Doctoral Thesis

Perfectness notions related to polarity

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Perfectness Notions related to Polarity

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Abstract

A significant achievement in polyhedral combinatorics has been the proof of the perfect graph theorem (Lovász [1972]). A major step thereof consisted in characterizing the notion of perfectness of graphs in terms of polyhedra and systems of inequalities, and led to the development of the anti-blocking theory of Fulkerson [1972]. We study notions of perfectness which generalize notions of perfectness of graphs, and which are related to classical polarity rather than to Fulkerson's anti-blocking theory.

For pairs \(((A,R),(\overline{A},\overline{R}))\) with \(\{0,1\}\)-matrices \((A,R)\) and \(\{1,0,-1,...,-k\}\)-matrices \((\overline{A},\overline{R})\), we define different notions of perfectness by means of the following properties:

1. \((P,Q)\) with \(P := \{x \in \mathbb{R}^V | Ax \leq 1, Rx \leq 0\}\), \(Q := \{x \in \mathbb{R}^V | \overline{A}x \leq 1, \overline{R}x \leq 0\}\) is a polar pair (i.e. \(P^* := \{x \in \mathbb{R}^V | x^Tz \leq 1 \ \forall \ x \in P\} = Q\) and \(Q^* = P\));
2. the describing system of \(P\), \(Ax \leq 1, Rx \leq 0\) is tdi (i.e. \(\min 1^Ty\) subject to \(y^TA + w^TR = c^T\), \(y, w \geq 0\) has an integer optimum solution for integral \(c\), if the optimum exists).
3. the describing system of \(Q\), \(x \geq 0, \overline{A}x \leq 1, \overline{R}x \leq 0\) is tdi;
4. For all \(J \subseteq V\), a pair \(((A^J,R^J),(\overline{A}^J,\overline{R}^J))\) is derived from \(((A,R),(\overline{A},\overline{R}))\), which inherits the properties among (1)-(3) required for \(((A,R),(\overline{A},\overline{R}))\).

The various perfectness notions read then:

5. \(((A,R),(\overline{A},\overline{R}))\) has the integrality property if it satisfies (1) and (4);
6. \(((A,R),(\overline{A},\overline{R}))\) is called A-perfect if it satisfies (1), (2) and (4);
(7) \(((A,R),(\overline{A},\overline{R}))\) is called \(A\)-perfect if it satisfies (1), (3) and (4); 
(8) \(((A,R),(\overline{A},\overline{R}))\) is called \((A,\overline{A})\)-perfect if it is \(A\)-perfect and \(\overline{A}\)-perfect.

Observe that (4), called heredity property, appears in each definition with a different meaning. This property parallels an essential feature of perfect graphs, where by definition any node induced subgraph inherits the perfection property of a perfect graph.

Notions (5) to (8) include the perfect graph case in the following sense: The pair \(((A,0),(\overline{A},0))\) with \([0,1]\)-matrices \(A\), \(\overline{A}\) has the integrality property if and only if \(A\), \(\overline{A}\) are perfect matrices (Padberg [1984]), i.e. \(A\) (\(\overline{A}\)) is essentially the incidence matrix of cliques (stable sets) of a perfect graph. Moreover, (5) implies (8) in this case.

This work is then articulated along the following three topics:

Characterizations: We characterize perfect pairs \(((A,R),(\overline{A},\overline{R}))\) by properties on \(((A,R),(\overline{A},\overline{R}))\) only (and which imply heredity). For this purpose, we introduce a concept of homogeneity and establish the following results: \(((A,R),(\overline{A},\overline{R}))\) has the integrality property (is \(A\)-perfect) if and only if \(((A,R),(\overline{A},\overline{R}))\) satisfies (1) ((1) and (2)) and the system \(x \geq 0\), \(\overline{A}x \leq 1\), \(\overline{R}x \leq 0\) has the so-called homogeneity property. Moreover, if \(((A,R),(\overline{A},\overline{R}))\) satisfies (1) and (3) ((1),(2) and (3)) and the system \(x \geq 0\), \(\overline{A}x \leq 1\), \(\overline{R}x \leq 0\) has the so-called homogeneous-tdi property, then \(((A,R),(\overline{A},\overline{R}))\) is \(\overline{A}\)-perfect ((\(A,\overline{A}\))-perfect). If \((\overline{A},\overline{R})\) are \([1,0,-1]\)-matrices, the homogeneous-tdi property is also necessary.

Some polyhedral descriptions: A pair \(((A,0),(\overline{A},\overline{R}))\) with property (1) describes a family of combinatorial objects in the following way: A can be interpreted as the incidence matrix of a family \(\mathcal{F}\) and \(x \geq 0\), \(\overline{A}x \leq 1\), \(\overline{R}x \leq 0\) is the polyhedral description of \(\mathcal{F}\). We consider families \(\mathcal{F}\) whose incidence matrices \(A_{\mathcal{F}}\) are so-called lattice matrices and derive a pair \(((A_{\mathcal{F}},0), (A_{\mathcal{F}},R_{\mathcal{F}}))\), which describes \(\mathcal{F}\) and which is \((A,\overline{A})\)-perfect. Examples of \(\mathcal{F}\) are dicuts in a graph, convex sets of bounded length in a poset, and families of so-called intersections.

Classes of integer polyhedra: Polyhedra of the form \(Q := \{x \geq 0 \mid A_{\mathcal{F}}x \leq 1, R_{\mathcal{F}}x \leq 0\}\) are integer polyhedra with homogeneous-tdi systems, which, if \(R_{\mathcal{F}} = 0\), belong to the switching paths polyhedra (Gröflin [1987]). We generalize this concept in order to contain polyhedra of the form \(Q\) as well as the coflow polyhedra (Cameron [1982]).

The family \(\mathcal{L} := \{x \mid A_{\mathcal{F}}x \leq 1\} \mid A_{\mathcal{F}}\) is a subclass of lattice polyhedra (Hoffman & Schwartz [1978], Hoffman [1976,1978], Gröflin & Hoffman [1982]). For a subclass \(\mathcal{S}\) of the generalized switching paths polyhedra (with elements of the form \(x \geq 0 \mid \overline{A}x \leq 1, \overline{R}x \leq 0\) for \([1,0,-1]\)-matrices \((\overline{A},\overline{R}))\), we show that \(\mathcal{L}\) and \(\mathcal{S}\) are polar classes, i.e. \(P \in \mathcal{L}\) implies \(P^* \in \mathcal{S}\) and \(Q \in \mathcal{S}\) implies \(Q^* \in \mathcal{L}\). Hence for \(x \mid Ax \leq 1\) \(\in \mathcal{L}\), there exists \((\overline{A},\overline{R})\) such that \(((A,0),(\overline{A},\overline{R}))\) is \((A,\overline{A})\)-perfect, and for \((x \geq 0 \mid \overline{A}x \leq 1, \overline{R}x \leq 0) \in \mathcal{S}\), there exists \(A\) such that \(((A,0),(\overline{A},\overline{R}))\) is \((A,\overline{A})\)-perfect.

Für Paare \(((A,R),(\overline{A},\overline{R}))\), mit \(\{0,1\}\)-Matrizen \((A,R)\) und \(\{1,0,-1,\ldots,-k\}\)-Matrizen \((\overline{A},\overline{R})\), werden diverse Perfektheitsbegriffe definiert, die auf folgenden Eigenschaften basieren:

1. \(P := \{x \in \mathbb{R}^V \mid Ax \leq 1, Rx \leq 0\}\) und \(Q := \{x \in \mathbb{R}^V \mid \overline{Ax} \leq 1, \overline{Rx} \leq 0\}\) sind polare Polyeder (d.h. \(P^* := \{x \in \mathbb{R}^V \mid x^Tz \leq 1 \ \forall x \in P\} = Q\) und \(Q^* = P\));
2. \(Ax \leq 1, Rx \leq 0\) ist tdg (i.e. \(\min \ 1^Ty\), so dass \(y^TA + w^TR = c^T\), \(y, w \geq 0\) besitzt für alle ganzzahligen \(c\) eine ganzzahlige optimale Lösung, sofern eine solche existiert);
3. \(x \geq 0, \overline{Ax} \leq 1, \overline{Rx} \leq 0\) ist tdg;
4. Für alle \(J \subseteq V\) wird ausgehend von \(((A,R),(\overline{A},\overline{R}))\) ein Paar \(((A^J,R^J),(\overline{A}^J,\overline{R}^J))\) definiert, das die Eigenschaften erbt, die für \(((A,R),(\overline{A},\overline{R}))\) verlangt werden.

Die verschiedenen Perfektheitsbegriffe lauten dann:

5. \(((A,R),(\overline{A},\overline{R}))\) besitzt die Ganzzahligkeitseigenschaft (GE), falls es (1) und (4) erfüllt;
6. \(((A,R),(\overline{A},\overline{R}))\) ist A-perfekt, falls es (1), (2) und (4) erfüllt;
(7) \(((A,R),(\overline{A},\overline{R}))\) ist \(\overline{A}\)-perfekt, falls es (1), (3) und (4) erfüllt;  
(8) \(((A,R),(\overline{A},\overline{R}))\) ist \((A,\overline{A})\)-perfekt, falls es \(A\)-perfekt und \(\overline{A}\)-perfekt ist.

Die sogenannte Erb-Eigenschaft (4) kommt in allen Definitionen mit einer unterschiedlichen Bedeutung vor. Sie ist das Analogon der Vererbung der Perfektheit bei Graphen, die definitionsgemäß gegeben ist (jeder induzierte Untergraph eines perfekten Graphen ist perfekt).

Die Begriffe (5) bis (8) enthalten den Fall der perfekten Graphen: \(((A,0),(\overline{A},0))\), mit \{0,1\}-Matrizen \(A\), \(\overline{A}\), besitzt die GE genau dann, wenn \(A\), \(\overline{A}\) perfekte Matrizen sind (Padberg [1984]), d.h. \(A\) ist im wesentlichen die Inzidenzmatrix der Cliquen, \(\overline{A}\) die der unabhängigen Mengen eines perfekten Graphen. Weiter gilt in diesem Fall: (5) \(\Leftrightarrow\) (8).

Die Schwerpunkte der vorliegenden Arbeit sind:

Charakterisierungen: Wir charakterisieren perfekte Paare \(((A,R),(\overline{A},\overline{R}))\) durch Eigenschaften, die nur von \(((A,R),(\overline{A},\overline{R}))\) abhängen (und die damit die Vererbung implizieren). Dazu führen wir das Homogenitäts-Konzept ein und erhalten folgendes Resultat: \(((A,R),(\overline{A},\overline{R}))\) besitzt die GE (ist \(A\)-perfekt) genau dann, wenn \(((A,R),(\overline{A},\overline{R}))\) (1) (resp. (1),(2)) erfüllt und \(x \geq 0\), \(\overline{Ax} \leq 1\), \(\overline{Rx} \leq 0\) die sogenannte Homogenitäts-Eigenschaft besitzt. Falls weiter \(((A,R),(\overline{A},\overline{R}))\) (1) und (3) (resp. (1),(2),(3)) erfüllt und \(x \geq 0\), \(\overline{Ax} \leq 1\), \(\overline{Rx} \leq 0\) die sogenannte Homogen-tdg-Eigenschaft besitzt, ist \(((A,R),(\overline{A},\overline{R}))\) \(\overline{A}\)-perfekt (resp. \((A,\overline{A})\)-perfekt). Für \{1,0,-1\}-Matrizen \((A,R)\) ist homogen-tdg auch eine notwendige Bedingung.

Polyedrische Beschreibungen: \(((A,0),(\overline{A},\overline{R}))\) mit Eigenschaft (1) beschreibt eine Familie kombinatorischer Objekte \(\mathcal{F}\): \(A\) kann als Inzidenzmatrix, \(x \geq 0\), \(\overline{Ax} \leq 1\), \(\overline{Rx} \leq 0\) als polyedrische Beschreibung von \(\mathcal{F}\) aufgefasst werden. Wir betrachten Familien \(\mathcal{F}\), dessen Inzidenzmatrizen \(A^F\) sogenannte Verbandsmatrizen sind, und leiten \((A,\overline{A})\)-perfekte Paare \(((A^F,0),(\overline{A},\overline{R}))\) her, die \(\mathcal{F}\) beschreiben. Schnitte von Graphen, konvexe Mengen beschränkter Längen in Halbordnungen und Durchschnitte von Ringfamilien sind Beispiele für solche \(\mathcal{F}\).

Klassen von ganzzahligen Polyedern: Die Polyeder \(Q := \{x \geq 0 \mid \overline{A}^Fx \leq 1\}, \overline{R}^Fx \leq 0\})\) sind ganzzahlige Polyeder mit homogen-tdg Systemen. Falls \(\overline{R}^F = 0\), gehören sie zu den switching-paths Polyedern (SPP) (Gröflin [1987]). Wir verallgemeinern die SPP, so dass die neue Klasse die Polyeder \(Q\) sowie die Coflow-Polyeder (Cameron [1982]) enthalten.

Die Familie \(L := \{x \mid A^Fx \leq 1\} \mid A^F\) ist eine Verbandsmatrix\) ist eine Unterklasse der Verbands-Polyeder (Hoffman & Schwartz [1978], Hoffman [1976,1978], Gröflin & Hoffman [1982]). Für eine Unterklasse \(S\) der verallgemeinierten SPP (mit Elementen der Form \(x \geq 0 \mid \overline{Ax} \leq 1\), \(\overline{Rx} \leq 0\}) für \{1,0,-1\}-Matrizen \((A,R)\)) zeigen wir, dass \(L\) und \(S\) polare Klassen sind, d.h. \(P \in L\) impliziert \(P^* \in S\) und \(Q \in S\) impliziert \(Q^* \in L\). Also existiert \((\overline{A},\overline{R})\) zu \(x \mid Ax \leq 1\} \in L\), so dass \(((A,0),(\overline{A},\overline{R}))\) \((A,\overline{A})\)-perfekt ist, und zu \(x \geq 0 \mid \overline{Ax} \leq 1\), \(\overline{Rx} \leq 0\}) \in S\) existiert A, so dass \(((A,0),(\overline{A},\overline{R}))\) \((A,\overline{A})\)-perfekt ist.
This work is situated within the scope of polyhedral combinatorics, the field of applying the theory of polyhedra and linear programming to combinatorial problems.

A significant achievement in this field has been the proof of the perfect graph theorem by Lovász [1972]. A major step thereof consisted in translating the notion of perfectness of a graph into properties of related polyhedra and systems of inequalities and led to the development of the anti-blocking theory of Fulkerson [1972]. The main goal of our work is to study notions analogous to perfectness, related to classical polarity rather than to Fulkerson's anti-blocking theory. In order to specify this idea, we first introduce basic concepts of polyhedral combinatorics and definitions associated with perfect graphs.

Basic concepts

Let $V$ be a finite set and $F$ a family of subsets of $V$. $F$ can be represented by its so-called incidence matrix $A$, whose rows are the incidence vectors $x^F \in \mathbb{R}^V$ of the sets $F \in F$. A central problem arising in polyhedral combinatorics is that of finding the polyhedral description of such a family $F$, i.e. describing $\text{CONV}(A)$, the convex hull of the rows of $A$, by a system of inequalities $\bar{A}x \leq b$, i.e.

$$P := \text{CONV}(A) = \{x \in \mathbb{R}^V \mid \bar{A}x \leq b\}.$$
Notice that by definition, \( P \) is an integral polyhedron (\( P \) is integral if any face of \( P \) contains an integral point, i.e. if there exist integral matrices \( A \) and \( R \), such that \( P = \text{CONV}(A) + \text{CONE}(R) \), where \( \text{CONE}(R) \) is the cone generated by the rows of \( R \)).

Proving integrality of polyhedra is an essential step when establishing polyhedral descriptions. For doing so, the concept of totally dual integrality (tdi) is a powerful tool and it plays an important role in our work.

A system \( Ax \leq b \) is called tdi, if

\[
\min \ b^T y \quad \text{subject to} \quad y^T A = c^T, \ y \geq 0
\]

admits an integer optimum solution for any integral vector \( c \) such that it has an optimum solution. A consequence of tdi-ness of the system \( Ax \leq b \) is that, if \( b \) is integral, also

\[
\max \ c^T x \quad \text{subject to} \quad Ax \leq b
\]

has an integer optimum solution and \( P := \{ x \mid Ax \leq b \} \) is integral.

The concept of tdi-ness is due to Hoffman [1974] and has been generalized to the above form by Edmonds and Giles [1977]. However, for the special case where \( A \) is a \( \{0,1\} \)-matrix and all components of \( b \) are equal to 1, it already appears within the scope of the anti-blocking theory of Fulkerson [1972], which has been developed for proving the perfect graph theorem. In fact, perfect graphs, anti-blocking theory and tdi-ness are closely related; to provide a framework and to contrast classical perfectness with our notions of perfectness, we now examine these relations more closely.

**Perfect graphs and matrices**

Let \( G = (V,E) \) be an undirected (simple) graph. A set \( S \subseteq V \) is a stable set if there exists no edge between two nodes of \( S \) and a set \( C \subseteq V \) is a clique if any two nodes of \( C \) are joined by an edge. Two numbers can be associated with \( G \), namely \( \omega(G) \), the maximum cardinality of a clique in \( G \), and \( \gamma(G) \), the minimum number of stable sets needed to cover \( V \). Clearly \( \omega(G) \leq \gamma(G) \).

A graph is called perfect if \( \omega(G^{'}) = \gamma(G^{'}) \) for any node induced subgraph \( G^{'} \) of \( G \). \( (G^{'},E^{'}) \) is a node-induced subgraph of \( G \) if \( E^{'} \) contains all edges of \( E \) with both end nodes in \( V^{'} \subseteq V \). The perfect graph theorem asserts that:

\[
G \text{ is perfect} \iff \overline{G} \text{ is perfect}
\]

where \( \overline{G} := (V,\overline{E}) \) is the complementary graph of \( G = (V,E) \) (for any \( v, w \in V \), \( (v,w) \in \overline{E} \) if and only if \( (v,w) \notin E \)).

This theorem was proved with a polyhedral approach by Lovász [1972]. Let \( A \) be the incidence matrix of all stable sets of \( G \). The proof can be seen as consisting of the following ingredients:
(5) G is perfect $\iff x \geq 0, Ax \leq 1$ is tdi,

(6) $x \geq 0, Ax \leq 1$ tdi $\iff P := \{x \in \mathbb{R}_+^y \mid Ax \leq 1\}$ integral,

(7) If $P := \{x \in \mathbb{R}_+^y \mid Ax \leq 1\}$ is integral, then its anti-blocker (according to Fulkerson) $Q := P^+_F := \{z \in \mathbb{R}_+^y \mid z^T x \leq 1 \text{ for all } x \in P\}$ is an integer polyhedron of the same type, i.e. there exists a $\{0,1\}$-matrix $\bar{A}$ with $Q = \{z \in \mathbb{R}_+^y \mid \bar{A} z \leq 1\}$.

In (5), the difficult implication is "$\Rightarrow$", i.e. perfectness implies tdi-ness, and was proved by Lovász [1972], while the other implication ($\Leftarrow$) follows easily from the definition of perfect graphs and the part "$\Rightarrow$" of (6). Properties (6) and (7) are due to Fulkerson [1972] and are valid for any $\{0,1\}$-matrix $A$. Moreover, from the anti-blocking theory of Fulkerson, the following results hold:

(8) If $P := \{x \in \mathbb{R}_+^y \mid Ax \leq 1\}$ for a matrix $A$ with non-negative entries, then $(P^+_F)_F = P$;

(9) If $P^*_F := \{x \in \mathbb{R}_+^y \mid \bar{A} x \leq 1\}$ for a matrix $\bar{A}$ with non-negative entries, then $P = (\text{CONV}(\bar{A}) - \mathbb{R}_+^y) \cap \mathbb{R}_+^y$.

Let us notice that if $G$ is perfect, the matrix $\bar{A}$ in (7) can be chosen to be the incidence matrix of all cliques of $G$. Then by (9), $P = \text{CONV}(\bar{A})$ and $P^*_F = \text{CONV}(A)$ (if $A$ and $\bar{A}$ both contain the 0 row). Hence $x \geq 0, Ax \leq 1$ and $z \geq 0, \bar{A} z \leq 1$ are the polyhedral descriptions of the cliques and the stable sets of $G$, respectively. Moreover, an optimum (integer) solution of

(10) $\max 1^T x$ subject to $Ax \leq 1, x \geq 0$

is the incidence vector of a clique of maximum size $\omega(G)$, an optimum (integer) solution of its dual

(11) $\min 1^T y$ subject to $y^T A \geq 1^T, y \geq 0$

yields a covering of $V$ with a minimal number ($= \gamma(G)$) of stable sets. In fact, $\omega(G) = \gamma(G)$ by the duality theorem for linear programming.

To see that (5) to (7) prove the perfect graph theorem, notice that $\bar{A}$ is also the incidence matrix of the stable sets of $\bar{G}$. If $G$ is perfect, by (5) to (7), $\{x \in \mathbb{R}_+^y \mid \bar{A} x \leq 1\}$ is integral and, by (6) and (5), $\bar{G}$ is perfect.

Padberg [1984] introduced the notion of perfect matrices which are $\{0,1\}$-matrices $A$ such that $P := \{x \in \mathbb{R}_+^y \mid Ax \leq 1\}$ is integral. As mentioned above, incidence matrices of all stable sets of a perfect graph are perfect matrices ((5) and (6)). Conversely, by the following remarks, every perfect matrix is closely related to a perfect graph:

(12) $A$ can be seen as the incidence matrix of a family $S'$ of subsets of $V$. Consider the independence system $S := \{S \subseteq V \mid \exists T \in S', S \subseteq T\}$ ($S$ is an independence system if $S \subseteq T \in S \Rightarrow S \in S$) and let $A^S$ be the incidence matrix of $S$. Then $\{x \in \mathbb{R}_+^y \mid A^S x \leq 1\} = P$. Moreover, $S$ is the family of all stable sets of an appropriate perfect graph (Fulkerson [1973]).
Let \( P := \{ x \in \mathbb{R}^V \mid Ax \leq 1 \} \) and \( \bar{A} \) be a \( \{0,1\} \)-matrix such that \( P^p := \{ z \in \mathbb{R}^V \mid Az \leq 1 \} \) (see (7)). An independence system \( \mathcal{C} \) can be associated with \( \bar{A} \) as in (12), as well as its incidence matrix \( \bar{A}^\mathcal{C} \). By (9) and (7), \( P = \text{CONV}(\bar{A}^\mathcal{C}) \) and \( P^p = \text{CONV}(\bar{A}^\mathcal{S}) \). In other words, \( x \geq 0, Ax \leq 1 \) is the polyhedral description of the family \( \mathcal{C} \) and \( x \leq 0, \bar{A}x \leq 1 \) that of the family \( \mathcal{S} \).

Example

Observe that an essential feature of perfect matrices is that they correspond to incidence matrices of independence systems \( \mathcal{S} \) (but not every independence system has an incidence matrix which is perfect). Our aim is to study notions analogue to perfectness for matrices associated with families of subsets of \( V \) which are not independence systems.

Before speaking about this notion of perfection, let us underline that the polyhedral description of the independence system \( \mathcal{S} \) (with a perfect incidence matrix \( A \)) is related to the vertices of \( P := \{ x \in \mathbb{R}^V \mid Ax \leq 1 \} \) (see (13)). The following example illustrates which form such a relation takes when \( A \) is the incidence matrix of a family which is not an independence system.

Let \( G = (V,E) \) be a directed graph, call a set \( F \subseteq V \) a path-closed set if \( v, w \in F \) and \( x \) is on a path from \( v \) to \( w \) implies \( x \in F \). Let \( \mathcal{F} \) be the family of path-closed sets and \( A \) its incidence matrix. Clearly, \( \mathcal{F} \) is not an independence system. To describe \( \text{CONV}(A) \), Gröflin [1984] showed that there exist two appropriate \( \{1,0,-1\} \)-matrices \( \bar{A} \) and \( \bar{R} \), such that

\[
\begin{align*}
P^* := \{ x \in \mathbb{R}^V \mid \bar{A}x \leq 1, \bar{R}x \leq 0 \} &= \text{CONV}(A), \\
P := \{ x \in \mathbb{R}^V \mid Ax \leq 1 \} &= \text{CONV}(\bar{A}) + \text{CONE}(\bar{R}) - \mathbb{R}^V.
\end{align*}
\]

Notice that \( P \) and \( P^* \) are related through (classical) polarity, i.e. \( P^* = Q \) and \( Q^* = P \), where \( P^* := \{ z \in \mathbb{R}^V \mid z^Tx \leq 1, \forall x \in P \} \) is the polar polyhedron of \( P \).

Comparing the pair of polyhedra related to path-closed sets to the pair related to perfect matrices, we see that the polyhedron \( P := \{ x \in \mathbb{R}^V \mid Ax \leq 1 \} \) is associated with the matrix \( A \) instead of \( \{ x \in \mathbb{R}^V \mid Ax \leq 1 \} \) and that classical polarity has replaced the anti-blocking theory of Fulkerson. Furthermore, if \( A \) is a perfect matrix, \( A \) and \( \bar{A} \) share the same basic properties. This symmetry is lost in the context of path-closed sets, since already \( A \) is a \( \{0,1\} \)-matrix, whereas \( (\bar{A}, \bar{R}) \) are \( \{1,0,-1\} \)-matrices.

New perfection notions

We would like to have a notion of perfection in relation to the classical polarity similar to the one related to perfect graphs and the anti-blocking theory of Fulkerson. In this latter case, many properties are equivalent (see (5)-(7)). Consequently various possibilities for
pointing out important attributes related to perfectness are at hand so that different notions of
perfectness may be inferred in the context of classical polarity. Moreover, given a perfect
matrix A, there always is another perfect matrix $\overline{A}$ involved by (7). For our purposes, it is
useful to think of perfect matrices in terms of pairs $(A, \overline{A})$ related by (7).

We now characterize those properties which are present in the perfect graph case and which
seem important to us:

The considered object is a pair $(A, \overline{A})$ of $\{0,1\}$-matrices.

(16) $P := \{x \in R^V \mid Ax \leq 1\}$, $Q := \{x \in R^V \mid \overline{A}x \leq 1\}$ are the polyhedra associated with
$(A, \overline{A})$ and $(P,Q)$ is a pair of anti-blocking polyhedra of Fulkerson, i.e. $P^*_F = Q$ and $Q^*_F = P$;

(17) The describing system of $P$, $x \geq 0$, $Ax \leq 1$ is tdi.

(18) The describing system of $Q$, $x \geq 0$, $\overline{A}x \leq 1$ is tdi.

(19) Let $A^J$ and $\overline{A}^J$ be the matrices obtained from A and $\overline{A}$ by deleting a set of columns $\mathcal{W}$.

Then $(A^J, \overline{A}^J)$ has the same properties as $(A, \overline{A})$, i.e. properties (16) to (19).

Property (16) reflects the definition of perfect matrices since it implies by (9) that P and Q
are integral. Conversely, if A is perfect, a matrix $\overline{A}$ always exists by (7) such that $(A, \overline{A})$
satisfies (16). While (17) and (18) follow from (6), (19) is related to the definition of per¬
fected graphs and will be called heredity-property: A and $\overline{A}$ are essentially the incidence matri¬
ces of stable sets and cliques of a perfect graph, $A^J$ and $\overline{A}^J$ are those of the subgraph $G^J$
induced by $J$, which, by definition of perfectness, is also perfect.

In our work, we consider two sorts of objects, pairs $(A, R)$ of $\{0,1\}$-matrices and pairs $(\overline{A},
\overline{R})$ of $(1,0,-1,...,-k)$-matrices for some $k \in Z_+$, which we will call for short $(1,0,-k)$-matrices.
The above properties of perfectness then read:

for a pair $((A,R),(\overline{A},\overline{R}))$, with $\{0,1\}$-matrices $(A,R)$ and $(1,0,-k)$-matrices
$(\overline{A},\overline{R})$:

(16') $P := \{x \in R^V \mid Ax \leq 1, Rx \leq 0\}$, $Q := \{x \in R^V \mid \overline{A}x \leq 1, \overline{R}x \leq 0\}$
are the polyhedra associated with $((A,R),(\overline{A},\overline{R}))$ and
$(P,Q)$ is a pair of polar polyhedra, i.e. $P^* = Q$ and $Q^* = P$;

(17') the describing system of $P$, $Ax \leq 1$, $Rx \leq 0$ is tdi;

(18') the describing system of $Q$, $x \geq 0$, $\overline{A}x \leq 1$, $\overline{R}x \leq 0$ is tdi;

(19') $((A,R),(\overline{A},\overline{R}))$ has some heredity-property.

From classical polarity results, (16') implies $P = CONV(\overline{A}) + CONE(\overline{R}) - R^V$ and $Q =
CONV(A) + CONE(R)$ (assuming that A and $\overline{A}$ both contain the 0 row). Therefore P and Q
are integral.

In (19) the matrices $A^J$ and $\overline{A}^J$ have been obtained from A and $\overline{A}$ by the same operation. In
our context, the operations are not the same since we have no symmetry between $(A,R)$ and
$(\overline{A},\overline{R})$ any more. For $J \subseteq V$ we define again $(A^J,R^J)$ as the matrices obtained from $(A,R)$ by
deleting the set of columns $\forall J$. If possible, the operation to obtain the submatrices $(\overline{A}^J, \overline{R}^J)$ of $(\overline{A}, \overline{R})$ should be defined so that the polyhedra associated with $((A^J, R^J), (\overline{A}^J, \overline{R}^J))$ in (16') form again a pair of polar polyhedra. In fact, it will turn out that this is not always possible, but if it is, then the operation (called negative deletion of $\forall J$) is the following: Delete the columns of $\forall J$ in $\overline{A}$, $\overline{R}$ and all rows $i$ of $\overline{A}$ and rows $j$ of $\overline{R}$ for which $\overline{A}_{ik} = -1$ for some $k \in \forall J$ and $\overline{R}_{je} = -1$ for some $e \in \forall J$. Clearly, if $\overline{A}$, $\overline{R}$ are $(0,1)$-matrices (like in the perfect graph case), the negative deletion is equal to deletion of the columns $\forall J$.

Property (19’) takes now the following form:

(19’) Let $A^J$ and $R^J$ be obtained from $A$ and $R$ by deleting the columns in $\forall J$, $\overline{A}^J$, $\overline{R}^J$ from $\overline{A}$, $\overline{R}$ by negative deletion of $\forall J$. Then $((A^J, R^J), (\overline{A}^J, \overline{R}^J))$ inherits the properties among (16’) to (19’) required for $((A, R), (\overline{A}, \overline{R}))$.

Clearly, an additional condition could have been included in (19’) based on the deletion of columns in $\overline{A}$ and $\overline{R}$. The reason for not doing it is that, as we shall show, such a condition is automatically satisfied.

In the perfect graph case, condition (16) implies (17), (18), and (19). In our case, none of the above properties follows from another. Therefore we define different types of perfectness for pairs $((A, R), (\overline{A}, \overline{R}))$ with $(0,1)$-matrices $A$, $R$ and $(1,0,-k)$-matrices $\overline{A}$, $\overline{R}$:

(20) $((A, R), (\overline{A}, \overline{R}))$ has the integrality property if it satisfies (16') and (19');
(21) $((A, R), (\overline{A}, \overline{R}))$ is called $A$-perfect if it satisfies (16'), (17') and (19');
(22) $((A, R), (\overline{A}, \overline{R}))$ is called $\overline{A}$-perfect if it satisfies (16'), (18') and (19');
(23) $((A, R), (\overline{A}, \overline{R}))$ is called $(A, \overline{A})$-perfect if it is $A$-perfect and $\overline{A}$-perfect.

Observe that in any of the above definitions, the heredity property (19') has a different meaning.

If $A$ and $\overline{A}$ are $(0,1)$-matrices and $((A, 0), (\overline{A}, 0))$ has the integrality property, we are in the "perfect graph case": In fact, $A$ and $\overline{A}$ are perfect in the sense of Padberg. Hence they are essentially the incidence matrices of the cliques and stable sets of a perfect graph (see (12)). Moreover, it follows from the perfect graph theorem and (5) that $((A, 0), (\overline{A}, 0))$ is $(A, \overline{A})$-perfect. Consequently, in the perfect graph case, the integrality property implies $(A, \overline{A})$-perfectness. For our more general case however, all properties are different.

**Overview**

Our work is now centred around three topics related to the just defined notions of perfectness, namely: characterizations, polyhedral descriptions of combinatorial objects, and classes of integer polyhedra.
Characterizations: We try to characterize perfect pairs \(((A,R),(\overline{A},\overline{R}))\) by properties on \(((A, R),(\overline{A},\overline{R}))\) only. A fundamental problem hereby is to ensure heredity and the following question arises: If \(((A,R),(\overline{A},\overline{R}))\) satisfies the set of properties \{\((16')\}\} \cup T with \(T \subseteq \{\(17\), \(18')\}\), under which conditions does the heredity property \((19')\) hold?

Polyhedral descriptions: A pair \(((A,0),(\overline{A},\overline{R}))\) with property \((16')\) describes a family of combinatorial objects in the following way: \(A\) can be interpreted as the incidence matrix of a family \(\mathcal{F}\) and \(x \geq 0, \overline{A}x \leq 1, \overline{R}x \leq 0\) is the polyhedral description of \(\mathcal{F}\). We consider families which are described in this manner by \((A,\overline{A})\)-perfect pairs \(((A,0),(\overline{A},\overline{R}))\).

Classes of integer polyhedra: A pair \(((A,R),(\overline{A},\overline{R}))\) having the integrality property yields integer polyhedra \(P\) and \(Q\) by \((16')\). We develop a class of integer polyhedra (switching paths polyhedra), described by pairs of \([1,0,-1]\)-matrices \((\overline{A},\overline{R})\) and a general right hand side. A subclass thereof yields polyhedra of the form \(Q := \{x \in \mathbb{R}^V \mid \overline{A}x \leq 1, \overline{R}x \leq 0\} = \text{CONV}(A) + \text{CONE}(R)\) with \((A,R)\) being \([0,1]\)-matrices. For this subclass, the pairs \(((A,R),(\overline{A},\overline{R}))\) will be shown to be \((A,\overline{A})\)-perfect.

The motivation to consider the third topic (classes of integer polyhedra) stems essentially from examples of combinatorial objects described by such systems (polyhedral descriptions). For this reason, the organization of our work will not necessarily follow the three topics, but will be guided by the motivation behind them. It is organized as follows:

After discussing in chapter 2 the main mathematical concepts our work is based on, chapters 3 and 4 are concerned with characterizations and heredity.

Chapter 3 deals with geometric questions of heredity. Let \(((A,R),(\overline{A},\overline{R}))\) be a pair of matrices with property \((16')\) and \((P,Q)\) its associated pair of polar polyhedra. Moreover, let \(A^J\) and \(R^J\) be the matrices obtained from \(A\) and \(R\) by deleting the set of columns \(V \setminus J\). Then there are again two polyhedra \(P^J\) and \(Q^J\) associated with \(A^J\) and \(R^J\), namely

\[
P^J := \{x \in \mathbb{R}^J \mid A^Jx \leq 1, R^Jx \leq 0\}
\]

The relations between \(P\) and \(P^J\), \(Q\) and \(Q^J\) are studied in this chapter. Clearly \(P^J = \{x \in \mathbb{R}^J \mid (x,0) \in P\}\) is the cut of \(P\) with the euclidean subspace \(H := \{z \in \mathbb{R}^V \mid z_i = 0, i \in J\}\). In the case of classical polarity, \(Q^J\) is the orthogonal projection of \(Q\) on \(H\) and in this spirit we can say that cut and projection are polar operations. In the context of a generalized notion of polarity, the anti-blocking theory of Gröflin [1982], we study under which conditions cut and projection remain polar operations.

Chapter 4 focuses on characterizations of perfect pairs \(((A,R),(\overline{A},\overline{R}))\) by means of properties on \(((A,R),(\overline{A},\overline{R}))\) only. A major step thereof consists in finding necessary and sufficient conditions for a pair \(((A,R),(\overline{A},\overline{R}))\) with property \((16')\) (respectively \((16')\) and \((17')\) )
to also satisfy the heredity property (19'). A property of the system $x \geq 0$, $Ax \leq 1$, $Rx \leq 0$, called homogeneity, is defined. It plays a crucial role in this context since the following result holds: $((A,R),(\bar{A},\bar{R}))$ has the integrality property (respectively is $A$-perfect) if and only if $((A,R),(\bar{A},\bar{R}))$ satisfies (16') (respectively (16') and (17')) and the system $x \geq 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$ is homogeneous.

Furthermore, a sufficient condition is derived for $((A,R),(\bar{A},\bar{R}))$ having (16') and (17') (respectively (16'), (17'), (18')) to be $\bar{A}$-perfect (respectively $(A,\bar{A})$-perfect). It is also necessary if $\bar{A}$ and $\bar{R}$ are $\{1,0,-1\}$-matrices. Again, this condition is a property required for the system $x \geq 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$. It is related to homogeneity and tdi-ness and called homogeneous-tdi property.

In chapter 5, the following family $F$ of combinatorial objects is considered: Given two proper ring families $C$ and $D$ over a ground set $V$, let $F := \{C \cap D \mid C \in C, D \in D \text{ and } C \cup D = V\}$. $F$ is called a family of intersections and it generalizes the path-closed sets. On one hand, it will turn out that its incidence matrix $A$ and the matrices $(A,\bar{R})$ involved in the polyhedral description of $F$ form a $(A,\bar{A})$-perfect pair $((A,0),(\bar{A},\bar{R}))$.

On the other hand, the related polyhedra $P$ and in some cases $Q$ of (16') are members of known classes of integer polyhedra with tdi systems, namely:

$P$ belongs to a subclass of lattice polyhedra, a class introduced by Hoffman & Schwartz [1978] and studied further in Hoffman [1976,1978], Gröflin & Hoffman [1982].

For those special cases of intersections where $\bar{R}$ is not present in the description of $Q$, they belong to a subclass of switching paths polyhedra, a class introduced by Gröflin [1987].

These facts have been the motivation for the sequel.

A new notion of switching paths polyhedra is defined in chapter 6 which mostly generalizes the concept of Gröflin and furthermore includes the coflow polyhedra introduced by Cameron [1982]. Moreover, it contains the polyhedra describing the intersections.

The family of intersections gives an example for a lattice polyhedron and a (generalized) switching paths polyhedron to be a pair of polar polyhedra. This polarity relation is established in chapter 7 for a whole class $L$ of lattice polyhedra and $S$ of switching paths polyhedra. More precisely, it is shown that $P \in L$ implies $P^* \in S$ and $Q \in S$ implies $Q^* \in L$. The main tool is given by intersections, since it can be shown that the considered subclasses can be obtained from pairs of polyhedra associated with intersections through projection and cut, respectively. In this sense, intersections can be viewed as prototypes for these subclasses.

The above mentioned polarity relation holds between a polyhedron $P \in L$ and $Q \in S$, where $P$ and $Q$ are the associated polyhedra of some $(A,\bar{A})$-perfect pair $((A,0),(\bar{A},\bar{R}))$. 
Finally, the matrix $A$ used to describe $P \in \mathcal{L}$ mentioned above can be considered as the incidence matrix of a family of objects $\mathcal{F}$. Chapter 8 is concerned with finding the polyhedral description of $\mathcal{F}$ and uses the approach traced by the proof of the polarity relation in chapter 7. As applications, the polyhedral descriptions of dicuts in a directed graph and convex sets of bounded length in a partially ordered set are given.
The purpose of this chapter is to introduce notations and to survey the mathematical concepts we shall need in the sequel. It is intended to provide the basic definitions and results. Proofs of the theorems are only sketched if analogous ideas will be used in the sequel. For more informations we give references to appropriate literature.

2.1 Notations and conventions

Let $E = \{e_1, \ldots, e_n\}$ be a finite ordered set. The components of a vector $x \in \mathbb{R}^E$ are indexed by the members of $E$. In the special case where $E = \{1, \ldots, n\}$ we use $\mathbb{R}^n$ instead of $\mathbb{R}^E$.

Let $A \in \mathbb{R}^{I \times J}$ be a matrix with rows indexed by the finite set $I$, columns indexed by the finite set $J$. A row $i \in I$ and a column $j \in J$ of $A$ will be denoted respectively by

\[ A_i \text{ and } A_j \text{ for } i \in I, j \in J. \]

For the submatrix of $A$ containing the rows $A_i$, $i \in I_0 \subseteq I$, respectively the columns $A_j$, $j \in J_0 \subseteq J$ we write

\[ A_{I_0} \text{ or } A_{J_0} \quad \text{respectively } A^{I_0} \text{ or } A^{J_0}. \]

Then

\[ A_{I_0J_0} := (A_{I_0})^{J_0} = (A^{I_0})^{J_0}. \]
The lower and upper integer part of $\alpha \in \mathbb{R}$ are given by
\[
\lfloor \alpha \rfloor := \max \{ z \in \mathbb{Z} \mid z \leq \alpha \} \quad \text{and} \quad \lceil \alpha \rceil := \min \{ z \in \mathbb{Z} \mid z \geq \alpha \}.
\]

The vector $a \in \mathbb{R}^n$ can either be a row or a column vector, depending on the context. Let $a := (a_1, \ldots, a_n)$, $b := (b_1, \ldots, b_n) \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, then
\[
\lfloor a \rfloor := (\lfloor a_1 \rfloor, \ldots, \lfloor a_n \rfloor)
\]
(analogously for $\lceil a \rceil$), $\alpha a := (\alpha a_1, \ldots, \alpha a_n)$ and we write $a \leq b$ if $a_i \leq b_i$ for $i = 1, \ldots, n$.

If $A$ is a matrix and $x$, $y$, $b$, $c$ are vectors, then, when using notations like
\[
Ax = b, \quad Ax \leq b, \quad yA = c
\]
we implicitly assume compatibility of sizes of $A$, $x$, $y$, $b$ and $c$. Similarly if $c$ and $x$ are vectors and if we use $cx$ then $c$ is a row vector and $x$ a column vector both of the same size. $A^T$ means the transpose of the matrix $A$.

The identity matrix is denoted by $I$, where the order is usually clear from the context. If $k$ is a real number, then $k$ denotes the vector or the matrix with all entries equal to $k$. Specifically $0$ stands for all-$0$-vectors or matrices and $1$ for all-$1$-vectors or matrices, all of appropriate dimensions. For $T \subseteq \mathbb{R}$ a vector (matrix) is called a $T$-vector ($T$-matrix) if its entries all belong to $T$. For a number $\beta$, vectors $a$, $b$, and a matrix $A$, $ax = \beta$, $ax \leq \beta$, $Ax = b$, $Ax \leq b$ are said to be rational (integral) if all entries of $\beta$, $a$, $b$, and $A$ are rational (integral).

The $k$-unit vector in $\mathbb{R}^E$, $k \in E$, denoted by $e_k$ is given by
\[
e_k := \begin{cases} 
1 & \text{if } j = k \\
0 & \text{if } j \neq k.
\end{cases}
\]

Let $E$ be a finite set, $y \in \mathbb{R}^E$ and $S \subseteq E$. By $y(S)$ or $\sum \{ y_e \mid e \in S \}$ we mean the sum
\[
y(S) \equiv \sum \{ y_e \mid e \in S \} := \sum_{e \in S} y_e.
\]

If $S = \emptyset$, the sum $\sum \{ y_e \mid e \in S \}$ is defined to be equal to 0.

The incidence vector $x^S$ of $S$ is the \{0,1\}-vector in $\mathbb{R}^E$ defined by
\[
x_e := \begin{cases} 
1 & \text{if } e \in S \\
0 & \text{if } e \notin S.
\end{cases}
\]

If $S$ is a collection of subsets of $E$, the incidence matrix of $S$ is a matrix $A \in \mathbb{R}^{S \times E}$ whose rows $A_S$, $S \in S$ are the incidence vectors of the sets $S \in S$.

Let $E$ be a ground set and $(p_1, \ldots, p_n)$ a sequence of elements of $E$. The sequence containing no elements of $E$ is denoted by $()$ and for $1 \leq i < k \leq n$, $(p_k, \ldots, p_i) := ()$.

For a vector $x \in \mathbb{R}^E$ the support of $x$ is denoted by
\[
supp(x) := \{ e \in E \mid x_e \neq 0 \}.
\]

Let $X$, $Y$ be sets of vectors in $\mathbb{R}^n$. By $X + Y$ we mean the set
\[
X + Y := \{ x + y \mid x \in X, y \in Y \}.
\]
The convex hull and the cone of the set \( X \), denoted by \( \text{CONV}(X) \) and \( \text{CONE}(X) \), are given by
\[
\text{CONV}(X) := \{ \lambda_1 x_1 + \ldots + \lambda_t x_t \mid t \geq 1; x_1, \ldots, x_t \in X; \lambda_1, \ldots, \lambda_t \in \mathbb{R}_+; \lambda_1 + \ldots + \lambda_t = 1 \}
\]
\[
\text{CONE}(X) := \{ \lambda_1 x_1 + \ldots + \lambda_t x_t \mid t \geq 0; x_1, \ldots, x_t \in X; \lambda_1, \ldots, \lambda_t \in \mathbb{R}_+ \}.
\]

For \( S \subseteq \mathbb{R} \) and matrices \( A \in S^{l \times E} \), \( B \in S^{j \times E} \), we shall sometimes use \( A \) to denote the set \( \{ A_i \mid i \in I \} \) of the rows of the matrix \( A \). Hence we write \( A \subseteq S^E \) instead of \( \{ A_i \mid i \in I \} \subseteq S \) and \( A \subseteq B \) instead of \( \{ A_i \mid i \in I \} \subseteq \{ B_j \mid j \in J \} \). Moreover, we define
\[
\text{CONV}(A,B) := \text{CONV}(\{ A_i \mid i \in I \} \cup \{ B_j \mid j \in J \})
\]
(analogously for \( \text{CONE}(A,B) \)).

Let \( a \in \mathbb{R}^n \), \( a \neq 0 \) and \( b \in \mathbb{R} \), the set \( \{ x \in \mathbb{R}^n \mid ax \leq 0 \} \) is called a linear halfspace, the set \( \{ x \in \mathbb{R}^n \mid ax \leq b \} \) an affine halfspace of \( \mathbb{R}^n \).

Let \( E \) be a ground set and \( T, S \subseteq E \). For the singleton \( \{ v \} \), \( v \in E \), we sometimes write \( v \), \( T \cap S := \{ v \in E \mid v \in T, v \in S \} \) and \( T \Delta S := (T \setminus S) \cup (S \setminus T) \). The complement of \( S \) in \( E \) is denoted by
\[
\overline{S} := E \setminus S.
\]

### 2.2 Graph theory

Some combinatorial applications will be given in terms of graphs. Moreover, graphs will be sometimes used as a tool for proofs and algorithms. We restrict ourselves to introduce those elementary concepts, results and problems in graph theory, which will be used in the sequel. Standard texts in graph theory are for example Berge [1983], and Bondy and Murty [1976].

**Undirected graphs**

An (undirected) graph is a pair \( G = (V,E) \), where \( V \) is a finite set and \( E \) is a family of unordered pairs of elements of \( V \). The elements of \( V \) are called vertices or nodes of \( G \) and the elements of \( E \) are called edges of \( G \). By this definition loops, i.e. edges of the form \( \{ v,v \} \), \( v \in V \), and multiple edges, i.e. pairs occurring more than once in \( E \), are allowed. Distinct edges may therefore be represented in \( E \) by the same pair of vertices. Nevertheless, we often speak of the edge \( \{ v,w \} \), where "an edge of type \( \{ v,w \} \)" would be more correct. A graph without loops and multiple edges is called a simple graph.

An edge \( \{ v,w \} \) connects the vertices \( v \) and \( w \). \( v \) and \( w \) are adjacent if there is an edge connecting \( v \) and \( w \). The edge \( \{ v,w \} \) is said to be incident with the vertex \( v \) and \( w \), and conversely. The vertices \( v \) and \( w \) are called the ends of the edge \( \{ v,w \} \).
The complementary graph $G = (V, E)$ of a simple graph $G = (V, E)$ is a simple graph with $V = V$ and $E$ containing all pairs of vertices $\{v, w\}$, $v \neq w$, which are not in $E$.

A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. The subgraph $G' = (V', E(V'))$ of $G = (V, E)$ induced by $V' \subseteq V$ is the subgraph with vertices $V'$ and edges $E(V')$, containing exactly those edges of $E$ having both ends in $V'$. $G'$ is node-induced (by $V'$). Similarly, the subgraph $G' = (V(E'), E')$ of $G = (V, E)$ induced by $E' \subseteq E$ is the subgraph with edges $E'$ and vertices $V(E')$, containing all ends of the edges in $E'$.

A path $P$ from $v_0$ to $v_t$ in the graph $G = (V, E)$ is a sequence of the form

$$ P = (v_0, e_1, v_1, ..., v_{t-1}, e_t, v_t) $$

where $v_0, ..., v_t \in V$ and $e_1, ..., e_t \in E$ such that $e_i = \{v_{i-1}, v_i\}$ for $i = 1, ..., t$. $P$ is also called a $v_0$-$v_t$-path. The vertex $v_0$ is called the starting node and $v_t$ the end node or sometimes both are just called end nodes of $P$. We denote by $V(P) := \{v_0, ..., v_t\}$ the set of vertices and by $E(P) := \{e_1, ..., e_t\}$ the set of edges of the path $P$ and identify $P$ sometimes with its vertex sequence, i.e. $P = (v_0, ..., v_t)$, or its edge sequence, i.e. $P = (e_1, ..., e_t)$. The length of $P$ is equal to $t$. A path is elementary if all its vertices and edges are distinct. If $v_0 = v_t$ the path is closed and is called a circuit. A closed path with $t \geq 1$ and without repeated edges or vertices (except the end nodes) is called an elementary circuit. A graph without circuits is called acyclic.

**Directed graphs**

A directed graph or a digraph is a pair $G = (V, E)$, where $V$ is a finite set and $E$ is a family of ordered pairs of elements of $V$. We call again the elements of $V$ vertices or nodes of $G$ and the elements of $E$ edges of $G$. The terms loops, multiple edges, simple graph, subgraphs and induced subgraphs are defined analogously to those in undirected graphs.

We say that the edge $e = (v, w)$ leaves $v$ and enters $w$, and call $t(e) := v$ the tail, $h(e) := w$ the head of $e$. For $S \subseteq V$ we define

$$ \delta_S(S) = \delta(S) := \{e \in E \mid e = (v, w); v \in S, w \in \overline{S}\}, $$

$$ \gamma_S(S) = \gamma(S) := \{e \in E \mid e = (v, w); v \in S, w \in S\}. $$

A (directed) path or dipath $P$ from $v_0$ to $v_t$ or a $v_0$-$v_t$-path in the digraph $G = (V, E)$ is a sequence of the form

$$ P = (v_0, e_1, v_1, ..., v_{t-1}, e_t, v_t) $$

where $v_0, ..., v_t \in V$ and $e_1, ..., e_t \in E$ such that $e_i = (v_{i-1}, v_i)$ for $i = 1, ..., t$. The vertex $v_0$ is called the starting node and $v_t$ the end node of $P$. As for undirected graphs, $V(P)$ is the vertex set and $E(P)$ the edge set of $P$, and $P$ is sometimes identified with its vertex sequence or edge sequence. The length of $P$ is equal to $t$. $P$ is elementary if all its vertices and edges are distinct. If $v_0 = v_t$ the path is closed and is called a circuit. A closed path with $t \geq 1$ and
without repeated edges or vertices (except for its starting and endpoint) is called an *elementary circuit*. A graph without circuits is called *acyclic*.

Given $c \in \mathbb{R}^E$ and fixed vertices $v_0$ and $v_t$, the *shortest path problem* consists of finding a $v_0$-$v_t$-path of minimum value $c(E(P))$. This problem is solvable by polynomial time algorithms.

A digraph $G = (V,E)$ is *transitive* if $(v,x) \in E$, $(x,w) \in E$ implies $(v,w) \in E$. The *transitive closure* $E'$ of $E$ for a graph $G = (V,E)$ contains $(v,v) \in E'$ if and only if $(v,v) \in E$ and $(v,w) \in E'$, $v \neq w$ if and only if there exists a path $P$ in $G$ with starting node $v$ and end node $w$.

Given a digraph $G = (V,E)$ and distinct vertices $s$, $t \in V$. A set of edges of the form $\delta(S)$ is called a *dicut*, if $\emptyset \subseteq S \subseteq V$ and $\delta(S) = \emptyset$. If for $S \subseteq V$, $s \in S$ and $t \notin S$, the edge set $\delta(S) \cup \delta(\bar{S})$ is called an $s$-$t$-*cut*. A *flow from $s$ to $t$* or *$s$-$t$-flow* is a vector $x \in \mathbb{R}^E$ satisfying

$$x(\delta(v)) - x(\delta(v)) = 0, \quad \forall v \in V \setminus \{s,t\}$$

$$x_e \geq 0, \quad e \in E.$$ 

The first set of constraints characterizes the flow conservation, i.e. the total amount of flow entering $v$ is equal to the total amount of flow leaving $v$. The *value* $\sigma$ of the flow is the net amount of flow leaving $s$ or equivalently the net amount of flow entering $t$, i.e.

$$\sigma := x(\delta(s)) - x(\delta(s)) \equiv - x(\delta(t)) + x(\delta(t)).$$

A *circulation* is a vector $x \in \mathbb{R}^E$ satisfying the flow conservation for each vertex $v \in V$ (i.e. an $s$-$t$-flow of value 0 for arbitrary vertices $s$ and $t$).

Every $s$-$t$-flow admits a decomposition in "paths" and "circuits" in the following sense:

**Lemma 2.1**

Let $G = (V,E)$ and $z$ be an $s$-$t$-flow of value $\sigma$. There exist elementary $s$-$t$-paths $P_i$, $i \in I$, circuits $Q_j$, $j \in J$, and positive vectors $\alpha \in \mathbb{R}_+^I$, $\beta \in \mathbb{R}_+^J$ with $\Sigma\{\alpha_i | i \in I\} = \sigma$, such that

$$z = \Sigma\{\alpha_i x_{P_i} | i \in I\} + \Sigma\{\beta_j y_{Q_j} | j \in J\},$$

where $x_{P_i}$ ($y_{Q_j}$) denotes the incidence vector of $E(P_i)$ ($E(Q_j)$). Furthermore, if $z$ is integer, $\alpha$ and $\beta$ can be chosen to be integer too.

A flow is said to be subject to given capacities $a, u \in \mathbb{R}_+^E$ if $a_e \leq x_e \leq u_e$ for all $e \in E$. The *capacity of an $s$-$t$-cut $\delta(S) \cup \delta(\bar{S})$* for $S \subseteq V$, $s \in S$, $t \notin S$, is given by $u(\delta(S)) - a(\delta(\bar{S}))$. In the context of flow problems with given capacities $a$, $u \in \mathbb{R}_+^V$, $[G,a,u]$ is called a *network*.

Polynomial time algorithms are available for the flow problems below. Furthermore, if the capacities are integer, they provide integer optimum flows.
Maximum flow problem: Given $G = (V,E)$, $s, t \in V$, find an $s$-$t$-flow of maximum value subject to given capacities $a, u \in \mathbb{R}^+_E$, if there exists one.

Minimum cost flow problem: Given $G = (V,E)$, $s, t \in V$, capacity vectors $a, u \in \mathbb{R}^+_E$ and a cost vector $c \in \mathbb{R}^E$, find an $s$-$t$-flow of a given value $\sigma$ subject to the capacities $a$ and $u$, which minimizes the costs $cx$.

In relation with the maximum flow problem we mention the following famous theorem of Ford and Fulkerson:

**Theorem 2.2 (Max-flow - min-cut theorem)**

Given a directed graph $G = (V,E)$, $s, t \in V$ and capacity vectors $a, u \in \mathbb{R}^+_E$, then the maximum value of an $s$-$t$-flow subject to the capacities $a$ and $u$ is equal to the minimum capacity of an $s$-$t$-cut.

Sometimes the capacities are given on the vertices of $G$ rather than on the edges. In this case we speak of a node constraint network $G = (V,E)$: Let $s, t \in V$ be two distinct vertices and $a, u \in \mathbb{R}^+_V$. The vector $x \in \mathbb{R}^E$ is a (feasible) flow subject to the capacities $a, u$ if it is an $s$-$t$-flow and

$$a_v \leq x(\delta(v)) \leq u_v, \forall v \in V \setminus \{s,t\}.$$ 

Notice that a node constraint networks can easily be reduced to networks with capacities on the edges. However, the former provide short treatments of problems for which the capacities are naturally given on the vertices.

Finally, notice that for each directed graph there is an underlying undirected graph in which we forget the orientation of the edges. An undirected path (undirected circuit) is a path (circuit) in the underlying undirected graph. In a natural way, an undirected path or circuit in a directed graph has forward edges and backward edges.

### 2.3 Posets, ring families and lattices

These three families will often be used in our work and we herewith give their definitions and the main standard results related to them.

**Posets**

A partially ordered set or poset is a set $V \equiv (V,\leq)$ with a partial order $\leq$, i.e. a binary relation which is:

(i) reflexiv: $v \in V, v \leq v$;
antisymmetric: \( v \leq w \) and \( w \leq v \) implies \( v = w \);

transitive: \( v \leq x \) and \( x \leq w \) implies \( v \leq w \).

In this work all considered posets are finite and, if not mentioned otherwise, the order is denoted by \( \leq \). Moreover we write \( v < w \) if \( v \leq w \) and \( v \neq w \).

A poset \( V \) can be represented in a digraph \( G = (V,E) \), called comparability graph of \( V \). It has the node set \( V \) and \((v,w) \in E\) if and only if \( w < v \).

Notice that \( G \) is transitive and acyclic. Conversely any acyclic, transitive simple graph \( G = (V,E) \) is the comparability graph of a poset \( V \).

A set \( S \subseteq V \) is called an antichain of \( V \) if no two nodes of \( S \) are connected by an edge of \( G \), and \( C \subseteq V \) is called a chain of \( V \) if it is the node set of a path in \( G \). A famous theorem about chains and antichains is:

\[\text{Theorem 2.3 (Dilworth's theorem [1950])}\]

\text{For any poset } V, \text{ the maximum cardinality of an antichain is equal to the minimum number of chains needed to partition } V.\]

\[\text{Ring families}\]

Let \( V \) be a finite set. A ring family \( C \) of \( V \) is a family of subsets of \( V \) closed with respect to intersection and union, i.e. \( A, B \in C \) implies \( A \cup B \in C, A \cap B \in C \). The ring family is proper if it includes as members \( \emptyset \) and \( V \).

Any proper ring family can be represented in a digraph:

\[\text{Proposition 2.4}\]

\text{Let } C \text{ be a proper ring family on } V \text{ and define the following graph } G^C := (V,E) \text{ on } V: \text{ for any distinct } v, w \in V, (v,w) \in E \text{ if and only if } v \in C \in C \text{ implies } w \in C. \text{ Then } C \text{ is the family of node sets } S \subseteq V \text{ such that } \delta(S) = \emptyset.\]

We shall refer to \( G^C \) as the graph representing \( C \). Notice that \( G^C \) is transitive. Conversely, any transitive simple graph \( G = (V,E) \) is the graph representing the proper ring family \( \{S \subseteq V \mid \delta(S) = \emptyset\} \).

\[\text{Lattices}\]

A lattice is a partially ordered set \( \mathcal{F} \equiv (\mathcal{F}, \leq) \) such that for any two elements \( S, T \in \mathcal{F} \):

(i) the subset \( \{U \in \mathcal{F} \mid U \leq S, U \leq T\} \) of \( \mathcal{F} \) has a unique maximal element, denoted by
S∧T and called meet of S and T;
(ii) the subset \{U ∈ F | S ≤ U, T ≤ U\} of F has a unique minimal element, denoted by S∨T and called join of S and T.

It is distributive if for S, U, T ∈ F: (S∧U)∨T = (S∨T)∧(U∨T), (S∨U)∧T = (S∧T)∨(U∧T).

In this work all considered lattices are finite and the smallest element will be denoted by min or m, the greatest element by max or M.

A sublattice \(F'\) of F is a subset of F with the same partial order, such that S, T ∈ F' implies S∧T, S∨T ∈ F'. Two lattices F and F' are isomorphic, if there exists a bijection \(\Omega: F \rightarrow F'\) such that for all S, T ∈ F, \(\Omega(S∨T) = \Omega(S)∨\Omega(T)\) and \(\Omega(S∧T) = \Omega(S)∧\Omega(T)\).

Notice that a proper ring family \(C\) on \(V\) is a distributive lattice with order, meet and join being the usual set operations \(\subseteq, \cap\) and \(∪\). Moreover:

**Theorem 2.5**
Any finite distributive lattice is isomorphic to a proper ring family.

### 2.4 Polyhedra, linear programming and duality theory

A standard text book on linear programming is for example Chvátal [1983]. For results on polyhedra we essentially follow Schrijver [1986], where the proofs of the theorems mentioned in this section can be found.

**Polyhedra**

A (convex) cone \(C ⊆ \mathbb{R}^n\) is a set of vectors with \(λx + μy ∈ C\) for any \(x, y ∈ C\) and \(λ, μ ≥ 0\). C is polyhedral, if \(C = \{x ∈ \mathbb{R}^n | Ax ≤ 0\}\) for some matrix A, i.e. C is the intersection of finitely many linear halfspaces. C is finitely generated if \(C = \text{CONE}(X)\) for some finite set of vectors \(X = \{x_1,...,x_n\}\).

**Theorem 2.6** (Farkas-Minkowsky-Weyl theorem)
Let \(C ⊆ \mathbb{R}^n\) be a convex cone, then C is polyhedral if and only if C is finitely generated.

A (convex) polyhedron \(P ∈ \mathbb{R}^n\) is a set of vectors \(P = \{x ∈ \mathbb{R}^n | Ax ≤ b\}\) for some matrix A and vector b, i.e. P is the intersection of finitely many affine halfspaces. We say that the system \(Ax ≤ b\) describes P. There may be different systems describing the same polyhedron P and for a polyhedron P := \(\{x ∈ \mathbb{R}^E | Ax ≤ b, Bx = d, Cx ≥ e, x^{E_0} ≥ 0, E_0 ≤ E\}\) we will
also say that the system $Ax \leq b$, $Bx = d$, $Cx \geq e$, $x^E \geq 0$ describes $P$. Obviously each polyhedral cone is a polyhedron.

A *polytope* $P \subseteq \mathbb{R}^n$ is a set of vectors $P = \text{CONV}(X)$ for a finite set of vectors $X \subseteq \mathbb{R}^n$.

**Theorem 2.7 (Decomposition theorem for polyhedra (Motzkin))**

Let $P \subseteq \mathbb{R}^n$, then $P$ is a polyhedron if and only if $P = Q + C$ for some polytope $Q$ and polyhedral cone $C$.

If $P = Q + C$ and $Q = \text{CONV}(X)$, $C = \text{CONE}(Y)$ for finite sets $X, Y \subseteq \mathbb{R}^n$, we say that $P$ is generated by the points $x, x \in X$ and by the directions $y, y \in Y$. $C$ is called the recession cone of $P$. If $P := \{x | Ax \leq b\}$, $C$ is given by

$$C = \{x \in \mathbb{R}^n | Ax \leq 0\}$$

The following relation between polytopes and polyhedra is derived directly from the decomposition theorem:

**Corollary 2.8 (Finite basis theorem (Minkowski, Steinitz, Weyl))**

Let $P \subseteq \mathbb{R}^n$, then $P$ is a polytope if and only if $P$ is a bounded polyhedron.

Let $P := \{x \in \mathbb{R}^n | Ax \leq b\}$ be a polyhedron. A *subsystem* $A'x \leq b'$ of $Ax \leq b$ is derived from $Ax \leq b$ by deleting some of the inequalities of $Ax \leq b$.

A *face* $F$ of a polyhedron $P := \{x \in \mathbb{R}^n | Ax \leq b\}$ is a subset $F$ of $P$ with $F := \{x \in P | A'x = b'\}$, where $A'x \leq b'$ is a subsystem of $Ax \leq b$. Another characterization of nonempty faces is given by the following theorem:

**Theorem 2.9**

$F$ is a nonempty face of the polyhedron $P$ if and only if there is a vector $c$ for which $F \neq \emptyset$ is the set of vectors of $P$ attaining $\max \{cx | x \in P\}$.

A *minimal face* $F$ of a polyhedron $P$ is a nonempty face not containing any other nonempty face of $P$.

**Theorem 2.10 (Hoffman, Kruskal)**

Let $P := \{x \in \mathbb{R}^n | Ax \leq b\}$. $F$ is a minimal face of $P$ if and only if $\emptyset \neq F \subseteq P$ and there exists a subsystem $A'x \leq b'$ of $Ax \leq b$ with $F = \{x \in \mathbb{R}^n | A'x = b'\}$.
All minimal faces have the same dimension. P is called pointed if the minimal faces have dimension 0, i.e. have the form \( F = \{ x \in \mathbb{R}^n | A'x = b' \} = \{ x_0 \} \) Such a face is called vertex of P. Equivalently \( x_0 \in P \) is a vertex of P if and only if \( x_0 = \alpha x_1 + (1-\alpha)x_2 \) with \( x_1, x_2 \in P, 0 < \alpha < 1 \), implies \( x_1 = x_2 = x_0 \).

An integral polyhedron P is a polyhedron, where every nonempty face of it contains an integral vector.

