = \alpha_k \quad (1.4.184) \]

The following iteration scheme is used to solve (1.4.184):

step 0: \( k=0 \), choose initial value \( \alpha_k \)
step 1: calculate \( \lambda_L(\alpha_k) \) from (1.4.180)
step 2: calculate \( \tilde{\alpha}(\alpha_k) \) from (1.4.183)
step 3: if \( |\tilde{\alpha} - \alpha_k| < \text{tolerance} \), then \( 1\alpha^i_{\text{opt}} = \tilde{\alpha} \)

otherwise: \( k=k+1, \alpha_k = \tilde{\alpha}, \) go to 1

*table 1.4.7: determination of optimal angle of attack*
A sufficient condition for the convergence of the above iteration scheme on the interval $I = [\alpha_{opt}^i - \Delta \alpha, \alpha_{opt}^i + \Delta \alpha]$ is [58]:

$$-1 < \frac{d\alpha^i_k}{d\alpha_k} < 1 \quad \forall \alpha_k \in I \quad (1.4.185)$$

In the following it is shown that under the assumptions

a) $\alpha(\alpha_k)$ is at least once continuously differentiable and

b) $\alpha_{opt}^i$ exists

there is always $\Delta \alpha > 0$ such that (1.4.185) is satisfied, guaranteeing local convergence of the iteration scheme.

Proof: With (1.4.151) $1H^i$ from (1.4.144) may be written as:

$$1H^i(\alpha, 1_\gamma^i) = 1H^0(\alpha, 1_\gamma^i) + \lambda^i_4 \frac{1L^i(\alpha)}{mv_0} \quad (1.4.186)$$

For arbitrary $\alpha$ and fixed $1_\gamma^i$ (1.4.152) renders:

$$1\lambda^i_4(\alpha) = -\frac{1H^0(\alpha)}{1L^i(\alpha)} \quad (1.4.187)$$

The optimal value is given by:

$$\lambda^*_4 = 1\lambda^i_4(\alpha_{opt}) \quad (1.4.188)$$

Since $\alpha_{opt}^i$ minimizes $1H^i$ one has:

$$1H^0(\alpha, 1_\gamma^i) + \lambda^*_4 \frac{1L^i(\alpha)}{mv_0} \geq 1H^0(\alpha_{opt}^i, 1_\gamma^i) + \lambda^*_4 \frac{1L^i(\alpha_{opt}^i)}{mv_0} = 0 \quad (1.4.189)$$

With (1.4.187) it follows immediately:

$$\lambda^*_4 \geq 1\lambda^i_4(\alpha) \quad (1.4.190)$$

Conclusion: $\lambda^*_4$ maximizes the function $1\lambda^i_4(\alpha)$. With assumption a) this implies locally:

$$\frac{\partial 1\lambda^i_4}{\partial \alpha}(\alpha_{opt}^i) = 0 \quad (1.4.191a)$$
or with (1.4.180):

$$\frac{\partial \lambda_L}{\partial \alpha} (^{1}\alpha_{opt}^{i}) = 0 \quad (1.4.191b)$$

From (1.4.183) follows with (1.4.191b):

$$\frac{\partial \alpha}{\partial \alpha_k} (^{1}\alpha_{opt}^{i}) = \frac{\partial \alpha}{\partial \lambda_L} \frac{\partial \lambda_L}{\partial \alpha_k} (^{1}\alpha_{opt}^{i}) = 0 \quad (1.4.192)$$

Existence of the convergence interval I given by (1.4.185) is guaranteed by (1.4.192) with assumption a).

**Discussion**

In order to perform the iteration given in table (1.4.7) the following information must be available:

a) the missile aerodynamic coefficients (missile model) for lift and drag evaluation

b) \( \gamma^0_{opt} \) and \( v^0_f \) for calculation of \( e_\gamma \) and \( \Delta v \)

c) the values of \( ^1\gamma^i \) (current value of \( \gamma \)) and \( v^0_0 \) (current value of \( v \) as discussed in section 1.4.6.1) for the calculation of \( e_\gamma, \Delta v, \) and \( D_0 \)

The main obstacles to the implementation of the boundary control law are due to the lack of information about the missile aerodynamic coefficients (which are usually subject to large inaccuracies) and about the missile heading angle \( \gamma \). \( \gamma \) should be known in the reference system \( Z^5(k) \) used for the determination of \( \gamma^0_{opt} \). Since a direct measurement is impossible \( \gamma \) could be obtained via (1.4.148) with \( \alpha \) - and \( \theta \) - measurements. Measuring \( \alpha \) is, however, costly and not common in short range missiles. Moreover the accuracy of these measurements must be enhanced for example by use of a Kalman filter. Since this approach is based on the missile model, the accuracy of the \( \alpha \) - estimates is linked to the accuracy of the missile-data. Therefore the boundary layer control law is very sensitive to errors in the missile model which affect the minimization process directly via (1.4.182) and indirectly via the \( \alpha \) - estimation.
1.4.6.4 Summary of the Guidance Scheme and Introduction of a Simplified First Boundary Layer (FBL)-Control Law

The guidance scheme derived in sections 1.4.5/6 consists of a predictor and the FBL control law. The predictor output is the desired heading angle $\gamma^0_{opt}$ based on the predicted point of intercept. Prediction is performed by assuming the unknown future target maneuver and using the missile's straight-flight characteristics for the flight-path calculations. In addition to the optimal heading angle a weighting factor $\lambda_L$ is determined which influences the performance index of the static minimization in the first boundary layer.

The FBL control law serves the correction of the course error $e_\gamma$ in such a way that a weighted sum of lift and drag is minimized. Its output is the commanded lateral acceleration $a_c$. The choice of $\lambda_L$ leads to a compromise between minimizing the time to correct $e_\gamma$ (i.e. maximizing lift) and minimizing induced drag losses.

The information requirements include the current and future target state as well as the knowledge of the missile state and aerodynamic data. The structure of the guidance loop is depicted in figure 1.4.5.

The guidance scheme described above will be referred to as OCE (Optimal Course Error Control) since it evaluates an optimal acceleration command based on the course error. The target state estimator shown in figure 1.4.5 will be discussed in chapter 2 of this thesis.

An obvious approach leading to substantial simplifications of the FBL-calculations is the use of proportional course error control (PCE), i.e. the optimization process (1.4.182) is replaced by the proportional controller:

$$a_c = K e_\gamma$$  (1.4.193)

where $K$ is a navigation gain which depends on the intercept geometry. A guidance law with the same structure is also investigated in [22]. Clearly (1.4.193) does not optimize induced drag losses any more and hence will result in a longer time-to-go than OCE. On the other hand the missile aerodynamic coefficients need not be known for implementation of PCE. Comparison of OCE and PCE in the subsequent simulations will show how much time can be gained by taking into account induced drag losses in the FBL-control.
figure 1.4.5: structure of guidance loop based on singular perturbation analysis
1.4.7 Simulations

In this section the (near) optimal guidance law OCE will be compared with the simpler but not time optimal guidance laws PCE and PE (section 1.3.5). PCE uses the same predictor as OCE and therefore still tries to establish the time optimal outer solution given by a straight line through the predicted point of intercept. Induced drag losses are, however, not considered any more during course error corrections. Therefore the comparison of OCE and PCE will reveal the savings of time due to shaping the acceleration profile in such a way that drag losses are minimized.

The guidance law PE does not perform any prediction but tries to keep the LOS-rate zero which is a sufficient condition for intercept. The resulting flight path is in general not time optimal and the comparison with OCE will show the time losses which result if no predictor is used.

Two intercept scenarios will be discussed. In scenario A the initial range is large and the problem duration long. The scaling assumptions inherent in the guidance laws OCE and PCE are satisfied (see section 1.4.4). In scenario B the initial range and problem duration are short and the missile maneuvers with its maximum acceleration capability most of the time. Here the scaling assumptions are violated.

First consider scenario A depicted in figure 1.4.6 with

\[ v_0 = 0.9 \text{ Ma initial missile velocity} \]
\[ v_T = 0.9 \text{ Ma target velocity (constant)} \]
\[ a_T = -6g \text{ target lateral acceleration (constant)} \]
\[ R_0 = 7.5 \text{ km initial range} \]
\[ h = 10 \text{ km altitude} \]

\[ \text{figure 1.4.6: scenario A} \]
The initial missile-target configuration is "head-on". The target performs an evasive maneuver with constant speed and constant lateral acceleration. This maneuver is assumed to be known for collision point prediction by OCE and PCE. For PE the current target state \((v_T, v_T, a_T)\) is assumed to be known. The main simulation results are summarized in the following table:

<table>
<thead>
<tr>
<th>guidance law</th>
<th>miss distance [m]</th>
<th>problem duration [s]</th>
<th>figures</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCE</td>
<td>1.21</td>
<td>14.29</td>
<td>1.4.7</td>
</tr>
<tr>
<td>PCE</td>
<td>1.16</td>
<td>14.80</td>
<td>1.4.8</td>
</tr>
<tr>
<td>PE</td>
<td>1.05</td>
<td>16.05</td>
<td>1.4.9</td>
</tr>
</tbody>
</table>

*table 1.4.8: simulation results of scenario A*

**Discussion of results:**
Before the simulation results are analyzed, some remarks on guidance law activation and missile lateral acceleration limits are necessary. For the type of missile under consideration here the guidance law is activated after fuel-burn-out which occurs about 2.2s after launch. Consequently there is no missile maneuver during the initial boost phase. The missile velocity profile is characterized by a steep increase in velocity until fuel-burn-out and a subsequent decrease due to drag losses (figure 1.4.11f). Since the aerodynamic forces are proportional to the squared velocity, the missile's maximum lateral acceleration is determined by the velocity profile and hence decreases after fuel-burn-out. If the commanded acceleration exceeds the current maximum possible acceleration flipper saturation occurs (figure 1.4.8c). The flipper deflection limits are +26 deg, -26 deg. During the period of flipper saturation the missile performs a damped pitch oscillation which is seen in the acceleration profile (figure 1.4.8d).

A look at the missile flight path obtained with OCE (figure 1.4.7a) shows that the missile heading angle \(\gamma\) reaches the outer solution after an initial transient. The missile maneuvers on a nearly straight line with low but nonzero lateral acceleration (figure 1.4.7c,d). Exact straight-flight conditions (i.e. zero course error and zero lateral
acceleration) are, however, never reached. This is due to the induced drag minimization in the first boundary layer which prohibits the use of excessive accelerations that would be necessary to correct the course error completely. Only at final time, when the missile velocity and therefore drag losses are minimal, complete course error correction occurs (figure 1.4.7e) in order to achieve zero miss distance.

In contrast to OCE, PCE nulls the course error in a strong initial maneuver with maximum acceleration (figure 1.4.8c,d,e). After course error correction the missile moves on a straight line with zero lateral acceleration. Clearly, induced drag losses are not minimized because the maneuver takes place during the period of maximum velocity. The comparison of the problem durations of OCE and PCE in table 1.4.8 shows, however, that the time loss due to increased drag losses in PCE is neglegible. On the other hand a significant increase to flight time results with application of PE. Since PE tries to null the LOS-rate the resulting LOS-rate profile (figure 1.4.9b) and flight path (figure 1.4.9a) are far from the time optimal trajectories (figures 1.4.7a,b).

The above simulations suggest that savings in flight time are mainly due to the flight path planning based on the prediction of the point of intercept, while shaping the acceleration profile for induced drag minimization has practically no effect on the problem duration.
MISSILE AND TARGET POSITION

MISSILE

TARGET

LOS-RATE

figure 1.4.7a,b
FLIPPER DEFLECTION

LATERAL ACCELERATION

figure 1.4.7c,d
COURSE ERROR

![Graph showing course error over time]

figure 1.4.7e
MISSILE AND TARGET POSITION

LOS-RATE

figure 1.4.8a,b
FLIPPER DEFLECTION

LATERAL ACCELERATION

figure 1.4.8c,d
COURSE ERROR

\[ \text{figure 1.4.8e} \]
FLIPPER DEFLECTION

LATERAL ACCELERATION

figure 1.4.9c,d
The guidance laws will now be investigated for scenario B (figure 1.4.10) with

\[ \gamma_{T0} = 45^\circ \] initial target heading

\[ R_0 = 5 \text{km} \] initial range

all other parameters as in scenario A

\[ \begin{align*}
\gamma_0 & \quad \text{始点} \\
R_0, \text{LOS} & \\
M & \quad \text{中心} \\
\gamma_{T0} &
\end{align*} \]

\textit{figure 1.4.10: scenario B}

Table 1.4.9 shows the main simulation results:

<table>
<thead>
<tr>
<th>guidance law</th>
<th>miss distance [m]</th>
<th>problem duration [s]</th>
<th>figures</th>
</tr>
</thead>
<tbody>
<tr>
<td>OCE</td>
<td>27.64</td>
<td>9.34</td>
<td>1.4.11</td>
</tr>
<tr>
<td>PCE</td>
<td>0.38</td>
<td>9.33</td>
<td>1.4.12</td>
</tr>
<tr>
<td>PE</td>
<td>0.03</td>
<td>9.27</td>
<td>1.4.13</td>
</tr>
</tbody>
</table>

\textit{table 1.4.9: simulation results of scenario B}
MISSILE AND TARGET POSITION

\[ X \text{ [M]} \times 10^3 \]
\[ Y \text{ [M]} \times 10^3 \]

\[ T. \text{TRAJ. B} \]
\[ \text{JOB : Y3213PO} \]
\[ \text{DATE : 22/06/86} \]
\[ \text{G.LAW : 5} \]
\[ \text{FILTER : 0} \]
\[ \text{TIME : 13.42.27.} \]

\( \circ \text{ MISSILE} \)
\( \triangle \text{ TARGET} \)

LOS-RATE

\[ \text{RAD/S} \]
\[ \text{TIME [S]} \]

\( \text{figure 1.4.11a,b} \)
FLIPPER DEFLECTION

LATERAL ACCELERATION

figure 1.4.11c,d
MISSILE AND TARGET POSITION

figure 1.4.12a,b
FLIPPER DEFLECTION

LATERAL ACCELERATION

figure 1.4.12c,d
COURSE ERROR

figure1.4.12e
MISSILE AND TARGET POSITION

LOD-RATE

figure1.4.13ab
FLIPPER DEFLECTION

LATERAL ACCELERATION

figure 1.4.13c,d
Discussion of results:
In this scenario OCE yields a large miss distance. The reason becomes clear from figures (1.4.11c,d,e). The missile maneuver starts with maximum acceleration. In order to reduce drag losses the acceleration is reduced after about 5.5 seconds (after launch) resulting in a considerable final course error (figure 1.4.11e). The effort to null this error close to final time fails, because the required acceleration exceeds the maneuver capability of the missile (flipper saturation).

The guidance law PCE applies maximum acceleration until the course error vanishes (figure 1.4.12c,d,e) and thus achieves low miss distance. Note that the periodic disturbance in the flipper deflection and acceleration profiles of OCE and PCE are due to the update of the outer solution (course error update).

The guidance law PE does not only yield the lowest miss distance but also the shortest flight time in this scenario, implying that trying to reach the straight-line intercept course is not time optimal in this case. From figure 1.4.12e it becomes clear that the main part of the trajectory consists of the $\gamma$ - boundary layer and only the last second is on the outer solution ($e_\gamma = 0$). Hence the scaling assumptions, especially the decoupling of the $\gamma$ - dynamics from the position dynamics according to (1.4.109/110) are no longer valid. Therefore the SP guidance scheme is not time optimal any more.

In summary the following conclusions can be drawn from the simulation results:

i) For long ranges time-optimality of a guidance law is mainly achieved by predicting (correctly) the point of intercept and choosing the time optimal path through this point. Taking into account induced drag losses does not yield significant savings in flight time. However, it results in a final course error entailing large miss distances if the correction requires a lateral acceleration that exceeds the missile's maneuver capability. In fact the path planning in the outer solution should be done by taking into account the missile's lateral acceleration limit. But since this limit depends on the missile velocity and the velocity loss depends on the missile maneuver, the flight path planning is coupled to the rotational dynamics. Therefore acceleration limits cannot be treated with the SP-approach described in section 1.4.5.

ii) The guidance law PE yields time losses for "long-range" intercept scenarios since the associated flight path is not time optimal. In short range scenarios with high maneuvers PE is superior to OCE and PCE since the scaling assumptions ensuring optimality of OCE and PCE are not valid due to the missile's acceleration limits.
iii) The savings in flight time obtained by OCE and PCE in certain scenarios are due to the accurate prediction of the target maneuver as mentioned before. Since accurate prediction is usually impossible the theoretical savings in time will be strongly reduced if not offset by errors in the predicted target motion (chapter 2).

1.4.8 Remarks on Interception in a Vertical Plane

The preceding analysis of the planar intercept problem was restricted to a horizontal plane. In this section the main aspects of interception in a vertical plane will be discussed.

The equations of the planar intercept problem in a vertical plane differ from (1.4.91-93) in the appearance of gravitational terms and the altitude $h$ as an additional state variable. They are given below without the missile pitch dynamics which are not needed for guidance law derivation as discussed before.

missile-target relative motion:

\[ \Delta \dot{x} = v_T \cos \gamma_T - v \cos \gamma \]  
(1.4.194a)

\[ \Delta \dot{y} = v_T \sin \gamma_T - v \sin \gamma \]  
(1.4.194b)

missile dynamics:

\[ \dot{v} = \frac{T - D}{m} - g \sin \gamma \]  
(1.4.195a)

\[ \dot{\gamma} = \frac{L}{mv} - \frac{g}{v} \cos \gamma \]  
(1.4.195b)

altitude:

\[ \dot{h} = v \sin \gamma \]  
(1.4.196)

target trajectory:

\[ v_T = v_T(t) \]  
(1.4.197a)

\[ \gamma_T = \gamma_T(t) \]  
(1.4.197b)
The main differences between vertical and horizontal interception stem from the dependence of the missile aerodynamic coefficients on the altitude \( h \). The type of solution obtained by an SP-analysis depends on the scaling of the equations (1.4.194-197). There are two important scaling assumptions that concern

a) the coupling of the \( h- \) and \( \Delta x- \) dynamics

b) the coupling of the \( h- \) and \( v- \) dynamics

If the distance travelled in \( x \)- direction (figure 1.4.14) is much larger than the altitude range of the missile, \( h \) may be treated in a faster time scale than \( x \). This decoupling is justified for long-range missiles. The coupling of \( h \) and \( v \) is governed by the exchange of kinetic and potential energy. The investigation of the minimum time-to-climb problem \([50,51,52,57]\) has shown that the use of the specific energy

\[
E = h + \frac{1}{2} \frac{v^2}{g}, \quad \dot{E} = \frac{T - D}{mg} \quad v
\]

(1.4.198a,b)

as a state variable rather than \( v \) is very appropriate to describe this effect and allows a decoupling of the \( h- \) and \( E- \) dynamics since changes in \( E \) are much slower than changes in \( h \). The missile model with (1.4.198) instead of (1.4.195a) is called energy-state model.
The intercept problem using the energy-state model has been solved in [45, 46, 48, 53, 56] with $E$ decoupled from $\Delta x$ and in [49] using the same time scale for $\Delta x$ and $E$. [45, 48, 49, 56] treat the three dimensional case which is a straightforward extension of the planar problem. Qualitatively the optimal missile flight path consists of an initial $h$-boundary layer to the optimal cruise altitude $h^*$. On the cruise path the missile has reached straight flight conditions with lift compensating weight. If $E$ is decoupled from $\Delta x$, $h^*$ is determined by the maximum missile velocity. Near the point of intercept I the missile leaves the cruise arc on a terminal boundary layer through I. While this solution is valid for long range missiles it does not apply to short range scenarios where the dynamics of $\Delta x$, $\Delta y$, $h$, and $E$ are coupled. This case will shortly be discussed in the sequel. Consider (1.4.194-197) with the following scaling:

\[
\Delta x = v_T \cos \gamma_T - v \cos \gamma \quad (1.4.199a)
\]
\[
\Delta y = v_T \sin \gamma_T - v \sin \gamma \quad (1.4.199b)
\]
\[
\dot{v} = -\frac{D}{m} \quad (1.4.199c)
\]
\[
h = v \sin \gamma \quad (1.4.200)
\]
\[
\epsilon \gamma = \frac{L}{mv} - \frac{g}{v} \cos \gamma \quad (1.4.201)
\]

In (1.4.199c) the thrust is zero according to (1.4.95) and the gravitational force has been neglected compared to the drag losses. The Hamiltonian of the outer solution is:

\[
H^O = 1 + \lambda^O_1 (v_T \cos \gamma_T - v^O \cos \gamma^O) + \lambda^O_2 (v_T \sin \gamma_T - v^O \sin \gamma^O)
- \lambda^O_3 \frac{D^O}{m} + \lambda^O_4 v^O \sin \gamma^O = 0 \quad (1.4.202)
\]

with $\lambda^O_1$, $\lambda^O_2$ constant according to (1.4.125) and

\[
\lambda^O_4 = -\frac{\partial H^O}{\partial h} = \frac{\lambda^O_3}{m} \frac{\partial D^O}{\partial h} \quad (1.4.203a)
\]

where

\[
\lambda^O_4(t_f) = 0 \quad (1.4.203b)
\]

since the final altitude is free.
The optimal missile heading is given by:

\[
\frac{\partial H^o}{\partial \gamma^o} = 0 \tag{1.4.204}
\]

yielding:

\[
\lambda_1^0 \sin \gamma_{\text{opt}}^o - \lambda_2^0 \cos \gamma_{\text{opt}}^o + \lambda_4^0 \cos \gamma_{\text{opt}}^o = 0 \tag{1.4.205}
\]

\[
\Rightarrow \quad \tan \gamma_{\text{opt}}^o = \frac{\lambda_2^0 - \lambda_4^0}{\lambda_1^0} \tag{1.4.206}
\]

In contrast to (1.4.130) \( \gamma_{\text{opt}}^o \) is not constant any more since \( \lambda_4^o \) is time-varying. The outer solution could be determined by estimating \( v(t_f) \) and \( h(t_f) \), integrating backward (1.4.199/200) with \( \Delta x(t_f) = \Delta y(t_f) = \lambda_4^o(t_f) = 0 \), and \( \gamma_{\text{opt}}^o \) from (1.4.206) and iteratively correcting \( v(t_f) \) and \( h(t_f) \) until the given initial conditions \( \Delta x(t_0) \), \( \Delta y(t_0) \), \( v(t_0) \), \( h(t_0) \) are satisfied. Estimates for \( h(t_f) \) and \( v(t_f) \) to start the iteration can easily be obtained from the outer solution of section (1.4.5.1). Note that the angle of attack is not zero in the outer solution (in contrast to the horizontal problem) because of the nonzero lift needed for weight compensation in (1.4.201).

With (1.4.203b) follows that the flight path given in (1.4.206) approaches the straight line defined by (1.4.130) near final time. Note that (1.4.130) is the flight path determined by the shortest distance between the current missile position and the point of intercept. It is therefore obvious that significant deviations from this path will only take place for long problem durations that allow excursions to optimal altitude levels for periods long enough to offset the time losses due to the increased range-to-go.

As the simulations of the last section suggest, the outer solution is never reached for many scenarios, i.e. all the motion takes place in the \( \gamma \)-boundary layer. It also turned out that the effect of drag minimization is minor compared to the savings in time obtained by choosing the shortest path to the collision point. For the type of missile investigated here the intercept problem in the vertical plane may therefore be treated like the constant-altitude case.
1.5 Summary and Conclusions

In the first part of this thesis two approaches to guidance law derivation via optimal control theory have been discussed. First, guidance laws based on linearization of the intercept problem were reviewed, then the time scale separation by SP-theory was investigated. It was pointed out the the main drawback of the resulting guidance schemes is the need to predict the future target maneuver. This is a direct consequence of formulating the intercept problem as an optimal control problem which inevitably leads to a two-point-boundary-value problem requiring knowledge of the future target maneuver. Since the target maneuver is unknown, assumptions must be made that degrade guidance law performance if the actual target behaviour is different. A guidance law (PE) was derived that avoids these difficulties. It is based on a linearization around the sufficient condition for intercept

$$\dot{\varphi} = 0$$ \hspace{1cm} (1.5.1)

and subsequent solution of an LQ-problem. The nominal control needed to establish the nominal trajectory associated with (1.5.1) is a target maneuver compensation (TMC) requiring knowledge of the current maneuver only. The solution of the LQ-Problem yields the well known Proportional Navigation (PN). Hence PE is a superposition of TMC and PN. Simulations demonstrated the superiority of PE over PN and showed that, in contrast to PN, the navigation gain of PE is practically independent of the intercept scenario.

The application of SP-theory leads to near time optimal solutions of the planar intercept problem for scenarios with long initial ranges. In these cases the underlying scaling assumptions are satisfied. The missile flight path is given by a boundary layer from the initial conditions to the outer solution and a subsequent straight flight to the predicted point of intercept. Simulations comparing the SP solution with PE show that time savings can be achieved through time optimal flight path planning. PE turns out to be superior in the case of high maneuvers during most of the trajectory, i.e. if the scaling assumptions of the SP-approach are violated. The simulations also revealed that the time savings are mainly due to the optimal flight path planning rather than drag minimization in the FBL-control law. The problems of implementing the SP control law which are due to substantial information requirements about the missile aerodynamic coefficients and the target maneuver have been discussed. Investigation of the intercept problem in a vertical plane showed that for short range missiles the
missiles the solution for the horizontal plane is valid here as well since there is no time for altitude transitions on energy-climb and -descent paths.

The results suggest that the SP-approach does not lead to implementable guidance laws for short range missiles: In scenarios with high maneuvers the scaling assumptions are invalid which sharply reduces SP-guidance-law performance. For long ranges the need to predict the point of intercept is likely to offset the advantages over PE in flight time. On the other hand PE has been found a robust (with respect to changing intercept scenarios) and simple guidance law with minimum information requirements. Its implementability depends on the availability of information on the current target maneuver. An estimator for this information will be derived in the next chapter.
2. The Tracking Problem

The tracking problem consists of estimating the state (position, velocity, acceleration) of a target by processing information gathered by a fixed or moving observer. The discussion in section 1.5 has shown that the solution of the tracking problem must be part of the development of implementable modern guidance laws. In the intercept problem discussed here, the missile plays the role of the observer. The tracking problem poses considerable difficulties if the information about the missile-target relative geometry is obtained by passive sensors which provide bearing- or bearing-rate measurements only. This is the case for the intercept problem discussed here. The planar Bearing (-rate) -Only Measurement Problem (BOMP) is stated in the next section.

2.1 Statement of the Tracking Problem

Consider the scenario depicted in figure 2.1.1. The target T is tracked by the observer M. It is assumed that the observer maneuver is completely known. The bearing angle $\phi$ or the bearing rate $\dot{\phi}$, respectively, is the only information available about the missile-target relative motion. The task is to estimate the current target state (for example given by $\Delta x, \Delta y, \Delta x, \Delta y$) based on the known observer maneuver and noisy $\phi$- or $\dot{\phi}$- measurements.

*figure 2.1.1: geometry of the planar tracking problem*
A straightforward approach to the BOMP is the use of an Extended Kalman Filter (EKF). Prior to the discussion of applications of the EKF to the BOMP its equations are summarized for later use [75,76]:

Consider a nonlinear dynamic system governed by the following discrete stochastic equations:

\[ y(k+1) = f[y(k), u(k+1)] + w(k+1) \]  

subject to the initial condition:

\[ y(0) = y_0 \]

with

- \( y(k) \in \mathbb{R}^n \) state vector at time \( t_k \)
- \( u(k) \in \mathbb{R}^m \) deterministic input at time \( t_k \)
- \( w(k) \in \mathbb{R}^n \) Gaussian white noise sequence with:
  \[ E[w(k)] = 0 \]
  \[ E[w(i)w^t(j)] = \delta_{ij} Q(i) \]
- \( y(k+1/k) \) estimated value of \( y(k+1) \) based on the measurements \( m(i), i=1, \ldots, k \)
- \( y(k+1/k+1) \) estimated value of \( y(k+1) \) based on the measurements \( m(i), i=1, \ldots, k+1 \)

\[ m(k) = h[y(k)] + s(k) \in \mathbb{R}^p \]

where

- \( h \in \mathbb{R}^p \) is a nonlinear function of \( y \)
- \( s \in \mathbb{R}^p \) is a Gaussian white measurement noise with:
  \[ E[s(k)] = 0 \]
  \[ E[s(i)s^t(j)] = \delta_{ij} S(i) \]
  \[ E[w(i)s^t(j)] = 0 \]

notations:

\( \hat{y}(k+1/k) \) : estimated value of \( y(k+1) \) based on the measurements \( m(i), i=1, \ldots, k \)

\( \hat{y}(k+1/k+1) \) : estimated value of \( y(k+1) \) based on the measurements \( m(i), i=1, \ldots, k+1 \)
The filter equations can now be summarized as follows:

**state propagation:**
\[
\hat{y}(k+1/k) = f[\hat{y}(k/k), u(k+1)] \tag{2.1.9}
\]
initial condition: \(\hat{y}(0/0) = \hat{y}_0 \tag{2.1.10}\)

**covariance propagation:**
\[
P(k+1/k) = W(k+1,k)P(k/k)W^t(k+1,k) + Q(k+1) \tag{2.1.11}
\]
with
\[
W(k+1,k) = \frac{\partial f[\hat{y}(k/k), u(k+1)]}{\partial \hat{y}(k/k)} \tag{2.1.12}
\]
initial condition: \(P(0/0) = P_0 \tag{2.1.13}\)
\[
P = P^t > 0 \in R^{n \times n} ; Q = Q^t \geq 0 \in R^{n \times n} \tag{2.1.14 a,b}
\]

**state update:**
\[
\hat{y}(k+1/k+1) = \hat{y}(k+1/k) + G(k+1)[m(k+1) - \hat{m}(k+1)] \tag{2.1.15}
\]
with
\[
\hat{m}(k) = h[\hat{y}(k)] \tag{2.1.16}
\]

**Kalman gains:**
\[
G(k+1) = P(k+1/k)C^t(k+1)[C(k+1)P(k+1/k)C^t(k+1) + S(k+1)]^{-1} \tag{2.1.17}
\]
\[
C(k) = \frac{\partial h}{\partial \hat{y}}[\hat{y}(k+1/k)] \in R^{m \times n} \tag{2.1.18}
\]

**covariance update:**
\[
P(k+1/k+1) = [I - G(k+1)C(k+1)] P(k+1/k) [I - G(k+1)C(k+1)]^t + \]
\[
+ G(k+1)S(k+1)G^t(k+1) \tag{2.1.19}
\]

**Definitions:**
\[
r(k) := m(k) - \hat{m}(k) \quad \text{measurement residual} \tag{2.1.20}
\]
\[
e(k/k) := \hat{y}(k/k) - y(k) \quad \text{estimation error} \tag{2.1.21}
\]
Remarks:

i) For linear systems (2.1.1) and linear measurements (2.1.5) the estimation errors are zero-mean and Gaussian. In this case, $P$ is the error covariance matrix given by:

$$P(k/k) = E[e(k/k)e^T(k/k)] \quad (2.1.22)$$

In the nonlinear case $P$ is, at best, an approximation to (2.1.22). An important reason for filter divergence is the underestimation of estimation errors which occurs if the estimated error variances (diagonal elements of $P$) are smaller than the actual error variances. Note that as $P$ tends to zero the filter gains become zero and the filter works as a mere predictor without taking into account new measurements. For more details see [75,76] and section 2.6.4.2.

ii) The filter equations of a specific system are determined if the functions $f$, $h$ and the matrices $W$ and $C$ are known. For filter implementation the initial conditions $\hat{y}_0$, $P_0$, and the noise statistics $S$ and $Q$ must be specified.

2.2 Applications of the EKF to the BOMP - A Review

In this section the main results and problems arising in the application of the EKF to the BOMP will be discussed. On this basis a new tracking filter will be developed in later sections.