**Linear programming and duality theory**

*Linear programming* concerns maximization problems of the form
\[
\begin{align*}
\text{(23)} \quad \max & \quad cx \\
\text{subject to} & \quad x \in \mathbb{R}^{I_0 \cup J_1}, \ x_{I_0} \geq 0, \ Bx \leq b^{I_0}, \ Cx = b^{J_1}
\end{align*}
\]
with \( J_0 \cap J_1 = \emptyset \) and \( I_0 \cap I_1 = \emptyset \). Such a problem will be referred to as a *linear program, LP* for short. Any maximization or minimization of a linear function over a polyhedron P can be formulated as an LP. For example \( \min \{ by \mid yA \geq c \} \) can be naturally reduced to the LP
\[
\begin{align*}
\text{(24)} \quad \max & \quad (-b)y \\
\text{subject to} & \quad (-A)^Ty \leq (-c).
\end{align*}
\]
When speaking of "the LP \( \min \ by \ subject to \ yA \geq c \)", the LP (24) is meant.

Given an LP of the form (23) the *polyhedron P associated with the LP* (23) is defined by \( P := \{ x \in \mathbb{R}^{I_0 \cup J_1} \mid x_{I_0} \geq 0, \ Bx \leq b^{I_0}, \ Cx = b^{J_1} \} \). A vector \( x \in P \) is called a *feasible solution* for the LP and a vector \( x_0 \in P \) with \( cx_0 = \max \{ cx \mid x \in P \} \) is called an *optimum solution*.

Let (PLP) be an LP of the form (23), referred to as *primal LP* in the following context, and define: \( I := I_0 \cup J_1, J := J_0 \cup J_1, b := (b^{I_0}, b^{J_1}), \) and \( A \in \mathbb{R}^{J \times I}, \ A = \begin{bmatrix} B \\ C \end{bmatrix} = [D|E] \) with \( A^{I_0} := B, \ A^{J_1} := C, \ D := A^{J_0}, \ E := A^{I_1} \). We define the *dual* (DLP) of (PLP) in the following way:
\[
\begin{align*}
\text{(25)} \quad \text{(PLP)} & \quad \text{(DLP)} \\
\maximize & \quad cx & \minimize & \quad by \\
\subjectto & \quad Bx \leq b^{I_0} & \subjectto & \quad y_{J_0} \geq 0 \\
& \quad Cx = b^{J_1} & & y_{J_1} \text{ free} \\
& \quad x_{I_0} \geq 0 & & y_{D} \geq c^{I_0} \\
& \quad x_{I_1} \text{ free} & & y_{E} = c^{J_1}.
\end{align*}
\]
Notice that (DLP) is again an LP and that the dual of (DLP) is (PLP). Given a primal LP (PLP) and its dual (DLP), a *primal* (respectively *dual*) *feasible solution* is a feasible solution of (PLP) (respectively (DLP)). *Primal* and *dual optimum solutions* are defined analogously.

The following lemma and theorem give some relations between primal and dual solutions.

**Lemma 2.11**

*Given a primal LP and its dual, then \( cx \leq by \) for any primal feasible solution \( x \) and dual feasible solution \( y \).*
Theorem 2.12 (Duality theorem (von Neumann, Gale, Kuhn, Tucker))
Let (PLP) be the primal LP of (25) and (DLP) be its dual. Then $cx_0 = by_0$ for any primal optimum solution $x_0$ and dual optimum solution $y_0$, provided (PLP) and (DLP) both have feasible solutions.

Theorem 2.12 gives an optimality criterion for primal and dual feasible solutions. Another criterion is given by the following lemma:

Lemma 2.13 (Complementary slackness)
Let (PLP) be the primal LP of (25) and (DLP) be its dual, $x^0$ a primal feasible solution and $y^0$ a dual feasible solution. $x^0$ and $y^0$ are optimum solutions of (PLP) and (DLP) if and only if (i) and (ii) are satisfied.

(i) $y^0_j > 0$, $j \in J_0$ \implies $B_j x^0 = b_j$

(ii) $x^0_i > 0$, $i \in I_0$ \implies $y^0 D_i = c_i$.

2.5 Polyhedral combinatorics

Polyhedral combinatorics is the field of applying the theory of polyhedra and linear programming to combinatorial problems. Pulleyblank [1983] and Schrijver [1983] provide a comprehensive survey of polyhedral combinatorics and combinatorial min-max theorems obtained via linear programming. Proofs of theorems mentioned in this section can be found in Schrijver [1986].

The LOCO problem

We shall consider a so called linear objective combinatorial optimization problem, LOCO problem for short, which can be formulated in the following way: Given a finite set $E$, a collection of subsets $\mathcal{F} \subseteq 2^E$ and a vector $c \in \mathbb{R}^E$, find a set $F^* \in \mathcal{F}$ with

$$c(F^*) = \max \{ c(F) \mid F \in \mathcal{F} \}.$$ 

The shortest path problem or the problem of finding a clique of maximum cardinality are examples of LOCO problems.

We shall show how the theory of linear programming can be used in relation with LOCO problems. Consider the LOCO problem

$$\begin{align*}
\text{Given} & \quad \mathcal{F} \subseteq 2^E, \ c \in \mathbb{R}^E \\
\text{Find} & \quad F^* \in \mathcal{F}: \ c(F^*) = \max \{ c(F) \mid F \in \mathcal{F} \}
\end{align*}$$
For each set \( F \in \mathcal{F} \), denote by \( x^F \in \mathbb{R}^E \) its incidence vector. Then problem (27) is equivalent to

\[(28) \text{ Given } X := \{ x^F \in \mathbb{R}^E \mid F \in \mathcal{F} \}, c \in \mathbb{R}^E \quad \text{Find } \quad \max \ c x \text{ subject to } x \in X.\]

Let \( P := \text{CONV}(X) \). \( P \) is a polytope, hence a polyhedron and by the decomposition theorem of Motzkin (theorem 2.7), there exists a matrix \( A \) and a vector \( b \), such that \( P = \text{CONV}(X) = \{ x \in \mathbb{R}^E \mid Ax \leq b \} \). Consider the problem

\[(29) \text{ Given } X, \text{CONV}(X) = \{ x \in \mathbb{R}^E \mid Ax \leq b \}, c \in \mathbb{R}^E \quad \text{Find } \quad \max \ c x \text{ subject to } x \in \mathbb{R}^E, Ax \leq b \]

The LP (29) is equivalent to problem (28) in the sense that there exists always a vertex \( x_0 \) of the polyhedron \( P \), which is an optimum solution of the LP (theorem 2.9). As \( x_0 \in P = \text{CONV}(X) \), it follows that \( x_0 \in X \), hence \( x_0 \) is the incidence vector of a set \( F \in \mathcal{F} \). The system \( x \in \mathbb{R}^E, Ax \leq b \) is called a polyhedral description of \( F \).

The major difficulties arise by finding a polyhedral description of \( F \). Frequently the necessary linear system is very large and finding the required inequalities may be very difficult. Nevertheless, even an incomplete system can be helpful in order to solve approximatively problems (see for example cutting plane algorithms for the travelling salesman problem in Lawler et al. [1985] or heuristic for the steiner tree problem, Gaillard & Gröflin [1985]).

However if the polyhedral description is known, the formulation (29) as an LP together with duality theory can be helpful for

- optimality criterion
- algorithms
- min-max relations.

The optimality criterion is given by the duality theorem 2.12 which says that

\[(30) \max \{ cx \mid Ax \leq b \} = \min \{ by \mid yA = c, y \geq 0 \}.\]

Hence for proving optimality of (an incidence vector \( x^F \) of) a set \( F \in \mathcal{F} \), it is sufficient to construct a solution of the dual of (29) with the same value. Consider for example the maximum flow problem subject to a rational capacity function. The algorithm of Ford and Fulkerson either improves the current flow (primal feasible solution) or finds an s-t-cut (dual feasible solution) with capacity equal to the value of the flow.

Another category of algorithms, for example Kuhn's hungarian algorithm for the weighted matching, constructs successively feasible primal and dual solutions, satisfying finally the
complementary slackness conditions, an optimality criterion too (theorem 2.13). (See for example Lawler [1976].)

Relation (30) already gives a min-max relation. Often a combinatorial min-max relation amounts to the fact that one or both sides of (30) have integer optimum solutions. We consider now an example, illustrating this point.

**Example 1**

Consider the problem of finding an antichain of maximum cardinality in a poset V. Let \( B \in \mathbb{R}^{k \times V} \), \( A \in \mathbb{R}^{l \times V} \) be the incidence matrix of all antichains (chains) of V. It is easy to see that

\[
\{ x \in \mathbb{R}^V \mid x \geq 0, Ax \leq 1, x \text{ integral} \} = \{ B_i \mid i \in I \} \cup \{0\}.
\]

Assume that for a given poset V

(i) \( P := \{ x \in \mathbb{R}^V \mid x \geq 0, Ax \leq 1 \} \) is integral;

(ii) \( \min \ y \text{ subject to } yA \geq 1, y \geq 0 \) has an integer optimum solution.

Then by (i) \( P = \text{CONV}(B, \{0\}) \). Moreover the LP's corresponding to the problem of finding an antichain of maximum cardinality and its dual are given by

\[
\begin{align*}
\text{(PLP)} & \quad \text{maximize } & 1x \\
\text{subject to } & Ax & \leq 1 \\
& x & \geq 0 \\
\text{(DLP)} & \quad \text{minimize } & 1y \\
\text{subject to } & yA & \geq 1 \\
& y & \geq 0
\end{align*}
\]

An integer optimum solution \( x^* \) of (PLP) is the incidence vector of an antichain of maximum cardinality and by assumption (ii), there exists an integer optimum solution \( y^* \) of (DLP) which is \( \{0,1\} \)-valued. Hence \( y^* \) can be interpreted as a minimum cardinality collection of chains covering V. Now, any subset of a chain is again a chain, therefore \( y^* \) can also be interpreted as a collection partitioning V. By the duality theorem we get \( 1x^* = 1y^* \) or the Dilworth theorem 2.3: The maximum cardinality of an antichain is equal to the minimum number of chains needed to partition V.

In fact (i) and (ii) are satisfied for incidence matrices of chains in posets.

**Total unimodularity and total dual integrality**

In example 1, integrality of the polyhedron \( P \) and integer optimum solutions of dual problems were required. Two important concepts, total unimodularity and total dual integrality, play here a role and will now be discussed.
Total unimodularity

A matrix $A$ is called \textit{totally unimodular} if each square submatrix of $A$ has determinant 0, +1 or -1. In particular $A$ is a \{0,+1,-1\}-matrix. Seymour [1980] showed that each totally unimodular matrix arises by certain compositions from so-called \textit{network matrices} and from certain 5x5-matrices (see for example Schrijver [1986] for definitions and a survey related to this topic). One consequence of this result is a polynomial time method for testing total unimodularity.

In this section we will not enter into these interesting aspects but give some alternative characterizations of totally unimodular matrices which will be used in the sequel.

\textbf{Theorem 2.14}

Let $A$ be a \{0,+1,-1\}-matrix. Then the following are equivalent:

(i) $A$ is totally unimodular;

(ii) for $b \in \mathbb{Z}^n$ the polyhedron $\{x \in \mathbb{R}^n | x \geq 0, Ax \leq b\}$ is integral;

(iii) for all vectors $a$, $b$, $d$, $e \in \{\mathbb{Z} \cup \{\pm \infty\}\}^n$ the polyhedron $\{x \in \mathbb{R}^n | d \leq x \leq e, a \leq Ax \leq b\}$ is integral;

(iv) each collection of columns of $A$ can be split into two parts so that the sum of the columns in one part minus the sum of the columns in the other part is a \{0,+1,-1\}-vector.

Characterizations (ii) and (iii) are due to Hoffman and Kruskal and characterization (iv) to Ghouila-Houri.

If $A$ is totally unimodular then both matrices $A^T$ and $[\begin{array}{cc} I & -A \\ -I & A \end{array}]$ are also totally unimodular. Furthermore the LP (PLP) and its dual (DLP)

\begin{equation}
(33) \text{(PLP)}: \max \ cx \ \text{subject to} \ d \leq x \leq e, a \leq Ax \leq b,
\end{equation}

\begin{equation}
(33) \text{(DLP)}: \min \ yb - za +ue - vd \ \text{subject to} \ y, z, u, v \geq 0, (y - z)A + u - v = c
\end{equation}

have both integer optimum solutions for arbitrary integer vectors $c$, $a$, $b$, $d$, and $e$, provided they have feasible solutions.

Total dual integrality

Another useful concept for integrality of primal and dual solutions is the concept of total dual integrality. First used by Hoffman [1974] it was formalized by Edmonds and Giles [1977]. Consider an LP and its dual
A rational system $Ax \leq b$ of linear inequalities is called *totally dual integral*, abbreviated *tdi*, if (35) has an integer optimum solution for each integral vector $c$ for which the minimum is finite. Moreover, $Ax \leq b$ is *box-tdi* if $e \leq x \leq d$, $Ax \leq b$ is tdi for all rational vectors $e$ and $d$.

If $A$ is totally unimodular, it follows by theorem 2.14 that the system $Ax \leq b$ is box-tdi for any rational vector $b$. The motivation for this tdi-concept comes from the following result:

**Theorem 2.15** *(Edmonds, Giles)*

If $Ax \leq b$ is a tdi-system and $b$ is integral, then the polyhedron \( \{ x \mid Ax \leq b \} \) is integral.

Hence showing the existence of integer optimum dual solutions of (35) (for any integral $c$) suffices to guarantee the existence of integer optimum primal solutions (if $b$ is integral).

Observe that tdi-ness is not a property of a polyhedron, but one of a system describing it. For one polyhedron there may be different describing systems, some tdi, some not - but a tdi-system always exists:

**Theorem 2.16** *(Giles and Pulleyblank, Schrijver)*

For each polyhedron $P := \{ x \mid A^\prime x \leq b^\prime \}$, with $A^\prime x \leq b^\prime$ being a rational system, there exists a tdi-system $Ax \leq b$ with $A$ integral and $P = \{ x \mid Ax \leq b \}$. Moreover $b$ can be chosen to be integral if and only if $P$ is integral.

### 2.6 Anti-blocking polyhedra

Fulkerson [1970, 1971, 1972] introduced blocking and anti-blocking relations between polyhedra as a tool in combinatorial optimization. They threw a new light on combinatorial min-max relations obtained through the classical polarity of vertices and facets of polyhedra. Moreover, the anti-blocking theory was the main tool for proving the perfect graph theorem (see introduction). We will only consider the anti-blocking relation. It has been extended in various directions, in particular by Gröflin [1982], where his generalization include the anti-blocking relation of Fulkerson and the classical polarity.

Here we will use the *anti-blocking theory of Gröflin* and provide the proofs of the theorems, since they can not be found in standard text books.

First we need some definitions: given matrices $D, D \subseteq \mathbb{R}^n$, ...
the reverse polar cone, \( \text{CONE}(D)^+ \) of \( \text{CONE}(D) \) is given by
\[
(36) \quad \text{CONE}(D)^+ := \{ x \in \mathbb{R}^n \mid x \leq 0 \text{ for all } z \in \text{CONE}(D) \} = \{ x \in \mathbb{R}^n \mid D x \geq 0 \}.
\]

For any polyhedron \( P \subseteq \mathbb{R}^n \), the **anti-blocker** \( P^*_D \) **with respect to** \( D \) is defined by
\[
(37) \quad P^*_D := \{ x \in \mathbb{R}^n \mid x z \leq 1 \text{ for all } z \in P \} \cap \text{CONE}(D)^+.
\]

and \( P \) is of **anti-blocking type with respect to** \( (D,D) \) if
\[
(38) \quad P = (P^*_D)^*.
\]

Moreover, \( (P,Q) \) is a **pair of anti-blocking polyhedra with respect to** \( (D,D) \) if
\[
(39) \quad P^*_D = Q, \quad Q^*_D = P.
\]

Clearly, if \( (P,Q) \) is a pair of anti-blocking polyhedra with respect to \( (D,D) \), \( Q^*_P \) is one with respect to \( (D,D) \) and \( P \) is of anti-blocking type with respect to \( (D,D) \), \( Q \) with respect to \( (D,D) \).

If the underlying matrices \( (D,D) \) are clear from the context, the indices \( D \) and \( D \) as well as the mention "with respect to \( (D,D) \)" are omitted.

The next lemma gives the relation between \( P, P^*_D \) and \( (P^*_D)^* \):

**Lemma 2.17**

Let \( P \subseteq \mathbb{R}^n \), \( P \neq \emptyset \) be a polyhedron and \( P = \text{CONV}(\overline{A}) + \text{CONE}(\overline{R}) \). Then

(i) \( P^*_D = \{ x \in \mathbb{R}^n \mid \overline{A} x \leq 1, \overline{R} x \leq 0 \} \cap \text{CONE}(D)^+ \);

(ii) \( (P^*_D)^* = (\text{CONV}(\overline{A}, 0) + \text{CONE}(\overline{R}) - \text{CONE}(D)) \cap \text{CONE}(D)^+ \).

**Proof:**

(i): Let \( z \in P^*_D \), then \( z \in \text{CONE}(D)^+ \) and \( xz \leq 1 \) for all \( x \in P \). Any row of \( \overline{A} \) is in \( P \), hence \( \overline{A} z \leq 1 \). Moreover, for any row \( r \) of \( \overline{R} \), \( \overline{A}_1 + \lambda r \in P \), \( \lambda \geq 0 \), hence \( \overline{A}_1 z + \lambda rz \leq 1 \) for all \( \lambda \geq 0 \) implying \( rz \leq 0 \). Therefore \( \overline{R} z \leq 0 \).

Let now \( z \in \{ z \in \mathbb{R}^n \mid \overline{A} z \leq 1, \overline{R} z \leq 0 \} \cap \text{CONE}(D)^+ \) and \( x \in P \). Then \( x = \lambda \overline{A} + \mu \overline{R} \) for \( \lambda, \mu \geq 0 \), \( 1 \lambda = 1 \) and \( xz = \lambda \overline{A} z + \mu \overline{R} z \leq 1 \).

(ii): Let \( \lambda, \mu, \nu \geq 0 \), \( 1 \lambda = 1 \), \( x := \lambda \overline{A} + \mu \overline{R} - \nu \overline{D} \in \text{CONE}(D)^+ \) and \( z \in P^*_D \). By (i), \( xz = \lambda \overline{A} z + \mu \overline{R} z - \nu \overline{D} z \leq 1 \).

Let now \( x \in (P^*_D)^* \), then \( 1 \geq \max \{ xz \mid z \in P^*_D \} \overset{(1)}{=} \max \{ xz \mid \overline{A} z \leq 1, \overline{R} z \leq 0 \} \overset{(*)}{=} \min \{ 1 \lambda \mid \overline{A} z + \mu \overline{R} - \nu \overline{D} = x, \lambda, \mu, \nu \geq 0 \} \), where (*) is given by the duality theorem 2.12.

Hence there exist \( \lambda, \mu, \nu \geq 0 \), \( 1 \lambda = 1 \) and \( x = \lambda \overline{A} + \mu \overline{R} - \nu \overline{D} \).

**Lemma 2.17** (ii) gives a criterion for a polyhedron \( P \) to be of anti-blocking type with respect to \( (D,D) \). We now provide another one.
Proposition 2.18

Let \( 0 \in P \subseteq \mathbb{R}^n \) be a polyhedron. Then (i), (ii), (iii) are equivalent:

(i) \( P \) is of anti-blocking type with respect to \((D, \overline{D})\);
(ii) \( P = (P - \text{CONE}(\overline{D})) \cap \text{CONE}(D)^+ \);
(iii) There exist matrices \( A, R \in \text{CONE}(\overline{D})^+ \) such that
\[
P = \{ x \in \mathbb{R}^n | A x \leq 1, R x \leq 0 \} \cap \text{CONE}(D)^+.
\]

Proof:

The equivalence between (i) and (ii) is a consequence of the definition of anti-blocking type and lemma 2.17 (ii).

(i) \( \Rightarrow \) (iii): There exist matrices \( A, R \in \text{CONE}(\overline{D})^+ \) such that \( P = \text{CONV}(A,0) + \text{CONE}(R) \).

Then by lemma 2.17 (i), \( P = (P^*_D)^* = \{ x \in \mathbb{R}^n | A x \leq 1, R x \leq 0 \} \cap \text{CONE}(D)^+ \).

(iii) \( \Rightarrow \) (i): Let \( P := \{ x \in \mathbb{R}^n | A x \leq 1, R x \leq 0 \} \cap \text{CONE}(D)^+ = \text{CONV}(\overline{A}) + \text{CONE}(R) \), with \( A, R \subseteq \text{CONE}(\overline{D})^+ \), \( \overline{A}, R \subseteq \text{CONE}(D)^+ \). By lemma 2.17 (ii), \( P \subseteq (P^*_D)^* \). We show \( P \subseteq (P^*_D)^* \):

Let \( x \in (P^*_D)^* \), i.e. \( x := \lambda \overline{A} + \mu R - v D ; \lambda, \mu, v \geq 0, 1\lambda \leq 1 \) (lemma 2.17). Then \( x \in P \), since for any row \( a \) of \( A \), \( r \) of \( R \) and \( d \) of \( D \): \( A a \leq 1, R r \leq 0 \) (\( a \in P \)), \( A r \leq 0, R r \leq 0 \) (\( a \in P \) for \( a \geq 0 \)) and \( A(-d) \leq 0, R(-d) \leq 0 \) (\( a \in \text{CONE}(D)^+ \)).

If \( P \subseteq \text{CONE}(D)^+ \), then \( P = \text{CONV}(\overline{A}) + \text{CONE}(R) \) for some matrices \( \overline{A}, R \subseteq \text{CONE}(D)^+ \).

As a corollary of lemma 2.17 and proposition 2.18 (iii) we get:

Corollary 2.19

If \( P \subseteq \text{CONE}(D)^+ \), then \( P^*_D \) is of anti-blocking type with respect to \((\overline{D}, D)\).

The next theorem shows a relation between a system of \( P \) and \( P^*_D \), provided \( P \) is of anti-blocking type.

Theorem 2.20

Let \( P \subseteq \mathbb{R}^n \) be a polyhedron of anti-blocking type with respect to \((D, \overline{D})\). If

(40) \( P = (\text{CONV}(\overline{A},0) + \text{CONE}(R) - \text{CONE}(\overline{D})) \cap \text{CONE}(D)^+ \), with \( \overline{A}, R \subseteq \text{CONE}(D)^+ \) and
(41) \( P = \{ x \in \mathbb{R}^n | A x \leq 1, R x \leq 0 \} \cap \text{CONE}(D)^+ \) with \( A, R \subseteq \text{CONE}(\overline{D})^+ \),

then

(42) \( P^*_D = (\text{CONV}(A,0) + \text{CONE}(R) - \text{CONE}(D)) \cap \text{CONE}(\overline{D})^+ \)

\[= \{ x \in \mathbb{R}^n | \overline{A} x \leq 1, \overline{R} x \leq 0 \} \cap \text{CONE}(\overline{D})^+ \].
Proof:
Notice that $\overline{A}$, $\overline{R} \subseteq \text{CONE}(D)^+$ and $A, R \subseteq \text{CONE}(D)^+$ always exist such that (40) and (41) are satisfied (lemma 2.17 (ii), proposition 2.18). For the first equality of (42), define $Q := \text{CONV}(A) + \text{CONE}(R)$. By lemma 2.17, $Q^* = \{ x \in \mathbb{R}^n | Ax \leq 1, Rx \leq 0 \} \cap \text{CONE}(D)^+ = P$ and therefore $(Q_D)^* = P_D^* = (\text{CONV}(A,0) + \text{CONE}(R) - \text{CONE}(D)) \cap \text{CONE}(D)^+$. To establish the second equality of (42), let $Q := \{ x \in \mathbb{R}^n | \overline{A}x \leq 1, \overline{R}x \leq 0 \} \cap \text{CONE}(\overline{D})^+$. Since $\overline{A}$, $\overline{R} \subseteq \text{CONE}(D)^+$, $Q$ is of anti-blocking type with respect to $(\overline{D}, D)$ (proposition 2.18). Using the just proved equality $Q_D^* = (\text{CONV}(\overline{A},0) + \text{CONE}(\overline{R}) - \text{CONE}(\overline{D})) \cap \text{CONE}(D)^+ = P$ and therefore $Q = (Q_D)^* = P_D^*$.

Remark
The anti-blocking theory of Fulkerson is obtained by taking $D := \overline{D} := I$ (i.e. $\text{CONE}(D) = \text{CONE}(\overline{D}) = \text{CONE}(D)^+ = \text{CONE}(\overline{D})^+ = \mathbb{R}^n$), the classical polarity by taking $D := \overline{D} := 0$ (i.e. $\text{CONE}(D) = \text{CONE}(\overline{D}) = 0$, $\text{CONE}(D)^+ = \text{CONE}(\overline{D})^+ = \mathbb{R}^n$). In the context of classical polarity we say that $P_D^*$ is the polar polyhedron of $P$.

Theorem 2.20 can be useful to find a polyhedral description of a family $\mathcal{F} \subseteq 2^V$ of combinatorial objects which includes the empty set: Let $A$ be the incidence matrix of the family $\mathcal{F}$. Take $R := 0$ and use for example the classical polarity, i.e. $\text{CONE}(D) = \text{CONE}(\overline{D}) = 0$, $\text{CONE}(D)^+ = \text{CONE}(\overline{D})^+ = \mathbb{R}^V$. Then $P$ defined by (41) is equal to $P := \{ x \in \mathbb{R}^V | Ax \leq 1 \}$. To find a polyhedral description of $\mathcal{F}$ it is sufficient to find points and directions which generate $P$, i.e. matrices $\overline{A}$ and $\overline{R}$ with $P = \text{CONV}(\overline{A}) + \text{CONE}(\overline{R})$, because this implies by theorem 2.20 that $P_D^* = \text{CONV}(A) = \{ x \in \mathbb{R}^V | \overline{A}x \leq 1, \overline{R}x \leq 0 \}$. Hence $\overline{A}x \leq 1, \overline{R}x \leq 0$ is a polyhedral description of the family $\mathcal{F}$.

Notice that by the definition of $P$ and $A \subseteq \mathbb{R}^V_+$, $\text{CONE}(\overline{R}) \supseteq -\mathbb{R}^V_+ = \text{CONE}(-I)$. It follows that $P := \{ x \in \mathbb{R}^V | Ax \leq 1 \} = \text{CONV}(\overline{A}) + \text{CONE}(\overline{R}) - \mathbb{R}^V_+$ and $P_D^* = \{ x \in \mathbb{R}^V_+ | \overline{A}x \leq 1, \overline{R}x \leq 0 \} = \text{CONV}(A)$.

In the introduction we used the forms (43) and (44) for the polyhedra associated to the pairs of matrices $(A,R)$ and $(\overline{A}, \overline{R})$ with $A, R \subseteq \mathbb{R}^V_+$, $\overline{A}, \overline{R} \subseteq \mathbb{R}^V$.

The same polyhedra are obtained by using the anti-blocking theory with $D := 0, \overline{D} := I$ and therefore $\text{CONE}(D)^+ = \mathbb{R}^V$, $\text{CONE}(\overline{D})^+ = \mathbb{R}^V$: $P$ in (43) is of anti-blocking type with respect to $(D,\overline{D})$ (proposition 2.18 (iii)) and it is given in the form (40). Hence $P_D^*$ is equal to $P_D^*$ in (44).

If $A$ (and $R$) are in $\mathbb{R}^V_+$, either $(D,\overline{D}) = (0,0)$ (classical polarity) or $(D,\overline{D}) = (0,I)$ can be used. In such cases, we choose the classical polarity, however we represent the related polyhedra in the form (43) and (44), respectively.
Let \((P,Q)\) be a pair of anti-blocking polyhedra in \(\mathbb{R}^n\) with respect to given matrices \((D,\overline{D})\), i.e. \(P^*_D = Q\) and \(Q^*_D = P\), and \(H := \{x \in \mathbb{R}^n | dx = 0\} = \text{CONE}(\pm d)^+\) the subspace orthogonal to a given vector \(d \in \mathbb{R}^n\), \(d \neq 0\).

We define two operations: The first, denoted by cut of \(P\) with \(H\) is the intersection of \(P\) with \(H\) and gives the polyhedron
\[
P^\prime := P \cap H = P \cap \text{CONE}(\pm d)^+.
\]
The second, denoted projection of \(Q\) on \(H\), is the orthogonal projection of \(Q\) on \(H\) cut with \(\text{CONE}(\overline{D})^+\) and gives the polyhedron
\[
Q^\prime := ((Q - \text{CONE}(\pm d)) \cap H) \cap \text{CONE}(\overline{D})^+ = (Q - \text{CONE}(\pm d)) \cap \text{CONE}(\overline{D}, \pm d)^+.
\]

Notice that the orthogonal projection of \(Q\) on \(H\) is equal to \((Q - \text{CONE}(\pm d)) \cap H\). The reason for cutting it with \(\text{CONE}(\overline{D})^+\) is the following: When considering anti-blocking pairs \((P,Q)\) with respect to \((D,\overline{D})\) we are only interested in those points being in \(\text{CONE}(D)^+\) for \(P\) and in those being in \(\text{CONE}(\overline{D})^+\) for \(Q\). While the cut of \(P\) with \(H\) remains in \(\text{CONE}(D)^+\), the orthogonal projection of \(Q\) on \(H\) is in general not in \(\text{CONE}(\overline{D})^+\). Therefore we cut the projection of \(Q\) with \(\text{CONE}(\overline{D})^+\). Observe that in the classical polarity \(\text{CONE}(\overline{D})^+ = \mathbb{R}^n\), in which case \(Q^\prime\) is simply the orthogonal projection of \(Q\) on \(H\).

The question we investigate is: are cut and projection polar operations or in other words, is \((P^\prime, Q^\prime)\) again a pair of anti-blocking polyhedra with respect to \(((D,\pm d), (\overline{D}, \pm d))\)?
This is not true in general, however it is true for the classical polarity \((D = \bar{D} = 0)\) and for the anti-blocking relation of Fulkerson \((\text{CONE}(D)^+ = \text{CONE}(\bar{D})^+ = R^n)\) if \(H\) is of the form \(\{x \in R^n| x_i = 0\}\) for some \(i, 1 \leq i \leq n\).

Our goal is to work out sufficient conditions for \((P', Q')\) to be a pair of anti-blocking polyhedra.

All along this chapter, examples will be given to motivate conditions required in lemmas and theorems. In order to conserve the thread of the thoughts, we have collected these examples in the last section of this chapter.

3.1 General pairs of anti-blocking polyhedra

Let \(P \subset R^n\) be a polyhedron and \(D\) a matrix. The anti-blocker \(P_D^*\) of \(P\) with respect to \(D\) is equal to

\[
P_D^* = \{z \in R^n | zx \leq 1 \text{ for all } x \in P\} \cap \text{CONE}(D)^+
\]

and \(P^*_0\) is the polar polyhedron of \(P\) (classical polarity).

We first consider some properties of polar polyhedra and their relation to anti-blocker with respect to \(\bar{D}\) and \(D\). In the sequel we introduce for better legibility the notation \(P_{0^*}^*\) instead of \((P_{0^*}^* )^*\).

**Lemma 3.1**

Let \(P \subset R^n\) be a nonempty polyhedron; \(D, \bar{D},\) and \(B\) matrices. Then

(i) \((P - \text{CONE}(B))^*_0 = P^*_0 \cap \text{CONE}(B)^+\)

(ii) if \(0 \in P\), then \((P \cap \text{CONE}(B))^*_0 = P^*_0 - \text{CONE}(B)\)

(iii) if \(P\) is of anti-blocking type with respect to \((D, \bar{D})\), then \(P^*_0 = P_D^* - \text{CONE}(D)\).

**Proof:**

(i) "\(\supseteq\)" Let \(z \in P^*_0 \cap \text{CONE}(B)^+\) and \(x = x - \nu B \in P - \text{CONE}(B), x \in P, \nu \geq 0, x^* z = x z - \nu B z \leq x z \leq 1\), hence \(z \in (P - \text{CONE}(B))^*_0\).

"\(\subseteq\)" Let \(z \in (P - \text{CONE}(B))^*_0\). Then \(x z \leq 1\) for all \(x \in P \subset P - \text{CONE}(B)\), hence \(z \in P_0^*\). There exists \(x^* \in P \neq \emptyset\) and for any \(j\) and all \(\nu \geq 0, z(x - \nu B_j) \leq 1\). This implies \(B_j z \geq 0\) for all \(j\) or equivalently \(z \in \text{CONE}(B)^+\). Hence \(z \in P_0^* \cap \text{CONE}(B)^+\).

(ii) \(P\) and \((P_0^* - \text{CONE}(B))\) are of anti-blocking type with respect to \((0,0)\), because they
contain the origin (proposition 2.18). Then by (i) we have: \( P_0^* \cap \text{CONE}(B) = (P_0^* \cap \text{CONE}(B))^* = (\text{CONE}(B))^*_0 = (\text{CONE}(B))^*_0. \)

(iii): \( P \) and \( P_D^* \cap \text{CONE}(D) \) contain the origin, hence they are of anti-blocking type with respect to \((0,0)\) and \( P_D^* \cap \text{CONE}(D) = (P_D^* \cap \text{CONE}(D))^* (\text{CONE}(D))^* (\text{CONE}(D))^* (\text{CONE}(D))^* (\text{CONE}(D))^* (\text{CONE}(D))^* \) \( = (P \cap \text{CONE}(D))^* \). The last equality is true because \( P \) is of anti-blocking type with respect to \((D,D)\) (proposition 2.18).

Notice that conditions "0 \( \in \) P" for (ii) and "P is of anti-blocking type with respect to \((D,D)\)" for (iii) can not be omitted (see examples 1 and 2 in section 3.3).

Consider now a pair of anti-blocking polyhedra \((P,Q)\) with respect to given matrices \((D,D)\) and a hyperplane \( H := \{ x \in \mathbb{R}^n \mid dx = 0 \}, d \neq 0 \). Let \( P' \) and \( Q' \) be respectively the cut of \( P \) with \( H \) and the projection of \( Q \) on \( H \) given in (1) and (2). The relations between \( (P')_*(D+\pm d) \) and \( Q, (Q')_*(D+\pm d) \) and \( P \) are given by the following lemma:

**Lemma 3.2**

Let \((P,Q)\) be a pair of anti-blocking polyhedra with respect to given matrices \((D,D)\), \( d \neq 0 \) a vector and \( P' := P \cap \text{CONE}(\pm d)^+ \), \( Q' := (Q \cap \text{CONE}(\pm d)) \cap \text{CONE}(D,\pm d)^+ \). Then

\[
\begin{align*}
(P')^*_{(D,\pm d)} &= (P \cap \text{CONE}(\pm d))^* \cap \text{CONE}(D,\pm d)^+ \\
\quad (P^*_{(D,\pm d)})^* \cap \text{CONE}(\pm d)^+ \\
\quad = (P \cap \text{CONE}(\pm d))^* \cap \text{CONE}(\pm d)^+ \cap \text{CONE}(D,\pm d)^+ \\
= (Q \cap \text{CONE}(\pm d))^* \cap \text{CONE}(D,\pm d)^+ \\
(Q')^*_{(D,\pm d)} &= ((Q \cap \text{CONE}(\pm d))^* \cap \text{CONE}(D,\pm d)^+ \\
\quad (Q^*_{(D,\pm d)})^* \cap \text{CONE}(\pm d)^+ \\
\quad = (Q \cap \text{CONE}(\pm d))^* \cap \text{CONE}(\pm d)^+ \cap \text{CONE}(D,\pm d)^+ \\
\quad = (P \cap \text{CONE}(\pm d))^* \cap \text{CONE}(\pm d)^+ \cap \text{CONE}(D,\pm d)^+.
\end{align*}
\]

We now investigate the question: Is \((P',Q')\) a pair of anti-blocking polyhedra with respect to \((D,\pm d),(D,\pm d)\)? In general this is not true. Even the following situations can occur:
(i) $P'$ is not of anti-blocking type with respect to $((D,\pm d),(\bar{D},\pm d))$.

(ii) $P'$ is of anti-blocking type with respect to $((D,\pm d),(\bar{D},\pm d))$, $Q'$ is not of anti-blocking type with respect to $((D,\pm d),(\bar{D},\pm d))$ and $(Q')^*_*(D,\pm d) = P'$.

(iii) $P'$ is of anti-blocking type with respect to $((D,\pm d),(\bar{D},\pm d))$, $Q'$ is of anti-blocking type with respect to $((\bar{D},\pm d),(D,\pm d))$, but $(P',Q')$ is not a pair of anti-blocking polyhedra with respect to $((D,\pm d),(\bar{D},\pm d))$.

Examples 3, 4 and 5 in section 3.3 illustrate these facts.

We derive now sufficient conditions for $(P',Q')$ to be a pair of anti-blocking polyhedra. First we work out properties which guarantee that $P'$ is of anti-blocking type with respect to $((D,\pm d),(\bar{D},\pm d))$ and that $P' = (Q')^*_*(D,\pm d)$ (lemma 3.3), then we establish properties such that similar results hold for $Q'$ (lemma 3.4).

**Lemma 3.3**

Let $(P,Q)$ be a pair of anti-blocking polyhedra with respect to given matrices $(D,\bar{D})$, $d \neq 0$ a vector, and $P' := P \cap \text{CONE}(\pm d)^+$, $Q' := (Q - \text{CONE}(\pm d)) \cap \text{CONE}(\bar{D},\pm d)^+$. If $(Q - \text{CONE}(\pm d)) \cap \text{CONE}(\pm d)^+ \subseteq \text{CONE}(\bar{D})^+$, then

(i) $P' = (Q')^*_*(D,\pm d)$

(ii) $P'$ is of anti-blocking type with respect to $((D,\pm d),(\bar{D},\pm d))$.

**Proof:**

Let $Q = \text{CONV}(A,0) + \text{CONE}(R) \subseteq \text{CONE}(\bar{D})^+$ for some matrices $A$ and $R$ and let $A'$ and $R'$ be the matrices with rows being the orthogonal projection on $\{x \in \mathbb{R}^n \mid dx = 0\}$ of the rows of $A$ respectively $R$, i.e.

$$A_i = A_i' + \alpha_i d, \text{ with } A_i'd = 0, \text{ for all rows } A_i \text{ of } A, \quad \alpha_i d = 0, \text{ for all rows } R_i \text{ of } R.$$ 

Then $A', R' \subseteq \text{CONE}(\bar{D},\pm d)^+$ and $Q' = \text{CONV}(B,A',0) + \text{CONE}(S,R')$ for some matrices $B$ and $S$. By lemma 2.17

$$(Q')^*_*(D,\pm d) = \{x \in \mathbb{R}^n \mid Bx \leq 1, A'x \leq 1, Sx \leq 0, R'x \leq 0\} \cap \text{CONE}(D,\pm d)^+$$

$$\subseteq \{x \in \mathbb{R}^n \mid A'x \leq 1, R'x \leq 0\} \cap \text{CONE}(D,\pm d)^+$$

$$= \{x \in \mathbb{R}^n \mid A'x \leq 1, R'x \leq 0\} \cap \text{CONE}(D,\pm d)^+$$

$$= P',$$

as $A'x = Ax$, $R'x = Rx$ for $x \in \text{CONE}(\pm d)^+$. The other inclusion follows through simple inclusions and by lemma 3.2:

$$P' = P \cap \text{CONE}(D,\pm d)^+ = (P \cap \text{CONE}(\pm d)^+) \cap \text{CONE}(D,\pm d)^+$$

$$\subseteq ((P \cap \text{CONE}(\bar{D})) \cap \text{CONE}(\pm d)^+) \cap \text{CONE}(D,\pm d)^+$$

$$\subseteq (((P \cap \text{CONE}(\bar{D})) \cap \text{CONE}(\pm d)^+) - \text{CONE}(\bar{D},\pm d)) \cap \text{CONE}(D,\pm d)^+$$
By corollary 2.19, \((Q_{(D,\pm d)}^{\ast})\) and therefore \(P^\prime\) is of anti-blocking type with respect to \(((D,\pm d),(D,\pm d))\).

**Lemma 3.4**

Let \((P,Q)\) be a pair of anti-blocking polyhedra with respect to given matrices \((D,\bar{D})\), \(d \neq 0\) a vector and \(P^\prime := P \cap \text{CONE}(\pm d)^{\ast}\), \(Q^\prime := (Q - \text{CONE}(\pm d)) \cap \text{CONE}(\bar{D},\pm d)^{\ast}\).

If for all rows \(D_j\) of \(\bar{D}\) with \(\bar{D}_j.d \neq 0\), \((P \cap \text{CONE}(\pm d)^{\ast}) \subset \text{CONE}(D)^{\ast}\), then

(i) \(Q^\prime = (P^\prime)_{(\bar{D},\pm d)^{\ast}}\)

(ii) \(Q^\prime\) is of anti-blocking type with respect to \(((\bar{D},\pm d),(D,\pm d))\).

**Proof:**

Let \(P = \text{CONV}(A,0) + \text{CONE}(R) \subset \text{CONE}(D)^{\ast}\) for some matrices \(A\) and \(R\) and \(\Delta := \begin{bmatrix} R \\ -D \end{bmatrix}\).

By the definition of \(P^\prime\) and the condition on \(P\) above, for:

\[
A_i.d - A_j.d < 0, \exists 0 < \lambda < 1 \text{ with } x_1 = \lambda A_i + (1-\lambda)A_j \in P^\prime, \text{ i.e. } x_1 = \frac{(A_j.d)A_i - (A_i.d)A_j}{A_j.d - A_i.d};
\]

\[
A_i.d - \Delta_j.d < 0, \exists 0 < \lambda \text{ with } x_2 = A_i + \lambda \Delta_j \in P^\prime, \text{ i.e. } x_2 = \frac{(\Delta_j.d)A_i - (A_i.d)\Delta_j}{\Delta_j.d};
\]

\[
R_i.d - \Delta_j.d < 0, \exists 0 < \lambda \text{ with } x_3 = R_i + \lambda \Delta_j \in P^\prime, \text{ i.e. } x_3 = \frac{(\Delta_j.d)R_i - (R_i.d)\Delta_j}{\Delta_j.d},
\]

and \(x_4 = \mu x_3 \in P^\prime\) for all \(\mu \geq 0\),

where \(A_i, A_j, \Delta_j\) (respectively \(R_i\)) are rows of \(A\) (\(\Delta\), respectively \(R\)).

Let \(Q = \{x \in \mathbb{R}^n | Ax \leq 1, \Delta x \leq 0\} = \{x \in \mathbb{R}^n | Bx \leq \beta, Sx \leq \gamma, Tx \leq \delta\},\) where

\(B \subseteq \mathbb{R}^{1 \times n}\), and \(Bx \leq \beta\) is the maximal subsystem of \(\begin{bmatrix} A \\ \Delta \end{bmatrix}x \leq \begin{bmatrix} 1 \\ 0 \end{bmatrix}\), with \(B_i.d > 0, \ i \in I,\)

\(S \subseteq \mathbb{R}^{1 \times n}\), and \(Sx \leq \gamma\) is the maximal subsystem of \(\begin{bmatrix} A \\ \Delta \end{bmatrix}x \leq \begin{bmatrix} 1 \\ 0 \end{bmatrix}\), with \(S_i.d < 0, \ i \in J,\)

\(T \subseteq \mathbb{R}^{1 \times n}\), and \(Tx \leq \delta\) is the maximal subsystem of \(\begin{bmatrix} A \\ \Delta \end{bmatrix}x \leq \begin{bmatrix} 1 \\ 0 \end{bmatrix}\), with \(T_i.d = 0, \ i \in K.\)

For the inclusion \((P^\prime)_{(\bar{D},\pm d)^{\ast}} \subset Q^\prime\), let \(y \in (P^\prime)_{(\bar{D},\pm d)^{\ast}}\). We show that there exists \(\sigma \in \mathbb{R}\) with \(y + \sigma d \in Q:\)

\[
\exists \sigma: y + \sigma d \in Q \iff \exists \sigma: \sigma \leq \frac{\beta_i - B_i.y}{B_i.d}, \forall i \in I; \ \sigma \geq \frac{\gamma_j - S_j.y}{S_j.d}, \forall j \in J; \ Ty \leq \delta
\]

\[
\iff \frac{\gamma_j - S_j.y}{S_j.d} \leq \frac{\beta_i - B_i.y}{B_i.d}, \forall j \in J, i \in I; \ Ty \leq \delta
\]

\[
\iff (B_i.d)(S_j.y) - (S_j.d)(B_i.y) \leq \gamma_j(B_i.d) - \beta_i(S_j.d), \forall j \in J, i \in I; \ Ty \leq \delta
\]
\begin{align*}
&\Leftrightarrow (a) \quad (B_i d)S_j - (S_j d)B_i \quad \gamma_j(B_i d) - \beta_i(S_j d) \quad y \leq 1, \forall j \in J, i \in I: \gamma_j + \beta_i \neq 0; \\
&\quad (b) ((B_i d)S_j - (S_j d)B_i) y \leq 0, \forall j \in J, i \in I: \gamma_j = \beta_i = 0; \\
&\quad (c) Ty \leq \delta.
\end{align*}

(a) is satisfied, because \( x = (B_i d)S_j - (S_j d)B_i \gamma_j(B_i d) - \beta_i(S_j d) \) is a point of the type \( x_1 \) or \( x_2 \) above. Therefore \( x \) is in \( P' \) and \( xy \leq 1 \).

(b) is satisfied: this is true if:

\( B_i \) or \( S_j \) is a row of \( R \), because \( x = (B_i d)S_j - (S_j d)B_i \) is a point of the type \( x_4 \) above and \( \mu x \in P', \forall \mu \geq 0 \). Hence \( xy \leq 0 \).

\( B_i \) and \( S_j \) are rows of \((-D)\), because \( ye \in \text{CONE}(D)^+ \). Therefore \( S_j y \leq 0, B_i y \leq 0 \) and \( ((B_i d)S_j - (S_j d)B_i) y \leq 0 \).

(c) is satisfied: this is true if:

\( T_k \) is a row of \( A \), because \( T_k \in P' \), which implies \( T_k y \leq 1 = \delta_k \).

\( T_k \) is a row of \( R \), because \( \lambda T_k \in P', \forall \lambda \geq 0 \), which implies \( T_k y \leq 0 = \delta_k \).

\( T_k \) is a row of \((-D)\), because \( ye \in \text{CONE}(D)^+ \), which implies \( T_k y \leq 0 = \delta_k \).

The other inclusion \( Q' \subseteq (P')_0 \) follows directly by lemma 3.2, hence \( Q' = (P')_0 \) and \( Q' \) is of anti-blocking type with respect to \( ((D',\pm d'),(D',\pm d')) \) (corollary 2.19).

Combining lemma 3.3 and lemma 3.4 we get a sufficient condition for \( (P',Q') \) to be a pair of anti-blocking polyhedra with respect to \( ((D',\pm d'),(D',\pm d')) \):

**Theorem 3.5**

Let \( (P, Q) \) be a pair of anti-blocking polyhedra with respect to given matrices \( (D,D) \), \( d \neq 0 \) a vector and \( P' := P \cap \text{CONE}(\pm d)^+ \), \( Q' := (Q - \text{CONE}(\pm d)) \cap \text{CONE}(D,\pm d)^+ \). If

(i) \( (Q - \text{CONE}(\pm d)) \cap \text{CONE}(\pm d)^+ \subseteq \text{CONE}(D)^+ \),

(ii) for all rows \( D_j \) of \( D \) with \( D_j d \neq 0 \), \( (P - \text{CONE}(D_j)) \cap \text{CONE}(\pm d)^+ \subseteq \text{CONE}(D)^+ \),

then \( (P',Q') \) is a pair of anti-blocking polyhedra with respect to \( ((D',\pm d'),(D',\pm d')) \).

Notice that conditions (i) and (ii) of theorem 3.5 are not necessary, as shows example 6 in section 3.3.

We give now a special case for which conditions (i) and (ii) of theorem 3.5 are satisfied:
Corollary 3.6
Let \((P,Q)\) be a pair of anti-blocking polyhedra with respect to given matrices \((D,D')\), \(d \neq 0\) a vector with \(Dd = 0\), \(P' = P \cap \text{CONE}(\pm d)^{+}\) and \(Q' = (Q - \text{CONE}(\pm d)) \cap \text{CONE}(D,\pm d)^{+}\). Then \((P',Q')\) is a pair of anti-blocking polyhedra with respect to \(((D,\pm d),(D',\pm d))\).

Proof:
As \(Dd = 0\), it is sufficient to show that condition (i) of theorem 3.5 is satisfied. Let \(x' := x - v \in (Q - \text{CONE}(\pm d)) \cap \text{CONE}(\pm d)^{+}\) with \(x \in Q \subseteq \text{CONE}(D)^{+}\) and \(v \in \mathbb{R}\). Then \(D_{j}x' = D_{j}x \geq 0\) for all rows \(D_{j}\) of \(D\), hence \(x' \in \text{CONE}(D)^{+}\) and \((Q - \text{CONE}(\pm d)) \cap \text{CONE}(\pm d)^{+} \subseteq \text{CONE}(D)^{+}\).

3.2 Classical polarity, anti-blocking theory of Fulkerson
We apply now the results of the previous section to two special cases of anti-blocking relations, namely the classical polarity and the anti-blocking theory of Fulkerson.

We consider first the case where \(P\) and \(Q\) are related through the classical polarity, i.e. \(D = D' = 0\). For any vector \(d \in \mathbb{R}^{n}\), \(d \neq 0\), we have \(Dd = 0\) and by corollary 3.6 we get:

Corollary 3.7
Let \((P,Q)\) be a pair of polyhedra, \(d\) a vector, \(P' := P \cap \text{CONE}(\pm d)^{+}\) and \(Q' := (Q - \text{CONE}(\pm d)) \cap \text{CONE}(\pm d)^{+}\). Then \((P',Q')\) is a pair of anti-blocking polyhedra with respect to \(((\pm d), (\pm d))\).

In other words, if \((P,Q)\) is a pair of polar polyhedra and \(H = \{x \in \mathbb{R}^{n} \mid dx = 0\}\) is an euclidean subspace of \(\mathbb{R}^{n}\), then \((P',Q')\), considered as polyhedra in \(H\) is a pair of polar polyhedra again.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.1.png}
\caption{Figure 3.1}
\end{figure}
In the spirit of this diagram we can say, that in the context of classical polarity the operations "cut with $H$" and "projection on $H$" are polar operations.

Consider now the case where $P$ and $Q$ are related through the anti-blocking relation of Fulkerson, i.e. both matrices $D$ and $\overline{D}$ are equal to the identity matrix. Let $d \in \mathbb{R}^n$, $d \neq 0$, $P' := P \cap \text{CONE}(\pm d)^+$ and $Q' := (Q - \text{CONE}(\pm d)) \cap \text{CONE}(I,\pm d)^+$. The pair $(P',Q')$ is not always a pair of anti-blocking polyhedra with respect to $((I,\pm d),(I,\pm d))$, as we already saw in example 4.

The peculiarities of the matrices involved in the anti-blocking theory of Fulkerson are

1. $D = \overline{D}$;
2. The rows of $D$ form an orthogonal basis of $\mathbb{R}^n$;
3. $\text{CONE}(D) = \text{CONE}(D)^+$.

Our interest is to investigate situations where the following holds: if $(P',Q')$ is a pair of anti-blocking polyhedra, then this anti-blocking relation holds in the context of the theory of Fulkerson. More precisely, we want the involved matrices $((I,\pm d),(I,\pm d))$ to retain the features (4) to (6).

Consequently we are interested in those vectors $d \in \mathbb{R}^n$ for which

1. There exists an orthogonal basis $\{f_1,...,f_{n-1}\}$ of $H$ with $\text{CONE}((f_1,...,f_{n-1})) = \text{CONE}(I,\pm d)^+$;
2. $\text{CONE}(I,\pm d)^+ = \text{CONE}(I,\pm d) \cap \text{CONE}(\pm d)^+$.

(In fact (5') and (6') are equivalent.) Let $d \in \mathbb{R}^n$ such that (6') is satisfied and $K^+ := \{k \mid d_k > 0\}$, $K^- := \{k \mid d_k < 0\}$. Without loss of generality we assume that $|K^+| \geq 1$ and $d^2 = 1$. Since for $1 \leq k \leq n$, $y^k := e_k - d_k \in \text{CONE}(I,\pm d) \cap \text{CONE}(\pm d)^+$, $y^k \geq 0$ for $k = 1,...,n$ and therefore $-d_k^*d_i \geq 0$ for $i \neq k$. It follows that $|K^+| = 1$, $|K^-| \leq 1$ and essentially $d$ is of one of the following types:

1. $d = e_i$ for $i \in \{1,...,n\}$;
2. $d = d_i e_i - d_k e_k$ for $i \neq k$, $i, k \in \{1,...,n\}$, $d_i, d_k > 0$, $d^2 = 1$.

It is easy to verify that any $d \in \mathbb{R}^n$ of type (7) or (8) satisfies (6') (and (5')). For $d \in \mathbb{R}^n$ being of type (7) or (8), cut and projection are polar operations:

**Corollary 3.8**

Let $(P,Q)$ be a pair of anti-blocking polyhedra according to Fulkerson, $d \neq 0$ a vector of type (7) or (8), $P' := P \cap \text{CONE}(\pm d)^+$ and $Q' := (Q - \text{CONE}(\pm d)) \cap \text{CONE}(\overline{D},\pm d)^+$. Then $(P',Q')$ is a pair of anti-blocking polyhedra with respect to $((I,\pm d),(I,\pm d))$. 

Proof:
Denote either by \( i \) the index in \( \{1,\ldots,n\} \) for which \( d = e_i \) (type (7)) or by \( i \) and \( k \) the indices in \( \{1,\ldots,n\} \) for which \( d = d_i e_i - d_k e_k \) (type (8)). We prove that conditions (i) and (ii) of theorem 3.5 are satisfied.

(i): Let \( x \in \mathbb{Q} \cap \mathbb{R}^n \) and \( y := x - (dx)d \in \text{CONE}(\pm d)^+ \). We show \( y \geq 0 \): If \( d \) is of type (7), i.e. \( d = e_i \), then \( y_j = x_j \geq 0 \) for \( j \neq i \) and \( y_i = 0 \). If \( d \) is of type (8), \( d = d_i e_i - d_k e_k \) and \( y_j = x_j \geq 0 \) for \( j \in \{i,k\} \), \( y_i = x_i - (x_i d_i - x_k d_k) d_i = x_i(1 - d_i^2) + x_k d_i d_k = x_i d_k^2 + x_k d_i d_k \geq 0 \), and analogously \( y_k \geq 0 \).

(ii): Let \( D_j, j \in \mathbb{D} = 1 \) with \( D_j.d \neq 0 \) and \( x \in \mathbb{P} \cap \mathbb{R}^n \), \( y := x - \lambda D_j \in \text{CONE}(\pm d)^+ \) with \( \lambda \geq 0 \). Then \( \lambda = (xd)/(D_j.d) \). We show \( y \geq 0 \): If \( d \) is of type (7), i.e. \( d = e_i \), then \( D_j = e_i \) and \( \lambda = x_i \). It follows that \( y \geq 0 \). If \( d \) is of type (8), i.e. \( d = d_i e_i - d_k e_k \), either \( D_j = e_i \) or \( D_j = e_k \). For \( D_j = e_i \), \( \lambda = (x_i d_i - x_k d_k)/d_i \) and \( y_e = x_e \geq 0 \) for \( e \neq i \), \( d_i y_i = d_i x_i - (x_i d_i - x_k d_k) = x_k d_k \geq 0 \). In the same way one shows \( y \geq 0 \), if \( D_j = e_k \).

3.3 Examples

In this section all examples used in the previous sections are collected. At the beginning of each example, written in italic, we mention what this example shows.

Example 1:

\( P \) is a nonempty polyhedron, not containing the origin, \( B \) a matrix and

\[
(P \cap \text{CONE}(B)^+)_0^* \neq P_0^* - \text{CONE}(B):
\]

\( P \subseteq \mathbb{R}^2, P := \text{CONV}((1,1/2)) + \text{CONE}((1,1)) \) and \( B := \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \).
Then $P \cap \text{CONE}(B)^+ = \emptyset$, $P^*_0 = \{ x \in \mathbb{R}^2 \mid x_1 + \frac{1}{2} x_2 \leq 1, x_1 + x_2 \leq 0 \}$ by lemma 2.17, and $(P \cap \text{CONE}(B)^+)^*_0 = \mathbb{R}^2 \neq P^*_0 - \text{CONE}(B) = \{ x \in \mathbb{R}^2 \mid x_1 + x_2 \leq 0 \}$.

Example 2:

$D, \bar{D}$ are matrices, $P$ a nonempty polyhedron with $0 \in P \subseteq \text{CONE}(D)^+$ and

$$P^*_0 \neq P^*_D - \text{CONE}(D):$$

$D := \bar{D} := \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$ and $P := \{ x \in \mathbb{R}^2 \mid x_1 \leq 1, -x_1 + x_2 \geq 0, 2x_1 - x_2 \geq 0 \}$

$= \text{CONV}((0,0),(1,1),(1,2))$.

Example 3:

$P$ is of anti-blocking type with respect to given matrices $(D, \bar{D})$, $d$ a vector and $P^\prime := P \cap \text{CONE}(\pm d)^+$. $P^\prime$ is not of anti-blocking type with respect to $((D, \pm d), (\bar{D}, \pm d))$:

$D := \bar{D} := \begin{bmatrix} -1 & 0 & 2 \\ \end{bmatrix}$, $d := (0,0,1)$ and

$P := \{ x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 \leq 1 \} \cap \text{CONE}(D)^+$.

Then $P$ is of anti-blocking type with respect to $(D, \bar{D})$ by proposition 2.18 (iii) and

$P^\prime = \{ x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 \leq 1, x_3 = 0, -x_1 + 2x_3 \geq 0 \}$

$= \{ x \in \mathbb{R}^3 \mid x_3 = 0, x_1 + x_2 \leq 1, x_1 \leq 0 \}$

$= \text{CONV}((0,1,0)) + \text{CONE}((0,-1,0),(-1,1,0)))$. 

Figure 3.3
Hence by lemma 2.17
\[(P')^\ast_{(\bar{D},\pm d)} = \{x \in \mathbb{R}^3 \mid 0 \leq x_2 \leq 1, -x_1 + x_2 \leq 0, x_3 = 0, -x_1 + 2x_3 \geq 0\} = \{0\}\] and
\[(Q')^\ast_{(\bar{D},\pm d)} = \text{CONE}(D,\pm d)^+ = \{x \in \mathbb{R}^3 \mid x_3 = 0, x_1 \leq 0\} \neq P'.\]

Example 4:

\((P,Q)\) is a pair of anti-blocking polyhedra with respect to given matrices \((D,\bar{D})\), \(d\) a vector, 
P' := \(P \cap \text{CONE}(\pm d)^+\), \(Q' := (Q \setminus \text{CONE}(\pm d)) \cap \text{CONE}(\bar{D},\pm d)^+\). 
P' is of anti-blocking type with respect to \(((D,\pm d),(\bar{D},\pm d))\), \(Q'\) is not of anti-blocking type with respect to \(((\bar{D},\pm d),(D,\pm d))\) and \((Q')^\ast_{(\bar{D},\pm d)} = P'\):

\[
D := \bar{D} := \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad d := (2,2,-1),
\]

\(P := \{x \in \mathbb{R}^3 \mid x_i \leq 1, i = 1,2,3\} \cap \text{CONE}(D)^+\) and
\(Q := \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 \leq 1\} \cap \text{CONE}(\bar{D})^+\).

\((P,Q)\) is a pair of anti-blocking polyhedra with respect to \((D,\bar{D})\) (theorem 2.20) and
\[
P' = \{x \in \mathbb{R}^3 \mid 0 \leq x_i \leq 1, i = 1,2,3; \ x_3 = 2x_1+2x_2\} = \{x \in \mathbb{R}^3 \mid x_3 = 2x_1+2x_2, 0 \leq x_1, x_2; 2x_1+2x_2 \leq 1\} = \text{CONV}(\{(0,0,0),(0,\frac{1}{2},1),(\frac{1}{2},0,1)\});
\]
\[
Q' = \{x \in \mathbb{R}^3 \mid x_i \geq 0, i = 1,2,3; \ x_3 = 2x_1+2x_2, \ \exists \sigma: x + \sigma d \in Q\} = \{x \in \mathbb{R}^3 \mid x_i \geq 0, i = 1,2,3; \ x_3 = 2x_1+2x_2, \ \exists \sigma: \sigma \geq \frac{1}{2}x_1; \ \sigma \geq \frac{1}{2}x_2; 2x_1+2x_2 \geq \sigma; \ \frac{1}{3}x_1-x_2 \geq \sigma\} = \{x \in \mathbb{R}^3 \mid x_i \geq 0, i = 1,2,3; \ x_3 = 2x_1+2x_2, \ \exists \sigma: \sigma \geq \frac{1}{2}x_1; \ \sigma \geq \frac{1}{2}x_2; 2x_1+2x_2 \geq \sigma; 3\frac{3}{2}x_1+3x_2 \leq 1, 3x_1+\frac{3}{2}x_2 \leq 1\} = \text{CONV}(\{(0,0,0),(0,\frac{1}{3},\frac{1}{3}),(\frac{2}{3},0,\frac{2}{3}),(\frac{2}{3},\frac{2}{3},\frac{2}{3})\}).
\]
orthogonal projection of $P'$ on \( \{x_3 = 0\} \)

Figure 3.5

Then by lemma 2.17

\[
(P')_{(D_\pm d)(D_\pm d)} = \{x \in \mathbb{R}^3 \mid x_i \geq 0, i = 1,2,3; x_3 = 2x_1 + 2x_2, \frac{5}{3}x_1 + 2x_2 \leq 1, 2x_1 + \frac{5}{2}x_2 \leq 1\}
\]

= \text{CONV}((0,0,0),(0,\frac{2}{3},\frac{2}{3}),(\frac{2}{3},0,\frac{2}{3}),(\frac{2}{3},\frac{2}{3},\frac{2}{3})) \neq Q' and

\[
(Q')_{(D_\pm d)} = \{x \in \mathbb{R}^3 \mid x_i \geq 0, i = 1,2,3; x_3 = 2x_1 + 2x_2, \frac{4}{3}x_1 + \frac{4}{3}x_2 \leq 1, \frac{5}{3}x_1 + \frac{4}{3}x_2 \leq 1, 2x_1 + 2x_2 \leq 1\} = P'.
\]

Furthermore

\[
((P')_{(D_\pm d)(D_\pm d)})_{(D_\pm d)} = \{x \in \mathbb{R}^3 \mid x_i \geq 0, i = 1,2,3; x_3 = 2x_1 + 2x_2, \frac{8}{3}x_1 + 2x_2 \leq 1, 2x_1 + \frac{8}{3}x_2 \leq 1, 2x_1 + 2x_2 \leq 1\}
\]

= $P'$ and

\[
((Q')_{(D_\pm d)(D_\pm d)})_{(D_\pm d)} = (P')_{(D_\pm d)} \neq Q'.
\]
Hence $P'$ is of anti-blocking type with respect to $((D,\pm d),(D,\pm d))$ and $Q'$ is not of anti-blocking type with respect to $((D,\pm d),(D,\pm d))$.

Example 5:

$(P,Q)$ is a pair of anti-blocking polyhedra with respect to given matrices $(D,\overline{D})$, $d$ a vector, $P' := P \cap \text{CONE}(\pm d)^+$, $Q' := (Q - \text{CONE}(\pm d)) \cap \text{CONE}(\overline{D},\pm d)^+$. $P'$ and $Q'$ are of anti-blocking type with respect to $((D,\pm d),(D,\pm d))$ and $((D,\pm d),(D,\pm d))$, respectively, but $(P',Q')$ is not a pair of anti-blocking polyhedra with respect to $((D,\pm d),(\overline{D},\pm d))$.

\[
D := \overline{D} := \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad d := (10,1,-1),
\]

\[
P := \{ x \in \mathbb{R}^3 \mid 0 \leq x_3, x_1+x_3 \leq 1 \} \cap \{ x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 = 0 \}
= \text{CONV}(\{(0,0,0),(1,0,0),(0,0,1)\}) \text{ and }
\]

\[
Q := \{ x \in \mathbb{R}^3 \mid x_1 \leq 1, x_3 \leq 1 \} \cap \{ x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 = 0 \}
= \text{CONV}(\{(0,0,1),(1,0,1)\}) + \text{CONE}(\{(0,0,-1)\}).
\]

$(P,Q)$ is a pair of anti-blocking polyhedra with respect to $(D,\overline{D})$ by theorem 2.20.

\[
(P') := \{ x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 = 0, x_3 = 10x_1, 10x_1 \leq 1 \}
= \text{CONV}(\{(0,0,0),(\frac{1}{10},0,\frac{10}{11})\})
\]

\[
(Q') := \{ x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 = 0, x_3 = 10x_1; \exists \sigma: x + \sigma d \in Q \} = Q \cap \{ dx = 0 \}
= \{ x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 = 0, x_3 = 10x_1, 10x_1 \leq 1 \}
= \text{CONV}(\{(0,0,0),(\frac{1}{10},0,1)\})
\]

and by lemma 2.17

\[
(P')^* := \{ x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 = 0, x_3 = 10x_1, 101x_1 \leq 11 \}
= \text{CONV}(\{(0,0,0),(\frac{11}{101},0,\frac{110}{101})\}) \neq Q',
\]
(Q')(D,±d) = \{x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 = 0, x_3 = 10x_1, 101x_1 \leq 101\} = \text{CONV}\((\{(0,0,0),(\frac{10}{101},0,\frac{100}{101})\}) \neq P'.

Furthermore

\((P')(D,±d)) = \{x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 = 0, x_3 = 10x_1, 111x_1 \leq 101\} = P'\) and

\((Q')(D,±d)) = \{x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 = 0, x_3 = 10x_1, 1010x_1 \leq 101\} = Q', \) hence

P' and Q' are of anti-blocking type with respect to \((D,±d),(D,±d))\) and \((D,±d),(D,±d))\), respectively, and \((P',Q')\) is not a pair of anti-blocking polyhedra with respect to \((D,±d),(D,±d))\), since \((P')(D,±d) \neq Q'\) and \((Q')(D,±d) \neq P'.

Example 6:

\((P,Q)\) is a pair of anti-blocking polyhedra with respect to given matrices \((D,\bar{D})\), d a vector such that

(i) \((Q - \text{CONE}(±d)) \cap \text{CONE}(±d)^* \cap \text{CONE}(D)^*\)

(ii) There exists a row \(\bar{D}_j\) of \(\bar{D}\), with \(\bar{D}_j.d \neq 0\) and

\((P - \text{CONE}(\bar{D}_j)) \cap \text{CONE}(±d)^* \cap \text{CONE}(D)^*\).

Let \(P' := P \cap \text{CONE}(±d)^*\) and \(Q' := (Q - \text{CONE}(±d)) \cap \text{CONE}(\bar{D},±d)^*\). \((P',Q')\) is a pair of anti-blocking polyhedra with respect to \((D,±d),(\bar{D},±d))\):

\[D := \bar{D} := \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \ d := (1,1),
\]

\[P := \{x \in \mathbb{R}^2 \mid x_1 \leq 1\} \cap \{x \in \mathbb{R}^2 \mid x_2 \geq 0, x_1 - x_2 \geq 0\}
= \text{CONV}\((\{(0,0),(1,0),(1,1)\})\) and
\]

\[Q := \{x \in \mathbb{R}^2 \mid x_1+x_2 \leq 1\} \cap \{x \in \mathbb{R}^2 \mid x_2 \geq 0, x_1 - x_2 \geq 0\}
= \text{CONV}\((\{(0,0),(1,0),(\frac{1}{2},\frac{1}{2})\})\).
\]

(P,Q) is a pair of anti-blocking polyhedra with respect to \((D,\bar{D})\) by theorem 2.20.

Figure 3.8
Conditions (i) and (ii) are satisfied:

\[(P - \text{CONE}((0,1))) \cap \text{CONE}(\pm d)^+, \quad Q \subseteq \text{CONE}(\pm d)^+\]

\[\implies \text{CONE}(D)^+ \subseteq \text{CONE}(D)^+ \subseteq \{0\},\]

As \(\text{CONE}(D, \pm d)^+ = \text{CONE}(\overline{D}, \pm d)^+ = \{0\}\),

\(P' := P \cap \text{CONE}(\pm d)^+ = \{0\},\)

\(Q' := (Q - \text{CONE}(\pm d)) \cap \text{CONE}(\overline{D}, \pm d)^+ = \{0\}\) and

\((P', Q')\) is a pair of anti-blocking polyhedra with respect to \((D, \pm d), (\overline{D}, \pm d)\).
In chapter 1 various notions of perfectness have been defined for a pair \( ((A,R), (\overline{A}, \overline{R})) \) with \( \{0,1\} \)-matrices \( A, R \) and \( \{1,0,-k\} \)-matrices \( \overline{A}, \overline{R} \). All of these notions involve the so-called heredity property which says: If the pair \( ((A,R), (\overline{A}, \overline{R})) \) has some properties, then other pairs \( ((A^J,R^J), (\overline{A}^J, \overline{R}^J)) \) inherit those properties.

The main goal of this chapter is to characterize "perfect" pairs \( ((A,R), (\overline{A}, \overline{R})) \) by means of properties on \( ((A,R), (\overline{A}, \overline{R})) \) only. A basic step thereof consists in finding conditions which guarantee the heredity property, and will be achieved by means of the homogeneity concept which will be defined below.

The organization of this chapter will then be as follows: In section 4.1 we recall the various definitions of perfectness. In order to characterize them, we discuss in section 4.2 the heredity property for each case. Although perfectness is defined for \( \{0,1\} \)-matrices \( (A,R) \) and \( \{1,0,-k\} \)-matrices \( (\overline{A}, \overline{R}) \), we also consider matrices \( A, R \subseteq \mathbb{Q}^n \) and \( \overline{A}, \overline{R} \subseteq \mathbb{Q}^n \) in order to get more general results. Sometimes however, we have to restrict ourselves to matrices \( (A,R) \) and/or \( (\overline{A}, \overline{R}) \) with certain properties (like being \( \{1,0,-1\} \)-matrices) and, when possible, we give examples to illustrate the necessity of such restrictions. These examples are collected in the last section of this chapter.

As mentioned above, a major step in finding characterizations and therefore conditions which guarantee heredity has been the development of the concept of homogeneity.
concept plays a fundamental role in relation with perfectness and will therefore be introduced now.

Homogeneity

The concept of homogeneity describes a property of a system of inequalities which, as will be shown, insures that integrality of associated polyhedra is hereditary. Interestingly, this new notion requires that whenever the dual LP associated with this system has an optimum solution, it has an optimum solution with a certain homogeneity property. Notice that this definition displays a formal analogy to the concept of tdi-ness.

Let \( B \subseteq \mathbb{Q}^{\times n} \); \( b, c \in \mathbb{Q}^n \); \( d, e \in (\mathbb{Q} \cup \{ +\infty \})^n \), and consider the pair of dual LP's

\[
\begin{align*}
(1) \quad & \text{max} \ cx \text{ subject to } Bx \leq b, \ d \leq x \leq e, \\
(2) \quad & \text{min} \ by + eu - dv \text{ subject to } yB + u - v = c, \ y, u \geq 0.
\end{align*}
\]

A homogeneous solution \((y,u,v)\) of (2) is a feasible solution for which the submatrix \( B' \) of \( B \), whose rows correspond to the support of \( y \), has each of its columns either non negative or non positive.

We define the inhomogeneity vector \( a \in \mathbb{R}^n_+ \) of a feasible solution \((y,u,v)\) of (2), by

\[
(3) \quad a_j := \min \{a^+_j, a^-_j\}, \quad \text{where} \quad a^+_j := \Sigma \{y_i(\star B_{ij}) \mid (\star B_{ij}) \geq 0, i \in I\}, \quad \text{for} \quad \star = +, -.
\]

Clearly \((y,u,v)\) is a homogeneous solution if and only if \( a = 0 \).

If \( e = +\infty \) and \( d = 0 \), we sometimes consider the dual LP

\[
(2') \quad \text{min} \ by \text{ subject to } yB \geq c, \ y \geq 0
\]

instead of (2) and say that \( y \) is a homogeneous solution of (2'), if \((y,0,yB-c)\) is a homogeneous solution of (2).

The rational system \( d \leq x \leq e, Bx \leq b \) is a homogeneous system if LP (2) has a homogeneous optimum solution for each rational vector \( c \) for which the minimum exists.

The system \( d \leq x \leq e, Bx \leq b \) is called homogeneous-tdi if (2) has an integer homogeneous optimum solution for any integral vector \( c \) for which the minimum exists. Clearly, every homogeneous-tdi system is homogeneous as well as tdi.

4.1 Notions of perfectness

Before discussing the various notions of perfectness, we introduce three operations on matrices \( B \subseteq \mathbb{Q}^n \) which will be used in the sequel.
Let $J \subseteq \{1, \ldots, n\}$. The *contraction of coordinates* $\bar{J}$ in $B$ consists in deleting the columns $B_{j}$, $j \in \bar{J}$. Hence the resulting matrix after contraction of the coordinates $\bar{J}$ is the matrix $B_{\bar{J}}$. The *positive deletion of coordinates* $\bar{J}$ in $B$ is given by deleting the columns $B_{j}$, $j \in \bar{J}$ of $B$ as well as deleting all rows $i$, for which there exists $j \in \bar{J}$ with $B_{ij} \neq 0$. The *negative deletion of coordinates* $\bar{J}$ in $B$ is defined by deleting the columns $B_{j}$, $j \in \bar{J}$ of $B$ as well as deleting all rows $i$, for which there exists $j \in \bar{J}$ with $B_{ij} < 0$. (The reason for adopting the denomination positive deletion rather than "non-zero"-deletion is that we are going to apply it on non-negative matrices exclusively.)

The terms "contraction of a coordinate $j \in \{1, \ldots, n\}" and "deletion of a coordinate $j"$ have already been introduced in Fulkerson [1970] and correspond to the contraction and positive deletion for non-negative matrices defined above.

We recall now the definitions of the various notions of perfectness for a pair $((A,R),(\bar{A},\bar{R}))$ with $\{0,1\}$-matrices $A$, $R$, $\{1,0,-k\}$-matrices $\bar{A}$, $\bar{R}$, and associated polyhedra

\begin{align*}
P & := \{x \in \mathbb{R}^n | Ax \leq 1, Rx \leq 0\}, \\
Q & := \{x \in \mathbb{R}^n | \bar{A}x \leq 1, \bar{R}x \leq 0\}.
\end{align*}

Consider the following properties:

\begin{enumerate}
\item[(5)] $(P,Q)$ of (4) is a pair of polar polyhedra, i.e. $P^* = Q$ and $Q^* = P$;
\item[(6)] the describing system of $P$, $Ax \leq 1$, $Rx \leq 0$ is tdi;
\item[(7)] the describing system of $Q$, $x \geq 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$ is tdi;
\item[(8)] for all $J \subseteq \{1, \ldots, n\}$, let $(A^J,R^J)$ be obtained from $(A,R)$ by contraction of $J$, $(\bar{A}^J,\bar{R}^J)$ be obtained from $(\bar{A},\bar{R})$ by negative deletion of $J$. Then $((A^J,R^J),(\bar{A}^J,\bar{R}^J))$ inherits the properties among (5) to (8) required for $((A,R),(\bar{A},\bar{R}))$.
\item[(9)] $((A,R),(\bar{A},\bar{R}))$ has the integrality property if it satisfies (5) and (8);
\item[(10)] $((A,R),(\bar{A},\bar{R}))$ is called $A$-perfect if it satisfies (5), (6) and (8);
\item[(11)] $((A,R),(\bar{A},\bar{R}))$ is called $\bar{A}$-perfect if it satisfies (5), (7) and (8);
\item[(12)] $((A,R),(\bar{A},\bar{R}))$ is called $(A,\bar{A})$-perfect if it is $A$-perfect and $\bar{A}$-perfect.
\end{enumerate}
Notice that the heredity property (8) is involved in all of the above definitions and has a different meaning in each case.

Moreover, all notions share the integrality property and we discuss it now, because it is this common point which did motivate us to define in (8) that:

\[(A^j, R^j) \text{ is obtained by contraction of } \mathcal{J} \text{ from } (A, R);\]
\[(\overline{A}^j, R^j) \text{ is obtained by negative deletion of } \mathcal{J} \text{ from } (\overline{A}, R).\]

Observe that contraction seems a natural choice when trying to generalize the concept of perfectness defined for graphs (see chapter 1). In order to justify the choice of negative deletion we discuss now the basic features of the integrality property. They are illustrated in diagram 4.2 which should be interpreted as follows:

![Diagram 4.2](image)

Given is the pair \(((A, R), (\overline{A}, R))\) with associated polyhedra \(P\) and \(Q\) defined by (4). For \(J \subseteq \{1, \ldots, n\}\), we are given operations \(\Psi_A\) and \(\Psi_{\overline{A}}\) on matrices. Let \((A^j, R^j) := \Psi_A(A, R), (\overline{A}^j, \overline{R}^j) := \Psi_{\overline{A}}(\overline{A}, \overline{R})\) and

\[
\begin{align*}
P^j &= \{x \in \mathbb{R}^J | A^j x \leq 1, R^j x \leq 0\}; \\
Q^j &= \{x^j \in \mathbb{R}^J_+ | A^j x^j \leq 1, R^j x^j \leq 0\}.
\end{align*}
\]

The geometric operations on the associated polyhedra will be denoted by \(\Phi_A\) and \(\Phi_{\overline{A}}\), that is \(\Phi_A(P) := P^j, \Phi_{\overline{A}}(Q) := Q^j\). The integrality property says that \((P, Q)\) as well as \((P^j, Q^j)\) are polar polyhedra for \(J \subseteq \{1, \ldots, n\}\).

Hence, if the integrality property holds, we must have

\[
Q^j = (\Phi_A(P))^* = \Phi_A(P^*).
\]

Given a polar pair \((P, Q)\) and \(\Psi_A := \text{"contraction of } \mathcal{J} \text{"},\) the following arguments lead to negative deletion for \(\Psi_{\overline{A}}\): we would like to define \(\Psi_{\overline{A}}\) such that \((P^j, Q^j)\) associated to \(((A^j, R^j), (\overline{A}^j, \overline{R}^j))\) is again a pair of polar polyhedra, i.e. such that (14) holds. We study
first the operations $\Phi_A$, given by $\Phi_A(P) := P^J$, and $\Phi_A^-$, given by the right part of (14), i.e. $\Phi_A^-(Q) := (\Phi_A(P))^*$. We have

$\text{(15)} \quad P^J := \{x \in R^J \mid A^J x \leq 1, R^J x \leq 0\} = \{x^J \in R^J \mid (x^J, 0^J) \in P\}$,

that is, $P^J$ is the cut of $P$ with $H := \{x \in R^n \mid x^J = 0\}$, i.e. $\Phi_A^-$ is "cut of $P$ with $H"$. By corollary 3.7, the polar operation $\Phi_A^-$ of $\Phi_A$ is $\Phi_A^+ := "projection of $Q$ on $H"$ and

$\text{(16)} \quad (P^J)^* = \{x^J \in R^J \mid \exists x^J \in R^J: (x^J, x^J) \in Q\}$.

Consequently, if possible, we want to define $\Psi_A$ such that the resulting system $x^J \geq 0$, $A^J x \leq 1$, $R^J x \leq 0$ is a describing system of $(P^J)^*$. The choice of $\Psi_A := "negative deletion ofJI"$ is natural, because in this case, for any row $a$ of $A^J$ and row $p$ of $R^J$, $a \in P^J$ and $p \in P^J$ for all $\lambda \geq 0$. Hence, by the definition of $(P^J)^*$,

$\text{(17)} \quad (P^J)^* \subseteq \{x^J \in R^J \mid A^J x^J \leq 1, R^J x^J \leq 0\}.

Conversely, for a row $a$ of $A^J$ not in $A^J$ and a row $p$ of $R^J$ not in $R^J$, $a$ and $p$, for $\lambda \geq 0$ are in general not in $P^J$. (By polarity, for a vector $a \in P^J$, $a x \leq 1$ is not a valid restriction for $(P^J)^*$, i.e. $(P^J)^* \not\subseteq \{x \in R^n \mid a x \leq 1\}$.)

However, to obtain equality in (17), additional properties of the system $x^J \geq 0$, $A^J x \leq 1$, $R^J x \leq 0$ are necessary. They will be discussed in the next section.

We have seen that if we choose $\Psi_A$ in figure 4.2 to be the contraction of $J$, then a natural choice for $\Psi_A^-$ is the negative deletion of $J$. In the same way one can see that if we choose $\Psi_A^+$ to be the contraction of $J$ then $\Psi_A$ is the positive deletion of $J$ (see also theorem 4.2).

A stronger version of the heredity property could have been required by replacing (8) by

(8') For $J \subseteq \{1, \ldots, n\}$ let

$(A^J, R^J)$ and $(\bar{A}^J, \bar{R}^J)$ be obtained from $(A, R)$ and $(\bar{A}, \bar{R})$ by contraction and negative deletion of $J$, respectively;

$(A'^J, R'^J)$ and $(\bar{A}'^J, \bar{R}'^J)$ be obtained from $(A, R)$ and $(\bar{A}, \bar{R})$ by positive deletion and contraction of $J$, respectively.

Then $\left((A^J, R^J), (\bar{A}^J, \bar{R}^J)\right)$ and $\left((A'^J, R'^J), (\bar{A}'^J, \bar{R}'^J)\right)$ inherit from $\left((A, R), (\bar{A}, \bar{R})\right)$ those properties among (5) to (7) and (8') which are required for the considered notion of perfectness.

The reason for not taking (8') in the definitions of the integrality property and of the different perfectness notions is that (8') is automatically satisfied when (8) is satisfied:

**Proposition 4.1**

*Let $((A, R), (\bar{A}, \bar{R}))$ be a pair with $\{0, 1\}$-matrices $A$, $R$ and $\{1, 0, -k\}$-matrices $\bar{A}$, $\bar{R}$. If $((A, R), (\bar{A}, \bar{R}))$ has the integrality property or if it is $\ast$-perfect with $\ast \in \{A, \bar{A}, (A, \bar{A})\}$, then $((A, R), (\bar{A}, \bar{R}))$ satisfies also (8').*
Proof:
Notice first that for $\mathcal{J}$, $\mathcal{J}' \subseteq \{1,...,n\}$ with $\mathcal{J} \cap \mathcal{J}' = \emptyset$, a matrix $B \subset \mathbb{Q}^n$, and $\pi^J$, $\sigma^J$ each being one of the operations contraction, positive deletion, negative deletion of $\mathcal{J}$, we have
\[
\pi^J(\pi^J(B)) = \pi^{J \cap J'}(B) = \pi^J(\pi^J(B)) \quad \text{and} \quad \pi^J(\sigma^J(B)) = \sigma^J(\pi^J(B)).
\]
Hence (8) is equivalent to
\[(8')\]
For $\mathcal{J}$, $\mathcal{J}' \subseteq \{1,...,n\}$ with $\mathcal{J} \cap \mathcal{J}' = \emptyset$, let $(A(J,J'),R(J,J'))$ and $(\bar{A}(J,J'),\bar{R}(J,J'))$ be obtained from $(A(J),R(J))$ and $(\bar{A}(J),\bar{R}(J))$ by positive deletion of $\mathcal{J}$ and contraction of $\mathcal{J}'$, respectively. Then $((A(J,J'),R(J,J')),(\bar{A}(J,J'),\bar{R}(J,J')))$ inherits from $((A,J),(\bar{A},\bar{R}))$ those properties among (5) to (7) which are required for the considered notion of perfectness,
where $A^J$, $R^J$, $\bar{A}^J$, $\bar{R}^J$ are defined as in (8).

Theorem 4.2 below then immediately implies that $((A,J),(\bar{A},\bar{R}))$ satisfies (8').

The next theorem states that properties (5) to (7) are hereditary under operations $(\Psi_A,\Psi_A^J)$ := (positive deletion of $\mathcal{J}$, contraction of $\mathcal{J}$) (see figure 4.2). More precisely, given a pair $((A,J),(\bar{A},\bar{R}))$, define $((A',J',R',J'),(\bar{A}',J',\bar{R}',J')) := (\Psi_A^{J'},\Psi_A^J(\bar{A},\bar{R}))$. Then $((A',J',R',J'),(\bar{A}',J',\bar{R}',J'))$ inherits properties (5) to (7) from $((A,J),(\bar{A},\bar{R}))$.

Theorem 4.2
Let $((A,J),(\bar{A},\bar{R}))$ be a pair with $A \subset \mathbb{Q}^n$, $\bar{A} \subset \mathbb{Q}^n$ such that $(P,Q)$ defined in (4) is a pair of polar polyhedra. For $J \subseteq \{1,...,n\}$, let $A^J$, $R^J$, $\bar{A}^J$, $\bar{R}^J$ be defined as in (8'). Then
(i) For all $J \subseteq \{1,...,n\}$,
\[
P_J := \{x^J \in \mathbb{R}_+^J \mid A^Jx \leq 1, R^Jx \leq 0\} = \{x^J \in \mathbb{R}_+^J \mid \exists x^J \in \mathbb{R}_+^J : (x^J,x^J) \in P\},
\]
\[
Q_J := \{x^J \in \mathbb{R}_+^J \mid A^Jx \leq 1, R^Jx \leq 0\} = \{x^J \in \mathbb{R}_+^J \mid (x^J,0^J) \in Q\}
\]
and $(P_J,Q_J)$ is a pair of polar polyhedra.
(ii) If $x \geq 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$ is tdi (homogeneous-tdi), then $x \geq 0$, $\bar{A}^Jx \leq 1$, $\bar{R}^Jx \leq 0$ is tdi (homogeneous-tdi) for all $J \subseteq \{1,...,n\}$.
(iii) If $Ax \leq 1$, $Rx \leq 0$ is tdi, then $A^Jx \leq 1$, $R^Jx \leq 0$ is tdi for all $J \subseteq \{1,...,n\}$.

Proof:
Let $J \subseteq \{1,...,n\}$, and denote by $I$ and $K$ the indices of the rows of $A$ and $R$, which have been deleted in the positive deletion of $\mathcal{J}$, and by $I$ and $K$ the remaining ones.
(i): We show $P^J = \{x^J \in \mathbb{R}^J \mid \exists x^J \in \mathbb{R}^J: (x^J, x^J) \in P\}$: For the inclusion "\(\supseteq\)", let $(x^J, x^J) \in P$. Since $A^J x^J = A^J x^J = A^J (x^J, x^J) \leq 1$ and $R^J x^J = R^J (x^J, x^J) \leq 0$, $x^J \in P^J$. The other inclusion is given by: $x^J \in P^J$ implies $(x^J, -M) \in P$ for a large positive integer $M$.

$Q^J := \{x^J \in \mathbb{R}^J \mid A^J x^J \leq 1, R^J x^J \leq 0\} = \{x^J \in \mathbb{R}^J \mid (x^J, 0) \in Q\}$ since $A^J x^J = A(x^J, 0)$ and $R^J x^J = R(x^J, 0)$. The polarity of $P^J$ and $Q^J$ follows then from corollary 3.7.

(ii): Let $c^J \in \mathbb{Z}^J$ be an integer vector, such that
\begin{equation}
\min_{y} \ y^A^J + w^R^J - u = c^J, \quad y, w, u \geq 0
\end{equation}
has an optimum solution. Then, for a large integer positive number $M$,
\begin{equation}
\min_{y'} \ y'^A + w'^R - u' = (c^J, -M), \quad y', w', u' \geq 0,
\end{equation}
has an optimum solution with the same optimum value as (18). There exists a (homogeneous) integer optimum solution $(y', w', u')$ of (19) and $(y', w', u')$ is a (homogeneous) integer optimum solution of (18).

(iii): Let $c^J \in \mathbb{Z}^J$ be an integer vector, such that
\begin{equation}
\min_{y^1} \ y^1A^J + w^K^J = c^J, \quad y^1, w^K \geq 0,
\end{equation}
has an optimum solution. Then
\begin{equation}
\min_{y'^1} \ y'^1A + w'^R = (c^J, 0), \quad y'^1, w'^K \geq 0,
\end{equation}
has an optimum solution with the same optimum value as (20), because any feasible solution $((y^1, y^1), (w^K, w^K))$ of (21) has $y^1 = 0$ and $w^K = 0$. There exists an integer optimum solution $((y^1, y^1), (w^K, w^K))$ of (21) and $(y^1, w^K)$ is an integer optimum solution of (20).

4.2 Characterizations

This section focuses on characterizations of perfect pairs $((A, R), (\overline{A}, \overline{R}))$ by means of properties on $((A, R), (\overline{A}, \overline{R}))$ only. Recall that by definition any perfectness notion involves heredity, that is properties for each $((A^J, R^J), (\overline{A}^J, \overline{R}^J))$. Consequently, characterizing perfectness amounts to express the heredity property in terms of attributes on $((A, R), (\overline{A}, \overline{R}))$ exclusively. Our main concern in this section will therefore be heredity.

It will appear that in most cases our results are not confined to pairs $((A, R), (\overline{A}, \overline{R}))$ with $\{0, 1\}$-matrices $A, R$ and $\{1, 0, k\}$-matrices $\overline{A}, \overline{R}$. Consequently, throughout this section we consider pairs $((A, R), (\overline{A}, \overline{R}))$ with $A, R \subseteq \mathbb{Q}^n$, $\overline{A}, \overline{R} \subseteq \mathbb{Q}^n$, and with associated polyhedra
\begin{equation}
P := \{x \in \mathbb{R}^n \mid Ax \leq 1, Rx \leq 0\},
\end{equation}
\begin{equation}
Q := \{x \in \mathbb{R}^n_+ \mid \overline{A}x \leq 1, \overline{R}x \leq 0\}.
\end{equation}
As in the previous section, for $J \subseteq \{1, \ldots, n\}$, $(A^J, R^J)$ and $(\overline{A}^J, \overline{R}^J)$ denotes the matrices obtained from $(A, R)$ and $(\overline{A}, \overline{R})$ by contraction and negative deletion of $J$, respectively.
Moreover, the set of indices of the remaining rows of $\bar{A}$ and $\bar{R}$ is called $I(J)$ and $K(J)$, the set of the deleted rows $\bar{T}(J)$ and $\bar{R}(J)$.

\[
A = \begin{bmatrix} A_J \\ \end{bmatrix} \quad \quad \quad R = \begin{bmatrix} R_J \end{bmatrix}
\]

contraction of coordinates $\bar{J}$

\[
\begin{array}{c|c|c|c|c}
I(0) & A_J & \geq 0 & J & \hline
\bar{I}(0) & \bar{A}_J & \leq 0 & \bar{J} & \hline
K(0) & R_J & \geq 0 & J & \hline
\bar{K}(0) & \bar{R}_J & \leq 0 & \bar{J} & \end{array}
\]

negative deletion of coordinates $\bar{J}$

Figure 4.3

The polyhedra associated with the pair $((A^J,R^J),(\bar{A}^J,\bar{R}^J))$ are then given by

\[
\begin{align*}
P^J & := \{ x^J \in \mathbb{R}^J \mid A^J x^J \leq 1, R^J x^J \leq 0 \}, \\
Q^J & := \{ x^J \in \mathbb{R}_+^J \mid \bar{A}^J x^J \leq 1, \bar{R}^J x^J \leq 0 \}.
\end{align*}
\]

Integrality property

In this paragraph we characterize pairs $((A,R),(\bar{A},\bar{R}))$ having the integrality property. In fact, we establish a more general result: Let $((A,R),(\bar{A},\bar{R}))$ with $A, R \subseteq \mathbb{Q}^n$, $\bar{A}, \bar{R} \subseteq \mathbb{Q}^n$ be given such that the associated polyhedra $P$ and $Q$ of (22) form a pair of polar polyhedra, we derive a necessary and sufficient condition for all pairs $(P^J,Q^J)$ of (23) to be polar pairs.

**Theorem 4.3**

Let $((A,R),(\bar{A},\bar{R}))$ be a pair with $A, R \subseteq \mathbb{Q}^n$, $\bar{A}, \bar{R} \subseteq \mathbb{Q}^n$ such that $(P,Q)$ defined in (22) is a pair of polar polyhedra. For $J \subseteq \{1,\ldots,n\}$ let $P^J$ and $Q^J$ be given by (23). Then

(i) $P^J = \{ x^J \in \mathbb{R}^J \mid (x^J,0^J) \in P \}$ and its polar $(P^J)^* = \{ x^J \in \mathbb{R}^J \mid \exists x^J, x^\bar{J} \in \mathbb{R}^J : (x^J,x^\bar{J}) \in Q \}$.

(ii) $(P^J,Q^J)$ is a pair of polar polyhedra for all $J \subseteq \{1,\ldots,n\}$ if and only if

\[
x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0 \text{ is a homogeneous system.}
\]

**Proof:**

(i) is given by corollary 3.7.

(ii): We show first that if $x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0$ is homogeneous then, $(P^J)^* = Q^J := \{ x^J \in \mathbb{R}_+^J \mid \bar{A}^J x^J \leq 1, \bar{R}^J x^J \leq 0 \}$ for all $J \subseteq \{1,\ldots,n\}$: Let $J \subseteq \{1,\ldots,n\}$ and $I := I(J)$, $K := K(J)$. By theorem 2.20, it is sufficient to show that $P^J = \text{CONV}(\bar{A}^J,0) + \text{CONE}(\bar{R}^J) - \mathbb{R}_+^J$. The inclusion "$\supseteq"$ is trivial. To show "$\subseteq$", let $(x^J,0^J) \in P$ and consider the pair of dual LP's

\[
\begin{align*}
\omega & = \max \{ x^J,0^J \} z \text{ subject to } \bar{A}z \leq 1, \bar{R}z \leq 0, z \geq 0, \\
\min \{ 1\lambda \} \text{ subject to } \lambda \bar{A} + \mu \bar{R} \geq (x^J,0), \lambda, \mu \geq 0.
\end{align*}
\]

As $(x^J,0^J) \in P$, the LP's in (24) have optimum solutions with $\omega \leq 1$. There exists a homoge-
neous optimum solution \((\lambda, \mu)\) of (24), hence \(\lambda^I = 0\) and \(\mu^R = 0\). Then \(\lambda^J A^J + \mu^K R^J \geq x^J, \lambda^I, \mu^K \geq 0, 1\lambda^I \leq 1\) and therefore \(x^J \in \text{CONV}(A^J, 0) + \text{CONE}(R^J) - R^J_+\).

For the other direction of (ii), assume that there exists an example for which \((P^J)^* = Q^J := \{x \in R^J_+ | \bar{A}^J x \leq 1, \bar{R}^J x \leq 0\}\) for all \(J \subseteq \{1, \ldots, n\} =: N\) and \(x \geq 0, \bar{A} x \leq 1, \bar{R} x \leq 0\) is not homogeneous. We choose such an example with minimal \(n\) (clearly \(n \geq 2\)).

For all \(J \subseteq N\), let \(D^J x \leq d(J)\) be the system \(\bar{A}^J x \leq 1, \bar{R}^J x \leq 0\), with \(D^J := [\bar{A}^J \bar{R}^J], d(J) := [1 \ 0], D := DN, d := d(N)\), and consider for \(J \subseteq N\) the dual LP's

\[
(25) \text{DLP}(J,c^J): \quad \min \{d(J)u(J) \mid u(J)D^J \geq c^J, u(J) \leq 0\}
\]

By the choice of the example, the systems \(x \geq 0, D^J x \leq d(J)\) are homogeneous for all \(J \subseteq N, J \neq N\) and there exists a rational vector \(c\), such that DLP\((N,c)\) has no homogeneous optimum solution. Let \(u^*\) be an optimum solution of DLP\((N,c)\), which is maximal with respect to \(u^D\) (i.e. there exists no optimum solution \(u'\) with \(u^D \geq u^D\) and \(u^D \neq u^D\)) and which (among the maximal optimum solutions) has a minimal inhomogeneity vector \(a\).

We will derive a contradiction to our assumption by constructing another optimum solution \(u'\), maximal with respect to \(u^D\) and with inhomogeneity vector smaller than the one of \(u\).

Since \(a \neq 0\), there exists \(t \in N\) with \(a_t > 0\). Hence there exists \(i\) and \(j\) with \(u_i, u_j > 0\) and \(D_{it} > 0, D_{jt} < 0\). Define \(\bar{c} := D_i + (\frac{D_{it}}{D_{jt}}) D_j, J := \mathbb{N}\{t\}\). Notice that \(\bar{c}_t = 0\). Furthermore, DLP\((J,\bar{c})\) has an optimum solution, because \(Q^J = \text{CONV}(A^J, 0) + \text{CONE}(R^J)\) and for any row \(R_e^J\) of \(R^J\), \(\bar{c}R_e^J = \bar{c}R_e = D_i R_e + (\frac{D_{it}}{D_{jt}}) D_j R_e \leq 0\).

\[
\min \{d(J)u(J) \mid u(J)D^J \geq \bar{c}^J, u(J) \geq 0\} = \max \{\bar{c}^J A^J_e, A^J_e, \text{ is a row of } A^J\} \cup \{0\} = \max \{\bar{c}^J A^J_e, A^J_e, \text{ is a row of } A\} \cup \{0\} = \min \{d^J u \mid u^D \geq \bar{c}, u \geq 0\}
\]

\[
\leq d_i + (\frac{D_{it}}{D_{jt}}) d_j.
\]

There exists a homogeneous optimum solution \(u(J) = (y^J, w^J)\) of DLP\((J,\bar{c})\) and \(\Delta u := ((y^J, 0T(J)), (w^J, 0\bar{R}(J)))\) is a homogeneous feasible solution of DLP\((N,\bar{c})\) with \(d\Delta u = d(J)u(J) \leq d_i + (\frac{D_{it}}{D_{jt}}) d_j\). Then, for \(0 < \varepsilon \leq \min \{u_i, \frac{D_{it}}{D_{jt}} u_j\}\),

\[
\bar{u} := u - \varepsilon(e^J + (\frac{D_{it}}{D_{jt}}) e^J) + \varepsilon \Delta u \geq 0 \text{ satisfies } \bar{u}^D \geq u^D \text{ and } d\bar{u} \leq du.
\]

Hence \(\bar{u}\) is an optimum solution of DLP\((N,\bar{c})\) and by the maximality of \(u\), \(\bar{u}^D = u^D\). Because \(\Delta u\) is a homogeneous solution of DLP\((N,\bar{c})\), the inhomogeneity vector \(\bar{a}\) of \(u\) and the inhomogeneity vector \(a\) of \(u\) satisfy

\[
\bar{a}_v < a_v \quad \text{for } v \text{ with } D_{iv} D_{jv} < 0,
\]

\[
\bar{a}_v = a_v \quad \text{otherwise},
\]

as can be seen by straightforward verification, taking into account the definition of \(\bar{u}\) and the equality \(\Delta u^D = \bar{c}\). Since \(D_{it} D_{jt} < 0, \bar{a} \leq a, \bar{a} \neq a\) and \(\bar{u}\) is an optimum solution, maximal
with respect to \( \mathbf{u} \mathbf{D} \), with a strictly smaller inhomogeneity vector than \( \mathbf{u} \), a contradiction to our assumption.

Theorem 4.3 yields the following characterization of the integrality property:

**Corollary 4.4**

Let \( ((\mathbf{A}, \mathbf{R}), (\overline{\mathbf{A}}, \overline{\mathbf{R}})) \) be a pair with \( \{0,1\} \)-matrices \( \mathbf{A}, \mathbf{R} \) and \( \{1,0,\ldots,k\} \)-matrices \( \overline{\mathbf{A}}, \overline{\mathbf{R}} \). Then (i) and (ii) are equivalent:

(i) \( ((\mathbf{A}, \mathbf{R}), (\overline{\mathbf{A}}, \overline{\mathbf{R}})) \) has the integrality property;

(ii) \( (\mathbf{P}, \mathbf{Q}) \) defined in (22) is a pair of polar polyhedra and

\[
\mathbf{x} \geq 0, \quad \overline{\mathbf{A}} \mathbf{x} \leq 1, \quad \overline{\mathbf{R}} \mathbf{x} \leq \mathbf{0}
\]

is a homogeneous system.

Observe that by theorem 4.3 the property of homogeneity is hereditary:

**Corollary 4.5**

Let \( \overline{\mathbf{A}}, \overline{\mathbf{R}} \subseteq \mathbb{Q}^n \) be matrices. If \( \mathbf{x} \geq 0, \mathbf{A} \mathbf{x} \leq 1, \mathbf{R} \mathbf{x} \leq \mathbf{0} \) is a homogeneous system, then \( \mathbf{x} \geq 0, \mathbf{A}^j \mathbf{x} \leq 1, \mathbf{R}^j \mathbf{x} \leq \mathbf{0} \) is a homogeneous system for any \( j \subseteq \{1, \ldots, n\} \).

**Proof:**

If \( \mathbf{x} \geq 0, \mathbf{A} \mathbf{x} \leq 1, \mathbf{R} \mathbf{x} \leq \mathbf{0} \) is a homogeneous system, then by theorem 4.3 (ii), \( (\mathbf{P}^j, \mathbf{Q}^j) \) is a pair of polar polyhedra for all \( j \subseteq \{1, \ldots, n\} \) and especially for all \( j' \subset j \) for a given \( j \subseteq \{1, \ldots, n\} \). Hence by the other direction of theorem 4.3 (ii), the system \( \mathbf{x} \geq 0, \mathbf{A}^j \mathbf{x} \leq 1, \mathbf{R}^j \mathbf{x} \leq \mathbf{0} \) is a homogeneous system.

Another consequence of theorem 4.3 is that, for any pair of polar polyhedra \( (\mathbf{P}, \mathbf{Q}) \) with \( \mathbf{P} \) defined as in (22) and \( \mathbf{A}, \mathbf{R} \subseteq \mathbb{Q}^n \), a homogeneous system for \( \mathbf{Q} \) can be constructed:

**Corollary 4.6**

Let \( (\mathbf{P}, \mathbf{Q}) \) be a pair of polar polyhedra with \( \mathbf{P} := \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{A} \mathbf{x} \leq 1, \mathbf{R} \mathbf{x} \leq \mathbf{0} \} \), \( \mathbf{A}, \mathbf{R} \subseteq \mathbb{Q}^n \). For \( j \subseteq \{1, \ldots, n\} \) := \( \mathbb{N} \), let \( \mathbf{P}^j := \{ \mathbf{x}^j \in \mathbb{R}^j \mid (\mathbf{x}^j, \mathbf{0}^j) \in \mathbf{P} \} = \text{CONV}(\mathbf{B}(j)) + \text{CONE}(\mathbf{S}(j)) - \mathbb{R}^j \),

with \( \mathbf{B}(j) \subseteq \mathbb{Q}^{(j) \times n} \), \( \mathbf{S}(j) \subseteq \mathbb{Q}^{M(j) \times n} \). Denote by \( \mathbf{A} \) the matrix, consisting of the rows \( \{ (\mathbf{B}(j)e_j, \mathbf{0}^j) \mid j \subseteq \{1, \ldots, n\}, \mathbf{e} \in \mathbb{L}(j) \} \) and by \( \mathbf{R} \) the matrix, consisting of the rows \( \{ (\mathbf{S}(j)e_j, \mathbf{0}^j) \mid j \subseteq \{1, \ldots, n\}, \mathbf{e} \in \mathbb{M}(j) \} \).

Then \( \mathbf{Q} = \{ \mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{A} \mathbf{x} \leq 1, \mathbf{R} \mathbf{x} \leq \mathbf{0} \} \) and \( \mathbf{x} \geq 0, \mathbf{A} \mathbf{x} \leq 1, \mathbf{R} \mathbf{x} \leq \mathbf{0} \) is a homogeneous system.
Proof:
For any J and any vector B(J)e., e∈L(J) and S(J)f., f∈M(J) we have B(J)e,e∈PJ and λS(J)f. e∈PJ for all λ ≥ 0. Hence (B(J)e.,0J)∈P, λ(S(J)f.,0J)∈P for all λ ≥ 0, and by polarity
Qc{x∈R^n | Ax ≤ 1, Rx ≤ 0} ⊆ {x∈R^n | B(N)x ≤ 1, S(N)x ≤ 0} = Q.
Moreover, by ĀJ=ōB(J), ṚJ=ōS(J) and PJ=CONV(ĀJ) + CONE(ṚJ) - ṚJ,
(PJ)* ⊆ {x∈R^n | ĀJx ≤ 1, ṚJx ≤ 0} ⊆ {x∈R^n | B(J)x ≤ 1, S(J)x ≤ 0} = (PJ)*
for all J⊂N. Hence (PJ)* = QJ := {x∈R^n | ĀJx ≤ 1, ṚJx ≤ 0} for all J⊂N, which implies by theorem 4.3 (ii) that x ≥ 0, Āx ≤ 1, Ṛx ≤ 0 is a homogeneous system.

Given {0,1}-matrices (A,R) and the polyhedron P defined in (22), Corollary 4.6 says that the integrality property is in some sense a property of (A,R) and P:

**Corollary 4.7**
Let A and R be {0,1}-matrices and P be defined as in (22). If for any J⊂{1,...,n} there exists {1,0,-k}-matrices B(J), S(J) with PJ = {xJ∈R^n | (xJ,0J)∈P} = CONV(B(J)) + CONE(S(J)) - ṚJ, then for Ā and Ṛ defined as in corollary 4.6, ((A,R),(Ā,Ṛ)) has the integrality property.

Proof:
ā and Ṛ are {1,0,-k}-matrices and P = CONV(Ā) + CONE(Ṛ) - Ṛ. By definition of P, 0∈P and therefore for Q := {x∈R^n | Āx ≤ 1, Ṛx ≤ 0} = CONV(A,0), (P,Q) is a pair of polar polyhedra (theorem 2.20). It follows from corollary 4.4 and the fact that x ≥ 0, Āx ≤ 1, Ṛx ≤ 0 is homogeneous that ((A,R),(Ā,Ṛ)) has the integrality property.

**A-perfectness**
In this paragraph we characterize A-perfect pairs. To achieve this goal we consider a pair ((A,R),(Ā,Ṛ)) with A, R⊂Q^n, Ā, Ṛ⊂Q^n, such that (P,Q) defined in (22) is a pair of polar polyhedra and investigate under which conditions A^Jx ≤ 1, Ṛ^Jx ≤ 0 inherits tdi-ness from Ax ≤ 1, Ṛx ≤ 0.

As we will see, tdi-ness of Ax ≤ 1, Ṛx ≤ 0 is not hereditary in general. However in some special cases, like in the following one, it is.
Theorem 4.8

Let \((A, R), (\overline{A}, \overline{R})\) be a pair such that \(A, R \subseteq \mathbb{Q}^n\), \(\overline{A}, \overline{R}\) are \(\{1,0,-k\}\)-matrices, and \((P,Q)\) defined as in (22) is a pair of polar polyhedra. If \(x \geq 0, \overline{A}x \leq 1, \overline{R}x \leq 0\) is a homogeneous system, then \(Ax \leq 1, Rx \leq 0\) tdi implies \(\overline{A}Jx \leq 1, \overline{R}Jx \leq 0\) tdi for all \(J \subseteq \{1,\ldots,n\}\).

Proof:

Let \(j \in \{1,\ldots,n\} =: N\) and \(J := N \setminus \{j\}\). Notice that by theorem 4.3 and corollary 4.5 it is enough to show the assertion for such a \(J\).

Let \(c^J \geq 0\) be an integral vector such that

\[
\min \ 1y \quad \text{subject to} \quad yA^J + wR^J = c^J, \ y, w \geq 0
\]

has an optimum solution and let \(\omega\) be its optimum value. We show that there exists \(z \in \mathbb{N}\) such that for \(c := (c^J, z)\), \(\min \ 1y \quad \text{subject to} \quad yA + wR = c; \ y, w \geq 0\) has a solution of value \(\omega\).

By theorem 4.3, \((P^I, Q^I)\) of (23) is a pair of polar polyhedra. Therefore \(P^I = \text{CONV}(\overline{A}^I, 0) + \text{CONE}(\overline{R}^I)\) (theorem 2.20), \(\omega = \max \{c^Ix \mid x \in P^I\} = \max \{\{0\} \cup \{c^J\overline{A}^I_i \mid i \in I(J)\}\}\) and \(c^J\overline{R}^I_i \leq 0\) for all \(i \in K(J)\). Hence \(\omega\) is integral, \(c^J_\omega \in Q^I\), and there exists \(x^I\), such that \((c^J_\omega, x^I) \in Q\). \(x^I\) is a solution of the system

\[
(27) \quad \overline{A}^Iz \leq 1 - \overline{A}^Ic^J_\omega, \ \overline{R}^Iz \leq - \overline{R}^Ic^J_\omega, \ z \geq 0.
\]

Conversely, for any solution \(z\) of (27), \((c^J_\omega, z)\) \(\in Q\). Let \(M\) be a large integer positive number and \(u := \min \{\{M\} \cup \{1 - \overline{A}^Ic^J_\omega \mid \overline{A}ii = 1\} \cup \{-\overline{R}^Ic^J_\omega \mid \overline{R}ii = 1\}\}\), then \((c^J_\omega, u) \in Q\). Hence for \(c := (c^J, u\omega)\), \(\max \{cx \mid x \in P\} = \max \{cx \mid x \in \text{CONV}(\overline{A}, 0) + \text{CONE}(\overline{R}) - \overline{R}^I\} =: \omega_1 \leq \omega\).

Since \(c\) is integral \((\overline{A}, \overline{R} \subseteq \mathbb{Z}^n)\), there exists an integer optimum solution \((y^*, w^*)\) of

\[
(28) \quad \min \ 1y \quad \text{subject to} \quad yA + wR = c; \ y, w \geq 0
\]

with value \(\omega_1 \leq \omega\). Then \((y^*, w^*)\) is an integer optimum solution of (26).

On one hand the theorem remains valid if the condition "\(\overline{A}, \overline{R}\) are \(\{1,0,-k\}\)-matrices" is replaced by "\(\overline{A}, \overline{R}\) are \(\{\ldots, 2,1,0,-1\}\)-matrices" (the proof is analogue to that of theorem 4.8). On the other hand the condition "\(\overline{A}, \overline{R}\) are \(\{1,0,-k\}\)-matrices" is tight in the sense that even its relaxation to "\(\overline{A}, \overline{R}\) are \(\{2,1,0,-1,-2\}\)-matrices" is not allowed:

Proposition 4.9

Consider pairs \(((A,R),(\overline{A},\overline{R}))\) with \(\{0,1\}\)-matrices \(A, R, \{2,1,0,-1,-2\}\)-matrices \(\overline{A}, \overline{R}, \) and such that \((P,Q)\) defined in (22) is a polar pair and \(x \geq 0, \overline{A}x \leq 1, \overline{R}x \leq 0\) is homogeneous. Then it is false that \(Ax \leq 1, Rx \leq 0\) tdi implies \(\overline{A}Jx \leq 1, \overline{R}Jx \leq 0\) tdi for \(J \subseteq \{1,\ldots,n\}\).
Theorem 4.8 gives the following characterization of $A$-perfectness:

**Corollary 4.10**

Let $((A,R),(\overline{A},\overline{R}))$ be a pair with $\{0,1\}$-matrices $A$, $R$ and $\{1,0,-k\}$-matrices $\overline{A}$, $\overline{R}$. Then (i), (ii) and (iii) are equivalent.

(i) $((A,R),(\overline{A},\overline{R}))$ is $A$-perfect;

(ii) $(P,Q)$ defined in (22) is a pair of polar polyhedra,

\[ Ax \leq 1, Rx \leq 0 \text{ is tdi, and} \]

\[ x \geq 0, \overline{A}x \leq 1, \overline{R}x \leq 0 \text{ is homogeneous.} \]

(iii) $((A,R),(\overline{A},\overline{R}))$ has the integrality property and

\[ Ax \leq 1, Rx \leq 0 \text{ is tdi.} \]

**Proof:**

It follows from corollary 4.4 that (ii) $\iff$ (iii), and (i) $\implies$ (ii) (since $A$-perfectness implies the integrality property and herewith homogeneity of $x \geq 0, \overline{A}x \leq 1, \overline{R}x \leq 0$). Finally, (ii) $\implies$ (i) follows from theorem 4.8 and corollary 4.4.

By the above equivalence (i) $\iff$ (iii), $A$-perfectness can be seen as a property of $(A,R)$ and $P := \{x \in \mathbb{R}^n \mid Ax \leq 1, Rx \leq 0\}$, in the same way as the integrality property (see corollary 4.7).

### $\overline{A}$-perfectness and $(A,\overline{A})$-perfectness

In this paragraph we derive sufficient conditions for $((A,R),(\overline{A},\overline{R}))$ to be $\overline{A}$-perfect respectively $(A,\overline{A})$-perfect. For $\{1,0,-1\}$-matrices $(\overline{A},\overline{R})$, these conditions will be seen to also be necessary, providing a characterization of $\overline{A}$-perfectness respectively $(A,\overline{A})$-perfectness.

Let $((A,R),(\overline{A},\overline{R}))$ be a pair with $A, R \subseteq \mathbb{Q}^n$, $\overline{A}, \overline{R} \subseteq \mathbb{Q}^n$, such that $(P,Q)$ defined in (22) is a pair of polar polyhedra. We investigate if the system $x \geq 0, \overline{A}^Jx \leq 1, \overline{R}^Jx \leq 0$ inherits tdi-ness from $x \geq 0, \overline{A}x \leq 1, \overline{R}x \leq 0$. In the context of $\overline{A}$-perfectness, this question is in interest only if $x \geq 0, \overline{A}Jx \leq 1, \overline{R}Jx \leq 0$ is a describing system of $(P,J)^* = \{x \in \mathbb{R}^n \mid A^Jx \leq 1, R^Jx \leq 0\}^*$. Hence by theorem 4.3 (ii) we assume that the system $x \geq 0, \overline{A}x \leq 1, \overline{R}x \leq 0$ is homogeneous.
We already saw in corollary 4.5, that the homogeneity property is hereditary. The next proposition states the heredity of homogeneous-tdi-ness.

Proposition 4.11

For $A, R \subseteq \mathbb{Q}^n$, let $x \geq 0, \overline{A}x \leq 1, \overline{R}x \leq 0$ be a homogeneous-tdi system.

Then the system $x \geq 0, \overline{A}Jx \leq 1, \overline{R}Jx \leq 0$ is homogeneous-tdi for any $J \subseteq \{1, \ldots, n\}$.

Proof:

Let $J \subseteq \{1, \ldots, n\}$, $I := I(J)$ and $K := K(J)$. Then $x \geq 0, \overline{A}Jx \leq 1, \overline{R}Jx \leq 0$ is a homogeneous system (corollary 4.5). Consider the dual LP

$$
\text{min } 1y \text{ subject to } y^\top \overline{A}J + w^\top \overline{R}J - u^\top J = c^\top J, \quad y^\top, w^\top, u^\top J \geq 0.
$$

Let $c^\top J$ be an integral vector, such that (29) has an optimum solution. For $c := (c^\top J, 0)$, (30) has an optimum solution too.

$$
\text{min } 1y \text{ subject to } y^\top \overline{A} + w^\top \overline{R} - u = c, \quad y^\top, w^\top, u \geq 0.
$$

There exists an integer homogeneous optimum solution $((y^\top, w^\top), (w^K, w^\overline{K}), u)$ of (30). Hence $y^\top = 0$, $w^\overline{K} = 0$ and $(y^\top, w^K, u^\top)$ is an integer homogeneous optimum solution of (29).

An $\overline{A}$-perfect pair $((A, R), (\overline{A}, \overline{R}))$ is a pair, which by definition has the integrality property and for which all systems $S(J): x \geq 0, \overline{A}Jx \leq 1, \overline{R}Jx \leq 0, J \subseteq \{1, \ldots, n\}$ are tdi. Hence, by corollary 4.4, a necessary and sufficient condition for a pair to be $\overline{A}$-perfect is that all systems $S(J)$ are homogeneous and tdi. Proposition 4.11 provides therefore a sufficient condition for $\overline{A}$-perfectness, namely homogeneous-tdi-ness of $x \geq 0, \overline{A}x \leq 1, \overline{R}x \leq 0$. We investigate, if it is also a necessary condition for all systems $S(J)$ to be homogeneous and tdi. In some cases, namely if $\overline{A}, \overline{R}$ are $\{1,0,-1\}$-matrices, the answer is affirmative:

Theorem 4.12

Let $\overline{A}, \overline{R}$ be $\{1,0,-1\}$-matrices. Then the system $x \geq 0, \overline{A}x \leq 1, \overline{R}x \leq 0$ is homogeneous-tdi iff for all $J \subseteq \{1, \ldots, n\}$ the systems $x \geq 0, \overline{A}Jx \leq 1, \overline{R}Jx \leq 0$ are homogeneous and tdi.

Proof:

One direction is given by proposition 4.11. For the other direction the proof is similar to the one of theorem 4.3 (ii):

Assume that there exists an example, for which $x \geq 0, \overline{A}Jx \leq 1, \overline{R}Jx \leq 0$ is homogeneous and tdi for all $J \subseteq \{1, \ldots, n\} =: N$, and $x \geq 0, \overline{A}x \leq 1, \overline{R}x \leq 0$ is not homogeneous-tdi. We
choose such an example with minimal \( n \) (clearly \( n \geq 2 \)) and let \( Q := \{ x \in \mathbb{R}_+^n \mid A x \leq 1, A_0 x \leq 0 \} = \text{CONV}(A,0) + \text{CONE}(R) \) for some matrices \( A, R \).

For all \( J \subseteq N \), let \( D^J x \leq d(J) \) be the system \( A^J x \leq 1, R^J x \leq 0 \), with \( D^J := \begin{bmatrix} A^J \\ R^J \end{bmatrix} \), \( d(J) := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), \( D := D^N, d := d(N) \), and consider for \( J \subseteq N \) the dual LP's

\[
(31) \quad \text{DLP}(J,c^J): \min d(J)v(J) \text{ subject to } v(J)D^J - u(J) = c^J, v(J), u(J) \geq 0.
\]

By the choice of the example, the systems \( x \geq 0, D^J x \leq d(J) \) are homogeneous-tdi for all \( J \subseteq N \), \( J \neq N \) and there exists an integral vector \( c \), such that DLP(N,c) has no integral homogeneous optimum solution. Let \( (v,u) \) be an integer optimum solution of DLP(N,c), which is maximal with respect to \( vD \) (i.e. there exists no integral optimum solution \( (v',u') \) with \( v'D \geq vD \) and \( v'D \neq vD \)) and which (among the maximal integer optimum solutions) has a minimal inhomogeneity vector \( a \).

We derive a contradiction to our assumption by constructing another integer optimum solution \( (v',u') \), maximal with respect to \( vD \) and with smaller inhomogeneity vector as \( (v,u) \).

Since \( a \neq 0 \), there exists \( t \in N \) with \( a_t > 0 \) and therefore \( i \) and \( j \) with \( v_i, v_j \geq 1 \) and \( D^t_i = 1, D^t_j = -1 \). Define \( \overline{c} := D^t_i + D^t_j \) and \( J := N \setminus \{ t \} \), then \( \overline{c}_i = 0 \). Furthermore, DLP(J,\( c^J \)) has an optimum solution, because \( Q^J = \text{CONV}(A^J,0) + \text{CONE}(R^J) \) (theorem 2.20 and theorem 4.3) and for any row \( R^J_e \) of \( R^J \), \( c^J R^J_e = c^J R^J_e = \overline{c} R^J_e = D^t_i R^J_e + D^t_j R^J_e \leq 0 \). Then

\[
\min \{ d(J)v(J) \mid v(J)D^J - u(J) = c^J, v(J), u(J) \geq 0 \} = \max \{ \{ \overline{c}^J A^J_e \mid A^J_e \text{ is a row of } A^J \} \cup \{0\} \} = \max \{ \{ \overline{c} A^J_e \mid A^J_e \text{ is a row of } A \} \cup \{0\} \} = \min \{ d v \mid vD - u = \overline{c}, v, u \geq 0 \} \leq d_i + d_j.
\]

There exists an integer homogeneous optimum solution \( (v(J),u(J)) = ((y^J,w^J),u(J)) \) of DLP(J,\( c^J \)), and for \( \Delta v := ((y^J,0\overline{c}(J)),(w^J,0\overline{c}(J))) \), \( (\Delta v,\Delta vD-c) \) is an integer homogeneous feasible solution of DLP(N,\( c \)) with \( d \Delta v = d(J)v(J) \leq d_i + d_j \). Then, for

\[
\overline{v} := v - (e^i + e^j) + \Delta v \geq 0, \text{ we have } \overline{v}D \geq vD \text{ and } d \overline{v} \leq d v.
\]

Hence \( (\overline{v},\overline{u}) := (\overline{v},vd - c) \) is an integer optimum solution of DLP(N,c) and by the maximality of \( (v,u) \), \( \overline{v}D = vD \). Because \( (\Delta v,0) \) is a homogeneous solution of DLP(N,\( c \)), the inhomogeneity vector \( \overline{a} \) of \( (\overline{v},\overline{u}) \) and the inhomogeneity vector \( a \) of \( (v,u) \) satisfy

\[
\overline{a}_v < a_v \quad \text{for } v \text{ with } D^v_i D^v_j = -1, \quad \overline{a}_v = a_v \quad \text{otherwise}.
\]

As \( D^t_i D^t_j = -1, \overline{a} \leq a, \overline{a} \neq a \) and \( (\overline{v},\overline{u}) \) is an integer optimum solution, maximal with respect to \( \overline{v}D \) with a strictly smaller inhomogeneity vector than \( (v,u) \), a contradiction to our assumption.

The condition "\( A, R \) are \( \{1,0,-1\}\)-matrices" of theorem 4.12 can not be relaxed to "\( \overline{A}, \overline{R} \),
\[ \mathbb{R} \subseteq \mathbb{Z}^n \]. In example 2 of section 4.3, \( \bar{A}, \bar{R} \) are \( \{2,1,0,-1,\ldots,-4\} \)-matrices and every system \( x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0, J \subseteq \{1,\ldots,n\} \) is homogeneous and tdi, but \( x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0 \) is not homogeneous-tdi.

This example gives also a first negative answer to the question: does homogeneity and tdi-ness of a system \( x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0 \) imply homogeneous-tdi-ness of it? Even if we restrict the matrices \( \bar{A} \) and \( \bar{R} \) to be \( \{1,0,-1\} \)-matrices this implication is not true:

**Proposition 4.13:**
Consider \( \{1,0,-1\} \)-matrices \( \bar{A}, \bar{R} \). Then it is false that \( x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0 \) homogeneous and tdi implies \( x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0 \) homogeneous-tdi.

**Proof:** Example 3 of section 4.3.

The following question is still open: Can "\( \bar{A}, \bar{R} \) are \( \{1,0,-1\} \)-matrices" in theorem 4.12 be replaced by "\( \bar{A}, \bar{R} \) are \( \{1,0,-k\} \)-matrices"?

Proposition 4.11 gives a sufficient condition for \( ((A,R),(\bar{A},\bar{R})) \) to be \( \bar{A} \)-perfect respectively \( (A,\bar{A}) \)-perfect and it is summarized in corollary 4.14. Moreover, if the matrices \( (\bar{A},\bar{R}) \) are \( \{1,0,-1\} \)-matrices, it is also a necessary condition and the resulting characterization is given in corollary 4.15.

**Corollary 4.14**
Let \( ((A,R),(\bar{A},\bar{R})) \) be a pair with \( \{0,1\} \)-matrices \( A, R \) and \( \{1,0,-k\} \)-matrices \( \bar{A}, \bar{R} \). Consider the following properties on \( ((A,R),(\bar{A},\bar{R})) \):

(i) \((P,Q)\) of (22) is a pair of polar polyhedra;

(ii) The system \( x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0 \) is homogeneous-tdi;

(iii) The system \( Ax \leq 1, Rx \leq 0 \) is tdi.

Then

(iv) \( ((A,R),(\bar{A},\bar{R})) \) is \( \bar{A} \)-perfect if it satisfies (i) and (ii);

(v) \( ((A,R),(\bar{A},\bar{R})) \) is \( (A,\bar{A}) \)-perfect if it satisfies (i), (ii) and (iii).

**Corollary 4.15**
Let \( ((A,R),(\bar{A},\bar{R})) \) be a pair with \( \{0,1\} \)-matrices \( A, R \), \( \{1,0,-1\} \)-matrices \( \bar{A}, \bar{R} \), and consider (i) to (iii) as in corollary 4.14. Then

(iv) \( ((A,R),(\bar{A},\bar{R})) \) is \( \bar{A} \)-perfect if and only if it satisfies (i) and (ii);

(v) \( ((A,R),(\bar{A},\bar{R})) \) is \( (A,\bar{A}) \)-perfect if and only if it satisfies (i), (ii), and (iii).
Relations between properties

Let \( ((A,R),(\bar{A},\bar{R})) \) be a pair with \( A, R \subseteq \mathbb{Q}^n \), \( \bar{A}, \bar{R} \subseteq \mathbb{Q}^n \) and \( P, Q \) the associated polyhedra of (22). We investigate whether or not there exist relations (not directly related to perfectness) between properties of \( P, Q \), the system \( Ax \leq 1, Rx \leq 0 \) of \( P \), and \( x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0 \) of \( Q \).

We give first a positive result in this context, namely that for a special case tdi-ness of \( x \geq 0, \bar{A}x \leq 1 \) implies homogeneity of it.

**Proposition 4.16**

Let \( ((A,R),(\bar{A},0)) \) be a pair with \( A, R \subseteq \mathbb{Q}^n \), \( \bar{A} \subseteq \mathbb{Q}^n \), and such that \( (P,Q) \) of (22) is a polar pair. If \( P^I := \{ x \in \mathbb{R}^I \mid A^Jx \leq 1, R^Jx \leq 0 \} \) is integral for all \( J \subseteq \{1,\ldots,n\} \), then \( x \geq 0, \bar{A}x \leq 1 \) tdi implies \( x \geq 0, \bar{A}x \leq 1 \) homogeneous.

**Proof:**

We show that \( P^I = \{ x^I \in \mathbb{R}^I \mid (x^I,0^I) \in P \} = \text{CONV}(\bar{A}^I,0) - R^I_+ \) for all \( J \subseteq \{1,\ldots,n\} \). Then, by theorem 2.20, \( (P^I)^* = Q^J := \{ x \in \mathbb{R}^J \mid \bar{A}^Jx \leq 1 \} \) for all \( J \subseteq \{1,\ldots,n\} \) implying \( x \geq 0, \bar{A}x \leq 1 \) is homogeneous (theorem 4.3). Let \( J \subseteq \{1,\ldots,n\} \) and define \( I := I(J) \).

(i) \( P^I \subseteq \text{CONV}(\bar{A}^J,0) - R^I_+ \): Let \( x^I = \lambda \bar{A}^J - d \in \text{CONV}(\bar{A}^J,0) - R^I_+ \) with \( \lambda, d \geq 0, 1\lambda \leq 1 \). Then \( x := (x^I,0^J) = (\lambda,0^J)\bar{A} - (d,\bar{A}^J) \in \text{CONV}(\bar{A},0) - R^I_+ = P \). Hence \( x \in P \) and \( x^I \in P^I \).

(ii) \( P^I \subseteq \text{CONV}(\bar{A}^J,0) - R^I_+ \): By assumption \( P^I = \{ x^I \in \mathbb{R}^I \mid (x^I,0^J) \in \text{CONV}(\bar{A},0) - R^I_+ \} = \text{CONV}(C,0) + \text{CONE}(D) - R^I_+ \), with \( C, D \subseteq \mathbb{Z}^J \).

For any row \( d \) of \( D \), \( \mu d \in P^I \) for all \( \mu \geq 0 \) and \( \mu(d,0^J) \in P = \text{CONV}(\bar{A},0) - R^I_+ \). Hence \( d \leq 0 \) and \( P^I = \text{CONV}(C,0) - R^I_+ \).

To show (ii) it is sufficient to show that for any row \( c^J \) of \( C \), \( c^J \in \text{CONV}(\bar{A}^J,0) - R^I_+ \). Let \( c^J \) be a row of \( C \), then \( c := (c^J,0^J) \in P \) and by polarity of \( P \) and \( Q \), \( 1 \geq \max \{ cx \mid \bar{A}x \leq 1, x \geq 0 \} \). Hence,

\[ 1 \geq \min 1\lambda \text{ subject to } \lambda \bar{A} - d = c, \ d, \lambda \geq 0 \]

and, since \( c \) is integer, there exists an integer optimum solution \( (\lambda,d) \) with \( 1\lambda \leq 1 \) integer.

Case 1: \( 1\lambda = 0 \), then \( c \leq 0 \), especially \( c^J \leq 0 \) and \( c^J \in \text{CONV}(\bar{A}^J,0) - R^I_+ \).

Case 2: \( 1\lambda = 1 \), then there exists exactly one \( i \) with \( \lambda_i = 1 \). Then \( 0^J = (\lambda \bar{A} - d)^J \leq \bar{A}^J_i \), hence \( i \in I, c^J \leq \bar{A}^J_i \) and \( c^J \in \text{CONV}(\bar{A}^J,0) - R^I_+ \).
This result is not true if $R \neq 0$, and in general there is no relation between homogeneity and tdi-ness of $x \geq 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$:

(32) Let $((A,0),(\bar{A},\bar{R}))$ be a pair with $\{0,1\}$-matrix $A$, $\{1,0,-1\}$-matrices $\bar{A}$, $\bar{R}$, and such that $(P,Q)$ of (22) is a polar pair of polyhedra. For all $J \subseteq \{1,\ldots,n\}$, $P_J := \{x \in \mathbb{R}^J \mid A^Jx \leq 1\} = \text{CONV}(B(J)) + \text{CONE}(S(J))$, with $\{1,0,-1\}$-matrices $B(J)$, $S(J)$. Then $x \geq 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$ tdi $\not\Rightarrow$ $x \geq 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$ homogeneous.

(example 5 of section 4.3)

(33) Let $\bar{A}$, $\bar{R}$ be $\{1,0,-1\}$-matrices. Then $x \geq 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$ homogeneous $\not\Rightarrow$ $x \geq 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$ tdi.

(example 4 of section 4.3)

In corollary 4.7 we saw that the integrality property could be defined on $(A,R)$ and its associated polyhedra $P$ of (22) in the following sense:

If for $J \subseteq \{1,\ldots,n\}$, $P_J := \{x \in \mathbb{R}^J \mid A^Jx \leq 1, R^Jx \leq 0\} = \text{CONV}(B(J)) + \text{CONE}(S(J)) - R*$ with $\{1,0,-k\}$-matrices $B(J)$ and $S(J)$, then there exists $(\bar{A},\bar{R})$ such that $((A,R),(\bar{A},\bar{R}))$ has the integrality property.