A prime source of divergence of tracking filters based on bearing (-rate) - only measurements is the lack of complete observability (see for example [65]). Moreover, due to the nonlinearity of the filtering problem, the behaviour of the filter is dependent on the coordinates used to formulate the filter equations [61]. In a cartesian coordinate frame the system dynamics (observer-target relative motion) are linear and the measurement equation is nonlinear. Formulating the filter equations in polar coordinates yields nonlinear system dynamics and a linear measurement equation.

First, the formulation in cartesian coordinates is considered. Defining the cartesian state vector:

$$z^t = (\Delta x, \Delta y, \Delta \dot{x}, \Delta \dot{y}) \quad (2.2.1)$$
the equations of motion become (see figure 2.1.1):

\[ \begin{align*}
\dot{z}_1 &= z_3 \quad (2.2.2a) \\
\dot{z}_2 &= z_4 \quad (2.2.2b) \\
\dot{z}_3 &= \Delta a_x \quad (2.2.2c) \\
\dot{z}_4 &= \Delta a_y \quad (2.2.2d)
\end{align*} \]

with

\[ \begin{align*}
\Delta a &= a_T - a \quad \text{relative acceleration} \quad (2.2.3) \\
a_T^t &= (a_{Tx}, a_{Ty}) \quad \text{target acceleration} \quad (2.2.4) \\
a^t &= (a_x, a_y) \quad \text{missile acceleration} \quad (2.2.5)
\end{align*} \]

The BOMP has been intensively studied for non-maneuvering targets. The quantities \( \Delta a_x \) and \( \Delta a_y \) then are determined by the observer maneuver and therefore known. Hence, (2.2.2) may easily be integrated yielding the discrete version of the equations of motion. Unfortunately, the measurement equation associated with (2.2.2) is nonlinear. For bearing-only measurements one obtains:

\[ m = \varphi + s = \arctan \frac{z_2}{z_1} + s \quad (2.2.6) \]

for bearing-rate-only measurements the measurement equations is:

\[ m = \dot{\varphi} + s = \frac{(z_1 + z_2)^2}{z_1^2} \frac{z_1 z_4 - z_2 z_3}{z_1^2} + s \quad (2.2.7) \]

where \( s \) is a zero-mean white measurement noise according to (2.1.6/7).

The EKF using the cartesian state \( z \) will be called cartesian EKF (CEKF). This filter, based on the measurement (2.2.6) has been found to exhibit serious stability problems. In [60] and [65] it was shown that the filter behaviour is very sensitive to the selection of the initial error covariance matrix \( P_0 \) (see section 2.1). Linearization of (2.2.6) according to (2.1.17) results, via the filter gain computations (2.1.16), in a feedback of estimation errors into the covariance calculations (2.1.18) and may therefore cause divergence. Filter stability may be considerably improved by decoupling the
covariance equations from the state estimates. This can be done by replacing the nonlinear measurement equation by pseudolinear measurements [59, 60, 63, 71]. The pseudolinear measurement equation associated with (2.2.6) is:

\[ \tilde{m} = C(m) z + \tilde{s} = 0 \]  \hspace{1cm} (2.2.8)

with

\[ C(m) = [-\sin(m), \cos(m), 0, 0] \]  \hspace{1cm} (2.2.9)

\[ \tilde{s} = R \sin(s) \]  \hspace{1cm} (2.2.10)

The new measurement noise \( \tilde{s} \) is, however, non-Gaussian, which results in biased estimates of the pseudolinear tracking filter [63, 71].

An important reason for filter instability is the lack of complete observability of \( z \) prior to the first observer maneuver [59, 62, 64]. This explains the sensitivity problems associated with covariance initialization mentioned above. It is shown in [64] that observable and unobservable states can be decoupled if the filter states are chosen to be modified polar coordinates (MPC):

\[ y_m^t = (\varphi, \frac{R}{R}, \varphi, \frac{1}{R}) \]  \hspace{1cm} (2.2.11)

While \( y_{m1}, y_{m2}, y_{m3} \) are always observable for \( \varphi \neq 0 \), \( y_{m4} \) is unobservable for a non-maneuvering observer. The covariance equations associated with (2.2.11) are, however, decoupled in the observable and unobservable part of \( y_m \). Therefore the EKF based on (2.2.11) remains stable in the observable states even if the \( y_{m4} \)-estimate diverges. Due to the coupling this is not true for the CEKF. Note that the measurement equation associated with (2.2.11) is linear for both cases (2.2.6) and (2.2.7). The estimates of the MPC-filter are therefore asymptotically unbiased.

The observability of the filter states is linked to the observer maneuver. In addition to the case of a non-maneuvering observer there are other maneuvers for which the filter states are not completely observable. They are discussed in [62]. The most important result for practical applications is that an observer maneuver resulting in constant bearing angle throughout the scenario results in unobservable filter states. Note that this type of maneuver is a typical nominal condition for many guidance laws (chapter 1).
In principle there are two approaches to the design of tracking filters in the presence of target maneuvers. One method is to interpret the target maneuver as an external disturbance. The target acceleration components are therefore not included in the filter state and hence will not be estimated. The second method includes the target acceleration in the filter state. Essentially, the first approach is based on two hypotheses:

\[ H_0: \text{non-maneuvering target} \]
\[ H_1: \text{maneuvering target} \]

The target maneuver must be specified in \( H_1 \). Typically the following model is used:

\[
\tau_T \ddot{a}_T = \ddot{a}_T - \ddot{a}_T^* + w
\]  

(2.2.12)

\( \tau_T \): correlation time  
\( \ddot{a}_T^* \): maneuver level  
\( w \): zero-mean white noise process

The task consists now of a detection problem (choose \( H_0 \) or \( H_1 \)) and state estimation by an EKF. It is obvious that this approach is not suitable for tracking continuously time-varying target maneuvers but is restricted to piecewise constant maneuvers. In order to track arbitrary maneuvers several hypotheses, each associated with a different maneuver level are tested [68,69]. The selection of a specific hypothesis can be done by a likelihood test conditioned on a finite number of past measurement residuals [66,69,72] or by ad hoc methods testing the size and correlation of the residuals [67]. All of these procedures require the use of filter banks and result in excessive computing and storage requirements. Moreover stability problems occur if none of the chosen target models matches the actual target behaviour [69].

If the target maneuver is included in the filter state, maneuver detection becomes obsolete. However, a model of the acceleration dynamics is needed in order to solve the extrapolation equations of the EKF. Since the target maneuver is unknown, the target dynamics are usually formulated as a random process. The following models have been investigated in [70] in conjunction with an EKF based on cartesian coordinates and bearing-only measurements:
a) \[ \dot{a}_T = w \]

b) \[ \tau_T \dot{a}_T = a_T - \ddot{a}_T + w \]

c) \[ \ddot{a}_T = w \]

with \( w \) from (2.2.12)

It was found that the EKF performance is generally poor for all three models. Model c) yielded slightly worse results (in terms of estimation accuracy) than models a) and b) which behaved similarly. The low estimation accuracy of the EKF may partly be attributed to the use of cartesian coordinates as discussed before.

Up to now the basic problems of target tracking have been discussed. For the solution of the intercept problem the estimated target information is used by the guidance law. The interaction between the guidance law and the tracking filter is therefore an important aspect affecting the design of both guidance law and tracking filter. Due to the low information about the missile-target relative geometry provided by bearing (rate) -only measurements stability of the tracking filter cannot always be guaranteed. It is therefore possible that the guidance law receives wrong information about the target maneuver. This may result in large miss distances. Hence, it is desirable to have guidance laws that are "robust" with respect to tracking errors.

The stability of the tracking filter depends on the missile maneuver as mentioned before. Unfortunately observability is lost if the bearing-angle is constant which is a nominal condition of many guidance laws, for example PE (see section 1.3.5). A remedy could be trajectory modulation [73,74]. However, the resulting guidance laws cannot be obtained in closed form but involve on-line solution of a TPBVP. This topic has already been discussed in chapter 1. Moreover the missile maneuvers required to enhance observability result in increased drag losses and thus diminish the missile's maneuver capability (section 1.4.7) which is undesirable for the type of missile investigated here. A different approach was taken in [72] where the uncertainty of the target state was coupled to an LQ-design based guidance law by adding the associated covariance matrix to the control weighting matrix in the quadratic cost criterion. In this way the bandwidth of the guidance law is adapted to the accuracy of the estimated target maneuver. This scheme is, however, only meaningful if the covariance estimates computed by the tracking filter are accurate, i.e. are a measure of the true estimation errors. Due to the weak observability this condition is often not satisfied for the
BOMP. In addition, the guidance law in [72] requires backward integration of the Riccati equation, implying that the characteristics of the estimation errors must be known in advance which is not very realistic.

In summary it can be stated that the BOMP is well understood for non-maneuvering targets. The important sources of tracking filter instability have been identified and effective counter measures (such as pseudolinear measurements, use of non-cartesian coordinates) were proposed. For maneuvering targets, approaches based on maneuver detection (multiple model filters) are not adequate to track targets that perform continuous maneuvers. The inclusion of the target acceleration as an additional filter state is in principle the right way to track general target maneuvers but the question of modeling the target dynamics is still open. The most common model is a first-order Gauss-Markov process. The observability analysis carried out in [64] has not yet been extended to maneuvering targets. Only very simple target maneuvers, such as constant acceleration and jumps in the acceleration have been investigated. Filter performance was not tested for continuously time-varying (for example sinusoidal) maneuvers.

The purpose of the subsequent investigation is to close this gap. An observability analysis for the BOMP with maneuvering targets is carried out. It is shown that the EKF designed to track piecewise constant-acceleration maneuvers does not track general time-varying maneuvers. An adaptive multiple time scale filter is developed for this case. Questions of coupling between guidance law and filter will be discussed. It will be shown that the guidance law PE exhibits a certain robustness with respect to tracking errors and is therefore ideally suited for use in conjunction with the tracking filter derived here. The filter design will be done under the restrictions already considered for the guidance law design: no inertial angular measurements available, passive seeker. Therefore the filter will be based on measurements of the bearing rate rather than the bearing angle.

2.3 The Target Model and Filter States

One of the main goals of the subsequent tracking filter design is to obtain estimates of the target maneuver, i.e. the target acceleration. These estimates are needed for the implementation of modern guidance laws such as PE. Target maneuver estimation by an extended Kalman filter requires a model of the acceleration dynamics in order to
solve the propagation equations (2.1.9/11). Since the target maneuvers are not arbitrary it is clear that the more knowledge about the anticipated target behaviour is exploited to derive the model equations, the better are the chances of filter convergence since unrealistic target motions can be excluded a-priori. This is especially important for the BOMP because the information contents of the bearing (rate) measurement may be very low. Moreover, the restriction of the possible target maneuvers also results in a reduction of the number of filter states, thus simplifying the filter algorithm.

According to figure 2.3.1 the planar target motion can be described by the states \( v_T, \gamma_T, \dot{\gamma}_T, a_T \).

\[ \text{figure 2.3.1: planar target motion} \]

The target model used here will be based on the following facts:

a) the target is an aerodynamically controlled vehicle and has certain acceleration characteristics. Especially, in intercept scenarios with evasive maneuvers, the lateral acceleration is much higher than the axial acceleration.

b) target maneuvers are usually highly correlated

Therefore the following assumptions will be made:

a') The target velocity is constant yielding: \( \dot{\gamma}_T = \frac{\kappa}{2} \) (2.3.1)

b') The target maneuver is deterministic (although unknown).
From a') follows that $v_T$ and $\tilde{v}_T$ are no filter states any more. $v_T$ may be considered an unknown filter parameter (which might motivate the design of a filter bank based on several velocity levels $v_T$). In the sequel, $v_T$ will be assumed known (for example from radar measurements processed by a tracking filter of the launching aircraft). Inaccurate $v_T$-estimates will be considered in the simulations (section 2.9). Hence the target model simplifies to:

$$\ddot{v}_T = \frac{a_T}{v_T} \quad (2.3.2)$$

$$\dddot{v}_T = \frac{a_T}{v_T} \quad (2.3.3)$$

The function $a_T(t)$ is, of course, unknown. Therefore, for the solution of the propagation equations, it will be assumed:

$$\dddot{v}_T = 0 \quad (2.3.4)$$

(2.3.4) is exact for constant-acceleration maneuvers but corrections are necessary in the case of time-varying maneuvers (section 2.8). The filter state consists of the target model (2.3.2/3) and quantities describing the observer-target relative motion. The latter may be formulated in cartesian and (modified) polar coordinates. Since all versions will be needed in the sequel they are summarized below (see figure 2.1.1):

filter state in cartesian coordinates (CC):

$$z^t = (\Delta x, \Delta y, \Delta \dot{x}, \Delta \dot{y}, v_T, \dot{v}_T) \quad (2.3.5)$$

measurement equations:

$$m = \arctg \frac{z_2}{z_1} + s \quad \text{bearing-only measurements} \quad (2.3.6)$$

$$m = \frac{(z_1 + z_2)^2}{z_1^2} \frac{z_1 z_4 - z_2^2}{z_1^2} + s \quad \text{bearing rate-only measurements} \quad (2.3.7)$$

The EKF based on CC will be denoted as CEKF.
filter states in ordinary polar coordinates (OPC):

\[ y^t = (\phi, \dot{R}, R, \gamma_T, \dot{\gamma}_T) \]  \hspace{1cm} (2.3.8)

measurement equations:

\[ m = y_3 + s \] \hspace{1cm} bearing-only measurements \hspace{1cm} (2.3.9)

\[ m = y_1 + s \] \hspace{1cm} bearing rate-only measurements \hspace{1cm} (2.3.10)

The EKF based on OPC will be denoted as OPEKF.

filter states in modified polar coordinates (MPC):

\[ y_m^t = (\phi, \dot{R}/R, \dot{\phi}/R, 1/R, \gamma_T, \dot{\gamma}_T) \]  \hspace{1cm} (2.3.11)

measurement equations:

\[ m = y_{m3} + s \] \hspace{1cm} bearing-only measurements \hspace{1cm} (2.3.12)

\[ m = y_{m1} + s \] \hspace{1cm} bearing rate-only measurements \hspace{1cm} (2.3.13)

The EKF based on MPC will be denoted as MPEKF.

2.4 Observability Analysis

When a filter has been designed, the behaviour of the estimation errors is of prime interest. The estimates delivered by the filter are only meaningful if the estimation errors are at least bounded. In the most favourable (but unrealistic) case they vanish asymptotically and the estimates become exact after some time. If the filter is divergent, the estimation errors grow without bounds. For linear Gaussian systems complete observability and complete controllability are sufficient conditions for filter stability [75-77]. Verifiable conditions guaranteeing filter stability do not, in general, exist for nonlinear systems. Often it is possible, however, to isolate situations (for example certain initial conditions) that will certainly lead to divergence of the nonlinear filter. For the BOMP, these cases will be discussed in the subsequent observability analysis.
2.4.1 Stochastic Observability and Filter Divergence

Before entering the analysis some explanations and definitions concerning stochastic observability are necessary. A condition for divergence of the EKF is derived which will be the basis for the investigation of the BOMP.

The diagonal elements of the covariance matrix $P$ may be regarded as the filter's estimates of the magnitude of the estimation errors. If - for a system with zero input noise - $P$ tends to zero asymptotically, stability of the linear filter may be concluded. For nonlinear systems this is not true because $P$ does usually not represent the actual estimation error statistics of the nonlinear filter. However, if $P$ tends to infinity, the filter is certainly unstable. Therefore, the investigation of the covariance matrix is suitable for the determination of (sufficient) conditions that result in filter divergence. These conditions are of great practical importance for filter implementation because they provide a useful guideline on how to avoid certain stability problems.

The analysis of $P$ is carried out for a noise free system, i.e.

$$ Q = 0 \quad (2.4.1) $$

With (2.4.1) $P(k/k)$ from (2.1.19) may be expressed in non-recursive form as follows (see [75]):

$$ P^{-1}(k/k) = W^t(0,k) P^{-1}_0 W(0,k) + I(k,0) \quad (2.4.2) $$

where

$$ I(k,0) = \sum_{i=0}^{k} W^t(i,k)C^t(i)S^{-1}(i)C(i)W(i,k) \quad (2.4.3) $$

is the information matrix.

Assuming infinite uncertainty about the initial system state results in

$$ P^{-1}_0 = 0 \quad (2.4.4) $$

and (2.4.2) renders

$$ P^{-1}(k/k) = I(k,0) \quad (2.4.5) $$

Hence, the behaviour of $P$ is determined by the information matrix.

Definition 1: The EKF (2.1.9 - 19) is said to be divergent if the information matrix is singular.
Definition 2 (see [75]):

The filter state \( y(k) \) is completely observable with respect to the measurements \( \{m_1, ..., m_k\} \) if and only if

\[
I(k,i) > 0 \tag{2.4.6}
\]

From (2.4.3) and (2.4.5) it is easy to see that for \( I(k,i) = 0 \) processing of the measurements \( \{m_1, ..., m_k\} \) does not result in any decrease of uncertainty in the state estimates expressed by \( P \), i.e. these measurements do not contain any information about \( y(k) \). In this case \( y(k) \) is called unobservable with respect to \( \{m_1, ..., m_k\} \).

In view of (2.4.5) the observability analysis is concerned with the singularity of the information matrix. For simplification of the analysis it is convenient to investigate the observability of the state vector at initial time with respect to the measurements \( \{m_0, ..., m_k\} \), i.e. to discuss the estimates \( \hat{y}(0/k) \) rather than \( \hat{y}(k/k) \). \( \hat{y}(0/k) \) is easily found from \( \hat{y}(k/k) \) by backward solution of the propagation equations (2.1.9). Consequently, the error covariance matrix \( P(0/k) \) associated with \( \hat{y}(0/k) \) is given by :

\[
P(0/k) = W(0,k)P(k/k)W^t(0,k) \tag{2.4.7}
\]

and hence

\[
P^{-1}(0/k) = W^t(k,0)P^{-1}(k/k)W(k/0) \tag{2.4.8}
\]

Substitution of (2.4.3) into (2.4.8) yields:

\[
P^{-1}(0/k) = P_0^{-1} + W^t(k,0)I(k,0)W(k,0) \tag{2.4.9}
\]

Since the initial time may be any time \( t_j \leq t_k \), (2.4.9) may be generalized to

\[
P^{-1}(j/k) = P^{-1}(j/j) + \tilde{I}(k,j) \tag{2.4.10}
\]

with

\[
\tilde{I}(k,j) = W^t(k,j)I(k,j)W(k,j) = \sum_{i=0}^{k} W^t(i,j)C^t(i)S^{-1}(i)C(i)W(i,j) \tag{2.4.11}
\]

Obviously, \( y(j) \) and hence \( y(k) \) are completely observable with respect to \( \{m_j, ..., m_k\} \) if and only if

\[
\tilde{I}(k,j) > 0 \tag{2.4.12}
\]
Now, consider the scalar measurement equation
\[ m = y_q + s \]  \hspace{1cm} (2.4.13)

with
\[ s \text{ according to (2.1.6/7)} \]
\[ y_q = \text{q-th component of } y \]

The measurement matrix associated with (2.4.13) is:
\[ C = [0,0, \ldots ,0,1,0, \ldots ,0] \] \hspace{1cm} (2.4.14)

where all except the q-th component of C are zero.

Let \( w_{qr} \) denote the element of W in the q-th row and r-th column. With (2.4.14) follows:
\[ C(i)W(i,j) = [w_{q1}(i,j), \ldots , w_{qn}(i,j)] = : D(i,j) \]  \hspace{1cm} (2.4.15)

According to (2.1.12) \( w_{qr}(i,j) \) is given by:
\[ w_{qr}(i,j) = \frac{\partial f_q[y(j/i-1),u(i)]}{\partial y_r(j/i-1)} \] \hspace{1cm} (2.4.16)

i.e. \( w_{qr}(i,j) \) is the sensitivity of the measured state \( y_q \) with respect to changes in \( y_r \). If \( w_{qr} \) vanishes for all i, the measurements contain no information about \( y_r \). In this case \( y_r \) is unobservable. To see this, assume that \( y \) has been partitioned in an observable part \( y^o \) and unobservable part \( y^u \):
\[ y^t = [y^o, y^u] \] \hspace{1cm} (2.4.17)

\( D(i,j) \) may be partitioned accordingly, yielding:
\[ D(i,j) = [D^o(i,j), D^u(i,j)] \] \hspace{1cm} (2.4.18)

With (2.4.18) and noting that S is a scalar quantity for the measurement (2.4.13) \( I^t \) from (2.4.11) becomes:
\[ I^t(k,j) = \sum_{i=j}^{k} \frac{1}{S(i)} D^t(i,j)D(i,j) = \]
where $I^o$ and $I^u$ are the information matrices associated with $y^o$ and $y^u$, respectively.

If the sensitivities with respect to $y^u$ vanish (2.4.15) yields:

$$D^u = 0$$  \hspace{1cm} (2.4.20)

which results via (2.4.19) in:

$$I^u = 0$$  \hspace{1cm} (2.4.21)

Hence, $y^u$ is completely unobservable and (2.4.20) is a sufficient condition for filter divergence. It is this equation that will be used for the analysis of the BOMP. The same conditions were used in [64] to analyze the BOMP for non-maneuvering targets.

2.4.2 Filter Equations in Modified Polar Coordinates

In order to evaluate the divergence condition established in the previous section the equations of the BOMP are derived here using MPC according to equation (2.3.11). The use of MPC is motivated by their decoupling property already mentioned in section 2.2 and discussed in more detail in section 2.5 and [64].

Determination of the filter equations amounts to specifying the functions $f$, $h$ and the matrices $W$ and $C$ in (2.1.9-19). $h$ is given by (2.3.12/13). With (2.1.18) $C$ is easily found to be:

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \text{ for bearing-only measurements}$$  \hspace{1cm} (2.4.22a)

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \text{ for bearing rate-only measurements}$$  \hspace{1cm} (2.4.22b)
For the determination of $f$ consider the differential equations of relative motion in MPC:

\[ \dot{y}_{m1} = y_{m4}[a_{T\phi} - a_{\Phi}] - 2y_{m1}y_{m2} \]  
\[ \dot{y}_{m2} = y_{m4}[a_{TR} - a_{R}] + y_{m1}^2 + y_{m2}^2 \]  
\[ \dot{y}_{m3} = y_{m1} \]  
\[ \dot{y}_{m4} = -y_{m4}y_{m2} \]  
\[ \dot{y}_{m5} = y_{m6} \]  
\[ \dot{y}_{m6} = 0 \] 

with (see figure 2.4.1):

\[ a_{T\phi} = v_T y_{m6} \cos(y_{m5} - y_{m3}) \]  
\[ a_{TR} = v_T y_{m6} \sin(y_{m5} - y_{m3}) \]

In order to obtain the discrete equations, (2.4.23) must be integrated over the interval $I^k = [t_k, t_{k+1}]$.

Equations (2.4.23) can, however, not be integrated analytically. A common approach is the approximation of $y_m(k+1)$ by an Euler-step:
\[ y_m(k+1) = T \dot{y}_m(k) \]  
with \[ T = t_{k+1} - t_k \] sampling period

However, this extrapolation may yield large inaccuracies for long sampling periods \( T \). Since the computation time is an important aspect of realistic filter design, \( T \) should not be restricted by accuracy requirements which may yield values for \( T \) too small for evaluation of the filter equations during one sampling interval. Fortunately, the equations of motion can be solved analytically in cartesian coordinates. Therefore, following the approach in [64], integration will be carried out in cartesian coordinates \( z \) defined in (2.3.5) and afterwards the results are transformed to MPC. The equations of motion in cartesian coordinates are:

\[ \begin{align*}
\dot{z}_1 &= z_3 \\
\dot{z}_2 &= z_4 \\
\dot{z}_3 &= \Delta a_x \\
\dot{z}_4 &= \Delta a_y \\
\dot{z}_5 &= z_6 \\
\dot{z}_6 &= 0
\end{align*} \]

where from (2.2.3), (2.3.2) and figure 2.4.2 one obtains:

\[ \begin{align*}
\Delta a_x &= -v_T z_6 \sin z_5 - a_x \\
\Delta a_y &= +v_T z_6 \cos z_5 - a_y
\end{align*} \]

\textit{figure 2.4.2: target acceleration in cartesian coordinates}
Note that due to the target maneuver in (2.4.29) equations (2.4.28) are nonlinear. Integration is, however, possible analytically yielding:

\[
\begin{align*}
    z_1(k+1) &= z_1(k) + T z_3(k) + u_{T1}(k+1) + u_{M1}(k+1) \\
    z_2(k+1) &= z_2(k) + T z_4(k) + u_{T2}(k+1) + u_{M2}(k+1) \\
    z_3(k+1) &= z_3(k) + u_{T3}(k+1) + u_{M3}(k+1) \\
    z_4(k+1) &= z_4(k) + u_{T4}(k+1) + u_{M4}(k+1) \\
    z_5(k+1) &= z_5(k) + u_{T5}(k+1) + u_{M5}(k+1) \\
    z_6(k+1) &= z_6(k) + u_{T6}(k+1) + u_{M6}(k+1)
\end{align*}
\]  
(2.4.30a-f)

The vectors

\[
\begin{align*}
    u_M^t &= (u_{M1}, \ldots, u_{M6}) \\
    u_T^t &= (u_{T1}, \ldots, u_{T6})
\end{align*}
\]  
(2.4.31)

describe the missile and target maneuver in \( I_k \) and are given below.

**Missile maneuver terms:**

\[
\begin{align*}
    u_{M1}(k+1) &= -\int_{t_k}^{t_{k+1}} dr \int a_x(\sigma) d\sigma, & u_{M2}(k+1) &= -\int_{t_k}^{t_{k+1}} dr \int a_y(\sigma) d\sigma \\
    u_{M3}(k+1) &= -\int_{t_k}^{t_{k+1}} a_x(\tau) d\tau, & u_{M4}(k+1) &= -\int_{t_k}^{t_{k+1}} a_y(\tau) d\tau \\
    u_{M5}(k+1) &= u_{M6}(k+1) = 0
\end{align*}
\]  
(2.4.32a-f)

Note that \( u_M \) is known via measurements of the missile acceleration. Details are given in section 2.6.

**Target maneuver terms:**

\[
\begin{align*}
    u_{T1}(k+1) &= T v_T \left[ -\cos z_5(k) + \frac{\sin[z_5(k) + T z_6(k)] - \sin z_5(k)}{T z_6(k)} \right]
\end{align*}
\]  
(2.4.34a)
\[ u_{T2}(k+1) = -Tv_T \left[ \sin z_5(k) + \frac{\cos[z_5(k) + Tz_6(k)] - \cos z_5(k)}{Tz_6(k)} \right] \]  

(2.4.34b)

\[ u_{T3}(k+1) = v_T \{ \cos[z_5(k) + Tz_6(k)] - \cos z_5(k) \} \]  

(2.4.34c)

\[ u_{T4}(k+1) = v_T \{ \sin[z_5(k) + Tz_6(k)] - \sin z_5(k) \} \]  

(2.4.34d)

\[ u_{T5}(k+1) = Tz_6(k) \]  

(2.4.34e)

\[ u_{T6}(k+1) = 0 \]  

(2.4.34f)

**Definition:**

Let \( TM_{zy} \) be the nonlinear transformation mapping modified polar coordinates to cartesian coordinates i.e.

\[ z = TM_{zy} (y_m) \]  

(2.4.35)

The inverse transformation is denoted by \( TM_{yz} \) and is given by:

\[ y_m = TM_{yz}^{-1}(z) = TM_{zy}(z) \]  

(2.4.36)

The transformations are easily derived using the following relations between cartesian and polar coordinates (see figure 2.1):

\[ R = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{z_1^2 + z_2^2} \]  

(2.4.37)

\[ \dot{R} = \frac{z_1 \dot{z}_1 + z_2 \dot{z}_2}{\sqrt{z_1^2 + z_2^2}} = \frac{z_1 z_3 + z_2 z_4}{\sqrt{z_1^2 + z_2^2}} \]  

(2.4.38)

\[ \varphi = \arctg \frac{\Delta y}{\Delta x} = \arctg \frac{z_2}{z_1} \]  

(2.4.39)

\[ \dot{\varphi} = \frac{z_1 \dot{z}_2 + z_2 \dot{z}_1}{z_1^2 + z_2^2} = \frac{z_1 z_4 + z_2 z_3}{z_1^2 + z_2^2} \]  

(2.4.40)
\[
\Delta x = R \cos \varphi = \frac{1}{y_{m4}} \cos y_{m3} \tag{2.4.41}
\]

\[
\Delta \dot{x} = \dot{R} \cos \varphi - R \dot{\varphi} \sin \varphi = \frac{1}{y_{m4}} (y_{m2} \cos y_{m3} - y_{m1} \sin y_{m3}) \tag{2.4.42}
\]

\[
\Delta y = R \sin \varphi = \frac{1}{y_{m4}} \sin y_{m3} \tag{2.4.43}
\]

\[
\Delta \dot{y} = \dot{R} \sin \varphi + R \dot{\varphi} \cos \varphi = \frac{1}{y_{m4}} (y_{m2} \sin y_{m3} + y_{m1} \cos y_{m3}) \tag{2.4.44}
\]

Inserting (2.4.37-40) into (2.3.11) yields:

\[
\begin{bmatrix}
y_{m1} \\
y_{m2} \\
y_{m3} \\
y_{m4} \\
y_{m5} \\
y_{m6}
\end{bmatrix} =
\begin{bmatrix}
\frac{z_1 z_4 + z_2 z_3}{z_1^2 + z_2^2} \\
\frac{z_1 z_3 + z_2 z_4}{z_1^2 + z_2^2} \\
\arctg \frac{z_2}{z_1} \\
(z_1^2 + z_2^2)^{-1/2} \\
z_5 \\
z_6
\end{bmatrix} = T_{M_{yz}}(z) \tag{2.4.45}
\]
With (2.4.41-44) and (2.3.5) $\text{TM}_{zy}$ becomes:

\[
\begin{bmatrix}
  z_1 \\
  z_2 \\
  z_3 \\
  z_4 \\
  z_5 \\
  z_6
\end{bmatrix} =
\begin{bmatrix}
  \frac{\cos y_m}{y_m} \\
  \frac{\sin y_m}{y_m} \\
  \frac{1}{y_m} (\frac{y_m^2 \cos y_m - y_m \sin y_m}{y_m}) \\
  \frac{1}{y_m} (\frac{y_m^2 \sin y_m + y_m \cos y_m}{y_m}) \\
  y_m^5 \\
  y_m^6
\end{bmatrix}
= \text{TM}_{zy}(y_m)
\]

(2.4.46)

Expressing $z(k)$ in (2.4.30) in terms of $y_m(k)$ via (2.4.46) yields:

\[
z(k+1) = f_z[TM_{zy} \{y_m(k), u_M(k+1)\}]
\]

(2.4.47)

Substitution of (2.4.47) into (2.4.45) renders:

\[
y_m(k+1) = TM_{yz}[z(k+1)] = f[y_m(k), u_M(k+1)]
\]

(2.4.48)

The transition matrix $W(k+1,k)$ according to (2.1.12) follows through application of the chain rule to (2.4.48):

\[
W(k+1,k) = \frac{\partial f[z(k+1/k), u_M(k+1)]}{\partial \hat{y}_m(k/k)}
\]

\[
= \frac{\partial TM_{yz}[\hat{z}(k+1/k)]}{\partial \hat{z}(k+1/k)} \frac{\partial f_z[TM_{zy} \{y_m(k/k), u_M(k+1)\}]}{\partial \hat{y}_m(k/k)}
\]

\[
= GZ[\hat{z}(k+1/k)] \quad HY[\hat{y}_m(k/k)]
\]

(2.4.49)
with 
\[ GZ = [g_{ij}], \quad HY = [h_{ij}]; \quad i, j = 1, \ldots, 6 \]  
and
\[ g_{ij} = \frac{\partial T_{M_{y_{z_{i}}}}}{\partial z_{j}} \]  
\[ h_{ij} = \frac{\partial f_{2}}{\partial y_{m_{j}}} \]  

The elements \( g_{ij} \) and \( h_{ij} \) are given in the appendix 2.1.