The following fact shows that it is not possible to define in the same way the integrality property on $(\bar{A},\bar{R})$ via properties of points and directions generating $Q_J := \{x \in \mathbb{R}^J \mid \bar{A}^Jx \leq 1\}:

(34) Let $\bar{A}$ be a $\{1,0,-1\}$-matrix and $\bar{R} = 0$. Then for all $J \subseteq \{1,\ldots,n\}$, $Q_J := \{x \in \mathbb{R}_+^J \mid \bar{A}^Jx \leq 1\} = \text{CONV}(C(J))$ for a $\{0,1\}$-matrix $C(J)$ $\not\Rightarrow$ $x \geq 0$, $\bar{A}x \leq 1$ homogeneous.

(example 6 of section 4.3).

To conclude this section let us mention that for a pair $((A,R),(\bar{A},\bar{R}))$ with $A$, $R$ $\{0,1\}$-matrices, $\bar{A}$, $\bar{R}$ $\{1,0,-k\}$-matrices and corresponding systems $Ax \leq 1$, $Rx \leq 0$ and $x \geq 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$, tdi-ness of of one system does not imply tdi-ness of the other one.

(35) Let $((A,R),(\bar{A},\bar{R}))$ be a pair with $\{0,1\}$-matrices $A$, $R$, $\{1,0,-1\}$-matrices $\bar{A}$, $\bar{R}$, and such that $(P,Q)$ of (22) is a polar pair. Then $x \geq 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$ homogeneous-tdi $\not\Rightarrow$ $Ax \leq 1$, $Rx \leq 0$ tdi.

(example 7 of section 4.3)

(36) Let $((A,R),(\bar{A},\bar{R}))$ be a pair with $\{0,1\}$-matrices $A$, $R$, $\{1,0,-1\}$-matrices $\bar{A}$, $\bar{R}$, and such that $(P,Q)$ of (22) is a polar pair and $x \geq 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$ is homogeneous. Then $Ax \leq 1$, $Rx \leq 0$ tdi $\not\Rightarrow$ $x \geq 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$ tdi.

(example 4 of section 4.3)
4.3 Examples

In this section all examples used in section 4.2 are collected. For pairs \( ((A,R),(\overline{A},\overline{R})) \), the matrices \( A^J, R^J, A^I, R^I, J \subseteq \{1, \ldots, n\} \) are defined as above.

Example 1:

\( ((A,0),(\overline{A},\overline{R})) \) is a pair with a \( \{0,1\} \)-matrix \( A \) and \( \{2,1,0,-1,-2\} \)-matrices \( \overline{A}, \overline{R} \). Moreover, \((P,Q)\) of (22) is a polar pair, \( x \geq 0, \overline{A}x \leq 1, \overline{R}x \leq 0 \) is homogeneous and \( Ax \leq 1 \) is tdi. There exists \( J \subseteq \{1, \ldots, n\} \) for which the system \( A^Jx \leq 1 \) is not tdi:

\[
A := \begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

\[
\overline{A} := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & 0 \\
0 & -1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & -2 \\
0 & 0 & 1 & 1 & -1 \\
0 & 1 & 0 & 1 & -1 \\
0 & 1 & 1 & 0 & -1 \\
1 & -1 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & 1 \\
1 & 0 & 0 & -1 & 1
\end{bmatrix}
\]

\[
\overline{R} := \begin{bmatrix}
1 & -1 & -1 & -1 & 2 \\
-1 & 1 & 1 & 1 & -2 \\
-1 & 1 & 1 & -1 & 0 \\
-1 & -1 & 1 & 1 & 0 \\
-1 & 1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 1 & 0 \\
-1 & 0 & 1 & 0 & 0 \\
-1 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & -1 \\
-1 & 0 & 1 & 1 & -1 \\
-1 & 1 & 0 & 1 & -1 \\
-1 & 1 & 1 & 1 & 0
\end{bmatrix}
\]

(1) \( Ax \leq 1 \) is a tdi system:

Proof: Let \( c \) be an integral vector, such that

\[
\min \ y \ \text{subject to} \ \ yA = c, \ y \geq 0
\]

has an optimum solution. Then \( (c_4 - c_5, c_3 - c_5, c_2 - c_5, c_5) \) is the unique optimum solution and it is integral.

(2) For \( J := \{1,2,3,4\} \), \( A^Jx \leq 1 \) is not tdi:

Proof: For \( c := (2,1,1,1) \) the only optimum solution of

\[
\min \ y \ \text{subject to} \ \ yA^J = c, \ y \geq 0
\]

is \( y = \begin{bmatrix} 1 \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \end{bmatrix} \), which is not integral.
(3) \( (P, O) \) is a pair of polar polyhedra:
Proof: We show that \( P := \{ x \in \mathbb{R}^5 \mid Ax \leq 1 \} = \text{CONV}(\bar{A}) + \text{CONE}(\bar{R}) - \mathbb{R}^5_+ \). Notice that \( \{ x \in \mathbb{R}^5 \mid Ax = 0 \} = \text{CONE}(\pm d) \) with \( d = (1,-1,-1,-1,2) \). To find points and directions generating \( P \), it is enough to find matrices \( B' \) and \( S' \), such that \( P \cap \{ x \in \mathbb{R}^5 \mid dx = 1 \} = \text{CONV}(B') + \text{CONE}(S') \). Then \( P = \text{CONV}(B') + \text{CONE}(\pm d, S') \).

We consider the extreme points and extremal rays of \( P \cap \{ x \in \mathbb{R}^5 \mid dx = 1 \} \): The unique extreme point is the unique vector in the minimal face \( \{ x \in \mathbb{R}^5 \mid Ax = 1, dx = 1 \} = \{(1,0,0,0,0)\} \). The extremal rays are given by the unbounded faces of dimension 1:

\[
\begin{align*}
F_1 &:= \{ A^2,3,4 \cdot x = 1, A_1 x \leq 1, dx = 1 \} = \{(1,0,0,0,0) + \lambda (-3,3,3,-5,2) \mid \lambda \geq 0 \} \\
F_2 &:= \{ A^1,3,4 \cdot x = 1, A_2 x \leq 1, dx = 1 \} = \{(1,0,0,0,0) + \lambda (-3,-3,-5,3,2) \mid \lambda \geq 0 \} \\
F_3 &:= \{ A^1,2,4 \cdot x = 1, A_3 x \leq 1, dx = 1 \} = \{(1,0,0,0,0) + \lambda (1,-1,-1,-1,-2) \mid \lambda \geq 0 \} \\
F_4 &:= \{ A^1,2,3 \cdot x = 1, A_4 x \leq 1, dx = 1 \} = \{(1,0,0,0,0) + \lambda (1,-1,-1,-1,-2) \mid \lambda \geq 0 \}.
\end{align*}
\]

Hence \( P = \text{CONV}(B') + \text{CONE}(S') \), with

\[
B' = \begin{bmatrix}
1 & -1 & -1 & -1 & 2 \\
-1 & 1 & 1 & 1 & -2 \\
-3 & 3 & 3 & -5 & 2 \\
-3 & 3 & -5 & 3 & 2 \\
-3 & -5 & 3 & 3 & 2 \\
1 & -1 & -1 & -1 & -2
\end{bmatrix}
\]

and for

\[
S' = \begin{bmatrix}
1 & -1 & -1 & -1 & 2 \\
-1 & 1 & 1 & 1 & -2 \\
-1 & 1 & -1 & 1 & 0 \\
-1 & 1 & 1 & 1 & 0
\end{bmatrix},
\]

\( P = \text{CONV}(B) + \text{CONE}(S) - \mathbb{R}^5_+ \), since \( S_i \in \text{CONE}(S) - \mathbb{R}^5_+ \) and \( S_i \in \text{CONE}(S') \) for \( 3 \leq j \leq 6, 3 \leq i \leq 5 \). Moreover, \( B \subseteq \bar{A}, \bar{S} \subseteq \bar{R} \) and \( \bar{A}_i \in P, \lambda \bar{R}_i \in P \) for \( \lambda \geq 0, 1 \leq i \leq 15 \), hence \( P = \text{CONV}(\bar{A}) + \text{CONE}(\bar{R}) - \mathbb{R}^5_+ \).

(4) \( x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0 \) is homogeneous
Proof: Consider

\[
\text{(37) } \min \ y \text{ subject to } y\bar{A} + w\bar{R} \geq c, \ y, w \geq 0.
\]

It can easy be verified that for \( c := \bar{A}_i + \bar{A}_j \), or \( c := \bar{A}_i + \bar{R}_j \), or \( c := \bar{R}_i + \bar{R}_j \), \( 1 \leq i, j \leq 15, i \neq j \), (37) has a homogeneous optimum solution, implying \( x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0 \) homogeneous.

Example 2:
\( \bar{A}, \bar{R} \) are \((-4,\ldots,2)\)-matrices. The system \( x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0 \) is not homogeneous-tdi and for all \( J \subseteq \{1,\ldots,n\} \) the system \( x \geq 0, \bar{A}^J x \leq 1, \bar{R}^J x \leq 0 \) is homogeneous and tdi:
For $J \subseteq \{1, \ldots, 4\} =: N$, consider the dual LP's:

(38) $\text{DLP}1(J,c^J)$: \[
\begin{array}{cccc}
-1 & 1 & 0 & -1 \\
2 & 0 & 1 & -1 \\
2 & 2 & 2 & -4 \\
0 & 2 & 1 & -3 \\
0 & 1 & 1 & -2
\end{array}
\]

(1) $x \geq 0. \ R^x \leq 0$ is homogeneous:
Proof: Let $c \in \mathbb{R}^4$ be a vector such that $\text{DLP}1(N,c)$ has a feasible solution $w$. Then for $\varepsilon_1 := \min \{w_1, w_2\}$, $w' := (w_1 - \varepsilon_1, w_2 - \varepsilon_1, w_3 + \frac{\varepsilon_1}{2}, w_4, w_5)$ is a feasible solution of (38) with $w_1 \cdot w_2 = 0$. For $\varepsilon_2 := \min \{2w_2, w_3, w_4, w_5\}$, $w'' := (w_1 - \varepsilon_2, w_2, w_3 - \frac{\varepsilon_2}{2}, w_4 + \varepsilon_2, w_5)$ is a feasible solution with $w_1''(w_2'' + w_3'') = 0$, hence $w''$ is a homogeneous feasible solution of (38).

(2) $x \geq 0. \ R^x \leq 0$ is homogeneous for all $J \subseteq N$:
Proof: By corollary 4.5, homogeneity is hereditary.

(3) $x \geq 0. \ R^x \leq 0$ is tdi for all $J \subseteq N$:
Proof: Notice that if (38) has a solution and if $K(J) = \{1, \ldots, 5\}$ (or equivalently $\{1,4\} \subseteq J$), (38) has also a feasible solution with $w_3 = w_4 = w_5 = 0$: If $w$ is a feasible solution of (38), then $w := (w_1 + 2w_3 + 2w_4 + w_5, w_2 + 2w_3 + w_4 + w_5, 0, 0, 0)$ is a feasible solution too. In these cases we consider the dual LP's:

(39) $\text{DLP}2(J,c^J)$: \[
\begin{array}{cccc}
w_1 R_1^J + w_2 R_2^J \geq c^J, \ w_1, w_2 \geq 0
\end{array}
\]

and show that (39) has an integer feasible solution for all integer vectors $c$ such that (39) has a feasible solution.

Case 1: $J \supseteq \{1,2,4\}$ and $K(J) = \{1, \ldots, 5\}$. Let $c^J$ be an integer vector such that (39) has a feasible solution. Then $(w_1, w_2)$, with $w_1 := \max \{c_1^J, 0\}$, $w_2 := \max \{-c_4^J - w_1, 0\}$ is an integer feasible solution of (39).

Case 2: $4 \notin J$, then $R^J = \emptyset$ and $x \geq 0$, $R^J x \leq 0$ is tdi.

Case 3: $J = \{1,3,4\}$ and $K(J) = \{1, \ldots, 5\}$. Let $c^J$ be an integer vector such that (39) has a feasible solution. Then $(w_1, w_2)$, with $w_1 := 0$, $w_2 := \max \{-c_4^J, 0\}$ is a homogeneous integer feasible solution of (39). Hence the system $x \geq 0$, $R^J x \leq 0$ is homogeneous-tdi.

Case 4: $J \subseteq \{1,4\}$, $\{3,4, \{4\}\}$. The system $x \geq 0$, $R^J x \leq 0$ is homogeneous-tdi for $L = \{1,3,4\}$ (case 3), hence by heredity of homogeneous-tdi (proposition 4.11) $x \geq 0$, $R^J x \leq 0$ is homogeneous-tdi too.

Case 5: $J \subseteq \{2,3,4\}$. Consider the matrix
B := \begin{bmatrix} -1 & 1 & 0 & -1 \\ 2 & 0 & 1 & -1 \\ 1 & 1 & 1 & -2 \\ 0 & 2 & 1 & -3 \end{bmatrix}, \text{ then } \{x \in \mathbb{R}_+^4 \mid Bx \leq 0\} = \{x \in \mathbb{R}_+^4 \mid Rx \leq 0\}. \\

By case 1 for J = \{1,2,3,4\}, the system \(x \geq 0, Bx \leq 0\) is tdi. We show first, that it is homogeneous-tdi: Let \(c\) be an integral vector, such that 
\begin{equation}
 wB \geq c, \ w \geq 0
\end{equation}
has a feasible solution. Let \(w\) be an integer solution of (40), then for \(\varepsilon_1 := \min \{w_1, w_2\}, w := (w_1 - \varepsilon_1, w_2 - \varepsilon_1, w_3 + \varepsilon_1, w_4)\) is an integer feasible solution of (40) with \(w_1 \cdot w_2 = 0\). For \(\varepsilon := \min \{w_1, w_2\}, w^\prime := (w_1 - \varepsilon, w_2 - \varepsilon, w_3 - \varepsilon, w_4 + \varepsilon)\) is a homogeneous integer feasible solution, because \(w_1(w_2 + w_3^\prime) = 0\). Hence \(x \geq 0, Bx \leq 0\) is homogeneous-tdi. As \(B^\prime \subseteq R^J\) for \(J \subseteq \{2, 3, 4\}\) the systems \(x \geq 0, R^Jx \leq 0\) are homogeneous-tdi for \(J \subseteq \{2, 3, 4\}\).

(4) \(x \geq 0, Rx \leq 0\) is not homogeneous-tdi:
Proof: For the integral vector \(c := (1,1,1,-2)\), DLP1(\(N, c\)) has a feasible solution \(w := \frac{1}{2}e^3\). But there exists no homogeneous integer solution: Otherwise \(w_1 = 0\) (homogeneity of \(w\)), \(w_3 = w_4 = 0\) (integrality of \(w\)), \(w_5 \geq 1\) (because \(c_2 = 1\) and \(c_3 = -2\)). But \(w_2 \geq \frac{1}{2}\) (by \(c_1 = 1\)), a contradiction.

Example 3:
\(A, R\) are \(\{1,0,-1\}\)-matrices. The system \(x \geq 0, Ax \leq 1, Rx \leq 0\) is homogeneous and tdi but not homogeneous-tdi:
\[A := \begin{bmatrix} 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \end{bmatrix} \text{ and } R := 0.
\]

(1) \(x \geq 0, A x \leq 1\) is a homogeneous system:
Proof: Let \(c \in \mathbb{R}^9\), such that
\begin{equation}
\min \ 1y \text{ subject to } yA - u = c, \ y, u \geq 0
\end{equation}
has an optimum solution \((y, u)\). Define \(c := \min \{y_1, y_2\}\) and \(y^\prime := (y_1 - c, y_2 - c, y_3 + \frac{c}{2}, y_4 + \frac{c}{2}, y_5 + \frac{c}{2}, y_6 + \frac{c}{2})\). Then \((y', u)\) is an optimum solution of (41) and since \(y_1'y_2 = 0\), \((y', u)\) is a homogeneous optimum solution.

(2) \(x \geq 0, A x \leq 1\) is a tdi system:
Proof: Let \(c\) be integral, such that DLP(c) has an optimum solution. Let \((y', u)\) be an optimum homogeneous solution \((y_1'y_2 = 0\) and define
\begin{equation}
 y := y' - [y'], c := c' - [y'] A.
\end{equation}
Then \((y, u)\) is a homogeneous optimum solution of DLP(c) and for any solution \((y, u)\) of
DLP(c), \((\bar{y} + \lfloor y' \rfloor, u)\) is an optimum solution of DLP(c'). Hence it is sufficient to construct an integer optimum solution for DLP(c) to show tdi-ness of the system \(x \geq 0, \bar{A}x \leq 1\).

Let \(\omega := 1y\), then as \((y, u)\) is a homogeneous optimum solution of DLP(c) and \(0 \leq y_i < 1, i = 1, \ldots, 6, 0 \leq \omega < 5\). Furthermore

(a) \(c \leq \lfloor \bar{y} \bar{A} \rfloor \) (c integral).

(b) \(c_1 \leq \lfloor y \bar{A}_1 \rfloor = \lfloor 1y \rfloor = \lfloor \omega \rfloor, c_2 \leq \begin{cases} 0 & \text{if } y_2 = 0 \\ -1 & \text{if } y_2 > 0 \end{cases}, c_3 \leq \begin{cases} 0 & \text{if } y_1 = 0 \\ -1 & \text{if } y_1 > 0 \end{cases} \) \((0 \leq y_1, y_2 < 1, y_1 y_2 = 0)\).

(c) \(I := \{i \mid y_i > 0\}\). Then \(\|I\| > \omega\) \((0 \leq y_i < 1, i = 1, \ldots, 6)\) and for \(J \subseteq I, \|J\| > \Sigma(y_i \mid i \in J) = \omega - \Sigma(y_i \mid i \in I) > \omega - \|I\|.\) Hence
\[- \Sigma(y_i \mid i \in J) \leq \|I\| - \lfloor \omega \rfloor - 1.\]

(d) for \(\forall \epsilon \in \{4, \ldots, 9\}, c_\epsilon \leq 0 \) and \(c_\epsilon \leq -1\) if there exists \(i \in I\) with \(\bar{A}_{i\epsilon} = -1\).

We consider the different possibilities for \(\omega\) and give for each case an integer vector \(\bar{y}\), with
\[y \geq 0, y \bar{A} \geq c, 1\bar{y} = \lfloor \omega \rfloor.\]

Hence \((\bar{y}, c - \bar{y} \bar{A})\) is an integer optimum solution of DLP(c).

**Case 1:** \(0 \leq \omega < 1\): Then \(c \leq 0\) by (b) and (d) and \(\bar{y} := 0\) satisfies (42).

**Case 2:** \(1 \leq \omega < 2\): Let \(i \in I\), then by (b) and (d), \(c \leq \bar{A}_i\) and \(\bar{y} := e^i\) satisfies (42).

**Case 3:** \(2 \leq \omega < 3\):

- **Case 3.1:** For all \(k \in \{4, \ldots, 9\}\) there exists \(i \in I\) with \(\bar{A}_{ik} = -1\):

  Then \(c \leq (2, 0, 0, -1, -1, -1, -1, -1, -1) = \bar{A}_1 + \bar{A}_2\) by (b) and (d). \(\bar{y}_d := \begin{cases} 1 & \text{if } d = 1, 2 \\ 0 & \text{otherwise} \end{cases}\) satisfies (42).

- **Case 3.2:** There exists \(k \in \{4, \ldots, 9\}\) with \(\bar{A}_{ik} = 0\) for all \(i \in I\):

  Then \(\|I\| = 3\) by \(|\{i \mid \bar{A}_{ik} = 0\}| \leq 3\) for \(k \in \{4, \ldots, 9\}\) and (c). Let \(I := \{i, j, k\}\), then

  \[\bar{y}_d := \begin{cases} 1 & \text{if } d = i, j \\ 0 & \text{otherwise} \end{cases} \text{ satisfies (42):} \]

  For \(d \in \{1, 2, 3\}\), \(c_d \leq \bar{A}_{id} + \bar{A}_{jd}\) by (b);
  for \(d \in \{4, \ldots, 9\}\), let \(J := \{m \in I \mid \bar{A}_{md} = -1\}\). Then \(c_d \leq \lfloor y_i \bar{A}_{id} + y_j \bar{A}_{jd} + y_k \bar{A}_{kd} \rfloor = -\Sigma(y_m \mid m \in J)\) \leq \(\bar{A}_{id} + \bar{A}_{jd}\).

**Case 4:** \(3 \leq \omega < 4\), then \(\|I\| \geq 4\).

- **Case 4.1** \(\|I\| = 4\): Let \(I := \{i, j, k, s\}\), then \(\bar{y}_d := \begin{cases} 1 & \text{if } d = i, j, k \\ 0 & \text{otherwise} \end{cases}\) satisfies (42):

  For \(d \in \{1, 2, 3\}\), \(c_d \leq \bar{A}_{id} + \bar{A}_{jd} + \bar{A}_{kd}\) by (b);
  for \(d \in \{4, \ldots, 9\}\), let \(J := \{m \in I \mid \bar{A}_{md} = -1\}\). Then

  \[c_d \leq \lfloor y_i \bar{A}_{id} + y_j \bar{A}_{jd} + y_k \bar{A}_{kd} + y_s \bar{A}_{sd} \rfloor = -\Sigma(y_m \mid m \in J)\]

  \[\begin{cases} -3 & \text{if } \|J\| = 3 \text{ (by (c))} \\ -2 & \text{if } \|J\| = 2 \text{ (by (c))} \\ -1 & \text{if } \|J\| = 1 \text{ (by (c))} \\ 0 & \text{if } \|J\| = 0 \end{cases} \leq \bar{A}_{id} + \bar{A}_{jd}.\]
\[
\begin{align*}
-3 & \text{ if } |I| = 3 \quad \text{(by (c))} \\
-2 & \text{ if } |I| = 2 \quad \text{(by (c))} \\
-1 & \text{ if } |I| = 1 \quad \text{(by (c))} \\
\end{align*}
\]

\[\bar{A}_{id} + \bar{A}_{jd} + \bar{A}_{kd}.\]

Case 4.2: \( |I| = 5 \), then \( I = \{1, \ldots, 6\}\{s\} \) with \( s \in \{1, 2\} \). Then

\[
\begin{align*}
c_d & \leq \\
&= \begin{cases}
3 & \text{if } d = 1 \\
-1 & \text{if } (d = 2 \text{ and } s = 1) \text{ or } (d = 3 \text{ and } s = 2) \\
0 & \text{if } (d = 3 \text{ and } s = 1) \text{ or } (d = 2 \text{ and } s = 2) \\
-1 & \text{if } (d \in \{4, 5, 6\} \text{ and } s = 1) \text{ or } (d \in \{7, 8, 9\} \text{ and } s = 2) \\
-2 & \text{if } (d \in \{7, 8, 9\} \text{ and } s = 1) \text{ or } (d \in \{4, 5, 6\} \text{ and } s = 2) \\
\end{cases} \quad \text{(by (b))}
\]

\[
\leq \begin{cases}
2\bar{A}_{2d} + \bar{A}_{1d} & \text{if } s = 1 \\
2\bar{A}_{1d} + \bar{A}_{2d} & \text{if } s = 2
\end{cases}
\]

Hence \( \bar{y}_d := \begin{cases}
2 & \text{if } d \in \{1, 2\}\{s\} \\
1 & \text{if } d = s \\
0 & \text{otherwise}
\end{cases} \) satisfies (42).

Case 5: \( 4 \leq \omega < 5 \): Then \( |I| = 5 \) and \( I = \{1, \ldots, 6\}\{s\} \) with \( s \in \{1, 2\} \). Then,

\[
\begin{align*}
c_d & \leq \\
&= \begin{cases}
4 & \text{if } d = 1 \\
-1 & \text{if } (d = 2 \text{ and } s = 1) \text{ or } (d = 3 \text{ and } s = 2) \\
0 & \text{if } (d = 3 \text{ and } s = 1) \text{ or } (d = 2 \text{ and } s = 2) \\
-2 & \text{if } (d \in \{4, 5, 6\} \text{ and } s = 1) \text{ or } (d \in \{7, 8, 9\} \text{ and } s = 2) \\
-3 & \text{if } (d \in \{7, 8, 9\} \text{ and } s = 1) \text{ or } (d \in \{4, 5, 6\} \text{ and } s = 2) \\
\end{cases} \quad \text{(by (c))}
\]

\[
\leq 2(\bar{A}_{2d} + \bar{A}_{1d}).
\]

Hence \( \bar{y}_d := \begin{cases}
2 & \text{if } d = 1, 2 \\
0 & \text{otherwise}
\end{cases} \) satisfies (42).

(3) \( x \geq 0, \bar{A}x \leq 1 \) is not a homogeneous-tdi system:

Proof: For \( c := (2, 0, 0, -1, -1, -1, -1, -1) \), (41) has an optimum solution \((y, u)\) with \( y := (1, 1, 0, 0, 0, 0) \), \( u := 0 \) but it has no homogeneous integer optimum solution.

Example 4:

\(((A, R), (\bar{A}, 0))\) is a pair with \(\{0,1\}\)-matrices \(A, R\), a \(\{1,0,-1\}\)-matrix \(\bar{A}\), and such that \((P, Q)\) of (22) is a polar pair. The system \(Ax \leq 1, Rx \leq 0\) is tdi, \(x \geq 0, \bar{A}x \leq 1, R x \leq 0\) is homogeneous and not tdi:

\[
A := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad R := \begin{bmatrix} R^k \\ R^j \end{bmatrix}
\]
(1) \( A_{x} \leq 1 \) \( R_{x} \leq 0 \) is a tdi system:

Proof: Let \( K := \{1, \ldots, 7\} \), \( J := \{1, \ldots, 6\} \) and \( c' \) be integral such that

\[
\text{DLP}(c') := \min \ y \ \text{subject to } y A + w R^K + u R^J = c', \ y, w, u \geq 0
\]

has an optimum solution.

Let \( (y', w', u') \) be a solution with \( w'_i w'_j = 0 \) for \( 4 \leq i < j \leq 7 \). (Such a solution exists: assume for example \( w'_4 w'_5 \neq 0 \) and let \( \varepsilon := \min \{w'_4, w'_5\} \). Then \( (y'', w'', u'') \) with \( y'' := y', \ w'' := w' - \varepsilon(e^4 + e^5) + \varepsilon(e^2 + e^3), \ u'' := u' + 2\varepsilon e', \) is a solution with \( w'_i w'_j = 0 \).)

Let \( c := c' - \sum_{i \in K} w_i A - \sum_{j \in J} R^K - \sum_{j \in J} R^J \). We show that \( \text{DLP}(c) \) has an integer optimum solution, implying \( \text{DLP}(c') \) has an integer optimum solution. Let \( y := y' - \sum_{i \in K} y_i A, \ w := w' - \sum_{j \in J} w_i R^K, \ u := u' - \sum_{j \in J} u_j R^J, \) then \((y, w, u)\) is an optimum solution of \( \text{DLP}(c) \) with \( 0 < y, w, u < 1, \ k \in K, j \in J \), hence \( c_1 \leq 4 \).

Let \( I := \{k \in K \mid w_k > 0\} \), we show that we find

\[
\text{M} \subseteq I \text{ such that } c_i \geq \sum_{k \in I} R_{ki} =: r_i, \ 2 \leq i \leq 7, \ c_1 = \sum_{k \in I} R_{k1}, \ 
\]

implying \((y^* := 0, w^* := \sum_{k \in I} w_k, u^* := \sum_{k \in I} (c_1 - r_i)e^{i-1} \mid 2 \leq i \leq 7))\) is an integer optimum solution of \( \text{DLP}(c) \).

If \( c_1 = 0 \), then \( M := \emptyset \) satisfies (44).

If \( c_1 = 1 \), then \( \|M\| \geq 1 \). Let \( i \notin I \), then \( M := \{i\} \) satisfies (44), since \( c \geq \sum_{k \notin I} w_k R_{ki} \), \( k \in I \).

If \( c_1 = 2 \), then \( \|M\| \geq 2 \). If \( \|M\| > 2 \), then there exists \( 1 \leq i < j \leq 3 \) with \( \{i, j\} \subseteq I \) and \( M := \{i, j\} \) satisfies (44) by the same arguments as for \( c_1 = 1 \). If \( \|M\| = 2 \), then \( M := I \) satisfies (44), since \( 2 = c_1 = y + \sum_{k \in I} w_k \) and therefore \( \sum_{k \in I} w_k > 1 \).

If \( c_1 = 3 \), then \( \|M\| \geq 3 \). If \( I \supseteq \{1, 2, 3\} \) then \( M := \{1, 2, 3\} \) satisfies (44). Otherwise \( \|M\| = 3 \).

Then \( M := I \) satisfies (44), since \( 2 \geq \sum_{k \in I} R_{ki} \) for \( 2 \leq i \leq 7 \) and \( w_k + w_j > 1 \) for \( k \neq j; k, j \in I \).

If \( c_1 = 4 \), then \( \|M\| = 4 \), \( 1w > 3 \) and \( M := I \) satisfies (44), since \( 2 \geq \sum_{k \in I} R_{ki} \) for \( 2 \leq i \leq 7 \) and \( w_k + w_j > 1 \) for \( k \neq j; k, j \in I \).
(2) \( P, Q \) is a pair of polar polyhedra:

Proof: We show that \( P := \{ x \in \mathbb{R}^7 \mid Ax \leq 1, Rx \leq 0 \} = \text{CONV}(\overline{A}, 0) - \mathbb{R}^7_+. \) Notice first that if \( P = \text{CONV}(B) + \text{CONE}(S), \text{CONE}(S) = \{ x \in \mathbb{R}^7 \mid Ax \leq 0, Rx \leq 0 \} = - \mathbb{R}^7_+. \) Hence \( P = \text{CONV}(B) - \mathbb{R}^7_+. \)

We consider the vertices of \( P. \) \( x \) is a vertex if it is a feasible solution, i.e. \( Ax \leq 1, Rx \leq 0 \) and if there exists \( I = \emptyset \) or \( I = \{ 1 \}, M \subseteq \{ 1, \ldots, 7 \}, N \subseteq \{ 2, \ldots, 7 \} \) and \( \overline{N} := \{ 2, \ldots, 7 \} \setminus N \) such that \( x \) is the unique solution of

\[
(A^I)x = 1, (R^M)x = 0, x^{\overline{N}} = 0.
\]

Consider a minimal system of the form (45) such that there exists a unique feasible solution \( x \) with \( x_i > 0 \) if \( i \in \overline{N} \). Hence \( |M| = 7 - |N| - |I| \).

If \( I = \emptyset \), then \( x = 0 \) is a solution of (45). Let now \( I = \{ 1 \} \), then \( x_1 = 1 \) and \( |N| \leq 3 \), otherwise \( \{ x_1 = 1, R^Kx \leq 0, x^{\overline{N}} = 0 \} \) is empty. Moreover, since \( Ax \leq 1, Rx \leq 0 \) is tdi, a vertex of \( P \) is integral. Therefore \( x_i \leq -1 \) for \( i \in \overline{N} \) and

\[
\sum \{ R_{ki} \mid i \in \overline{N} \cup I \} = 1 \quad \text{for } k \in M.
\]

Case 1: \( |N| = 3 \). Let \( \overline{N} = \{ i, j, k \} \) with \( i < j < k \). We enumerate the possibilities:

If \( i = 2 \) and \( j = 3 \), then \( k \in \{ 4, 5, 6 \} \) (by \( R_3x \leq 0, R_7x \leq 0 \)). Then \( k = 7 \) and \( x = (1, -1, -1, 0, 0, 0, -1) \). By the same arguments, the only possibilities for \( (i, j, k) \) are

\[
(i, j, k) = \begin{cases} (2, 4, 5) & \text{and } x = (1, -1, 0, -1, 0, 0), \\ (3, 5, 6) & \text{and } x = (1, 0, -1, 0, -1, 0), \\ (4, 6, 7) & \text{and } x = (1, 0, 0, -1, 0, -1). \end{cases}
\]

Case 2: \( |N| \leq 2, |N| \geq 4 \) and \( |M| \geq 4 \). Assume \( 7 \in M \), then by (46), \( \{ 2, 3, 5 \} \subseteq \overline{N} \) and therefore (again by (46)), \( 4, 5, 6 \in M \). Hence \( M = \{ 1, 2, 3, 7 \} \) and (46) implies: \( 6 \in N \) for \( k = 1 \), \( 4 \in N \) for \( k = 2 \), and \( 7 \in N \) for \( k = 3 \), a contradiction to \( |N| \leq 2 \). In the same way \( \{ 4, 5, 6 \} \cap M = \emptyset \), a contradiction to \( |M| \geq 4 \).

(3) \( x \geq 0, \overline{A}x \leq 1 \) is homogeneous and not tdi;

Proof: Clearly \( x \geq 0, \overline{A}x \leq 1 \) is homogeneous. For \( c := (2, -1, -1, -1, -1, -1, -1) \),

\[ \min \ y \text{ subject to } y\overline{A} \geq c, \ y \geq 0 \]

has an optimum solution \( y = \frac{1}{2} \) but no integer optimum solution.

Example 5:

Let \( ((A,R), (\overline{A}, \overline{R})) \) be a pair with a \( \{ 0, 1 \} \)-matrix \( A, \{ 1, 0, -1 \} \)-matrices \( \overline{A}, \overline{R} \), and such that \( (P,Q) \) of (22) is a polar pair. For all \( J \subseteq \{ 1, \ldots, n \} \), \( P^J := \{ x \in \mathbb{R}^7 \mid A^Jx \leq 1 \} = \text{CONV}(B(J)) + \text{CONE}(S(J)) \), with \( \{ 1, 0, -1 \} \)-matrices \( B(J), S(J) \). The system \( x \geq 0, \overline{A}x \leq 1, \overline{R}x \leq 0 \) is tdi and not homogeneous:
Let $A$ be the incidence matrix of the intersections (see chapter 5, (1) and theorem 5.4), $\bar{A}$ (respectively $\bar{R}$) the matrices with its rows corresponding to the alternating vectors of paths (respectively circuits) of the following graph $G = (V, E_b \cup E_r)$:

![Figure 4.4]

Then $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $\bar{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\bar{R}' = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.

Let $R := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = (\bar{R})^{(1,2)}$.

Notice that since $R_3' = R_1 + R_2$, $Q := \{ x \in \mathbb{R}_+^3 | \bar{A}x \leq 1, Rx \leq 0 \} = \{ x \in \mathbb{R}_+^3 | \bar{A}x \leq 1, (\bar{R})x \leq 0 \} = Q'$.

We know from theorem 5.4, that

(i) $(P, Q')$ is a pair of polar polyhedra,

(ii) $x \geq 0, \bar{A}x \leq 1, (\bar{R})x \leq 0$ is homogeneous-tdi.

By theorem 4.3, $(P^J, Q^J)$ is a pair of polar polyhedra, where $P^J := \{ x \in \mathbb{R}_+^J | A^Jx \leq 1 \}$ and $Q^J := \{ x \in \mathbb{R}_+^J | \bar{A}^Jx \leq 1, (\bar{R})^Jx \leq 0 \}$. Hence for $B(J) := \bar{A}^J$ and $S(J) := \begin{bmatrix} (\bar{R})^J \\ -I \end{bmatrix}$, $P^J = CONV(B(J), 0) + CONE(S(J))$.

Furthermore

(iii) $x \geq 0, \bar{A}x \leq 1, Rx \leq 0$ is a tdi system, because $x \geq 0, \bar{A}x \leq 1, (\bar{R})x \leq 0$ is tdi and $R_3' = R_1 + R_2$.

(iv) $x \geq 0, \bar{A}x \leq 1, Rx \leq 0$ is not homogeneous:

For $c := (1,0,-1) = R_3'$, $(y, w, u)$ with $y := 0$, $w := (1,1)$, $u := 0$ is an optimal solution of $\min 1y$ subject to $y\bar{A} + wR - u = c$; $y, w, u \geq 0$. For any optimal solution, $y = 0$ and there exists no homogeneous solution of $wR \geq c$. 
Example 6:

Let $\bar{A}$ be a $\{1,0,-1\}$-matrix and $R = 0$. For all $J \subseteq \{1,...,n\}$, $Q^J := \{x \in \mathbb{R}_+^n \mid \bar{A}^J x \leq 1\} = \text{CONV}(C(J))$ for some $\{0,1\}$-matrices $C(J)$ and the system $x \geq 0$, $\bar{A} x \leq 1$ is not homogeneous:

Let $A$ be the incidence matrix of the intersections (see chapter 5, (1) and theorem 5.4), $\bar{A}'$ the matrix with its rows corresponding to the alternating vectors of paths of the following graph $G = (V, E_b \cup E_r)$:

![Figure 4.5](image_url)

Then $(\bar{A}^\gamma) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & -1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 & 1 \\
0 & 0 & 1 & -1 & 1
\end{bmatrix}$.

Denote by $\bar{A} := (\bar{A}^\gamma)^I$ with $I := \{1,...,9\}$.

We know from theorem 5.4, that $(P, Q')$ is a pair of polar polyhedra, where

$P := \{x \in \mathbb{R}_5^5 \mid Ax \leq 1\}$ and $Q' := \{x \in \mathbb{R}_5^5 \mid (\bar{A}^\gamma) x \leq 1\}$.

Furthermore $x \geq 0$, $(\bar{A}^\gamma) x \leq 1$ is homogeneous-tdi and $Q' = \{x \in \mathbb{R}_5^5 \mid \bar{A} x \leq 1\} = Q$, as $\frac{1}{2}(\bar{A}_8 + \bar{A}_9) = \bar{A}_{10}$.

(1) For all $J \subseteq \{1,...,5\}$, $Q^J = \text{CONV}(C(J))$ for some $\{0,1\}$-matrices $C(J)$.

Proof: Let $J \subseteq \{1,...,5\}$.

Case 1: $\{1,2\} \subseteq J$ or $4 \in J$, then $\{x \in \mathbb{R}_+^5 \mid (\bar{A}^\gamma)^J x \leq 1\} = \{x \in \mathbb{R}_+^5 \mid \bar{A}^J x \leq 1\}$, because $\bar{A}_{10} = \frac{1}{2}(\bar{A}_8 + \bar{A}_9)$. As $x \geq 0$, $(\bar{A}^\gamma) x \leq 1$ is homogeneous-tdi, $Q^J = \text{CONV}(A^J, 0)$ by theorem 4.3.

Case 2: $J = \{2,3,4,5\}$, then
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-1 & 1 & 0 & 0 \\
-1 & 1 & -1 & 1
\end{bmatrix}
\]

Define \( D :=
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-1 & 1 & -1 & 1
\end{bmatrix}
\)

Then \( \{ x \in \mathbb{R}^4_+ \mid \overline{A}^j x \leq 1 \} = \{ x \in \mathbb{R}^4_+ \mid D x \leq 1 \} \). \( D \) is totally unimodular, hence a \( \{0,1\} \)-matrix \( C(J) \) exists.

Case 3: \( J = \{2,3,4\}, J = \{2,4,5\}, J = \{3,4,5\} \), then \( \overline{A}^j =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & 1 & 0 \\
-1 & 1 & -1
\end{bmatrix}
\)

\( J = \{2,3,4\}, J = \{2,4,5\}, J = \{3,4,5\} \).

Hence \( \{ x \in \mathbb{R}^4_+ \mid \overline{A}^j x \leq 1 \} = \{ x \in \mathbb{R}^4_+ \mid I x \leq 1 \} \) and a \( \{0,1\} \)-matrix \( C(J) \) exists.

Case 4: \( J = \{2,4\}, J = \{3,4\}, J = \{4,5\}, J = \{4\} \), then \( \overline{A}^j =
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
-1 & 0 \\
-1 & -1
\end{bmatrix}
\)

\( J = \{2,4\}, J = \{3,4\}, J = \{4,5\}, J = \{4\} \).

Hence \( \{ x \in \mathbb{R}^4_+ \mid \overline{A}^j x \leq 1 \} = \{ x \in \mathbb{R}^4_+ \mid I x \leq 1 \} \) and a \( \{0,1\} \)-matrix \( C(J) \) exists.

Case 5: \( 1 \in J, 2 \in J \): For \( L := J \cup \{2\} \setminus \{1\} \), \( \overline{A}^j = \overline{A}^L \). Hence by cases 2 to 4, a \( \{0,1\} \)-matrix \( C(J) \) exists.

(2) \( x \geq 0, \overline{A} x \leq 1 \) is not homogeneous:

Proof: For \( c := (0,0,1,-1,1) = \frac{1}{2}(\overline{A}_8 + \overline{A}_9) \) we have \( 1 \geq \omega(c) := \min \ y \) subject to \( y\overline{A} \geq c, y \geq 0 \). For a homogeneous solution \( y \) we have \( y_6, y_7, y_8, y_9 = 0 \) by \( c_1 = c_2 = 0 \), and \( y_3, y_5 \geq 1 \) by \( c_3 = c_5 = 1 \). Hence \( 1 y > \omega(c) \) and \( x \geq 0, \overline{A} x \leq 1 \) is not homogeneous.

Example 7:

Let \(((A,0),(\overline{A},\overline{R}))\) be a pair with a \( \{0,1\} \)-matrix \( A \), \( \{1,0,-1\} \)-matrices \( \overline{A} \), \( \overline{R} \), and such that \((P,Q)\) of \((22)\) is a polar pair. The system \( x \geq 0, \overline{A} x \leq 1, \overline{R} x \leq 0 \) is homogeneous-tdi and \( Ax \leq 1 \) is not tdi:
(1) \((P, Q)\) is a pair of polar polyhedra and \(x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0\) is homogeneous:

Proof: Let \(A', \bar{A}', \bar{R}'\) be the matrices \(A, \bar{A}, \bar{R}\) of example 1. Then for \(J := \{1, 2, 3, 4\}\), \(A = (A')^T, \bar{A} = (A')^T, \bar{R} = (\bar{R}')^T\), and for any row \((\bar{A}')_j^T \leq 0\) and \((\bar{R}')_k^T \leq 0\) there exists a row \(\bar{A}_m\) and \(\bar{R}_n\) with \((\bar{A}')_j^T \leq \bar{A}_m\) and \((\bar{R}')_k^T \leq \bar{R}_n\). Hence \(Q := \{x \in \mathbb{R}^4_+ | \bar{A}x \leq 1, \bar{R}x \leq 0\} = \{x \in \mathbb{R}^4_+ | (\bar{A}')_j^T x \leq 1, (\bar{R}')_j^T x \leq 0\}\) and by (3) and (4) of example 1, \((P, Q)\) is a polar pair and \(x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0\) is homogeneous.

(2) \(x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0\) is tdi:

Proof: Let \(B := [1, 0, 0, 0]\), then \(\{x \in \mathbb{R}^4_+ | \bar{A}x \leq 1, \bar{R}x \leq 0\} = \{x \in \mathbb{R}^4_+ | Bx \leq 1, \bar{R}x \leq 0\}\). Consider the dual LP's

\[
\text{DLP1}(c^\prime): \quad \text{min} \quad 1y \quad \text{subject to} \quad y\bar{A} + w\bar{R} - u = c^\prime, \quad y, w, u \geq 0
\]

\[
\text{DLP2}(c^\prime): \quad \text{min} \quad y \quad \text{subject to} \quad yB + w\bar{R} - u = c^\prime, \quad y, w, u \geq 0.
\]

We show that (48) has an integer optimum solution for any integer vector \(c^\prime\), such that it has an optimum. Clearly such a solution is also an integer optimum solution of (47). Let \(c^\prime\) be integral, \(\omega(c^\prime)\) the optimum value of (48) and \((y, w, u)\) an optimum solution of (48). Then

\[
\omega(c^\prime) = \max \{c^\prime x | x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0\} = \max \{\{0\} \cup \{c^\prime A_i | i \in \{1, \ldots, 4\}\}\}.
\]

Hence \(y = \omega(c^\prime)\) and \(y\) is integral for any optimum solution. Moreover,

\[
\text{DLP2}: \quad w\bar{R} - u = c^\prime, \quad w, u \geq 0
\]

has a solution for \(c := c^\prime - yB\) and \(\omega(c) = 0\). By (49)

\[
c_1 + c_4 \leq 0; \quad c_1 + c_3 \leq 0; \quad c_1 + c_2 \leq 0; \quad c_1 + c_2 + c_3 + c_4 \leq 0.
\]

We assume without loss of generality that \(c_2 \leq c_3 \leq c_4\) and that \((w, u)\) is a homogeneous solution of (50). We construct an integer solution \((w', u')\) of (50) implying that \((y, w', u')\) is an optimum integer solution of (48).

Case 1: \(c_4 \leq 0\): Then \((w', c-w\bar{R})\) with \(w' := \max \{0, c_1\}e^4\) is an integer solution of DLP2 by (51).

Case 2: \(c_3 \leq 0, c_4 > 0\): Then \((w', c-w\bar{R})\) with \(w' := c_4e^7\) is an integer solution of DLP2 by (51).

Case 3: \(c_2 \leq 0, c_3 > 0\): Since \((w, u)\) is a homogeneous solution of DLP2, \(w_1 = w_2 = w_4 = 0\) and \(w_3w_5 = 0\). Then \((w', c-w\bar{R})\), with \(w' := w - w_5e^5\) is a solution too and \(J := \{j | w'_j > 0\} \subseteq \{3, 6, 7\} =: I\). As \(((w')_j^T, c-w\bar{R})\) is a solution of
and $\mathbb{R}_1$ is totally unimodular, there exists an integer solution $(w',u')$ of (52) (Theorem 2.14) and therefore of DLP2.

Case 4: $c_2 > 0$: Since $(w,u)$ is a homogeneous solution of DLP2, $w_1 = w_2 = w_3 = w_4 = 0$. Then $(w',c-w\cdot R)$ with $w' := c_2e^5 + c_3e^6 + c_4e^7$ is an integer solution of DLP2.

(3) $x > 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$ is homogeneous-tdi:
Proof: Notice that for $i,j \in \{1,\ldots,7\}$, there exists $m,k \in \{1,\ldots,7,8\}$ and $u \geq 0$, such that for

$\bar{A}_i + \bar{A}_j = \bar{A}_m + \bar{A}_k - u$ and $(e^m+e^k,0,u)$ is a homogeneous solution of DLP1($\bar{A}_i + \bar{A}_j$),
$\bar{A}_i + \bar{R}_j = \bar{A}_m + \bar{R}_k - u$ and $(e^m,e^k,u)$ is a homogeneous solution of DLP1($\bar{A}_i + \bar{R}_j$),
$\bar{R}_i + \bar{R}_j = \bar{R}_m + \bar{R}_k - u$ and $(0,e^m+e^k,u)$ is a homogeneous solution of DLP1($\bar{R}_i + \bar{R}_j$).

Therefore any integral solution $(y,w,u)$ of (47) can be transformed successively with help of these equalities to a homogeneous integer solution.

(4) $x \geq 0$, $Ax \leq 1$ is not tdi:
Proof: See example 1, (2).
5

Intersections

In this chapter we consider a family \( \mathcal{L} \) of subsets of a finite ground set \( V \), called the family of intersections. It is defined by

\[
\mathcal{L} := \{ L \subseteq V \mid L = C \cap D, \ C \in \mathcal{C}, \ D \in \mathcal{D} \text{ and } C \cup D = V \},
\]

where \( \mathcal{C} \) and \( \mathcal{D} \) are two proper ring families on \( V \).

We already met an example of intersections in the introduction, namely the path-closed sets of a digraph \( G = (V,E) \) introduced by Gröflin [1984]. Recall that a set \( T \subseteq V \) is path-closed if \( w \in V \) is on a (directed) path from \( v \) to \( v' \) for \( v, v' \in T \) implies \( w \in T \) (\( v, v' \) need not be distinct nodes). Clearly, if \( \mathcal{C} \) consists of those subsets \( C \subseteq V \) such that no edge leaves \( C \) and \( \mathcal{D} \) consists of those sets \( D \subseteq V \) such that no edge enters \( D \), then the family of path-closed sets is the family \( \mathcal{L}^G := \{ L \subseteq V \mid L = C \cap D, \ C \in \mathcal{C}, \ D \in \mathcal{D}, \ C \cup D = V \} \).

The main goal of this chapter is to find the polyhedral description of \( \mathcal{L} \). Let \( A \) be the incidence matrix of \( \mathcal{L} \) (containing the 0-row). Appropriate \( \{1,0,-1\} \)-matrices \( \bar{A} \) and \( \bar{R} \) will be found such that

\[
(P) \quad P := \{ x \in \mathbb{R}^V \mid Ax \leq 1 \} = \text{CONV}(\bar{A}) + \text{CONE}(\bar{R}) - \mathbb{R}^V,
\]

\[
(P^*) \quad P^* := \{ x \in \mathbb{R}^V_+ \mid \bar{A}x \leq 1, \ \bar{R}x \leq 0 \} = \text{CONV}(A),
\]

i.e. \( x \geq 0, \ \bar{A}x \leq 1, \ \bar{R}x \leq 0 \) is a polyhedral description of \( \mathcal{L} \). Moreover, the system \( Ax \leq 1 \) is tdi and \( x \geq 0, \ \bar{A}x \leq 1, \ \bar{R}x \leq 0 \) is homogeneous-tdi. Hence by corollary 4.15
((A,0),(\bar{A},\bar{R})) \text{ is } (A,\bar{A})\text{-perfect, where}

A \text{ is the incidence matrix of a family of intersections, and}

x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0 \text{ a polyhedral description of it.}

In this sense, we say that the intersections are described by a 
(A,\bar{A})\text{-perfect pair }((A,0), (\bar{A}, \bar{R})). \text{ In chapter 8, we will exhibit a more general class of combinatorial objects which is}

\text{described by a } (A,\bar{A})\text{-perfect pair }((A,0), (\bar{A}, \bar{R})).

The tdi-properties of the systems \(Ax \leq 1\) and \(x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0\) are related to two facts:

On one hand \(P\) belongs to the class of lattice polyhedra, a class of integer polyhedra with

box-tdi systems introduced by Hoffman, Schwartz [1978] and studied further in Hoffman

[1976,1978] and Gröflin, Hoffman [1982]. On the other hand, in the case where \(\bar{R}\) is not

present, \(P^*\) belongs to the class of switching paths polyhedra introduced by Gröflin [1987],

another class of integer polyhedra with box-tdi systems. When \(\bar{R}\) is present, \(P^*\) belongs to

the generalized class of switching paths polyhedra to be developed in chapter 6.

Furthermore, the following optimization problems will be considered:

(5) \textit{Optimum intersection problem}

Given a weight vector \(c \in \mathbb{R}^V\), find an intersection \(L\) of maximum weight.

(6) \textit{Minimum partitioning problem}

Given a set \(S \subseteq V\), find a partition of \(S\) into a minimal number of intersections (if such

a partition exists).

Problem (5) will be shown to be equivalent to the problem of finding a set of maximum

weight in a proper ring family, a problem solved by Picard [1976]. Problem (6) is a special

case of the submodular flow problem of Edmonds and Giles [1977] for which Cunningham

and Frank [1985] developed a polynomial time algorithm, provided an oracle is available.

We will give a more direct algorithm for this partitioning problem (without oracle), based on

longest path computation.

Finally, we discuss the condition \(C \cup D = V\) imposed in the definition of intersections.

Consider

\[
L_0 := \{L \subseteq V \mid L = C \cap D, C \in \mathcal{C}, D \in \mathcal{D} \}
\]

instead of \(L\). Observe that in the case of path-closed sets \(L^G\), the condition \(C \cup D = V\) is not really a restriction, i.e. \(L_0 = L^G\). This is not the case in general, i.e. \(L_0 \supseteq L\) is to be expected. A natural step is to also address the intersection and partitioning problem for \(L_0\)

instead of \(L\). For this case we show that both problems are NP-hard in general.

We notice that most of the content of this section can be found in Cochand, Gaillard, Gröflin [1988,1990].
5.1. Representation of intersections in a graph

Let $\mathcal{C}$ and $\mathcal{D}$ be two proper ring families on a ground set $V$ and consider the family of intersections

$$L := \{L \subseteq V \mid \text{\textnormal{there exists } } C \in \mathcal{C}, D \in \mathcal{D} \text{ such that } C \cap D = V \}$$

(where multiple copies of a same set in $L$ are discarded).

We make use of the fact that a proper ring family can be represented in an associated digraph (proposition 2.4) in order to represent $\mathcal{C}$ and $\mathcal{D}$ simultaneously in a digraph $H = (V,E)$, where the edges are bicolored blue and red, i.e. $E = E_b \cup E_r$, and

$$(a,b) \in E_b \text{ iff } a \in C \cap D \text{ and } b \in C \cap D,$$

$$(a,b) \in E_r \text{ iff } a \in C \cap D \text{ and } b \in C \cap D.$$

With this definition of $H$, it is easy to see that

$$C = \{ C \subseteq V \mid \delta(C) \cap E_b = \emptyset \}, \quad D = \{ D \subseteq V \mid \delta(D) \cap E_r = \emptyset \}.$$

Observe that $H$ is blue-transitive and red-transitive, i.e. $(a,b) \in E_b$ and $(b,c) \in E_b$ imply $(a,c) \in E_b$, and analogously for $E_r$.

Conversely, to each bicolored digraph $H = (V,E = E_b \cup E_r)$ with transitive blue edges and transitive red edges correspond two proper ring families defined by $(8)$, of which $H$ is the associated digraph.

**Remark:** When $\mathcal{L}$ is the family of path-closed sets of a digraph $G = (V,E)$, the associated graph $H = (V,E_b \cup E_r)$ has a particularly simple structure: To obtain $H$, take the transitive closure $E'$ of $E$, duplicate each edge of $E'$ and color the two parallel copies blue and red, respectively.

From now on, we assume that the two ring families $\mathcal{C}$ and $\mathcal{D}$ are given by their associated (or representing) digraph $H = (V,E_b \cup E_r)$ and definition $(8)$. $H$ is then also called the graph associated with (or representing) the intersections of $\mathcal{C}$ and $\mathcal{D}$. Furthermore, all mentioned paths and circuits in the various digraphs of this chapter shall be directed and elementary.

In order to characterize the intersections in $H$ we need some additional notations.

The ordered node set $(v_1, v_2, \ldots, v_{2n+1}) \subseteq V$, $n \geq 0$, is called a

- **b-r-path** if it is a path in $H$ and, if $n \geq 1$, $(v_{2i-1}, v_{2i}) \in E_b$ and $(v_{2i}, v_{2i+1}) \in E_r$, $i = 1, \ldots, n$;
- **b-r-circuit** if it is a circuit in $H$, $n \geq 1$, $v_{2n+1} = v_1$, $(v_{2i-1}, v_{2i}) \in E_b$ and $(v_{2i}, v_{2i+1}) \in E_r$, $i = 1, \ldots, n$;
- **b-r-handle** of a set $A \subseteq V$ if it is a b-r-path with $n \geq 1$ or a b-r-circuit, such that $v_1, v_{2n+1} \subseteq A$ and $(v_2, \ldots, v_{2n}) \cap A = \emptyset$. 


Theorem 5.1

$\mathcal{L} \subseteq \mathcal{V}$ is an intersection if and only if it has no b-r-handle.

Proof:

(i) Let $\mathcal{L} = \mathcal{C} \cap \mathcal{D}$ be an intersection and suppose it has a b-r-handle $P$ with (ordered) node set $(v_1, \ldots, v_{2n+1})$, $n \geq 1$ (if the handle is a circuit, $v_1 = v_{2n+1}$). Let $S := \{v_2, \ldots, v_{2n}\}$. $\mathcal{S} \cap \mathcal{C} \cap \mathcal{D} = \emptyset$ and, since $\mathcal{C} \cup \mathcal{D} = \mathcal{V}$, $(\mathcal{S} \cap \mathcal{C}) \cup (\mathcal{S} \cap \mathcal{D})$ is a partition of $S$. Observe next that $v_i \in \mathcal{C}$, $i$ odd, $1 \leq i \leq 2n$ implies $v_{i+1} \in \mathcal{C}$ since $\delta(\mathcal{C}) \cap E_b = \emptyset$, and $v_i \in \mathcal{D}$, $i$ odd and $2n+1 \geq i > 1$ implies $v_{i-1} \in \mathcal{D}$ since $\delta(\mathcal{D}) \cap E_r = \emptyset$. Since $v_1$, $v_{2n+1} \in \mathcal{C} \cap \mathcal{D}$, we have $v_2 \in \mathcal{S} \cap \mathcal{C}$ and $v_{2n} \in \mathcal{S} \cap \mathcal{D}$. Hence, by the above observation, there exists $v_i \in \mathcal{S} \cap \mathcal{C}$, $i$ even, $1 < i < 2n$, and $v_{i+1} \in \mathcal{S} \cap \mathcal{D}$; but then $v_{i+1} \in \mathcal{S} \cap \mathcal{D}$, and by the second part of the observation, $v_i \in \mathcal{S} \cap \mathcal{D}$. Therefore $v_i \in \mathcal{S} \cap \mathcal{C} \cap \mathcal{D}$, a contradiction.

(ii) Suppose $\mathcal{L} \subseteq \mathcal{V}$ has no b-r-handle. For any $v \in \mathcal{L}$, call a path with nodes $(v_1, \ldots, v_k)$, $k > 1$, a path from $L$ to $v$ if $v_1 \in \mathcal{L}$, $v_k = v$ and $v_i \in \mathcal{L}$, $k \geq i > 1$, and a path from $v$ to $L$ if $v_1 = v$, $v_k \in \mathcal{L}$ and $v_i \in \mathcal{L}$, $1 \leq i < k$. Let $\mathcal{C} := L \cup \{v \in \mathcal{L} \mid \exists$ a path from $L$ to $v$ whose first edge is blue$\}$ and $\mathcal{D} := L \cup \{v \in \mathcal{L} \mid \exists$ a path from $v$ to $L$ whose last edge is red$\}$. Then $\mathcal{C} \subseteq \mathcal{C}$, $\mathcal{D} \subseteq \mathcal{D}$, $\mathcal{D} \cap \mathcal{C} = \emptyset$, $\mathcal{C} \cup \mathcal{D} = \mathcal{V}$ and $\mathcal{L} \subseteq \mathcal{C} \cap \mathcal{D} = \mathcal{C} \cap \mathcal{D}^\prime$.

Suppose there is $v \in (\mathcal{C} \cap \mathcal{D}^\prime) \setminus \mathcal{L}$. Then there is a path $P$ from $L$ to $v$ whose first edge is blue and a path $P'$ from $v$ to $L$ whose last edge is red. Hence $P \cup P'$ contains a path $P''$ with first edge blue, last edge red, extremities in $L$ and intermediary nodes outside of $L$, or a circuit $Q$ with exactly one node in $L$. Applying transitivity to successive edges of $P''$ (of $Q$) of same color, one obtains a b-r-handle of $L$, a contradiction. Hence $\mathcal{L} = \mathcal{C} \cap \mathcal{D}$ and it is an intersection.
5.2 Lattice matrices and switching paths polyhedra

In relation to the polyhedral description of intersections, two classes of integer polyhedra occur, namely lattice polyhedra and switching paths polyhedra. In this section we state all definitions and results which will be used in the sequel.

**Lattice polyhedra**

Let $\mathcal{F}$ be a finite distributive lattice with partial order $\leq$, meet $\land$ and join $\lor$, minimal and maximal element $m$ and $M$. The function $f: \mathcal{F} \rightarrow \mathbb{R}$ is said to be submodular if for any $S, T \in \mathcal{F}$, $f(S \land T) + f(S \lor T) \leq f(S) + f(T)$, and modular if equality holds for any $S, T \in \mathcal{F}$. The $(0,1)$-valued function $f: \mathcal{F} \rightarrow \{0,1\}$ is consecutive if for any $S, T, R \in \mathcal{F}$ with $S \land T \leq R$, $f(S) = f(R) = 1$ implies $f(T) = 1$.

We consider matrices $A \in \{0,1\}^{F \times V}$ which are formed as follows: $\mathcal{F}$ is a lattice as above, $V$ is a finite set and the columns of $A$, say $f_v, v \in V$, are non-zero and define $(0,1)$-valued functions on $\mathcal{F}$ which are consecutive, modular, and such that $f_v(m) = f_v(M) = 0$. For brevity, let us call such a matrix $A$ a lattice matrix. Notice that $A$ contains rows with all elements zero (for the elements $m$ and $M$ of $\mathcal{F}$).

Lattice matrices together with a suitable right hand side vector describe a subclass of lattice polyhedra, and the following holds (Gröflin, Hoffman [1982]):

**Theorem 5.2**

Let $A \in \{0,1\}^{F \times V}$ be a lattice matrix and $r: \mathcal{F} \rightarrow \mathbb{Z}$ be a submodular function, $e, d \in (\mathbb{Z} \cup \{\pm \infty\})^V$. Then $P := \{x \in \mathbb{R}^V \mid e \leq x \leq d, Ax \leq r\}$ is integral and its system is tdi.

**Switching paths polyhedra**

The following definitions and results are due to Gröflin [1987].

Let $V$ and $\mathcal{A}$ be finite sets and $s := \{f(a) \in \mathbb{R}^V \mid a \in \mathcal{A}\}$ a family of $(1,0,-1)$-vectors. Denote for any vector $f \in \mathbb{R}^V$, by $supp^+(f) := \{v \in V \mid f_v > 0\}$ its positive support and by $supp^-(f)$ its negative support. Let further be given for any $a \in \mathcal{A}$ a linear ordering $\preceq$ of $supp(f(a))$ and define for any $a, b \in \mathcal{A}$ with $v \in supp(f(a)) \cap supp(f(b))$ the following sets:

\[
\{av\} := \{w \in supp(f(a)) \mid w \preceq v\}, \{vb\} := \{w \in supp(f(b)) \mid v \preceq w\} \text{ and } \{avb\} := \{av\} \cup \{vb\}.
\]
The family \( s \) is called a switching family if for any \( a, b \in \mathcal{A} \) and \( v \in \text{supp}(f(a)) \cap \text{supp}(f(b)) \) there exist \( c, d \in \mathcal{A} \), denoted by \( avb \) and \( bva \), such that the following properties hold:

9) \( \text{supp}(f(avb)) \cap \text{supp}(f(a)) \cap \text{supp}(f(b)) \subseteq \{ avb \} \)

10) \( \text{supp*}(f(avb)) \cap \{ bv \} \subseteq \text{supp*}(f(a)) \cap \text{supp*}(f(b)) \)

11) \( \text{supp*}(f(avb)) \cap \{ va \} \subseteq \text{supp*}(f(b)) \)

similar properties for \( bva \), to which we refer later as (9'), (10'), (11'), and

12) \( \text{supp}(f(avb)) \cap \text{supp}(f(bva)) \subseteq \text{supp}(f(a)) \cap \text{supp}(f(b)) \).

A function \( h: \mathcal{A} \to \mathbb{R} \) is said to be submodular on \( \mathcal{A} \) if for any \( a, b \in \mathcal{A} \) and \( v \in \text{supp}(f(a)) \cap \text{supp}(f(b)) \), \( h(avb) + h(bva) < h(a) + h(b) \), supermodular on \( \mathcal{A} \) if \(-h\) is submodular on \( \mathcal{A} \) and modular on \( \mathcal{A} \) if \( h \) is both super- and submodular on \( \mathcal{A} \).

Given a switching family \( s = \{ f(a) \in \mathbb{R}^V \mid a \in \mathcal{A} \} \), let \( \bar{A} \) be the \( \mathcal{A} \times V \)-matrix with rows \( f(a) \), \( a \in \mathcal{A} \). \( s \) is called submodular (modular) if all columns \( \bar{A}_v, v \in V \) of \( \bar{A} \) are submodular (modular) functions on \( \mathcal{A} \).

The main result then reads:

**Theorem 5.3**

Let \( s = \{ f(a) \in \mathbb{R}^V \mid a \in \mathcal{A} \} \) be a switching family, \( \bar{A} \in \mathbb{R}^\mathcal{A} \times V \) the matrix with rows \( f(a) \), \( a \in \mathcal{A} \), \( \mathcal{A} \to \mathbb{Z} \) a function and \( e, d \in \{ \mathbb{Z} \cup \infty \}^V \). Then the following polyhedra are integral with homogeneous-tdi systems:

13) \( P_1 := \{ x \in \mathbb{R}^V \mid e \leq x \leq d, \bar{A}x \geq r \} \) for \( s \) submodular and \( r \) supermodular on \( \mathcal{A} \).

14) \( P_2 := \{ x \in \mathbb{R}^V \mid e \leq x \leq d, \bar{A}x \leq r \} \) for \( s \) modular and \( r \) submodular on \( \mathcal{A} \).

**5.3 Polyhedral description**

Let \( \mathcal{C} \) and \( \mathcal{D} \) be proper ring families on \( V \), \( H = (V, E_b \cup E_r) \) be the graph representing \( \mathcal{C} \) and \( \mathcal{D} \), and consider the intersections

\[ L := \{ L \subseteq V \mid L = C \cap D, C \in \mathcal{C}, D \in \mathcal{D} \text{ and } C \cup D = V \} \]

Let \( (v_1, \ldots, v_{2n+1}) \) be a b-r-path \( P \) (b-r-circuit \( Q \)) of \( H \). The \( \{1,0,-1\} \)-vector \( x \in \mathbb{R}^V \) will be called the alternating vector of \( P \) (of \( Q \)) if \( x_{v_{2i+1}} = 1 \) for \( i = 0, \ldots, n \), \( x_{v_{2i}} = -1 \) if \( n \geq 1 \) and for \( i = 1, \ldots, n \), and \( x_v = 0 \) otherwise. For convenience we will also call an empty path a b-r-path and define its alternating vector to be equal to 0.