### 2.4.3 Observability Via Bearing • Only Measurements

According to section 2.4.2 and equation (2.4.22a) observability of the filter states \( y_{m} \) depends on the sensitivity of the measured state \( y_{m3} \) with respect to changes in \( y_{m} \). Since \( y_{m3} \) is measured and \( y_{m1} \) is its derivative, only the observability of the components \( y_{m2}, y_{m4}, y_{m5} \) and \( y_{m6} \) is of interest.

**Observability of \( y_{m2} \) (\( = \frac{\dot{y}}{R} \))**:

The observability of \( y_{m2} \) is determined by the sensitivity

\[ w_{32}(i, j) = \frac{\partial \hat{\dot{y}}_{m}(j/i-1), u_{M}(i)}{\partial \hat{y}_{m2}(j/i-1)} \]  

If \( w_{32}(i, j) \) vanishes for all \( i > j \), \( y_{m2} \) is unobservable according to (2.4.20/21). From (2.4.49) follows:

\[ w_{32}(i, j) = \sum_{k=1}^{6} g_{3k} \hat{\dot{y}}_{m}(j/i-1) \]  

Substituting \( g_{3k} \) and \( h_{k2} \) from appendix 2.1 into (2.4.54) yields:

\[ w_{32}(i, j) = \frac{\dot{y}_{m4}}{\dot{y}_{m4}} T_{ij} \left[ \tilde{z}_{2} \cos \tilde{y}_{m3} + \tilde{z}_{1} \sin \tilde{y}_{m3} \right] \]  

(2.4.55)
with
\[ \ddot{x} = \dot{x}(i/i-1) \quad (2.4.56a) \]
\[ \dddot{x} = \dot{x}(j/i-1) \quad (2.4.56b) \]
\[ T_{ij} = t_i - t_j \quad (2.4.57) \]

Since \( y_{m4} \) and \( T_{ij} \) are always nonzero, \( w_{32} \) can only vanish if
\[ \tan \bar{y}_{m3} = \frac{z_2}{z_1} = \tan \bar{y}_{m3} \quad \text{for all } i > j \quad (2.4.58) \]
i.e. if the bearing angle remains constant on the interval \([t_j, t_i]\).

**Conclusion:**

\( y_{m2} \) is unobservable on the interval \([t_j, t_i]\) if the bearing rate vanishes identically on this interval.

**Observability of \( y_{m4} \) (\( = \frac{1}{R} \)):**

The sensitivity of the measurement with respect to \( y_{m4} \) is:
\[ w_{34}(i, j) = \frac{\bar{y}_{m4}^2}{\bar{y}_{m4}} \{ \bar{z}_2 \{ \cos \bar{y}_{m3} + T_{ij} \bar{D}x \} - \bar{z}_1 \{ \sin \bar{y}_{m3} + T_{ij} \bar{D}y \} \} \quad (2.4.59a) \]
\[ = \frac{\bar{y}_{m4}^2}{\bar{y}_{m4}} \{ - \bar{z}_2 [u_{T1}(i) + u_{M1}(i)] + \bar{z}_1 [u_{T2}(i) + u_{M2}(i)] \} \quad (2.4.59b) \]

with
\[ \bar{D}x = \bar{y}_{m2} \cos \bar{y}_{m3} - \bar{y}_{m1} \sin \bar{y}_{m3} \quad (2.4.60a) \]
\[ \bar{D}y = \bar{y}_{m2} \sin \bar{y}_{m3} + \bar{y}_{m1} \cos \bar{y}_{m3} \quad (2.4.60b) \]

There are three cases resulting in vanishing \( w_{34} \) in \([t_j, t_i]\):

a) \( u_T(i) = u_M(i) = 0 \quad \text{for all } i > j \quad (2.4.61) \)
i.e. \( y_{m4} \) is unobservable if neither the observer nor the target maneuvers.
b) It may be assumed without loss of generality that the cartesian reference system in figure 2.1.1 is chosen in such a way that

\[ \bar{y}_{m3} = 0 \]  \hspace{1cm} (2.4.62)

and hence

\[ \hat{z}_2(i/j-1) = 0 \]  \hspace{1cm} (2.4.63)

i.e. the x-axis is directed along the initial LOS. This reference system will be denoted as \( Z^0 \) in the sequel. If the bearing angle remains constant it follows:

\[ \bar{y}_{m1} = 0 \]  \hspace{1cm} (2.4.64)

\[ \hat{z}_2 = 0 \quad \text{for all } i > j \]  \hspace{1cm} (2.4.65)

Substitution of (2.4.62/64/65) into (2.4.59a) and (2.4.60b) results in

\[ w_{34}(i,j) = 0 \quad \text{for all } i > j \]  \hspace{1cm} (2.4.66)

Hence, \( y_{m4} \) is unobservable for vanishing bearing rate.

c) From (2.4.59b) follows immediately that \( w_{34} \) vanishes for

\[ \frac{z_2}{z_1} = \frac{u_{T2}(i) + u_{M2}(i)}{u_{T1}(i) + u_{M1}(i)} \]  \hspace{1cm} (2.4.67)

This is a generalization of the result found in [62] where a geometric interpretation of (2.4.67) is given. Since these maneuvers are not of practical interest they will not be discussed here.

**Observability of \( y_{m5} \) (= \( \gamma_T \)):**

The sensitivity associated with \( y_{m5} \) is:

\[
w_{35}(i,j) = - \bar{y}_{m4} \hat{z}_2 T_{ij} v_T \left[ \sin \bar{y}_{m5} + \frac{\cos \bar{y}_{m5} + T_{ij} \bar{y}_{m6} - \cos \bar{y}_{m5}}{T_{ij} \bar{y}_{m6}} \right] + \bar{y}_{m4} \hat{z}_1 T_{ij} v_T \left[ \cos \bar{y}_{m5} - \frac{\sin \bar{y}_{m5} + T_{ij} \bar{y}_{m6} - \sin \bar{y}_{m5}}{T_{ij} \bar{y}_{m6}} \right]
\]  \hspace{1cm} (2.4.68)
Assume that
\[ T_{ij} \tilde{y}_{m6} \ll 1 \]  
(2.4.69) is valid for weak target maneuvers or for small values of \( T_{ij} \), i.e. during the period shortly after filter initialization. It is easy to see that in this situation one obtains from (2.4.68):
\[ w_{35}(i,j) = 0 \]  
(2.4.70)

**Conclusion:**
For non-maneuvering targets \( \tilde{y}_{m6} = 0 \) the target heading is unobservable. Otherwise observability is only weak.

**Observability of \( y_{m6} (= \dot{\tilde{y}}_T) \):**

\[ w_{36} \] is given by:

\[
w_{36}(i,j) = \frac{T_{ij} v_T}{\tilde{y}_{m6}} \left[ \cos(\tilde{y}_{m5} + T_{ij} \tilde{y}_{m6}) - \frac{\sin(\tilde{y}_{m5} + T_{ij} \tilde{y}_{m6}) - \sin \tilde{y}_{m5}}{T_{ij} \tilde{y}_{m6}} \right] \\
+ \frac{T_{ij} v_T}{\tilde{y}_{m6}} \left[ \sin(\tilde{y}_{m5} + T_{ij} \tilde{y}_{m6}) + \frac{\cos(\tilde{y}_{m5} + T_{ij} \tilde{y}_{m6}) - \cos \tilde{y}_{m5}}{T_{ij} \tilde{y}_{m6}} \right]
\]

(2.4.71)

with
\[ \tilde{y}_{31} = -\frac{\tilde{y}_{m4} \tilde{z}_2}{\tilde{z}_1} \]  
(2.4.72)
\[ \tilde{y}_{32} = \frac{\tilde{y}_{m4} \tilde{z}_1}{\tilde{z}_2} \]  
(2.4.73)

Using (2.4.69) the first-order approximation of \( w_{36} \) becomes:
\[ w_{36}(i,j) \approx T_{ij}^2 v_T \tilde{y}_{m4} \tilde{z}_2 \tilde{z}_1 \cos \tilde{y}_{m5} \]  
(2.4.74)

\( w_{36}(i,j) \) vanishes in \([t_j, t_i]\) for
\[ \tilde{z}_2 \sin \tilde{y}_{m5} = -\tilde{z}_1 \cos \tilde{y}_{m5} \quad \text{for all } i > j \]  
(2.4.75)

For zero bearing rate in \([t_j, t_i]\) equation (2.4.65) holds in the reference system \( Z^0 \).
Since $\tilde{z}_1$ is nonzero equation (2.4.75) renders:

$$\cos \tilde{y}_{m5} = 0 \quad \text{for all } i > j \quad (2.4.76)$$

or

$$|\tilde{y}_{m5}| = \frac{\pi}{2} \quad \text{for all } i > j \quad (2.4.77)$$

If (2.4.77) holds for any $j$, $y_{m6}$ is unobservable at any time $t_j$. The observer-target motion associated with this situation is depicted in the following figure:

![Figure 2.4.3](image)

Clearly, (2.4.65/77) are only satisfied if the target does not maneuver and the component $v_y$ of the observer velocity always equals the target velocity $v_T$. Note that this is the nominal condition associated with (extended) proportional navigation (chapter 1.3.5) for the geometry given by (2.4.77).

In practice divergence problems will also occur, if the sensitivities are nonzero but low. Since $w_{36}$ is proportional to $T_{ij}^2$, low observability has to be expected in the initial period of the observation interval. Obviously, the initial behavior of $w_{36}$ depends on the initial filter state, i.e. the initial geometry of the intercept scenario. Assuming again that (2.4.65) holds in $Z^0$, $w_{36}$ during the initial observation period becomes:

$$w_{36}(i,j) \approx T_{ij}^2 v_T \tilde{y}_{m4}^2 \tilde{z}_1 \cos \tilde{y}_{m5} \quad (2.4.78)$$

Hence, observability of $y_{m6}$ is maximal for $\tilde{y}_{m5} = 0$ and minimal for $\tilde{y}_{m5} = \frac{\pi}{2}$.

These situations are depicted in figure 2.4.4a/b:
Figure 2.4.4 suggests that the observability of \( y_{m6} \) or \( a_T \) is maximal if \( a_T \) is directed perpendicular to the LOS and minimal if \( a_T \) is directed along the LOS. This is also clear from intuition since maneuvers along the LOS do not influence the bearing angle.

In summary the following statements about the observability of the states \( y_{m2}, y_{m4}, y_{m5}, y_{m6} \) with respect to bearing-only measurements can be made:

- \( y_{m2} \): unobservable for zero bearing rate
- \( y_{m4} \): unobservable if both observer and target are non-maneuvering, and for certain nonzero relative maneuvers
- \( y_{m5} \): unobservable for non-maneuvering target, otherwise weakly observable
- \( y_{m6} \): unobservable only for the scenario depicted in figure 2.4.3
  - minimum observability for \( a_T \parallel \text{LOS} \)
  - maximum observability for \( a_T \perp \text{LOS} \)
2.4.4 Observability via Bearing Rate-Only Measurements

The analysis of section 2.4.3 is repeated here for bearing rate-only measurements. The sensitivities of the measured state $y_{m1}$ with respect to $y_{m2}$, $y_{m4}$, $y_{m5}$ and $y_{m6}$ are given below:

$$w_{12(i,j)} = \frac{\ddot{y}_{m4}^2}{y_{m4}} \left\{ [T_{ij} (\ddot{z}_4 - 2 \ddot{z}_1 \ddot{y}_{m1}) - \ddot{z}_2] \cos \dot{y}_{m3} \right. \
\left. - [T_{ij} (\ddot{z}_3 + 2 \ddot{z}_2 \ddot{y}_{m1}) + \ddot{z}_1] \sin \dot{y}_{m3} \right\}$$  (2.4.79)

$$w_{14(i,j)} = \frac{\ddot{y}_{m4}^2}{y_{m4}^2} \left\{ (\ddot{z}_4 - 2 \ddot{z}_1 \ddot{y}_{m1}) (\cos \dot{y}_{m3} + T_{ij} \overline{Dx}) \right. \
\left. + (\ddot{z}_3 + 2 \ddot{z}_2 \ddot{y}_{m1}) (\sin \dot{y}_{m3} + T_{ij} \overline{Dy}) \right. \
\left. + \ddot{z}_2 \overline{Dx} - \ddot{z}_1 \overline{Dy} \right\}$$  (2.4.80a)

$$= \frac{\ddot{y}_{m4}^2}{y_{m4}} \left\{ [\ddot{z}_4 - 2 \ddot{z}_1 \ddot{y}_{m1}] [u_{T1(i)} + u_{M1(i)}] \right. \
\left. - [\ddot{z}_3 + 2 \ddot{z}_2 \ddot{y}_{m1}] [u_{T2(i)} + u_{M2(i)}] \right. \
\left. - \ddot{z}_2 [u_{T3(i)} + u_{M3(i)}] + \ddot{z}_1 [u_{T4(i)} + u_{M4(i)}] \right\}$$  (2.4.80b)

with

$$\overline{Dx} = \ddot{y}_{m2} \cos \ddot{y}_{m3} - y_{m1} \sin \ddot{y}_{m3}$$  (2.4.81a)

$$\overline{Dy} = \ddot{y}_{m2} \sin \ddot{y}_{m3} + y_{m1} \cos \ddot{y}_{m3}$$  (2.4.81b)

$$w_{15(i,j)} = \ddot{y}_{11} T_{ij} v_T [\sin \ddot{y}_{m5} + \frac{\cos(\ddot{y}_{m5} + T_{ij} \ddot{y}_{m6}) - \cos \ddot{y}_{m5}}{T_{ij} \ddot{y}_{m6}}]$$

$$- \ddot{y}_{12} T_{ij} v_T [\cos \ddot{y}_{m5} - \frac{\sin(\ddot{y}_{m5} + T_{ij} \ddot{y}_{m6}) - \sin \ddot{y}_{m5}}{T_{ij} \ddot{y}_{m6}}]$$
- \( \dot{y}_{13} v_T [\sin(\bar{y}_{m5} + T_{ij} \bar{y}_{m6}) - \sin \bar{y}_{m5}] \) \\
+ \( \dot{y}_{14} v_T [\cos(\bar{y}_{m5} + T_{ij} \bar{y}_{m6}) - \cos \bar{y}_{m5}] \) 

(2.4.82)

\[
\begin{align*}
\dot{w}_{16(i,j)} &= \dot{y}_{11} \frac{T_{ij} v_T}{y_{m6}} [\cos(\bar{y}_{m5} + T_{ij} \bar{y}_{m6}) - \\
&\quad \sin(\bar{y}_{m5} + T_{ij} \bar{y}_{m6}) - \sin \bar{y}_{m5}] \\
&\quad \frac{T_{ij} \bar{y}_{m6}}{
+ \dot{y}_{12} \frac{T_{ij} v_T}{y_{m6}} [\sin(\bar{y}_{m5} + T_{ij} \bar{y}_{m6}) + \\
&\quad \cos(\bar{y}_{m5} + T_{ij} \bar{y}_{m6}) - \cos \bar{y}_{m5}] \\
&\quad \frac{T_{ij} \bar{y}_{m6}}{
- \dot{y}_{13} T_{ij} v_T \sin(\bar{y}_{m5} + T_{ij} \bar{y}_{m6}) \\
+ \dot{y}_{14} T_{ij} v_T \cos(\bar{y}_{m5} + T_{ij} \bar{y}_{m6}) 
\end{align*}
\] 

(2.4.83)

with 
- \( \dot{y}_{11} = \bar{y}_{m4} (\bar{z}_4 - 2 \bar{z}_1 \bar{y}_{m1}) \) 
- \( \dot{y}_{12} = \bar{y}_{m4} (\bar{z}_3 + 2 \bar{z}_2 \bar{y}_{m1}) \) 
- \( \dot{y}_{13} = -\bar{y}_{m4} \bar{z}_2 \) 
- \( \dot{y}_{14} = \bar{y}_{m4} \bar{z}_1 \) 

(2.4.84a, 2.4.84b, 2.4.84c, 2.4.84d)

and \( \bar{z}, \bar{y} \), as defined in (2.4.56).

Consider first the case of vanishing bearing rate in \([t_j, t_i]\). Then, in the reference system \( Z^0 \) one has:

- \( \bar{y}_{m1} = \bar{y}_{m3} = 0 \) for all \( i > j \) 
- \( \bar{y}_{m3} = 0 \) for all \( i > j \)

(2.4.85, 2.4.86)
Substituting (2.4.85/86) into (2.4.43/44) yields:

\[ \tilde{z}_2 = \tilde{z}_4 = 0 \]  

(2.4.87)

Inserting (2.4.85-87) into (2.4.79), (2.4.80a) and (2.4.81b) results in:

\[ w_{12}(i,j) = w_{14}(i,j) = 0 \quad \text{for all } i > j \]  

(2.4.88)

Hence, \( y_{m2} \) and \( y_{m4} \) are unobservable for zero bearing rate. Moreover from (2.4.80b) follows that \( w_{14} \) vanishes identically if neither observer nor target perform any maneuver.

For the discussion of \( w_{15} \) its first order approximation with respect to (2.4.69) is considered. From (2.4.82) one obtains:

\[ w_{15}(i,j) \approx - T_{ij} v_T \tilde{y}_{m6} (\tilde{y}_{13} \cos \tilde{y}_{m5} + \tilde{y}_{14} \sin \tilde{y}_{m5}) \]  

(2.4.89)

Substitution of \( \tilde{y}_{13}, \tilde{y}_{14} \) from (2.4.84c,d) into (2.4.89) renders:

\[ w_{15}(i,j) \approx - y_{m4}^2 T_{ij} v_T \tilde{z}_1 \tilde{y}_{m6} \left[ - \frac{\tilde{z}_2}{\tilde{z}_1} \cos \tilde{y}_{m5} + \sin \tilde{y}_{m5} \right] \]  

(2.4.90)

Obviously \( w_{15} \) vanishes for all \( i > j \) if

\[ \tilde{y}_{m6} = 0 \quad \text{for all } i > j \]  

(2.4.91)

i.e. for zero target maneuver (note that because of the assumption \( \dot{y}_{m6} = 0 \) equation (2.4.91) implies \( \tilde{y}_{m6} = 0 \)).

Another condition for vanishing \( w_{15} \) is:

\[ \tan \tilde{y}_{m5} = \frac{\tilde{z}_2}{\tilde{z}_1} = \tan \tilde{y}_{m3} \quad \text{for all } i > j \]  

(2.4.92)

According to the definitions of \( \tilde{y} \) and \( \tilde{\tilde{y}} \) in (2.4.56) equation (2.4.92) can only be satisfied for constant bearing angle. In this case (2.4.92) implies that the target heading angle is unobservable if the target velocity vector is directed along the LOS (see figure 2.4.4a).
In order to obtain some qualitative statements about the observability of $y_{m5}$ during the initial observation period, consider again the cartesian reference system $Z^0$. Recall that in this system $z_1$ is the initial range and that the x-axis is identical to the LOS (figure 2.4.5). Since $z_2$ is zero in $Z^0$ it follows that for small $T_{ij}$:

$$z_2 \ll z_1 \quad (2.4.93)$$

Hence, $w_{15}(i,j)$ is approximately:

$$w_{15}(i,j) \approx -\gamma_{m4} T_{ij} v_T z_1 y_{m6} \sin y_{m5} \quad (2.4.94)$$

Thus the observability of $y_{m5}$ is maximal if the target heading is perpendicular to the LOS ($\gamma_{m5} = \frac{\pi}{2}$) and minimal if it is parallel to the LOS ($\gamma_{m5} = 0$). This is intuitively clear because the target motion along the LOS does not influence the bearing rate.

In summary it can be stated that the target heading is unobservable for non-maneuvering target. Observability is minimal/maximal if the target heading is directed parallel/perpendicular to the LOS.

$$Z^0 = (x, y)$$

$\beta_1 = y_{11} T_{ij} + y_{13} + T_{ij} y_{14} y_{m6}$

$\beta_2 = y_{12} T_{ij} + y_{14} + T_{ij} y_{13} y_{m6}$

The investigation of $w_{16}$ is carried out using the first order approximation of (2.4.83) with respect to (2.4.69):

$$w_{16}(i,j) \approx -T_{ij} v_T \beta_1 \sin y_{m5} + T_{ij} v_T \beta_2 \cos y_{m5} \quad (2.4.95)$$

with

$$\beta_1 = y_{11} T_{ij} + y_{13} + T_{ij} y_{14} y_{m6} \quad (2.4.96)$$

$$\beta_2 = y_{12} T_{ij} + y_{14} + T_{ij} y_{13} y_{m6} \quad (2.4.97)$$
The coefficients $\beta_1$, $\beta_2$ will now be approximated for small $T_{ij}$. From figure 2.4.5 follows (also observe equation 2.3.5):

\[
\begin{align*}
\tilde{z}_1 & \approx R_0 \\
\tilde{z}_2 & \ll \tilde{z}_1 \\
T_{ij} \tilde{z}_3 & \ll R_0 \\
T_{ij} \tilde{y}_m & \ll 1 \\
T_{ij} \tilde{y}_m & \ll 1 \\
\tilde{z}_4 T_{ij} & \approx \tilde{z}_1 \tilde{y}_m \quad T_{ij} \approx \tilde{z}_2
\end{align*}
\]

(2.4.98) with (2.4.84d) yields:

\[
|\tilde{y}_{14}| \approx R_0 \tilde{y}_m^2
\]  
(2.4.104)

(2.4.100/102) in (2.4.84b) and (2.4.99) in (2.4.84c) yield:

\[
\begin{align*}
T_{ij} |\tilde{y}_{12}| & \ll R_0 \tilde{y}_m^2 \\
|\tilde{y}_{13}| & \ll R_0 \tilde{y}_m^2
\end{align*}
\]  
(2.4.105, 2.4.106)

With (2.4.104-106) follows:

\[
|\beta_2| \approx |\tilde{y}_{14}| \approx R_0 \tilde{y}_m^2
\]  
(2.4.107)

(2.4.103) in (2.4.84b,c) yields:

\[
\tilde{y}_{11} T_{ij} = c(T_{ij}) \tilde{y}_{13}
\]  
(2.4.108)

where

\[
c(T_{ij}) = O(1)
\]  
(2.4.109)

With (2.4.108/109) $\beta_1$ may be written as:

\[
\begin{align*}
\beta_1 &= [1 + c(T_{ij})] \tilde{y}_{13} + T_{ij} \tilde{y}_{14} \tilde{y}_m \\
&= \tilde{y}_{14} \{[1 + c(T_{ij})] \tilde{e} + T_{ij} \tilde{y}_m\}
\end{align*}
\]  
(2.4.110, 2.4.111)

with (see figure 2.4.5):

\[
\tilde{e} = \frac{\tilde{y}_{13}}{\tilde{y}_{14}} = \frac{\tilde{z}_2}{\tilde{z}_1} = \sin \varphi(i/i-1)
\]  
(2.4.112)
Observing (2.4.99) equation (2.4.112) renders:

$$\tilde{e} \approx \tilde{\varphi}(i/i-1) \ll 1$$  \hspace{1cm} (2.4.113)

and with (2.4.111) follows:

$$[1 + c(T_{ij})] \tilde{e} \ll 1$$  \hspace{1cm} (2.4.114)

With (2.4.114) and (2.4.102) the comparison of (2.4.111) and (2.4.107) yields:

$$\beta_2 \gg \beta_1$$  \hspace{1cm} (2.4.115)

Therefore it may be concluded from (2.4.95) that $|w_{16}|$ is maximal for $\bar{y}_{m5} = 0$ and minimal for $|\bar{y}_{m5}| = \frac{\pi}{2}$. In other words: Observability of $y_{m6}$ is maximal if $a_T$ is directed perpendicular to the LOS and minimal if $a_T$ is directed along the LOS (figures 2.4.4a/b). The same result was found for bearing-only measurements in section 2.4.3. A comparison with the results obtained for $y_{m5}$ reveals that the target heading with maximum observability of $y_{m6}$ results in minimum observability of $y_{m5}$ and vice versa. Moreover it is evident from (2.4.107) that the observability of $y_{m6}$ is practically independent of the value of $\bar{y}_{m6}$ for $\bar{y}_{m5} = 0$. In contrast, if $|\bar{y}_{m5}| = \frac{\pi}{2}$ the observability depends strongly on the target maneuver. From (2.4.111) follows:

Observability of $y_{m6}$ is minimal for zero target maneuver, i.e. $\bar{y}_{m6} = 0$.

Observability of $y_{m6}$ is maximal if:

$$\bar{y}_{m6} \gg \frac{\tilde{\varphi}(i/i-1)}{T_{ij}}$$  \hspace{1cm} (2.4.116)

i.e. observability of $y_{m6}$ is maximal if the target heading rate is much larger than the average bearing rate.

The results on observability via bearing rate-only measurements may be summarized as follows:

$y_{m2}$: unobservable for zero bearing rate

$y_{m4}$: unobservable for zero bearing rate and zero relative observer-target maneuver
$y_{m5}$: unobservable for non-maneuvering target
- minimum observability if target heading is parallel to LOS
- maximum observability if target heading is perpendicular to LOS

$y_{m6}$: minimum observability if target acceleration is directed along LOS; in this scenario observability depends on the heading rate and is maximal for strong target maneuvers (equation 2.4.116)
- maximum observability if target acceleration is directed perpendicular to the LOS; in this case observability is practically independent of the value of the heading rate.

The main results of the observability analysis of this section is that the tracking filter is "blind" towards target maneuvers along the LOS and that the target heading is unobservable for zero target acceleration. Target maneuvers can only be tracked if they influence the bearing rate directly, i.e. if they have a component perpendicular to the LOS. Clearly the "blindness" with respect to motions along the LOS could be avoided by range- or range rate measurements. According to the preconditions of this study these measurements are, however, not available.

2.5 Selection of Coordinates for Filter Implementation

The final goal of developing the tracking filter is the use of this filter in conjunction with any of the guidance laws derived in chapter 1. The information of interest are estimates of range, range rate, target heading, and target lateral acceleration (or heading rate). In principle this information can be obtained from any of the state vectors (2.3.5/8/11). The behaviour of the filter depends, however, on the coordinates used for the filter design. The basic differences between cartesian and polar coordinates were discussed in section 2.2. Based on the results of the observability analysis of the previous section some additional conclusions will be delivered in the following.

In [64] it was shown that the tracking filter based on MPC (MPEKF) performs better than the CEKF in scenarios with zero initial maneuver because the unobservable range is decoupled from the other states in the MPC-formulation. To reveal this decoupling property for the more general problem discussed here, observe first that with (2.4.20/21) the information matrix (2.4.19) becomes:
\[
I(k,j) = \begin{bmatrix}
I^{oo} & 0 \\
0 & 0
\end{bmatrix}
\]  

(2.5.1)

Now assume that at initial time \( t_j \) the covariance matrix is of the form

\[
P(j/j) = \begin{bmatrix}
P^{oo}(j/j) & 0 \\
0 & P^{uu}(j/j)
\end{bmatrix}
\]  

(2.5.2)

Substitution of (2.5.1/2) into (2.4.10) renders:

\[
P(j/k) = P^{oo}(j/j) \{ [P^{oo}(i,j)]^{-1} + I^{oo} \}^{-1} \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  

(2.5.3)

Hence, the covariance equations of the observable and unobservable states are decoupled. Note that so far no assumptions about the filter states have been made. The decoupling is just a consequence of the vanishing sensitivities \( D_u \) associated with the unobservable states. (2.5.3) does not imply a decoupling of the \( y^o \)- and \( y^u \)-estimates because from (2.4.16) follows:

\[
I^{oo} = I^{oo} \{ \hat{y}(j/i-1), u(i) \} = I^{oo} \{ \hat{y}(j/i-1), \hat{y}(j/i-1), u(i) \}
\]  

(2.5.4)

i.e. \( I^{oo} \) depends on both observable and unobservable states. Therefore, filter divergence in the unobservable states may in general cause divergence in \( y^o \) as well. If, however, the information matrix depends on the observable states alone, i.e.

\[
I^{oo} = I^{oo} \{ \hat{y}(j/i-1), u(i) \}
\]  

(2.5.5)

the estimates of the observable states remain unaffected by wrong estimates of \( y^u \). For the MPEKF this is the case for zero-maneuver scenarios. Recall that the states \( y_{m4} \) and \( y_{m5} \) are unobservable in this situation. From (2.4.23a -c/f) follows that the states \( y_{mi}, i = 1,2,3,6 \) are decoupled from the unobservable components for vanishing accelerations \( a_M \) and \( a_T \). Of course the same is valid for the discrete version of (2.4.23). With (2.4.16) this results in (2.5.5). Thus, wrong estimates of \( y_{m4} \) and \( y_{m5} \) do not affect the estimates of the other states. This is not true for cartesian coordinates. From figure 2.1.1 follows for the non-maneuvering case:
\[ \Delta x = R \cos \varphi \]  
\[ \Delta y = R \sin \varphi \]  
\[ \dot{\Delta x} = v_T \cos \gamma_T - v \cos \gamma \]  
\[ \dot{\Delta y} = v_T \sin \gamma_T - v \sin \gamma \]  

Obviously all cartesian components are influenced by estimation errors in either the range or the target heading.

Consider now the equations of relative motion in ordinary polar coordinates (2.3.8): With (2.4.23) one obtains:

\[ \begin{align*}
\dot{y}_1 &= \frac{a_{T\varphi} - a\varphi}{y_4} - 2 \frac{y_1 y_2}{y_4} \\
\dot{y}_2 &= a_{TR} - a_R + y_4 y_1^2 \\
\dot{y}_3 &= y_1 \\
\dot{y}_4 &= y_2 \\
\dot{y}_5 &= y_6 \\
\dot{y}_6 &= 0
\end{align*} \]

where

\[ a_{T\varphi} = v_T y_6 \cos(y_5 - y_3) \]  
\[ a_{TR} = v_T y_6 \sin(y_5 - y_3) \]

Clearly, the OPC-estimates are not decoupled from range errors for zero maneuvers as was the case for MPC because (2.5.7a/b) always depend explicitly on \( y_4 \). However, the decoupling property with respect to target heading errors is maintained.

In view of the central result of the observability analysis (i.e. minimum observability of target motions along the LOS and maximum observability of target motions normal to the LOS) it may be concluded that polar coordinates are preferable to cartesian coordinates for designing a tracking filter for the BOMP because they yield a representation of the observer-target relative motion in the direction of maximum and minimum observability. Thus observable and unobservable states are at least partially decoupled according to (2.5.3).
There is another aspect that must be considered for filter state selection:

Since the BOMP is an intercept problem here, the observer-target range will approach zero at final time. Hence, according to (2.3.11) the MPC-components $y_{m2}$ and $y_{m4}$ will become indefinite and may cause a breakdown of the filter algorithm prior to intercept. Implementation of the filter will therefore be done in ordinary polar coordinates.

### 2.6 The Basic Tracking Algorithm (BTA)

In view of the conclusions in the previous section the EKF for the BOMP will be based on ordinary polar coordinates. Due to the absence of an inertial reference system only the bearing rate can be measured in addition to the observer acceleration. Therefore the tracking algorithm is basically the application of (2.1.9-19) to the state vector (2.3.8) with the measurement equation (2.3.10). The lack of an inertial reference has some consequences concerning the evaluation of the filter equations and the processing of the measurement data. These problems will be discussed in the following.

#### 2.6.1 Equations of the OPEKF

The equations of the EKF based on OPC (OPEKF) are derived in the same way as was done for the MPEKF (section 2.4.2). The central point is the solution of the propagation equations (2.1.9/11). Equation (2.1.9) can be solved analytically in a cartesian reference frame yielding (2.4.30). The propagated cartesian solution is then transformed to OPC via the transformation

$$y = TO_{yz}(z) \quad (2.6.1)$$

In analogy to (2.4.49) the transition matrix associated with $y$ is given by:

$$W(k+1,k) = GZ[\hat{z}(k+1/k)]H_{y}(\hat{y}(k/k)) \quad (2.6.2)$$

with

$$g_{zij} = \frac{\partial TO_{yz,i}}{\partial z_j} \quad (2.6.3)$$

$$h_{yij} = \frac{\partial f_{zi}}{\partial y_j} \quad (2.6.4)$$

$GZ, HY, TO_{yz}, TO_{zy}$ are given in the appendix 2.2.
In principle the choice of the cartesian reference system for the computation of (2.4.30) is arbitrary. However, the computational effort for the calculation of the elements $gz_{ij}$ and $hy_{ij}$ can be substantially reduced if the reference system is chosen in such a way that in the sampling interval

$$I_k = [t_k, t_{k+1}]$$

one has

$$\hat{y}_3(k/k) = 0$$

(2.6.6)

i.e. the x-axis of the reference system is directed along the estimated LOS at time $t_k$ (see figure 2.6.1). This results in the simplification of many expressions involving $\sin[\hat{y}_3(k/k)]$ and $\cos[\hat{y}_3(k/k)]$ (see appendix 2.2). It is, however, necessary to carry out some simple transformations at the beginning of each sampling interval in order to adapt the filtering algorithm to each new reference frame. This entails a scheduling problem and has an impact on the sequence of computations of the tracking algorithm. Therefore, before summarizing the algorithm, the question of data transformation is addressed in the next section.

### 2.6.2 The Measurement Module

In addition to the bearing rate the observer acceleration has to be measured in order to determine the observer maneuver $u_M$ according to (2.4.32) which is used in (2.4.30). The acceleration components are measured by accelerometers in the observer’s body fixed reference system. They must be transformed to the (inertial) reference system defined by (2.6.6). Before discussing the necessary computations some notations are introduced:

- $Z(k) = [x(k), y(k)]$: cartesian reference frame with axes $x(k), y(k)$ at time $t_k$
- $Z^S = [x_s, y_s]$: cartesian reference frame defined by (2.6.6)
- $Z^M = [x_M, y_M]$: observer fixed reference frame; $x_M \parallel$ center line
- $Z^s = [x_s, y_s]$: seeker frame; $x_s \parallel$ seeker axis (see section 1.4.6.2)

$$y_{\mid Z}$$

$y$ expressed in the reference frame $Z$

The geometry associated with the sampling interval $I_k$ is depicted in figure 2.6.1.
$M =$ missile, $CL =$ missile center line

$a_l$ : lateral acceleration

$a_a$ : axial acceleration

$\theta_s(\tau)$ : missile pitch angle with respect to current LOS

$\theta(\tau)$ : missile pitch angle with respect to $Z^S(k)$

$\xi(\tau,t_k)$: bearing angle increment in $[t_k, \tau]$

$\xi(k)$ : estimation error in bearing angle

$\psi(k)$ : seeker axis orientation

*figure 2.6.1: reference frames*
For the observer acceleration components $a_x, a_y$ in $Z^S$ one obtains:

\[
\begin{align*}
    a_x(r) |_{Z^S(k)} &= a_a(r) \cos \theta(r) - a_i(r) \sin \theta(r) \quad (2.6.7a) \\
    a_y(r) |_{Z^S(k)} &= a_a(r) \sin \theta(r) + a_i(r) \cos \theta(r) \quad (2.6.7b)
\end{align*}
\]

with

\[
\theta(r) = \xi(k) + \xi(r, t_k) + \theta_s(r) \quad (2.6.8)
\]

$\theta_s(r)$ can be easily measured and $\xi(r, t_k)$ may be obtained by integrating the measured bearing rate:

\[
\xi(r, t_k) = \int_{t_k}^{r} m(\sigma) \, d\sigma \quad (2.6.9)
\]

However, the reference frame $Z^S(k)$ and hence $\xi(k)$ are unknown at time $t_k$ because $\hat{y}_3(k/k)$ must first be computed from the update equations associated with $I_{k-1}$. These calculations can only be started at time $t_k$ because the bearing rate measurement $m(k)$ is needed in (2.1.15). Hence $\hat{y}_3(k/k)$ is available only at some time $r^*$ with

\[
t_k < r^* < t_{k+1} \quad (2.6.10)
\]

This problem is circumvented by first computing the components $u_m$ with respect to the reference system $Z^S(k)$ and transforming them to $Z^S(k)$ at time $t_{k+1}$. For this purpose the acceleration components in $Z^S(k)$ are needed. They are given by:

\[
\begin{align*}
    a_x(r) |_{Z^S(k)} &= a_a(r) \cos \theta'(r) - a_i(r) \sin \theta'(r) \quad (2.6.11a) \\
    a_y(r) |_{Z^S(k)} &= a_a(r) \sin \theta'(r) + a_i(r) \cos \theta'(r) \quad (2.6.11b)
\end{align*}
\]

with

\[
\theta'(r) = \xi(r, t_k) + \theta_s(r) \quad (2.6.12)\]
Inserting
\[ a_x = a_x \big|_{Z^S(k)} \]  \hspace{1cm} (2.6.13a)
\[ a_y = a_y \big|_{Z^S(k)} \]  \hspace{1cm} (2.6.13b)

into (2.4.36) yields the components of \( u_M \) in \( Z^S(k) \). At time \( t_{k+1} \) they are transformed to \( Z^S(k) \) via:
\[
\begin{pmatrix}
  u_{M,j+1} \\
  u_{M,j+2}
\end{pmatrix}
\big|_{Z^S(k)} = -TR[\xi(k)]
\begin{pmatrix}
  u_{M,j+1} \\
  u_{M,j+2}
\end{pmatrix}
\big|_{Z^S(k)}, j = 0, 1 \]  \hspace{1cm} (2.6.14)

with \( TR(\gamma) = \begin{bmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{bmatrix} \) \hspace{1cm} (2.6.15)
and from figure 2.6.1:
\[
\xi(k) = \psi(k) - \hat{y}_3(k/k) \big|_{Z^S(k-1)} \]  \hspace{1cm} (2.6.16)

where \( \psi(k) = \xi(k-1) + \xi(k,k-1) \) \hspace{1cm} (2.6.17)

Note that all quantities in (2.6.17) are known from the previous sampling interval. The determination of \( u_M \big|_{Z^S(k)} \) may be regarded as measurement data preprocessing in order to obtain the correct filter input data. The associated computations do not involve any quantities estimated by the filter and are therefore summarized in a separate module called measurement module. The signal flow of the measurement module is shown in figure 2.6.2.
$S_1, ..., S_5$ : integrators

*figure 2.6.2: measurement module*
2.6.3 Summary of BTA

The computations for the determination of \( \hat{y}(k+1/k+1) \) from the previous estimate \( \hat{y}(k/k) \) and the new measurement \( m(k+1) \) are now summarized. Consider the sampling interval \( I_k = [t_k, t_{k+1}] \). At time \( t_k \) the following data are available:

- from measurement module: \( u_M \rvert_{Z^S(k)} \), \( \xi(k+1,k) \), \( m(k+1) \)
- from previous cycle: \( \hat{y}(k/k) \rvert_{Z^S(k)} \), \( \xi(k) \)

The following calculations have to be carried out in \( I_k \):

**step 1:**
- a) store output data of measurement module
- b) reset integrators \( S_1, \ldots, S_5 \) (figure 2.6.2) for measurement data evaluation in \( I_{k+1} \)

**step 2:** compute \( u_M \rvert_{Z^S(k)} \) from \( u_M \rvert_{Z^s(k)} \) according to (2.6.14)

**step 3:** compute seeker axis orientation in \( Z^S(k) \):

\[
\psi(k+1) = \xi(k) + \xi(k+1,k) \tag{2.6.18}
\]

**step 4:** compute

\[
\hat{z}(k/k) \rvert_{Z^S(k)} = T_{O_zy} [\hat{y}(k/k) \rvert_{Z^S(k)}] \tag{2.6.19}
\]

**step 5:** compute estimated target maneuver \( u_T \rvert_{Z^S(k)} \) from (2.4.34) with

\[
z(k) = \hat{z}(k/k) \rvert_{Z^S(k)} \tag{2.6.20}
\]

**step 6:** compute \( \hat{z}(k+1/k) \rvert_{Z^S(k)} \) from (2.4.30) with (2.6.19) and

\[
u_T = u_T \rvert_{Z^S(k)}, \quad u_M = u_M \rvert_{Z^S(k)}
\]
step 7: compute
\[
\hat{y}(k+1/k)\big|_{Z^S(k)} = T_{O_yz}[\hat{z}(k+1/k)\big|_{Z^S(k)}] 
\] (2.6.21)

step 8: compute \(W(k+1,k)\) from (2.6.2)

step 9: compute \(P(k+1/k)\) from (2.1.11)

step 10: compute \(G(k+1)\) from (2.1.17)

step 11: compute \(P(k+1/k+1)\) from (2.1.19)

step 12: compute \(\hat{y}(k+1/k+1)\big|_{Z^S(k)}\) from (2.1.15)

step 13: define new cartesian reference frame \(Z^S(k+1)\) for next sampling interval \(I_{k+1}\)

a) compute seeker axis at \(t_{k+1}\) in \(Z^S(k+1)\):
\[
\xi(k+1) = \psi(k+1) - \hat{y}_3(k+1/k+1)\big|_{Z^S(k)} 
\] (2.6.22)

with \(\psi(k+1)\) from (2.6.18)

b) compute \(\hat{y}(k+1/k+1)\) in \(Z^S(k+1)\):

\[
\text{Note that only } y_3 \text{ and } y_5 \text{ depend on } Z^S. \text{ Hence only these components are corrected; the others remain unchanged.}
\]

estimated target heading in \(Z^S(k+1)\):
\[
\hat{y}_5(k+1/k+1)\big|_{Z^S(k+1)} = \hat{y}_5(k+1/k+1)\big|_{Z^S(k)} - \hat{y}_3(k+1/k+1)\big|_{Z^S(k)} 
\] (2.6.23)

estimated bearing angle in \(Z^S(k+1)\) according to (2.6.6):
\[
\hat{y}_3(k+1/k+1)\big|_{Z^S(k+1)} = 0 
\] (2.6.24)

With \(\hat{y}_5\) and \(\hat{y}_3\) from (2.6.23/24) and the other components from step 12 the desired
2.6.4 Simulations

In this section simulation results obtained with the tracking algorithm described in section 2.6.3 are presented. The simulations illustrate the main conclusions on observability derived in section 2.4.4. Questions of guidance law-filter interaction will also be addressed.

2.6.4.1 Filter Initialization

In order to execute the tracking algorithm a number of filter parameters must be initialized. The values used for the subsequent simulations are summarized below:

- a) sampling period: \[ T = 0.05 \text{ s} \] (2.6.25)
- b) variance of measurement noise: \[ S = 10^{-6} \text{ rad}^2 \text{s}^{-2} \] (2.6.26)
- c) initial covariance matrix: \[ P(0/0) = \text{diag}[p_{01}(0/0)] \] (2.6.27)

with

\[
\begin{align*}
p_{01} &= 10^{-4} \text{ rad}^2 \text{s}^{-2}, & p_{02} &= 10^4 \text{ m}^2 \text{s}^{-2}, & p_{03} &= 10^{-6} \text{ rad}^2 \quad (2.6.28a,b,c) \\
p_{04} &= 5 \times 10^4 \text{ m}^2, & p_{05} &= 0.16 \text{ rad}^2, & p_{06} &= 0.05 \text{ rad}^2 \text{s}^{-2} \quad (2.6.28d,e,f)
\end{align*}
\]

The elements \( p_{0i} \) are a measure of the range of the expected squared initial estimation errors \( e_0(0/0) \) with \( e \) defined in (2.1.21). The value of \( p_{06} \) is based on the assumption that the maximal initial estimation error of the target acceleration is

\[ \Delta a_T = 6g \quad (2.6.29) \]

Assuming a target velocity of

\[ v_T = 270 \text{ m s}^{-1} \quad (2.6.30) \]

this renders with (2.3.2/8):

\[ \Delta e_6(0/0) = \frac{\Delta a_T}{v_T} \approx 0.22 \text{ rad s}^{-1} = \sqrt{p_{06}} \quad (2.6.31) \]

- d) covariance matrix of input noise:

According to (2.4.30-34) the system input is the missile maneuver \( u_M \) which is obtained from the measurement module (figure 2.6.2). These measurements are modeled noise free here, yielding

\[ Q = 0 \quad \text{(noise free system)} \quad (2.6.32) \]

For a further discussion of this assumption see section 2.10. The estimation of \( Q \) in order to account for uncertainties in the target maneuver is discussed in section 2.7.
e) initial estimation errors:
   The assumed values for the initial estimation errors will be given in the discussion of the simulation results. All errors not explicitly mentioned are assumed zero, i.e. the associated filter states are initialized with their exact values.

f) target velocity:
   As mentioned in section 2.3 the target velocity is a parameter of the filter. Unless explicitly mentioned, the error
   \[ e_v = \hat{v}_T - v_T \]  
   is assumed zero.

2.6.4.2 Assessment of Filter Behaviour and Presentation of Filtering Results

The results produced by the tracking filter will be judged by the behaviour of the estimation errors and their variances. Visualization of the estimation errors is done by plotting the time histories of the estimated and exact filter states. The diagonal elements \( p_{ii} \) of the covariance matrix \( P \) are the filter's estimates of the error variance associated with the state \( y_i \). Therefore the expected squared estimation error should always be lower than the estimated variance, i.e.

\[ \mathbb{E}[e_i^2(t)] \leq p_{ii}(t) \]  

(2.6.34)

If (2.6.34) is violated the filter may diverge because it underestimates its estimation accuracy and does not take into account new information by incoming measurements. Note that as \( P \to 0 \) the filter degenerates to a predictor (equations 2.1.15/17/19). In order to check (2.6.34) the squared estimation error \( e_i^2 \) is plotted in addition to \( p_{ii} \).

From (2.4.2) follows that the subsystem \( (y_5, y_6) \) is unstable. Hence any estimation error in \( y_5 \) or \( y_6 \) can only be reduced via the update equations but not by merely solving the propagation equations. Because only the observable states are influenced by new measurements according to (2.5.3) observability of \( y_5 \) and \( y_6 \) is indicated by the behaviour of \( p_{55} \) and \( p_{66} \). These variances can only decrease if the respective states are observable.
The filter states and variances are plotted in a normalized scale. The associated scaling factor (SCALE) is written on the plots. The filter states, variances, and associated dimensions are summarized in the following table.

<table>
<thead>
<tr>
<th>state no.</th>
<th>meaning</th>
<th>name</th>
<th>dimension</th>
<th>variance</th>
<th>dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>bearing rate</td>
<td>$y_1$</td>
<td>rad/s</td>
<td>$P_{11}$</td>
<td>rad²/s²</td>
</tr>
<tr>
<td>2</td>
<td>range rate</td>
<td>$y_2$</td>
<td>m/s</td>
<td>$P_{22}$</td>
<td>m²/s²</td>
</tr>
<tr>
<td>3</td>
<td>bearing angle</td>
<td>$y_3$</td>
<td>rad</td>
<td>$P_{33}$</td>
<td>rad²</td>
</tr>
<tr>
<td>4</td>
<td>range</td>
<td>$y_4$</td>
<td>m</td>
<td>$P_{44}$</td>
<td>m²</td>
</tr>
<tr>
<td>5</td>
<td>target heading angle</td>
<td>$y_5$</td>
<td>rad</td>
<td>$P_{55}$</td>
<td>rad²</td>
</tr>
<tr>
<td>6</td>
<td>target heading rate</td>
<td>$y_6$</td>
<td>rad/s</td>
<td>$P_{66}$</td>
<td>rad²/s²</td>
</tr>
</tbody>
</table>

* $y_3$ and $y_5$ are computed in the current reference system according to (2.6.23/24). On each plot the associated scenario is written in the lower right corner.

2.6.4.3 Observability of Target Maneuver

In section 2.4.4 it was shown that the observability of the target maneuver is highly dependent on the intercept scenario. This is illustrated by comparing the estimated covariance histories $P_{55}(t)$, $P_{66}(t)$ generated by the tracking filter for different missile-target flight paths. In order to avoid coupling effects between filter and guidance law the tracking algorithm will be applied to precomputed and stored missile-target trajectories. The measurement data processed by the filter are noise free. Hence the resulting filter trajectories are deterministic. Moreover, if the initial estimation errors are zero they remain zero for constant target maneuvers because the state propagation equations are exact. In this case $P_{55}(t)$ and $P_{66}(t)$ reflect the observability of $y_5$ and $y_6$ with respect to the exact missile-target relative motion.
Consider first the following scenario (see figure 2.6.3):

**scenario A1:**

\[ \gamma_{T0} = 0.25 \pi \]
\[ \gamma_0 = 0 \]
\[ a_T = 0 \]
\[ R_0 = 5 \text{ km} \]

\[ v_0 = v_T = 270 \text{ ms}^{-1} \]

*figure 2.6.3: initial intercept geometry*

The flight paths and LOS-rate profile associated with scenario A1 are shown in figures 2.6.4a,b. The filtering results are depicted in figures 2.6.5a,b.

It was shown that the target heading is unobservable for non-maneuvering target. This is reflected by the \( p_{55} \) - history in figure 2.6.5a: \( p_{55} \) remains constant throughout the observation interval. In contrast the target heading rate is observable as indicated by the fast decay of \( p_{66} \) in figure 2.6.5b.
MISSILE AND TARGET POSITION

T.TRAJ. A1
DATE : 19/08/86
FILTER: 0

G.LAW : 3
TIME : 17.29.37.

○ MISSILE
△ TARGET

LOS-RATE

T.TRAJ. A1
DATE : 19/08/86
FILTER: 0

G.LAW : 3
TIME : 17.29.37.

$\text{figure 2.6.4a,b}$
ERROR VARIANCE OF STATE NO. 5

\[ \Delta \text{ VARIANCE (ESTIMATED)} \]

\[ \text{SCALE: } 1.60310^{-01} \]

\[ \text{TIME [S]} \]

---

ERROR VARIANCE OF STATE NO. 6

\[ \Delta \text{ VARIANCE (ESTIMATED)} \]

\[ \text{SCALE: } 5.300000E-02 \]

\[ \text{TIME [S]} \]

---

\textit{figure 2.6.5a,b}
Initial observability of \( y_5 \) and \( y_6 \) depends on the initial target heading \( \gamma_{T0} \). This is seen by comparing the histories of \( p_{55} \) and \( p_{66} \) for the following scenarios (see figure 2.6.3):

**scenario A2** (figures 2.6.6a,b and 2.6.8a,b):
\[
\gamma_{T0} = 0, \quad \gamma_0 = 0 \\
\alpha_T = 6g, \quad R_0 = 5 \text{ km}
\]

**scenario A3** (figures 2.6.7a,b and 2.6.9-11a,b):
\[
\gamma_{T0} = 0.5 \pi, \quad \gamma_0 = 0 \\
\alpha_T = 6g, \quad R_0 = 5.5 \text{ km}
\]

According to section 2.4.4 the initial observability of \( y_5 \) is minimal in scenario A2 and maximal in scenario A3. This is reflected by the initial behaviour of \( p_{55} \) in figures 2.6.8a (increasing \( p_{55} \) indicates low observability) and 2.6.9a (decreasing \( p_{55} \) indicates high observability). On the other hand the initial observability of \( y_6 \) is maximal in scenario A2 and minimal in scenario A3 which is confirmed by figures 2.6.8b and 2.6.9b.

The observability analysis has also shown that for scenario A3 the initial observability of \( y_6 \) depends on the initial estimate \( \hat{y}_6(0) \). For low absolute values of \( \hat{y}_6(0) \) observability of \( y_6 \) is lost. Figures 2.6.10a-d show the results of the filter obtained with an initial estimation error in \( y_6 \). The value of \( \hat{y}_6(0) \) is about half the exact value of \( y_6(0) \) (figure 2.6.10d). Comparison of figures 2.6.10a,b and 2.6.9a,b reveals the loss of observability in both \( y_5 \) and \( y_6 \). The filter converges, however, in the second half of the observation interval rendering bias free estimates of \( y_5 \) and \( y_6 \).

If \( \hat{y}_6(0) \) is decreased to about 25\% of the exact initial value (figures 2.6.11 a-d), \( y_5 \) and \( y_6 \) are practically unobservable. The heading rate error is not corrected any more (figure 2.6.11d) and entails a growing heading error (figure 2.6.11c). The filter finally diverges because the variances evaluated for the estimated target maneuver do not approximate the true estimation errors, i.e. (2.6.34) is violated as seen in figures 2.6.11a,b. The influence of the initial estimation errors on the future filter behaviour is very evident from these simulations. Their impact on filter stability is especially critical in scenarios with low observability of both \( y_5 \) and \( y_6 \), i.e. for type A3 scenarios with low target maneuver.
MISSILE AND TARGET POSITION

LOS-RATE

Figure 2.6.6a,b
MISSILE AND TARGET POSITION

LOS-RATE

*figure 2.6.7a,b*
ERROR VARIANCE OF STATE NO. 5

T.TRAJ. A2  
F.TRAJ.  
G.LAV : 3  
FILTER: 2

△ VARIANCE (ESTIMATED)
SCALE: $2.04350E-01$

Figure 2.6.8a,b

ERROR VARIANCE OF STATE NO. 6

T.TRAJ. A2  
F.TRAJ.  
G.LAV : 3  
FILTER: 2

△ VARIANCE (ESTIMATED)
SCALE: $5.30000E-02$

Figure 2.6.8a,b
ERROR VARIANCE OF STATE NO. 5

T. TRAJ. A3  
F. TRAJ. DATE: 22/08/86  
G. LAW : 3  TIME: 20.55.16.  
FILTER: 2 FIGURE  

Δ VARIANCE (ESTIMATED)  
SCALE: 1.60760E-01

ERROR VARIANCE OF STATE NO. 6

T. TRAJ. A3  
F. TRAJ. DATE: 22/08/86  
G. LAW : 3  TIME: 20.55.16.  
FILTER: 2 FIGURE  

Δ VARIANCE (ESTIMATED)  
SCALE: 5.30000E-02

figure 2.6.9a,b
ERROR VARIANCE OF STATE NO. 5

T. TRAJ. A3  JOB : Y3213YT
F. TRAJ. DATE : 22/08/86
G. LAW : 3  TIME : 21.01.04.
FILTER : 2  FIGURE:

\[ \text{VARIANCE (ESTIMATED)} \]
\[ \text{SQUARED ESTIM. ERROR} \]
\[ \text{SCALE: } 5.46666E-01 \]

TIME [S]

A3

ERROR VARIANCE OF STATE NO. 6

T. TRAJ. A3  JOB : Y3213YT
F. TRAJ. DATE : 22/08/86
G. LAW : 3  TIME : 21.01.04.
FILTER : 2  FIGURE:

\[ \text{VARIANCE (ESTIMATED)} \]
\[ \text{SQUARED ESTIM. ERROR} \]
\[ \text{SCALE: } 5.30000E-02 \]

TIME [S]

A3

figure 2.6.10a,b
FILTER STATE NO. 5

T.TRJ. A3  JOB : Y3213YT
F.TRJ. DATE : 22/08/86
FILTER : 2

A  ESTIMATED STATE
+ EXACT STATE

SCALE: 2.73050E-00

TIME [S]

FILTER STATE NO. 6

T.TRJ. A3  JOB : Y3213YT
F.TRJ. DATE : 22/08/86
FILTER : 2

+ ESTIMATED STATE
X EXACT STATE

SCALE: 2.26520E-01

TIME [S]

figure 2.6.10c,d
ERROR VARIANCE OF STATE NO.  5

A3

ERROR VARIANCE OF STATE NO.  6

A3

*figure 2.6.11a,b*
FILTER STATE NO. 5

T. TRAJ. A3  JOB: Y3213DU
F. TRAJ.  DATE: 22/08/86
FILTER: 2  FIGURE:

Δ ESTIMATED STATE  X EXACT STATE
SCALE: 2.78160E-00

FILTER STATE NO. 6

T. TRAJ. A3  JOB: Y3213DU
F. TRAJ.  DATE: 22/08/86
FILTER: 2  FIGURE:

+ ESTIMATED STATE  X EXACT STATE
SCALE: 2.18300E-01

figure 2.6.11c,d
2.6.4.4 Observability and Guidance Law Performance

In section 1.4.7 the guidance laws PE (section 1.3.5), OCE, and PCE (section 1.4.6.4) were compared assuming exact knowledge of all information required by the guidance algorithms. The sensitivity of OCE and PCE with respect to inaccurate prediction of the point of intercept were mentioned. The purpose of the following simulations is to illustrate the drastic degradation of guidance law performance caused by estimation errors in conjunction with target flight path prediction used by OCE and PCE. This is done by comparing the guidance laws PE and PCE with the required target information delivered by the tracking filter according to figure 2.6.12.

\[ a_1 : \text{missile lateral acceleration} \]
\[ a_c : \text{commanded acceleration} \]
\[ m : \text{LOS-rate measurement (figure 2.4.7)} \]
\[ u'_m : \text{missile maneuver (figure 2.4.7)} \]
\[ \hat{y} : \text{filter state (equ. 2.3.8a)} \]
\[ \delta : \text{flipper deflection} \]

*figure 2.6.12: guidance loop with tracking filter*
Consider the following scenario (see figure 2.6.3):

scenario A4: \[ \gamma_T^0 = 0.25 \, \pi, \quad \gamma_0 = 0 \]
\[ a_T = 0, \quad R_0 = 3 \text{ km} \]

The initial estimation errors of the tracking filter are:

- \[ e_4(0/0) = 175 \text{ m} \] (range error)
- \[ e_5(0/0) = -0.1 \text{ rad} \] (target heading error)

Both guidance laws were simulated twice, once using exact information (simulations 1 and 3) and once using estimated information generated by the tracking filter (simulations 2 and 4). The results are summarized in the following table:

<table>
<thead>
<tr>
<th>simul. no.</th>
<th>guid.law</th>
<th>miss distance [m]</th>
<th>problem duration [s]</th>
<th>figures</th>
<th>remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>PCE</td>
<td>0.04</td>
<td>8.87</td>
<td>2.6.13</td>
<td>exact information</td>
</tr>
<tr>
<td>2</td>
<td>PCE</td>
<td>52.8</td>
<td>8.85</td>
<td>2.6.14/15</td>
<td>estimated inf.</td>
</tr>
<tr>
<td>3</td>
<td>PE</td>
<td>0.52</td>
<td>8.74</td>
<td>2.6.16</td>
<td>exact information</td>
</tr>
<tr>
<td>4</td>
<td>PE</td>
<td>0.01</td>
<td>8.73</td>
<td>2.6.17</td>
<td>estimated inf.</td>
</tr>
</tbody>
</table>

*table 2.6.2: influence of estimation errors on miss distance*

As seen from table 2.6.2 the use of estimated information results in a large increase of the miss distance obtained with PCE whereas the results of PE in simulations 3 and 4 do not differ significantly. Figures 2.6.15a-d show the estimates computed by the tracking filter in simulation 2. The filtering results of simulation 4 are similar. Note that the time interval depicted in these figures is less than the problem duration given in table 2.6.2. Since the bearing rate becomes practically indefinite at final time (as seen in figures 2.6.13b/14b/16b/17b) the filter algorithm often breaks down immediately before intercept resulting in indefinite values for the state estimates. These are suppressed in the plots in order to guarantee a reasonable scaling. For additional explanations concerning the scaling and the dimensions of the filter states see table 2.6.1 in section 2.6.4.2.
It can be seen that the estimation errors in $y_2$, $y_4$, and $y_6$ are very low whereas there is a high permanent target heading error because $y_5$ is unobservable (no target maneuver). Due to the heading error the prediction algorithm described in section 1.4.6.2 generates wrong values for the collision course $\gamma_0^{opt}$ via equations (1.4.170c, d, e) resulting in wrong values for the course error $e_\gamma$ (equation 1.4.176). Since the commanded acceleration computed by PCE is proportional to $e_\gamma$ according to (1.4.193) the influence of the target heading error on the missile flight path is obvious. The flight paths and LOS-rate profiles associated with simulations 1 and 2 are depicted in figures 2.6.13 and 2.6.14, respectively.

In order to analyse the behaviour of PE the guidance law is rewritten here in terms of the filter state $y$. From (1.3.68) and figure 2.4.1 follows:

$$a_c = c (a_{T \Phi} - \lambda_0 \hat{R} \hat{\varphi})$$  \hspace{1cm} (2.6.35)

where $c$ is a constant and $a_{T \Phi}$ is the target acceleration normal to the current LOS. Substitution of (2.3.8) and (2.5.8a) into (2.6.35) renders:

$$a_c = c [v_T y_6 \cos (y_5 - y_3) - \lambda_0 y_1 y_2]$$  \hspace{1cm} (2.6.36)

Replacing the exact values of $y$ in (2.6.36) by their estimates computed in section 2.6.3 and observing (2.6.24) yields at time $t_k$:

$$a_c(t_k) = c [v_T \hat{y}_6(k/k) \cos \hat{y}_5(k/k) - \lambda_0 \hat{y}_1(k/k) \hat{y}_2(k/k)]$$  \hspace{1cm} (2.6.37)

where $\hat{y}_5(k/k)$ is the estimated target heading with respect to the LOS at time $t_k$.

If $y_6$ is observable $\hat{y}_6$ vanishes for non-maneuvering target. According to (2.6.37) $a_c$ is independent of $\hat{y}_5$ in this case and hence target heading errors have no effect on the missile flight path. This is reflected by the results of simulations 3 and 4 (figures 2.6.16/17). Estimation errors in $\hat{y}_6$ may occur for unobservable $y_6$ which is the case for $\hat{y}_5 \approx \frac{\pi}{2}$. However, $\cos \hat{y}_5 \to 0$ as $\hat{y}_5 \to \frac{\pi}{2}$ and therefore estimation errors in $y_6$ again have no effect on $a_c$. In other words, if either of the states $y_5$ or $y_6$ is unobservable, $a_c$ becomes independent of the unobservable quantity and is not
influenced by the associated estimation errors. This is also evident from equation (2.6.35): \( a_c \) depends on the observable component of the target acceleration \( a_{T\varphi} \) only. In this sense the guidance law PE is robust with respect to tracking errors. Since PCE uses unobservable information for the solution of the prediction equations the sensitivity with respect to estimation errors is plausible. OCE exhibits the same sensitivity problems because the computation of the course error is carried out exactly in the same way as by PCE.

Finally it is noted that the range and range-rate estimates are not influenced by the heading error as seen from figures 2.6.15a/b. This demonstrates the decoupling of \( y_2 \) and \( y_4 \) from \( y_5 \) for zero target maneuvers as mentioned in section 2.5.
MISSILE AND TARGET POSITION

LOS-RATE

figure 2.6.13a,b
MISSILE AND TARGET POSITION

MISSILE

TARGET

LOS-RATE

figure 2.6.14a,b
FILTER STATE NO. 2

![Graph showing estimated and exact states for Filter State No. 2]

FILTER STATE NO. 4

![Graph showing estimated and exact states for Filter State No. 4]

*figure 2.6.15a,b*
FILTER STATE NO. 5

FILTER STATE NO. 6

figure 2.6.15c,d
MISSILE AND TARGET POSITION

T.TRAJ. A4 JOB: Y3213YK
DATE: 23/08/86
FILTER: 0 FIGURE:

O MISSILE
△ TARGET

LOS-RATE

T.TRAJ. A4 JOB: Y3213YK
DATE: 23/08/86
FILTER: 0 FIGURE:

RAD/S

TIME [S]

figure 2.6.16a,b
MISSILE AND TARGET POSITION

LOS-RATE

Figure 2.6.17a,b
2.6.4.5 Summary

The simulation results obtained in this section confirm the conclusions of the observability analysis. They also reveal the sensitivity of the tracking algorithm with respect to initial estimation errors in certain intercept scenarios. Because the sensitivity problems are linked to observability of the target maneuver with respect to the bearing rate, they can only be avoided if additional information is available by measurements complementary to the bearing rate (for example range or range rate) enhancing observability of the target motion along the LOS. Nonetheless the guidance law PE has been found applicable in conjunction with the tracking filter because only the observable part of the target maneuver is used. With the information required for target maneuver compensation made available by the tracking filter the main obstacle to the realization of PE is in principle overcome (see section 1.3.6).