Let \( \bar{A}_i, i \in I \), be the alternating vectors of the b-r-paths of \( H \), \( \bar{R}_j, j \in J \), the alternating vectors of the b-r-circuits of \( H \). Denote by \( \bar{A} \) and \( \bar{R} \) the matrices with rows \( \bar{A}_i, i \in I \), and \( \bar{R}_j, j \in J \) and by \( A \) the incidence matrix of the intersections \( L \).

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Theorem 5.4

\[ P := \{ x \in R^V | Ax \leq 1 \} = \text{CONV}(\bar{A}) + \text{CONE}(\bar{R}) - \bar{R}^V, \]

\[ P^* := \{ x \in R^V | \bar{A}x \leq 1, \bar{R}x \leq 0 \} = \text{CONV}(A), \]

and the system \( Ax \leq 1 \) is tdi, the system \( x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0 \) is homogeneous-tdi.

Proof:

We first establish that

(i) \( \{ x \in R^V | \bar{A}x \leq 1, \bar{R}x \leq 0, x \text{ integer} \} = \{ x_L | x_L \text{ is the incidence vector of } L \in L \}. \)

Then we prove that

(ii) \( x \geq 0, \bar{A}x \leq 1, \bar{R}x \leq 0 \) is homogeneous-tdi,

hence \( P^* = \text{CONV}(A) \) and by polarity (theorem 2.20) \( P = \text{CONV}(\bar{A}) + \text{CONE}(\bar{R}) - \bar{R}^V. \)

Finally we show that

(iii) \( Ax \leq 1 \) is tdi.

Proof of (i):

"\( \supseteq \)" Let \( x_L \) be the incidence vector of an intersection \( L \) and \( \bar{A}_i \) be a row of \( \bar{A} \), i.e. the alternating vector of a b-r-path \( P \) with node set \( V(P) \). We show \( \bar{A}_i x_L = \bar{A}_i (L) \leq 1 \): If \( L \cap V(P) = \emptyset, \bar{A}_i x_L = 0 \), otherwise denote by \( P_1,...,P_m \) the maximal subpaths of \( P \) contained in \( L \) and ordered in the natural way. For \( 1 \leq k \leq m \), let \( v_k \) and \( w_k \) be the first and last node of \( P_k \). For any \( v \in V(P) \) with \( \bar{A}_{iv} = 1 \), the edge of \( P \) entering \( v \) is red and the edge of \( P \) leaving \( v \) is blue (if the edge is present). Therefore, for any \( 1 \leq k < m, \bar{A}_{iw_k} = 1 \) implies \( \bar{A}_{iv_{k+1}} = -1 \), otherwise the subpath of \( P \) from \( w_k \) to \( v_{k+1} \) is a b-r-handle of \( L \). Hence, the sequence of the components of \( \bar{A}_i \) which are in \( L \) does not contain any two consecutive 1's and \( \bar{A}_i x_L \leq 1 \). Analogously, \( \bar{R}_j x_L \leq 0 \) is shown for an alternating vector \( \bar{R}_j \) of a b-r-circuit.

"\( \subseteq \)" Let \( x \) be an integer vector in \( P^* \). Since each singleton \( \{ v \} \) is a b-r-path, the identity matrix \( I \) is contained in \( \bar{A} \). Therefore, \( x \) has all its components 0 or 1 and is the incidence vector of some set \( S \subseteq V \). If \( S \in L \), there exists a b-r-handle of \( S \). Let \( \bar{A}_i \) be its alternating vector if the handle is a path (\( \bar{R}_j \) if the handle is a circuit). Then \( \bar{A}_i x = \bar{A}_i (S) = 2 > 1 (\bar{R}_j x = \bar{R}_j (S) = 1 > 0), \) contradicting \( x \in P^* \).

Proof of (ii):

In chapter 6, example 1, we will show that \( P^* \) belongs to a generalized class of switching paths polyhedra and the proof is then given by corollary 6.2. The generalization is not necessary if \( \bar{R} \) is absent (i.e. \( \bar{R} = 0 \)), in which case \( P^* \) is a switching paths polyhedron of type (14). We now prove this fact (and hence (ii) for this case), because it was the basic motivation for developing the generalization of chapter 6.
Assume that \( H = (V, E_b \cup E_r) \) has no \( b \)-\( r \)-circuits and let \( I \) be the set of all \( b \)-\( r \)-paths. Then \( s := \{ A_P \mid P \in I \} \) is the set of all alternating vectors of \( b \)-\( r \)-paths. \( s \) is a modular switching family if the ordering \( \prec \) of the support of the alternating vector of the \( b \)-\( r \)-path \( P \) is the natural order on the node set of \( P \) and if the switching is defined as follows: Let \( P_1, P_2 \) be \( b \)-\( r \)-paths with ordered node sets \( V(P_1) := (p_1, \ldots, p_{2n+1}) \) and \( V(P_2) := (q_1, \ldots, q_{2m+1}) \) and \( v = p_i = q_j \). Observe that if for \( r, t \geq 1 \), \( p_{i+r} = q_{j+t} \), then \( r = t \), \( p_{i+k} = q_{j+k} \) and \( A_{P_1P_1+k} = A_{P_2P_2+k} \) for all \( 0 \leq k \leq t \), otherwise a \( b \)-\( r \)-circuit would exist. Denote by \( t \geq 0 \) the largest integer with \( p_{i+t} = q_{j+t} \). (By the above observation, \( t = 0 \) if \( A_{P_1v} = A_{P_2v} \)). Hence \( P_1vP_2 \) with ordered node set \( (p_1, \ldots, p_{i-t+1}, q_{j+t+1}, \ldots, q_{2m+1}) \) if \( i,j \) is odd and \( (p_1, \ldots, p_i = q_j, \ldots, q_{2m+1}) \) otherwise is a \( b \)-\( r \)-path. Let \( P_2vP_1 \) be defined analogously. Then it is easy to verify that \( s \) is a modular switching family. As \( r = 1 \) is submodular on \( I \), \( P^* = \{ x \in \mathbb{R}_+^V \mid A x \leq 1 \} \) is of type (14) and \( x \geq 0 \), \( A x \leq 1 \) is homogeneous-tdi.

Proof of (iii):
Let \( \mathcal{F} = \{(C,D) \mid C \in \mathcal{C}, D \in \mathcal{D} \text{ and } C \cup D = V\} \). \( \mathcal{F} \) with ordering, meet and join defined by \((C,D) \leq (C',D') \) if \( C \subseteq C' \) and \( D \supseteq D' \), \((C,D) \wedge (C',D') := (C \cap C', D \cap D') \) and \((C,D) \vee (C',D') := (C \cup C', D \cup D') \) is a distributive lattice. For all \( v \in V \), define \( f_v : \mathcal{F} \to \{0,1\} \) by \( f_v(C,D) = 1 \) if \( v \in C \cap D \) and \( f_v(C,D) = 0 \) otherwise, \( (C,D) \in \mathcal{F} \). It is easy to verify that \( f_v, v \in V \), is consecutive and modular and have value zero on the minimum \((\emptyset, V)\) and the maximum \((V, \emptyset)\) of \( \mathcal{F} \). Then the matrix \( A^\mathcal{F} \) with columns \( f_v, v \in V \), is a lattice matrix and by theorem 5.2 the system \( A^\mathcal{F} x \leq 1 \) is tdi. As \( A^\mathcal{F} \) is equal to \( A \) up to duplicate rows, \( P \) belongs to the class of lattice polyhedra defined in theorem 5.2 and \( A x \leq 1 \) is also tdi.

Notice that by theorem 5.4, \( x \geq 0 \), \( A x \leq 1 \), \( R x \leq 0 \) is a polyhedral description of the intersections. Moreover, \((P,P^*)\) is a pair of polar polyhedra (theorem 2.20) and \((A,0), (A,\overline{A})\) is \((A,\overline{A})\)-perfect (corollary 4.15 ).

5.4 Optimization problems

Optimum intersection problem

Let \( \mathcal{L} := \{ L \in \mathcal{V} \mid L = C \cap D, C \in \mathcal{C}, D \in \mathcal{D} \text{ and } C \cup D = V\} \) and \( c \in \mathbb{R}_+^V \) be a weight vector.

We consider the problem
\[(15) \text{ Find an intersection } L_{\text{max}} \text{ of maximum weight, i.e. } c(L_{\text{max}}) = \max \{ c(L) \mid L \in \mathcal{L} \}. \]

We show that this problem can be reduced to the one of finding a set of maximum weight in a proper ring family, which has been solved by Picard [1976] via a flow problem.
Let $V^* := V_1 \cup V_2$, where $V_1$ and $V_2$ are two distinct copies of $V$. For $v \in V$ and $S \subseteq V$, we denote by $v_1$, $S_1$ and $v_2$, $S_2$ the corresponding copies in $V_1$ and $V_2$, respectively. Then

\[(16) \quad \mathcal{B} := \{C_1 \cup (V_2 \setminus D_2) \mid C \in \mathcal{C}, D \in \mathcal{D} \text{ and } C \cup D = V\}\]

is a proper ring family on $V^*$. For the weight vector $d \in \mathbb{R}^{V^*}$ defined by $d_{v_1} := c_v$ and $d_{v_2} := -c_v$ for $v \in V$, and $L = C \cap D \in \mathcal{L}$, we have

\[
c(L) = c(C) + c(D) - c(V) = d(C_1) - d(D_2) + d(V_2) = d(C_1 \cup (V_2 \setminus D_2)).
\]

Hence an intersection of maximum weight with respect to the weight vector $c$ corresponds to an element of $\mathcal{B}$ of maximum weight with respect to $d$ and conversely.

**Partitioning problem**

We consider the problem

\[(17) \quad \text{Given a set } S \subseteq V, \text{ find a partition of } S \text{ into a minimal number of intersections (if such a partition exists).}
\]

Recall that $A$ is the incidence matrix of the family $\mathcal{L}$. For $c \in \mathbb{N}^V$, consider the LP and its dual

\[
\begin{align*}
(18) & \quad \max \ cx \quad \text{subject to } \ Ax \leq 1 \\
(19) & \quad \min \ 1y \quad \text{subject to } \ yA = c, \ y \geq 0.
\end{align*}
\]

By theorem 5.4, both LP's can be solved with integer solutions $x^*$ and $y^*$ (provided (19) has a feasible solution). In particular, if $c$ is the incidence vector of a set $S \subseteq V$, $y^*$ solves the minimum partitioning problem (17) on $S$.

**Algorithm**

An algorithm based on a longest path computation in a digraph is now developed, which solves both LP's.

We use the fact that the incidence matrix $A^F$ of $\mathcal{F} := \{(C,D) \mid C \in \mathcal{C}, D \in \mathcal{D} \text{ and } C \cup D = V\}$ is a lattice matrix and is equal to $A$ up to duplicate rows (see proof (iii) of theorem 5.4). Hence we know from Gröflin, Hoffman [1982] that

\[(20) \quad \min \ 1y' \quad \text{subject to } \ y' A^F = c, \ y' \geq 0 \]

has an optimal solution $\bar{y}$, whose support corresponds to a chain of $\mathcal{F}$, say $(C_1, D_1) > (C_2, D_2) > ... > (C_n, D_n)$, i.e. $C_1 \supseteq C_2 \supseteq ... \supseteq C_n$ and $D_1 \subseteq D_2 \subseteq ... \subseteq D_n$. We shall use the idea introduced by Ford and Fulkerson [1962] that a chain of sets can be represented by a so-called potential function $u \in \mathbb{R}^V$. More precisely, given $\bar{y}$, define $u: V \to \mathbb{R}$ by

\[
u_v := \sum \{\bar{y}_{(C_i, D_j)} \mid v \in C_i \text{ and } 1 \leq i \leq n\}
\]

and let $\pi_0 := 0, \pi_i := \sum \{\bar{y}_{(C_j, D_j)} \mid j \leq i\}, i = 1, ..., n$. 

Then one verifies that

\[(21) \quad C_i = \{ v \mid u_v \geq \pi_i \}, \quad D_i = \{ v \mid u_v - c_v < \pi_i \}, \quad i = 1, \ldots, n,\]

\[(22) \quad \mathcal{Y}(C_i, D_i) = \pi_i - \pi_{i-1}, \quad i = 1, \ldots, n, \text{ and} \]

\[(23) \quad 1\mathcal{Y} = \max \{ u_v \mid v \in V \}. \]

(The expression for \(D_i\) can be shown as follows: given \(v \in V\), let \(k\) be the largest index with \(v \in C_k\) and \(m\) be the smallest index with \(v \in D_m\), convening \(k = 0\) if \(v\) is in no \(C_i\) and \(m = n + 1\) if \(v\) is in no \(D_i\). \(c_v = \sum \{ \mathcal{Y}(C_i, D_j) \mid v \in C_i \cap D_j, \quad 1 \leq i \leq k \} = \pi_k - \pi_{m-1} = u_v - \pi_{m-1}\), hence \(\pi_{m-1} = u_v - c_v\). Since \(v \in D_i\) iff \(n \geq i > m-1\), we have \(v \in D_i\) iff \(\pi_i > \pi_{m-1} = u_v - c_v\). Hence \(D_i = \{ v \mid u_v - c_v < \pi_i \}\).)

Conversely, given a suitable function \(u\), the relations (21) and (22) define a feasible solution \(\mathcal{Y}\) of (20):

**Lemma 5.5**

Given \(u : V \to \mathbb{R}\), associate to \(u\) the values \(\pi_0 < \ldots < \pi_n\) such that \(\pi_0, \ldots, \pi_n\) are the distinct elements of \(\{ u_v, u_v - c_v \mid v \in V \}\) and assume \(\pi_0 = 0\). If

\[(24) \quad (v, w) \in E_b \Rightarrow u_w \geq u_v \quad \text{and} \quad (v, w) \in E_r \Rightarrow u_w - c_w \geq u_v - c_v \]

then \(\mathcal{Y}\) defined by (21) and (22) is a feasible solution of (20) with \(1\mathcal{Y} = u^* = \max \{ u_v \mid v \in V \}\). Moreover, \(y\) given by

\[(25) \quad y_L := \sum \{ \mathcal{Y}(C_i, D_j) \mid L = C_i \cap D_j, \quad 1 \leq i \leq n \} \quad \text{for all } L \in \mathcal{L} \]

is a feasible solution of (19) with \(y = u^*\).

**Proof:**

Conditions (24) are necessary and sufficient for the sets \(C_i\) and \(D_i\) defined by (21) to be members of \(C\) and \(D\): Sufficiency is straightforward. For the necessity, assume that \(C_i \in \mathcal{C}\), \(1 \leq i \leq n\) and \(u_w < u_v\) for \((v, w) \in E_b\). Then there exists \(i \geq 1\) with \(\pi_i = u_v\). Hence \(v \in C_i\) and \(w \notin C_i\), a contradiction. Now assume that \(D_i \in \mathcal{D}\), \(1 \leq i \leq n\) and \(u_w - c_w < u_v - c_v\) for \((v, w) \in E_r\). Then there exist \(1 \leq i \leq n\) with \(\pi_i = u_v - c_v\). Hence \(w \notin D_i\) and \(v \notin D_i\), a contradiction.

For \(v \in V\), \(\sum \{ \mathcal{Y}(C_i, D_j) \mid v \in C_i \cap D_j, \quad 1 \leq i \leq n \} = \sum \{ \mathcal{Y}(C_i, D_j) \mid u_v - c_v < \pi_i \leq u_v, \quad 1 \leq i \leq n \} = u_v - (u_v - c_v) = c_v\) and \(\mathcal{Y} \geq 0\), hence \(\mathcal{Y}\) is feasible and \(1\mathcal{Y} = u^*\). The derivation of \(y\) from \(\mathcal{Y}\) is obvious. Notice that a function \(u\) satisfying (24) is defined up to an additive constant, so that \(\pi_0 = 0\) can be assumed.

The following algorithm constructs a potential function \(u\) through a longest path computation in an extended graph \(G'\) and determines a dual solution \(y\), or proves that no such solu-
tion exists. In the former case, it derives a primal solution \( x \) with value \( cx = 1y \), proving optimality of both \( x \) and \( y \).

**Partitioning Algorithm**

(0) Given \( H = (V, E_b \cup E_r) \) and \( c \in \mathbb{R}^\chi \), construct the graph \( G' = (V', E') \) where

\[
V' := V \cup \{s\} \quad \text{and} \quad E' := E_b \cup E_r, \quad E'_r := E_r \cup \{(s, v) \mid v \in V\},
\]

and define for all \((v, w) \in E'\) the length

\[
d(v, w) := \begin{cases} 
c_w - c_v & \text{for } (v, w) \in E'_r \quad \text{(with the convention } c_s := 0) \\
0 & \text{for } (v, w) \in E_b
\end{cases}
\]

(1) Find a longest path from \( s \) to each node \( v \in V \). One of cases (a) or (b) occurs:

(a) Either there is a circuit of positive length \( \varepsilon \). Using transitivity for consecutive edges of same color, obtain a b-r-circuit \( Q \) of \( H \), whose alternating vector \( R_Q \) satisfies \( cR_Q = \varepsilon > 0 \). (18) has no finite optimum and (19) no solution. Stop.

(b) Else, let \( u_v \) be the length of a longest path from \( s \) to \( v \) for any \( v \in V \), \( u^* := \max \{u_v \mid v \in V\} \) and \( P^* \) be a longest path of length \( u^* \), ending in \( v^* \).

(2) Dual solution

Let \( 0 = \pi_0 < \ldots < \pi_n = u^* \) be the different values of \( \{u_v, u_v - c_v \mid v \in V\} \) and \( C_i := \{v \mid u_v \geq \pi_i\} \), \( D_i := \{v \mid u_v - c_v < \pi_i\} \), \( i = 1, \ldots, n \). Define

\[
y_L := \sum \{\pi_i - \pi_{i-1} \mid L = C_i \cap D_i, \ 1 \leq i \leq n\} \quad \text{for all } L \in L.
\]

\( y \) is an optimal solution of (19).

(3) Primal solution

Obtain a b-r-path \( P \) of \( H \) from \( P^* \), using transitivity for consecutive edges of same color and deleting the first edge of \( P^* \) and terminal blue edges if present. The alternating vector \( x \) of \( P \) is an optimal solution of (18).

**Proof:**

Observe first that \( H = G \setminus \{s\} \), that any s-v path (from \( s \) to \( v \)) has its first edge colored red and that any circuit of \( G' \) is a circuit of \( H \). Moreover, by the choice of \( d \) and transitivity for edges of same color:

\[
d(v, w) + d(w, z) = d(v, z) \quad \text{for any } (v, w), (w, z), (v, z) \in E_b \cup E_r.
\]

Hence for any s-v path or circuit in \( G' \), say \( Q' \) with node set \( V(Q') \) and weight \( d(Q') \), there is an s-v path, respectively a circuit, say \( Q \), such that \( V(Q) \subseteq V(Q') \), \( d(Q) = d(Q') \) and the edges of \( Q \) are alternatively red and blue.

In step 1 (a), a circuit \( Q' \) in \( G' \) of length \( d(Q') > 0 \) is identified and hence a b-r-circuit \( Q \) of \( H \) with \( d(Q) = d(Q') \) is readily found. The alternating vector \( R_Q \) of \( Q \) satisfies \( cR_Q = d(Q) > 0 \).
0. As by theorem 5.4, \( P := \{ x \in \mathbb{R}^V \mid Ax \leq 1 \} = \text{CONV}(\mathbb{A}) + \text{CONE}(\mathbb{R}) - \mathbb{R}^V \), there is no finite optimum of (18) and no solution for (19). Notice that the converse also holds, i.e. if (19) has no solution, there is an alternating vector \( \bar{R}_j \) of a b-r-circuit with \( c\bar{R}_j > 0 \), and therefore a circuit in \( G' \) of positive length.

To show that \( y \) of step 2 and \( x \) of step 3 are optimal, we first establish that the \( u_v \)'s satisfy (24) and \( \pi_0 = 0 \). This follows from the choice of \( d \) and the \( u_v \)'s being longest \( s-v \) path lengths. Then \( y \) is feasible and \( 1y = u^* \) by lemma 5.5. Next, if \( P^* \) is a longest path of length \( u^* \) from \( s \) to \( v^* \), its first edge \( (s,v) \) is red. We can assume that its edges are alternatively red and blue by the above observation, and that the last one is red, since a last blue edge has length zero and can be deleted. Deleting the first edge \( (s,v) \) yields a b-r-path \( P \). Its alternating vector \( x \) satisfies \( cx = d(P^*) = u^* \). \( x \) and \( y \) are optimal by duality.

Relation to submodular flows

We establish now that the polyhedron \( P := \{ x \in \mathbb{R}^V \mid Ax \leq 1 \} \) is a member of the class of submodular flow polyhedra introduced by Edmonds and Giles [1977]. A general algorithm for solving the LP's associated to this class has been developed by Frank [1982,1984a, 1984b] and Cunningham and Frank [1985]. This algorithm applied to our special case is compared to our algorithm above.

Let \( G^* = (V^*,E^*) \) be a digraph, \( S \) a crossing family of subsets of \( V \) (i.e. \( S, T \in \mathcal{S}, S \cap T \neq \emptyset, S \cup T = V \Rightarrow S \cup T, S \cap T \in \mathcal{S} \), \( f : \mathcal{S} \to \mathbb{R} \) submodular on crossing pairs (i.e. \( f(S) + f(T) \geq f(S \cap T) + f(S \cup T) \) for \( S, T \in \mathcal{S} \), \( S \cap T \neq \emptyset, S \cup T = V \)) and \( a, b \in \{ 0, \pm \infty \}^E \). A theorem of Edmonds and Giles [1977] states that the following linear system is tdi:

\[
z(\delta(\mathcal{S})) - z(\delta(S)) \leq f(S), \quad S \in \mathcal{S}, \quad a \leq z \leq b.
\]

In Schrijver [1984], the equivalence between the so-called submodular flow model (26) and a lattice polyhedron (of a more general form than the one mentioned in theorem 5.2) is shown. We give here a formulation of problem (18) and (19) using a system of the form (26).

Define the digraph \( G^* = (V^*,E^*) \) as follows:

Let \( V_1 \) and \( V_2 \) be two copies of \( V \) as in the paragraph "optimum intersection problem". \( V^* := V_1 \cup V_2 \), \( E^* := E_v \cup E_b \cup E_r \), where \( E_v := \{(v_2,v_1) \mid v \in V\}, E_b^1 := \{(v_1,w_1) \mid v, w \in V, (v,w) \in E_b\} \) and \( E_r^2 := \{(v_2,w_2) \mid v, w \in V, (v,w) \in E_r\} \).

Let \( \mathcal{S} := \{ S \subseteq V^* \mid \delta(S) = \emptyset \} \). Then \( S = C \cup F_2 \), with \( C \subseteq V_1 \) and \( F_2 \subseteq V_2 \), is a member of \( \mathcal{S} \) iff \( C \cap (V \setminus F) \) is an intersection and \( v \in C \cap (V \setminus F) \) iff \( (v_2,v_1) \in \delta(S) \).
The system

\[(27) \quad z(\delta(S)) \leq 1, \quad S \in \mathcal{S}; \quad z_e = 0 \quad \text{for all } e \in E^1_0 \cup E^2_0\]

is of form (26) and problems (18) and (19) are equivalent to

\[(28) \quad \max \; c^*z \; \text{subject to} \; \sum_{e \in E} \{ y_S \mid e \in \delta(S) \} = c^*_e, \quad e \in E_V; \quad \sum_{e \in E} \{ y_S \mid e \in \delta(S) \} + w_e - q_e = 0, \quad e \in E^1_0 \cup E^2_0, \]

\[(29) \quad \min \; y \; \text{subject to} \; y, w, q \geq 0; \quad \sum \{ y_S \mid e \in \delta(S) \} = c^* e, \quad e \in E^1_0 \cup E^2_0, \]

where \(c^* \in \mathbb{R}^E^* \) is given by \(c^*_e = c_v \) for \(e = (v_2, v_1) \in E_V, \) \(c^*_e = 0 \) otherwise.

For the general problem: \(\max c^*z \; \text{subject to} \; (26), \) Cunningham and Frank [1985] developed a primal-dual algorithm which constructs a primal feasible solution \(z\) and a potential \(p: V^* \rightarrow \mathbb{R}\) satisfying the following conditions:

\[(30) \quad (v, r) \in E^*, \quad c_{(v, r)} + p_v - p_r < 0 \Rightarrow z(v, r) = a(v, r) \ (> -\infty), \]

\[(31) \quad (v, r) \in E^*, \quad c_{(v, r)} + p_v - p_r > 0 \Rightarrow z(v, r) = b(v, r) \ (< +\infty), \]

\[(32) \quad V^* \rightarrow \mathbb{R} \; \text{and} \; S = \{ v \in V^* \mid p_v > p_{v_1} \} \Rightarrow \min \{ f(S) - z(\delta(S)) \mid S \in \mathcal{S}, \ v \in S, \ r \in S \} = 0. \]

An optimal solution \(\bar{y}\) of the dual LP is then derived as follows. Let \(\pi_0 < \ldots < \pi_n\) be the different values of \(\{ p_v \mid v \in V^* \} \), then \(S_i := \{ v \in V^* \mid p_v \geq \pi_i \} \) and \(\bar{y}_S := \pi_i - \pi_{i-1}\) if \(S = S_i\) \((1 \leq i \leq n)\) and \(\bar{y}_S := 0\) otherwise.

Our algorithm above gives a dual solution \(y\), a potential \(u \in \mathbb{R}^V\) and a primal solution \(x\). Using the equivalence between problems, it is easy to see that solution \(z\) of (28) corresponding to \(x\) and the potential function \(p \in \mathbb{R}^V^*\) defined by

\[p_{v_1} := u_v; \quad p_{v_2} = u_v - c_v \quad \text{for all } v \in V,\]

satisfies conditions (30), (31). Notice that for \(C_i \cap D_i\) defined in (21) with the help of \(\pi_i,\)
\(y_{C_i \cap D_i} > 0,\) hence \(x(C_i \cap D_i) = 1.\) To verify (32), it suffices to find \(i\) such that for \(v, r \in V^*, \ v \in S := (C_i)_1 \cup V_2 (D_i)_2, \ r \in S.\) Taking \(i\) such that \(\pi_i = p_v\) satisfies this condition. Moreover, the sets \(C_i \cap D_i\) and \(S_i = \{ v \in V^* \mid p_v \geq \pi_i \} \) correspond to each other, as well as \(y\) and the dual solution \(\bar{y}\) of (29).

The primal-dual algorithm starts with a feasible solution \(z\) and a potential satisfying condition (32). Successively, flow \(z\) and potential \(p\) are changed in order to also satisfy (30) and (31). For each flow change, an auxiliary network has to be constructed, an operation which requires an oracle for each pair \((v, r) \in V^* \times V^*.\) In our case, the oracle is available and consists in solving the problem of finding an intersection of maximum weight. Of course, this general method would require a substantially larger computational effort than our partitioning algorithm, since already the oracle has about the same complexity as our algorithm.
5.5 About the condition $C \cup D = V$

Instead of the family of intersections

$$L = \{ C \cap D : C \in \mathcal{C}, D \in \mathcal{D} \text{ and } C \cup D = V \},$$

consider

(33) $$L_0 = \{ C \cap D : C \in \mathcal{C}, D \in \mathcal{D} \}.$$

Again, associate to $\mathcal{C}$ and $\mathcal{D}$ the digraph $H = (V,E_b \cup E_r)$. Then $L$ is a member of $L_0$ iff it admits no $b$-$r$-handle $(q_1,q_2,q_3)$ with exactly one blue and one red edge: If $L = C \cap D \in L_0$ and $(q_1,q_2,q_3)$ is a $b$-$r$-handle, then $q_2 \in C$ and $q_2 \in D$, a contradiction. Conversely if $L$ contains no $b$-$r$-handle $(q_1,q_2,q_3)$, then $C := L \cup \{ w \in V \mid \exists v \in L \text{ with } (v,w) \in E_b \} \in \mathcal{C}$, $D := L \cup \{ w \in V \mid \exists v \in L \text{ with } (w,v) \in E_r \} \in \mathcal{D}$ and $L = C \cap D \in L_0$.

Consider

(34) Given $c \in \mathcal{R}^V$, find $L \in L_0$ of maximum weight.

(35) Given $S \subseteq V$, find a partitioning of $S$ into a minimal number of sets $L \in L_0$.

**Proposition 5.6:**

Problems (34) and (35) are NP-hard.

**Proof:**

We show that the problems of finding a maximum weight stable set and a minimum coloring in an arbitrary undirected graph $G'$ can be reduced to (34) and (35) (these problems are known to be NP-hard).

Let $G' = (V',E')$ and orient its edges arbitrarily. Subdivide each (oriented) edge $(v,w) \in E'$ obtaining the oriented edges $(v,x)$ and $(x,w)$ and color $(v,x)$ blue and $(x,w)$ red. Let $H := (V' \cup V_E,E_b \cup E_r)$ be the resulting digraph, $\mathcal{C}$ and $\mathcal{D}$ be the associated ring families, and $L_0$ be defined as above. By the characterization of the members of $L_0$, $S$ is a stable set in $G'$ iff $S \subseteq V'$ and $S \in L_0$. The proposition follows.
Chapter 5 has been devoted to find a polyhedral description of a family of intersections in the form
\[ P^* := \{ x \in \mathbb{R}^V_+ \mid \bar{A}x \leq 1, \bar{R}x \leq 0 \} = \text{CONV}(A), \]
where \( V \) is the ground set of the intersections, \( A \) their incidence matrix and \( \bar{A}, \bar{R} \) \((1,0,-1)\)-matrices. Recall that the support of the rows of \( \bar{A} \) and \( \bar{R} \) correspond to paths and circuits in a directed graph. We already saw in theorem 5.4 that when \( \bar{R} \) is absent (i.e. \( \bar{R} = 0 \)), \( P^* \) belongs to the family of switching paths polyhedra defined by Gröflin [1987].

Models involving switching of (abstract) paths have been first introduced by Hoffman [1974] as a generalization of max flow - min cut: He established a class of integer polyhedra of the type \( \{ x \geq 0 \mid \bar{A}x \geq r \} \), where \( \bar{A} \) is the incidence matrix of a family of abstract paths closed with respect to "switching", and where \( r \) is in a certain sense "supermodular". Gröflin generalized this model in order to contain also polyhedra of the form (1) for the case where \( \bar{R} \) is not present. His new model contains a class of integer polyhedra, called "switching paths polyhedra", of the type \( \{ e \leq x \leq d \mid \bar{A}x \leq r \} \), where the rows of \( \bar{A} \) are \((1,0,-1)\)-vectors associated with abstract paths (again closed under "switching") and \( r \) is "submodular". In his paper he mentioned that it would be interesting to find a common frame for his switching paths polyhedra and the so-called "cfow polyhedra" of Cameron [1982]. This
latter polyhedra are integer and of the type \( \{ e \leq x \leq d \mid R \leq x \leq r \} \), where \( R \) is the incidence matrix of circuits of a digraph and \( r \) is "modular".

Our goal was to find such a frame which additionally includes polyhedra of the type (1).

We define a new notion of switching family in which objects related to paths and objects related to circuits can be considered simultaneously. Given such a family, two \( \{1,0,-1\} \)-matrices \( \bar{A} \) and \( R \) are defined, associated with paths and circuits, respectively, yielding integer polyhedra of the form

\[
Q := \{ x \in \mathbb{R}^V \mid \bar{A}x \leq r_1, R \leq r_2, e \leq x < d \}
\]

(for appropriate vectors \( e \), \( d \), and submodular functions \( r_1 \), \( r_2 \)). Moreover, their describing systems are homogeneous-tdi.

The adopted approach is very close to the one used by Gröflin. We extract the essence of his arguments to apply them in our more general context. In order to distinguish between his switching paths polyhedra and the ones to be developed, we adopt the notation \( G \)-switching paths polyhedra and \( G \)-switching family for the terms referring to his concept.

Our goal will almost be achieved. The new class of polyhedra does in fact contain polyhedra of the type (1) and more general ones, as well as the class of coflow polyhedra. Concerning the \( G \)-switching paths polyhedra, we include the most important part of it, especially all examples, but not all cases.

In relation with perfectness, observe that if \( \bar{A} \) contains the identity matrix, \( e := 0 \), \( d := \infty \) and \( r_1 := 1 \), \( r_2 := 0 \), then

\[
Q = \text{CONV}(A)
\]

for some \( \{0,1\} \)-matrix \( A \). Since \( x \geq 0 \), \( \bar{A}x \leq 1 \), \( R \leq 0 \) is homogeneous-tdi, the pair \(((A,0),(\bar{A},R))\) is \( \bar{A} \)-perfect (corollary 4.15). This aspect of switching family will be the central object of chapter 7, where for a special case the related pair will be shown to be \((A,\bar{A})\)-perfect.

### 6.1 Switching family

Let \( V \) be a finite set. We consider notions similar to paths and circuits in a directed graph, where additionally a number +1 or -1 is associated with each node of them. Such entities will again be called paths or circuits. The main object we are interested in is a family of paths and circuits called a switching family. For such a family, a completeness property is required which generalizes in some sense natural properties of paths and circuits in a digraph, like "decomposition", i.e. any non-elementary path decomposes into an elementary path and elementary circuits, or "switch", i.e. given two crossing paths with node set \( (v_1, \)

...,v_n), respectively (w_1,...,w_m), and v_i = w_j for some i and j, 1 ≤ i ≤ n, 1 ≤ j ≤ m, then (v_1,...,v_i = w_j,...,w_m) is again a path.

In relation to switching families several definitions are introduced in this section. In order to give an intuitive support to these rather technical definitions, we shall illustrate them in the context of intersections. Recall that intersections can be represented in an edge-bicolored graph H = (V,E_b∪E_r), where both E_b and E_r are transitive. Furthermore, they are described by means of b-r-paths and b-r-circuits together with related alternating vectors. Actually, the latter are typical examples of the kind of paths and circuits we are interested in.

Paths, circuits and partial paths

We define a path as an object [(f_1,F_1),...,(f_n,F_n)] with f_i∈ V, F_i∈ {1,-1} for all i, 1 ≤ i ≤ n, n ≥ 1. It is said to be an elementary path if f_i ≠ f_j for all i and j, 1 ≤ i < j ≤ n.

A path is called a circuit if n ≥ 2 and (f_1,F_1) = (f_n,F_n) and it is an elementary circuit if additionally f_i ≠ f_j for all i and j, 1 ≤ i < j < n.

The set of elements of V in a path f := [(f_1,F_1),...,(f_n,F_n)] will be denoted by \{f\} := \{f_1,...,f_n\}.

For convenience we will also allow empty paths / ] and call them circuits, too. Furthermore if f := [(f_1,F_1),...,(f_n,F_n)] is a path and 1 ≤ i < k ≤ n, then [(f_k,F_k),...,(f_i,F_i)] := [ ].

Given f := [(f_1,F_1),...,(f_n,F_n)], a partial path f' of f is a path f' := [(f'_1,F'_1),...,(f'_p,F'_p)] such that there exists 1 ≤ i(1) < ... < i(p) ≤ n with (f'_k,F'_k) = (f_i(k),F_i(k)) for k = 1,...,p. Two kinds of partial paths are of special interest, namely for v∈ \{f\}, [fv] := [(f_i(v),F_i(v)),...,(f_n,F_n)] and [vf] := [(f_i(v),F_i(v)),...,(f_v,F_v)], where f_i(v) is the first occurrence of v in the sequence (f_1,...,f_n), i.e. i(v) := min{i | f_i = v, 1 ≤ i ≤ n}.

An important operation on paths is the so-called composition of two paths. Let a := [(a_1,A_1),...,(a_n,A_n)] and b := [(b_1,B_1),...,(b_m,B_m)] be paths with a_n = b_1. The composition of a and b, denoted by [a;b], is the path [a;b] := [(a_1,A_1),...,(a_n,A_n) = (b_1,B_1),...,(b_m,B_m)] if A_n = B_1 or [a;b] := [(a_1,A_1),...,(a_{n-1},A_{n-1}), (b_2,B_2),...,(b_m,B_m)] if A_n ≠ B_1.

Example 1.1

The b-r-path P in H = (V,E_b∪E_r) (figure 6.1) with (ordered) node set (1,2,3,4,5) and alternating vector x^P = (1,-1,1,-1,1,0) corresponds to the path p = [(1,1),(2,-1),(3,1), (4,-1),(5,1)] and the b-r-circuit Q with node set (1,2,3,6,1) and x^Q = (1,-1,0,0,-1) to the circuit [(1,1),(2,-1),(3,1),(6,-1),(1,1)]. For v = 4, [pv] = [(1,1),(2,-1),(3,1),(4,-1)], [[pv]; [vp]] = p and the path d := [[pv];[(4,1),(6,-1),(1,1)]] = [(1,1),(2,-1),(3,1),(6,-1),(1,1)]. d
corresponds to a b-r-circuit as well as to a non-elementary b-r-path, i.e. a non-elementary path whose edges are alternately blue and red; the first one blue, the last one red.

\[ H = (V, E_b \cup E_r), \quad P: b-r-path, \quad Q: b-r-circuit. \]

**Figure 6.1**

Notice that the notion of b-r-paths introduced in the context of intersections was related to elementary paths. As seen in the above example, this notion extends in an obvious way to non-elementary b-r-paths. Moreover, if \((b_1, ..., b_{2n+1})\) is the node set of a b-r-path, its corresponding path is defined by \([((b_1,B_1), ..., (b_{2n+1},B_{2n+1})) \] with \(B_i := (-1)^{i+1}, 1 \leq i \leq 2n+1.\)

**Families \((B, Z)\), associated vectors and matrices**

A family of paths and circuits is a pair \((B, Z)\) with \(B := \{b^i \mid b^i \text{ is a path}, i = 1, ..., \alpha \} \) and \(Z := \{z^j \mid z^j \text{ is a circuit}, j = 1, ..., \beta \}\). \(\alpha, \beta \in \mathbb{Z}^+ \cup \{+\infty\}\). Moreover, for any, \(z^j = [(z_1, Z_1), ..., (z_n, Z_n)]\), \(1 \leq j \leq \beta\), and for any s, \(1 \leq s \leq n\), there exists \(1 \leq k(s) \leq \beta\) with \(z^{k(s)} = [(z_s, Z_s), ..., (z_n, Z_n) = (z_1, Z_1), ..., (z_s, Z_s)] \in Z\) (i.e. any cyclic permutation of \(z^j\) is again in \(Z\)). Notice that a particular path or circuit can occur more than once in \(B\) or \(Z\). Furthermore, a circuit of \(Z\) may but does not have to be in \(B\) (recall that by definition a circuit is a path). The meaning of \(b \equiv [(b_1, B_1), ..., (b_n, B_n)] \in B (\in Z)\) is that there exists i, \(1 \leq i \leq \alpha (1 \leq i \leq \beta), \) with \(b^i = [(b_1, B_1), ..., (b_n, B_n)] (z^i = [(b_1, B_1), ..., (b_n, B_n)])\). By \(B \subset B\) and \(Z \subset Z\), we denote the elementary elements of \(B\) and \(Z\), respectively.

Several vectors are associated in a natural way to a path \(a = [(a_1, A_1), ..., (a_n, A_n)] \in B (\in Z)\), namely its inhomogeneity vectors \(inh(a)^* \in Z^V, * = +, -\), and value vector \(val(a) \in Z^V\) defined by

\[
inh(a)^v := \Sigma (*A_i \mid 1 \leq i \leq n \ (1 \leq i < n), \ A_i = *1, \ a_i = v), \ v \in V, \ * = +, -;
\]

\[
val(a)^v := inh(a)^v - inh(a)^v.
\]

The matrices \(\overline{A} \in (1,0,-1)^B \times V\) and \(\overline{R} \in (1,0,-1)^Z \times V\) of the family \((B, Z)\) are matrices,
whose rows are the value vectors of $B^\prime$ and $Z^\prime$, i.e. $\bar{A}_b := \text{val}(b)$, be $B^\prime$ and $\bar{R}_b := \text{val}(b)$, be $Z^\prime$.

Example 1.2

Let $H = (V, E_b \cup E_r)$ be the graph associated with a family of intersections and $B$ be the set of all paths corresponding to (possibly non-elementary) b-r-paths in $H$, $Z$ the set of all circuits corresponding to (possibly non-elementary) b-r-circuits together with cyclic permutations of them. $B^\prime$ and $Z^\prime$ correspond to the elementary b-r-paths and b-r-circuits of $H$ and the matrices $\bar{A}$, $\bar{R}$ of $(B, Z)$ are up to duplicate rows exactly the matrices whose rows are the alternating vectors of the elementary b-r-paths and b-r-circuits.

In the sequel, we will simply call b-r-path a "path corresponding to a (possibly non-elementary) b-r-path", and analogously for b-r-circuits.

![Figure 6.2](image)

For the b-r-path $P_1$ and $P_2$ represented in figure 6.2,

inh($P_1$)$^+$ = $(1, 0, 1, 0, 1, 1)$, inh($P_2$)$^+$ = $(1, 0, 1, 0, 1, 1)$,

inh($P_1$)$^-$ = $(0, 2, 0, 1, 0, 0)$, inh($P_2$)$^-$ = $(0, 1, 1, 0, 0)$,

val($P_1$) = $(1, -2, 1, -1, 1, 1)$, val($P_2$) = $(1, -1, 0, -1, 1, 1)$.

Decomposition and switching

The family $(B, Z)$ of paths and circuits is said to be closed under decomposition, if for any $b := [(b_1, B_1), ..., (b_n, B_n)] \in B$ (respectively $Z$), and any $i$ and $j$ with $1 \leq i < j \leq n$ (respectively $1 \leq i < j < n$), $b_i = b_j$, and $b_e \neq b_k$ for all $e$ and $k$ such that $i \leq e < k < j$, we have

(2) if $b_i = b_j$, then

$r := [(b_i, B_i), ..., (b_j, B_j)] \in Z'$,

$b' := [(b_1, B_1), ..., (b_i, B_i) = (b_j, B_j), ..., (b_n, B_n)] \in B$ (resp. $Z$),
if \( B_i \neq B_j \), then

\[
\begin{align*}
\text{if } i < j - 1 & \quad \Rightarrow \quad \{[(b_{i+1}, B_{i+1}), \ldots, (b_j, B_{j-1}), (b_{j+1}, B_{i+1})]\} \in Z' \\
\text{otherwise} & \quad \Rightarrow \quad \{[(b_{1}, B_{1}), \ldots, (b_{i-1}, B_{i-1}), (b_{i+1}, B_{i+1}), \ldots, (b_n, B_n)]\} \text{ if } i \neq 1 \text{ or } b \in \mathcal{B}
\end{align*}
\]

\[
\begin{align*}
b' := & \quad \{[(b_{j+1}, B_{j+1}), \ldots, (b_{n-1}, B_{n-1}), (b_{n}, B_{n})]\} \text{ if } be \in Z, i = 1, j < n - 1 \\
& \quad \{[(b_{j+1}, B_{j+1}), \ldots, (b_{n-1}, B_{n-1}), (b_{n}, B_{n})]\} \text{ if } be \in Z, i = 1, j = n - 1
\end{align*}
\]

\( \in \mathcal{B} \) (respectively Z).

For \( a \in \mathcal{B} \) (respectively Z), \( a' \Theta R^a \) (respectively \( \Theta R^a \)) is a decomposition of \( a \), if \( a' \in \mathcal{B}' \) and \( R^a \subseteq Z' \), where the same circuit can occur more than once, and \( a' \) and \( R^a \) can be obtained by the following algorithm:

\[
\begin{align*}
b := a &= [(b_1, B_1), \ldots, (b_n, B_n)], R^a := \emptyset; \\
\text{while } b \notin \mathcal{B}' \text{ (respectively Z') do} \\
\text{Choose } 1 \leq i < j < n \text{ (respectively } 1 \leq i < j < n) \text{ with } b_i = b_j \text{ and} \\
\quad b_e \neq b_k \text{ for all } e \text{ and } k, i \leq e < k < j; \\
\text{Let } b' \text{ and } r \text{ be the elements obtained in (2);} \\
\quad R^a := R^a \cup r, b := b'; \\
\text{end; (*while*)} \\
\text{if } a \in \mathcal{B} \text{ then } a' := b \text{ else } R^a := R^a \cup b.
\end{align*}
\]

Notice that if \( a = a' \Theta R^a, a \in \mathcal{B} \) then

\[
\begin{align*}
\text{val}(a) &= \text{val}(a') + \Sigma \{\text{val}(c) \mid c \in R^a\} \text{ and} \\
\text{inh}(a)^* & \geq \text{inh}(a')^* + \Sigma \{\text{inh}(c)^* \mid c \in R^a\} \text{ for } * = +,-,
\end{align*}
\]

and analogously for \( a \in Z \).

**Example 1.3**

The family \( (\mathcal{B}, Z) \) of (possibly non-elementary) b-r-paths and b-r-circuits in \( H = (V, E_b \cup E_r) \) is closed under decomposition as \( E_b \) and \( E_r \) are transitive.
Notice that for a given path P, there may be several decompositions. In figure 6.4, two decompositions of the path \( p = [(p_1,P_1),..., (p_9,P_9)] \) are illustrated.

For the inhomogeneity vector \( \text{inh}^* \), \( * = +, - \), different values are obtained for the different decompositions, namely

\[
\begin{align*}
\text{inh}(p)^+ &= (1,1,1,0,0,1,1), \quad \text{inh}(p)^- = (0,1,0,2,1,0,0), \\
\text{inh}(q)^+ + \text{inh}(z)^+ &= (1,1,1,0,0,1,1), \quad \text{inh}(q)^- + \text{inh}(z)^- = (0,1,0,2,1,0,0), \\
\text{inh}(p)^+ + \text{inh}(r)^+ &= (1,0,1,0,0,1,1), \quad \text{inh}(p)^- + \text{inh}(r)^- = (0,0,0,2,1,0,0).
\end{align*}
\]
The family \((\mathcal{B}, \mathcal{Z})\) is said to be **closed under switching** if

(i) For all \(a \in \mathcal{B}\), \(b \in \mathcal{B}'\) and all \(v \in \{a\} \cap \{b\}\), there exist \(avb := [(c_1, C_1), \ldots, (c_p, C_p)]\) and 
\(bva := [(d_1, D_1), \ldots, (d_q, D_q)] \in \mathcal{B}\) with the following properties:
- There exist partial paths \(c'\) of \(avb\), \(d'\) of \(bva\), \(e'\) of \(e := [[av];[vb]]\), \(f'\) of \(f := [[bv];[va]]\) \(c' = e'\), \(d' = f'\).
- For any element \((c_i, C_i)\) of \(avb\), \(1 \leq i \leq p\), which does not belong to the partial path \(c'\), we have \(c_i \in \{bva\}\). Furthermore, \(c_i \notin \{a\} \cup \{b\}\) or there exists an element \((e_k, E_k)\) of \(e\) not belonging to \(e'\) with \(C_i \neq E_k\). Moreover, if \(c_i = c_k\) for some \(k\), \(1 \leq k \leq p\), \(i \neq k\) and \((c_k, C_k)\) does not belong to \(c'\), then \(C_i = C_k\). (Analogously for elements not belonging to \(d'\).)

(ii) For all \(a \in \mathcal{B}\) (respectively \(\mathcal{Z}\)), \(b := [(b_1, B_1), \ldots, (b_n, B_n)] \in \mathcal{B}'\) such that \(v := b_1 \in \{a\}\) and no decomposition of \(a\) contains \(b\) (or a cyclic permutation of it), there exists \(avb := [(c_1, C_1), \ldots, (c_p, C_p)] \in \mathcal{B}\) (respectively \(\mathcal{Z}\)), with the following properties:
- There exist partial paths \(c'\) of \(avb\), \(e'\) of \(e := [[av];b];[va]\) with \(c' = e'\).
- For any element \((c_i, C_i)\) of \(avb\), \(1 \leq i \leq p\) (respectively \(1 \leq i < p\)), which does not belong to \(c'\), we have \(c_i \notin \{a\} \cup \{b\}\) or there exists an element \((e_k, E_k)\) not belonging to \(e'\) with \(C_i \neq E_k\). Furthermore, if \(c_i = c_k\) for some \(k\), \(1 \leq k \leq p\), \(i \neq k\) and \((c_k, C_k)\) does not belong to \(c'\), then \(C_i = C_k\).

**Example 1.4**

Let \((\mathcal{B}, \mathcal{Z})\) be the family of b-r-paths and b-r-circuits in \(H = (V, E_b \cup E_r)\). Define the switching for \((\mathcal{B}, \mathcal{Z})\) by \(avb := [[av];[vb]]\), \(bva := [[bv];[va]]\) if \(a \in \mathcal{B}\), \(b \in \mathcal{B}'\), \(v \in \{a\} \cap \{b\}\); and \(avb := [[av];b];[va]\) if \(a \in \mathcal{B} \cup \mathcal{Z}\), \(b \in \mathcal{B}'\), \(v \in \{a\} \cap \{b\}\) such that \(avb\) has to be defined (i.e. \(b = [(b_1, B_1), \ldots, (b_n, B_n)]\) with \(v = b_1 \in \{a\}\) and no decomposition of \(a\) contains \(b\) or a cyclic permutation of it).

By transitivity of \(E_b\) and \(E_r\), \(avb\) and \(bva\) are again b-r-paths or b-r-circuits. Hence \((\mathcal{B}, \mathcal{Z})\) is a switching family.
A family \((\mathcal{B}, \mathcal{Z})\) of paths and circuits is called a *switching family*, if it is closed under decomposition and switching. Notice that the set \(V\) is implicit in this definition and we call \(V\) the *ground set* of \((\mathcal{B}, \mathcal{Z})\).

We now introduce the notion of (sub)modularity for a switching family \((\mathcal{B}, \mathcal{Z})\):

A function \(g: (\mathcal{B}, \mathcal{Z}) \to \mathbb{R}\) is said to be *submodular* on \((\mathcal{B}, \mathcal{Z})\), if

(i) For any \(a \in \mathcal{B}\), \(b \in \mathcal{B}'\), \(v \in \{a\} \cap \{b\}\) and any decomposition of \(a = a' \Theta R^a\), \(avb = avb' \Theta R^{avb}\), and \(bva = bva' \Theta R^{bva}\),

\[
g(a') + g(b) + \sum \{g(r) \mid r \in R^a\} \geq g(avb') + g(bva') + \sum \{g(r) \mid r \in R^{avb} \cup R^{bva}\}\]

(Recall that a circuit can occur more than once in \(R^{avb}\) and in \(R^{bva}\) in which case each occurrence contributes to the summation on the right hand side.)

(ii) For any \(a \in \mathcal{B}\) (respectively \(Z\)), \(b = [(b_1, B_1), \ldots, (b_n, B_n)] \in \mathcal{Z}\), \(v \in \{a\} \cap \{b\}\), such that \(avb\) is defined, and any decomposition of \(a = a' \Theta R^a\) (respectively \(a = \Theta R^a\)) and \(avb = avb' \Theta R^{avb}\) (respectively \(avb = \Theta R^{avb}\)),

\[
g(a') + \sum \{g(r) \mid r \in R^a \cup b\} \geq g(avb') + \sum \{g(r) \mid r \in R^{avb}\}\] if \(a \in \mathcal{B}\),

\[
\sum \{g(r) \mid r \in R^a \cup b\} \geq \sum \{g(r) \mid r \in R^{avb}\}\] if \(a \in \mathcal{Z}\).

\(g\) is called *supermodular* on \((\mathcal{B}, \mathcal{Z})\), if \((-g)\) is submodular on \((\mathcal{B}, \mathcal{Z})\), and *modular* on \((\mathcal{B}, \mathcal{Z})\), if \(g\) is submodular and supermodular on \((\mathcal{B}, \mathcal{Z})\).

\((\mathcal{B}, \mathcal{Z})\) is said to be a *modular* (submodular) *switching family*, if all columns of its matrices \(\overline{A}\) and \(R\), i.e.

\[
g_v: (\mathcal{B}', \mathcal{Z}') \to \{1, 0, -1\}, v \in V\text{ with } g_v(b) := \overline{A}_{bv}, b \in \mathcal{B}'\text{ and } g_v(b) := \overline{R}_{bv}, b \in \mathcal{Z}'\]

are modular (submodular) on \((\mathcal{B}, \mathcal{Z})\).

Notice that a switching family is submodular if and only if for all \(a \in \mathcal{B} \cup \mathcal{Z}\), \(b \in \mathcal{B}' \cup \mathcal{Z}'\), \(v \in \{a\} \cap \{b\}\) such that \(avb\) is defined,
val(a) + val(b) ≥ val(avb) + val(bva) if \( a \in \mathcal{B}, b \in \mathcal{B}' \),
val(a) + val(b) ≥ val(avb) otherwise,

and modular if and only if equality holds.

**Example 1.5**

Let \((\mathcal{B}, \mathcal{Z})\) be the family of \(b\)-r-paths and \(b\)-r-circuits in \(H = (V, E_B \cup E_U)\). \((\mathcal{B}, \mathcal{Z})\) is a switching family and by the definition of the switching rule for \(a \in \mathcal{B} \cup \mathcal{Z}, b \in \mathcal{B}' \cup \mathcal{Z}'\), \(v \in \{a\} \cup \{b\}\) such that \(avb\) is defined (example 1.4),

\[
val(a) + val(b) = val(avb) + val(bva) \quad \text{if} \quad a \in \mathcal{B}, \quad b \in \mathcal{B}',
\]
\[
val(a) + val(b) = val(avb) \quad \text{otherwise}.
\]

Hence \((\mathcal{B}, \mathcal{Z})\) is a modular switching family.

### 6.2 Theorem

In this section, the main theorem of this chapter will be proved, namely that the matrices \(\bar{A}, R\) of a modular or submodular switching family \((\mathcal{B}, \mathcal{Z})\) together with a suitable right hand side define integer polyhedra with homogeneous-tdi systems.

**Theorem 6.1**

Let \((\mathcal{B}, \mathcal{Z})\) be a switching family, \(\mathcal{B}', \mathcal{Z}'\) its elementary paths and circuits and \(\bar{A} \in \{1, 0, -1\}^{\mathcal{B} \times V}, \bar{R} \in \{1, 0, -1\}^{Z \times V}\) its matrices. For \(r = (r_1, r_2): (\mathcal{B}', \mathcal{Z}') \rightarrow \mathbb{Q} \) and \(e, d \in \{\mathbb{Q} \cup \pm \infty\}^V\) the following systems are homogeneous-tdi:

(i) \(e \leq x \leq d, \bar{A}x \geq r_1, \bar{R}x \geq r_2\), for \((\mathcal{B}, \mathcal{Z})\) submodular and \(r\) supermodular on \((\mathcal{B}, \mathcal{Z})\).

(ii) \(e \leq x \leq d, \bar{A}x \leq r_1, \bar{R}x \leq r_2\), for \((\mathcal{B}, \mathcal{Z})\) modular and \(r\) submodular on \((\mathcal{B}, \mathcal{Z})\).

**Corollary 6.2**

Let \((\mathcal{B}, \mathcal{Z})\) be a switching family, \(\mathcal{B}', \mathcal{Z}'\) its elementary paths and circuits and \(\bar{A} \in \{1, 0, -1\}^{\mathcal{B} \times V}, \bar{R} \in \{1, 0, -1\}^{Z \times V}\) its matrices. For \(r = (r_1, r_2): (\mathcal{B}', \mathcal{Z}') \rightarrow \mathbb{Z} \) and \(e, d \in \{\mathbb{Z} \cup \pm \infty\}^V\) the following polyhedra are integral with homogeneous-tdi systems:

(3) \(P_1 := \{x \in \mathbb{R}_+^V \mid e \leq x \leq d, \bar{A}x \geq r_1, \bar{R}x \geq r_2\}\), for \((\mathcal{B}, \mathcal{Z})\) submodular and \(r\) supermodular on \((\mathcal{B}, \mathcal{Z})\).

(4) \(P_2 := \{x \in \mathbb{R}_+^V \mid e \leq x \leq d, \bar{A}x \leq r_1, \bar{R}x \leq r_2\}\), for \((\mathcal{B}, \mathcal{Z})\) modular and \(r\) submodular on \((\mathcal{B}, \mathcal{Z})\).
Notice that if in theorem 6.1 homogeneous-tdi is replaced by the weaker term tdi, theorem 6.1 says that the systems $x \geq 0$, $\overline{A}x \geq r^1$, $\overline{R}x \geq r^2$ in case (i) respectively $\overline{A}x \leq r^1$, $\overline{R}x \leq r^2$ in case (ii) are box-tdi.

In relation to perfectness, observe that switching families can be used for generating $\overline{A}$-perfect pairs:

Let $(B, Z)$ be a submodular switching family and $\overline{A}$, $\overline{R}$ its matrices. Clearly $(r^1 := -1, r^2 := 0)$ is modular on $(B, Z)$. If $(-\overline{A})$ contains the identity matrix, $e := 0$, $d := \infty$, then

$$P^1 = \{x \in \mathbb{R}^V \mid (-\overline{A})x \leq 1, (-\overline{R})x \leq 0\} = \text{CONV}(A)$$

for some $\{0,1\}$-matrix $A$. Hence by theorem 2.20, $(P, Q)$ with $P := \{x \in \mathbb{R}^V \mid Ax \leq 1\}$, $Q := P^1$ is a pair of polar polyhedra and since $x \geq 0$, $(-\overline{A})x \leq 1$, $(-\overline{R})x \leq 0$ is homogeneous-tdi (theorem 6.1), $((A, 0), (\overline{A}, \overline{R}))$ is $\overline{A}$-perfect (corollary 4.15). Analogously, special cases of modular switching families $(B, Z)$ in relation with polyhedra of type (4) yield $\overline{A}$-perfect pairs.

In relation to modular switching families, we first give two lemmas which will be used in the proof of theorem 6.1. The first deals with a property of the switching operation. The second lemma considers a feasibility problem for which there exists an integer solution whenever there is a solution; it will be useful since any linear programming problem can be reduced by means of the complementary slackness conditions to a feasibility problem.

**Lemma 6.3**

Let $(B, Z)$ be a switching family.

(i) Let $a \in B$, $b \in B^\dagger$, $v \in \{a\} \cup \{b\}$, $e = [(e_1, E_1), \ldots, (e_n, E_n)] := [[av]; [vb]]$ and $f = [(f_1, F_1), \ldots, (f_m, F_m)] := [[bv]; [va]]$ such that for any $p \in V$, either $p$ occurs at most twice in $e$ and $f$ (i.e. $\|i \mid e_i = p, 1 \leq i \leq n\| + \|i \mid f_i = p, 1 \leq i \leq m\| \leq 2$), or all occurrences of $p$ in $e$ and $f$ have the same value (i.e. $e_i = e_k = p$ or $f_i = f_k = p$) for some indices $i$ and $k$ implies $E_i = E_k$ ($F_i = F_k$ or $F_i = F_k$).

If $\text{val}(a) + \text{val}(b) = \text{val}(avb) + \text{val}(bva)$, then $avb$ is a partial path of $e$ and $bva$ a partial path of $f$.

(ii) Let $a \in B \setminus (Z)$, $b = [(b_1, B_1), \ldots, (b_n, B_n)] \in Z^\dagger$, $v := b_1 \in \{a\}$, such that $avb$ is defined, and $e = [(e_1, E_1), \ldots, (e_n, E_n)] := [[av]; b]; [vb]]$. Moreover, for any $p \in V$, either $p$ occurs at most twice in $e$ (i.e. $\|i \mid e_i = p, 1 \leq i \leq n\| \leq 2$), or all occurrences of $p$ have the same value (i.e. $e_i = e_k = p$ for some indices $i$ and $k$ implies $E_i = E_k$).

If $\text{val}(a) + \text{val}(b) = \text{val}(avb)$, then $avb$ is a partial path of $e$. 
Proof:
We prove only case (i), case (ii) being similar. Let \( avb = [(c_1,C_1),..., (c_p,C_p)] \), \( bva = [(d_1,D_1),..., (d_q,D_q)] \) and \( c', d', e', f' \) be the partial paths of \( avb, bva, e \) and \( f \) used in the definition of switching. We have to show that \( c' = avb \) and \( d' = bva \).

By assumption
\[
(*) \quad \text{val}(avb) + \text{val}(bva) = \text{val}(a) + \text{val}(b) = \text{val}(e) + \text{val}(f).
\]

Assume there is an element \((q,Q) \notin c' \) (i.e. \((c_i,C_i) \) does not belong to \( c' \)), then \( c_i \notin \{bva\} \). If \( c_i \notin \{a\} \cup \{b\} \), then
\[
0 = \text{val}(a)_{c_i} + \text{val}(b)_{c_i} = \text{val}(avb)_{c_i} = \Sigma \{C_k | c_k = c_i, 1 \leq k \leq p\}
\]
\[
= C_i \{k | c_k = c_i, 1 \leq k \leq p\} \neq 0,
\]
a contradiction.

Hence \( c_i \in \{a\} \cup \{b\} \) and there exists \( 1 \leq j \leq n \), \((e_j,E_j) \in e' \) with \( C_i = -E_k \). By \((*)\) and \( c' = e', d' = f' \), \( c_i \notin \{bva\} \) we have
\[
\Sigma \{C_j | c_j = c_i, 1 \leq j \leq p, (c_j,C_j) \in c'\} = \Sigma \{E_j | e_j = c_i, 1 \leq j \leq n, (e_j,E_j) \in e'\} + \Sigma \{F_j | f_j = c_i, 1 \leq j \leq q\}.
\]
The left side is equal to \( C_i \{j | c_j = c_i, 1 \leq j \leq p, (c_j,C_j) \in c'\} =: C_i K, K \geq 1 \). Consider the possibilities for the number of terms on the right hand side, which we denote by \( S \). It contains:
- one element. Then \( S = E_k = -C_i \neq C_i K \), a contradiction,
- two elements, namely \( S = E_k + E_j \) or \( S = E_k + F_j \) for an index \( j \). Then \( S = 0 \) or \( S = -C_i 2 \) and in both case \( S \neq C_i K \), a contradiction,
- \( h \geq 2 \) elements. Then \( S = -C_i h \neq C_i K \), a contradiction.

Hence \((c_i,C_i) \notin c' \) does not exist and \( c' = avb \). By the same arguments \( d' = bva \).

Lemma 6.4
Let \( c \in \mathbb{Z}^V \) be an integral vector, \((B,Z) \) a modular switching family conforming to \( c \) (i.e. for any \( v \in V \) and \( b = [(b_1,B_1),..., (b_n,B_n)] \in B \cup Z \), \( b_i = v, 1 \leq i \leq n \) implies \( B_i : c_v \geq 0 \)), and \( \bar{A}, \bar{R} \) be the matrices of \((B,Z) \). Moreover, let \( J_1 \cup J_2 \cup J_3 = V, J_i \cap J_k = \emptyset, i \neq k, 1 \leq i, k \leq 3 \).

Then the system of inequalities:
\[
(5) \quad \begin{align*}
y \bar{A} J_1 + w \bar{R} J_1 &= c J_1 \\
y \bar{A} J_2 + w \bar{R} J_2 &\geq c J_2 \\
y \bar{A} J_1 + w \bar{R} J_3 &\leq c J_3 \\
y, w &\geq 0
\end{align*}
\]
has an integer solution \((y,w)\) whenever it has a solution.
Proof:
We can assume that $c \geq 0$, otherwise let $V^+ := \{ v \in V | c_v \geq 0 \}$ and define a family $(B^*, Z^*)$ by

$$ B^*(Z^*) := \left\{ [\{(b_1, C_1), \ldots, (b_n, C_n)\} \in B(Z), C_i := B_i \right\}$$

if $b_i \in V^+$, $C_i := -B_i$ otherwise,

c^* := \{c_v, v \in V \} and $J_1^* := J_1, J_2^* := J_2 \cap V^+ \cup (J_3 \setminus V^+), J_3^* := (J_3 \cap V^+) \cup (J_2 \setminus V^+)$. Then

$$(B^*, Z^*)$$

is a modular switching family conforming to $c^* \geq 0$ and $(y, w)$ is a solution of (5) with respect to $(B, Z)$ and $c$ iff $(y, w)$ is a solution of (5) with respect to $(B^*, Z^*)$ and $c^*$.

As $c \geq 0$, be $B \cup Z$ with

$$ (6) \quad b = [(b_1, B_1), \ldots, (b_n, B_n)] \text{ implies } B_i = 1, 1 \leq i \leq n. $$

Consider now the switch operation for $a \in B, b \in B', v \in \{a\} \cap \{b\}$. By lemma 6.3, $a b v$ is a partial path of $e := [(a v);[v b]]$, $b v a$ of $f := [(v b);[v a]]$. By modularity, $\text{val}(a b v) + \text{val}(b v a) = \text{val}(a) + \text{val}(b) = \text{val}(e) + \text{val}(f)$, and by (6) any element of $e (f)$ has to belong to $a b v (b v a)$.

Hence $a b v = [(a v);[v b]]$ and $b v a = [(v b);[v a]]$. By the same arguments, for $a \in B(Z), b \in B', v \in \{a\} \cup \{b\}$ such that $a b v$ is defined, $a b v = [(a v);[v b]]$.

We now construct a node constraint network $G = (V', E)$ with integral capacities such that any solution $(y, w)$ of (5) corresponds to a feasible flow in $G$ and conversely, any decomposition of an integral feasible flow in paths and circuits yields an integer solution of (5).