The main drawback of the present form of the filtering algorithm is its restriction to constant target maneuvers. In the next section an adaption scheme will be derived that allows tracking of time-varying target maneuvers.

2.7 An Adaption Scheme for Tracking Time-Varying Target Maneuvers

The BTA is based on the assumption that the target heading rate is constant. The extrapolation equations (2.4.30 - 34) are exact only if

\[ \dot{z}_6 = 0 \]  

(2.7.1)

according to (2.4.28f).

Assume that the true target maneuver is given by

\[ \dot{z}_6 = g(t) \]  

(2.7.2)

Let \( z^0(k+1) \) denote the propagated states computed by the filter based on (2.7.1) and \( z(k+1) \) the exact values associated with (2.7.2). Thus there will be a propagation error
\begin{align*}
  p_z(k+1) &= z^0(k+1) - z(k+1) \quad (2.7.3)
\end{align*}

The error \( p_z \) in the cartesian states \( z \) translates to an error \( p_y \) in the polar states \( y \) via (2.6.1). Filter divergence is likely to occur if \( p_y \) is not taken into account for covariance propagation in (2.1.11) because the accumulation of propagation errors may result in a violation of (2.6.34). This is demonstrated by applying the tracking algorithm to intercept scenario 2 of section 1.3.5.2. The target heading rate in this scenario is given by (see equation 1.3.70):

\begin{align*}
  \dot{\gamma}_T(t) &= 0 \quad 0 \leq t \leq t_e \quad (2.7.4a) \\
  \dot{\gamma}_T(t) &= \frac{a_{T0}}{v_T} \left[ 1 - \exp\left( -\frac{t - t_e}{t_v} \right) \right] \quad t > t_e \quad (2.7.4b)
\end{align*}

with

\[ a_{T0} = 6g, \quad v_T = 270 \text{ m/s}, \quad t_e = 2.5 \text{ s}, \quad t_v = 0.1 \text{ s} \]

The simulation results are shown in figures 2.7.1/2. While the initial estimation error in \( \gamma_6 \) (\( = \dot{\gamma}_T \)) is corrected in \([0, t_e]\) (figure 2.7.2), the filter fails to track the acceleration jump at \( t = t_e \) because the propagation errors due to the sudden change of \( \gamma_6 \) exceed the error level given by \( p_{66} \) (figure 2.7.2b).

In the following an adaption algorithm is derived that allows correction of the error variances in such a way that (2.6.34) is satisfied.

### 2.7.1 Covariance Matching

The propagation errors produce inconsistency between the error statistics computed by the filter and the true error statistics. Since the actual estimation errors are of course unknown the only means of obtaining information about their statistics is by analyzing the measurement residuals. Comparison of the residual statistics predicted by the filter and their "true" statistics obtained from a data window of \( N \) measured residuals allows to reestablish consistency between the computed and measured error statistics by appropriate covariance correction. This procedure is known as covariance matching [75,78,79].
FILTER STATE NO.  

ERROR VARIANCE OF STATE NO.  

figure 2.7.1a,b
FILTER STATE NO. 6

T.TRAJ. A2
F.TRAJ. DATE : 21/09/86
G.LAV. 3 TIME : 12.59.25.
FILTER. 2 FIGURE:

+ ESTIMATED STATE
X EXACT STATE
SCALE : 2.18300E-01

TIME [S]

ERROR VARIANCE OF STATE NO. 6

T.TRAJ. A2
F.TRAJ. DATE : 21/09/86
G.LAV. 3 TIME : 12.59.25.
FILTER. 2 FIGURE:

Δ VARIANCE (ESTIMATED)
○ SQUARED ESTIM. ERROR
SCALE : 5.30000E-02

TIME [S]

figure 2.7.2a,b
Covariance matching can be used for adaptive estimation of the input noise $Q$ as well as the measurement noise $S$. $Q$ acts on $P$ via the propagation equations (2.1.11) whereas $S$ appears in the update equations (2.1.19). An application of adaptive measurement noise estimation for homing missiles is reported in [80]. The purpose here is to stabilize the tracking algorithm in the presence of propagation errors suggesting covariance matching by adaptive $Q$-estimation. The relation between $Q$ and the propagation errors is established from the consistency requirement mentioned above.

Consider the linear measurement equation

$$m(k) = C(k) y(k) + s(k) \in \mathbb{R}^p$$

$$C = p \times n \text{ - measurement matrix where } n = \text{dim}(y)$$

$s(k) \sim N(0, S(k)) \text{ measurement noise}$

According to (2.1.20) the measurement residual associated with (2.7.5) is given by

$$r(k+1) = m(k+1) - \hat{m}(k+1) =$$

$$= C(k+1)[y(k+1) - \hat{y}(k+1/k)] + s(k+1) \quad (2.7.6)$$

Let $e^0$ denote the estimation error (2.1.21) in the absence of propagation errors. Hence, the exact value of $y$ at time $t_{k+1}$ is:

$$y(k+1) = \hat{y}(k+1/k) - e^0(k+1/k) - py(k+1) \quad (2.7.7)$$

Substitution of (2.7.7) into (2.7.6) yields:

$$r(k+1) = - C(k+1)[e^0(k+1/k) + py(k+1)] + s(k+1) \quad (2.7.8)$$

For further analysis a stochastic model of the propagation errors is needed. Therefore the following assumptions are made:

i) $E\{py(k+1)\} = 0 \quad (2.7.9)$

ii) $E\{e^0(k+1/k) py^t(k+1)\} = 0 \quad (2.7.10)$

iii) $E\{py(k) s^t(j)\} = 0 \quad \forall \ k, j \quad (2.7.11)$

Assumption i) is motivated by the desire to design an adaption scheme that results in compensation of the propagation errors in the update equations. Hence one is interested in keeping the measurement residuals bias free.
From (2.7.8-11) follows:

\[ M(k+1) = E\{ r(k+1) r^t(k+1) \} = \]
\[ = C(k+1) P^0(k+1/k) C^t(k+1) + \]
\[ + C(k+1) E\{ p_y(k+1) p_y^t(k+1) \} C^t(k+1) + S(k+1) \]  
(2.7.12)

where

\[ P^0(k+1/k) = E\{ e^0(k+1/k) e^{0t}(k+1/k) \} \]  
(2.7.13)

is the solution of (2.1.11) with \( Q = 0 \).

The residual covariance predicted by the filter is:

\[ M^0(k+1) = C(k+1) P^0(k+1/k) C^t(k+1) + S(k+1) \]  
(2.7.14)

Matching \( M \) and \( M^0 \) requires correction of \( P^0 \) with

\[ Q(k+1) = E\{ p_y(k+1) p_y^t(k+1) \} \]  
(2.7.15)

and the corrected value of \( P \) becomes:

\[ P(k+1/k) = P^0(k+1/k) + Q(k+1) \]  
(2.7.16)

(2.7.15) is the desired relation between \( Q \) and \( p_y \). An equation for \( Q(k+1) \) is obtained by substituting (2.7.14/15) into (2.7.12) rendering:

\[ C(k+1) Q(k+1) C^t(k+1) = M(k+1) - M^0(k+1) \]  
(2.7.17)

Since \( M(k+1) \) is unknown it is replaced by the sample covariance of the \( N \) most recent measurement residuals

\[ \overline{M}(k+1) = \frac{1}{N-1} \sum_{i=k+1-N}^{k+1} r(i) r^t(i) \]  
(2.7.18)

Thus one obtains:

\[ C(k+1) Q(k+1) C^t(k+1) \approx \overline{M}(k+1) - M^0(k+1) \]  
(2.7.19)

(2.7.19) is a set of \( p \) equations for the \( \frac{n(n+1)}{2} \) unknown elements of \( Q \) (note that \( Q \) is symmetric and positive definite). For the tracking problem discussed here (2.7.19) is a scalar equation (\( p = 1 \)) and the filter dimension is \( n = 6 \). Therefore, additional conditions are necessary to determine \( Q \). In view of (2.7.15) this amounts to adding further specifications to the stochastic model of the propagation errors.
2.7.2 Stochastic Model of Propagation Errors

In order to reduce the number of unknown parameters in the stochastic model (2.7.15) an approximate relation between the variances associated with the components of \( py \) is established. For this purpose an upper bound for the propagation errors produced in the sampling interval \( I_k = [t_k, t_{k+1}] \) is derived. From (2.7.1-3) follows that the propagation error associated with \( z_6 \) in \( I_k \) is given by:

\[
pz_6(\tau) = \int_0^\tau g(t_k + \sigma) \, d\sigma
\]

(2.7.20)

Hence

\[
|pz_6(\tau)| \leq \int_0^\tau |g(t_k + \sigma)| \, d\sigma \leq g_{\text{max}} \tau
\]

(2.7.21)

where

\[
g_{\text{max}} = \max_{0 \leq \sigma \leq \tau} |g(t_k + \sigma)|
\]

(2.7.22)

From (2.7.21) follows:

\[
|pz_6(k+1)| \leq g_{\text{max}} T =: \mu_6(k+1)
\]

(2.7.23)

\[
T = t_{k+1} - t_k
\]

(2.7.24)

For any given value \( 0 < \mu_6 < \infty \) the target maneuver that maximizes \(|pz_6(\tau)|\) is

\[
g(t_k + \sigma) = \delta(\sigma) \mu_6(k+1)
\]

(2.7.25)

Therefore an upper bound for the remaining components of \( pz \) is obtained by solving (2.4.28) with \( z_6 \) from (2.7.2) and \( g \) from (2.7.25). The solution is easily found by replacing \( z_6(k) \) with \( z_6(k) + \mu_6 \) in (2.4.30-34) and is of the following form:

\[
z(k+1) = f_z[z_1^0(k), ..., z_3^0(k), z_6^0(k) + \mu_6(k+1), u_M(k+1)]
\]

(2.7.26)

Substitution of (2.7.26) into (2.6.1) yields:

\[
y(k+1) = T_0 y_2[z_1^0(k), ..., z_3^0(k), z_6^0(k) + \mu_6(k+1), u_M(k+1)]
\]

(2.7.27)
Expansion of (2.7.27) in terms of $\mu_6$ using $y_6 = z_6$ according to (2.3.5/8) renders:

$$y(k+1) \approx TO_{yz}[z^0(k), u_M(k+1)] + \frac{\partial TO_{yz}[z^0(k), u_M(k+1)]}{\partial y_6^0(k)} \mu_6(k+1)$$

(2.7.28)

Observing that

$$TO_{yz}[z^0(k), u_M(k+1)] = y^0(k+1)$$

(2.7.29)

the propagation errors $p_y$ are found to be

$$p_y(k+1) = \frac{\partial y^0(k+1)}{\partial y_6^0(k)} \mu_6(k+1)$$

(2.7.30)

Note that (2.7.30) is valid for the target maneuver (2.7.25) only. Since this maneuver maximizes the extrapolation error (2.7.30) yields the first approximation of the upper bound of $|p_{y_i}|$:

$$|p_{y_i}(k+1)|_{\text{max}} \approx b_i(k+1) \left| \mu_6(k+1) \right|, \quad i = 1, \ldots, 6$$

(2.7.31)

with

$$b_i(k+1) = \frac{\partial y_i^0(k+1)}{\partial y_6^0(k)}$$

(2.7.32)

According to (2.7.9) it is assumed that

$$p_{y_6}(k) \sim N[0, q_6(k)]$$

(2.7.33)

Using (2.7.31) the stochastic properties of $p_{y_i}, i = 1, \ldots, 5$ can now be approximated in terms of (2.7.33). Noting that

$$b_6 = 1$$

(2.7.34)

$q_6$ is approximated by:

$$q_6 = \mu_6^2$$

(2.7.35)
With (2.7.35) the estimated variance associated with \( py_i \) is of the form:

\[
q_i = E[py_i^2] = b_i^2 q_6
\]  

(2.7.36)

Since (2.7.31) does not allow for conclusions about the cross-correlation between the components of \( py \), the \( py_i \) are modelled as uncorrected processes yielding:

\[
E\{py_i(k) py_j(k)\} = 0 \quad \forall \ i \neq j; \ i, j = 1, ..., 6
\]  

(2.7.37)

(2.7.36/37) with (2.7.10) and (2.7.15) result in:

\[
Q(k+1) = \text{diag}[q_i(k+1)] = \text{diag}[b_i^2(k+1)] q_6(k+1)
\]  

(2.7.38a)\(=\)  

(2.7.38b)

For the computation of \( b_i \) the values of \( y^0(k) \) must be known. Because the exact values are not available they are replaced by their estimates \( \hat{y}(k/k) \). Hence one obtains from (2.7.32) with (2.1.9/12):

\[
b_i \approx \frac{\partial \hat{y}_i(k+1/k)}{\partial \hat{y}_6(k/k)} = w_{i6}(k+1, k)
\]  

(2.7.39)

Substitution of (2.7.38) into (2.7.37) renders finally:

\[
Q(k+1) = \text{diag} [w_{i6}^2(k+1, k)] q_6(k+1)
\]  

(2.7.40)

Discussion:

(2.7.40) represents the stochastic model of the propagation errors. The \( \frac{n}{2}(n+1) \) unknowns in (2.7.15) have been replaced by the single unknown quantity \( q_6 \). Covariance matching has therefore been reduced to the determination of \( q_6 \) from the scalar equation (2.7.19).

In contrast to (2.7.37) the \( py_i \) are highly correlated. The cross-correlations depend on the target maneuver and have been used to compute the variances \( q_i \) associated with the maneuver (2.7.25). Due to the special choice of this maneuver the true variances are bounded by \( q_i \).
Neglection of the cross-correlations in Q is necessary because the actual target maneuver is unknown. With (2.7.37) the single "noise"-source \( \nu_6 \) is replaced by \( n \) uncorrelated noise channels. Consequently, filter performance is affected only in so far as the variance estimates computed by (2.1.11/19) will be conservative. This is desirable because it enhances robustness with respect to other modeling error sources such as changing target velocity \( v_T \).

### 2.7.3 The Adaption Algorithm (AA)

In the following the computation of \( Q(k+1) \) according to (2.7.19) and (2.7.40) is summarized. Some precautions have to be taken in order to ensure that \( Q \) remains positive semi-definite according to (2.7.15) and to keep the variances \( \sigma_i \) bounded.

For bearing-rate only measurements according to (2.3.10) the measurement matrix is:

\[
C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}
\]  

(2.7.41)

Inserting (2.7.41) into (2.7.19) renders:

\[
\sigma_q(k+1) = M(k+1) - p_{11}(k+1/k) - S(k+1)
\]  

(2.7.42)

From (2.7.36/39) follows:

\[
\sigma_q(k+1) = \frac{q_1(k+1)}{w_{16}^2(k+1/k)}
\]  

(2.7.43)

and \( Q \) may be computed from (2.7.40). In order to ensure that \( Q \) remains positive semi-definite adaption is carried out only if \( q_1 \) is positive. Therefore \( q_1 \) is computed as follows:

\[
q_1'(k+1) = M(k+1) - p_{11}(k+1/k) - S(k+1)
\]  

(2.7.44)

\[
q_1(k+1) = \max[0, q_1'(k+1)]
\]  

(2.7.45)

If \( w_{16} \) in (2.7.43) tends to zero \( Q \) becomes indefinite. This situation occurs if the heading rate \( \nu_6 \) is unobservable (see section 2.4). Due to (2.7.40/43) the error variances then grow without bounds, indicating complete uncertainty in the state estimates. This is physically reasonable because no information about the propagation
errors is available in the measurements. Typically, conditions with (nearly) zero
sensitivity \( w_{16} \) occur during short periods only, because of maneuvers and the
changing relative geometry. Hence, it is desirable to avoid unrealistically high values
of \( p_{66} \) during these intervals in order to ensure convergence of the filter on subsequent
arcs with improved observability. A straightforward approach is the limitation of \( q_6 \) by
\( q_{6\text{max}} \) in (2.7.43) and the suppression of adaption if \( p_{66} \) crosses an upper bound
\( p_{66\text{max}} \). With this modifications one obtains:

\[
q_6(k+1) = \min\{q_{6\text{max}}, \max[0, q_1'(k+1)] \over w_{16}^2(k+1,k) \}, \quad \text{if } p_{66} < p_{66\text{max}} \quad (2.7.46a)
\]

\[
q_6(k+1) = 0, \quad \text{if } p_{66} \geq p_{66\text{max}} \quad (2.7.46b)
\]

\( q_{6\text{max}} \) is a measure of the maximum possible propagation error in \([t_k, t_{k+1}]\)
associated with the target heading rate. It is determined by the target's maximum
heading angle acceleration \( \dot{y}_6 \) which is related to the dynamics of the lateral
acceleration via (2.3.3). \( p_{66\text{max}} \) approximates the maximal quadratic estimation error
in \( y_6 \) and is a tuning parameter of the adaption algorithm.

In summary the adaption proceeds as follows:

step 1 (prediction):
- compute propagated state from (2.1.9) and the measurement residual \( r(k+1) \)
  according to (2.7.6)
- compute predicted covariance matrix \( P^0(k+1/k) \) from (2.1.11)

step 2 (matching):
- compute \( \tilde{M}(k+1) \) according to (2.7.18) with \( r(k+1) \) known from step 1
- compute \( q_1'(k+1) \) from (2.7.44)
- compute \( q_6(k+1) \) from (2.7.46) and \( Q(k+1) \) from (2.7.40)
- compute \( P(k+1/k) \) from (2.7.16)

\textit{table 2.7.1: adaption scheme}
Remarks:
The output of the adaption algorithm is the matched covariance matrix $P(k+1/k)$ which is input to the update equations (2.1.15/19). Hence the propagation errors are accounted for during update.

From the equations (2.7.44/45) it follows that the variance $S$ of the measurement noise determines the sensitivity of the adaption algorithm with respect to the propagation errors: According to (2.7.46) adaption is carried out for positive $q_1'$ only. For large values of $S$ $q_1'$ is positive only for large values of $\overline{M}$. Hence substantial propagation errors are tolerated without adaption which may result in filter divergence. In other words, increasing $S$ decreases the sensitivity of the adaption scheme. If $S$ is low adaption is activated at low error levels (high sensitivity) resulting in high values for $q_i$ and $p_{II}$ which may result in a loss of estimation accuracy.

In view of its influence on the adaption algorithm $S$ may be regarded a tuning parameter (especially if the true measurement noise is unknown). The selection of $S$ is a trade-off between sufficient sensitivity necessary to adapt to changing target maneuvers and estimation accuracy.

The influence of each new measurement residual on $\overline{M}$ is inversely proportional to $N-1$ (equation 2.7.18). Hence, the adaption scheme reacts slowly to changing target maneuvers if the data window used to compute $\overline{M}$ is long. Therefore, for fast adaption $N$ should be chosen low. However, statistical significance of $\overline{M}$ may be lost if only few residuals are taken into account in the averaging process (2.7.18). The "optimal" value for $N$ has to be found by experiments.

The statistical significance of $\overline{M}$ is also determined by the length of the sampling period $T$. Because the innovations process is instationary the current statistics of the measurement residuals at $t_{k+1}$ can only be approximated by $\overline{M}$ for small $T$. $T$ is, however, constrained by the computing time required to solve the filter and adaption equations. Motivated by the experiences in section 1.4 singular perturbation theory will be applied to the tracking problem in the next section in order to reduce the dimension of the filter. In this way a substantial reduction of computing time can be achieved.
2.8 A Singly Perturbed Adaptive Tracking Filter

In this section an adaptive tracking algorithm for maneuvering targets is developed. The algorithm is based on the BTA of section 2.6.3 and the adaption scheme derived in the previous section. Advantage is taken of the time scale separation revealed in section 1.4.4. It was shown that the dynamics of the missile heading angle are decoupled from the missile-target relative motion in many scenarios. The same may be expected for the target heading angle. This is confirmed by the scaling of the tracking problem in the following section. Due to the time scale separation, two filters are obtained: a high dimensional filter associated with a slow time scale and a low dimensional fast filter. The low dimension of the fast filter allows high sampling rates as required for adaption because the number of computations in the fast time scale is considerably lower than for the BTA.

2.8.1 Scaling of the Tracking Problem

A scaled representation of the tracking problem is found by introducing dimensionless variables (·) and appropriate reference values (·)max in an analog manner as was done for the intercept problem in section 1.4.4:

\[
\begin{align*}
\dot{y}_1 &= \dot{y}_1 \varphi_{\text{max}} \quad (2.8.1a) \\
\dot{y}_2 &= \dot{y}_2 \varphi_{\text{max}} \quad (2.8.1b) \\
\dot{y}_3 &= \dot{y}_3 \varphi_{\text{max}} \quad (2.8.1c) \\
\dot{y}_4 &= \dot{y}_4 R_{\text{max}} \quad (2.8.1d) \\
\dot{y}_5 &= \dot{y}_5 \varphi_{\text{Tmax}} \quad (2.8.1e) \\
\dot{y}_6 &= \dot{y}_6 \varphi_{\text{Tmax}} \quad (2.8.1f) \\
\end{align*}
\]

\[
\begin{align*}
\dot{a}_T \varphi &= \dot{a}_T \varphi_{\text{max}} \quad (2.8.2a) \\
\dot{a}_T R &= \dot{a}_T R_{\text{max}} \quad (2.8.2b) \\
\end{align*}
\]

\[
\begin{align*}
\dot{a}_\varphi &= \dot{a}_\varphi a_{\text{max}} \quad (2.8.2c) \\
\dot{a}_R &= \dot{a}_R a_{\text{max}} \quad (2.8.2d) \\
\end{align*}
\]

\[
t = t_r t_f \\
\left(\cdot\right)' = \frac{d}{dr} = t_f \frac{d}{dt} ; \left(\cdot\right) = \frac{d}{dt} \\
\]

Substitution of (2.8.1-4) into (2.5.7) yields:
\[
y'_{1} = \frac{1}{\epsilon_{11}^{*}} \frac{a_{T\Phi}^{*} - a_{\Phi}^{*}}{y_{4}^{*}} - \frac{1}{\epsilon_{12}^{*}} 2 \frac{y_{1}^{*}y_{2}^{*}}{y_{4}^{*}} \\
y'_{2} = \frac{1}{\epsilon_{21}^{*}} (a_{TR}^{*} - a_{R}^{*}) + \frac{1}{\epsilon_{22}^{*}} \frac{y_{2}^{*2}}{y_{4}^{*}} \\
y'_{3} = \frac{1}{\epsilon_{3}^{*}} y_{1}^{*} \\
y'_{4} = \frac{1}{\epsilon_{4}^{*}} y_{2}^{*} \\
y'_{5} = \frac{1}{\epsilon_{5}^{*}} y_{6}^{*} \\
y'_{6} = \frac{1}{\epsilon_{6}^{*}} f_{6}(r) \\
\] (2.8.5a, 2.8.5b, 2.8.5c, 2.8.5d, 2.8.5e, 2.8.5f)

with \( a_{T\Phi} \) and \( a_{TR} \) according to (2.5.8a,b) and figure 2.4.1 (see p. 127), and

\[
\epsilon_{11} = \frac{\frac{R_{\text{max}} \dot{\varphi}_{\text{max}}}{a_{\text{max}} t_{f}}} \sim 0.05 ..... 0.005 \\
\epsilon_{12} = \frac{\frac{R_{\text{max}}}{\dot{R}_{\text{max}} t_{f}}} \sim 1 \\
\epsilon_{21} = \frac{\frac{\dot{R}_{\text{max}}}{a_{\text{max}} t_{f}}} \sim 1 \\
\epsilon_{22} = \frac{\frac{R_{\text{max}}}{t_{f}} \frac{1}{\dot{R}_{\text{max}} \varphi_{\text{max}}^{2}}} \sim 1 ..... 100 \\
\epsilon_{3} = \frac{\frac{\varphi_{\text{max}}}{t_{f}} \frac{1}{\dot{\varphi}_{\text{max}}}} \sim 1 \\
\epsilon_{4} = \frac{\frac{R_{\text{max}}}{t_{f}} \frac{1}{\dot{R}_{\text{max}}}} \sim 1 \\
\] (2.8.6a, 2.8.6b, 2.8.6c, 2.8.6d, 2.8.6e, 2.8.6f)
The numerical values for $\varepsilon$ are obtained with the following reference values, which are "typical" for the scenarios investigated here:

- $a_{\text{max}} = 10g = 100\text{m/s}^2$
- $\dot{\varphi}_{\text{max}} = 0.1\text{s}^{-1} \ldots 0.01\text{s}^{-1}$
- $R_{\text{max}} = 5000\text{m}$
- $\dot{R}_{\text{max}} = 500\text{m/s}$
- $t_f = 3\text{s} \ldots 9\text{s}$

\[
\frac{R_{\text{max}}}{\dot{R}_{\text{max}}} \sim t_f
\]

Discussion

Comparison of the scaling factors $\varepsilon$ reveals that the states $y_5$ and $y_6$ are candidates for selection as fast variables compared to $y_2, \ldots, y_4$. The bearing rate $y_1$ is composed of a fast and a slow part according to:

\[
y_1 = y_{11} + y_{12}
\]

(2.8.7)

with

\[
\varepsilon_{11} y_{11} = \frac{a_T \varphi - a \varphi}{y_4} \quad \text{fast part}
\]

(2.8.8)
Note that $y_3$ is a slow variable mainly because the guidance law tries to keep the LOS-rate low resulting in only slow changes of the bearing angle. Hence the following scaling is suggested:

**Fast states:** $y_{11}, y_5, y_6$  \hspace{1cm}  \text{(2.8.10)}

**Slow states:** $y_{12}, y_2, y_3, y_4$  \hspace{1cm}  \text{(2.8.11)}

If it is feasible to assume that the slow states remain constant in the fast time scale (which is not obvious as discussed in section 2.8.2) the dimension of the fast filter is only $n_F = 3$ according to equation (2.8.10) instead of $n = 6$ in the BTA. The substantial reduction of computing time becomes obvious by observing that the transition matrix $W$ required for propagation of the covariance matrix according to (2.1.11) has 36 elements for the BTA but only 9 elements for the fast filter based on (2.8.10).

Finally it is noted that the design of a fast filter is possible only if the measurement equation contains information about the fast states in the fast time scale, i.e. the fast subsystem must be boundary layer observable. Fortunately this observability condition is satisfied (with the restrictions discussed in section 2.4) because the fast states appear in the fast part of the bearing rate as seen from (2.8.8) and (2.5.8). Note that for bearing only measurements the fast subsystem is not observable in the fast time scale because the bearing angle is not part of the fast state.

Before designing a multiple time scale filter based on the scaling given by (2.8.10/11) some general remarks on the application of singular perturbations in filtering theory are necessary.

**Remark:**
The scaling of the fast states is due to the missile-target relative acceleration according to (2.8.8). Since the relative acceleration occurs in (2.8.5b) as well one might split $y_2$ in a fast and a slow part, too. It turns out, however, that the dynamics of the fast states
(y_{11}, y_5, y_6) are decoupled from y_2 in the fast time scale. Since y_2 is not measured the same applies to the associated reduced filter. Therefore a split of the dynamics of y_2 is of no advantage and y_2 is treated as a slow variable in order to keep the dimension in the fast time scale minimal. Hence the scaling (2.8.6c) is more a matter of convenience rather than a physical fact. In cases where y_2 is measured a split of the associated dynamics is justified because it provides additional information in the fast time scale thus enhancing observability of the fast states.

2.8.2 Singularly Perturbed Stochastic Systems

One of the main results of SP-theory for deterministic systems is the following procedure to construct the zero-th order approximation to the solution of a singularly perturbed system (see section 1.4.3):

a) Solve the reduced problem in the time scale of the slow variables. The fast variables may be considered quasistationary in this time scale provided that the fast subsystem is stable.

b) Solve the boundary layer equations in a stretched time scale. In this time scale the slow variables remain constant.

In the following, some comments on the applicability of the above procedure to stochastic systems are made.

2.8.2.1 Linear Systems

Consider the linear system

\[ \dot{x} = A_{11} x + A_{12} z + B_1 u \quad \in \mathbb{R}^{n_1} \]  
\[ \epsilon \dot{z} = A_{21} x + A_{22} z + B_2 u \quad \in \mathbb{R}^{n_2} \]  
\[ u \sim \mathcal{N}(0, Q) \quad \in \mathbb{R}^p \quad \text{input noise} \]  

All matrices may be time variant and A_{22}(t) is assumed to be stable for all t. As usual \( \epsilon \) denotes a small perturbation parameter. Inner and outer solution will be denoted by (\( )^i \) and (\( )^o \), respectively as introduced in section 1.4.2. The autocorrelation function of \( z \) is defined as
\[ \Sigma_{zz}(t,\sigma) = E[z(t)z(t)] \quad (2.8.15) \]

It can be shown [81] that:

\[ \lim_{\varepsilon \to 0} \Sigma_{zz}(t,\sigma) = \Sigma_{zz}^0(t,\sigma) = \begin{bmatrix} A_{22}^{-1} A_{21} A_{22}^{-1} & A_{22}^{-1} B_2 \end{bmatrix} \begin{bmatrix} t^{-1} & 0 \\ 0 & t^{-2} \end{bmatrix} \begin{bmatrix} A_{21} & A_{22}^{-1} B_2 \end{bmatrix} \quad (2.8.16) \]

Setting \( \varepsilon = 0 \) in (2.8.13) yields the quasistationary solution

\[ z^0 = - A_{22}^{-1} A_{21} x - A_{22}^{-1} B_2 u \quad (2.8.17) \]

Using (2.7.17) it is easily shown that

\[ \Sigma_{zz} z^0(t,\sigma) = \Sigma_{zz}^0(t,\sigma) \quad (2.8.18) \]

Moreover it is proved in [82] that:

\[ \lim_{\varepsilon \to 0} \int_{t-\Delta t}^{t+\Delta t} \Sigma_{zz} e_z e_z'(t,\sigma) \, d\sigma = 0 \quad (2.8.19) \]

where

\[ e_z = z - z^0 \quad (2.8.20) \]

Hence, for \( \varepsilon \to 0 \) \( z \) converges versus the quasistationary solution \( z^0 \) in the sense of (2.8.19/20) justifying the computation of \( x^0 \) by replacing \( z \) with \( z^0 \) in (2.8.12) (a rigorous proof can be found in [82]). The relation between the zero-order outer solution and the exact solution is given by

\[ x = x^0 + O(\varepsilon) \quad (2.8.21) \]

\[ z = z^0 + O(\varepsilon) \quad (2.8.22) \]

Note that for deterministic systems the corresponding approximations are \( O(\varepsilon) \).

Application of the time scale transformation

\[ \tau = \frac{t}{\varepsilon} \quad (2.8.23) \]

\[ (\cdot)' = \frac{d}{d\tau} = \frac{1}{\varepsilon} \frac{d}{dt} \quad (2.8.24) \]

to (2.8.12/13) yields the boundary layer equations
\[ x' = \varepsilon A_{11} x + \varepsilon A_{12} z + \varepsilon B_1 u \]  
\[ z' = A_{21} x + A_{22} z + B_2 u \]

For deterministic \( u \) (2.8.25) yields constant \( x \) in the boundary layer as \( \varepsilon \) tends to zero. However, this is not true if \( u \) is a white noise according to (2.8.14) because \( u \) has infinite variance. Further conclusions may be obtained by analyzing the Ito-differential equation associated with (2.8.12):

\[ dx = A_{11} x \, dt + A_{12} z \, dt + B_1 \, dw \]

with

\[ E[ dw] = 0 \]  
\[ E[ dw \, dw^t] = Q \, dt \]

With (2.8.27) the covariance matrix of \( dx \) becomes:

\[ E[dx \, dx^t] = A_{11} P_{xx} A_{11}^t \, dt^2 + A_{11} P_{xz} A_{12}^t \, dt^2 + \] 
\[ A_{12} P_{xz} A_{11}^t \, dt^2 + A_{12} P_{zz} A_{12}^t \, dt^2 + B_1 Q B_1^t \, dt \]

with

\[ P_{xy} = \Sigma_{xy}(t, t) \]

According to (2.8.23) \( E[dx \, dx^t] \) may be expressed in the fast time scale by substituting

\[ dt = \varepsilon \, d\tau \]

into (2.8.28) yielding:

\[ E[dx \, dx^t] = \varepsilon^2 A_{11} P_{xx} A_{11}^t \, d\tau^2 + \varepsilon^2 A_{11} P_{xz} A_{12}^t \, d\tau^2 + \] 
\[ \varepsilon^2 A_{12} P_{xz} A_{11}^t \, d\tau^2 + \varepsilon^2 A_{12} P_{zz} A_{12}^t \, d\tau^2 + \varepsilon B_1 Q B_1^t \, d\tau \]

Letting \( \varepsilon \) tend to zero one obtains:

\[ \lim_{\varepsilon \to 0} E[dx \, dx^t] = 0 \]  
in the boundary layer
Conclusion:
In the sense of (2.8.32) the slow variables may be considered constant in the fast time scale. From (2.8.32) follows:

\[ E[dx \ dx'] = O(\epsilon) \quad \text{as } \epsilon \to 0 \]  
(2.8.33)

Hence for the inner solutions obtained from setting \( dx = 0 \) in (2.8.25/26) one has:

\[ x = x^1 + O(\sqrt{\epsilon}) \]  
(2.8.34)
\[ z = z^1 + O(\sqrt{\epsilon}) \]  
(2.8.35)

The results of the above discussion may be summarized as follows: The procedure for constructing the zero-th order approximation to the solution of a singularly perturbed deterministic system may be applied to linear stochastic systems (2.8.12-14). However, the notions "quasistationary" and "constant" have to be interpreted in the sense of (2.8.18/19) and (2.8.33), respectively. Therefore the resulting approximations are of order \( O(\sqrt{\epsilon}) \) rather than \( O(\epsilon) \).

For application of the above result to the filtering problem consider (2.8.12-14) with the measurement equation

\[ y = C_1 x + C_2 z + s \]  
(2.8.36)
\[ s \sim N(0,S), \ E[u(t) s(\tau)] = 0 \]  
(2.8.37)

Determination of the (zero-th order) outer solution is done by setting \( \epsilon = 0 \) in (2.8.13) yielding \( z^0 \) according to (2.8.17). Substitution of \( z^0 \) into (2.8.12) and (2.8.36) renders:

\[ \dot{x}^0 = [A_{11} - A_{12} A_{22}^{-1} A_{21}] x^0 + [B_1 - A_{12} A_{22}^{-1} B_2] u \]  
(2.8.38)
\[ y^0 = [C_1 - C_2 A_{22}^{-1} A_{21}] x^0 - C_2 A_{22}^{-1} B_2 u + s \]  
(2.8.39)

Note that the noise input of the fast subsystem, \( B_2 u \), appears as an additional measurement noise in the outer solution. Defining

\[ A^0 = A_{11} - A_{12} A_{22}^{-1} A_{21} \]  
(2.8.40a)
\[ B^0 = B_1 - A_{12} A_{22}^{-1} B_2 \]  
(2.8.40b)
\[ C^0 = C_1 - C_2 A_{22}^{-1} A_{21} \]  
(2.8.40c)
\[ D^0 = C_2 A_{22}^{-1} B_2 \]  
(2.8.40d)
the equations of the Kalman-filter associated with (2.8.38/39) become:

\[ \tilde{x}^0 = A^0 \tilde{x}^0 + K_1^0 [y^0 - C^0 \tilde{x}^0] \quad (2.8.41) \]

\[ K_1^0 = P_{11}^0 C_{11}^o [S^o]^{-1} \quad (2.8.42) \]

\[ \dot{P}_{11}^o = P_{11}^o A_{11}^o + A_{11}^o P_{11}^o + B_{11}^o Q B_{11}^o - P_{11}^o C_{11}^o [S^o]^{-1} C^o P_{11}^o \quad (2.8.43) \]

with

\[ P_{11}^o = E[(x-x_0)(x-x_0)^t] \quad (2.8.44) \]

and

\[ S^o = S + D^o Q D^o t \quad (2.8.45) \]

\( S^o \) is the variance of the measurement noise in the outer solution which includes the contribution of \( u \) according to (2.8.39).

It can be shown [81, 82] that the exact Kalman filter associated with (2.8.12-14/36) converges versus (2.8.41-45) as \( \varepsilon \) tends to zero.

In the fast time scale (2.8.25/26) with (2.8.32) and (2.8.36) yield for \( \varepsilon = 0 \):

\[ z^i = A_{21} x_0 + A_{22} z^i + B_2 u \quad (2.8.46) \]

\[ y^i = C_1 x_0 + C_2 z^i + s \quad (2.8.47) \]

\[ x_0 = x(0) \quad (2.8.48) \]

The filter associated with (2.8.46-48) is:

\[ \hat{z}^i = A_{21} \hat{x}_0 + A_{22} \hat{z}^i + K_2^i [y^i - C_1 \hat{x}_0 - C_2 \hat{z}^i] \quad (2.8.49) \]

\[ K_2^i = P_{22}^i C_{22}^t S^{-1} \quad (2.8.50) \]

\[ P_{22}^i = P_{22}^i A^t_{22} + A_{22} P_{22}^i + B_{22} Q_{22}^i B_{22}^t - P_{22}^i C_{22}^t S^{-1} C_{22} P_{22}^i \quad (2.8.51) \]

The filter (2.8.49-51) is denoted as fast filter in the sequel. Again it can be proved [81, 82] that in the boundary layer the exact filter converges versus the fast filter as \( \varepsilon \) tends to zero.
Conclusion:
The zero-th order approximation of the Kalman filter associated with (2.8.12-14/36) may be constructed by first taking the limit of (2.8.12-14/36) as $\epsilon \to 0$, viz. deriving the differential equations of the slow system (2.8.38/39) and of the fast system (2.8.46/47) and then designing the filters associated with each subsystem. The original filter of dimension $n = n_1 + n_2$ is split into two lower order filters with dimensions $n_1$ and $n_2$, respectively.

Based on this procedure and the scaling given in (2.8.10/11) a multiple time scale filter for the tracking problem is designed in the subsequent sections.

2.8.2.2 On the Applicability of the SP-Concept to Extended Kalman Filters

The results found in the previous section are valid for linear systems with deterministic system, control, and measurement matrices. For linearized stochastic differential equations these matrices and hence covariance matrices and filter gains are themselves stochastic quantities. Therefore, the conclusions obtained for linear systems do not apply to the design of extended Kalman filters. Especially, it cannot be expected that the EKF associated with a scaled nonlinear system converges versus the reduced filters obtained from the decoupling of slow and fast states prior to filter design. In other words: The operations "\(\lim\)" and "filter design" are not commutative for nonlinear systems. Nevertheless it may be feasible to take advantage of the time scale separation and to carry out filter design for the decoupled systems. Although the resulting slow and fast EKFs are not the $O(\epsilon)$-approximation to the exact EKF, stability of the design is not excluded. Also, it is not clear whether there is a loss of "optimality" because the exact EKF is not optimal in the sense of a minimal-variance filter either. The validity of the SP-approach has to be verified after filter design by checking filter performance by means of simulations.

A conceptual difficulty of decoupled filter design for the tracking problem is due to the fact that the fast subsystem is unstable. This is evident from equations (2.8.5e,f). Hence, no quasistationary solution $y^0_{11}, y^0_{5}, y^0_{6}$ exists, implying that order reduction in the slow time scale is impossible. However, in the fast time scale the slow variables are constant (in the sense of 2.8.32). Therefore a reduced order fast filter $FF$ with state
may be designed. The information produced by FF can be used to decouple the estimation of the slow variables from the dynamics of the fast subsystem. The basic approach consists of using a full order slow filter (FS) with state vector

\[
y^t_S = (y_1, y_2, ..., y_6)
\]

(2.8.53)

(which is essentially the BTA) in the slow time scale and performing covariance matching with respect to the residual statistics in the fast time scale. In this way FS may be stabilized in the presence of variable target maneuvers without its sampling rate being dictated by the dynamics of the fast subsystem. Essentially the fast filter is used for prediction of the anticipated propagation errors in the slow time scale. Details of the procedure are discussed in the subsequent sections.

### 2.8.3 Synthesis of the Multiple Time Scale Tracking Filter (MTSTF)

The MTSTF is a combination of the following modules: A slow filter with the state vector (2.8.53), the measurement module (section 2.6.2), the adaption algorithm (section 2.7.3), and a fast filter with the state vector (2.8.52). The only one of these modules that has not yet been discussed is the fast filter. In the following the equations of the fast filter are derived. Subsequently questions of linking and synchronization of the different modules are addressed. For clarity the main parameters associated with each module are listed below:

**FS (slow filter):**
- state \( y^t_S = (y_1, ..., y_6) \)
- covariance matrices \( P_S, Q_S, S_S \)
- missile, target maneuver \( u_M^S, u_T^S \)
- sampling period \( T_S \)

**FF (fast filter):**
- state \( y^t_F = (y_1, y_5, y_6) \)
- covariance matrices \( P_F, Q_F, S_F \)
- missile, target maneuver \( u_M^F, u_T^F \)
- sampling period \( T_F \)
**MM (measurement module):**
measured bearing rate \( m \)
measured missile maneuver \( u_M' \)

**AA (adaptation algorithm):**
(applied to FF only)
input data : \( \hat{y}_F, \hat{y}_S \)
output data: \( Q_F \)

### 2.8.3.1 Equations of Fast Filter (FF)

Based on the time scale separation assumed in (2.8.10/11) the scaled equations of the tracking problem are formally:

\[
\begin{align*}
\dot{y}_1 &= -2 \frac{y_1 y_2}{y_4} \\
\dot{y}_2 &= a_{TR} - a_R + y_4^2 \\
\dot{y}_3 &= y_1 \\
\dot{y}_4 &= y_2 \\
\epsilon \dot{y}_{11} &= \frac{v_T y_6 \cos(y_5 - y_3) - a_\varphi}{y_{T4}} \\
\epsilon \dot{y}_5 &= y_6 \\
\epsilon \dot{y}_6 &= f_6(t)
\end{align*}
\]

Using the time scale transformation (2.8.4) the boundary layer equations become:

\[
\begin{align*}
\dot{y}_{12} &= \dot{y}_2 = \dot{y}_3 = \dot{y}_4 = 0 \\
y_1' &= y_{11}' = \frac{1}{y_{T4}^o} [v_T y_6 \cos(y_5 - y_3^o) - a_\varphi] \\
y_5' &= y_6 \\
y_6' &= f_6(t)
\end{align*}
\]

With (2.8.7) the measurement equation becomes:
\[ m(\tau) = y_1(\tau) + w = C_F y_F + y_1^O + s_F \]  
with \[ C_F = [1, 0, 0] \]  
\[ s_F \sim N(0, S_F) \]  

Recall that \((\cdot)^O\) denotes the outer solution which is constant in the boundary layer.

Let \( T_F \) denote the sampling period of \( FF \). Since the target maneuver is unknown it is assumed as in the BTA:

\[ f_6(\tau) = 0 \]  

The propagation equations of \( y_5 \) and \( y_6 \) in the sampling interval \( I_j = [t_j, t_j+1] \) then become:

\[ \hat{y}_5(j+1/j) = \hat{y}_5(j/j) + T_F \hat{y}_6(j/j) \]  
\[ \hat{y}_6(j+1/j) = \hat{y}_6(j/j) \]  

with \[ T_F = t_{j+1} - t_j \]  

Since \( T_F \) is assumed to be very small (which is one of the design goals) propagation of the bearing rate is approximated by an Euler step rather than using the exact equations as was done for the BTA. In this way the computational burden for solving the propagation equations in the fast time scale is further reduced. With (2.8.7), (2.8.56), (2.8.57a) one obtains:

\[ \hat{y}_1(j+1/j) = \hat{y}_1(j/j) + \frac{T_F}{\hat{y}_4} \{ v_T \hat{y}_6(j/j) \cos[\hat{y}_5(j/j) - y_3^O] - a_\varphi \} \]  

(2.8.65) may be simplified if it is evaluated in a reference system \( Z^F = (x_F, y_F) \) given by

\[ \hat{y}_3^O \bigg|_{Z^F} = 0 \]  

in a similar manner like \( Z^S \) was defined in section 2.6.1/2. (2.8.65) then becomes:
\[
\hat{y}_1(j+1/j) = \hat{y}_1(j/j) + \frac{1}{y_4^o} \{ T_F v_T \hat{y}_6(j/j) \cos \hat{y}_5(j/j) - u_{M4}(j+1) \} \tag{2.8.67}
\]

with \( u_{M4} \) according to (2.4.32d).

The transition matrix associated with (2.8.62/63/67) is:

\[
W_F(j+1,j) = \frac{d\hat{y}_F(j+1/j)}{d\hat{y}_F(j/j)} =
\begin{bmatrix}
1 & -\frac{v_T T_F}{y_4^o} \hat{y}_6(j/j) \sin \hat{y}_5(j/j) & \frac{v_T T_F}{y_4^o} \cos \hat{y}_5(j/j) \\
0 & 1 & T_F \\
0 & 0 & 1
\end{bmatrix}
\tag{2.8.68}
\]

With (2.8.68) the propagation equations of the error covariance matrix \( P_F \) of the fast system are:

\[
P_F(j+1/j) = \begin{bmatrix} P_{F11} & P_{F15} & P_{F16} \\ P_{F15} & P_{F55} & P_{F56} \\ P_{F16} & P_{F56} & P_{F66} \end{bmatrix} (j+1/j) =
W_F(j+1,j)P_F(j/j)W_F^t(j+1,j) + Q_F(j+1) \tag{2.8.69}
\]

where

\[
Q_F(j+1) = \text{diag}[q_{F1}(j+1), q_{F5}(j+1), q_{F6}(j+1)] \tag{2.8.70}
\]
denotes the output of the adaption algorithm according to (2.7.40) in the fast time scale (see also section 2.8.3.5).

Substitution of (2.8.58/59) into (2.1.16-18) yields the following update equations in the fast time scale:
filter gains:
\[
\begin{pmatrix}
 g_1(j+1) \\
g_5(j+1) \\
g_6(j+1)
\end{pmatrix}
= \frac{1}{p_{F11}(j+1/j) + s_F}
\begin{pmatrix}
p_{F11}(j+1/j) \\
p_{F15}(j+1/j) \\
p_{F16}(j+1/j)
\end{pmatrix}
\tag{2.8.71}
\]

state update:
\[
\begin{align*}
\hat{y}_1(j+1/j+1) &= \hat{y}_1(j+1/j+1) + g_1(j+1) I \\
\hat{y}_5(j+1/j+1) &= \hat{y}_5(j+1/j+1) + g_5(j+1) I \\
\hat{y}_6(j+1/j+1) &= \hat{y}_6(j+1/j+1) + g_6(j+1) I \\
I &= m(j+1) - \hat{y}_1(j+1/j) \text{ innovations process}
\end{align*}
\tag{2.8.72}
\]

covariance update:
\[
\begin{align*}
p_{F11}(j+1/j+1) &= p_{F11}(j+1/j) (1-g_1(j+1)) \tag{2.8.74a} \\
p_{F55}(j+1/j+1) &= p_{F55}(j+1/j) - g_5(j+1) p_{F15}(j+1/j) \tag{2.8.74b} \\
p_{F66}(j+1/j+1) &= p_{F66}(j+1/j) - g_6(j+1) p_{F16}(j+1/j) \tag{2.8.74c} \\
p_{F15}(j+1/j+1) &= p_{F15}(j+1/j) - g_5(j+1) p_{F11}(j+1/j) \tag{2.8.74d} \\
p_{F16}(j+1/j+1) &= p_{F16}(j+1/j) - g_6(j+1) p_{F11}(j+1/j) \tag{2.8.74e} \\
p_{F56}(j+1/j+1) &= p_{F56}(j+1/j) - g_5(j+1) p_{F16}(j+1/j) \tag{2.8.74f}
\end{align*}
\]

Comparison of the equations of FF given above and the BTA reveals the substantial reduction of the number of computations in the fast time scale.

The equations of FF are based on the assumption (2.8.61). Therefore the filter must be adapted in the presence of variable target maneuvers. As discussed in section 2.7.1 adaption is carried out by appropriate selection of $Q_F$ in (2.8.69). Only the error variances associated with $y_F$ are matched. Because the slow states are constant in the fast time scale they are not affected by extrapolation errors in the fast states.

Summary:
In the fast time scale the states $y_F$ are estimated by the fast filter FF in conjunction with the adaption algorithm AA (see section 2.7.3). Due to its low order this filter may operate with a high sampling rate which is required for adaption and allows tracking of rapid changes in the target maneuver.
Changes of $\hat{y}^0$ which are parameters of FF are easily taken into account by updating the equations of FF with the latest estimates obtained from FS in a similar manner as discussed for SP-guidance laws in section 1.4.6.1.

2.8.3.2 Adaption and Propagation in the Slow Time Scale

Consider the sampling interval $I_k = [t_k, t_{k+1}]$ in the slow time scale, with

$$T_S = t_{k+1} - t_k \quad (2.8.75)$$

Assume that FF performs $N$ cycles in $I_k$ according to figure 2.8.1, hence

$$T_S = N T_F \quad (2.8.76)$$

At $t_{k+1}$ the estimated propagation errors in $I_j$ measured by $Q_P$ according to (2.8.70) are available for all $j \in [J(k) + 1, J(k+1)]$ from FF/AA. Since the propagation errors are modeled as a white noise sequence according to sections 2.7.1/2 the variance of $p_{y_6}$ (propagation error in $y_6$) in $I_k$ is given by:

$$q_{S6}(k+1) = \sum_{j=J(k)+1}^{J(k+1)} q_{F6}(j) \quad (2.8.77)$$

It is noted that $q_{S6}$ is the zero-th order approximation to the variance of $p_{y_6}$ because $q_{F6}$ is computed for constant slow states. Adaption in the slow time scale can now be carried out by simply solving the propagation equations of $P_S$ with $Q_S$ computed according to (2.4.70) and $q_{S6}$ from (2.8.77).
The estimates \( y_F \) may be used to compute the target maneuver \( u_T^{F}(j+1) \) in \( I_j \) according to (2.4.34) with

\[
z_m(j) = \hat{y}_m(j/j) ; \quad m = 5, 6
\]

(2.8.78)

The target maneuver in \( I_k \) is then given by

\[
u_T^S(k+1) = \sum_{j=J(k)+1}^{J(k+1)} u_T^{F}(j)
\]

(2.8.79)

The accuracy of the propagated states in the slow time scale may be considerably improved by using (2.8.76) for state propagation rather than assuming a constant target maneuver in \( I_k \).

2.8.3.3 Initialization of FF

The use of the target maneuver estimated by FF for state propagation in FS suggests an order reduction in the slow time scale by regarding the fast states as parameters known from FF. However, since there is no need to minimize the sampling period \( T_S \) it is preferable to use the full order filter in the slow time scale allowing for periodical reinitialization of \( y_F \) and \( P_F \) with the corresponding estimates of FS. In this way the fast filter may be prevented from diverging if the scaling assumptions are violated (i.e. the slow states are not constant in the fast time scale) which may happen for large \( N \). Thus, the fast filter is used as a predictor for both propagation errors and target maneuver in \( I_k \), whereas FS may be interpreted as a corrector improving the accuracy of the preliminary estimates obtained from FF.

Initialization of FF occurs in the interval \( I_{J(k)} \). The procedure includes updating the parameter \( \hat{y}_4^O \) used in (2.8.67/68) and reinitializing \( \hat{y}_F[J(k)/J(k)] \), \( P_F[J(k)/J(k)] \) with the estimates found by FS. The best estimates associated with \( t_k \) are the updated values \( \hat{y}_S(k/k) \) and \( P_S(k/k) \). These quantities are, however, not available within \( I_{J(k)} \) because the computing time required to solve the propagation and update equations of FS is basically \( T_S \). Since the solution of the state propagation equations requires only few computations compared to covariance propagation it is assumed that the
propagated states $\hat{y}(k/k-1)$ are available in $I_J(k)$ whereas the covariance matrix is only known from the previous update. Hence FF is initialized as follows:

$$\hat{y}_0 = \hat{y}(k/k-1)$$

$$\hat{y}[J(k)/J(k)] = \hat{y}(k/k-1)$$

$$P_{Fi}[J(k)/J(k)] = P_{Si}(k-1/k-1) ; i, j = 1,5,6$$

(2.8.80)  
(2.8.81)  
(2.8.82)

2.8.3.4 Measurement Data Transformation

The equations of FS and FF are evaluated in the reference systems $Z^S$ defined by (2.6.6) and $Z^F$ defined by (2.8.66), respectively. The missile maneuver $u_M$ computed by the measurement module is given with respect to the seeker system $Z^S$. In order to exchange data among the three modules, transformations to the appropriate reference systems are necessary. They will be discussed in the following.

Consider the situation at time $t = t_k + \tau$, $0 < \tau < T_F$ (figures 2.8.1/2). The extrapolated states $y_S(k/k-1)$ are known with respect to the reference system $Z^S(k-1)$. For initialization of FF the slow states have to be known with respect to $Z^F(k)$. According to (2.8.66/80) $Z^F(k)$ is determined by

$$\hat{y}_3 = 0$$

(2.8.83)

Since $y_1$, $y_2$, $y_4$, and $y_6$ are independent of the reference system and $y_3$ is given in (2.8.83) only $y_5$ has to be transformed. From figure 2.8.2 follows:

$$\hat{y}_5[J(k)/J(k)]\bigg|_{Z^F(k)} = \hat{y}_5[k/k-1]\bigg|_{Z^S(k-1)} - \hat{y}_3[k/k-1]\bigg|_{Z^S(k-1)}$$

(2.8.84)

The measurement module renders the missile maneuver in $I_j$ with respect to $Z^S(k)$:

$$u^F_M(j+1)\bigg|_{Z^S(k)} = u^M_M(j+1)$$

(2.8.85)
In order to solve the propagation equation (2.8.63) \( u_M^F \) must be transformed to \( Z^F(k) \). The orientation of \( Z^S(k) \) with respect to \( Z^S(k-1) \) is given by the angle \( \psi(k) \) according to (2.6.17) and figure 2.8.2. Hence, the orientation of \( Z^S(k) \) with respect to \( Z^F(k) \) is:

\[
\xi^F(k) = \psi(k) - \hat{y}_3[k/k-1]
\]

For the missile maneuver in \( Z^F(k) \) follows:

\[
\begin{pmatrix}
  u_{M1+1}^F(j+1) \\
  u_{M1+2}^F(j+1)
\end{pmatrix}_{Z^F(k)} = -TR[\xi^F(k)]
\begin{pmatrix}
  u_{M1+1}^{F,j+1} \\
  u_{M1+2}^{F,j+1}
\end{pmatrix}_{Z^F(k)}
\]

\[i = 0,1; \quad j = J(k), ..., J(k+1)-1\]

with \( TR \) according to (2.6.15)

*figure 2.8.2: reference systems in the sampling interval \([t_k, t_{k+1}]\)*
Equations (2.62/67-74) may now be solved at $t_{j(k)+1}$. The maneuver terms $u^F_{M}(j+1), u^F_{T}(j+1)$ are stored. For the computations in $I_{j+1}$ the measurement module is reinitialized by resetting the integrators $S_1, ..., S_4$ (see figure 2.6.2). The contents of $S_5$ remains unchanged for determination of $Z^S(k)$. After termination of the update equations of FF at $t_{j(k)+1}$ the equations of FS are solved in $I_k$. The total missile and target maneuvers in $I_k$ required for state propagation are obtained by summing up the maneuvers in $I_j$:

$$u^S_{K}(k+1) = \sum_{j=1}^{J(k)+1} u^F_K(j+1) ; K = M, T \quad (2.8.88)$$

Because $u^F$ is given in $Z^F(k)$ the maneuvers (2.8.88) must be transformed to the reference system $Z^S(k)$ of FS. The orientation of $Z^S(k)$ with respect to $Z^S(k)$ is given by the known angle $\xi(k)$ according to (2.6.16). From figure 2.8.2 follows:

$$\Delta \xi(k) = \xi_F(k) - \xi(k) \quad (2.8.89)$$

where $\xi_F(k)$ is known from (2.8.86). $\Delta \xi(k)$ is the orientation of $Z^S(k)$ with respect to $Z^F(k)$. Hence in $Z^S(k)$ one obtains:

$$\begin{pmatrix} u^S_{K i+1(k+1)} \\ u^S_{K i+2(k+1)} \end{pmatrix} \bigg|_{Z^S(k)} = - TR[\Delta \xi(k)] \begin{pmatrix} u^S_{K i+1(k+1)} \\ u^S_{K i+2(k+1)} \end{pmatrix} \bigg|_{Z^F(k)} \quad i = 0, 1; K = M, T \quad (2.8.90)$$

where the maneuver terms on the right hand side of (2.8.90) are known from (2.8.88).

With $u^S$ and $Q_S$ known from (2.8.90) and (2.8.77), respectively, the equations of FS in $I_k$ can be evaluated. A summary of all computations in $I_k$ is given in the next section.

2.8.3.5 Summary of MTSTF-Algorithm

In the following the sequence of computations carried out by the MTSTF in the sampling interval $I_k = [t_k, t_{k+1}]$ is summarized. The cycle starts at time $t_{j(k)} + \tau$, with $\tau > 0$. The following data are available:
from measurement module: $u_M^*\{J(k)+1\}$
$m[J(k)+1]$

from slow filter: $\hat{y}_S(k/k-1), P_S(k-1/k-1)$
$\xi_F(k), \psi(k)$

step 1: Initialize FF:
  a) Initialize parameter $\phi_4^0$ and states $\hat{y}_F$ according to (2.80/81/84).
  b) Initialize covariance matrix $P_F$ according to (2.8.82).

step 2: Compute missile maneuver in $I_j$ with respect to $Z^F(k)$ according to (2.8.87).

step 3: Solve propagation equations:
  a) Propagate states according to (2.8.62/63/65).
     Output are the propagated states $\hat{y}_F(j+1/j)$ and the residual $r_F(j+1)$.
  b) Propagate covariance matrix according to (2.8.69) with $Q_F = 0$.
     Output is $P_F^0(j+1/j)$.

step 4: Compute estimated target maneuver $u^F_{t}(j+1)\big|_{Z^F(k)}$ according to (2.4.34)
  and (2.8.78).

step 5: Adaptation and update:
  a) compute $q_F$ from (2.7.46) and $Q_F$ from (2.7.40)
  b) compute $P_F(j+1/j)$ from (2.7.16)
  c) solve update equations (2.8.72/74)

step 6: Initialize measurement module for next sampling interval $I_{j+1}$:
$S_1 = S_2 = S_3 = S_4 = 0$

IF $j < J(k+1)$: $j = j + 1$, GO TO 2
IF $j = J(k+1)$: start slow filter

step 7: Reset MM for next sampling interval $I_{k+1}$:
  a) store $s(t_{k+1}, t_k)$
  b) reset $S_5$
step 8: Determine missile and target maneuver in $I_k^k$:
   a) compute $u_M^S(k+1)$ and $u_T^S(k+1)$ from (2.8.88)
   b) compute $u_M^S(k+1)$ and $u_T^S(k+1)$ from (2.8.90)

step 9: Adaption in slow time scale:
   compute $Q_S(k+1)$ from (2.7.40) with (2.8.75)

step 10: Solve propagation equations of slow filter.
   Output is: $\hat{y}_S(k+1/k)$, $P_S(k+1/k)$

step 11: Determine bearing angle at $t_{k+1}$ with respect to $Z^S(k)$:
   $\psi(k+1) = \xi(k) - \xi(t_{k+1}, t_k)$

step 12: Determine reference system $Z^F(k+1)$:
   $\xi_F(k+1) = \psi(k+1) - \hat{y}_3(k+1/k)_{Z^S(k)}$

step 13: Start FF at 1 with $k = k + 1$

step 14: Solve update equations of FS.
   Output is $\hat{y}_S(k+1/k+1)_{Z^S(k)}$ and $P_S(k+1/k+1)$.

step 15: Determine reference system for FS in $I_{k+1}^k$: $Z^S(k+1)$
   $\xi(k+1) = \psi(k+1) - \hat{y}_3(k+1/k+1)_{Z^S(k)}$
   $\Delta \xi(k+1) = \xi_F(k+1) - \xi(k+1)$
   $\hat{y}_3(k+1/k+1)_{Z^S(k+1)} = \hat{y}_3(k+1/k+1)_{Z^S(k)} - \hat{y}_3(k+1/k+1)_{Z^S(k)}$
   $k = k + 1$
   GOTO 10

The signal flow of the MTSTF is depicted in the following figure.
2.9 Simulations

In the following the performance of the MTSTF is tested. First, the influence of the tuning parameters on the behaviour of the filter is demonstrated. Subsequently guidance law performance with the MTSTF in the guidance loop according to figure 2.9.1 is investigated. For this purpose the miss distance obtained with exact and estimated target information is compared for several intercept scenarios. In view of the discussion in section 2.6.4.4 only the guidance law PE (see section 1.3.5) will be considered. The guidance loop is depicted in the following figure.

Note that the guidance law operates in the fast time scale according to the SP-analysis in section 1.4.5. In this time scale the estimates for $y_1$, $y_5$, and $y_6$ are obtained from the fast filter while the estimates of $y_2$ (which is the only required slow variable) are delivered from the slow filter according to equation 2.8.80. Substitution of the appropriate estimates into (2.6.36) yields:

$$a_c(t_j) = c [v_T \hat{y}_6(j/j) \cos \hat{y}_5(j/j) - \lambda_0 \hat{y}_1(j/j) \hat{y}_2^0]$$  \hspace{1cm} (2.9.1)

The investigated intercept scenarios are grouped in five series A–E according to figure 2.9.2. The following parameters are fixed for all simulations:
\[ v_0 = 270 \text{ m/s (initial missile velocity)} \]  
\[ v_T = 270 \text{ m/s (target velocity)} \]  
\[ R = 3.5 \text{ km} \]  

Two types of target maneuvers are considered:

**Type P**: periodical target maneuver

\[ a_T = a_{T0} \sin(2\pi\nu + \eta) \]  
with \( a_{T0} = 6g, \nu = 0.25 \text{ Hz} \)

**Type J**: sudden change of target acceleration ("jump")

\[ a_T = 0; \quad 0 \leq t \leq t_e \]  
\[ a_T = a_{T0} \left[ 1 - \exp\left(\frac{t - t_e}{t_v}\right) \right]; \quad t > t_e \]  
with \( a_{T0} = 6g, t_v = 0.1 \text{ s} \)

In all simulations the filter is initialized with the following estimation errors:

\[ e_4(0/0) = 175 \text{ m (initial range error)} \]  
\[ e_5(0/0) = 0.1 \text{ rad (initial target heading error)} \]  
\[ e_6(0/0) = 0.1 \text{ rad/s (initial target heading rate error)} \]  
\[ e_v = 0.1 v_T = 27 \text{ m/s (velocity error according to equ. 2.6.33)} \]

The variance of the LOS-rate measurement noise is:

\[ S = 0.3 \times 10^{-3} \text{ rad}^2 \text{s}^{-2} \]
2.9.1 Influence of the Tuning Parameters

According to section 2.7.3 the quantities $N$ (or $T_S$ via equation 2.8.76) and the measurement noise variances $S_F$ and $S_S$ in the fast and slow time scale, respectively, may be viewed as tuning parameters of the filter. Their influence on the behaviour of the MTSTF is discussed in the following.