Let $V^b := \{ v \in V | \exists [(b_1, B_1), \ldots, (b_n, B_n)] \in B', b_1 = v \}, V^e := \{ v \in V | \exists [(b_1, B_1), \ldots, (b_n, B_n)] \in B', b_n = v \}$ and define $G = (V', E)$ by

$$ V' := \{ s, t \} \cup V,$$

$$ E := \{ (s, v) \in V^b \} \cup \{ (v, t) \in V^e \} \cup \{ (v, w) \in V^b \cup V^e | \exists b = [(b_1, B_1), \ldots, (b_n, B_n)] \in B' \cup Z', 1 \leq i \leq n \text{ with } b_i = v, b_{i+1} = w \}. $$

We show that for any elementary dipath (dicircuit) of $G$ with node set $(p_1, \ldots, p_n), p_i \in V, 1 \leq i \leq n$, and $p_1 \in V^b, p_n \in V^e$ if it is a dipath, $[(p_1, 1), \ldots, (p_n, 1)] \in B^*(Z')$.

Let $(p_1, \ldots, p_n)$ be a dipath of $G$. Choose a path $a = [(a_1, 1), \ldots, (a_m, 1)] \in B$ with $i(a) := \text{max} \{ i | a_j = p_j, 1 \leq j \leq i \}$ maximal. Notice that we can assume that $[a_i(a)]a$ is elementary. Then $i(a) \geq 1$.

Assume $i(a) < n$. Then there exists an elementary path or circuit $b = [(b_1, 1), \ldots, (b_k, 1)] \in B \setminus Z'$ with $b_e = a_{i(a)}, b_{e+1} = p_{i(a)+1}$ for some $e, 1 \leq e < k$ if $b \in B'$, $e = 1$ otherwise. For $v := b_e, avb$ is defined and $avb = [(a v);[v b]]$ if $b \in B'$, $avb = [(a v);[v b]]$ if $b \in Z'$ and $i(awb) > i(a)$, a contradiction. Hence $i(a) = n$. As $p_n \in V^e$ there exists an elementary path $b = [(b_1, 1), \ldots, (b_k, 1)] \in B'$ with $b_k = p_n$. Hence $avb = [(a v);[v b]] = [(p_1, 1), \ldots, (p_n, 1)] \in B'$. 

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Let now \((p_1,...,p_n = p_1)\) be a dicircuit of \(G\). By similar arguments as above, there is a path \(a = [(a_1,1),...,(a_1 = p_1,1),...,(a_{i+n-1} = p_n,1),...,(a_m,1)] \in \mathcal{B} \cup \mathcal{Z}\) for which \(a_i\) is the first occurrence of \(v := p_1\). By the decomposition property \([(a_i = p_1,1),...,(a_{i+n-1} = p_n,1)] \in \mathcal{Z}'\).

The digraph \(G\), together with the hereafter defined lower and upper bounds \(\mathcal{L}_v\) and \(u_v\), \(v \in V'\), is a node constraint network. For \(L\) and \(u\) we take \(L_s := L_t := 0, u_s := u_t := \infty, L_v := u_v := c_v\) for all \(v \in J_1, L_v := c_v\) and \(u_v := \infty\) for all \(v \in J_2, L_v := 0\) and \(u_v := c_v\) for all \(v \in J_3\).

To any feasible solution \((y,w)\) of (5) corresponds a feasible s-t-flow in \(G\) in the following way: For any path \(b = [(b_1,B_1),...,(b_n,B_n)] \in \mathcal{B} \cup \mathcal{Z}'\), \((s,b_1,...,b_n)\) is the node set of a dipath \(P(b)\), if \(b \in \mathcal{B}'\) and, if \(b \in \mathcal{Z}'\), \((b_1,...,b_n)\) is the node set of a dicircuit \(P(b)\) of \(G\). Let \(x^{P(b)} \in \mathbb{R}^E\) be the edge-incidence vector of \(P\). Then

\[
\Sigma \{y_b \cdot x^{P(b)} \mid b \in \mathcal{B}'\} + \Sigma \{w_b \cdot x^{P(b)} \mid b \in \mathcal{Z}'\}
\]

is a feasible flow in \(G\). Since \(c\) is integral, there exists an integer feasible s-t-flow in \(G\) if (5) has a solution. Any integral path- and circuit decomposition of its integer flow yields then an integer solution \((y,w)\) of (5).

Proof of theorem 6.1:

We only prove case (i), case (ii) being similar. Consider \((\mathcal{B},\mathcal{Z})\) submodular, \(r = (r_1,r_2)\):

\((\mathcal{B}',\mathcal{Z}') \to \mathbb{Q}\) supermodular on \((\mathcal{B},\mathcal{Z})\), \(e, d \in \{\mathbb{Q} \cup \{\infty\}\}^V\) and

\[
(7) \quad \min cx \text{ subject to } \bar{A}x \geq r_1, \bar{R}x \geq r_2, \quad e \leq x \leq d,
\]

\[
(8) \quad \max yr_1 + wr_2 + q e - pd \text{ subject to } y\bar{A} + w\bar{R} + q - p = c, \quad y,w,q,p \geq 0.
\]

Let \(c\) be integral, and such that there exist an optimum solution \((y,w,q,p)\) of (8).

Notice that

\[
(9) \quad \text{if } d_v = \infty, \text{ then } p_v = 0, \forall v \in V.
\]

\[
(10) \quad q_v^* := \max\{0,c_v - y\bar{A}v - w\bar{R}v\}, \quad p_v^* := \max\{0,y\bar{A}v - w\bar{R}v - c_v\}, \quad \text{then } (y,w,q^*,p^*) \text{ is an optimal solution of (8)}.
\]

\[
(11) \quad \text{If } y^*, w^* \text{ satisfy }
\]

\[
y^*\bar{A} + w^*\bar{R} \leq y\bar{A} + w\bar{R},
\]

\[
y^*r_1 + w^*r_2 \geq yr_1 + wr_2
\]

then \((y^*,w^*,q^*,p^*)\) with \(q^*, p^*\) defined as in (10) is an optimum solution of (8).

In the sequel for any solution \((y,w,q,p)\) of (8) we assume that \(q\) and \(p\) are defined by (10).

Consider the set \(S_1\) of optimal solutions of (8) (with \(q\) and \(p\) defined by (10)). Let \(S_2 \subseteq S_1\) denote those solutions \((y,w,q,p)\) which are minimal with respect to \(y\bar{A} + w\bar{R}\) (i.e. there exists no other solution \((y^*,w^*,q^*,p^*)\) in \(S_1\) with \(y^*\bar{A} + w^*\bar{R} \leq y\bar{A} + w\bar{R}\) and \(y^*\bar{A} + w^*\bar{R} \neq y\bar{A} + w\bar{R}\)). Furthermore, denote by \(S_3 \subseteq S_2\) the set of solutions of \(S_2\) with minimal inho-
mogeneity vector and finally, let $S^4$ be the solutions $(y,w,q,p)$ of $S^3$ with maximum support of $(y,w)$ (with respect to inclusion). Let $(y,w,q,p) \in S^4$.

Our goal is to construct a sub-switching family $(\mathcal{A}, \mathcal{W})$ of $(\mathcal{B}, \mathcal{Z})$, whose elementary elements correspond to the support of $(y,w)$, and which is modular and conforming to a certain vector $c^*$. Furthermore, we will find a system of the form (5) for $(\mathcal{A}, \mathcal{W})$ such that any feasible solution of (5) yields an optimum solution of (8), ensuring the existence of an integer homogeneous optimum solution of (8).

In some sense $S^2$ guarantees modularity, $S^3$ conformity, and $S^4$ closure under switching required for the switching family $(\mathcal{A}, \mathcal{W})$.

We need first some properties of the chosen solution $(y,w,q,p)$.

Let $a \in \mathcal{B}$, $b \in \mathcal{B}'$, $v \in \{a\} \cap \{b\}$ and $a = a^* \Theta R^a$ a decomposition of a with $y_{a^*} : y_b > 0$, $w_r > 0$ for all $r \in R^a$ (case 1); or $a \in \mathcal{B}$, $b \in \mathcal{Z}'$, $v \in \{a\} \cap \{b\}$ and $a = a' \Theta R^a$ a decomposition of a with $y_{a'} : y_b > 0$, $w_r > 0$ for all $r \in R^a \cup b$ (case 2); or $a \in \mathcal{Z}$, $b \in \mathcal{Z}'$, $v \in \{a\} \cap \{b\}$ and $a = \Theta R^a$ a decomposition of a with $w_r > 0$, $r \in R^a \cup b$ (case 3). Furthermore $avb$ shall be defined. Let $avb = avb' \Theta R^{avb}$ (case 1,2), $bva = bva' \Theta R^{bva}$ (case 1) and $avb = \Theta R^{avb}$ (case 3) be any decomposition.

Then

\begin{align*}
(12) \quad & \text{case 1: } f(a') + f(b) + \Sigma \{f(r) \mid r \in R^a\} = f(avb') + f(bva') + \Sigma \{f(r) \mid r \in R^{avb} \cup R^{bva}\} \\
& \text{case 2: } f(a') + \Sigma \{f(r) \mid r \in R^a \cup b\} = f(avb') + \Sigma \{f(r) \mid r \in R^{avb}\} \\
& \text{case 3: } \Sigma \{f(r) \mid r \in R^a \cup b\} = \Sigma \{f(r) \mid r \in R^{avb}\}
\end{align*}

for any $f = g_i$, $i \in V$ (column-function of $\overline{A}$ and $\overline{R}$) and $f = r$. Moreover, for a small $\varepsilon$, $(y^*, w^*, q^*, p^*)$ with

\begin{align*}
(13)\quad y^* := \begin{cases} 
    y - \varepsilon (e_{a^*} + e_b - e_{avb'} - e_{bva'}) & \text{if case 1} \\
    y - \varepsilon (e_{a^*} - e_{avb'}) & \text{if case 2} \\
    y & \text{if case 3}
\end{cases} \\
w^* := \begin{cases} 
    w - \varepsilon (\Sigma \{e_r \mid r \in R^a\} - \Sigma \{e_r \mid r \in R^{avb} \cup R^{bva}\}) & \text{if case 1} \\
    w - \varepsilon (\Sigma \{e_r \mid r \in R^a \cup b\} - \Sigma \{e_r \mid r \in R^{avb}\}) & \text{if case 2 or 3}
\end{cases}
\end{align*}

is an optimum solution in $S^2$:

We have "\geq" in (12) for any $f = g_i$, $i \in V$, as $g_i$ is submodular, and "\leq" for $f = r$ as $r$ is supermodular. Hence $(y^*, w^*, q^*, p^*)$ is admissible and by (11) also optimal. Hence (12) holds for $f = r$. If (12) were not true for some $f = g_i$, $i \in V$, $(y^*, w^*, q^*, p^*)$ would be an optimum solution, smaller as $(y, w, q, p)$ with respect to $(y\overline{A} + w\overline{R})$. Hence (12) is true for any $f = g_i$, $i \in V$ and $(y^*, w^*, q^*, p^*)$ is an optimum solution in $S^2$.

We show next that $(y, w, q, p)$ is a homogeneous solution.

Let $h(y, w, q, p)^*$, $* = +, -$ and $h(y, w, q, p)$ be its inhomogeneity vectors, then
\[
(14) \quad h(y, w, q, p)^* = \Sigma \{ y_a \cdot \text{inh}(a)^* \mid a \in B \} + \Sigma \{ w_a \cdot \text{inh}(a)^* \mid a \in Z \}, \quad * = +, -.
\]
Assume that \((y, w, q, p)\) is not homogeneous. Then there exists \(i \in V, a, b \in B^*, y_a y_b > 0\) and \(\overline{A}_{ai} \overline{A}_{bi} = -1\) (case 1); or \(a \in B^*, b \in Z^*, y_a w_b > 0\) and \(\overline{A}_{ai} \overline{R}_{bi} = -1\) (case 2); or \(a, b \in Z^*, w_a w_b > 0\) and \(\overline{R}_{ai} \overline{R}_{bi} = -1\). Let \(v := i\) and in case 2, 3 consider the row \(\overline{R}_{bi}\), as the row corresponding to an elementary circuit starting with \(v\), i.e. \(b = [(b_1, B_1), ..., (b_n, B_n)]\), \(b_1 = v\).

For each decomposition of \(avb\) (and \(bva\) in case 1), let \((y^*, w^*, q^*, p^*)\) be the solution of (13). By (12), \(\text{val}(a) + \text{val}(b) = \text{val}(avb) + \text{val}(bva)\) in case 1 and \(= \text{val}(avb)\) in case 2, 3.

As \(a\) and \(b\) are elementary, by lemma 6.3, \(avb\) is a partial path of \(e := [[av];[vb]]\), \(bva\) a partial path of \(f := [[bv];[va]]\) in case 1; \(avb\) a partial path of \(e := [[av];[vb]]\) in case 2, 3.

Then for any \(j \in V\), and \(x = 1\) if \(j = v\), \(x = 0\) otherwise,

\[
\text{inh}(a)^* + \text{inh}(b)^* = \begin{cases} 
\text{inh}(e)^* + \text{inh}(f)^* + x & \text{if case 1} \\
\text{inh}(e)^* + x & \text{if case 2, 3} \\
\text{inh}(avb)^* + \text{inh}(bva)^* + \Sigma \{ \text{inh}(r)^* \mid r \in R^{avb} \cup R^{bva} \} + x & \text{if case 1} \\
\text{inh}(avb)^* + \Sigma \{ \text{inh}(r)^* \mid r \in R^{avb} \} + \Sigma \{ \text{inh}(r)^* \mid r \in R^{bva} \} + x & \text{if case 2} \\
\Sigma \{ \text{inh}(r)^* \mid r \in R^{avb} \} + x & \text{if case 3}. 
\end{cases}
\]

Then by \((14)\), \(h(y^*, w^*, q^*, p^*)j \leq h(y, w, q, p)j\) for any \(j \in V\) and \(h(y^*, w^*, q^*, p^*)j < h(y, w, q, p)j\) for \(j = v\), a contradiction to the choice of \((y, w, q, p)\). Hence \((y, w, q, p)\) is a homogeneous solution.

Then the vector \(\text{sign} \in \{1, 0, -1\}^V\) is well defined by

\[
(15) \quad \text{sign}_v := \begin{cases} 
1 & \text{if } h(y, w, q, p)_v > 0 \\
-1 & \text{if } h(y, w, q, p)_v > 0 \\
0 & \text{otherwise}. 
\end{cases}
\]

Let \(A^*\) be the support of \(y\), \(W^*\) the support of \(w\), and define

\[
A := \{ b \in B \mid \text{For any decomposition of } b = [(b_1, B_1), ..., (b_n, B_n)] = b' \oplus R^b, \ b' \in A^*, \ b \in W^* \text{ for all } r \in R^b, \text{ and } B_i = \text{sign}_i, 1 \leq i \leq n \},
\]

\[
W := \{ b \in Z \mid \text{For any decomposition of } b = [(b_1, B_1), ..., (b_n, B_n)] = \Theta^b, \ r \in W^* \text{ for all } r \in R^b, \text{ and } B_i = \text{sign}_i, 1 \leq i \leq n \}.
\]

We show that \((A, W)\) with the same switching rule as \((B, Z)\) is a modular switching family.

By definition, the elementary paths and circuits of \((A, W)\) are \(A^*, W^*\) and \((A, W)\) is closed under decomposition. \((A, W)\) is also closed under switching: Consider \(a \in A \cup W, b \in A^* \cup W^*, v \in \{a\} \cap \{b\}\) such that \(avb\) is defined, and any decomposition of \(avb = avb' \oplus R^{avb}\) (if \(a \in A\), \(avb = \Theta^b \oplus R^{avb}\) (if \(a \in W\), \(b = bva' \Theta^b \oplus R^{bva}\) (if \(a \in A, b \in A^*\)). Then \((y^*, w^*, q^*, p^*)\) defined by (13) is optimal and minimal with respect to \(y^*A + w^*W\).

Furthermore, by lemma 6.3, \(avb\) is a partial path of \(e := [[av];[vb]]\), \(bva\) a partial path of \(f := [[bv];[va]]\) if \(a \in A, b \in A^*\); and \(avb\) is a partial path of \(e := [[av];[vb]]\) otherwise.
Hence
\[ \text{inh}(avb)_j^* = 0 \text{ for } \ast - 1 = \text{sign}_j \text{ and if } a \in \mathcal{A}, b \in \mathcal{A}', \text{ inh}(bva)_j^* = 0 \text{ for } \ast - 1 = \text{sign}_j, j \in V, \]
and therefore \( h((y^*, w^*, q^*, p^*)) = 0 \) and \((y^*, w^*, q^*, p^*) \in \mathcal{S}^3 \). By (13) and the choice of \((y, w, q, p), \text{ supp}(y, w) = \text{ supp}(y^*, w^*) \) and hence \( avb \in \mathcal{A} \cup \mathcal{W}, bva \in \mathcal{A} \cup \mathcal{W} \) if \( a \in \mathcal{A}, b \in \mathcal{A}' \), and \((\mathcal{A}, \mathcal{W})\) is closed under switching. Furthermore, \((\mathcal{A}, \mathcal{W})\) is modular by (12).

Let \( x^* \) be an optimum solution of (7) and \( J_1 := \{j \in V | q_j = p_j = 0\}, J_2 := \{j \in V | p_j > 0\}, J_3 := \{j \in V | q_j > 0\}. \) Notice that by (10), \( J_2 \cap J_3 = \emptyset \). Furthermore, denote by \( \overline{\mathcal{A}} \) and \( \overline{\mathcal{W}} \) the matrices of \((\mathcal{A}, \mathcal{W})\) or equivalently, the matrices with rows corresponding to the support of \( y \) and \( w \).

Consider
\[
\begin{align*}
y^* \overline{\mathcal{A}} J_1 + w^* \overline{\mathcal{W}} J_1 &= c J_1 =: (c^*) J_1 \\
y^* \overline{\mathcal{A}} J_2 + w^* \overline{\mathcal{W}} J_2 &\geq \\left[ (y \overline{\mathcal{A}} J_2 + w \overline{\mathcal{W}} J_2) \right] =: (c^*) J_2 \\
y^* \overline{\mathcal{A}} J_3 + w^* \overline{\mathcal{W}} J_3 &\leq \\left[ (y \overline{\mathcal{A}} J_3 + w \overline{\mathcal{W}} J_3) \right] =: (c^*) J_3 \\
y^*, w^* &\geq 0.
\end{align*}
\]
Notice that \((y^*, w^*, q^*, p^*)\) is a solution of (16) and by the complementary slackness conditions for any solution \((y^*, w^*)\) of (16),
\[
(y', w', q', p') := ((y^*, 0 \overline{\mathcal{A}}), (w^*, 0 \overline{\mathcal{W}}), q', p')
\]
(with \( q', p' \) defined as in (10) ) is an optimum homogeneous solution of (8), as \( \text{ supp}(a^*) \subseteq \text{ supp}(a) \) for \( a = y, w, q, p \).

\((\mathcal{A}, \mathcal{W})\) is a modular switching family conforming to \( c^* \). Furthermore (16) has a feasible solution. Hence by lemma 6.4 there exists an integer solution \((y^*, w^*)\) of (16) and \((y', w', q', p')\) defined in (17) is an integer homogeneous optimum solution of (8).

The so-called transition (or \( t^- \)) phenomenon, already established for lattice polyhedra (Gröflin, Hoffman [1981]) and for the \( G^- \)switching paths polyhedra, also holds for polyhedra of type \( P^1 \) and \( P^2 \) of corollary 6.2:

**Theorem 6.5**
Consider \( P^1 \) and \( P^2 \) of corollary 6.2 and the corresponding dual LP's (18)_r to \( P^1 \) and (19)_r to \( P^2 \),
\[
\begin{align*}
\text{(18)}_r & \quad \max yr' + wr' + qe - pd \text{ subject to } y \overline{\mathcal{A}} + w \overline{\mathcal{W}} + q - p = c, \quad y, w, q, p \geq 0, \\
\text{(19)}_r & \quad \min yr' + wr' + qd - pe \text{ subject to } y \overline{\mathcal{A}} + w \overline{\mathcal{W}} + q - p = c, \quad y, w, q, p \geq 0,
\end{align*}
\]
for a given integer vector \( c, r^r = r^1 \) and \( r^r = r^1 + 1 \).

The two LP's (18)_r and (18)_r^1+1 respectively (19)_r and (19)_r^1+1 have a common homogeneous integer optimum solution whenever they have an optimum solution.
Proof:
We give the proof for the LP's (18) for the case for (19) being similar.

Let $c$ be integral such that $(18)_{r+1}$ and $(18)_{r+1}$ have optimum solutions, and $(\mathcal{B},Z)$ be the underlying submodular switching family on the set $V$. A new submodular switching family $(\mathcal{D},Z)$ on the set $V'$ is obtained by adding an element $s$ to $V$ and augmenting any path by the element $(s,-1)$ at the end of it, i.e. $V' := V \cup \{s\}$, $b = [(b_1,B_1),\ldots,(b_n,B_n)] \in \mathcal{B}$ iff $[(b_1,B_1),\ldots,(b_n,B_n),(s,-1)] \in \mathcal{D}$.

Therefore,

$$P' := \{(x,z) \mid x \in \mathbb{R}^V, z \in \mathbb{R}, \bar{A}x - I z \geq r, \bar{R}x \geq r^2, e \leq x \leq d, z \geq 0\}$$

is a polyhedron of type (3) and both LP's (21)

$$\begin{align*}
\min & \quad cx + (\omega_0 - \omega_1)z \\
\text{subject to} & \quad \bar{A}x - I z \geq r, \bar{R}x \geq r^2, e \leq x \leq d, z \geq 0,
\end{align*}$$

(22)

$$\begin{align*}
\max & \quad yr^1 + wr^2 + qe - pd \\
\text{subject to} & \quad y\bar{A} + w\bar{R} + q - p = c, -y1 \leq \omega_0 - \omega_1,
y, w, q, p \geq 0,
\end{align*}$$

have integer optimum solutions, where $\omega_z$ denotes the optimum value of $(18)_{r+1}$, $z \geq 0$ (if it exists). Let $Z := \{z \in \mathbb{Z}_+ \mid \omega_z \text{ exists}\}$.

The proof is now similar to the one of Gröflin, Hoffman [1981]. Define for all $z \in Z$,

$$m(z) := \omega_z + (\omega_0 - \omega_1)z,$$

then $m(0) = \omega_0 = m(1)$ and the optimum value of (21) is equal to $\omega := \min\{m(z) \mid z \in Z\}$. We show that (21) has an optimum solution with $z = 1$ which is equivalent to show $m(z) \geq \omega_0$ for all $z \in Z$. Let $x_z$ be an optimum solution of $(18)_{r+1}$. For all $z \geq 1$, $z \in Z$, $x := 1/z(x_z + (z - 1)x_0)$ is a feasible solution of $(18)_{r+1}$, hence $\omega_1 \leq cx = 1/z(m(z) + z\omega_1 - \omega_0)$ and equivalently $\omega_0 \leq m(z)$. Hence there exists a solution of (21) with $z = 1$.

Let $(y,w,q,p)$ be a homogeneous integer optimum solution of (22) and $(x^*,1)$ an optimum solution of (21). Then $(y,w,q,p)$ is feasible in $(18)_{r+1}$ and $(18)_{r+1}$ and $x^*$ is an optimum solution of the primal problem to $(18)_{r+1}$, implying $cx^* = \omega_1$.

$$\begin{align*}
yr^1 + wr^2 + qe - pd &= cx^* + (\omega_0 - \omega_1) = \omega_1 + (\omega_0 - \omega_1) = \omega_0 \\
y(r^1 + 1) + wr^2 + qe - pd &= \omega_0 + y1 \geq \omega_0 + \omega_1 - \omega_0 = \omega_1,
\end{align*}$$

hence $(y,w,q,p)$ is a homogeneous integer optimum solution of $(18)_{r+1}$ and $(18)_{r+1}$.

6.3 Examples

In this section, different models and examples contained in the switching paths polyhedra are considered. The relations between them are given in figure 6.6. As already mentioned the main models included are intersections, a subclass of the G-switching paths polyhedra.
and the coflow polyhedra (Cameron [1982]). The latter two are fairly abstract and include on their own other models and results, which mostly are related to paths and circuits in graphs, or chains in posets.

In order to establish the relation to well known classical results (like the theorem of Dilworth) and to give illustrative examples of the switching paths polyhedra, the models represented in figure 6.6 will be discussed. Of course, the list of the shown applications is not exhaustive.

Most of the related theorems are min-max relations due to tdi-ness of the systems of $P^1$ and $P^2$ in (3) and (4). Consider for example a switching family $(B, Z)$ and $P^2 = \{ x \in \mathbb{R}^V \mid A x \leq r^1, R x \leq r^2, e \leq x \leq d \}$ of type (4) with $e = 0$ and $d = 1$. Then any vertex of $P^2$ is the incidence vector of a set $S \subseteq V$ so that $P^2$ is the convex hull of a family $S \subseteq 2^V$. 
Consider the LP's

\[
\begin{align*}
(24) \quad \max c^T x & \text{ subject to } A^T x \leq r^1, R^T x \leq r^2, \quad 0 \leq x \leq 1, \\
(25) \quad \min y r^1 + w r^2 + q \mathbb{1} & \text{ subject to } y A^T + w R^T + q \geq c, \quad y, w, q \geq 0.
\end{align*}
\]

Both LP's have integer solutions as the system of $P^2$ is tdi. An integer solution of (24) corresponds to a set $S \in \mathcal{S}$, an integer solution of (25) to a collection $\mathcal{C}$ of paths of $\mathcal{B}$ and a collection $\mathcal{D}$ of circuits of $\mathcal{Z}$, where the same path can occur more than once.

The duality theorem yields then the following min-max result:

\[
(26) \quad \max \{ c(S) \mid S \in \mathcal{S} \} = \min \{ r^1(\mathcal{C}) + r^2(\mathcal{D}) + \sum \{ \max \{ 0, c_v - \text{val}(\mathcal{C})_v - \text{val}(\mathcal{D})_v \} \mid v \in V \} \mid (\mathcal{C}, \mathcal{D}) \text{ is a collection of elementary paths and circuits} \},
\]

where $r^1(\mathcal{C}) := \sum \{ r^1(b) \mid b \in \mathcal{C} \}$, $\text{val}(\mathcal{C}) := \sum \{ \text{val}(b) \mid b \in \mathcal{C} \}$ and analogously for $r^2(\mathcal{D})$ and $\text{val}(\mathcal{D})$.

In fact, by appropriately choosing the family $(\mathcal{B}, \mathcal{Z})$, $r^1$ and $r^2$, the min-max relation (26) gives exactly the theorem of Dilworth [1950] or the theorem of Greene-Kleitman [1976] (example 7).

In the same way a min-max result can be obtained for $e = -\infty$ and for the polyhedron $P^1$ of type (3).

In example 8 a generalization of the theorem of Gallai-Milgram [1960] for acyclic graphs is given, which includes the generalization derived by Cameron [1982].

**Example 1: Intersections**

Let $\mathcal{L}$ be the family of intersections of the ring families $\mathcal{C}$ and $\mathcal{D}$ on $V$, $A$ its incidence matrix and $H = (V, E_B \cup E_R)$ the graph representing $\mathcal{C}$ and $\mathcal{D}$. In examples 1.1 to 1.5, the family $(\mathcal{B}, \mathcal{Z})$ of (elementary and non-elementary) b-r-paths and b-r-circuits was defined and shown to be a modular switching family. Furthermore the matrices $\overline{A}$ and $\overline{R}$ of $(\mathcal{B}, \mathcal{Z})$ correspond to the alternating vectors of b-r-paths and b-r-circuits of $H$ (example 1.2). Hence

\[
(27) \quad P := \{ x \in \mathbb{R}_+^V \mid \overline{A} x \leq 1, \overline{R} x \leq 0 \} = \text{CONV}(A)
\]

is a switching paths polyhedron of type (4).

Let $k \geq 1$, $k$ integer, $\mathcal{B}^k$ be the family of $k$-intersections, i.e. subsets of $V$ which are partitionable into $k$ intersections or less, and $\mathcal{B}^k$ its incidence matrix.

**Proposition 6.6**

The following polyhedron is a switching paths polyhedron of type (4) and gives the polyhedral description of the $k$-intersections.

\[
(28) \quad P^k := \{ x \in \mathbb{R}_+^V \mid \overline{A} x \leq k, \overline{R} x \leq 0, 0 \leq x \leq 1 \} = \text{CONV}(\mathcal{B}^k).
\]
Proof:
This follows directly from theorem 5.4 and corollary 6.2. More precisely, \( P^k \) is a switching path polyhedra of type (4) since \((B, Z)\) is a modular switching family. The inclusion \( \subseteq \) in (28) is then given by (27). For the inclusion \( \subset \), let \( c^S \) be a vertex of \( P^k \). It is integral and therefore it is the incidence vector of a set \( S \subseteq V \).

Recall that \( A \) is the incidence matrix of the intersections. LP (30) has a solution because \( \bar{R}c^S \leq 0 \), \( c^S \geq 0 \) and by theorem 5.4

\[
\omega := \max \{ c^S \alpha x \mid Ax \leq 1 \} = \max \{ c^S \bar{A}_i \mid \bar{A}_i \text{ is a row of } \bar{A} \} \leq k,
\]

\[
\omega = \min 1 y \text{ subject to } yA = c^S, \quad y \geq 0.
\]

Moreover, (30) has an integer optimum solution, yielding a partition of \( S \) into less than \( k+1 \) intersections. Hence \( S \) is a \( k \)-intersection.

Recall that path-closed sets are a special case of intersections. Hence (28) describes also the sets partitionable into \( k \) path-closed sets or less, as Gröflin [1987] already showed.

Example 2: Modular G-switching families

Let \( V \) and \( \mathcal{A} \) be finite sets, \( s := \{ f(a) \in \mathbb{R}^V \mid a \in \mathcal{A} \} \) a modular G-switching family, and \( B \) the \( AxY \) matrix with rows \( f(a), a \in \mathcal{A} \) (see section 5.2). For any \( a \in \mathcal{A} \) a linear ordering \( \leq \) on the support of \( f(a) \) is given. Hence for any \( a \in \mathcal{A} \), the pair \( (a, f(a)) \) with support \( \text{supp}(f(a)) = \{ a_1, \ldots, a_n \} \) can be identified with the path \( p(a) := [(a_1, A_1), \ldots, (a_n, A_n)] \), where \( A_i := f(a)_{a_i}, 1 \leq i \leq n \). Define

\[
(31) \quad B := \{ p(a) \mid a \in \mathcal{A} \}, \quad Z := \{ [ ] \}.
\]

Proposition 6.7

\((B, Z)\) together with the same switching rule as in \( s \) (i.e. \( a, b \in \mathcal{A}, \forall e \in \{ p(a) \} \cap \{ p(b) \} \), \( p(a) \lor p(b) := p(\lor a, b), p(b) \lor p(a) := p(\lor b, a) \), avb, bva \( \in \mathcal{A} \)) is a modular switching family. Furthermore, a polyhedron \( P := \{ x \in \mathbb{R}^V \mid e \leq x \leq d, Bx \leq r \} \) of type (14), section 5.2, is a switching paths polyhedron of type (4).

Proof:

\((B, Z)\) is closed under decomposition as \( Z' = Z = \{ [ ] \}, B' = B \) and the matrices of \((B, Z)\) are \( \bar{A} = \bar{B}, \bar{R} = 0 \). We show now that \((B, Z)\) is also closed under switching.

Notice that if \( a, b \in \mathcal{A}, \forall e \in \{ p(a) \} \cap \{ p(b) \} \), then \( e \in \text{supp}(f(a)) \cap \text{supp}(f(b)) \) and avb, bva \( \in \mathcal{A} \) are well defined. Furthermore, by definition of \( \{ av \} \) and \( \{ bv \} \) (section 5.2), \( \{ av \} \equiv \{ [p(a)v] \}, \{ bv \} \equiv \{ [v(p)b] \} \).
Let $e'$ be the partial path of $e := [[p(a)v];[vp(b)]]$ containing all occurrences of $((av)\Delta \{vb\})\cap \{v\}$. Then by the following lemma 6.8, $p(avb)$ is a partial path of $e'$, hence a partial path of $e$. By symmetry, $p(bva)$ is a partial path of $[[p(b)v];[vp(a)]]$ and $p(a)vp(b) := p(avb), p(b)vp(a) := p(bva)$ satisfy the switching conditions; hence $(B,Z)$ is closed under switching.

Since $val(p(a)) = f(a)$ for all $a \in A$, $s$ modular (submodular) implies $(B,Z)$ modular (submodular). Furthermore, a function $r : A \rightarrow R$ corresponds to a function $(r^1, r^2) : (B \times Z) \rightarrow R$ by $r^1(p(a)) := r(a), a \in A, r^2(\emptyset) := 0$ and $r$ submodular (supermodular) on $s$ implies $(r^1, r^2)$ submodular (supermodular) on $(B,Z)$.

**Lemma 6.8**

Let $s = \{f(a) \mid a \in A\}$ be a modular $G$-switching family. Then for any $a, b \in A, v \in supp(f(a)) \cap supp(f(b))$,

\[
\begin{cases}
0 & i \in \{avb\} \\
0 \text{ and } f(a)_i + f(b)_i = 0 & i \in \{av\} \cap \{vb\} \\
= \frac{1}{2}(f(a)_i + f(b)_i) & i = v \\
\in \{0, f(b)_i\} & i \in \{vb\} \cap \{av\}
\end{cases}
\]

and the ordering is conserved, i.e.

\[
\begin{align*}
&i, j \in \{av\} \cap supp(f(avb)), \text{ then } (i \not\leq j \text{ iff } i \not\leq j) \\
i, j \in \{vb\} \cap supp(f(avb)), \text{ then } (i \not\leq j \text{ iff } i \not\leq j) \\
i \in \{av\} \cap supp(f(avb)), j \in \{vb\} \cap supp(f(avb)), \text{ then } i \not\leq j.
\end{align*}
\]

**Proof:**

Throughout this proof we use the notation (5.a) instead of "chapter 5, (a)". We first prove the following property:

\[
(34) \quad \text{If } a, b \in s, f(a)_i = f(b)_i, f(a)_j = f(b)_j, i \neq j, \text{ then } (i \not\leq j \text{ iff } i \not\leq j).
\]

Assume $i \not\leq j, j \not\leq i$ and consider $c := ajb, d := bja$. Then $i \in \{aj\} \cap \{jb\}$, hence $i \in \{bja\}$. Since $i \in \supp(f(a)) \cap \supp(f(b))$, by (5.9'), $f(d)_i = 0$ and by modularity $f(c)_i = f(a)_i + f(b)_i = 2f(a)_i$, a contradiction to $f(c) \in \{1, 0, -1\}$.

**Proof of (32):**

Let $i \in \{avb\}$ and assume $f(avb)_i \neq 0$. Then by (5.9), $i \in \supp(f(a)) \cap \supp(f(b))$, and by (5.12), $f(bva)_i = 0$, hence $f(avb)_i = f(a)_i + f(b)_i$. Either $f(a)_i \neq 0$ and $i \in \{va\}$, implying $f(avb)_i = 0$ by (5.11), or $f(b)_i \neq 0$ and $i \in \{bv\}$ implying $f(avb)_i = 0$ by (5.10), in both cases a contradiction.
Let \( i \in \{av\} \cap \{vb\} \setminus w \), then \( i \not\in \{bva\} \) and \( i \in \supp(f(a)) \cap \supp(f(b)) \) implies \( f(bva)_i = 0 \) by (5.9') and \( f(\text{avb})_i = f(a)_i + f(b)_i \). Hence \( f(\text{avb})_i = 0 = f(a)_i + f(b)_i \).

Let \( i \in \{av\} \setminus \{vb\} \), then \( f(bva)_i \in \{0,f(b)_i\} \) by (5.10'). If \( f(b)_i = 0 \) then \( f(\text{avb})_i = f(a)_i \), otherwise \( i \in \{bv\} \) and by (5.10), \( f(\text{avb})_i \in \{0,f(a)_i\} \).

Let \( i = v \), then \( i \in \{av\} \cap \{va\} \cap \{bv\} \cap \{vb\} \), hence by (5.10, 5.10', 5.11, 5.11'), \( f(\text{avb})_i \), \( f(bva)_i \in \{0,f(a)_i\} \cap \{0,f(b)_i\} \). Then \( f(a)_i \neq f(b)_i \) implies \( f(\text{avb})_i = f(bva)_i = 0 \) and \( f(a)_i = f(b)_i \) implies \( f(\text{avb})_i = f(bva)_i = f(a)_i \).

Let \( i \in \{vb\} \setminus \{av\} \), then \( f(bva)_i \in \{0,f(a)_i\} \) by (5.11'). If \( f(a)_i = 0 \), then \( f(\text{avb})_i = f(b)_i \), otherwise \( i \in \{va\} \) and by (5.11), \( f(a)_i \in \{0,f(b)_i\} \).

Proof of (33):
 Let \( i, j \in \{av\} \setminus \supp(f(\text{avb})) \), then by (32), \( f(\text{avb})_i = f(a)_i \), \( f(\text{avb})_j = f(a)_j \) and \( \langle i \lessdot j \rangle \iff (i \lessdot_{G^v(b)} j) \) by property (34).

Let \( i, j \in \{vb\} \setminus \supp(f(\text{avb})) \), then by (32), \( f(\text{avb})_i = f(b)_i \), \( f(\text{avb})_j = f(b)_j \) and \( \langle i \lessdot j \rangle \iff (i \lessdot_{G^v(b)} j) \) by property (34).

Let \( i \in \{av\} \setminus \supp(f(\text{avb})) \), \( j \in \{vb\} \setminus \supp(f(\text{avb})) \), then \( f(\text{avb})_i = f(a)_i \), \( f(\text{avb})_j = f(b)_j \) and \( i \not\in \{vb\} \) by (32). If \( v \in \{av\} \), then \( i \lessdot_{G^v(b)} v \) and \( v \lessdot_{G^v(b)} j \), hence \( i \lessdot_{G^v(b)} j \). Assume \( v \not\in \{av\} \) and \( j \lessdot_{G^v(b)} i \).

If \( i \in \supp(b) \), then \( i \lessdot_{G^v(b)} j \) and by (34), \( f(b)_i = -f(\text{avb})_i \). For \( c := (\text{avb})ib, j \in \{(avb)i\}, j \in \{ib\} \) and \( f(\text{avb})_j = f(b)_j \), a contradiction to (32). Hence \( i \not\in \supp(b) \). Consider \( c := bj(\text{avb}) \). Then using (32) and modularity

\[
\begin{align*}
\text{f(c)}_v &= \text{f(b)}_v, \text{since } v \in \{bj\} \text{ and } v \not\in \{(avb)jb\}, \\
\text{f(c)}_j &= \text{f(b)}_j = \text{f(avb)}_j, \\
\text{f(c)}_i &= \text{f(avb)}_i = \text{f(a)}_i, \text{since } i \in \{j(\text{avb})\} \text{ and } i \not\in \{(avb)jb\}, \\
\text{v } &\lessdot_j i, \text{by (34) and } (v \lessdot_j j) \text{ and } (j \lessdot_{G^v(b)} i).
\end{align*}
\]

For \( avc, i \in \{av\}, i \in \{vc\} \) and \( f(a)_i = f(c)_i \), a contradiction to (32). Hence \( i \lessdot_{G^v(b)} j \).

For applications see examples 5, 6, 7 and 8.

Example 3: Submodular G-switching families

We already saw in example 2 that a G-switching family (i.e. \( s := \{f(a) \in \mathbb{R}^V \mid a \in \mathcal{A}\} \) for given finite sets \( V \) and \( \mathcal{A} \)) corresponds to a family \((\mathcal{B}, \mathcal{Z})\) of paths and circuits with

\( \mathcal{B} := \{p(a) \mid a \in \mathcal{A}\}, \mathcal{Z} := \{\emptyset\}. \)

For \( a, b \in \mathcal{A}, v \in \supp(f(a)) \cap \supp(f(b)) \), the definition of switching in G-switching families does not contain any explicit condition on the order of the support of \( f(\text{avb}) \), \( f(bva) \). In the
modular case, this order is implicitly fixed (lemma 6.8), but this is not the case in the submodular case.

Our definition of switching requires some conservation of the order and therefore we only contain the subfamily of submodular G-switching families "conserving order" in the following sense:

The idea of conserving the order is that the order between elements of supp(f(avb)) which occur in \{[[p(a)v];[vp(b)]]\} should be the same as in \{[[p(a)v];[vp(b)]]\}. More precisely, a G-switching family \(\mathcal{S}\) conserves order if for any \(a, b \in \mathcal{A}\), \(v \in \text{supp}(f(a)) \cap \text{supp}(f(b))\) the following holds: Consider the partial path \(c^-\) of \(p(avb)\) containing those elements \((c_i, C_i)\) of \(p(avb)\) which occur in \(e := [[p(a)v];[vp(b)]]\) with the same value (i.e. any element \((e_k, E_k)\) of \(e\) with \(e_k = c_i\) has \(E_k = C_i\)), then \(c^-\) is a partial path of \(e\).

**Proposition 6.9**

Let \(\mathcal{S}\) be a submodular G-switching family conserving order and \((\mathcal{B}, \mathcal{Z})\) defined as above. \((\mathcal{B}, \mathcal{Z})\) together with the same switching rule as in \(\mathcal{S}\) (i.e. \(a, b \in \mathcal{A}\), \(v \in \{p(a)\} \cap \{p(b)\}\)), \(p(a)vp(b) := p(avb), p(b)vp(a) := p(bva)\), \(avb, bva \in \mathcal{A}\) is a submodular switching family. Furthermore, polyhedra \(P := \{x \in \mathbb{R}^Y \mid e \leq x \leq d, Bx \geq r\}\) of type (13) section 5.2 are switching paths polyhedra of type (3).

**Proof:**
Throughout this proof we use the notation (5.a) instead of "chapter 5, (a)". \((\mathcal{B}, \mathcal{Z})\) is closed under decomposition because \(\mathcal{Z}^- = \mathcal{Z} = \{\}\), \(\mathcal{B}^- = \mathcal{B}\) and the matrices of \((\mathcal{B}, \mathcal{Z})\) are \(\mathcal{A} = \mathcal{B}\), \(\mathcal{R} = 0\). We show now that \((\mathcal{B}, \mathcal{Z})\) is also closed under switching.

Notice that if \(a, b \in \mathcal{A}\), \(v \in \{p(a)\} \cap \{p(b)\}\), then \(v \in \text{supp}(f(a)) \cap \text{supp}(f(b))\) and \(avb, bva \in \mathcal{A}\) are well defined. Furthermore, \(\{av\} \equiv \{[p(a)v]\}, \{bv\} \equiv \{[vp(b)]\}\).

Let, as in the definition of "conserving order", \(c^-\) be the subpath of \(p(avb)\). By symmetry of \(a\) and \(b\) it remains to show that for an element \((c_i, C_i)\) not belonging to \(c^-\), \(c_i \in \text{supp}(f(bva))\) and moreover, \(c_i \in \text{supp}(f(a)) \cap \text{supp}(f(b))\) or there exists an element \((e_k, E_k)\) of \(e := [[p(a)v];[vp(b)]]\) with \(C_i = E_k\).

Let \((c_i, C_i)\) be an element of \(p(avb)\) not belonging to \(c^-\) (\(\Rightarrow c_i \neq v\)). Assume \(c_i \in \text{supp}(f(avb)) \cap \text{supp}(f(bva))\), then \(c_i \in \text{supp}(f(a)) \cap \text{supp}(f(b))\) by (5.12) and \(c_i \in \{avb\} \cap \{bva\}\) by (5.9) and (5.9'). Therefore \(c_i\) occurs exactly once in \(e\) and there exists an element \((e_k, E_k)\) of \(e\) with \(C_i \neq E_k\). Hence, either \(c_i \in \{av\}\) and \(E_k = f(a)_{c_i}\) or \(c_i \in \{bv\}\) and \(E_k = f(b)_{c_i}\). But since \(c_i \in \{bva\}\), \(c_i \in \{av\}\) implies \(c_i \in \{bv\}\) and \(C_i = f(a)_{c_i}\) by (5.10), \(c_i \in \{bv\}\) implies \(c_i \in \{va\}\) and \(C_i = f(b)_{c_i}\) by (5.11), a contradiction to \(C_i \neq E_k\).
Now, let \( c_i \in \text{supp}(f(a)) \cup \text{supp}(f(b)) \). Then \( c_i \in (avb) \setminus v = \{ e \} \setminus v \), otherwise \( c_i \in \text{supp}(f(a)) \cap \text{supp}(f(b)) \) by (5.9) and either \( c_i \in (bv) \setminus \text{supp}(f(a)) \) and by (5.10), \( 0 \neq f(ab)_c \neq f(a)_c = 0 \), or \( c_i \in (va) \setminus \text{supp}(f(b)) \) and by (5.11), \( 0 \neq f(ab)_c \neq f(b)_c = 0 \). Since \( s \) conserves order, an element \( (e_k, E_k) \) of \( e \) with \( C_i = E_k \) exists.

The rest of the proof is given by the last paragraph of the proof of proposition 6.7.

All but one example for G-switching families given by Gröflin [1987] are related to paths in a digraph (example 6) or chains in a poset (example 7), objects for which the order and switching rule are given in a natural way. Of course, these examples conserve order. The one exception is the example of the switching model of Hoffman [1974]. This is an abstract model developed to generalize the different max flow - min cut theorems. Again, all examples mentioned there conserve order as they are related to paths in graphs (see example 5).

Summarizing, all applications of interest of G-switching paths polyhedra seem to conserve order. Notice however, that examples can be constructed which do not conserve order and consequently do not belong to our switching paths polyhedra.

**Example 4: Coflow polyhedra (Cameron [1982])**

Let \( G = (V,E) \) be a digraph, \( a \in Z^V \), \( e, d \in (Z \cup \{-\infty, +\infty\})^V \) and \( C \) be the set of (node set of) elementary circuits of \( G \). The polyhedron

\[
P := \{ x \in R^V | e \leq x \leq d, x(C) \leq a(C) \text{ for all } C \in C \}
\]

is called a **coflow polyhedron**. Cameron showed that \( P \) is an integral polyhedron and that \( x(C) \leq a(C), C \in C \), is box-tdi.

**Proposition 6.10**

\( P \) is a switching paths polyhedron of type (4).

**Proof:**

Let \( \mathcal{B} := \{ [ ] \}, Z := \{ p(C) := [(c_1,1),\ldots,(c_n,1)] \mid C \text{ is a (non-)elementary circuit of } G \} \mid C \text{ with } V(C) = \{ c_1,\ldots,c_n \} \} \), and \( Z^\prime \) the elementary elements of \( Z \). Then \( (\mathcal{B},Z) \) together with the switching rule \( avb := [[av];b];[vb]] \) for any \( a \in Z, b \in Z^\prime, v \in V \) such that \( avb \) has to be defined, is a modular switching family with matrices \( \bar{A} = 0 \) and \( \bar{R} \). Furthermore the function \( (r^1,r^2) : (B^\prime,Z^\prime) \rightarrow IR \) defined by \( r^1([ ]) := 0, r^2(C) := a(C), C \in C \), is modular on \( (\mathcal{B},Z) \) and

\[
( x(C) \leq a(C), C \in C ) \equiv ( \bar{R}x \leq r^2 ).
\]

Hence \( P \) is a switching paths polyhedron of type (4).
Applications of coflow polyhedra given by Cameron include the derivation of the Gallai-Milgram theorem for acyclic graphs and a generalization of it (example 8), the Dilworth theorem and its generalization by Greene-Kleitman (example 7).

Example 5: Switching model of Hoffman [1974]

This model was developed to unify and generalize different versions of max flow - min cut theorems.

Let $V$ be a finite set and $S := \{S_0, S_1, ..., S_m\}$ a family of subsets $S_i \subseteq V$, with $S_0 := \emptyset$. Each non-empty set $S_i$, $1 \leq i \leq m$, is linearly ordered by a relation $\leq$ and for any $v \in S_i \cap S_j$, $0 \leq i, j \leq m$ the set

$$\{(i,v,j) := \{w \in S_i \mid w \leq v\} \cup \{w \in S_j \mid v \leq w\}\}$$

is defined. Moreover a function $r: S \to \mathbb{Z}^+$ satisfying $r(S_0) = 0$ and

$$\max \{r(S_k) \mid S_k \subseteq (i,v,j)\} + \max \{r(S_k) \mid S_k \subseteq (j,v,i)\} \geq r(S_i) + r(S_j)$$

for all $i, j, 0 \leq i, j \leq m$, and $v \in S_i \cap S_j$, is given.

Let $B$ be the incidence matrix of $S$ and consider

$$P := \{x \in \mathbb{R}_+^V \mid Bx \geq r\}.$$

Hoffman showed that $P$ is an integral polyhedron.

**Proposition 6.11 (Gröflin [1987])**

$P$ is a $G$-switching paths polyhedron.

As in examples 2 and 3, a family $(B, Z)$ of paths and circuits can be associated to $S$ by considering $p(S_i) := [(s_1, 1), ..., (s_n, 1)]$ for any $S_i \in S$ with $S_i = \{s_1, ..., s_n\}$, $s_1 \leq ... \leq s_n$, and $B := \{p(S_i) \mid 0 \leq i \leq m\}$, $Z := \{[\ ]\}$. Moreover, if for any $v \in S_i \cap S_j$ there exists a set $S \in S$, (denoted by $S_{ivj}$) with $r(S_{ivj}) = \max \{r(S) \mid S \in S, S \subseteq (i,v,j)\}$ and order naturally induced by $(i,v,j)$, (i.e. $p(S_{ivj})$ is a partial path of $[[p(S_i)v];[vp(S_j)]]$), then $(B, Z)$ is a submodular switching family and $r$ is supermodular on $(B, Z)$. In this case $P$ is a switching paths polyhedron of type (3).

Notice that this subclass of switching families is included in the family conserving order defined in example 3. Hoffman presented only examples conserving order and hence belonging to our switching families. We consider now one of his applications yielding the well known max flow - min cut theorem.

**Example:**

Let $G = (V, E)$ be a directed graph with a source $s$ and a sink $t$ in $V$, and $S \subseteq 2^E$ consists of
the empty set and the edge sets of (elementary) s-t-paths ordered in the natural way and \( n_k := 1, k \geq 0 \).

For any \( S_i, S_j \in \mathcal{S} \) and \( e \in S_i \cap S_j \), the edge set \((i, v, j)\) ordered in the natural way contains the edge set of an s-t-path, say \( S_{i,v,j} \), hence \( \mathcal{S} \) conserves order and the associated family \((\mathcal{B}, \mathcal{Z})\) is a submodular switching family.

Let \( \mathbf{A}, \mathbf{R} \) be the matrices of \((\mathcal{B}, \mathcal{Z})\), then \( \mathbf{A} = \mathbf{B}, \mathbf{R} = 0 \). Consider a capacity vector \( c \in \mathbb{Z}^E \) on the edges of \( G \) and the pair of dual LP's

\[
\begin{align*}
\text{(40)} & \quad \min \ c x \quad \text{subject to} \quad \mathbf{A} x \geq 1, \ x \geq 0, \\
\text{(41)} & \quad \max \ y \quad \text{subject to} \quad \mathbf{y} \mathbf{A} \leq c, \ y \geq 0.
\end{align*}
\]

Problem (41) is equivalent to find a flow of maximum value subject to the capacity vectors \( 0 \) and \( c \). In fact \( \mathbf{y} \mathbf{A} \) is a sum of flows of value \( y_j \) along the path \( S_i \) and any flow can be decomposed into paths and circuits, where the flow along the circuits does not affect the value of the flow.

Consider now problem (40). There is an optimal solution which is a vertex of \( \{ x \in \mathbb{R}_+^E \mid \mathbf{A} x \geq 1 \} \), hence \( \{0,1\} \)-valued and consequently the incidence vector of a set \( A \subseteq E \). Let \( S := \{ v \in V \mid \exists \text{ path from } s \text{ to } v \text{ in } G' := (V,E \setminus A) \} \), then \( s \in S, t \in S \) (as \( \mathbf{A} x \geq 1 \)), \( \mathbf{B}(S) \cup \mathbf{B}(\bar{S}) \) is an s-t-cut with \( \mathbf{B}(S) = \mathbf{A}, \) and \( \mathbf{c}(\mathbf{B}(S)) = \mathbf{c}(\mathbf{A}) = cx \). The duality theorem yields then the max flow - min cut theorem 2.2.

Example 6: Directed graphs

In this section we consider switching families related to paths and circuits in directed graphs.

To any (non-elementary) path (circuit) \( P \) of \( G \) with ordered node set \( V(P) = (p_1, \ldots, p_n) \), a path (circuit) \( p(P) := [(p_1, 1), \ldots, (p_n, 1)] \) can be associated. Denote by \((\mathcal{B}^V, \mathcal{Z}^V)\) the family of paths and circuits corresponding to node sets of paths and circuits of \( G \).

Another family can be obtained by considering the edges \( E(P) = (e_1, \ldots, e_n) \) of a path \( P \) of \( G \). Again the path associated to \( P \) is given by \( p(P) := [(e_1, 1), \ldots, (e_n, 1)] \). In our notation, the first and last element of a circuit has to be the same, hence for a circuit \( Z \) of \( G \) with \( E(Z) = (e_1, \ldots, e_n) \), \( p(Z) := [(e_1, 1), \ldots, (e_n, 1), (e_1, 1)] \). Denote by \((\mathcal{B}^E, \mathcal{Z}^E)\) the family of paths and circuits corresponding to edge sets of paths and circuits of \( G \).

Let first \( G \) be an acyclic graph and consider \((\mathcal{B}^V, \mathcal{Z}^V)\), where \( \mathcal{B}^V \) corresponds to the family of all paths in \( G \) and \( \mathcal{Z}^V = \{ [ ] \} \). For two (elementary) paths \( a \) and \( b \) of \( G \) with a common vertex \( v \), define the switching of \( p(a) \) and \( p(b) \) in \( v \) by the path corresponding to the path in
G which starts on a until v and ends along b, i.e. \( p(a)vp(b) := [[p(a)v];[vp(b)]] \). \((B^V,Z^V)\) with this switching rule is a modular switching family.

Applications of this switching family are given in example 7, where G will be the comparability graph of a poset \((P,\leq)\) and \(B^V\) the family corresponding to chains in \((P,\leq)\).

In the same way, \((B^E,Z^E)\) with \(Z^E = \{[]\}\) is a modular switching family if \(G\) is acyclic.

Let now \(G = (V,E)\) be a directed graph where circuits may occur. Consider \((B^V,\{[]\})\) respectively \((B^E,\{[]\})\), where \(B^V\), respectively \(B^E\), corresponds to the family of all elementary paths of \(G\). For two elementary paths \(a\) and \(b\) with common node \(v\), respectively common edge \(v\), the (non-elementary) path starting on \(a\) until \(v\) and ending along \(b\) contains an elementary path, say \(av\). Defining the switching between \(p(a)\) and \(p(b)\) in \(v\) by \(p(a)vp(b) := p(av)\), \((B^V,\{[]\})\) respectively \((B^E,\{[]\})\) is a submodular switching family.

In example 5, for given nodes \(s\) and \(t\), only elementary s-t paths of \(G\) were considered and the corresponding switching family \((B^E,\{[]\})\) was used to derive the max flow - min cut theorem.

Let now \((B^V,Z^V)\) (respectively \((B^E,Z^E)\)) correspond to the family of all (also non-elementary) paths and circuits of \(G\). Define the switching between two paths \(p(a)\) and \(p(b)\) with \(v \in \{p(a)\} \cap \{p(b)\}\) by \(p(a)vp(b) := [[p(a)v];[vp(b)]]\). Analogously for a path or circuit \(p(a)\) and a circuit \(p(b)\), \(v \in \{p(a)\} \cap \{p(b)\}\) such that the switch has to be defined, \(p(a)vp(b) := [[[p(a)v];[p(b)]]\). Then \((B^V,Z^V)\) (respectively \((B^E,Z^E)\)) is a modular switching family.

Let \(A, R\) be the matrices of \((B^V,Z^V)\), then
\[
\begin{align*}
P := \{x \in \mathbb{R}^V | A_x \leq 1, \; R_x \leq 0\}
\end{align*}
\]
is a switching paths polyhedron of type (4), hence integral. Moreover, its vertices are \(0, \pm 1\)-vectors associated with so-called node cut sets as can be found in Gröflin, Hoffman [1982]. A node cut set is a pair \((\Lambda^+(S),\Lambda^-(S))\) for some \(S \subseteq V\), with
\[
\begin{align*}
\Lambda^+(S) := \{v \in S \mid \exists w \in S: (v,w) \in E\}, \; \Lambda^-(S) := \{v \in S \mid \exists w \in S: (v,w) \in E\}
\end{align*}
\]
and its associated vector \(f(S) \in \{0,\pm 1\}^V\) is given by \(f(S)_v := 1\) if \(v \in \Lambda^+(S)\), \(f(S)_v := -1\) if \(v \in \Lambda^-(S)\) and \(f(S)_v = 0\) otherwise. The node cut sets are related to flows in node constraint networks (and play a similar role as s-t-cuts). The max flow-min cut theorem for such a network reads:

**Theorem 6.12 (Gröflin [1985])**

Let \(G = (V,E)\) be a node constraint network, with distinct vertices \(s, t \in V\) and capacity vectors \(a, u \in \{R_+ \cup \{\infty\}\}^V\). If \(G\) admits a feasible s-t-flow subject to \(a\) and \(u\), the minimum value \(\beta\) of an s-t-flow satisfy \(\beta = \max \{a(\Lambda^+(S)) - u(\Lambda^-(S)) \mid S \subseteq V, \; s \in S, \; t \notin S\}\).
Analogously to example 5, this min-max theorem can be obtained by considering the pair of dual LP’s associated to (42) (with $B^V$ consisting only of $s$-$t$-paths of $G$) and the duality theorem.

Notice that switching families $(B, Z)$ considered here together with the modular function $r = (r^1, r^2)$: $r^1(b) = 1$, be $B^-$, $r^2(b) = 0$, be $Z^-$ form switching paths polyhedra $P$ of type (3), if $(B, Z)$ is submodular, and of type (4), if $(B, Z)$ is modular.

In the cases where $(B, Z)$ is modular, $P$ of (42) is equivalent to a coflow polyhedra: Any path in $G = (V, E)$ can be considered as a circuit in $G' := (V \cup q, E \cup \{(q, v), (v, q) \mid v \in V\})$ through the vertex $q$. With $aeR^V \cup q$, $a_q := 1$, $a_v := 0$ otherwise and $C$ being the set of elementary circuits of $G'$,

$$\{x \in R^V \cup q \mid x(C) \leq a(C) \text{ for all } C \in C, x_q = 0\} \sim P.$$  

Finally, the cases where $Z = \{[ ]\}$, $(B, Z)$ corresponds also to a $G$-switching family.

**Example 7: Poset**

Let $(P, \leq)$ be a poset and $G = (P, E)$ its comparability graph. Then $G$ is acyclic and any node set of a path corresponds to a chain in $(P, \leq)$.

We first present the chain model of Hoffman-Schwartz [1977] and a generalization of it by Schrijver [1984]. Then the theorems of Dilworth [1950] and Greene-Kleitman [1976] will be derived from the min-max relation (26).

Hoffman and Schwartz [1977] considered the set of chains $C$ in a poset and defined for any $C, D \in C$ and $v \in C \cap D$ the chain denoted by $CvD := \{q \in C \mid q \leq v\} \cup \{q \in D \mid v \leq q\}$. For a function $r: C \rightarrow \mathbb{Z}_+$ with the properties

(i) $C \subseteq D \in C$ implies $r(C) \leq r(D),$

(ii) $C, D \in C, v \in C \cap D$ implies $r(C) + r(D) = r(CvD) + r(DvC)$

they obtained the following results:

$$x \in R^P, \ 0 \leq x \leq 1, \ x(C) \leq r(C), \ C \in C \text{ is tdi and the dual LP's }$$

$$\min r'y + 1w \text{ subject to } \Sigma\{y_C \mid v \in C \in C\} + w_v \geq c_v, \ v \in P, \ y, w \geq 0$$

for $r' = r$ and $r' = r+1$ have a common optimum solution (transition phenomenon).

The generalization of Schrijver [1984] concerns the function $r$: Let $h \in Q^P, g \in Q^E$, and for the path $C$ in $G$ with node set $V(C)$ and edge set $E(C),$

(iii) $r(C) := \Sigma h_v \mid v \in V(C) \} + \Sigma g_e \mid e \in E(C) \}.$

Any function $r$ with properties (i) and (ii) can be written in the form (iii) (Hoffman,
Schwartz [1977]). Then

\[(47) \quad \chi(C) \leq r(C), \quad C \in \mathcal{C} \quad \text{is box-tdi.} \]

Let \((\mathcal{B}, \mathcal{Z}) := (\mathcal{B}^V, \mathcal{Z})\) be the switching family corresponding to the node sets of elementary paths of \(G\) (see example 6) and \(p(C)\) be the path corresponding to the chain \(C\). Then \(p(C \cup D) \equiv p(C) \cup p(D)\) and \((r^1, r^2) : (\mathcal{B}, \mathcal{Z}) \rightarrow \mathbb{Q}\) defined by \(r^1(p(C)) := r(C), \quad r^2(\mathcal{Z}) := 0\) is a modular function on \((\mathcal{B}, \mathcal{Z})\). Hence (47) is a system describing a switching paths polyhedron of type (4), and is also included in the \(G\)-switching paths polyhedra.

Consider now the min-max relation (26) for the family \((\mathcal{B}, \mathcal{Z})\) of chains, where \((r^1, r^2) : (\mathcal{B}, \mathcal{Z}) \rightarrow \mathbb{Z}_+\) is defined by \(r^1(b) := k, \quad b \in \mathcal{B}, \quad r^2(\mathcal{Z}) := 0\) and \(c := 1:\)

\[(48) \quad \max \{|S| \mid S \in \mathcal{S}\} = \min \left\{ (k \cdot |C' \cup \mathcal{P} \cup \{C \mid C \in \mathcal{C}\}) \mid C' \subseteq \mathcal{C} \right\}, \quad \text{where} \quad \mathcal{S} \text{ is the family of sets } S \subseteq V \text{ whose incidence vectors are integral vectors in } \{x \in \mathbb{R}_+^k \mid \bar{A} x \leq k, \quad 0 \leq x \leq 1\}. \quad \text{Recall that } \bar{A}, \quad \bar{R} = 0 \text{ are the matrices of } (\mathcal{B}, \mathcal{Z}), \text{ i.e. } \bar{A} \text{ is the incidence matrix of the chains of } (P, \leq). \quad \mathcal{S} \text{ is the family of } S \subseteq V \text{ which are partitionable into less then } (k+1) \text{ antichains since no chain of size } (k+1) \text{ is contained in } S. \quad (\text{This last inference follows from the "polar" Dilworth theorem which is obtained from theorem 2.3 by switching "chains" and "antichains".) Hence the relation (48) is equivalent to:}

\[\text{Theorem 6.13 (Greene-Kleitman [1976])} \]

\[\quad \max \{|S| \mid S \text{ is partitionable in less then } (k+1) \text{ antichains} \} \]

\[\quad = \min \left\{ (k \cdot |C' \cup \mathcal{P} \cup \{C \mid C \in \mathcal{C}\}) \mid C' \subseteq \mathcal{C} \right\} \]

Notice that Dilworth's theorem 2.3 is obtained by taking \(k = 1\): For any collection \(\mathcal{C}^*\) of chains, \(\mathcal{P} \cup \{C \mid C \in \mathcal{C}'\}\) can be seen as a collection \(\mathcal{C}''\) of chains having all length 1 and \(\mathcal{C} \cup \mathcal{C}''\) is a collection covering \(P\). From \(\mathcal{C} \subseteq \mathcal{C}''\), a collection \(\mathcal{C}^*\) of chains partitioning \(P\) of cardinality not larger than \(\mathcal{C} \cup \mathcal{C}''\) can easily be derived.

A polar theorem to theorem 6.13 due to Greene [1976] is obtained by interchanging the words "chain" and "antichain". Cameron [1982] showed that it is also a (not immediate) implication of the duality theorem applied to an instance of coflow polyhedra.

Another theorem of Greene-Kleitman involves the transition phenomenon. Call a collection \(\mathcal{C}'\) of chains \(k\)-\textit{saturated} if the minimum in theorem 6.13 is achieved by \(\mathcal{C}'\). Recall that a \(k\)-saturated collection \(\mathcal{C}'\) corresponds to an optimum solution of the dual LP (25) and conversely. The transition phenomenon states that there exists a dual solution of (25) which is simultaneously optimal for \(r^1 = k\) and \(r^1 = k + 1\) or equivalently:
Theorem 6.14 (Greene-Kleitman [1976])
There exists a collection \( C \) of chains which is \( k \)-saturated and \( (k+1) \)-saturated.

Example 8: Gallai-Milgram Theorem

Let \( G = (V,E) \) be a directed graph. An independent set \( U \) of \( G \) is a set \( U \subseteq V \) such that \( i, j \in U \) implies \((i,j) \in E \) and \((j,i) \notin E \). A \( k \)-independent set is a set \( U \subseteq V \) for which the longest path \( P \) with \( V(P) \subseteq U \) has length smaller than \( k \), i.e. \( |V(P)| \leq k \). The maximum size of an independent \( (k \)-independent) set is then denoted by \( \alpha(G) \) (\( \alpha_k(G) \)). Furthermore, a collection \( D \) of paths is called a path partition of \( G \) if the node sets of the paths in \( D \) partition \( V \).