The sampling period $T_F$ of FF is chosen $0.05s$ for all subsequent simulations. The following "nominal" values of the tuning parameters yielded satisfactory estimation accuracy in most of the simulated engagement scenarios:

\[ T_S^n = 0.1 \text{ (i.e. } N = 2) \quad (2.9.7a) \]

\[ S_F^n = S \quad (2.9.7b) \]

\[ S_S^n = 10 S_F^n \quad (2.9.7c) \]

The simulation results summarized in table 2.9.1 illustrate the filter behaviour for the nominal parameters (2.9.7) and deviations from the nominal values.

<table>
<thead>
<tr>
<th>trajectory</th>
<th>$T_S$</th>
<th>$S_F$</th>
<th>$S_S$</th>
<th>figure</th>
<th>remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>scenario: A (fig. 2.9.2)</td>
<td>(T_S^n)</td>
<td>$S_F^n$</td>
<td>$S_S^n$</td>
<td>2.9.3</td>
<td>nominal trajectory</td>
</tr>
<tr>
<td>target maneuver: P (equ. 2.9.3)</td>
<td>(10 T_S^n)</td>
<td>$S_F^n$</td>
<td>$S_S^n$</td>
<td>2.9.4</td>
<td>increased $T_S$</td>
</tr>
<tr>
<td>name: AP</td>
<td>(T_S^n)</td>
<td>$S_F^n$</td>
<td>$S_S^n$</td>
<td>2.9.5</td>
<td>nominal trajectory</td>
</tr>
<tr>
<td>scenario: A</td>
<td>(T_S^n)</td>
<td>0.1 $S_F^n$</td>
<td>$S_S^n$</td>
<td>2.9.6</td>
<td>decreased $S_F$</td>
</tr>
<tr>
<td>target maneuver: J (equation 2.9.4)</td>
<td>(10 T_S^n)</td>
<td>$S_F^n$</td>
<td>$S_S^n$</td>
<td>2.9.7</td>
<td>increased $S_F$</td>
</tr>
<tr>
<td>name: AJ</td>
<td>(T_S^n)</td>
<td>$S_F^n$</td>
<td>0.1 $S_S^n$</td>
<td>2.9.8</td>
<td>nominal trajectory</td>
</tr>
<tr>
<td>scenario: C</td>
<td>(T_S^n)</td>
<td>$S_F^n$</td>
<td>$S_S^n$</td>
<td>2.9.9</td>
<td>decreased $S_S$</td>
</tr>
</tbody>
</table>

*table 2.9.1: influence of the tuning parameters*
**Remark:**
As in previous sections the time histories of the filter states and variances are depicted in a normalized scale. The associated dimensions are given in table 2.6.1 (section 2.6.4.2).

**Discussion of results:**
Comparison of the nominal trajectories depicted in figures 2.9.3a-c with figures 2.9.4a-c reveals the effect of an increased sampling period $T_s$. Estimation accuracy in the slow time scale remains practically unaffected as evident from figures 2.9.3c and 2.9.4c. This is due to performing the propagation of $y_S$ using the target maneuver estimated by FF according to (2.8.88) which results in high propagation accuracy even for long sampling periods $T_s$. There is, however, a loss of estimation accuracy in the fast states indicating a violation of the scaling assumptions.

Figures 2.9.5-7 demonstrate the influence of the tuning parameter $S_F$. According to section 2.7.3 this parameter determines the sensitivity of the adaption algorithm. In figures 2.9.6a-c the simulation results obtained with $S_F$ selected lower than the true measurement noise variance $S$ are shown. The sudden change of the target acceleration at $t = 2.5$ s is "detected" and tracked. However, due to the low value of $S_F$ adaption is activated at low error levels resulting in high error variances (compare figures 2.9.5c and 2.9.6c) and a loss of estimation accuracy (compare figures 2.9.5a/b and 2.9.6a/b). Note that the adaption to changing target maneuvers is always delayed because it starts only after the propagation errors exceed the level determined by $S_F$.

If $S_F$ is chosen much higher than $S$ the adaption algorithm becomes insensitive. High propagation errors are tolerated without adaption which takes place with a long delay at $t \approx 5$ s (see figure 2.9.7c).

Finally, figures 2.9.8/9 show that filter divergence in the slow time scale may occur if the noise variance $S_S$ is chosen too low. The filter underestimates the accuracy of the estimates of the slow variables (figure 2.9.9b). The necessity of selecting $S_S$ higher than the true measurement noise is plausible because the fast variables introduce additional measurement noise in the slow time scale as indicated by equation 2.8.45.
FILTER STATE NO. 2

Figure 2.9.3c
FILTER STATE NO.  5

T.TRAJ. AP6000  JOB : Y321381
F.TRAJ. DATE: 25/06/86
G.LAV : 3  TIME: 14.07.23.
FILTER: 2  FIGURE.

Δ ESTIMATED STATE
+ EXACT STATE
SCALE: 3.03990E-01

FILTER STATE NO.  6

T.TRAJ. AP6000  JOB : Y321381
F.TRAJ. DATE: 25/06/86
G.LAV : 3  TIME: 14.07.23.
FILTER: 2  FIGURE.

+ ESTIMATED STATE
× EXACT STATE
SCALE: 8.71100E-01

figures 2.9.4a,b
FILTER STATE NO. 2

TIME [S]

ESTIMATED STATE
EXACT STATE
SCALE: 4.77250E+02

figure 2.9.4c
FILTER STATE NO.  5

T.TRAJ.  AJ6000  JOB : Y32132J
F.TRAJ.  DATE : 25/06/86
G.LAW :  3  TIME : 13.50.49.
FILTER :  2  FIGURE.

ESTIMATED STATE
+ EXACT STATE

SCALE: 1.69390E+00

FILTER STATE NO.  6

T.TRAJ.  AJ6000  JOB : Y32132J
F.TRAJ.  DATE : 25/06/86
G.LAW :  3  TIME : 13.50.49.
FILTER :  2  FIGURE.

ESTIMATED STATE
+ EXACT STATE

SCALE: 4.36640E-01

figures 2.9.5a,b
figure 2.9.5c
FILTER STATE NO. 5

T.TRAJ. AJ6000
F.TRAJ. SF1
G.LAY : 3
FILTER: 2

JOB : Y3213G8
DATE : 25/06/86
TIME : 15.58.14.

ESTIMATED STATE
EXACT STATE
SCALE: 1.82490E+00

AJ

FILTER STATE NO. 6

T.TRAJ. AJ6000
F.TRAJ. SF1
G.LAY : 3
FILTER: 2

JOB : Y3213G8
DATE : 25/06/86
TIME : 15.58.14.

ESTIMATED STATE
EXACT STATE
SCALE: 1.33070E+00

AJ

figures 2.9.6a,b
ERROR VARIANCE OF STATE NO. 6

SAMPLE ERROR VARIANCE

T.TRAJ. AJ6000  JOB : Y213GI
F.TRAJ. SF1  DATE : 25/06/86
FILTER : 2  FIGURE:

© SQUARED ESTIM. ERROR
Δ VARIANCE (ESTIMATED)

SCALE: 1.56225E+00

TIME [S]

figure 2.9.6c
FILTER STATE NO.  5

+ ESTIMATED STATE
X EXACT STATE

SCALE: 1.33590E+00

FILTER STATE NO.  6

+ ESTIMATED STATE
X EXACT STATE

SCALE: 2.21020E-01

figures 2.9.7a,b
ERROR VARIANCE OF STATE NO.

\[ \text{figure 2.9.7c} \]
FILTER STATE NO. 4

ERROR VARIANCE OF STATE NO. 4

figures 2.9.8a,b
2.9.2 Performance of the Guidance Loop

The purpose of the following simulations is to show how the miss distance produced by PE is affected by the estimation errors fed into the guidance loop by the MTSTF. Since observability of the target maneuver depends on the intercept geometry the performance of the guidance loop shown in figure 2.9.1 varies for different intercept scenarios. Therefore, five scenarios according to figure 2.9.2 will be investigated for the target maneuvers P and J given in (2.9.3/4). The MTSTF is initialized according to (2.9.5/7).

2.9.2.1 Periodical Target Maneuvers

For each scenario A-E the guidance loop is simulated with the target maneuver (2.9.3) for the following values of the maneuver parameter \( \eta \):

\[
\eta = -\frac{\pi}{2}, -\frac{3\pi}{8}, -\frac{\pi}{4}, -\frac{1}{8}\pi, 0, \frac{1}{8}\pi, \frac{\pi}{4}, \frac{3}{8}\pi
\]

The miss distances obtained with exact and estimated target information are depicted in figures 2.9.10-14 as a function of \( \eta \). Typical trajectories of the important filter states are shown in figures 2.9.15-19. They are summarized in table 2.9.2.

<table>
<thead>
<tr>
<th>scenario</th>
<th>( \gamma_0 , [^\circ] )</th>
<th>( \gamma_{T0} , [^\circ] )</th>
<th>( \eta , [\text{rad}] )</th>
<th>figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>0</td>
<td>-( \pi/8 )</td>
<td>2.9.15a-d</td>
</tr>
<tr>
<td>B</td>
<td>8</td>
<td>45</td>
<td>0</td>
<td>2.9.16a-d</td>
</tr>
<tr>
<td>C</td>
<td>13</td>
<td>90</td>
<td>-( \pi/4 )</td>
<td>2.9.17a-d</td>
</tr>
<tr>
<td>D</td>
<td>14</td>
<td>135</td>
<td>-( \pi/8 )</td>
<td>2.9.18a-d</td>
</tr>
<tr>
<td>E</td>
<td>0</td>
<td>180</td>
<td>( \pi/4 )</td>
<td>2.9.19a-d</td>
</tr>
</tbody>
</table>

*table 2.9.2: typical filtering results for sinusoidal target maneuvers*
Discussion of Results:
Figures 2.9.10-14 show that the miss distance $R_f$ is practically independent of the maneuver parameter $\eta$ for exact target maneuver compensation. A substantial increase of $R_f$ due to estimation errors produced by the MTSTF is observed in scenarios B and C. In scenario C the miss distances lie outside the hit range $R_{\text{max}}$ defined in (1.3.71) for certain values of $\eta$. Here the target maneuver is not tracked any more (figure 2.9.17d) because the heading rate $y_6$ is unobservable (section 2.4.4 and 2.7.3). The strong dependency of $R_f$ on $\eta$ for the MTSTF is due to the different initial estimates $\hat{y}_6(0/0)$ associated with each $\eta$. If $\hat{y}_6(0/0)$ is close to zero observability of the target maneuver is low resulting in slow correction of the initial estimation error. Note that the target heading error remains practically uncorrected in all scenarios because the target heading is unobservable (zero average target maneuver).

In the table 2.9.3 the miss distances obtained with the guidance law PN (see section 1.3.5.2) are summarized for the scenarios A and B, respectively. A comparison of these results with figures 2.9.10/11 shows that in all cases where exact target maneuver compensation is required for low miss distances (indicated by a high miss distance of PN compared to PE using exact target information) the guidance scheme PE/MTSTF performs considerably better than PN. Hence, the merits of target maneuver compensation are not offset by the use of estimated data about the target motion. In the scenarios C, D, and E the miss distances of PN are comparable to those obtained with PE indicating that target maneuver compensation has no significant effect here.

Due to the low bearing rate there is no correction of the range error (figures 2.9.15b-19b). Only in the scenarios B and C the range error is corrected at final time because the bearing rate increases due to the high miss distances in these scenarios. Here the conflict between the guidance law which tries to establish zero bearing rate and the tracking filter requiring high bearing rate for observability of the filter states becomes evident. However, the range error has no direct influence on the guidance law because only the range rate is required (see equation 2.9.1). As can be seen from figures 2.9.15a-2.9.19a the range rate estimates are very accurate in all scenarios except C.
**Scenario A**

- **Exact Information**: 
  - Symbols: •

- **Filter**: 
  - Symbols: o

**Scenario B**

- **Exact Information**: 
  - Symbols: •

- **Filter**: 
  - Symbols: o

*Figure 2.9.10*

*Figure 2.9.11*
figure 2.9.12

scenario C

* : exact information
○ : filter
+ : miss; numbers in parentheses = miss distance in [m]

R_f [m]

-0.5 π -0.25 π 0 0.25 π 0.5 π

scenario D

* : exact information
○ : filter

R_f [m]

-0.5 π -0.25 π 0 0.25 π 0.5 π

figure 2.9.13
figure 2.9.14

<table>
<thead>
<tr>
<th>scenario</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$ [rad]</td>
<td>$R_f$ [m]</td>
<td>$R_f$ [m]</td>
</tr>
<tr>
<td>- $\pi/2$</td>
<td>1.4</td>
<td>0.2</td>
</tr>
<tr>
<td>- $3/8\pi$</td>
<td>2.3</td>
<td>0.1</td>
</tr>
<tr>
<td>- $\pi/4$</td>
<td>2.2</td>
<td>0.1</td>
</tr>
<tr>
<td>- $\pi/8$</td>
<td>0.2</td>
<td>0.6</td>
</tr>
<tr>
<td>0</td>
<td>0.3</td>
<td>1.6</td>
</tr>
<tr>
<td>$\pi/8$</td>
<td>0.05</td>
<td>2.6</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>0.02</td>
<td>3.3</td>
</tr>
<tr>
<td>$3/8\pi$</td>
<td>1.18</td>
<td>3.0</td>
</tr>
</tbody>
</table>

table 2.9.3: miss distances of PN
Figure 2.9.16

$\frac{a, c}{b, d}$
Figure 2.9.17

a, c
b, d
FILTER STATE NO. 2

FILTER STATE NO. 5

FILTER STATE NO. 4

FILTER STATE NO. 6

\begin{figure}[h]
\centering
\begin{tabular}{ll}
\textbf{FILTER STATE NO. 2} & \textbf{FILTER STATE NO. 5} \\
\includegraphics[width=0.4\textwidth]{filter_state_2.png} & \includegraphics[width=0.4\textwidth]{filter_state_5.png} \\
\textbf{FILTER STATE NO. 4} & \textbf{FILTER STATE NO. 6} \\
\includegraphics[width=0.4\textwidth]{filter_state_4.png} & \includegraphics[width=0.4\textwidth]{filter_state_6.png} \\
\end{tabular}
\caption{Figure 2.9.19}
\end{figure}

\textit{a, c}

\textit{b, d}
2.9.2.2 Sudden Change of Target Acceleration

The following simulation results were obtained for the target maneuver (2.9.4). For each scenario A-E the miss distance as a function of

\[ \Delta t = t_f - t_e \]  

(2.9.8)
is depicted in figures 2.9.20-24. \( \Delta t \) is the time-to-go at the beginning of the target maneuver (see section 1.3.5.2). In order to ensure target maneuver compensation by PE the filter must converge during this time interval. Typical trajectories of the main filter states are depicted in figures 2.9.25-29 (see table 2.9.4).

<table>
<thead>
<tr>
<th>scenario</th>
<th>( y_0 [^\circ] )</th>
<th>( y_{00} [^\circ] )</th>
<th>( \Delta t [s] )</th>
<th>figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>0</td>
<td>0.679</td>
<td>2.9.25a-d</td>
</tr>
<tr>
<td>B</td>
<td>9</td>
<td>45</td>
<td>3.212</td>
<td>2.9.26a-d</td>
</tr>
<tr>
<td>C</td>
<td>13</td>
<td>90</td>
<td>2.472</td>
<td>2.9.27a-d</td>
</tr>
<tr>
<td>D</td>
<td>13</td>
<td>135</td>
<td>1.023</td>
<td>2.9.28a-d</td>
</tr>
<tr>
<td>E</td>
<td>0</td>
<td>180</td>
<td>0.177</td>
<td>2.9.29a-d</td>
</tr>
</tbody>
</table>

_table 2.9.4: typical filtering results for sudden change of target maneuver_

Discussion of results:
For all scenarios the miss distances lie well within the hit range \( R_{\text{max}} \). The changing target maneuver is tracked in all cases except C where \( y_6 \) is unobservable (figure 2.9.22). In the interval \([0, t_e]\) the target heading \( y_5 \) is unobservable due to the vanishing heading rate \( y_6 \). The associated estimation error has no effect on \( R_f \) as discussed in section 2.6.4.4.
For the scenarios A and B the miss distances obtained with PN are shown in the tables 2.9.5a,b. As was the case for periodical target maneuvers the comparison with figures 2.9.20/21 reveals the substantial improvement of guidance law performance due to target maneuver compensation even in the presence of estimation errors produced by the MTSTF. In the scenarios C, D, and E the miss distances of PE and PN do not differ significantly indicating that target maneuver compensation is not required here.

As for periodical maneuvers the range error is not corrected due to the low bearing rate. Despite the range error the range rate estimates remain accurate. The only exception is scenario C where estimation errors in the slow variables $y_2$ and $y_4$ are caused by the wrong target maneuver estimates produced by the fast filter.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2.9.20}
\caption{figure 2.9.20}
\end{figure}
scenario B

- : exact information
o : filter

scenario C

- : exact information
o : filter

figure 2.9.21

figure 2.9.22
**figure 2.9.23**

- **Scenario D**
  - •: exact information
  - o: filter

- **Scenario E**
  - •: exact information
  - o: filter

---

**figure 2.9.24**

- •: exact information
- o: filter
### scenario A

<table>
<thead>
<tr>
<th>$\Delta t$ [s]</th>
<th>$R_f$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.50</td>
<td>31.7</td>
</tr>
<tr>
<td>6.35</td>
<td>38.6</td>
</tr>
<tr>
<td>5.54</td>
<td>38.8</td>
</tr>
<tr>
<td>4.16</td>
<td>33.21</td>
</tr>
<tr>
<td>3.71</td>
<td>23.94</td>
</tr>
<tr>
<td>2.70</td>
<td>13.93</td>
</tr>
<tr>
<td>1.67</td>
<td>5.78</td>
</tr>
<tr>
<td>0.64</td>
<td>0.68</td>
</tr>
<tr>
<td>0.14</td>
<td>0.23</td>
</tr>
</tbody>
</table>

### scenario B

<table>
<thead>
<tr>
<th>$\Delta t$ [s]</th>
<th>$R_f$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.36</td>
<td>0.1</td>
</tr>
<tr>
<td>4.71</td>
<td>2.3</td>
</tr>
<tr>
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<td>7.8</td>
</tr>
<tr>
<td>3.2</td>
<td>10.2</td>
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<tr>
<td>2.34</td>
<td>8.7</td>
</tr>
<tr>
<td>1.42</td>
<td>4.53</td>
</tr>
<tr>
<td>0.55</td>
<td>0.44</td>
</tr>
</tbody>
</table>

*tables 2.9.5a,b: miss distances of PN*
2.10 Conclusions and Extensions

In the second part of this thesis a new tracking filter for maneuvering targets based on bearing-rate-only measurements was derived. The filter was tested in different intercept scenarios for periodical target maneuvers and sudden changes of the target acceleration. It was shown that the filter is suitable for implementation in conjunction with extended proportional navigation (PE) derived in chapter 1.

There are two major problems associated with the tracking problem under investigation here:

a) the lack of information about the target maneuver in the bearing rate measurement
b) the lack of an accurate model of the target acceleration dynamics.

The observability analysis carried out in section 2.4 revealed that only target maneuvers normal to the current line of sight (LOS) are observable. As a consequence the behaviour of the tracking filter strongly depends on the intercept geometry. This fact is reflected by the simulation results in section 2.9.2. A loss of tracking accuracy is likely if both target heading and target heading rate are only weakly observable (scenario C with low target acceleration). Since the guidance law PE uses only the observable part of the target maneuver the miss distances obtained with the tracking filter in conjunction with PE are satisfactory in all investigated scenarios except the critical case C.

The target model used for the design of the filter is based on the following a priori knowledge about the target maneuver:

i) the target maneuver is essentially deterministic
ii) the target velocity is nearly constant during the engagement.

Assumption ii) allows to describe the planar target maneuver with two states (heading angle and heading rate) only. In this way unrealistic target motions are excluded supporting convergence of the tracking filter. Since the dynamics of the target heading rate are unknown it is assumed constant in the target model. Therefore the filter has to be adapted in the presence of variable target maneuvers. Adaption is carried out via covariance matching. The matching procedure is based on an approximation of the propagation errors caused by the wrong target model. An efficient implementation of the adaption algorithm is possible by taking advantage of the time scale separation of the missile-target relative motion. A singular perturbation-analysis of the tracking
problem results in a low dimensional fast filter with the target state and the bearing rate appearing as fast variables and a slow filter based on all states of the tracking problem. The source of the propagation errors is the heading rate propagation in the fast time scale. Hence covariance matching is performed with respect to the measurement residuals produced by the fast filter. This procedure reflects the physical properties of the system: Fast changes of the bearing rate are attributed to the fast changes of the target maneuver and are used to update the target states. Slow changes in the bearing rate are used for update of the slow variables and reinitialization of the fast filter. The fast filter works as a predictor for the propagation errors and the target maneuver in the sampling interval of the slow filter. In this way the sampling interval in the slow time scale may be decoupled from the dynamics of the fast subsystem without loss of estimation accuracy. This is an important aspect for the realization of the filtering algorithm because the computations in the slow time scale may be time consuming due to the high dimension of the slow filter.

The simulations in section 2.9 show that the adaptive multiple time scale tracking filter (MTSTF) developed here allows to track maneuvering targets in most intercept geometries. Moreover the advantages of target maneuver compensation used by PE are maintained if the exact target maneuver is replaced by the estimates produced by the MTSTF. Only in scenarios which result in unobservable target heading rate a loss of estimation accuracy in all filter states is likely because there is no information about the target maneuver in the measurement residuals.

Many approaches to tracking of maneuvering targets with bearing-only information have been based on measurements of the bearing angle rather than the bearing rate [67,69,70,71,73,80,83]. However, the scaling of the tracking problem in section 2.8.1 suggests that to zero-th order (with respect to the scaling parameter $\varepsilon$) the bearing angle does not contain any information about the target maneuver because it remains constant in the boundary layer. The appearance of the bearing angle as a slow variable is due to the low average bearing rate established by the guidance law. Hence, the guidance law and tracking filter should be designed jointly in the presence of bearing-only measurements in order to guarantee sufficient observability [73,74,83]. This conclusion does, however, not apply if bearing rate measurements are used because:

a) The bearing rate appears in the fast time scale.
b) The target maneuver normal to the LOS is observable even for zero bearing rate (see section 2.4.4).
Note however, that the observability of the slow states is not enhanced by bearing rate measurements because the dynamics in the slow time scale are determined by the average behaviour of the fast subsystem. It is therefore plausible that range errors are not corrected by the MTSTF as revealed by the simulations in section 2.9.

In order to stabilize the MTSTF in type C scenarios it is necessary to measure additional information which is complementary to the bearing rate. If no direct measurements of range and/or range rate are available one could conceive extracting the desired information by including the seeker characteristics in the filter design. A related approach is reported in [84], however no information about the motion along the LOS is provided here.

The target velocity $v_T$ is a parameter of the MTSTF. There is a certain robustness with respect to estimation errors in $v_T$ because they affect the residual statistics via the propagation of the bearing rate. Therefore they are taken into account by the adaption algorithm. An estimate for $v_T$ may be obtained from a filter in the launching aircraft prior to launch. Here range and range rate measurements are usually available justifying the introduction of $v_T$ as an additional (slow) filter state.

In addition to the input noise caused by the extrapolation errors there is another noise source which is due to the measurement module. According to figure 2.6.2 the missile maneuver $u_M$ is computed from the measurements of the missile acceleration and the rotation of the seeker axis measured by the angle $\theta$. Since these measurements are all noisy they produce an input noise via $u_M$ in 2.4.30. This input noise is non-Gaussian because the acceleration and angle measurements are related in a nonlinear way via (2.6.11). Therefore it is desirable to suppress this noise source prior to the evaluation of $u_M$ by passing the measured signals through appropriate low pass filters. These filters could themselves be Kalman filters based on the rotational dynamics of the seeker and the missile.
3. Summary and Conclusions

Optimal control theory and singular perturbation (SP) theory have been applied to analyse the planar intercept problem and the associated tracking problem. The main topic is the derivation of a new adaptive tracking filter for maneuvering targets.

In the first part of this thesis the structure of guidance laws based on optimal control theory is discussed. Two basic approaches to the derivation of guidance laws LQ-theory and order reduction via SP-theory are investigated.

A short review of Pontryagin's minimum principle and necessary conditions of optimality is given in section 1.1 for use in subsequent sections. Basic problems of the application of optimal control theory to the derivation of implementable missile guidance laws are discussed in section 1.2. A review of missile guidance laws based on LQ-theory is given in section 1.3. These guidance laws may be viewed as extensions of proportional navigation (PN). Their main drawback is the need to make assumptions about the target maneuver in order to solve the associated optimal control problem. This difficulty can be avoided by observing that a sufficient condition for intercept is vanishing bearing rate throughout the scenario. Based on this nominal condition a simple guidance law termed extended proportional navigation (PE) is derived. PE differs from PN in a compensation term for the target maneuver normal to the current line of sight (LOS). Simulations indicate that PE performs considerably better than PN against maneuvering targets.

An SP-analysis of the intercept problem is carried out in section 1.4. The optimal structure of the guidance scheme associated with a minimum time optimal control problem is obtained. There are essentially three modules operating in three different time scales:

a) The first module in the slow time scale predicts the collision point by extrapolation of the missile-target relative motion.

b) The nonlinear control law in the first boundary layer performs the correction of the course error, i.e., it tries to establish the collision course associated with the predicted collision point computed by the first module. The required lateral acceleration is the result of a static minimization process of a weighted sum of lift and drag. In this way drag losses are minimized during course error correction in order to reduce the time-to-go.

c) The third module which is associated with the fastest time scale is the autopilot.
A simulation study reveals that the savings in time-to-go obtained with the SP-guidance law are mainly due to the prediction of the point of intercept whereas the effect of induced drag minimization on flight time is negligible. The control law in the first boundary layer may therefore be replaced by a much simpler proportional controller for the course error. In many scenarios the scaling assumptions of the SP-approach are not satisfied due to saturation of the missile acceleration. Therefore the flight times of PE and SP-guidance do not differ significantly in most cases. The main obstacles to the implementation of SP-guidance laws is the prediction of the target maneuver. There is a conflict between satisfaction of the scaling assumptions (long initial range) and accurate prediction of the collision point which is realistic for short time-to-go only. This problem is not shared by the guidance law PE which uses information about the current target maneuver only. Implementation of PE is possible if a tracking filter can be designed which produces estimates of the target state. This problem is addressed in chapter 2.

The tracking problem is stated in section 2.1. The basic approach is the design of an extended Kalman filter. Since the observer (missile) is equipped with an infrared seeker and no inertial angular measurements are available the bearing rate is the only measured information about the missile-target relative motion. As a consequence serious stability problems of the filter arise because some filter states are unobservable in certain scenarios. In section 2.2 an overview of results in the literature on target tracking based on passive measurements is given. The results on observability of non-maneuvering targets via bearing-only measurements are extended to maneuvering targets and bearing rate-only measurements in section 2.4. It turns out that observability depends on the missile-target relative maneuver and that only target maneuvers normal to the current LOS are observable. Fortunately this is exactly the information required by the guidance law PE. The selection of a target model is discussed in section 2.3. In view of the low information about the missile-target relative motion the target model is based on certain a-priori knowledge about the target dynamics. The main assumption is that the target velocity is nearly constant during the engagement. Hence the target is assumed to maneuver with lateral acceleration only. For planar motions the target maneuver may then be described by two states, the target heading and target heading rate. The target heading rate is assumed constant because the true dynamics are unknown. Therefore, the filter has to be adapted in the presence of variable target maneuvers.
For the implementation of the filter algorithm a reference system has to be defined. Since inertial angular measurements are assumed to be unavailable polar coordinates are used to formulate the filter equations and a new reference system is selected for every sampling interval. Additional aspects of coordinate selection are discussed in section 2.5. The basic tracking algorithm is summarized in section 2.6. The simulation results in section 2.7 confirm the results of the observability analysis. They also demonstrate the robustness of PE with respect to certain estimation errors. The simulations of a SP-guidance law in conjunction with the tracking filter illustrate the high sensitivity with respect to estimation errors of guidance schemes based on prediction of the target maneuver.

As mentioned before an adaption scheme is necessary in order to track variable target maneuvers. If the target heading rate is observable the propagation errors due to the wrong target model affect the residual statistics. Therefore, adaption is possible via covariance matching. The adaption algorithm is described in section 2.7.

Motivated by the time scale separation of the intercept problem discussed in chapter 1 an SP-analysis of the tracking problem is carried out in section 2.8. The scaling of the tracking problem allows the identification of two time scales with slow and fast variables. After some general remarks on singularly perturbed stochastic systems a multiple time scale tracking filter (MTSTF) is synthesized. The filter consists of a low dimensional fast filter (FF) and a high dimensional slow filter (SF). Due to its low dimension FF operates with a high sampling rate which allows to track fast changes of the target maneuver. Since FF works as a predictor for the target maneuver and the propagation errors in the slow time scale stability of the basic tracking algorithm is enhanced and the sampling rate of SF is decoupled from the dynamics of the fast subsystem. Hence there are practically no restrictions on computing time in the slow time scale which is of great importance for filter implementation.

The simulations in section 2.9 confirm the capability of the MTSTF to track maneuvering targets and suggest that the filter is suitable for implementation in conjunction with the guidance law PE. Possible extensions of the MTSTF are discussed in section 2.10.
Appendix 1

A1.1 Equations of Planar Missile Motion

The missile simulations carried out in this thesis are based on the aerodynamic data of a realistic short range missile. The quantities describing the motion of this missile in a horizontal plane are depicted in the following figure.

\[(x_{\text{ref}}, y_{\text{ref}})\] : inertial reference system
\[(x_{w}, y_{w})\] : wind axis system
\[(x_{M}, y_{M})\] : missile body fixed system

- \(v\): velocity
- \(\alpha\): angle of attack
- \(\delta\): flipper deflection
- \(\theta\): pitch angle
- \(\gamma\): heading angle
- \(L\): lift
- \(D\): drag
- \(M\): aerodynamic torque
- \(T\): thrust
- \(A\): total aerodynamic force

L, D: components of A in \((x_{w}, y_{w})\); \(A_x, A_y\): components of A in \((x_{M}, y_{M})\)

*figure A1.1: planar missile motion*
The components of $A$ are of the following form:

$$A_x = -\frac{1}{2} \rho v^2 S c_x \quad (A1.1.1)$$
$$A_y = \frac{1}{2} \rho v^2 S c_y \quad (A1.1.2)$$

with

$$\rho = \rho(h): \text{air density} \quad (A1.1.3)$$
$$S: \text{reference cross section}$$

$$c_x = c_x(\alpha, h, Ma) \quad (A1.1.4)$$
$$c_y = c_y(\alpha, Ma, \delta) \quad (A1.1.5)$$

$$Ma = Ma(h): \text{Mach number} \quad (A1.1.6)$$
$$h: \text{altitude}$$

The functions $\rho, Ma, c_x, c_y$ are given in the form of tabulated data and are evaluated by linear interpolation.