Let \( D \) be a path partition of \( G \), then for \( k \geq 1 \), \( D^k \) denotes the subcollection of \( D \) with exactly \( k \) nodes; for \( k \geq 2 \), \( D^{\geq k} \subseteq D \) denotes the subcollection of \( D \) with \( k \) nodes or more. For conveniences, we define \( D^{-1} := D^{-2} \).

The theorem of Gallai-Milgram is the following:

Theorem 6.15 (Gallai-Milgram [1960])
\[
\min \{ |D| \mid D \text{ is a path partition of } G \} \leq \alpha(G)
\]

Cameron [1982] made the conjecture that
\[
\min \{ k \cdot |D^{\geq k}| + |D^1| \mid D \text{ is a path partition of } G \} \leq \alpha_k(G).
\]
(Notice that for \( k = 1 \), (49) is equivalent to theorem 6.15). However, C. Berge subsequently found a class of counterexamples, one of which is given in figure 6.7 (for \( k = 5 \)).

The graph \( G \) of the example is a hypotraceable graph (see Grötschel et al. [1980]), i.e. \( G \) contains no hamilton path (an elementary path with \( |V| \) nodes) and all induced subgraphs \( G^v := (\forall v \in V, E(\forall w)) \), \( v \in V \) contain a hamilton path. Hence \( \alpha_5(G) = 5 \); but the path partition with
smallest value in (49) has value 6, for example for \( \mathcal{D} := \{(6,4,5,2,3,1),(7)\} \), where the paths are identified with their ordered node sets.

However, Cameron [1982] showed that (49) is true for acyclic graphs. We extend this result by giving for graphs with circuits an inequality similar to (49) and which includes (49) for acyclic graphs (although it does not fully yield Gallai-Milgram's theorem).

A collection \( \mathcal{D} \) of elementary paths, \( \mathcal{C} \) of elementary circuits is called a path-circuit partition \( (\mathcal{D},\mathcal{C}) \) of \( G \), if their node sets partition \( V \). \( \mathcal{D}^{\geq k} \) and \( \mathcal{D}^k \) are defined as before and \( \beta_k(G) \) denotes the maximum size of a \( k \)-independent set not containing a circuit.

**Proposition 6.16**

\[
\min \{ k|D^{\geq k}| + |D^1| \mid (\mathcal{D},\mathcal{C}) \text{ is a path-circuit partition of } G \} \leq \beta_k(G).
\]

**Proof:**

Let \( (\mathcal{B},\mathcal{Z}) := (\mathcal{B}^V,\mathcal{Z}^V) \) be the modular switching family of node sets of all paths and circuits in \( G \) (see example 6) and \( \bar{A}, \bar{R} \) its matrices. \((r^1,r^2): (\mathcal{B}^V,\mathcal{Z}^V) \rightarrow \mathbb{Z} \) with \( r^1 := k \), \( r^2 := 0 \) is a modular function on \( (\mathcal{B},\mathcal{Z}) \). Hence both dual LP's

\[
\begin{align*}
(50) & \quad \max \ 1x \ \text{subject to} \ \bar{A}x \leq k, \ \bar{R}x \leq 0, \ x \leq 1, \\
(51) & \quad \min \ ky +lz \ \text{subject to} \ y\bar{A} + w\bar{R} + z = 1, \ y, w, z \geq 0
\end{align*}
\]

have integral solutions. Let \( x^* \) and \((y^*,w^*,z^*)\) be integer optimum solutions. Then \((x^*)_c := \max\{0,x^*_c\}\) is a \( \{0,1\} \)-vector and the incidence vector of a \( k \)-independent set not containing a circuit, because \( \bar{A}x^* \leq k, \bar{R}x^* \leq 0 \). (Conversely, for any set \( S \) with this property, \( x_v := 1 \) if \( v \in S \), \( x_v := -k \) otherwise, is a feasible solution of (50)). Hence \( \beta_k(G) \geq 1(x^*)_c + 1x^* \).

On the other hand \( y^* \) yields a collection of paths \( \mathcal{D}_1 \) with \(|V(P)| \geq k \) for all \( P \in \mathcal{D}_1 \) (by optimality), \( z^* \) corresponds to a collection \( \mathcal{D}^2 := \{P^v \mid P^v \text{ is a path with } V(P^v) = v, z_v = 1\} \) and \( w^* \) corresponds to a collection of circuits \( \mathcal{C} \). By \( y^*\bar{A} + w^*\bar{R} + z^* = 1, \ (\mathcal{D} := \mathcal{D}_1 \cup \mathcal{D}_2,\mathcal{C}) \) is a path-circuit partition of \( G \) with

\[
\beta_k(G) \geq 1(x^*)_c + 1x^* = ky^* + 1z^* = k|D^{\geq k}| + |D^1|.
\]
7

Polarity between
lattice and switching
paths polyhedra

In this chapter, we consider two classes of polyhedra $P^L$ and $Q^S$ which will be shown to be polar classes, i.e. $P \in P^L$ implies $P^* \in Q^S$ and $Q \in Q^S$ implies $Q^* \in P^L$ (where $P^*$ and $Q^*$ denote the polar polyhedron of $P$ and $Q$, respectively).

The first class $P^L$ is a subclass of lattice polyhedra and $P \in P^L$ is described in the form

$$P = \{ x \in \mathbb{R}^V \mid Ax \leq 1 \},$$

where $A$ is a $(0,1)$-matrix.

The second class $Q^S$ is a subclass of switching paths polyhedra and $Q \in Q^S$ is described in the form

$$Q = \{ x \in \mathbb{R}^V_+ \mid \bar{A}x \leq 1, \bar{R}x \leq 0 \},$$

where $\bar{A}$, $\bar{R}$ are $(1,0,-1)$-matrices.

By the above polarity relation, any polyhedron $P \in P^L$ yields a pair of matrices $((A,0), (\bar{A}, \bar{R}))$, $A$ involved in the description of $P$, $\bar{A}$ and $\bar{R}$ in the description of $P^*$. Since describing systems of lattice polyhedra are tdi, those of switching paths polyhedra are homogeneous-tdi, it follows that the pair $((A,0), (\bar{A}, \bar{R}))$ is $(A,\bar{A})$-perfect (corollary 4.15). Clearly such a pair can also be obtained starting from a polyhedron $Q \in Q^S$. 
Families of intersections (see chapter 5) play a central role in order to establish the considered polarity relation. Let $A^L$ be the incidence matrix of a family of intersections on a ground set $V$,

\[(1)\quad P(L) := \{ x \in \mathbb{R}^V \mid A^L x \leq 1 \} \quad \text{and} \quad Q(L) := \text{CONV}(A^L) = P(L)^* .\]

The following results, of interest on their own, will be essential for our purpose:

\[(2)\quad P(L) \in \mathcal{PL} \quad \text{and} \quad Q(L) \in \mathcal{QS} ,\]

any polyhedron $P \in \mathcal{PL}$ is a projection of some polyhedron $P(L)$,

any polyhedron $Q \in \mathcal{QS}$ is a cut of some polyhedron $Q(L)$.

Applications of the above polarity relation will be given in chapter 8, where it will be used as a tool for finding polyhedral descriptions of families $F$, whose incidence matrices are lattice matrices.

### 7.1 Lattice matrices and pairs of switching matrices

Lattice matrices were introduced in section 5.2 and we recall the definition: Given a distributive lattice $F$ with minimal and maximal element $m$ and $M$ and a finite set $V$, a matrix $A \in \{0,1\}^{F \times V}$ is called a lattice matrix if it is formed as follows: The columns of $A$, say $f_v$, $v \in V$, are non-zero and define $\{0,1\}$-valued functions on $F$ which are consecutive, modular and such that $f_v(m) = f_v(M) = 0$. The class $\mathcal{PL}$ of polyhedra we are interested in consists of those lattice polyhedra described by lattice matrices, namely

\[(3)\quad \mathcal{PL} := \{ P \mid P = \{ x \in \mathbb{R}^V \mid A x \leq 1 \} : A \text{ is a lattice matrix} \} .\]

A pair $(\bar{A}, \bar{R})$ of matrices with $\bar{A} \in \{1,0,-1\}^{B \times V}$, $\bar{R} \in \{1,0,-1\}^{Z \times V}$ is called a pair of switching matrices, if $\bar{A}$ and $\bar{R}$ are the matrices of a modular switching family $(B,Z)$ with ground set $V$, elementary elements $B^\prime$ and $Z^\prime$, and moreover, containing all paths $[ (v,1) ] \in B, \forall v \in V$. Notice that the condition $[ (v,1) ] \in B$ is equivalent to the fact that $\bar{A}$ contains the identity matrix. We consider the class of switching paths polyhedra $\mathcal{QS}$ described as follows by such matrices:

\[(4)\quad \mathcal{QS} := \{ Q \mid Q = \{ x \in \mathbb{R}^V \mid \bar{A} x \leq 1, \bar{R} x \leq 0 \} : (\bar{A}, \bar{R}) \text{ is a pair of switching matrices} \} .\]

Instances of lattice matrices and pairs of switching matrices occur in connection with intersections $L$ of ring families $C$ and $D$: Let $V$ be the ground set of $L$ and $H = (V, E_0 \cup E_7)$ its representing graph. Recall that considering the lattice $F(L) := \{ (C,D) \mid C \in C, D \in D, C \cup D = V \}$ and for all $v \in V$, functions $f_v$, defined by $f_v(C,D) := 1$ if $v \in C \cap D$ and $f_v(C,D) := 0$ otherwise, the matrix $A^L \in \{0,1\}^{F \times V}$ with columns $f_v$ is a lattice matrix (see proof of theo-
rem 5.4 (iii). Moreover, $A^L$ is up to duplicate rows the incidence matrix $A$ of $L$. We will refer to $A^L$ as the lattice matrix corresponding to $L$.

In section 5.3, a polyhedral description of $L$ was given by means of matrices $\overline{A}^L$ and $R^L$ the rows of which are the alternating vectors of the b-r-paths and b-r-circuits of $H$. Actually, $(\overline{A}^L, R^L)$ are exactly the matrices of the modular switching family $(B, Z)$ of paths and circuits corresponding to the b-r-paths and b-r-circuits (see section 6.3, example 1). By the above definition $(\overline{A}^L, R^L)$ is a pair of switching matrices and will be called the pair of switching matrices corresponding to $L$.

Given a family of intersections $L$, let

$$(5) \quad P(L) := \{ x \in \mathbb{R}^V \mid \overline{A}^L x \leq 1 \} \text{ and } Q(L) := \{ x \in \mathbb{R}^V_+ \mid \overline{A}^L x \leq 1, R^L x \leq 0 \},$$

then $P(L) \in \mathcal{P}^L$, $Q(L) \in \mathcal{Q}^S$ and $P(L)^* = Q(L)$, $Q(L)^* = P(L)$ by $Q(L) = \text{CONV}(\overline{A}^L)$ (theorem 5.4 and theorem 2.20). Hence $P(L)$ is a lattice polyhedra, the polar of which is a switching paths polyhedra $Q(L)$ and conversely. We claim that this polarity relation holds for the whole classes $\mathcal{P}^L$ and $\mathcal{Q}^S$:

**Theorem 7.1**

$\mathcal{P}^L$ and $\mathcal{Q}^S$ are polar classes of polyhedra in the following sense: $P^* \in \mathcal{Q}^S$ for $P \in \mathcal{P}^L$, $Q^* \in \mathcal{P}^L$ for $Q \in \mathcal{Q}^S$.

Consequently, any lattice matrix $A$ is involved in an $(A, \overline{A})$-perfect pair, as well as any pair of switching matrices $(\overline{A}, \overline{R})$:

**Corollary 7.2**

Let $A$ be a lattice matrix, then there exists a pair $(\overline{A}, \overline{R})$ of switching matrices such that $((A, 0), (\overline{A}, \overline{R}))$ is $(A, \overline{A})$-perfect.

Let $(\overline{A}, \overline{R})$ be a pair of switching matrices, then there exists a lattice matrix $A$ such that $((A, 0), (\overline{A}, \overline{R}))$ is $(A, \overline{A})$-perfect.

**Proof:**

The existence of a pair $((A, 0), (\overline{A}, \overline{R}))$ with $A$ a lattice matrix, $(\overline{A}, \overline{R})$ a pair of switching matrices, and $Q^* = P := \{ x \in \mathbb{R}^V \mid Ax \leq 1 \}$, $P^* = Q := \{ x \in \mathbb{R}^V_+ \mid \overline{A}x \leq 1, \overline{R}x \leq 0 \}$ follows directly from theorem 7.1. Moreover, since $Ax \leq 1$ is tdi (theorem 5.2) and $x \geq 0$, $\overline{A}x \leq 1$, $\overline{R}x \leq 0$ is homogeneous-tdi (theorem 6.1), the pair $((A, 0), (\overline{A}, \overline{R}))$ is $(A, \overline{A})$-perfect (corollary 4.15).
The proof of theorem 7.1 will be reduced to the fact that for intersections \( L \), the associated polyhedra \( P(L) \) and \( Q(L) \) of type (5) are polar polyhedra. The main work in this approach will be to show theorems 7.3 and 7.4. They say that for an appropriate family of intersections \( L \), lattice matrices can be essentially obtained from \( A_L \) by positive deletion, pairs of switching matrices from \( (\overline{A}_L, \overline{R}_L) \) by contraction of coordinates. We first state these theorems (which will be proved in sections 7.2 and 7.3, respectively) and then give the proof of theorem 7.1.

**Theorem 7.3**

For any lattice matrix \( A \), there exists a family of intersections \( L \) on some ground set \( V \cup J \) with corresponding lattice matrix \( A_L \) such that \( A \) is obtained from \( A_L \) by positive deletion of \( J \).

**Theorem 7.4**

For any pair \( (\overline{A}, \overline{R}) \) of switching matrices, there exists a family of intersections \( L \) on some ground set \( V \cup J \) with corresponding switching matrices \( (\overline{A}_L, \overline{R}_L) \) such that \( (\overline{A}', \overline{R}') \) obtained from \( (\overline{A}_L, \overline{R}_L) \) by contraction of coordinates \( J \) satisfies: \( \{ x \in \mathbb{R}^V \mid Ax \leq 1, \overline{R}x \leq 0 \} = \{ x \in \mathbb{R}^V \mid \overline{A}'x < 1, \overline{R}'x < 0 \} \).

**Proof of theorem 7.1:**

(i) Proof of \( P \in P_L \) implies \( P^* \in Q_S \):

Let \( A \) be a lattice matrix and \( P := \{ x \in \mathbb{R}^V \mid Ax \leq 1 \} \). By theorem 7.3, there exists a family of intersections \( L \) on some ground set \( V \cup J \) (with corresponding lattice matrix \( A_L \) and corresponding switching matrices \( (\overline{A}_L, \overline{R}_L) \) such that \( A \) is obtained from \( A_L \) by positive deletion of coordinates \( J \). By theorem 4.2, \( P^* = \{ x \in \mathbb{R}^V \mid \overline{A}'x \leq 1, \overline{R}'x \leq 0 \} \) where \( \overline{A}' \) and \( \overline{R}' \) are obtained from \( (\overline{A}_L, \overline{R}_L) \) by contraction of coordinates \( J \).

We show that \( P^* = \{ x \in \mathbb{R}^V \mid \overline{A}'x \leq 1, \overline{R}'x \leq 0 \} \in Q_S \):

Let \( (B_L, Z_L) \) be the modular switching family of b-r-paths and b-r-circuits associated to \( L \) with \( B_L := \{ b^i \mid i \in I \} \), \( Z_L := \{ z_k \mid k \in K \} \) for sets \( I \) and \( K \). Define \( K' := \{ k \in K \mid z_k = [(z_1, Z_1), ..., (z_n, Z_n)] \} \) and \( z_1 \in V \) and consider \( B := \{ c^i \mid i \in I \} \), \( Z := \{ w_k \mid k \in K' \} \) such that for \( i \in I \), \( c_i \) is the partial path of \( b^i = [(b_1, B_1), ..., (b_n, B_n)] \) containing all elements \( (b_j, B_j) \) of \( b^i \) with \( b_j \in V \), \( 1 \leq j \leq n \), and \( w_k \) is the analogous partial path of \( z_k \), \( k \in K' \). \( (B, Z) \) with the same modular switching rule as in \( (B_L, Z_L) \) (i.e. \( c_i v c_j := c_k \) for \( v \in \{ c_i \} \cap \{ c_j \} \)), if \( b_i v b_j = b_k \) is a modular switching family and \( \overline{A}' \), \( \overline{R}' \) are up to duplicate rows the matrices of \( (B, Z) \). Hence \( P^* \in Q_S \).
(ii) Proof of $Q \in Q^$ implies $Q^ \in P^L$:

Let $(\bar{A}, \bar{R})$ be a pair of switching matrices and $Q := \{x \in R^V | \bar{A}x \leq 1, \bar{R}x \leq 0\}$. By theorem 7.4, there exists a family of intersections $L$ on some ground set $V \cup J$ (with corresponding lattice matrix $A^L$ and corresponding switching matrices $(\bar{A}^L, \bar{R}^L)$) such that the matrices $(\bar{A}', \bar{R}')$ obtained from $(\bar{A}^L, \bar{R}^L)$ by contraction of coordinates $J$ satisfy $Q \equiv \{x \in R^V | \bar{A}x \leq 1, \bar{R}x \leq 0\} = \{x \in R^V | \bar{A}'x \leq 1, \bar{R}'x \leq 0\}$. By theorem 4.2, $Q^* = \{x \in R^V | A'x \leq 1\}$ where $A'$ is obtained from $A^L$ by positive deletion of coordinates $J$.

We show $Q^* = \{x \in R^V | A'x \leq 1\} \in P^L$:

Let $f_\nu, v \in V \cup J$ be the columns of the lattice matrix $A^L \in \{0, 1\}^{|V \cup J|}$, $F$ the distributive lattice, and consider $F' := \{a \in F | f_j(a) = 0 \text{ for all } j \in J\}$. By modularity of $f_j, j \in J$ and $f_j$ being $\{0, 1\}$-valued, $F'$ is a sublattice of $F$. Furthermore, since for $v \in V$ the columns $g_v$ of $A' \in \{0, 1\}^{|F'|^V}$ are equal to $(f_\nu)^{(F')}$, they define functions on $F'$ which are modular, consecutive and such that $g_v(m) = g_v(M) = 0$ for $m$ and $M$ being the minimal and maximal elements of $F'$ (and also of $F$). Hence $A'$ is a lattice matrix and $Q^* \in P^L$.

7.2 Proof of theorem 7.3

Theorem 7.3 states that intersections $L$ of ring families (with corresponding lattice matrix $A^L$) can be viewed as prototypes for generating lattice matrices since any lattice matrix $A$ can be obtained by positive deletion of a matrix $A^L$. Another prototype, namely the dicuts in a directed graph will be given by theorem 8.5.

To establish the proof of theorem 7.3, the following result about consecutive modular functions on a lattice is needed:

**Lemma 7.5 (Gröflin, Hoffman [1982])**

Let $f: F \rightarrow \{0, 1\}, f \neq 0$, be consecutive and modular on a distributive lattice $F$. Then the functions $g^1: F \rightarrow \{0, 1\}$ and $g^2: F \rightarrow \{0, 1\}$ defined by $g^1(S) := \max\{f(T) | T \leq S\}$ and $g^2(S) = \max\{f(T) | T \geq S\}$ for all $S \in F$, fulfill (i) to (iii):

(i) $g^1(S) \leq g^1(T)$ if $S \leq T$,
(ii) $g^2(S) \geq g^2(T)$ if $S \leq T$,
(iii) $g^1$ and $g^2$ are modular, and $f = g^1 + g^2 - 1$.

**Proof of theorem 7.3:**

Let $A \in \{0, 1\}^{F \times V}$ be a lattice matrix. For all columns $f_\nu, v \in V$, of $A$, define the functions $g^1_\nu$ and $g^2_\nu$ similarly to the way $g^1$ and $g^2$ are defined for $f$ in lemma 7.5. Let
(6) \( C(S) := \{ v \in V | g^1_v(S) = 1 \} \), \( D(S) := \{ v \in V | g^2_v(S) = 1 \} \) for all \( S \in \mathcal{F} \),

(7) \( C' := \{ C(S) | S \in \mathcal{F} \}, \quad D' := \{ D(S) | S \in \mathcal{F} \} \).

We show first that \( C' \) and \( D' \) are proper ring families on \( V \). Let \( S, T \in \mathcal{F} \). \( C(S) \cap C(T) = \{ v \in V | g^1_v(S) = g^1_v(T) = 1 \} \). Since for any \( v \), \( g^1_v \) is modular, \( g^1_v(S) = g^1_v(T) = 1 \) implies \( g^1_v(S \land T) = 1 \); and by (i), \( g^1_v(S \land T) = 1 \) implies \( g^1_v(S) = g^1_v(T) = 1 \). Therefore \( C(S) \cap C(T) = C(S \land T) \). Similarly, one shows \( C(S) \cup C(T) = C(S \lor T) \), \( D(S) \cap D(T) = D(S \land T) \), \( D(S) \cup D(T) = D(S \lor T) \). Hence \( C' \) and \( D' \) are ring families. Also, since \( f_v(m) = 0 \), \( g^1_v(m) = 0 \) for all \( v \in V \), hence \( C(m) = \emptyset \); since \( f_v \neq 0 \), \( g^2_v(m) = 1 \) for all \( v \), hence \( D(m) = V \). Similarly, \( C(M) = V \) and \( D(M) = \emptyset \), therefore \( C' \) and \( D' \) are proper.

Next, from (iii) of lemma 7.5 and the definition of \( C' \) and \( D' \) follows that \( A \) is the incidence matrix of the family \( \{ C(S) \cap D(S) | S \in \mathcal{F} \} \) and that \( C(S) \cup D(S) = V \) for all \( S \in \mathcal{F} \). Therefore, \( A \) is the incidence matrix of a subfamily of the intersections \( L' := \{ C \cap D | C \in C', D \in D' \} \). The intersections \( C(S) \cap D(T) \) for which no \( A \in \mathcal{F} \) exists with \( C(S) \cap D(T) = C(A) \cap D(A) \) are, however, unwanted. To be able to remove them through positive deletion, we shall extend \( V \) to a set \( V \cup J \) and \( C' \) and \( D' \) to ring families \( C \) and \( D \) so that the lattice matrix corresponding to the intersections of \( C \) and \( D \) precisely yields \( A \) after positive deletion of \( J \).

Since \( \mathcal{F} \) is distributive, consider \( \mathcal{F} \) as a proper ring family over a ground set \( U \), so that \( \leq, \land \) and \( \lor \) are the usual set inclusion and operations. Consider the "unwanted pairs"

(8) \( S := \{ (S, T) \in \mathcal{F} \times \mathcal{F} | C(S) \cap D(T) \in L', \not\exists A \in \mathcal{F}, C(S) \cap D(T) = C(A) \cap D(A) \} \)

and choose \( J \subseteq U \) with \( J \cap (S \land T) \neq \emptyset \) for all \( (S, T) \in S \). Notice that \( J := U \) satisfies this condition. Define:

(9) \( C := \{ C_S := C(S) \cup (S \cap J) | S \in \mathcal{F} \}, \quad D := \{ D_S := D(S) \cup (J \setminus S) | S \in \mathcal{F} \} \).

\( C \cap C_T = C \cap C_T = C_\emptyset = \emptyset \) and \( C \cap U = V \cup J \), hence \( C \) is a proper ring family. Similarly, \( D \) is a proper ring family. In the intersections \( L := \{ C \cap D | C \in C, D \in D \} \), all members \( C \cap D_T \) with \( S = T \) are disjoint from \( J \), while the members with \( (S, T) \in S \) have some element in \( J \) (by the choice of \( J \)). \( C \cap D_T \cap J = \emptyset \Rightarrow (J \cap S) \cap (J \cap T) = \emptyset \Rightarrow \emptyset \Rightarrow (J \cap S) \cap (J \cap T) = \emptyset \Rightarrow (J \cap S) \cap (J \cap T) = \emptyset \Rightarrow C \cap D_T \cap J \neq \emptyset \Rightarrow C \cap D_T \cap J \neq \emptyset \Rightarrow C \cap D_T \neq \emptyset \Rightarrow C \cap D_T \neq \emptyset \).

The positive deletion of coordinates \( J \) in the lattice matrix corresponding to \( L \) yields precisely \( A \) if \( J = U \) and \( A \) up to duplicate rows otherwise.

Notice that if \( A \) is itself a lattice matrix arising from intersections \( L^{\text{old}} \) of ring families \( C^{\text{old}} \) and \( D^{\text{old}} \) with corresponding lattice \( \mathcal{F} := \mathcal{F}(L^{\text{old}}) \), then \( C' = C^{\text{old}} \) and \( D' = D^{\text{old}} \) since \( g^1_v((C, D)) = 1 \) iff \( v \in C \) and \( g^2_v((C, D)) = 1 \) iff \( v \in D \), \((C, D) \in \mathcal{F} \). Hence the set \( S \) of (8) is empty and \( J \) can be chosen to be empty. For this choice \( L = L^{\text{old}} \).
7.3 Proof of theorem 7.4

Let \((\mathcal{B}, \mathcal{Z})\) on \(V\) be a modular switching family with \([(v,1)] \in \mathcal{B}\) for \(v \in V\) and \((\overline{A}, \overline{R})\) its matrices. The idea of finding intersections \(L\) satisfying the condition of theorem 7.4 is the following: We construct a graph \(H = (V \cup J, E_b \cup E_r)\) such that any path \(b \in \mathcal{B} \cup \mathcal{Z}'\) corresponds to a path \(b'\) in \(H\) with edges being alternately blue \((\in E_b)\) and red \((\in E_r)\). Consider the path \(b = [(b_1, B_1), \ldots, (b_n, B_n)] \in \mathcal{B} \cup \mathcal{Z}'\). The corresponding path \(b'\) in \(H\) can be obtained by starting with the ordered node set \((b_1, \ldots, b_n)\) and introducing a blue edge \((b_i, b_{i+1})\), if \(B_i = 1, B_{i+1} = -1\), and a red edge \((b_i, b_{i+1})\), if \(B_i = -1, B_{i+1} = 1\), \(1 \leq i < n\). Now if \(B_i = 1, B_{i+1} = 1\), a new node, denoted by \(b_ib_{i+1} \in J\), is introduced between \(b_i\) and \(b_{i+1}\), together with a blue edge \((b_i, b_ib_{i+1})\) and a red edge \((b_ib_{i+1}, b_{i+1})\). Analogously, if \(B_i = -1, B_{i+1} = -1\), a new node, denoted by \(b_ib_{i+1}^+ \in J\), is introduced between \(b_i\) and \(b_{i+1}\), together with a red edge \((b_i, b_ib_{i+1}^+)\) and a blue edge \((b_ib_{i+1}^+, b_{i+1})\). In figure 7.1, a path \(b \in \mathcal{B}'\) with its corresponding path \(b'\) in \(H\) is represented.

![Figure 7.1](image-url)

Starting from elementary paths and circuits \(b\), the union of their corresponding paths \(b'\), constructed as above, yields a graph with node set \(V \cup J\) and blue and red edges. This graph is however not blue- and red-transitive. Taking the transitive closure of the edges of the same color yields a graph \(H = (V \cup J, E_b \cup E_r)\) which is blue- and red-transitive, hence a graph representing a family of intersections \(L\). This family will be the one we are looking for.

Graph construction

For the following definitions, we assume that a path \(b \in \mathcal{B} \cup \mathcal{Z}\) is given by \(b = [(b_1, B_1), \ldots, (b_n, B_n)]\). We first formalize the construction of \(H\) discussed above.
Define $J := J^+ \cup J^-$, with

$$J^+ := \{vw^+ \mid \exists b \in \mathcal{B} \cup \mathcal{Z}_r, \exists 1 \leq i < n, v = b_i, w = b_{i+1}, v \neq w, B_i = B_{i+1} = -1\},$$

$$J^- := \{vw^- \mid \exists b \in \mathcal{B} \cup \mathcal{Z}_s, \exists 1 \leq i < n, v = b_i, w = b_{i+1}, B_i = B_{i+1} = 1\},$$

and

$$E_b^0 := \{(v,w) \mid v, w \in V, \exists b \in \mathcal{B} \cup \mathcal{Z}_s, \exists 1 \leq i < n, v = b_i, w = b_{i+1}, B_i = 1, B_{i+1} = -1\} \cup \{(v,vw^+) \mid v \in V, vw^+ \in J^+\} \cup \{(vw^+,w) \mid w \in V, vw^+ \in J^+\},$$

$$E_r^0 := \{(v,w) \mid v, w \in V, \exists b \in \mathcal{B} \cup \mathcal{Z}_r, \exists 1 \leq i < n, v = b_i, w = b_{i+1}, B_i = 1, B_{i+1} = 1\} \cup \{(v,vw^-) \mid v \in V, vw^- \in J^-\} \cup \{(vw^-,w) \mid w \in V, vw^- \in J^-\}.$$

Let $E_b$ ($E_r$) be the transitive closure of $E_b^0$ ($E_r^0$) and consider

$$H := (V \cup J, E_b \cup E_r).$$

Due to the correspondence between $b = [(b_1,B_1),\ldots,(b_n,B_n)] \in \mathcal{B} \cup \mathcal{Z}_s$ and its path or circuit in $H$ we will use the expression $b$ contains the edge $(v,w) \in E_b^0$ ( $(v,w) \in E_r^0$; or $(v,vw^+)$, $(vw^+,w)$; or $(v,vw^-),(vw^-,w) \in E_b^0 \cup E_r^0$) if there exists $1 \leq i < n, b_i = v, b_{i+1} = w$ and $(B_i, B_{i+1}) = (1,-1)$ ( $(B_i, B_{i+1}) = (-1,1)$; or $(B_i, B_{i+1}) = (-1,-1)$; or $(B_i, B_{i+1}) = (1,1)$ ). We shall also use the same expression for $b \in \mathcal{B} \cup \mathcal{Z}_s$.

Denote by $\mathcal{L}$ the intersections whose representing graph is $H$, by $(\mathcal{B}_L, \mathcal{Z}_L)$ its corresponding switching family of $b$-r-paths and $b$-r-circuits with matrices $(\mathcal{L}, \mathcal{R}_L)$ and by $(\mathcal{L}^*, \mathcal{R}^*)$ the matrices obtained from $(\mathcal{L}, \mathcal{R}_L)$ by contraction of $J$.

To prove theorem 7.4, we show the following facts:

(13) For any row $\alpha$ of $\mathcal{A}$ ($\mathcal{R}$), $\alpha \neq 0$, there exists a row $\beta$ of $\mathcal{A}^*$ ($\mathcal{R}^*$) with $\alpha \leq \beta$.

(14) For any row $\alpha$ of $\mathcal{A}^*$, $\alpha \neq 0$, there exist a row $\gamma$ of $\mathcal{R}$ and rows $\gamma_i$ of $\mathcal{R}$, $i \in I$ with $\alpha \leq \beta + \sum \{|\gamma_i|, i \in I\}$. For any row $\alpha$ of $\mathcal{R}^*$, $\alpha \neq 0$, there exist rows $\gamma_i$ of $\mathcal{R}$, $i \in I$ with $\alpha \leq \sum \{|\gamma_i|, i \in I\}$.

(13) is essentially given by the construction of $H$ and the proof will be given at the end of this paragraph. (14) is more involved. In fact, for any $b$-r-path or $b$-r-circuit $P$ in $H$ with node set $V(P) = (b_1,\ldots,b_n)$, $b^* = [(b_1,B_1),\ldots,(b_n,B_n)] \in \mathcal{B} \cup \mathcal{Z}_s$ (with $B_i = (-1)^{i+1}$, $1 \leq i \leq n$), and $P$ corresponds to a path $b$ with respect to the ground set $V$, namely the partial path $b = [(b_{i(1)},B_{i(1)}),\ldots,(b_{i(m)},B_{i(m)})]$ of $b^*$ containing all elements $(b_{i,B_i})$ of $b^*$ with $b_i \in V$. However, $b$ is not necessarily in $\mathcal{B} \cup \mathcal{Z}_s$. We will see that essentially, if $P$ uses only edges in $E_b^0 \cup E_r^0$, then $b \in \mathcal{B} \cup \mathcal{Z}_s$. Notice that in this case $\text{val}(b) = \text{val}(b^*)^V$ (where $\text{val}(b) \in \mathbb{Z}^V$, $\text{val}(b^*) \in \mathbb{Z}^{V \cup J}$) and therefore $\mathcal{A}_b = \mathcal{A}_{b^*}$ if $b \in \mathcal{B}^*$, $\mathcal{R}_b = \mathcal{R}_{b^*}$ if $b \in \mathcal{Z}_s$. The main problem arises therefore from $b$-r-paths and $b$-r-circuits in $H$ which use edges not in $E_b^0 \cup E_r^0$. The following example shows that such paths can occur.
Consider the modular switching family \((\mathcal{B}, \mathcal{Z})\) on \(V := \{a,b,c,d\}\) with \(\mathcal{Z} := \{[ ]\}\) and \(\mathcal{B}\) containing the paths \(P_v := [(v,l)]: v \in V\) and \(P_1 := [(a,1),(b,-1),(c,1)],\) \(P_2 := [(b,1),(c,-1),\) \((d,1)],\) \(P_3 := [(a,1),(d,1)].\) The paths \(P_wQ\) for \(P, Q \in \mathcal{B},\) we \(\{P\} \cap \{Q\}\) are defined by \(P_wQ := [ [P_w];[wQ] ]\) for the triples \((P_w,w,Q)e \{(P_v,v,P_i), (P_i,v,P_v) \mid v \in \{P_i\}, 1 \leq i \leq 3\} \cup \{(P_i,\) \(a,P_3), (P_3,a,P_i), (P_2,d,P_3), (P_3,d,P_2)\}\.\) Moreover \(P_1bP_2 := P_1cP_2 := P_3, P_2bP_1 := P_2cP_1 := [ ].\) It is easy to verify that \((\mathcal{B}, \mathcal{Z})\) is a modular switching family. Its graph \(H\) is represented in figure 7.2.

In \(H,\) two new b-r-paths are present, namely the paths corresponding to \(P := [(a,l),(c,-l),\) \((d,l)]\) and \(Q := [(a,l),(b,-l),(d,l)].\) \(P\) and \(Q\) are not in \((\mathcal{B}, \mathcal{Z}),\) but in the switching family \((\mathcal{B}_L, \mathcal{Z}_L)\) arising from the b-r-paths and b-r-circuits of \(H.\) However, they are "covered" by the path \(P_3,\) in the sense that for the value function \(\text{val}: (\mathcal{B}_L, \mathcal{Z}_L) \rightarrow \mathbb{Z}^\cup J,\) \(\text{val}(P)V, \text{val}(Q)V \leq \text{val}(P_3)V,\) hence \(\overline{A}_P \leq \overline{A}_{P_3}\) and \(\overline{A}_Q \leq \overline{A}_{P_3}.\)

In the sequel, lemma 6.3 will often be used and we give here a corollary of it.

**Corollary 7.6**

Let \((\mathcal{B}, \mathcal{Z})\) be a modular switching family.

(i) Let \(ae \mathcal{B}, be \mathcal{B}', v \in (a) \cup (b), e = [(e_1,E_1),...,,(e_n,E_n)] := [[av];[vb]]\) and \(f = [(f_1,F_1),\) \(...,,(f_m,F_m)] := [[bv];[va]].\) If for all \(p \in V,\) all occurrences of \(p\) in \(e\) and \(f\) have the same value (i.e. \(e_i = F_k = p (e_i = f_k = p, f_i = f_k = p)\) for some indices \(i\) and \(k\) implies \(E_i = E_k (E_i = F_k, F_i = F_k)\)), then \(avb = e\) and \(bva = f.\)

(ii) Let \(ae \mathcal{B} (Z), b = [(b_1,B_1),...,,(b_m,B_m)] \in \mathcal{B}', v := b_1 \in (a)\) such that \(avb\) is defined, \(e = [(e_1,E_1),...,,(e_n,E_n)] := [[av];[vb]].\) If for all \(p \in V,\) all occurrences of \(p\) in \(e\) have the same value (i.e. \(e_i = e_k = p\) for some indices \(i\) and \(k\) implies \(E_i = E_k\)), then \(avb = e.\)
Notice that if \( b = [(b_1,B_1),..., (b_n,B_n)] \in Z' \) with \( B_1 = 1 \), then \( [(b_1,B_1)]b_1b \) is equal to \( d := [(b_1,B_1),..., (b_n,B_n)] \in B \) by the corollary above. In the sequel we will identify \( b \) with \( d \) and write

\[
(15) \quad b = [(b_1,B_1),..., (b_n,B_n)] \in Z' \quad \text{with} \quad B_1 = 1, \quad \text{then} \quad b \in B.
\]

**Proof of (13):**

Let \( B'' (Z'') \) contain all elementary paths \( b \) of \( B (Z) \) with \( b = [(b_1,B_1),..., (b_n,B_n)], B_1 = B_n = 1, \) and \( \alpha \) be a row of \( \bar{A} (\bar{R}) \), \( \alpha \not\leq 0 \). Then \( \alpha \) corresponds to an elementary path (circuit) \( a = [(a_1,A_1),..., (a_n,A_n)] \) of \( B' (Z') \). If \( a \in B'' (Z'') \), \( a \) corresponds to a b-r-path (b-r-circuit) \( b \) in \( H \). For \( \beta \) being the corresponding row of \( \bar{A}L (\bar{R}L) \) we get \( \alpha = \beta V \).

Now, if \( a \in Z \cap Z'' \), there exists \( i, 1 \leq i < n, A_i = 1 \) and \( a' = [(a_i,A_i),..., (a_n,A_n)] = (a_1,A_1),..., (a_i,A_i)] \in Z''. \) By the arguments above there exists a row \( \beta \) of \( \bar{R}L \) with \( \beta V = \text{val}(a') = \text{val}(a) = \alpha \).

The last case to consider is \( a \in B \setminus B'' \). If \( A_1 = -1 \), let \( i \) be the smallest index such that \( A_i = -1, A_{i+1} = 1 \) and consider \( a' := [(a_i,1)]a = [(a_{i+1},A_{i+1}),..., (a_n,A_n)] \) by corollary 7.6. If \( A_1 = 1 \), define \( a' := a \) and \( i := 0 \). Analogously, if \( A_n = -1 \) let \( j \) be the largest index such that \( A_{j-1} = 1, A_j = -1 \) and define \( a'' := a'[(a_j,1)] = [(a_{i+1},A_{i+1}),..., (a_{j-1},A_{j-1})] \). If \( A_n = 1 \) define \( a' := a' \). By the arguments above there exists a row \( \beta \) of \( \bar{A}L \) with \( \beta V = \text{val}(a') \geq \text{val}(a) = \alpha \).

**Paths with edges in \( E_b \) and \( E_r \)**

In this paragraph, we show that any elementary b-r-path or b-r-circuit in \( H \), starting and ending with nodes of \( V \), and which uses only edges of \( E_b \cup E_r \), corresponds to a path or circuit in \( B' \) or \( Z' \).

In the sequel the notation \( b = [(b_1,B_1),..., (b_n,B_n),(z,Z)], z \in V \cup \emptyset \) will be used, which for \( z = \emptyset \) is defined as \( b := [(b_1,B_1),..., (b_n,B_n)] \).

We examine first which paths and circuits have to be in \( B \cup Z \) if certain edges are in \( E_b \cup E_r \).

**Lemma 7.7**

(i) Let \( (v,w) \in E_b, v, w \in V, \) then there exists a circuit in \( Z' \) containing \( (v,w) \) which is equal to \( [(v,1),(w,-1),(v,1)] \) or there exists \( z \in V \cup \emptyset \) with \( [(v,1),(w,-1),(z,1)] \in B' \).

(ii) Let \( (v,w) \in E_r, v, w \in V, \) then there exists a circuit in \( Z' \) containing \( (v,w) \) which is equal to \( [(w,1),(v,-1),(w,1)] \) or there exists \( z \in V \cup \emptyset \) with \( [(z,1),(v,-1),(w,1)] \in B' \).
(iii) Let \((v,vw^-) \in E'_0, (vw^-, w) \in E'_1\), then either \(v = w\) and there exists \([v, 1, (v, 1)] \in Z'\), or \([v, 1], (w, 1) \in \mathcal{B}'\).

(iv) Let \((v, vw^+) \in E'_0, (vw^+, w) \in E'_1\), then there exists a circuit in \(Z'\) containing \((v, vw^+), (vw^+, w)\) which is equal to \(b := [(v, -1, (w, 1), (z, 1), (v, -1)]\) for \(z \in V \cup \emptyset\) or there exists \(z, q \in V \cup \emptyset\) with \(b := [(z, 1), (v, 1), (w, 1), (q, 1)] \in \mathcal{B}'\). Moreover, if \(b \in Z', [(z, 1), (v, -1), [(w, -1), (z, 1)] \in \mathcal{B}'\); if \(b \in \mathcal{B}'\), [(z, 1), (v, -1), [(w, -1), (q, 1)] \in \mathcal{B}'\).

Proof:

Notice that if there exists \(b = [(b_1, B_1), \ldots, (b_n, B_n)] \in \mathcal{B}'\) and \(1 < i < n\) with \(b_i = v, b_{i+1} = w,\) and \((B_i, B_{i+1}) = (1, -1)\) in case (i), \((-1, 1)\) in (ii), \((1, 1)\) in (iii), \((-1, -1)\) in (iv), the required elementary paths can be obtained as follows using corollary 7.6:

Consider \(b' := [(b_i, 1)]b_i b\) if \(B_i = 1, b' := [(b_i, -1)]b_i b\) if \(B_i = -1\) and \(i > 1, b' := b\) otherwise; and \(b'' := b' b_{i+1} [(b_{i+1}, 1)]\) if \(B_{i+1} = 1, b'' := b' b_{i+2} [(b_{i+2}, 1)]\) if \(B_{i+1} = -1\) and \(i+1 < n, b'' := b\) otherwise. Then \(b''\) has the required properties.

(i), (iii): Assume there exists no path in \(\mathcal{B}'\) containing \((v, w)\), \((v, vw^-), (vw^-, w)\). Therefore there exists a circuit \(b = [(b_1, B_1), \ldots, (b_n, B_n)] \in Z'\) with \(b_1 = v, B_1 = 1, b_2 = w, B_2 = -1\) \((B_2 = 1)\). Consider \(b\) as a path in \(\mathcal{B}\) (see (15)). We have \(n = 3\) \((n = 2)\) otherwise \(a := bb_3 [(b_3, 1)]\) \((a := bb_2 [(b_2, 1)]\) is an elementary path containing \((v, w)\), \((v, vw^-), (vw^-, w)\). Hence \(b = [(v, 1), (w, -1), (v, 1)]\) \((b = [(v, 1), (v, 1)]\).

(ii): Analogous to (i): Assume there exists no path in \(\mathcal{B}'\) containing \((v, w)\), then there exists a circuit \(b = [(b_1, B_1), \ldots, (b_n, B_n)] \in Z'\) with \(b_1 = w, B_1 = 1, b_n = v, B_n = -1\). Consider \(b\) as a path in \(\mathcal{B}\) (see (15)). We have \(n = 3\) otherwise \(a := [(b_n-2, 1)]b_{n-2} b\) is an elementary path containing \((v, w)\). Hence \(b = [(w, 1), (v, -1), (w, 1)]\).

(iv): Assume there exists no path in \(\mathcal{B}'\) containing \((v, vw^+), (vw^+, w)\) and \([(v, -1), (w, 1), (v, -1)] \in Z'\). Then there exists \(b = [(b_1, B_1), \ldots, (b_n, B_n)] \in Z'\) with \(n \geq 4\) and \(b_2 = v, b_3 = w, B_2 = B_3 = -1\). Define \(a := [(b_1, 1)]b_1 b\) if \(B_1 = -1\) and \(a := [(b_1, 1)]b_1 b_b_{n-1}[[(b_{n-1}, 1)]\) if \(B_1 = 1\) and \(n > 4\). Then \(a\) is an elementary path containing \((v, vw^+), (vw^+, w)\), a contradiction. Hence \(n = 4, B_1 = 1,\) and for \(z := b_1, b = [(z, 1), (v, -1), (w, 1), (z, 1)]\). Moreover, for \(d := [(z, 1)]z b \in \mathcal{B}, dw[(w, 1)] = [(z, 1), (v, 1)]\) and \([(v, 1)]v d = [(w, -1), (z, 1)] \in \mathcal{B}'\).

If \(b = [(z, 1), (v, -1), (w, 1), (q, 1)] \in \mathcal{B}'\), then \(b w[(w, 1)] = [(z, 1), (v, -1)]\) and \([(v, 1)]v b = [(w, -1), (q, 1)] \in \mathcal{B}'\). If \(b = [(v, -1), (w, 1), (v, 1)] \in Z'\), then \(b' = [(w, -1), (v, 1),(w, -1)] \in Z'\) and \([(w, 1)]w b' = [(v, 1)]\) and \([(v, 1)]v b = [(w, 1)] \in \mathcal{B}'\).
We now come to the proposition which states that any elementary $b$-$r$-path or $b$-$r$-circuit starting and ending in nodes of $V$, and using only edges in $E^0 \cup E^r$ corresponds to a path or circuit in $\mathcal{B}'$ or $Z'$. In fact we show a slightly stronger version of it:

**Proposition 7.8**

Let $P = (a_1, \ldots, a_n)$ be an elementary path or circuit of $H$ with edges in $E^0 \cup E^r$ and $a_1, a_n \in V$.

(i) If $n$ is odd and $P$ is a $b$-$r$-path or $b$-$r$-circuit, then:

there exists a path $b \in \mathcal{B}' \setminus Z'$ with corresponding $b$-$r$-path in $H$ equal to $P$.

(ii) If $n$ is even, $P$ is elementary, $(a_1, \ldots, a_{n-1})$ a $b$-$r$-path, and $(a_{n-1}, a_n) \in E^0$, then:

there exists a path $b = [(b_1, B_1), \ldots, (b_m, B_m)] \in \mathcal{B}$, such that $P$ is the corresponding path of $[(b_1, B_1), \ldots, (b_m, B_m)]$ and $q \in V \cup \emptyset$, $q \neq b_i$ for all $i$ with $B_i = -1$, $1 \leq i \leq m$.

**Proof:**

We prove it by induction on $n$. If $n = 1$, $b = [(a_1, 1)] \in \mathcal{B}'$. Assume $b$ exists if $n \leq k-1$ and let $n = k$. In this proof we often switch paths and circuits and if not mentioned otherwise the resulting path is given by corollary 7.6.

**Case 1:** $a_{n-1} \in V$ and $n$ even. Hence there exists a path $b' = [(b_1, B_1), \ldots, (b_{m-1}, B_{m-1})]$ with $B_1 = B_{m-1} = 1$, $b_m = a_{n-1}$ and $b'$ corresponds to $(a_1, \ldots, a_{n-1})$ in $H$. There exists a circuit $c = [(a_{n-1}, 1), (a_{n-1}, 1)] \in Z'$ or a path $[(a_{n-1}, 1), (a_{n-1}, 1), (q, 1)] \in \mathcal{B}'$ with $q \neq b_i$, for all $i$, $1 < i < m$, $B_i = -1$. (Otherwise by lemma 7.7 there exists a path $c' = [(a_{n-1}, 1), (a_{n-1}, 1), (q, 1)]$ with $q = b_i$, for some $i$, $1 < i < m$, $B_i = -1$. Consider $d := c'b_i b_{i+1}' = [(a_{n-1}, 1), (a_{n-1}, 1), (b_{i+1}, B_{i+1}), \ldots, (b_{m-1}, 1)] \in \mathcal{B}$ and $c := db_{i+1}'[(b_{i+1}, 1)] = [(a_{n-1}, 1), (a_{n-1}, 1), (q, 1)]$, with $q = \emptyset$ if $B_{i+1} = -1$, $q = b_{i+1}$ otherwise, a contradiction.) Then we can choose $b := b' \cap c$.

**Case 2:** $a_{n-1} \in V$ and $n$ odd: Hence there exists a path $b' = [(b_1, B_1), \ldots, (b_{m-1}, B_{m-1}), (q, 1)] \in \mathcal{B}$ with $[(b_1, B_1), \ldots, (b_{m-1}, B_{m-1})]$ corresponding to the path $(a_1, \ldots, a_{n-1})$ in $H$, $q \in V \cup \emptyset$, $q \neq b_i$ for $B_i = -1$. There exists a circuit $c = [(a_{n-1}, 1), (a_{n-1}, 1), (a_{n-1}, 1)] \in Z'$ or a path $[(z, 1), (a_{n-1}, 1), (a_{n-1}, 1)] \in \mathcal{B}'$ with $z \in V \cup \emptyset$, $z \neq b_i$, for all $i$, $1 < i < m$, $B_i = -1$. (Otherwise by lemma 7.7 there exists a path $c' = [(z, 1), (a_{n-1}, 1), (a_{n-1}, 1)]$ with $z = b_i$, for some $i$, $1 < i < m$, $B_i = -1$. Consider $d := b' b_{i-1} c = [(b_1, B_1), \ldots, (b_{i-1}, B_{i-1}), (a_{n-1}, 1), (a_{n-1}, 1)]$ and $c := [(b_{i-1}, 1), b_{i-1}] = [(z, 1), (a_{n-1}, 1), (a_{n-1}, 1)]$, with $z = \emptyset$ if $B_{i-1} = -1$, $z = b_{i-1}$ otherwise, a contradiction.)

The desired path can then be obtained as follows: Consider first the case, where $c \in Z'$ and define $e := b' \cap c$. If $P$ is a circuit, we decompose $e$ in $a_1$ and get $e = [(a_1, 1), (a_{n-1}, 1)] \cap \emptyset b$, with $b = [(b_1, B_1), \ldots, (b_{m-1}, B_{m-1}), (b_1, B_1)] \in Z'$. If $P$ is not a circuit, then $b := e a_{n-1}[(a_{n-1}, 1)] \in \mathcal{B}'$. Let now $c \in \mathcal{B}'$. If $P$ is a circuit, we choose $b := b' \cap c \in Z'$ (according to (15)), if $P$ is not a circuit, $b := b' \cap c \in \mathcal{B}'$. 


Case 3: \( a_{n-1} \notin V \) and \( n \) even. Hence there exists a path \( b^' = [(b_1, B_1), \ldots, (b_{m-2}, B_{m-2}), (q, 1)] \in \mathcal{B} \) with \( b_{m-2} = a_{n-2}, B_{m-2} = -1, [(b_1, B_1), \ldots, (b_{m-2}, B_{m-2})] \) corresponding to the path \( (a_1, \ldots, a_{n-2}) \) in \( H \), and \( q \notin V \cup \emptyset, q \neq b_i \) for \( B_i = -1 \). By lemma 7.7 there exists a circuit \( c = [(a_{n-2}, -1), (a_{n-1}, l), (a_{n-2}, -1)] \in \mathcal{Z}' \) with \( z \notin V \cup \emptyset \) or a path \( c := [(z, 1), (a_{n-2}, -1), (a_{n-1}, -1), (p, 1)] \in \mathcal{B}' \) with \( z, p \notin V \cup \emptyset \).

Notice that \( b^' \) can be chosen such that \( q = \emptyset \): First \( d := [(z, 1), (a_{n-2}, -1)] \in \mathcal{B}' \) by lemma 7.7.

Next, if \( z = b_i, B_i = -1 \), define \( b^{''} := b^'a_{n-2}[b^'zd] \), otherwise \( b^{''} := b^'a_{n-2}c \in \mathcal{B}' \). In both cases \( b^{''} = [(b_1, B_1), \ldots, (b_{m-2}, B_{m-2})] \).

Let first \( c \in \mathcal{Z}' \), then \( c' := [(z, 1), (a_{n-2}, -1), (a_{n-1}, -1), (z, 1)] \in \mathcal{Z}' \) if \( z \neq \emptyset \). If \( z = b_i \), for some \( i, 1 \leq i < m-2, B_i = -1 \), then \( e := (b^'b_{i+1}[(b_{i+1}, 1)])zc' = [(b_1, B_1), \ldots, (b_{i-1}, B_{i-1}), (a_{n-2}, -1), (a_{n-1}, -1), (v, 1)], \) with \( v = b_{i+1} \) if \( B_{i+1} = 1, v = \emptyset \) otherwise. Then we can choose \( b := b^'a_n \).

If \( z = b_i \), for some \( i, 1 \leq i < m-2, B_i = 1 \), then \( e := (b^'z[(z, 1)])zc' = [(b_1, B_1), \ldots, (b_i, B_i), (a_{n-2}, -1), (a_{n-1}, -1), (b_i, 1)] \) and we can choose \( b := b^'a_n \).

If \( z = b_i \) for all \( i, 1 \leq i < m-2 \), then we choose \( b := (b^'a_n)z[(z, 1)] \) if \( z \neq \emptyset \) and \( b := (b^'a_n)z[(a_{n-1}, -1)] \) otherwise \( ([(a_{n-1}, -1) \in \mathcal{B}' \) by lemma 7.7].

Let now \( c \in \mathcal{B}' \), \( I := \{ b_i \mid B_i = -1, 1 \leq i < m-2 \} \). Define \( a := [(a_{n-1}, -1), (p, 1)] \in \mathcal{B}' \) (lemma 7.7) and recall that \( d = [(z, 1), (a_{n-2}, -1)] \). Assume \( ze I, z = b_j \). Then \( B_j = -1, 1 < i < m-2 \) and \( e := b^'d = [(b_1, B_1), \ldots, (b_{i-1}, B_{i-1}), (a_{n-1}, -1)] \). Define \( d^' := [(b_{i-1}, 1)]b_{i-1}e = [(v, 1), (a_{n-1}, -1)], \) with \( v = b_{i-1} \) if \( B_{i-1} = 1, v = \emptyset \) otherwise; then \( v \in I, v \neq a_n \). Let \( c^' := d^'a_{n-2}c \), where \( d^' := d \) if \( z \notin I \). Then \( c^' = [(v, 1), (a_{n-2}, -1), (a_{n-1}, -1), (p, 1)], \) \( v \in I \).

Analogously, if \( p \in I, p = b_j \), then \( B_j = -1, j < m-2 \) and \( e := ab^'b^' = [(a_{n-1}, -1), (b_{j+1}, B_{j+1}), \ldots, (b_{n-2}, -1)] \). Define \( a^' := cb_{j+1}[(b_{j+1}, 1)] = [(a_{n-1}, -1), (w, 1)] \), with \( w = b_{j+1} \) if \( B_{j+1} = 1, w = \emptyset \) otherwise; then \( w \notin I, w \neq a_n \). Let \( c^'' := c^'a^'a^' \), where \( a^' := a \) if \( p \notin I \). Then \( c^'' = [(v, 1), (a_{n-2}, -1), (a_{n-1}, -1), (w, 1)], \) and \( v, w \in I \).

Either \( v = w, [(v, 1), (a_{n-2}, -1), (a_{n-1}, -1), (v, 1)] \in \mathcal{Z}' \) and \( b \) exists by the part "\( c \in \mathcal{Z}'" or we can choose \( b := b^'a_{n-2}c^'' \).

Case 4: \( a_{n-1} \notin V \) and \( n \) odd. Hence there exists a path \( b^' = [(b_1, B_1), \ldots, (b_{m-2}, B_{m-2})] \in \mathcal{B}' \) with \( b_{m-2} = a_{n-2}, B_{m-2} = 1 \) and \( b^' \) corresponding to the path \( (a_1, \ldots, a_{n-2}) \) in \( H \). If \( n = 2 \) and \( P \) is a circuit, \( a_1 = a_n \) and by lemma 7.7 \( b := [(a_1, 1), (a_1, 1)] \in \mathcal{Z}' \) exists. Otherwise there exists \( c := [(a_{n-2}, 1), (a_1, 1)] \in \mathcal{B}' \). If \( P \) is a circuit, we choose \( b := b^'a_{n-2}c \in \mathcal{Z}' \) (according to (15)).

Transitivity

This paragraph examines the implications of an edge being in \( (E_1 \setminus E_0) \cup (E_2 \setminus E_0) \). The aim is to
get statements of the form "if an edge belongs to \((E_b \backslash E_r) \cup (E_r \backslash E_f)\), then other edges have to be in \(E_b \cup E_r\)". The proofs are in general very technical and reduce to case enumerations.

**Lemma 7.9**

Let \((q_0,...,q_m)\) be a sequence of nodes in \(V\) with \(m \geq 2\), \((q_i,q_{i+1}) \in E_b\) for all \(i\), \(0 \leq i < m\), and \((q_i,q_j) \in E_r\), \(0 \leq i < j-1 < m\). Then \([(q_{i-1},1),(q_i-1),(q_{i+1},1)] \in \mathcal{B}'\), \(0 < i < m\) and \([(q_i,1), (q_{m-1},1)] \in \mathcal{B}'\) for \(0 \leq i < m-2\). Moreover, \([(q_{i-1},1),(q_i-1)] \in \mathcal{B}'\) for \(0 < i < m\).

**Proof:**

We show first that \([(q_0,1),(q_1-1),(q_2,1)] \in \mathcal{B}'\), \([(q_0,1),(q_1-1)] \in \mathcal{B}'\), (which implies \([(q_{i-1},1),(q_i-1),(q_{i+1},1)] \in \mathcal{B}'\), \([(q_i-1),(q_{i+1},1)] \in \mathcal{B}'\) for \(0 < i < m\). Since \((q_0,q_1),(q_1,q_2) \in E_b\), by lemma 7.7, there exist \(b := [(q_0,1),(q_1-1),(z,1)], c := [(q_1,1),(q_2-1),(p,1)] \in \mathcal{B}' \cup \mathcal{Z}'\), \(z, p \in V \cup \emptyset\), \(z \neq q_1\), \(p \neq q_2\).

Assume \(b, c \in \mathcal{B}'\). Then by lemma 6.3, \(bq_1c\) is a partial path of \([(q_0,1),(q_2-1),(p,1)]\), \(cq_1b\) a partial path of \([(z,1)]\). Since \((q_0,q_2) \in E_b\), \((q_2-1)\) can not be an element of \(bq_1c\), hence \(q_2 = z\) by modularity, \(z = \emptyset\) is not possible and \(b = [(q_0,1),(q_1-1),(q_2,1)]\). If \(b \in \mathcal{B}'\), \(c \in \mathcal{Z}'\) and therefore \(p = q_2\), \(bq_1c\) is a partial path of \([(q_0,1),(q_2-1),(z,1)]\). By the same arguments as before \(q_2 = z\) and again \(z = \emptyset\) is not possible. The case \(b \in \mathcal{Z}'\) is not possible, otherwise \(d := cq_1[(q_1-1),(q_0,1),(q_1-1)]\) is a partial path of \(e := [(q_0,1),(q_2-1),(p,1)]\). By modularity and \(q_0 \neq q_2\), \(q_2 \neq p\) we have \(d = e\), a contradiction to \((q_0,q_2) \in E_b\).

Let now \(m > 2\) and define \(a_i := [(q_{i-1},1),(q_i-1),(q_{i+1},1)]\) for \(1 \leq i < m\). Then by the first part \(a_i \in \mathcal{B}'\). Moreover let \(b_{m-1} := a_{m-1}\) and \(b_i := a_iqb_{i+1}, 1 \leq i < m-1\). We show that \(b_i = [(q_{i-1},1),(q_{m-1},1)]\), \(1 \leq i < m-1\) by induction. For \(1 \leq i := m-2\), \(b_{m-2}\) is a partial path of \([(q_{m-3},1),(q_{m-1},1),(q_{m},1)]\), \((a_{m-1}q_{m-2}a_{m-2})\) a partial path of \([(q_{m-1},1)]\). Since \((q_{m-3},q_{m-1}) \in E_b\), \(b_{m-2} = [(q_{m-3},1),(q_{m},1)]\). For \(b_i := a_iqb_{i+1}\) with \(b_{i+1} = [(q_i,1),(q_{m},1)]\), \(b_i = [(q_{i-1},1),(q_{m},1)]\) by corollary 7.6.

In the last paragraph we saw that elementary \(b\)-\(r\)-paths and \(b\)-\(r\)-circuits with edges in \(E_b \cup E_r\) correspond to paths and circuits in \(\mathcal{B}' \cup \mathcal{Z}'\). In order to prove (14) we would like to "cover" an elementary \(b\)-\(r\)-path or \(b\)-\(r\)-circuit \(p\) in \(H\) by a path or circuit \(q\) using only edges in \(E_b \cup E_r\), in the sense that \(\text{val}(p)^V \leq \text{val}(q)^V\). Doing so, we hope that a decomposition of \(q\) into paths and circuits contains \(b\)-\(r\)-paths and \(b\)-\(r\)-circuits which themselves use only edges in \(E_b \cup E_r\), and which correspond to the desired paths and circuits in \(\mathcal{B}' \cup \mathcal{Z}'\).

The next lemma examines, if locally a part of a \(b\)-\(r\)-path containing an edge in \((E_b \backslash E_r)\) can be replaced by edges in \(E_b \cup E_r\) such that the new path covers the old one.
**Lemma 7.10**

Let \((q_0, \ldots, q_m)\) be a sequence of disjoint nodes with \(m \geq 2\), \(q_i \in V\) for all \(i\), \(1 \leq i \leq m-1\), \(q_0, q_m \in V \cup J\), and \((q_i, q_{i+1}) \in E_0^\pm\) for all \(i\), \(0 \leq i < m\), \((q_i, q_j) \in E_0^\pm\) for all \(i\) and \(j\), \(0 \leq i < j-1 < m\).

<table>
<thead>
<tr>
<th>Case a: (q_0, q_m \in V):</th>
<th>(E_b := {\rightarrow})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a.i) Let (n \in V), ((q_m, n) \in E_r^\mp). Then (q_0 n^+ \in J^+).</td>
<td>(E_r := {\rightarrow})</td>
</tr>
<tr>
<td>(a.ii) Let (n \in V \setminus q_0) with (q_m n^+ \in J^+). Then ((q_0, n) \in E_b^\mp).</td>
<td>(E_b \cap E_r := {\rightarrow})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case b: (q_0 \in V), (q_m = q_m-1 n^+):</th>
<th>(E_b \cap E_r := {\rightarrow})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Then (q_0 n^+ \in J^+).</td>
<td>(E_r \cap E_r := {\rightarrow})</td>
</tr>
</tbody>
</table>

| Case c: \(q_m \in V\), \(q_0 = k q_1^+ \in J^+\): | \(E_r := \{\rightarrow\}\) |
| (c.i) Let \(n \in V \setminus k\), \((q_m, n) \in E_r^\mp\). If \(m = 2\), then \(k q_m^+ \in J^+\) or \((k, n) \in E_r^\mp\); if \(m > 2\) then \((k, n) \in E_r^\mp\). | \(E_b := \{\rightarrow\}\) |
| (c.ii) Let \(n \in V \setminus k\) with \(q_m n^+ \in J^+\) and \((\delta(n) \cap E_r^\mp) \setminus \{(n, k), (n, q_m)\} \neq \emptyset\) or \((\delta(k) \cap E_r^\mp) \setminus \{(n, k), (q_1, k)\} \neq \emptyset\). Then \(k q_m^+ \in J^+\) or \(k n^+ \in J^+\). | \(E_r := \{\rightarrow\}\) |

| Case d: \(q_0 = k q_1^+, q_m = q_m-1 n^+, n \neq k\): | \(E_r := \{\rightarrow\}\) |
| Then \((k, n) \in E_r^\mp\). | \(E_r := \{\rightarrow\}\) |
Proof of lemma 7.10:

Proof of (a.i):

By lemma 7.9, there exists \( a := [(q_0, l), (w, -l), (q_m, l)] \) and \( w = q_i \), if \( m = 2 \), \( w = 0 \) otherwise, and by proposition 7.8 and (15), \( b := [(q_m, 1), (q_m, -1), (n, l)] \) possibly \( n = q_m - 1 \).

Let first \( m = 2 \) and consider \( aq_1b \) and \( bq_1a \) for \( b \in B \): \( aq_1b \) is a partial path of \([q_0, 1), (q_2, -1), (n, 1)] \) (lemma 6.3). Since \((q_0, q_2) \in E_b^0\), \( q_2 \neq q_0 \), \( q_2 \neq n \), we have \( aq_1b = [(q_0, 1), (n, 1)] \) by modularity. Hence \( q_0n \in J^* \). (Notice that if \( [(q_0, 1), (q_0, 1)] \in Z^* \), then \( q_0 q_0 \in J^* \), hence \( n = q_0 \) is theoretically possible.)

Let now \( m > 2 \). Then \( aq_m b = [(q_0, 1), (n, 1)] \) by corollary 7.6. Hence \( q_0n \in J^* \).

Proof of (a.ii):

Notice that by definition of \( H \), \( n \neq q_m \) and if \( n = q_1 \), \((q_0, n) \in E_b^0\). Hence assume \( n \neq q_1 \). Let first \( m = 2 \). By lemma 7.9, there exists \( a := [(q_0, 1), (q_1, -1), (q_2, 1)] \) and by proposition 7.8 a path \( b := [(q_1, 1), (q_2, -1), (n, -1), (x, l)] \) with \( x \in V \cup \emptyset \), \( x \neq \{q_2, n\} \).

Assume first \( x \neq q_1 \), then \( aq_2b \) is a partial path of \( e := [(q_0, 1), (q_1, -1), (n, -1), (x, 1)] \), \( bq_2a \) a partial path of \([q_1, 1)] \). The only element of \( e \) which may not occur in \( aq_2b \) is by modularity \( (q_1, -1) \) and actually it can not occur: otherwise \( aq_2b = e \) and \( en[n(n, 1)] = [(q_0, 1), (q_1, -1)] \) \( B^* \), which by lemma 7.9 is not possible. Hence \( aq_2b = [(q_0, 1), (n, -1), (x, 1)] \) and therefore \( (q_0, n) \in E_b^0 \).