Figure A1.1 yields:

$$L = A_x \sin \alpha + A_y \cos \alpha = \frac{1}{2} \rho v^2 S c_L(\alpha, h, Ma, \delta) \quad (A1.1.7)$$
$$D = -A_x \cos \alpha + A_y \sin \alpha = \frac{1}{2} \rho v^2 S c_D(\alpha, h, Ma, \delta) \quad (A1.1.8)$$

with

$$c_L(\alpha, h, Ma, \delta) = c_x(\alpha, h, Ma) \sin \alpha + c_y(\alpha, Ma, \delta) \cos \alpha \quad (A1.1.9)$$
$$c_D(\alpha, h, Ma, \delta) = -c_x(\alpha, h, Ma) \cos \alpha + c_y(\alpha, Ma, \delta) \sin \alpha \quad (A1.1.10)$$

The aerodynamic torque $M$ is given by:

$$M = \frac{1}{2} \rho v^2 S d c_M(\alpha, Ma, \delta, v, q) \quad (A1.1.11)$$

with

$$d: \text{reference distance}$$
$$q = \dot{\theta}: \text{pitch rate} \quad (A1.1.12)$$

The missile thrust contains all forces which are due to the mass loss of the missile. Hence the differential equation of the missile velocity becomes:
\[ \dot{v} = \frac{T - D}{m} \quad (A1.1.13) \]

where \( m \) is the current missile mass. During the boost phase \( T \) and \( m \) are given as functions of time. Afterwards \( T \) is zero and \( m \) remains constant. \((A1.1.13)\) then reduces to:

\[ \dot{v} = -\frac{D}{m} = -a_D \quad (A1.1.14) \]

Evaluation of the aerodynamic coefficients \( c_L, c_D, c_M \) requires knowledge of \( \alpha \). According to figure A1.1 one has:

\[ \alpha = \theta - \gamma \quad (A1.1.15) \]

where

\[ \dot{\gamma} = \frac{L}{mv} = \frac{a_L}{v} \quad (A1.1.16a) \]

with

\[ a_L = \frac{L}{m} \quad (A1.1.16b) \]

and

\[ \dot{\theta} = \dot{\phi} = \frac{M}{I} \quad (A1.1.17) \]

\( I \) is the mass moment of inertia. As \( m \) it is given as a function of time during the boost phase and remains constant afterwards.

The position of the center of gravity with respect to the inertial reference frame \( (x_{\text{ref}}, y_{\text{ref}}) \) is given by (see figure A1.1):

\[ \dot{x} = v \cos \gamma \quad (A1.18a) \]
\[ \dot{y} = v \sin \gamma \quad (A1.18b) \]

For horizontal motions \( h \) is a constant parameter. Equations \((A1.1.13-18)\) with \((A1.1.7-12)\) constitute the missile model. For a more detailed discussion of the missile aerodynamics and kinematics see [85-87].
A1.2 Remarks on the Domain of Attraction of a Stable Root

Consider the singularly perturbed system (1.4.22/23). It was shown that the asymptotic expansions (1.4.24/25) are nonuniform at initial time $t^0$. Here the behaviour of the fast variables is determined by the transient from the initial conditions $y_0^i$ to the outer solution $y_0^O$ according to (1.4.35) and figure 1.4.1. According to (1.4.55a), (1.4.56c), and (1.4.57) the transient is, to zero-th order, governed by:

\[ y_0^i' = g_0^i = g(x_0, y_0^i, \tau) \]  
\[ y_0^i(0) = y_0 \]  

(A1.2.1a)

(A1.2.1b)

Obviously, certain stability conditions must be satisfied in order to ensure that the solution to (A1.2.1) converges versus $y_0^O$ as $\tau \to \infty$. They are given in theorem 1.4 in section 1.4.2 which is due to Tikhonov [41]. In the following the basic definitions and some explanations taken from [42] are summarized:

Let

\[ \hat{y} = \Psi(\hat{x}, \hat{t}) \]  

(A1.2.2)

denote a root of

\[ g(\hat{x}, \hat{y}, \hat{t}) = 0 \]  

(A1.2.3)

in a closed bounded domain $D(\hat{x}, \hat{t})$ where $\hat{x}$ and $\hat{t}$ are regarded as parameters.

**Definition:**

The root $\hat{y} = \Psi(\hat{x}, \hat{t})$ is called stable in $D$ if $\forall \hat{x}, \hat{t} \in D$ the states $\hat{y}$ are asymptotically stable (in the sense of Lyapunov) with respect to the system

\[ y' = g(\tilde{x}, y, \tilde{t}) \]  

(A1.2.4)

Hence, if $\hat{y}$ is a stable root there exists a neighbourhood $N(\hat{y})$ such that the solution of (A1.2.4) tends to $\hat{y}$ as $\tau \to \infty$ if the initial condition $y(\tau = 0) = y_0$ is in $N$. $N$ is called the domain of attraction or domain of influence of $\hat{y}$.

For a geometric interpretation consider the case $g = g(x, t)$. Let $\Phi_1$, $\Phi_2$, $\Phi_3$ denote three roots of $g(x, t) = 0$ such that:
The roots and the field of directions associated with $g$ as well as two boundary layers associated with the initial conditions $y_0^1$ and $y_0^2$, respectively, are depicted in figure A1.2. Obviously $\phi_2$ is a stable root and its domain of attraction is bounded by $\phi_1$ and $\phi_3$.

As a final remark it is noted that in the presence of several stable roots discontinuous periodic solutions in the slow time scale may occur [42].

\[ g < 0 \quad \text{if} \quad \phi_1 > y > \phi_2 \quad \text{(A1.2.5a)} \]
\[ g > 0 \quad \text{if} \quad \phi_2 > y > \phi_3 \quad \text{(A1.2.5a)} \]
A1.3 Limiting Behaviour of $^1\lambda_4$

In order to investigate the limiting behaviour of $^1\lambda_4$ as $^1a_{opt}$ and $^1\gamma$ approach the outer solution $^1H^o$ and $^1L^i$ are expanded around their values in the outer solution:

$$1_{H^0}(^1a_{opt}, ^1\gamma) \approx H^0 + \left. \frac{\partial^1_{H^0}}{\partial ^1\gamma} \right|_o \ e_{\gamma} + \left. \frac{\partial^1_{H^0}}{\partial ^1a_{opt}} \right|_o \ e_{a}$$  \hspace{1cm} (A1.3.1)

$$1_{L^i}(^1a_{opt}) \approx L^0 + \left. \frac{\partial^1_{L^i}}{\partial ^1a_{opt}} \right|_o \ e_{a}$$  \hspace{1cm} (A1.3.2)

with

$$e_{\gamma} = \gamma - \gamma_{opt}^o$$  \hspace{1cm} (A1.3.3a)

$$e_{a} = a_{opt} - a^o$$  \hspace{1cm} (A1.3.3b)

All partial derivatives are evaluated along the outer solution as indicated by $()_o$.

Substitution of (1.4.136/139) into (1.4.151) yields:

$$\left. \frac{\partial^1_{H^0}}{\partial ^1\gamma} \right|_o = [\lambda_{10} v_0 \sin^1\gamma - \lambda_{20} v_0 \cos^1\gamma] \bigg|_{\gamma^1} = \gamma_{opt}^0 = 0$$  \hspace{1cm} (A1.3.4)

Using (A1.3.4), (1.4.119/123), and (A1.1.8) one obtains:

$$1_{H^0} \approx -\frac{\lambda_{30}}{m} c_{Da}|_o e_{a}$$  \hspace{1cm} (A1.3.5)

$$1_{L^i} \approx c_{La}|_o e_{a}$$  \hspace{1cm} (A1.3.6)

with

$$c_{Da} = \left. \frac{\partial^1_{D^i}}{\partial ^1a_{opt}} \right|_o$$  \hspace{1cm} (A1.3.7)

$$c_{La} = \left. \frac{\partial^1_{L^i}}{\partial ^1a_{opt}} \right|_o$$  \hspace{1cm} (A1.3.8)
Substitution of (1.3.5-8) into (1.4.152) renders:

\[
\lim_{\epsilon_{\alpha} \to 0, \epsilon_{\gamma} \to 0} 1 \lambda_4^i = v_0 \lambda_{30} \frac{c_{Da}|_0}{c_{La}|_0} \quad (A1.3.9)
\]

For the symmetric missile considered here (see appendix A1.1) one has:

\[
c_{Da}(1 \alpha_{opt}^i = 0) = 0 \quad (A1.3.10)
\]
\[
c_{La}(1 \alpha_{opt}^i = 0) \neq 0 \quad (A1.3.11)
\]

Substitution of (A1.3.10/11) into (A1.3.9) yields vanishing \(1 \lambda_4^i\) in the slow time scale.
A1.4 Derivation of $\frac{\partial \gamma_s^0(t_{go})}{\partial t_{go}}$

From (1.4.163) one obtains:

$$\gamma_s^0 = \arctan \left( \frac{\int_0^{t_{go}} \sin \gamma_{Ts}(\tau) \, d\tau}{R(t) + \int_0^{t_{go}} \cos \gamma_{Ts}(\tau) \, d\tau} \right) =: \arctan \eta$$  \hspace{1cm} (A1.4.1)

Application of the chain rule to (A1.4.1) yields:

$$\frac{\partial \gamma_s^0(t_{go})}{\partial t_{go}} = \frac{1}{1 + \eta^2} \frac{\partial \eta}{\partial t_{go}}$$ \hspace{1cm} (A1.4.2)

Definitions:

$$x_{Ts}(t_{go}) = R(t) + \int_0^{t_{go}} \cos \gamma_{Ts}(\tau) \, d\tau$$ \hspace{1cm} (A1.4.3)

$$y_{Ts}(t_{go}) = \int_0^{t_{go}} \sin \gamma_{Ts}(\tau) \, d\tau$$ \hspace{1cm} (A1.4.4)

Substitution of (A1.4.3/4) into (A1.4.2) yields:

$$\frac{\partial \gamma_s^0(t_{go})}{\partial t_{go}} = \frac{v_T^2 x_{Ts}(t_{go}) - v_T \cos \gamma_{Ts}(t_{go}) y_{Ts}(t_{go})}{x_{Ts}(t_{go})^2 + y_{Ts}(t_{go})^2}$$

$$= \frac{v_T}{R^2(t_{go})} [x_{Ts}(t_{go}) \sin \gamma_{Ts}(t_{go}) - y_{Ts}(t_{go}) \cos \gamma_{Ts}(t_{go})]$$ \hspace{1cm} (A1.4.5)
A1.5 Sign of $\lambda_{30}$

Equation (1.4.138) yields:

$$\lambda_{30} = \lambda_{30}^0 = \frac{m}{D^0} \left[ \frac{v_0^0 - v_f^0}{v_T \cos(\gamma_{Tf} - \gamma_{opt}^0) - v_f^0} \right]$$

(A1.5.1)

Obviously one has:

$$m > 0 \quad \text{(A1.5.2)}$$

$$D^0 > 0 \quad \text{(A1.5.3)}$$

Moreover from (1.4.109c) follows:

$$v_0^0 > v_f^0 \geq 0 \quad \text{(A1.5.4)}$$

According to figure A5.1

$$v_T' = v_T \cos(\gamma_{Tf} - \gamma_{opt}^0) \quad \text{(A1.5.5)}$$

is the projection of the target velocity onto the collision path MI at final time. It is evident that intercept is possible only if

$$v_f^0 > v_T' \quad \text{(A1.5.6)}$$

Substitution of (A1.5.2-6) into (A1.5.1) yields:

$$\lambda_{30} < 0 \quad \text{(A1.5.7)}$$
Appendix 2

A2.1 Transition Matrix Associated with Modified Polar Coordinates

Consider the initial time \( t_j \) and the sampling time \( t_i \). The filter has processed \( i-1 \) measurements prior to solving the update equations at \( t_i \). The state estimate at time \( t_j \) based on \( i-1 \) measurements is \( \hat{y}_m(j/i-1) \). It is easily obtained by solving backward the propagation equations (2.1.9) in \([t_j, t_{j-1}]\) with final condition \( \hat{y}_m(i-1/i-1) \). Propagating \( \hat{y}_m(j/i-1) \) from \( t_j \) to \( t_i \) yields the estimated value of \( y_m(i) \) based on \( i-1 \) measurements i.e. \( \hat{y}_m(i/i-1) \). The associated transition matrix is

\[
W(i,j) = \frac{\partial f[\hat{y}_m(j/i-1), u_M(i)]}{\partial y_m(j/i-1)}
\]

(A2.1.1)

where \( u_M(i) \) denotes the system input associated with the interval \([t_j, t_i]\). Equation (A2.1.1) is a generalization of (2.4.49). The propagation equations are linearized around \( \hat{y}_m(j/i-1) \) rather than \( \hat{y}_m(j/j) \) which turns out to be convenient to carry out the observability analysis. Obviously (2.4.49) is obtained by setting \( i = j+1 \) in (A2.1.1).

According to (2.4.49) the matrices \( G_Z \) and \( H_Y \) become:

\[
G_Z[\hat{z}(i/i-1)] = \frac{\partial f_Z[\hat{z}(i/i-1), u_M(i)]}{\partial y_m(j/i-1)}
\]

(A2.1.2)

\[
H_Y[\hat{y}_m(j/i-1)] = \frac{\partial f_Y[\hat{y}_m(j/i-1), u_M(i)]}{\partial y_m(j/i-1)}
\]

(A2.1.3)

Definitions:

\[
x : = \hat{x}(i/i-1)
\]

(A2.1.4)

\[
\bar{x} : = \hat{x}(j/i-1)
\]

(A2.1.5)

\[
\bar{D}_x := D_x(\bar{y}_m) : = \bar{y}_m \cos \bar{y}_m - \bar{y}_m \sin \bar{y}_m
\]

(A2.1.6a)

\[
\bar{D}_y := D_y(\bar{y}_m) : = \bar{y}_m \sin \bar{y}_m + \bar{y}_m \cos \bar{y}_m
\]

(A2.1.6b)

\[
T : = t_i - t_j
\]

(A2.1.7)
After replacing \( z(k+1) \) by \( \hat{z}(i/i-1) \) and \( z(k) \) by \( \hat{z}(j/i-1) \) in (2.4.30) the elements of \( GZ \) are calculated according to (A2.1.2). The results are:

\[
\begin{align*}
g_{z11} &= \frac{y_{m4}}{z_2 (z_4 - 2 z_1 y_1)} \\
g_{z12} &= -\frac{y_{m4}}{z_2 (z_3 + 2 z_2 y_1)} \\
g_{z13} &= -\frac{y_{m4}}{z_2} \\
g_{z14} &= \frac{y_{m4}}{z_1} \\
g_{z15} &= 0 \\
g_{z16} &= 0 \\
g_{z21} &= \frac{y_{m4}}{z_2 (z_3 - 2 z_1 y_2)} \\
g_{z22} &= \frac{y_{m4}}{z_2 (z_4 - 2 z_2 y_2)} \\
g_{z23} &= \frac{y_{m4}}{z_1} \\
g_{z24} &= \frac{y_{m4}}{z_2} \\
g_{z25} &= 0 \\
g_{z26} &= 0 \\
g_{z31} &= -\frac{y_{m4}}{z_2} \\
g_{z32} &= \frac{y_{m4}}{z_1} \\
g_{z33} &= 0 \\
g_{z34} &= 0 \\
g_{z35} &= 0 \\
g_{z36} &= 0 \\
g_{z41} &= -\frac{y_{m4}}{z_1} \\
g_{z42} &= -\frac{y_{m4}}{z_2} \\
g_{z43} &= 0 \\
g_{z44} &= 0 \\
g_{z45} &= 0 \\
g_{z46} &= 0 \\
g_{z51} &= 0 \\
g_{z52} &= 0 \\
g_{z53} &= 0 \\
g_{z54} &= 0 \\
g_{z55} &= 1 \\
g_{z56} &= 0 \\
g_{z61} &= 0 \\
g_{z62} &= 0 \\
g_{z63} &= 0 \\
g_{z64} &= 0 \\
g_{z65} &= 0 \\
g_{z66} &= 1 \\
\end{align*}
\]
Equation (A2.1.3) yields the elements of HY:

\[ h_{y11} = - T \frac{\sin \bar{y}_m3}{\bar{y}_m4} \quad h_{y12} = T \frac{\cos \bar{y}_m3}{\bar{y}_m4} \] (A2.1.14a, b)

\[ h_{y13} = - \frac{\sin \bar{y}_m3 + T \bar{D}y}{\bar{y}_m4} \] (A2.1.14c)

\[ h_{y14} = - \frac{\cos \bar{y}_m3 + T \bar{D}x}{\bar{y}_m4} \] (A2.1.14d)

\[ h_{y15} = T v_T [\sin \bar{y}_m5 + \frac{\cos(\bar{y}_m5 + T \bar{y}_m6) - \cos(\bar{y}_m5)}{T \bar{y}_m6}] \] (A2.1.14e)

\[ h_{y16} = \frac{T v_T}{\bar{y}_m6} [\cos(\bar{y}_m5 + T \bar{y}_m6) - \frac{\sin(\bar{y}_m5 + T \bar{y}_m6) - \sin(\bar{y}_m5)}{T \bar{y}_m6}] \] (A2.1.14f)

\[ h_{y21} = T \frac{\cos \bar{y}_m3}{\bar{y}_m4} \quad h_{y22} = T \frac{\sin \bar{y}_m3}{\bar{y}_m4} \] (A2.1.15a, b)

\[ h_{y23} = \frac{\cos \bar{y}_m3 + T \bar{D}x}{\bar{y}_m4} \] (A2.1.15c)

\[ h_{y24} = - \frac{\sin \bar{y}_m3 + T \bar{D}y}{\bar{y}_m4} \] (A2.1.15d)

\[ h_{y25} = - T v_T [\cos \bar{y}_m5 - \frac{\sin(\bar{y}_m5 + T \bar{y}_m6) - \sin(\bar{y}_m5)}{T \bar{y}_m6}] \] (A2.1.15e)
\[ hy_{26} = \frac{T v_T}{\bar{y}_{m6}} [\sin(\bar{y}_{m5} + T \bar{y}_{m6}) \]
\[ + \frac{\cos(\bar{y}_{m5} + T \bar{y}_{m6}) - \cos(\bar{y}_{m5})}{T \bar{y}_{m6}}] \]  
(A2.1.15f)

\[ hy_{31} = - \frac{\sin \bar{y}_{m3}}{\bar{y}_{m4}} \quad hy_{32} = \frac{\cos \bar{y}_{m3}}{\bar{y}_{m4}} \]  
(A2.1.16a,b)

\[ hy_{33} = - \frac{Dy}{\bar{y}_{m4}} \quad hy_{34} = - \frac{Dx}{\bar{y}_{m4}} \]  
(A2.1.16c,d)

\[ hy_{35} = - v_T [\sin(\bar{y}_{m5} + T \bar{y}_{m6}) - \sin(\bar{y}_{m3})] \]  
\[ hy_{36} = - T v_T \sin(\bar{y}_{m5} + T \bar{y}_{m6}) \]  
(A2.1.16e,f)

\[ hy_{41} = \frac{\cos \bar{y}_{m3}}{\bar{y}_{m4}} \quad hy_{42} = \frac{\sin \bar{y}_{m3}}{\bar{y}_{m4}} \]  
(A2.1.17a,b)

\[ hy_{43} = \frac{Dx}{\bar{y}_{m4}} \quad hy_{44} = - \frac{Dy}{\bar{y}_{m4}} \]  
(A2.1.17c,d)

\[ hy_{45} = v_T [\cos(\bar{y}_{m5} + T \bar{y}_{m6}) - \cos(\bar{y}_{m5})] \]  
\[ hy_{46} = T v_T \cos(\bar{y}_{m5} + T \bar{y}_{m6}) \]  
(A2.1.17e,f)

\[ hy_{51} = 0 \quad hy_{52} = 0 \]  
(A2.1.18a,b)

\[ hy_{53} = 0 \quad hy_{54} = 0 \]  
(A2.1.18c,d)

\[ hy_{55} = 1 \quad hy_{56} = T \]  
(A2.1.18e,f)

\[ hy_{61} = 0 \quad hy_{62} = 0 \]  
(A2.1.19a,b)

\[ hy_{63} = 0 \quad hy_{64} = 0 \]  
(A2.1.19c,d)

\[ hy_{65} = 0 \quad hy_{66} = 1 \]  
(A2.1.19e,f)
A2.2 Transformations and Transition Matrix Associated with Ordinary Polar Coordinates

The transformation $T_{0Y_2}(z)$ mapping the cartesian state vector $z$ according to (2.3.5) to ordinary polar coordinates $y$ according to (2.3.8) and the inverse transformation $T_{0Y_2}(y)$ are easily found from (2.4.37-44):

\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3 \\
  y_4 \\
  y_5 \\
  y_6 \\
\end{bmatrix} = \begin{bmatrix}
  \frac{z_1 z_4 - z_2 z_3}{\sqrt{z_1^2 + z_2^2}} \\
  \frac{z_1 z_3 + z_2 z_4}{\sqrt{z_1^2 + z_2^2}} \\
  \arctan \frac{z_2}{z_1} \\
  \sqrt{z_1^2 + z_2^2} \\
  z_5 \\
  z_6 \\
\end{bmatrix} = T_{0Y_2}(z) \quad (A2.2.1)
\]

\[
\begin{bmatrix}
  z_1 \\
  z_2 \\
  z_3 \\
  z_4 \\
  z_5 \\
  z_6 \\
\end{bmatrix} = \begin{bmatrix}
  y_4 \cos y_3 \\
  y_4 \sin y_3 \\
  y_2 \cos y_3 - y_1 y_4 \sin y_3 \\
  y_2 \sin y_3 + y_1 y_4 \cos y_3 \\
  y_5 \\
  y_6 \\
\end{bmatrix} = T_{0Y_2}(y) \quad (A2.2.2)
\]
Using the definitions (A2.1.4/5) the transition matrix associated with ordinary polar coordinates is given by:

\[ W(i,j) = GZ[\tilde{z}] \nabla[y] \]  

(A2.2.3)

with

\[ GZ(\tilde{z}) = \frac{\partial TO_{yz}[\tilde{z}]}{\partial \tilde{z}} \]  

(A2.2.4)

\[ \nabla(y) = \frac{\partial f_z\{TO_{zy}[\tilde{y}], u_M(i)\}}{\partial \tilde{y}} \]  

(A2.2.5)

Elements of \( GZ \):

\[ g_{11} = \frac{\tilde{z}_4 - 2 \tilde{z}_1 \tilde{y}_1}{\gamma_2 \gamma_4} \quad g_{12} = -\frac{\tilde{z}_3 + 2 \tilde{z}_2 \tilde{y}_1}{\gamma_2 \gamma_4} \]  

(A2.2.6a,b)

\[ g_{13} = -\frac{\tilde{z}_2}{\gamma_2 \gamma_4} \quad g_{14} = \frac{\tilde{z}_1}{\gamma_2 \gamma_4} \]  

(A2.2.6c,d)

\[ g_{15} = 0 \quad g_{16} = 0 \]  

(A2.2.6e,f)

\[ g_{21} = \frac{\tilde{z}_3 \gamma_4 - \tilde{z}_1 \gamma_2}{\gamma_2 \gamma_4} \quad g_{22} = \frac{\tilde{z}_4 \gamma_4 - \tilde{z}_2 \gamma_2}{\gamma_2 \gamma_4} \]  

(A2.2.7a,b)

\[ g_{23} = \frac{\tilde{z}_1}{\gamma_2 \gamma_4} \quad g_{24} = \frac{\tilde{z}_2}{\gamma_2 \gamma_4} \]  

(A2.2.7c,d)

\[ g_{25} = 0 \quad g_{26} = 0 \]  

(A2.2.7e,f)
\[ g_{31}^z = \frac{z_2}{y_4} \quad g_{32}^z = \frac{z_1}{y_4} \]  
\[ g_{33}^z = 0 \quad g_{34}^z = 0 \]  
\[ g_{35}^z = 0 \quad g_{36}^z = 0 \]  
\[ g_{41}^z = \frac{z_1}{y_4} \quad g_{42}^z = \frac{z_2}{y_4} \]  
\[ g_{43}^z = 0 \quad g_{44}^z = 0 \]  
\[ g_{45}^z = 0 \quad g_{46}^z = 0 \]  
\[ g_{51}^z = 0 \quad g_{52}^z = 0 \]  
\[ g_{53}^z = 0 \quad g_{54}^z = 0 \]  
\[ g_{55}^z = 1 \quad g_{56}^z = 0 \]  
\[ g_{61}^z = 0 \quad g_{62}^z = 0 \]  
\[ g_{63}^z = 0 \quad g_{64}^z = 0 \]  
\[ g_{65}^z = 0 \quad g_{66}^z = 1 \]  

Elements of HY:

\[ h_{11} = - T \bar{y}_4 \sin \bar{y}_3 \quad h_{12} = T \cos \bar{y}_3 \]  
\[ h_{13} = - \sin \bar{y}_3 (\bar{y}_4 + T \bar{y}_2) - T \bar{y}_1 \bar{y}_4 \cos \bar{y}_3 \]  
\[ h_{14} = \cos \bar{y}_3 - T \bar{y}_1 \sin \bar{y}_3 \]  
\[ h_{15} \text{ and } h_{16} \text{ according to (A2.1.14e/f)} \]  

\[ h_{21} = T \bar{y}_4 \cos \bar{y}_3 \quad h_{22} = T \sin \bar{y}_3 \]  
\[ h_{23} = \cos \bar{y}_3 (\bar{y}_4 + T \bar{y}_2) - T \bar{y}_1 \bar{y}_4 \sin \bar{y}_3 \]  
\[ h_{24} = \sin \bar{y}_3 + T \bar{y}_1 \cos \bar{y}_3 \]  
\[ h_{25} \text{ and } h_{26} \text{ according to (A2.1.15e/f)} \]
\[ hy_{31} = -\bar{y}_4 \sin \bar{y}_3 \quad hy_{32} = \cos \bar{y}_3 \]  
\[ hy_{33} = -\bar{y}_2 \sin \bar{y}_3 - \bar{y}_1 \bar{y}_4 \cos \bar{y}_3 \]  
\[ hy_{34} = -\bar{y}_1 \sin \bar{y}_3 \]  
\[ hy_{35} \text{ and } hy_{36} \text{ according to (A2.1.16e/f)} \]
\[ hy_{41} = \bar{y}_4 \cos \bar{y}_3 \quad hy_{42} = \sin \bar{y}_3 \]  
\[ hy_{43} = \bar{y}_2 \cos \bar{y}_3 - \bar{y}_1 \bar{y}_4 \sin \bar{y}_3 \]  
\[ hy_{44} = \bar{y}_1 \cos \bar{y}_3 \]  
\[ hy_{45} \text{ and } hy_{46} \text{ according to (A2.1.17e/f)} \]
\[ hy_{51} = 0 \quad hy_{52} = 0 \]  
\[ hy_{53} = 0 \quad hy_{54} = 0 \]  
\[ hy_{55} = 1 \quad hy_{56} = T \]  
\[ hy_{61} = 0 \quad hy_{62} = 0 \]  
\[ hy_{63} = 0 \quad hy_{64} = 0 \]  
\[ hy_{65} = 0 \quad hy_{66} = 1 \]  

Remark: It is easily seen that the evaluation of the elements of HY may be substantially simplified if the reference system is chosen in such a way that
\[ \bar{y}_{m3} = 0 \]  
(A2.2.18)

For the basic tracking algorithm in section 2.6 equation (A2.1.18) results in the reference system defined by (2.6.6).
Literature


76 A. Gelb (editor), *Applied Optimal Estimation.*


Symbols and Notations

a : acceleration
A : system matrix
B : control matrix
C : measurement matrix
D : drag
e : error
E : expectation
f : nonlinear function
g : gravitational acceleration, nonlinear function
G : filter gain matrix
GZ : sensitivity matrix
h : altitude, nonlinear function
H : Hamiltonian
HY : sensitivity matrix
I : information matrix, unity matrix, moment of inertia, interval
J : performance index, index
K : feedback matrix
L : penalty function, lift
m : mass
M : aerodynamic torque, covariance matrix of measurement residuals
py, pz : propagation errors
P : covariance matrix of estimation errors
q : pitch rate, variance of propagation errors
Q : covariance matrix of input noise
r : measurement residual
R : range
s : measurement noise
S : covariance matrix of measurement noise
t : time
T : thrust, sampling interval
TM : transformation relating cartesian and modified polar coordinates
TO : transformation relating cartesian and ordinary polar coordinates
TR : rotation matrix
u : system input, maneuver
v : velocity
w : system input noise
W : transition matrix
$\mathbf{x}$ : state vector
$\mathbf{y}$ : state vector, polar coordinates
$\mathbf{z}$ : state vector, cartesian coordinates
$\mathbf{Z}$ : cartesian reference frame

$\alpha$ : angle of attack
$\delta$ : flipper deflection, Dirac function
$\epsilon$ : perturbation parameter
$\varphi$ : bearing angle
$\gamma$ : heading angle
$\lambda$ : Lagrange multiplier
$\theta$ : pitch angle
$\tau$ : time in fast time scale

subscripts:
0 : value at initial time
f : value at final time
F : value in fast time scale
s : value in seeker frame
S : value in slow time scale
m : modified polar coordinates
M : missile
T : target

superscripts:
i : inner solution
o : outer solution
t : transposed

notations:
$\dot{}$ : total derivative with respect to time (in slow time scale)
$\dot{}^*$ : total derivative with respect to time in fast time scale
$^*$ : scaled quantity
$\mathbf{0}$ : vector

$\frac{\partial}{\partial x}$ : partial derivative with respect to $x$

$\frac{d}{dx}$ : total derivative with respect to $x$

abbreviations:

AA : adaption algorithm
BOMP : bearing-only measurements problem
BTA : basic tracking algorithm
CC : cartesian coordinates
CEKF : cartesian extended Kalman filter
EKF : extended Kalman filter
FBL : first boundary layer
FF : fast filter
FS : slow filter
IR : infrared
LOS : line of sight
MM : measurement module
MPC : modified polar coordinates
MPEKF: extended Kalman filter based on modified polar coordinates
MTSTF: multiple time scale tracking filter
OCE : optimal course error control
OPC : ordinary polar coordinates
OPEKF: extended Kalman filter based on ordinary polar coordinates
PCE : proportional course error control
PE : extended proportional navigation
PN : proportional navigation
SP : singular perturbation
TMC : target maneuver compensation
Summary

In this thesis the planar intercept and tracking problem involving a short range missile and a highly maneuverable target are analyzed. Optimal control theory and singular perturbation (SP) theory are applied for the derivation of several guidance laws and a tracking filter.

First the structure of guidance laws based on optimal control theory is discussed. As a consequence of the optimal control approach most of these guidance laws require some information about the future target maneuver which is a severe obstacle to their implementation. It turns out that an extended form of proportional navigation (PE) performs better than guidance schemes based on the prediction of the future target maneuver in most short range scenarios.

Implementation of PE requires a tracking filter that produces estimates of the current target maneuver. The tracking filter developed here is based on bearing-rate-only measurements. Scenarios that may yield unstable filter behaviour due to the lack of observability of certain filter states are analyzed. The implications of estimation errors on the performance of PE and guidance schemes that predict the target maneuver are discussed.

Basically, the tracking filter is an extended Kalman filter. Since the target dynamics are unknown the filter has to be adapted in the presence of time-varying target maneuvers. Motivated by the SP-analysis of the intercept problem a time scale separation of the tracking problem is proposed. As a result a low dimensional fast filter operating with a high sampling rate and a higher dimensional slow filter with a low sampling rate are obtained. The fast filter allows tracking fast changes of the target maneuver and may be viewed as a predictor for the target maneuver in the slow time scale. Adaption is carried out by covariance matching with respect to the measurement residuals in the fast time scale.

All of the results are illustrated by a large number of simulations.
Zusammenfassung


Das Filterungsproblem ist Inhalt des zweiten Teiles dieser Arbeit. Bei der Entwicklung des Tracking-Filters ergeben sich zwei grundlegende Schwierigkeiten:

a) Da als einzige Messung über die Relativbewegung von Lenkwaffe und Ziel die Peilwinkelgeschwindigkeit zur Verfügung steht, sind einige Filterzustände in gewissen Interzeptionsszenarios nicht beobachtbar. Daraus ergeben sich Stabilitätsprobleme.

b) Weil die Zieldynamik unbekannt ist, entstehen Modellfehler, die das Filter ebenfalls destabilisieren können.


Die Leistungsfähigkeit des Filters wird an verschiedenen Interzeptionsgeometrien für periodische und sprunghafte Zielbeschleunigungsprofile überprüft.
Curriculum Vitae

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