If \( x = q_1 \), then by lemma 6.3, \( aq_1b \) is a partial path of \([q_0, 1), (q_2, -1), (n, -1), (q_1, 1)] \), \( bq_1a \) a partial path of \([q_1, 1)] \). Since \((q_0, q_2) \in E_b^0\), \( aq_2b = [(q_0, 1), (n, -1), (q_1, 1)] \), implying \( (q_0, n) \in E_b^0 \).

Let now \( m > 2 \). Then by lemma 7.9 \( a := [(q_0, 1), (q_m, 1)] \) \( B^* \).

Assume first \( n \neq q_{m-1} \). Then by proposition 7.8, there exists a path \( b := [(q_{m-1}, 1), (q_m, -1), (n, -1), (x, 1)] \) \( B \) with \( x \in V \cup \emptyset \), \( x \neq \{q_m, n\} \). By corollary 7.6, \( aq_m b = [(q_0, 1), (n, -1), (x, 1)] \) and \( (q_0, n) \in E_b^0 \).

Let now \( n = q_{m-1} \). We show that this case is not possible. By lemma 7.7, there exists \( c := [(z, 1), (q_m, -1), (q_m, -1), (q, 1)] \) \( B^* \) with \( z, q \in V \cup \emptyset \) or \( c := [(q_m, 1), (q_m, -1), (z, 1), (q_m, -1)] \) \( B^* \) with \( z \in V \cup \emptyset \). If \( c \in Z^* \) then \( aq_m c = [(q_0, 1), (q_m, -1), (z, 1)] \) \( B^* \) by corollary 7.6 \( (z \neq q_0 \), \( (q_0, q_m) \in E_b^0 \) and \( (q_0, q_m) \in E_b^0 \), a contradiction. If \( c \in B^* \), then \( e := aq_m c = [(q_0, 1), (q_m, -1), (1), 1)] \) \( B \), implying either \( e \in B^* \) if \( q \neq q_0 \) or \( e \in Z^* \) if \( q = q_0 \) (according to (15) ), hence \( (q_0, q_0) \in E_b^0 \), a contradiction.

Proof of case b:

We show that \( c(i) := [(q_i, 1), (n, 1)] \) \( B^* \cup Z^* \) for \( 0 \leq i < m \) by induction. For \( i := m-1 \), \( c(i) \in \)
A'uZ' is given by proposition 7.8. Let \(0 \leq i < m-1\) and \(c(i+1) = [(q_i+1,1),(n,1)] \in \mathcal{B} \cup \mathcal{Z}'\). Since \((q_i,q_{i+1}) \in E_b', \) there exists \(b := [(q_i,1),(q_{i+1},1),(x,1)] \in \mathcal{B}'\) with \(x \in V \cup \emptyset\) or \(b := [(q_{i+1},1),(q_i,1),(q_i+1,1)] \in \mathcal{Z}'\) (lemma 7.7). Consider \(c(i+1) \in \mathcal{B}\) (see (15)). If \(b \in \mathcal{B}'\), then \(b q_{i+1} c(i+1) = [(q_i,1),(n,1)] \in \mathcal{B};\) if \(b \in \mathcal{Z}'\), then \(c(i+1) q_{i+1} b = [(q_i,1),(n,1)] \in \mathcal{B}\) (corollary 7.6), and in both cases \([(q_i,1),(n,1)] \in \mathcal{B} \cup \mathcal{Z}'\). Hence \([(q_i,1),(n,1)] \in \mathcal{B} \cup \mathcal{Z}'\) and \(q_{m+} \in J^+\) for \(0 \leq i < m\).

**Proof of (c.i):**

There exist \(a := [(q_1,1),(w,-1),(n,1)] \in \mathcal{B}\) with \(w = q_2\), if \(m = 2\), \(w = \emptyset\) otherwise (proposition 7.8, (a.i) and (15)), and \(c := [(z,1),(k,-1),(q_1,1),(x,1)] \in \mathcal{B}'\) for \(z \in V \cup \emptyset\) or \(c := [(q_1,1),(z,1),(k,-1),(q_1,1)] \in \mathcal{Z}'\) for \(z \in V \cup \emptyset\) (lemma 7.7). If \(c \in \mathcal{B}'\), consider \(b := cq_1 a\) and \(d := aq_1 c;\) if \(c \in \mathcal{Z}'\), \(b := aq_1 c\). In both cases \(b\) is a partial path of \(e := [(z,1),(k,-1), (w,-1),(n,1)]\) and \(d\) is a partial path of \([(x,1)]\) (lemma 6.3). Consider the case \(m = 2\): If \(b = e, k q_3 \in J^+,\) otherwise either \(z = q_2\) and \(b = [(k,-1),(n,1)];\) or \(c \in \mathcal{B}'\), \(x = q_2\) and \(b = [(z,1),(k,-1),(n,1)],\) which implies \((k,n) \in E_r^f\). If \(m > 2, b = [(z,1),(k,-1),(n,1)]\) and \((k,n) \in E_r^f\).

**Proof of (c.ii):**

If \(k = q_m\) or \(n = q_1\), then \(k n^+\) exists. Hence assume \(k \neq q_m, n \neq q_1\). Notice that \(n \neq q_{m-1}: n \neq q_1\) and for \(m \geq 3\), we have \((q_1,n) \in E_b^f\) by (a.ii) and therefore \(n \neq q_m\) for \(m > 3\). Let \(m = 3\) and assume \(n = q_2\). By lemma 7.7 and 7.9, \(q_m n^+ \in J\), \(c := [(n,-1),(v,1)] \in \mathcal{B}'\), \(v \in \{q_m,n\}, q := [(q_2,1),(q_3,-1),(w,1)] \in \mathcal{B}\) (\(w \neq q_3\), possibly \(w = q_2\)), and \(p := [(q_1,1),(q_2,-1),(q_3,1)] \in \mathcal{B}'\). Then \(p' := pq c = [(q_1,1),(q_2,-1),(v,1)] \in \mathcal{B} \cup \mathcal{Z}'\), \(v \neq q_3\) (corollary 7.6, (15)). If \(p \in \mathcal{B}'\), \(p' q_2 q = [(q_1,1),(q_3,-1),(w,1)]\), if \(p \in \mathcal{Z}'\), \(qq_2 [(q_2,-1),(q_1,1),(q_2,1)] = [(q_1,1),(q_3,-1), (w,1)]\) (corollary 7.6). Both cases imply \((q_1,q_3) \in E_b^f,\) a contradiction.

By lemma 7.7 and proposition 7.8, there exists \(a := [(q_m,1),(q_m,1),(n,1),(v,1)] \in \mathcal{B}\) with \(v \in V \cup \emptyset, v \in \{q_m,n\}\), \(b := [(q_m,1),(q_m,1)]\) and \(c := [(n,-1),(v,1)] \in \mathcal{B}'\). Moreover, there exists \(d := [(z,1),(k,-1),(q_1,1),(x,1)] \in \mathcal{B}'\) for \(z \in V \cup \emptyset\), or \(d := [(q_1,1),(z,1),(k,-1),(q_1,1)] \in \mathcal{Z}'\) for \(x \in V \cup \emptyset\). Let \(p := [(z,1),(k,-1)] \in \mathcal{B}'\) (lemma 7.7).

We show first, that \(k \neq q_m\). Assume \(k = q_m\), then \(m > 2\) and \(e := [(q_m,2,1),(q_m,1,-1),(q_m,1)] \in \mathcal{B}^+\) (lemma 7.9). Then \(eq_m p = [(q_m,2,1),(q_m,1,-1)] \in \mathcal{B}^+\) (corollary 7.6), which according to lemma 7.9 is not possible.

We show now, that \(v \neq k\) and \(z \neq n\) can be assumed. By \(\delta(n) \cap E_r^f \setminus \{(n,k),(n,q_m)\} \neq \emptyset\) or \(\delta(k) \cap E_b^f \setminus \{(n,k),(q_1,k)\} \neq \emptyset,\) we can choose \(v \neq k\) or \(z \neq n,\) (Consider \(\delta(n) \cap E_r^f \setminus \{(n,k), (n,q_m)\} \neq \emptyset,\) the case \(\delta(k) \cap E_b^f \setminus \{(n,k),(q_1,k)\} \neq \emptyset\) being similar. If \(n,w) \in E_r, w \in V \setminus \{q_k,k\},\) then by proposition 7.8 a can be chosen with \(v = w.\) If \(n w^+ \in J,\) then there exists \(f := [(r,1),(n,-1)] \in \mathcal{B}'\) (lemma 7.7) and \(c_{nf} = [(n,-1)]\). Hence replacing a by \(an [(n,-1)] = [(q_m,1),(q_m,1),(n,1)] \in \mathcal{B}^+\) we get \(v = \emptyset.\) Assume \(v = k\) and therefore \(z \neq n.\) Then cvp
\[ [(n,-1)] \in \mathcal{B}^- \text{ and we can choose } \text{an}(\text{cvp}) = [(q_{m-1},1),(q_{m-1},1),(n,-1)] \in \mathcal{B}^- \text{ as a new path } a. \]

Assume now \( z = n \) (and \( v \neq k \)), then \( \text{cnp} = [(k,-1)] \). If \( \text{de} \mathcal{B}^- \), then we can choose \((\text{cnp})kd = [(k,-1),(q_{m-1},1),(x,1)] \in \mathcal{B}^- \) as a new path \( d \). If \( \text{de} \mathcal{Z}' \), then \( d' := [(z,1),(k,-1),(q_m,1),(z,1)] \in \mathcal{Z}' \) and \((\text{cnp})k([(z,1)]zd') = [(k,-1),(q_{m-1},1),(z,1)] \in \mathcal{B}^- \) can be chosen as a new path \( d \) in \( \mathcal{B}^- \).

Consider first \( \text{de} \mathcal{B}^- \).

Let \( m = 2 \). Assume \( x = q_2 \), then \( dq_1a \) is a partial path of \( e := [(z,1),(k,-1),(q_2,-1),(n,-1),(v,1)] \), \( aq_1d \) a partial path of \([(q_2,1)] \). By modularity \( dq_1a = e \in \mathcal{B}^- \cup \mathcal{Z}' \) (\( \mathcal{Z}' \) if \( z = v \)) or \( dq_1a = [(z,1),(k,-1),(n,-1),(v,1)] \in \mathcal{Z}' \). Hence \( kq_2 \in J^+ \) or \( kn^+ \in J^+ \).

Assume \( x \neq q_2 \) and consider \( e := dq_1b \), which is a partial path of \( f := [(z,1),(k,-1),(q_2,-1)] \). Then either \( e = [(k,-1)] \) and \( z = q_2 \), or \( e = f \). In the first case \((ekd)q_1b = [(k,-1),(q_2,-1)] \). Hence \( kq_2 \in J^+ \).

Let \( m > 2 \). Then by (a.ii) there exists \((q_1,n) \in E_0 \) and therefore \( e := [(q_1,1),(n,-1),(v,1)] \in \mathcal{B}^- \) (possibly \( v = q_1 \)).

If \( x \neq n \), \( dq_{1e} \) is a partial path of \([(z,1),(k,-1),(n,-1),(v,1)] \), \( eq_{1d} \) a partial path of \([(x,1)] \) (lemma 6.3). By modularity \( dq_{1e} = [(z,1),(k,-1),(n,-1),(v,1)] \), hence \( kn^+ \in J^+ \).

If \( x = n \), then \( v = q_1 \) can be assumed, otherwise replacing \( d \) by \( dnc = [(z,1),(k,-1),(q_1,1),(n,-1),(v,1)] \) we get a new path \( d \) with \( x \neq n \), or a new circuit \( d \) (if \( v = z \neq n \)).

Let first \( m = 3 \), then \( f := [(q_1,1),(q_2,-1),(q_3,1)] \in \mathcal{B}^- \) (lemma 7.9). We show that \( z \neq q_2, z \neq q_3 \) can be assumed, otherwise define \( d' := bq_3d = [(q_2,1),(k,-1),(q_1,1),(x,1)] \in \mathcal{B}^- \) if \( z = q_3 \); \( d' := d \) if \( z = q_2 \); and \( p' := d'q_1[(q_1,1)] = [(q_2,1),(k,-1)] \). Consider \( p'' := f(q_2p' = [(q_1,1),(k,-1),(q_1,1),(x,1)] \in \mathcal{B}^- \) (lemma 7.7). Then \( qq_{1p''} = [(k,-1)] \) and we can replace \( d \) by \( [(k,-1)]kd = [(k,-1),(q_1,1),(x,1)] \).

Since \( n \neq q_2, z \neq q_3 \), \( g := dq_1f = [(z,1),(k,-1),(q_2,-1),(q_3,1)] \in \mathcal{B}^- \) and \( gq_2a \) is a partial path of \( h := [(z,1),(k,-1),(q_3,-1),(n,-1),(q_1,1)] \), \( aq_2g \) a partial path of \([(q_3,1)] \) (lemma 6.3). Either \( gq_2a = [(z,1),(k,-1),(n,-1),(q_1,1)] \in \mathcal{B}^- \), or \( gq_2a = h \in \mathcal{B}^- \). Hence \( kq_3 \in J^+ \) or \( kn^+ \in J^+ \).

Let now \( m > 3 \), then by lemma 7.9, \( f := [(q_1,1),(q_2,1),(q_3,1)] \in \mathcal{B}^- \). Consider \( g := dq_1f = [(z,1),(k,-1),(q_3,1)] \in \mathcal{B}^- \cup \mathcal{Z}' \). If \( g \in \mathcal{B}^- \), \( gq_{1m}a \) is a partial path of \( h := [(z,1),(k,-1),(n,-1),(q_1,1)] \), \( aq_{1m}g \) a partial path of \([(q_1,1)] \), hence \( gq_{1m}a = h \in \mathcal{B}^- \) and \( kn^+ \in J^+ \). If \( g \in \mathcal{Z}' \), \( aq_{1m}g = [(q_{m-1},1),(k,-1),(n,-1),(q_1,1)] \in \mathcal{B}^- \) and \( kn^+ \in J^+ \).

Let now \( d \in \mathcal{Z}' \) and \( d' := [(k,-1),(q_1,1),(z,1),(k,-1)] \in \mathcal{Z}' \).

Consider first \( m = 2 \) and let \( f := bq_1d \). Either \( f = [(z,1),(k,-1),(q_{m-1},1)] \) and \( kq_m^+ \in J^+ \), or \( f = [(k,-1)] \in \mathcal{B}^- \) and \( z = q_m \). In the latter case, \((fkd')z[(z,1)] = [(k,-1),(q_1,1),(z,1)] \in \mathcal{B}^- \) and \( kq_m^+ \in J^+ \) or \( kn^+ \in J^+ \) by the part "\( \text{de} \mathcal{B}^- \), \( m = 2 \)".
Let $m > 2$, then by (a.ii), $e := [(q_1,1),(n,-1),(v,1)] \in \mathcal{B}$. The path $f := eq_1d$ is a partial path of $h := [(z,1),(k,-1),(n,-1),(v,1)]$. Since $z \neq n, v \neq k, f = h \in \mathcal{B} \cup \mathcal{Z}$ and $kn^+ \in J^+$.

**Proof of case d:**

By case b there exists $a := [(q_1,1),(n,1)] \in \mathcal{B}$ ($q_i = n$ is possible) and by lemma 7.7 $d := [(z,1),(k,-1),(q_1,-1),(x,1)] \in \mathcal{B}'$ for $z, x \in V \cup \emptyset$ or $d := [(q_1,-1),(z,1),(k,-1),(q_1,-1)] \in \mathcal{Z}'$ for $z \in V \cup \emptyset$. If $d \in \mathcal{B}'$, $dq_1a = [(z,1),(k,-1),(n,1)] \in \mathcal{B} \cup \mathcal{Z}'$ and $(k,n) \in E_r$. If $d \in \mathcal{Z}'$, $aq_1d = [(z,1),(k,-1),(n,1)] \in \mathcal{B} \cup \mathcal{Z}'$ and $(k,n) \in E_r$.

Notice that lemma 7.10 considers only edges in $E_b \cup E'_b$. Of course a similar lemma exists for edges in $E \setminus E'_r$, however we will not need it. The reason for this is, that red edges can be reduced to blue edges by the following remark.

**Remark 7.11**

If the ordering of any path in a switching family $(\mathcal{B}, \mathcal{Z})$ is reversed, the resulting family $(\mathcal{A}, \mathcal{W})$ is again a switching family. Moreover if $(\mathcal{B}, \mathcal{Z})$ is modular, $(\mathcal{A}, \mathcal{W})$ is modular, too. The graph $H = (V \cup J, E_b \cup E_r)$ corresponding to $(\mathcal{A}, \mathcal{W})$ is obtained from the graph $H = (V \cup J, E_b \cup E_r)$ corresponding to $(\mathcal{B}, \mathcal{Z})$ by reversing all edges and changing their color. Hence a $b$-$r$-path in $H$ with node set $(p_1, \ldots, p_m)$ corresponds to a $b$-$r$-path in $\overline{H}$ with node set $(p_m, p_{m-1}, \ldots, p_1)$ and for $1 \leq i < m$, $(p_i,p_{i+1}) \in E_b (E_r)$ in $H$ iff $(p_{i+1},p_i) \in E_r (E_b)$.

**Proof**

Recall that theorem 7.4 is equivalent to (13) and (14). The proof of (13) has already been given and (14) will be proved here.

**Proof of theorem 7.4 and (14):**

Without loss of generality we assume that $\mathcal{B}$, $\mathcal{Z}$, $\mathcal{B}^L$, $\mathcal{Z}^L$ each contain the empty path $[$$. In order to prove (14) we will show (16):

(16) To any elementary path $p \in \mathcal{B}^L (\mathcal{Z}^L)$, $\text{val}(p)^V \neq 0$, there exists a path $q \in \mathcal{B}^L (\mathcal{Z}^L)$ with the following properties:

(17) the $b$-$r$-path in $H$ corresponding to $q$ uses only edges in $E_b \cup E_r$,

(18) $\text{val}(p)^V \leq \text{val}(q)^V$,

(19) $q = [(b_1,B_1), \ldots, (b_n,B_n)]$, $b_i = b_j$ for some $i \neq j$, $1 \leq i, j \leq n$, implies $b_i \in J$.

**Proof of (16) implies (14):**

Recall that $\text{val}(b) \in \mathcal{Z}^V$ for $b \in \mathcal{B} \cup \mathcal{Z}$, $\text{val}(b) \in \mathcal{Z}^V \cup J$ for $b \in \mathcal{B}^L \cup \mathcal{Z}^L$. 

Let $p \in \mathcal{B}(Z^C)$ be elementary. By (16) there exists $q = [(b_1,B_1),..., (b_n,B_n)] \in \mathcal{B}(Z^C)$ with properties (17)-(19). Let $q = q' \oplus Rq$ ($q = \oplus Rq$) be a decomposition of $q$. Since $q$ satisfies (19) and for $v \in J$, either $\delta(v) \subseteq E_b$ and $\overline{\delta(v)} \subseteq E_r$; or $\delta(v) \subseteq E_r$ and $\overline{\delta(v)} \subseteq E_b$, $q'$ and $r \in Rq$ use only edges contained in $q$. Hence $q'$ and $r \in Rq$ use themselves only edges in $E_b \cup E_r$.

Consider first $q'$. Either $\text{val}(q') = 0$ and we define $f(q') = [ \ ] \in \mathcal{B}$, or there exists a maximal b-r-path $q''$ contained in $q'$, starting and ending with nodes in $V$. Hence $\text{val}(q)' \leq \text{val}(q'')$ and by proposition 7.8, $q''$ corresponds to an elementary path, say $f(q' \in \mathcal{B}$, i.e. $\text{val}(f(q')) = \text{val}(q') \geq \text{val}(q)'$.

Consider now $r = [(r_1,R_1),..., (r_m,R_m)] \in Rq$. Either $\text{val}(r) \leq 0$ and define $f(r) = [ \ ]$, or an index $1 \leq i \leq m$ exists with $\overline{r_i} \in V$, $R_i = 1$. Let $r^{'^1} = [(r_1,R_1),..., (r_m,R_m)] \in Z^C$. By proposition 7.8, $r^{'^1}$ corresponds to a circuit, say $f(r) \in \mathcal{Z}$, and $\text{val}(r)' = \text{val}(f(r))$.

Then $f(q')$ and $f(r)$, $r \in Rq$ are the desired paths and circuits of (14): If $p \in \mathcal{B}(Z^C)$, then

$$\overline{A}' = \text{val}(p)' \leq \text{val}(q)' = \text{val}(q') + \Sigma \{ \text{val}(r)' | r \in Rq \}$$

$$\leq \text{val}(f(q')) + \Sigma \{ \text{val}(f(r)) | r \in Rq \}$$

$$= \overline{A}f(q') + \Sigma \{ \overline{A}f(r) | r \in Rq \}.$$ Analogously, $\overline{R}' \leq \Sigma \{ \overline{R}f(r) | r \in Rq \}$ if $p \in \mathcal{Z}$.

Proof of (16):

To prove (16) we proceed as follows: Given a b-r-path $p = [(p_1,P_1),..., (p_m,P_m)] \in \mathcal{B} \cup \mathcal{Z}$ with $p_1, p_m \in V$, satisfying property (19) and using at least one edge in $E_b \cup E_r \cup E'_r$, we construct a path $q \in \mathcal{B} \cup \mathcal{Z}$ with property (20):

(20) If $q = [(b_1,B_1),..., (b_n,B_n)]$, then $b_1, b_n \in V$, $q$ satisfies (18) and (19). Moreover, either $q$ has less edges in $E_b \cup E_r \cup E'_r$ then $p$, or $q$ has the same number of edges in $E_b \cup E_r \cup E'_r$, but totally less edges then $p$.

Clearly the existence of such a path $q$ for any path $p$ implies (16).

Let $p = [(p_1,P_1),..., (p_m,P_m)]$ be a b-r-path in $\mathcal{B} \cup \mathcal{Z}$ with $p_1 = p_m \in V$ and satisfying property (19). By the remark 7.11, we can assume without loss of generality that the first or the last edge in $E_b \cup E_r \cup E'_r$ is in $E_b$. Let $(p_i,p_{i+1}) \in E_b \cup E'_r$, $1 \leq i < m$, be such an edge.

Notice that if for $\alpha \in \{ b,r \}$, $(v,w) \in E_{\alpha} \cup E'_{\alpha}$, then

(21) there exists a sequence $(v = q_0,q_1,...,q_t = w)$ of disjoint nodes, $(q_e,q_{e+1}) \in E_{\alpha}$, $1 \leq e < t$, $(q_e,q_j) \in E'_{\alpha}$, $1 \leq e < j-1 < t$.

Case 1: $p_i, p_{i+1} \in V$ (implying $i < m-1$).

Case 1.1 $(p_{i+1},p_{i+2}) \in E'_r$, $p_{i+2} \in V$.

By lemma 7.10 (a.i), $p_ip_{i+2} \in J$ and therefore $q = [(p_1,P_1),..., (p_i,P_i), (p_ip_{i+2}$,
-1), \((p_{i+2}, p_{i+2}), \ldots, (p_m, p_m)\) satisfies (20), since it contains no new edge in \(E_b \cup E_b^c \cup E_t \cup E_t^c\) and no more \((p_i, p_{i+1}) \in E_b \setminus E_b^c\).

**Case 1.2**: \((p_{i+1}, p_{i+2}) \in E_t^c, p_{i+2} \notin V\) (implying \(i < m-3\)).

Then \(p_{i+2} = p_{i+1} n^+, n \in V\). Define \(q := [(p_1, p_1), \ldots, (p_i, p_i), (p_{i+3}, p_{i+3}), \ldots, (p_m, p_m)]\). We shall show that \(q \in B \cup Z\) and that it satisfies (20).

Consider first the case \(n = p_{i+1}\). By (19), \(n \neq p_i\) and therefore \((p_i, p_{i+1}) \in E_b^c\) (lemma 7.10, (a.ii)). Hence \(q \in B \cup Z\) satisfies (20), since it has less edges in \(E_b \cup E_b^c \cup E_t \cup E_t^c\).

Consider now \(n \neq p_{i+1}\), then \((p_{i+2}, p_{i+3}) \in E_b^c\) and in the sequence \((q_0, \ldots, q_t)\) of (21) with \((v, w) := (p_{i+2}, p_{i+3})\), \(q_1 = n\) and therefore by transitivity \((n, p_{i+3}) \in E_b\). We have \((p_{i+3}, p_{i+3}) \in E_b^c\), since if \(n \neq p_i\), \((p_i, n) \in E_b^c\) (lemma 7.10 (a.ii)) and \((n, p_{i+3}) \in E_b\). Hence \(q \in B \cup Z\) satisfies (20): \(q\) contains possibly \((p_{i+3}, p_{i+3})\) not in \(E_b\), but no more \((p_i, p_{i+1})\), \((p_{i+2}, p_{i+3}) \in E_b^c\).

**Case 1.3**: \((p_{i+1}, p_{i+2}) \in E_t^c\) (implying \(i < m-1\)).

Let \((q_0, \ldots, q_t)\) be the sequence of (21) for \((v, w) := (p_{i+1}, p_{i+2})\). Let \(n := q_1\), then \((n, p_{i+2}) \in E_t^c\) by transitivity and \(p_{i+1} \in J\) by lemma 7.10 (a.ii). Therefore we have \((p_{i+2}, p_{i+3}) \in E_b^c\) and \((p_{i+1}, p_{i+2}) \in E_t^c\), since \((p_{i+1}, n) \in E_t^c\) and \((n, p_{i+2}) \in E_t^c\) (\(p_{i+2} \neq p_i\) as \(\delta(p_{i+1}) \subseteq E_b\)). Hence \(q := [(p_1, p_1), \ldots, (p_i, p_i), (p_{i+3}, n), (p_{i+2}, p_{i+2}), \ldots, (p_m, p_m)]\) satisfies (20): \(q\) contains \((p_{i+3}, p_{i+2}) \in E_t^c\), but no more \((p_i, p_{i+1})\), \((p_{i+1}, p_{i+2}) \in E_b \cup E_b^c \cup E_t \cup E_t^c\).

**Case 2**: \(p_i \in V, p_{i+1} \notin V\) (implying \(i < m-1\)).

Let \(p_{i+1} = km \in J\). There exists \(p_{i+1} \in J\) by lemma 7.10, case b. Define \(q := [(p_1, p_1), \ldots, (p_i, p_i), (p_{i+3}, n), (p_{i+2}, p_{i+2}), \ldots, (p_m, p_m)]\). We shall show that \(q \in B \cup Z\) and that it satisfies (20).

Consider first the case \(n = p_{i+2}\), then \(q \in B \cup Z\) and it satisfies (20).

Let now \(n \neq p_{i+2}\), then \((p_{i+1}, p_{i+2}) \in E_t^c\). Let \((q_0, \ldots, q_t)\) be the sequence of (21) for \((v, w) := (p_{i+1}, p_{i+2})\). Then \(q_1 = n\), \((n, p_{i+2}) \in E_t^c\) and therefore \((p_{i+1}, p_{i+2}) \in E_t^c\) by transitivity. Hence \(q \in B \cup Z\) and it satisfies (20): \(q\) contains \((p_{i+2}, p_{i+2}) \in E_t \cup E_t^c\), but no more \((p_i, p_{i+1})\), \((p_{i+1}, p_{i+2}) \in E_b \cup E_b^c \cup E_t \cup E_t^c\).

**Case 3**: \(p_i \in V, p_{i+1} \in V\) (implying \(3 \leq i < m-1\)).

Let \((q_0, \ldots, q_t)\) be the sequence of (21) for \((v, w) := (p_{i+1}, p_{i+1})\). Then \(q_0 = p_i = kq^j\). If \(k \neq p_i\), then \((p_i, p_{i+1}) \in E_t^c\) and \(k = r_{i-1}\) for the sequence \((r_0, \ldots, r_e)\) of (21) with \((v, w) = (p_{i-1}, p_i)\). Moreover \((p_{i-1}, k) \in E_t^c\) by transitivity.
Case 3.1: \((p_{i+1}, p_{i+2}) \in E_r, p_{i+2} \in V\)

Consider first the case \(k = p_{i+2}\), then by (19) we have \((p_{i-1}, p_i) \in E_r, (p_{i-1}, k = p_{i+2}) \in E_r\) and \(q := [(p_1, P_1), ..., (p_{i-1}, P_{i-1}), (p_{i+2}, P_{i+2}), ..., (p_m, P_m)] \in B^L \cup Z^L\) satisfies (20): \(q\) contains possibly \((p_{i-1}, p_{i+2}) \in E_r\) but no more \((p_{j-1}, p_j)\) and \((P_{i+2}, P_{i+3}) \in E_b\).

If \(k \neq p_{i+2}\), then by lemma 7.10 (c.i), \(k, p_{i+1} \in J\) or \((k, p_{i+2}) \in E_r\).

If \(k, p_{i+1} \in J\), then \(q := [(p_1, P_1), ..., (p_{i-1}, P_{i-1}), (p_{i+1}, P_{i+1}), ..., (p_m, P_m)] \in B^L \cup Z^L\) (if \(k \neq p_{i-1}\), then \((p_{i-1}, k) \in E_r\), \((k, p_{i+1}) \in E_r\), hence \((p_{i-1}, k, p_{i+1}) \in E_r\) and it satisfies (20): \(q\) contains \((p_{i-1}, k, p_{i+1}) \in E_r\) iff \((p_{i-1}, p_{i+1}) \in E_r\).

If \((k, p_{i+2}) \in E_r\), define \(q := [(p_1, P_1), ..., (p_{i-1}, P_{i-1}), (p_{i+2}, P_{i+2}), ..., (p_m, P_m)] \in B^L \cup Z^L\) (if \(p_{i-1} \neq k\), then \((p_{i-1}, k) \in E_r\), \((k, p_{i+2}) \in E_r\) and \((p_{i-1}, p_{i+2}) \in E_r\)). It satisfies (20), since \((p_{i-1}, p_{i+2}) \in E_r\) implies \((p_{i-1}, p_{i+1}) \in E_r\).

Case 3.2: \((p_{i}, p_{i+1}) \in E_r, p_{i+2} \in V\) (implying \(3 \leq i < m-3\)).

Then \(p_{i+2} = p_{i+1} n^+\) for \(n \in V\). Recall that \((p_{i}, p_{i+1})\) is the first or the last edge in \(E_b \setminus E_r \cup E_r'\). Hence \(k = p_{i-1}\) or \(n = p_{i+3}\).

Consider first the case \(n = k\). If \(k = p_{i-1}\), then \(n \neq p_{i+3}\) and \((p_{i+2}, p_{i+3}) \in E_b\).

Moreover \((p_{i-2}, p_{i-1}) \in E_b, (p_{i-1}, n, p_{i+3}) \in E_b\), hence \((p_{i-2}, p_{i+3}) \in E_b\) and \(q := [(p_1, P_1), ..., (p_{i-2}, P_{i-2}), (p_{i+3}, P_{i+3}), ..., (p_m, P_m)] \in B^L \cup Z^L\) satisfies (20): It contains possibly \((p_{i-2}, p_{i+3}) \in E_b\), however no more \((p_{i-1}, p_{i+1})\) and \((p_{i+2}, p_{i+3}) \in E_b\).

If \(k \neq p_{i-1}\), then \(n = p_{i+3}\) and \((p_{i-1}, p_{i+1}) \in E_r\). By similar arguments as before, \((p_{i+2}, p_{i+4}) \in E_r\) and \(q := [(p_1, P_1), ..., (p_{i-1}, P_{i-1}), (p_{i+4}, P_{i+4}), ..., (p_m, P_m)] \in B^L \cup Z^L\) satisfies (20).

Let now \(n \neq k\) and \((\delta(n) \cap E_r') \setminus \{(n, k), (n, p_{i+1})\} = \emptyset\) or \((\delta(k) \cap E_b') \setminus \{(n, k), (q_1, k)\} = \emptyset\). Then by lemma 7.10 (c.ii), there exists \(k, p_{i+1} \in J\) or \(k, n^+ \in J\). The case \(k, p_{i+1} \in J\) has already been considered in case 3.1. If \(k, n^+ \in J\), then \(q := [(p_1, P_1), ..., (p_{i-1}, P_{i-1}), (k, n^+), (p_{i+3}, P_{i+3}), ..., (p_m, P_m)] \in B^L \cup Z^L\), since \((p_{i-1}, n^+) \in E_r\) (if \(k \neq p_{i-1}\), \((p_{i-1}, k) \in E_r\), \((k, n^+) \in E_r\) and \((k, n^+, p_{i+3}) \in E_b\) if \(n \neq p_{i+3}\), \((k, n^+) \in E_b\), \((n, p_{i+3}) \in E_b\) and \((p_{i+2}, p_{i+3}) \in E_b\). (20) is satisfied by \(q\), because \(q\) contains \((p_{i-1}, n^+) \in E_r\) iff \((p_{i-1}, p_i) \in E_r\) and \((n^+, p_{i+3}) \in E_b\) iff \((p_{i+2}, p_{i+3}) \in E_b\).

Let now \((\delta(n) \cap E_r') \setminus \{(n, k), (n, p_{i+1})\} = \emptyset\) and \((\delta(k) \cap E_b') \setminus \{(n, k), (q_1, k)\} = \emptyset\).

Consider first the case where \((p_{i}, p_{i+1})\) is the first edge of \(p\) in \(E_b \setminus E_b' \cup E_r \setminus E_r'\), then \(k = p_{i-1}\) and \((p_{i-2}, p_{i-1}) \in \delta(k) \cap E_b\). Hence \(p_{i-2} = q_1\) or \(p_{i-2} = n\). If \(p_{i-2} = q_1\), then \((p_{i-2}, p_{i+1}) \in E_b\) by transitivity and \(q := [(p_1, P_1), ..., (p_{i-2}, P_{i-2}), (p_{i+1}, P_{i+1}), ..., (p_m, P_m)] \in B^L \cup Z^L\) satisfies (20): Compared with \(p\), \(q\) contains equal or less...
edges in $E_b \setminus E_{\mathcal{E}} \cup E_{\mathcal{E}'}$ (no more $(p_i, p_{i+1}) \in E_b \setminus E_{\mathcal{E}}$, but possibly $(p_{i-2}, p_{i+1}) \in E_b \setminus E_{\mathcal{E}}$) and totally less edges. If $p_{i-2} = n$, then $n \neq p_i+3$, $(p_{i+2}, p_{i+3}) \in E_{\mathcal{E}}$ and $(p_{i-2} = n, p_{i+3}) \in E_b$. Hence $q := [(p_1, P_1), ..., (p_{i-2}, P_{i-2}), (p_{i+3}, P_{i+3}), ..., (p_m, P_m)] \in \mathcal{B} \cup \mathcal{Z}$ satisfies (20).

Let now $(p_i, p_{i+1})$ be the last edge of $p$ in $E_b \setminus E_{\mathcal{E}} \cup E_{\mathcal{E}'}$, then $n = p_{i+3}$ and $(p_{i+3}, P_{i+4}) \in \delta(n) \cap E_{\mathcal{E}'}$. Hence $p_{i+4} = k$, $k \neq p_{i-1}$, $(p_{i-1}, p_i) \in E_{\mathcal{E}}$ and $(p_{i-1}, k = p_{i+4}) \in E_{\mathcal{E}'}$. Then $q := [(p_1, P_1), ..., (p_{i-2}, P_{i-2}), (p_{i+4}, P_{i+4}), ..., (p_m, P_m)] \in \mathcal{B} \cup \mathcal{Z}$ satisfies (20).

**Case 3.3:** $(p_{i+1}, p_{i+2}) \in E_{\mathcal{E}'}$.

Let $(s_0, ..., s_a)$ be the sequence of (21) for $(v, w) = (p_{i+1}, p_{i+2})$ and define $n := s_1 \in V$.

If $n = k$ or $(k, n) \in E_{\mathcal{E}'}$, then $(p_{i-1}, p_{i+2}) \in E_{\mathcal{E}}$ by transitivity and $q := [(p_1, P_1), ..., (p_{i-1}, P_{i-1}), (p_{i+2}, P_{i+2}), ..., (p_m, P_m)] \in \mathcal{B} \cup \mathcal{Z}$ satisfies (20): It contains possibly $(p_{i-1}, p_{i+2}) \in E_{\mathcal{E}} \setminus E_{\mathcal{E}'}$, however no more $(p_i, p_{i+1}) \in E_{\mathcal{E}} \setminus E_{\mathcal{E}'}$.

If $n \neq k$, then by lemma 7.10 (c.i), there exists $k \neq p_{i+1} \in J$ or $(k, n) \in E_{\mathcal{E}'}$. Both cases have already been considered, $(k, n) \in E_{\mathcal{E}'}$ just before and $k \in J$ in case 3.1.

**Case 4:** $p_i, p_{i+1} \notin V$ (implying $3 \leq i < m-1$).

Let $(q_0, ..., q_d)$ be the sequence of (21) for $(v, w) = (p_i, p_{i+1})$ and let $p_i := kq_i$, $p_{i+1} := q_{i+1}$. If $p_{i-1} \neq k$, then $(p_{i-1}, p_i) \in E_{\mathcal{E}'}$ and $(p_{i-1}, k) \in E_{\mathcal{E}}$; if $p_{i+2} \neq n$, then $(p_{i+1}, p_{i+2}) \in E_{\mathcal{E}}$ and $(n, p_{i+2}) \in E_{\mathcal{E}}$. Define $q := [(p_1, P_1), ..., (p_{i-1}, P_{i-1}), (p_{i+2}, P_{i+2}), ..., (p_m, P_m)]$. Consider first $k = n$, then both $k = p_{i-1}$ and $n = p_{i+2}$ is not possible by (19) and $(p_{i-1}, p_{i+2}) \in E_{\mathcal{E}}$. Hence $q \in \mathcal{B} \cup \mathcal{Z}$ and it satisfies (20): Possibly $q$ contains $(p_{i-1}, p_{i+2}) \in E_{\mathcal{E}} \setminus E_{\mathcal{E}'}$, however no more $(p_i, p_{i+1}) \in E_{\mathcal{E}} \setminus E_{\mathcal{E}'}$ and $((p_{i-1}, p_i), (p_{i+1}, p_{i+2}))$, one of which is in $E_{\mathcal{E}} \setminus E_{\mathcal{E}'}$.

Let now $k \neq n$, then by lemma 7.10 case d, $(k, n) \in E_{\mathcal{E}'}$ and by transitivity $(p_{i-1}, p_{i+2}) \in E_{\mathcal{E}}$ or $(p_{i-1}, p_{i+2}) \in E_{\mathcal{E}'}$, if $p_{i-1} = k$ and $p_{i+2} = n$. In both cases $q \in \mathcal{B} \cup \mathcal{Z}$ and it satisfies (20): In the first case, $q$ contains possibly $(p_{i-1}, p_{i+2}) \in E_{\mathcal{E}} \setminus E_{\mathcal{E}'}$, however no more $(p_i, p_{i+1}) \in E_{\mathcal{E}} \setminus E_{\mathcal{E}'}$ and $((p_{i-1}, p_i), (p_{i+1}, p_{i+2}))$, one of which is in $E_{\mathcal{E}} \setminus E_{\mathcal{E}'}$. In the second case, $q$ contains no more $(p_i, p_{i+1}) \in E_{\mathcal{E}} \setminus E_{\mathcal{E}'}$. \(\square\)
One goal of our work was to consider families $\mathcal{A}$ of subsets of a ground set $V$ which are described by $(A,\overline{A})$-perfect pairs $((A,0), (\overline{A}, \overline{R}))$, i.e. $A$ is the incidence matrix of $\mathcal{A}$ and $\overline{A}$, $\overline{R}$ give the polyhedral description of $\mathcal{A}$ in the following sense:

$$\text{CONV}(A) = \{ x \in \mathbb{R}^V_+ \mid Ax \leq 1, \overline{R}x \leq 0 \}.$$

In this chapter, we consider families of combinatorial objects $\mathcal{A}$ whose incidence matrices $A$ are lattice matrices. By corollary 7.2, such families are described by $(A,\overline{A})$-perfect pairs. The main goal of this chapter is to find the polyhedral description of such families, that is, the matrices $(\overline{A}, \overline{R})$ such that $((A,0), (\overline{A}, \overline{R}))$ describes $\mathcal{A}$.

One example of such a family $\mathcal{A}$ has been treated in chapter 5, namely families of intersections $L$, and we shall denote by $\mathcal{A}^L$ lattice matrices arising from intersections. Two further examples are considered here: the dicuts $D$ of a directed graph and the convex sets of bounded length in a poset. The latter sets generalize antichains as well as convex sets of a poset.

In chapter 7, we saw that intersections can be viewed as prototypes for generating lattice matrices, since any lattice matrix $A$ can be obtained from some matrix $A^L$ by positive deletion (theorem 7.3). We shall see that dicuts constitute another prototype for lattice matrices, in some sense polar to intersections: Any lattice matrix can be obtained from some incidence matrix of dicuts by contraction.
In order to find polyhedral description of families \( \mathcal{A} \) described by lattice matrices, two different approaches, depending on the prototype chosen (intersections or dicuts), are therefore at hand. We choose the approach based on families of intersections, since their polyhedral description has already been established in chapter 5. After having described \( \mathcal{A} \), we also derive an explicit description for our examples, dicuts and convex sets of bounded length.

We conclude this chapter by considering the alternative approach (via dicuts) which constitutes a nice application of the polarity relation studied in chapter 4: Whereas the approach via intersections uses the polarity between the operations positive deletion and contraction, the approach via dicuts relies on polarity between contraction and negative deletion.

Observe that the approach via intersections mentioned above was also crucial in the proof of the polarity between subclasses of lattice polyhedra and switching paths polyhedra (theorem 7.1). It appears that the approach via dicuts yields an alternate proof of it.

(Remark: most of the content of this chapter can be found in Cochand, Gaillard, Gröflin [1990].)

### 8.1 Examples and prototypes

Any lattice matrix \( A \in \{0,1\}^{F \times V} \) can be interpreted as the *incidence matrix* of a family of combinatorial objects, namely the family of subsets of \( V \) whose incidence vectors are the rows of \( A \). We give now three examples of such families.

#### Intersections of ring families

Let \( \mathcal{C} \) and \( \mathcal{D} \) be two proper ring families on \( V \) and consider the family of intersections

\[
(1) \quad L := \{ L \subseteq V \mid L = C \cap D, C \in \mathcal{C}, D \in \mathcal{D} \text{ and } C \cup D = V \}.
\]

The following result has already been established in the proof of theorem 5.4 (iii):

**Proposition 8.1**

Let \( F := \{(C,D) \mid C \in \mathcal{C}, D \in \mathcal{D} \text{ and } C \cup D = V \} \). For all \( v \in V \), define \( f_v : F \rightarrow \{0,1\} \) by \( f_v(C,D) := 1 \) if \( v \in C \cap D \) and \( f_v(C,D) := 0 \) otherwise, for all \( (C,D) \in F \). Then \( A \in \{0,1\}^{F \times V} \) with columns \( f_v, \forall v \in V \), is a lattice matrix and, up to duplicate rows, is the incidence matrix of the intersections of \( \mathcal{C} \) and \( \mathcal{D} \).

#### Convex sets of bounded length in a poset

This example defines subsets of a poset which generalize convex sets as well as antichains.
Let $V$ be a finite poset. Recall that $S \subseteq V$ is convex if $u, v \in S$, $v \in V$ and $u < v < w$ implies $v \in S$. Let further be given a length vector $c \in \mathbb{R}_+^V$ and a bound $K \in \mathbb{R}$. For any set $S \subseteq V$, define the length $c(S)$ of $S$ by $c(S) := \max \{ \sum \{ c(v) \mid v \in B \} \mid B \subseteq S, B$ is a chain of $V \}$. In other words, the length of a chain is defined as usual by the sum of the lengths of its elements, and that of a set $S$ by the length of a longest chain contained in $S$. Consider the family of all convex sets of length less than or equal to $K$. Clearly, if $K$ is large enough, it consists of all convex sets of $V$. If $c$ is chosen to have all its components 1 and $K = 1$, it is the family of antichains of $V$.

Let a set $C \subseteq V$ be called a lower ideal if $u \in C$, $v \in V$ and $v < u$ implies $v \in C$, and a set $D \subseteq V$ be called an upper ideal if $u \in D$, $v \in V$ and $v \geq u$ implies $v \in D$.

**Proposition 8.2**

Let $C$ and $D$ be the families of lower and upper ideals of $V$, and let $\mathcal{F} := \{ (C,D) \mid C \in C, D \in D, C \cup D = V \text{ and } c(C \cap D) \leq K \}$. Define the $f_v$'s and $A \in \{0,1\}^{f \times V}$ as in proposition 8.1. Then $A$ is a lattice matrix and, up to duplicate rows, is the incidence matrix of the family of convex sets of length $\leq K$.

**Proof:**

Clearly, $C$ and $D$ are proper ring families on $V$, hence $\mathcal{F} := \{ (C,D) \mid C \in C, D \in D \text{ and } C \cup D = V \}$ is a distributive lattice with ordering, operations $\wedge$ and $\vee$ defined as for the intersections $(C,D) \leq (C',D')$ if $C \subseteq C'$ and $D \supseteq D'$, $(C,D) \wedge (C',D') := (C \cap C', D \cup D')$, $(C,D) \vee (C',D') := (C \cup C', D \cap D')$. Moreover the functions $f_v$ of proposition 8.1 have the required properties on $\mathcal{F}$.

We show that $\mathcal{F} := \{ (C,D) \in \mathcal{F} \mid c(C \cap D) \leq K \}$ is a sublattice of $\mathcal{F}$. Consider $(C,D)$ and $(C',D') \in \mathcal{F}$. Suppose that for $(C,D) \wedge (C',D') \equiv (C \cap C', D \cup D')$, $c((C \cap C') \cap (D \cup D')) > K$, i.e. $(C \cap C') \cap (D \cup D') = (C \cap C') \cup (C \cap C') \cap (D \cup D')$ contains a chain $B := \{ v_1, v_2, ..., v_q \}$ with $v_1 < v_2 < ... < v_q$ and $c(B) > K$. Without loss of generality, $v_1 \in (C \cap C') \cap D$, and therefore by definition of $D$, $v_i \in D$, $i = 1, ..., q$. Hence $B \subseteq C \cap C' \cap D \subseteq C \cap D$, contradicting $(C,D) \in \mathcal{F}$. Hence $(C,D) \wedge (C',D') \in \mathcal{F}$. Analogously, one shows $(C,D) \vee (C',D') \in \mathcal{F}$. The $f_v$'s are consecutive and modular on $\mathcal{F}$, and since the minimum $(\emptyset, V)$ and maximum $(V, \emptyset)$ of $\mathcal{F}$ are also in $\mathcal{F}$, the $f_v$'s are zero-valued on the minimum and maximum of $\mathcal{F}$. Hence $A \in \{0,1\}^{f \times V}$ with columns $f_v$ restricted to $\mathcal{F}$ is a lattice matrix.

Finally, any convex set can be represented as $C \cap D$ for some $C \in C$, $D \in D$ with $C \cup D = V$ (Gröfelin [1984]), and clearly, any intersection $C \cap D$, $(C,D) \in \mathcal{F}$ is a convex set. Therefore, $A$ is the incidence matrix of the family of all convex sets of length bounded by $K$ (up to duplicate rows).
Dicuts

Let \( G = (V, E) \) be a directed graph. Recall that a dicut is an edge set \( \delta(\mathcal{S}) \) such that \( \emptyset \subseteq S \subseteq V \) and \( \delta(S) = \emptyset \). Let us allow also \( S = \emptyset \) or \( V \). Observe that no edge of a circuit of \( G \) can be in any dicut. We can contract all circuits of \( G \) without affecting the family of dicuts and assume therefore that \( G \) is acyclic.

**Proposition 8.3**
The incidence matrix of the family of all dicuts of \( G \) is a lattice matrix.

**Proof:**
The proposition follows from more general results of Schrijver [1984]: let \( \mathcal{F} \) be the lattice of all subsets \( S \) of \( V \) with \( \delta(S) = \emptyset \) (with meet \( \cap \) and join \( \cup \)) and \( A \in \{0,1\}^{\mathcal{F} \times E} \) be the matrix whose rows are the incidence vectors of \( \delta(S) \), \( S \in \mathcal{F} \). The columns of \( A \) are consecutive, modular and zero-valued on the minimum and maximum of the lattice. Hence \( A \) is a lattice matrix and the incidence matrix of the dicuts of \( G \).

**Prototypes**

Some preliminary observations regarding the three examples above can be made. First, although similar arguments are used in the first and second examples, they differ in the following sense. On one hand, the family of convex sets of bounded length in a poset is not in general a family of intersections, since the latter always includes as a member the whole set \( V \). On the other hand, convex sets involve a pair of ring families which are related to each other in a particular and simple way, and therefore are not representative of the general intersection case.

Regarding the first and third examples, it can be shown that intersections of ring families are equivalent to a family of dicuts - however not all dicuts - in a certain graph (see proof of theorem 8.5).

This paragraph settles the question of how these examples as well as lattice matrices in general are related: in fact, it shall be shown that intersections of ring families as well as dicuts can be viewed as prototypes for generating any lattice matrix.

The role of intersections in this context has already been established in theorem 7.3 and we recall this result here to better underline the differences between the two prototypes, intersections and dicuts.
Theorem 8.4
For any lattice matrix $A$, there exists a family of intersections $L$ on $V \cup J$ with corresponding lattice matrix $A^L$ such that $A$ is obtained from $A^L$ by positive deletion of $J$.

Theorem 8.5
For any lattice matrix $A$, there exists a directed graph $G = (V,E \cup J)$ such that $A$ is obtained from the incidence matrix $A^D$ of the dicuts of $G$ by contraction of $J$.

Proof:
The proof immediately follows from the equivalence of certain lattice polyhedra with the submodular flow model (Edmonds, Giles [1977]) which was shown by Schrijver [1984]. As a consequence, any lattice matrix $A \in \{0,1\}^{F \times E}$ is the incidence matrix of a family of dicuts $\{\delta(S) \mid S \in F^*\}$ in a graph $G = (V,E)$, where $F^*$ is a proper ring family on $V$ isomorphic to $F$. Let $G_F = (V,J)$ be the graph representing $F$ (or $F^*$), i.e. $F \in F^*$ iff $\delta(F) = \emptyset$ (proposition 2.4), and form the graph $H = (V,E \cup J)$. $A$ is obtained from the incidence matrix of all dicuts of $H$ by contracting $J$. (Notice that in section 5.4, paragraph "relation to submodular flows", graph $H$ has explicitly been constructed for $A$ being the incidence matrix of intersections.)

The relation between incidence matrices of a prototype and lattice matrices gives hope to have a "polar" relation between polyhedral descriptions of a prototype and polyhedral descriptions of lattice matrices (see figure 8.1). In fact, such relations exist (chapter 4), however they are only useful if the description of the prototype is known. For this reason, we will use the intersections for our approach.

![Diagram](image-url)

Figure 8.1
Let $A \in \{0, 1\}^{F \times V}$ be a lattice matrix. We now derive a polyhedral description of $\text{CONV}(A)$. Let $L$ be the family of intersections on $V \cup J$ with incidence matrix $A^L \in \{0, 1\}^{F \times V \cup J}$ such that $A$ is obtained from $A^L$ by positive deletion of $J$ (theorem 8.4). Recall that by theorem 5.4, the polyhedral description of the intersection is given by

\[ Q := \text{CONV}(A^L) = \{ x \in \mathbb{R}^{V \cup J} \mid A^L x \leq 1, \, R^L x \leq 0 \}, \]

where $A^L$, respectively $R^L$, is the matrix whose rows are the alternating vectors of all $b$-$r$-paths, respectively $b$-$r$-circuits in the graph representing $L$. Moreover, the system is homogeneous-tdi.

Consider now the polar polyhedron of $Q$ (classical polarity), namely

\[ P := Q^* = \{ x \in \mathbb{R}^{V \cup J} \mid A^L x \leq 1 \}, \text{ and define} \]

\[ P^V := \{ x \in \mathbb{R}^V \mid A x \leq 1 \}. \]

By theorem 4.2, we obtain the following results:

\begin{enumerate}
\item $P^V$ is the projection of $P$ onto $\{ x \in \mathbb{R}^{V \cup J} \mid x^J = 0 \}$ (restricted to $\mathbb{R}^V$),
\item $(P^V)^* = \text{CONV}(A) =: Q^V$ is the cut of $Q$ with $\{ x \in \mathbb{R}^{V \cup J} \mid x^J = 0 \}$ (restricted to $\mathbb{R}^V$),
\item Let $\breve{A}$ and $\breve{R}$ be the matrices obtained by contraction of coordinates $J$ from $A^L$ and $R^L$, respectively, then $Q^V = \{ x \in \mathbb{R}^{V \setminus J} \mid \breve{A} x \leq 1, \, \breve{R} x \leq 0 \}$. Moreover, the system is homogeneous-tdi.
\end{enumerate}

As a result, an approach for describing $\text{CONV}(A)$ for any lattice matrix $A$ is the following: (i) construct as in the proof of theorem 8.4 (= theorem 7.3) intersections $L$ on $V \cup J$ such that the incidence matrix of $L$ after positive deletion of $J$ is $A$; (ii) take the alternating vectors of $b$-$r$-paths and circuits of the graph representing $L$, form the corresponding matrices $\breve{A^L}$ and $\breve{R^L}$ and contract $J$ in $\breve{A^L}$ and $\breve{R^L}$, obtaining $\breve{A}$ and $\breve{R}$. Then $\text{CONV}(A) = \{ x \in \mathbb{R}^{V \setminus J} \mid \breve{A} x \leq 1, \, \breve{R} x \leq 0 \}$ and moreover, the describing system is homogeneous-tdi. In the remainder of this section, by carefully examining this approach, we shall be able to derive $\breve{A}$ and $\breve{R}$ directly.

Let $A \in \{0, 1\}^{F \times V}$ be a lattice matrix and consider $F$ as a proper ring family on a ground set $J$. Recall that there were several families of intersections satisfying the condition of theorem 8.4. For our purpose, we choose $L$ as the intersections of the ring families $\mathcal{C} := \{ C_S := \{ v \in V \mid g_1(S) = 1 \} \cup S \mid S \in F \}$ and $\mathcal{D} := \{ D_S := \{ v \in V \mid g_2(S) = 1 \} \cup (NS) \mid S \in F \}$ on $V \cup J$. Let $H := (V \cup J, E_b \cup E_r)$ be the graph representing $L$, $\breve{A^L}$ and $\breve{R^L}$ the corresponding matrices of alternating vectors of $b$-$r$-paths and $b$-$r$-circuits.
Let $P$ be any path in $H$. Call a subpath $P'$ of $P$ with at least three nodes and having its extremities in $V$ and its intermediary nodes in $J$ a detour of $P$ (in $J$). Call further a detour of $P$ with exactly three nodes an elbow. Not all rows of $\overline{A}^L$ and $\overline{R}^L$ are necessary to describe $\text{CONV}(A)$; the following lemma allows to limit ourselves to a certain type of $b$-$r$-paths and $b$-$r$-circuits in $H$.

**Lemma 8.6**

Let $P$ be a $b$-$r$-path of $H$ with $V(P) \cap V \neq \emptyset$. Then there exists a $b$-$r$-path $P'$ of $H$, possibly $P' = \emptyset$, such that

(i) $V(P') \subseteq V(P)$ and the extremities of $P'$ are in $V$,

(ii) the detours of $P'$, if any, are elbows,

(iii) $\overline{A}^L_{P,v} \geq \overline{A}^L_{P,v}$ for all $v \in V$.

**Proof:**

If $V(P) \subseteq V$, let $P' = P$. So assume that $P$ has nodes in both $V$ and $J$. If $P$ starts in $J$, delete the beginning of $P$ up to the first node of $P$ in $V$ that is the tail of a blue edge of $P$. Analogously, if the end node of $P$ is in $J$, delete the nodes following the last node of $P$ in $V$ that is the head of a red edge of $P$. Clearly, if $P'$ denotes $P$ after these deletions, $P'$ is a $b$-$r$-path satisfying (i) and (iii). Suppose $P'$ has a detour with nodes $(v_i, \ldots, v_k)$, $k > 3$, and edges $(e_i, \ldots, e_{k-i})$. $H$ is blue-transitive and red-transitive. Moreover, for any edge $e \in \gamma(J)$:

$$e \in E_b \iff (\forall S \in \mathcal{F}: t(e) \in S \Rightarrow h(e) \in S) \iff e \in E_r.$$ 

Hence blue and red edges in $\gamma(J)$ are parallel. Therefore, if $e_1, e_{k-1} \in E_b(E_r)$, there exists $e \in E_b(E_r)$ with $t(e) = v_1$, $h(e) = v_k$. If $e_1 \in E_b(E_r)$ and $e_{k-1} \in E_r(E_b)$, there is an elbow with edges $e_1, e$ where $e \in E_r(E_b)$ and $t(e) = v_2$, $h(e) = v_k$, which we call a $b$-$r$-elbow ($r$-$b$-elbow). In any case, a detour can be bridged either directly or through an elbow, while maintaining a $b$-$r$-path, proving (ii).

Similarly, if $Q$ is a $b$-$r$-circuit of $H$ with $V(Q) \cap V \neq \emptyset$, there is a $b$-$r$-circuit $Q'$ of $H$ such that $V(Q') \subseteq V(Q)$, the detours of $Q'$ are elbows and $\overline{R}^L_{Q,v} = \overline{R}^L_{Q,v}$ for all $v \in V$. As a result, we can define a graph with node set restricted to $V$ and edges of four types: $H' := (V, E_b \cup E_r \cup E_{br} \cup E_{rb})$ where $E_b := E_b \cap \overline{\gamma}_H(V)$, $E_r := E_r \cap \overline{\gamma}_H(V)$, $e \in E_{br}(E_{rb})$ if there is a $b$-$r$-elbow ($r$-$b$-elbow) from $t(e)$ to $h(e)$ in $H$. We can think of a $b$-$r$-edge $e \in E_{br}$ to be colored blue from its tail to its middle, and red from thereon to its head, and, similarly, of a $r$-$b$-edge $e \in E_{rb}$ to be colored red and then blue. The graph $H'$ is called the graph representing the lattice matrix $A$. (We shall also give in coming lemma 8.8 an alternative and more direct construction of $H'$).
The notions of b-r-path, b-r-circuit and vectors of b-r-paths and b-r-circuits extend now as follows. A b-r-path of $H'$ is an (elementary) path starting with blue and ending with red and in which colors alternate at its intermediary nodes. A b-r-circuit is a closed b-r-path. The vector $x \in \{0, \pm 1\}^V$ is the vector of the b-r-path $P$ (of the b-r-circuit $Q$) if $x_v = 1$ if $P$ ($Q$) leaves $v$ with color blue or enters $v$ red or $V(P) = v$, $x_v = -1$ if $P$ ($Q$) leaves $v$ red, and $x_v = 0$ for $v \notin V(P)$. Summarizing, we have shown

**Theorem 8.7**

Let $A \in \{0, 1\}^{F \times V}$ be a lattice matrix, $H := (V, E_b \cup E_r \cup E_{br} \cup E_{rb})$ be the graph representing $A$, and $\bar{A}$, respectively $\bar{R}$, be the matrix whose rows are the vectors of all b-r-paths, respectively of all b-r-circuits of $H$. Then $\text{CONV}(A) = \{x \in \mathbb{R}_+^V \mid \bar{A}x \leq 1, \bar{R}x \leq 0\}$ and the system $x \geq 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$ is homogeneous-tdi.

A direct characterization of the graph representing $A$ is now derived. Given a lattice matrix $A \in \{0, 1\}^{F \times V}$, let $f_v$, $v \in V$, be the columns of $A$ and define for any $v \in V$:

$$
\begin{align*}
S^1_v &:= \{S \in F \mid f_v(S) = 1\}, \\
S^2_v &:= \{S \in F \mid f_v(S) = 1\}, \\
S^3_v &:= \{S \in F \mid S \subseteq S^2_v\}.
\end{align*}
$$

(These three characteristic elements of a column $f_v$ also come into play in Schrijver [1984].)

**Lemma 8.8**

Let $H := (V, E_b \cup E_r \cup E_{br} \cup E_{rb})$ be the graph representing the lattice matrix $A$. Then for any edge $e$ of $H$ with tail $v$ and head $w$, the following holds:

$$
\begin{align*}
e \in E_b & \iff S^1_w \leq S^1_v; \\
e \in E_r & \iff S^3_v \leq S^3_w; \\
e \in E_{br} & \iff S^1_w \leq S^1_v; \\
e \in E_{rb} & \iff S^3_v \leq S^3_w.
\end{align*}
$$

**Proof:**

Consider $F$ as a proper ring family over a ground set $J$. Notice that for $v \in V$ the following equivalences are valid:

$$
\begin{align*}
g^1_v(S) = 1 & \iff S \subseteq S^1_v, \\
g^2_v(S) = 1 & \iff S \subseteq S^2_v, \\
g^3_v(S) = 0 & \iff S \supseteq S^2_v.
\end{align*}
$$

The last equivalence is true because $\{S \in F \mid S \supseteq S^2_v\}$ is non-empty ($J \subseteq S^2_v$) and has a unique minimum: For suppose it contains two minimum elements $R$ and $T$. Then $S^2_v \subseteq S^2_v \cup R$, $S^2_v \subseteq S^2_v \cup T$, hence $f_v(S^2_v \cup R) = f_v(S^2_v \cup T) = 0$ and by modularity and distributivity $0 = f_v(S^2_v \cup R) + f_v(S^2_v \cup T) = f_v(S^2_v \cup (R \cup T))$. But $R \cup T \subseteq S^2_v$ by minimality of $R$ and $T$, therefore $0 = f_v(S^2_v \cup (R \cup T)) + f_v(S^2_v)$, contradicting $f_v(S^2_v) = 1$. Hence $S^1_v$ is the (unique) minimum of $\{S \in F \mid S \supseteq S^2_v\}$. 
Denote by $S_k$, $k \in I$, the minimum of $\{S \in \mathcal{F} \mid k \in S\}$. Then for all $v \in V$ and $k \in S_k$, $S_k \neq \emptyset$, $S_k = S^3$, because $S_k \subseteq S^3$ and $S^3 \subseteq S_k$ by $S_k \subseteq S^2$.

Let $e \in E_b \cup E_r \cup E_{br} \cup E_{rb}$ with $t(e) = v$ and $h(e) = w$. Then by construction of $H$:

$$(e \in E_b) \iff (\forall S \in \mathcal{F}: g^1_v(S) = 1 \Rightarrow g^2_w(S) = 1) \iff (\forall S \in \mathcal{F}: S^1_v \subseteq S \Rightarrow S^1_w \subseteq S)$$

$$(e \in E_{br}) \iff (\exists k \in I: [\forall S \in \mathcal{F}: (S^3_v \subseteq S \Rightarrow k \in S) \text{ and } (k \in S \Rightarrow g^2_v(S) = 0)])$$

$$(e \in E_{br}) \iff (\exists k \in I: [\forall S \in \mathcal{F}: (S^1_v \subseteq S \Rightarrow k \in S) \text{ and } (k \in S \Rightarrow S^3_w \subseteq S)])$$

$$(\exists k \in I: S_k \subseteq S^1_v \text{ and } S^3_v \subseteq S_k)$$

$$(S^3_w \subseteq S^1_v) \quad \text{(for }"\leq"\text{, take } k \in S^3_w \setminus S^2_v).$$

The cases $e \in E_r$ and $e \in E_{rb}$ are analogous to the cases $e \in E_b$ and $e \in E_{br}$.

**8.3 Applications**

The polyhedral description of a general lattice matrix given above is now applied to the remaining two examples of section 8.1.

**Convex sets of bounded length**

The lattice matrix of interest (proposition 8.2) is $A \in \{0,1\}^{F \times V}$ where $F := \{(C,D) \mid C \subseteq C, D \subseteq D, C \cup D = V \text{ and } c(C \cap D) \leq K\}$, the columns $f_v$ of $A$ are given by $f_v(C,D) := 1$ if $v \in C \cap D$ and $f_v(C,D) := 0$ otherwise, for all $(C,D) \in F$, and $C$ $(D)$ is the family of lower (upper) ideals of poset $V$. For any $v \in V$, define

$$C_v := \cap \{C \subseteq C \mid v \in C\}, \quad D_v := \cap \{D \subseteq D \mid v \in D\},$$

$$K_v := \{w \in V \mid \exists \text{ chain } B \text{ with extremities } v \text{ and } w, w \leq v, \text{ and } c(B) > K\},$$

$$L_v := \{w \in V \mid \exists \text{ chain } B \text{ with extremities } v \text{ and } w, w \geq v, \text{ and } c(B) > K\}.$$

We show that the three characteristic elements of column $f_v$ needed for the construction of the graph $H$ representing $A$ (see lemma 8.8) are:

$$S^1_v = (C_v,K_v),$$

$$S^2_v = (C_v,D_v),$$

$$S^3_v = (C_v,L_v).$$

First, $(C_v,K_v) \in F \cap C_v \subseteq C \subseteq F$ for any $C \subseteq C$; $K_v \subseteq C_v$, hence $C_v \cup K_v = V$. For any chain $B \subseteq C_v \cap K_v$, $B \cup V \subseteq C_v \cap K_v$ is also a chain of length $c(B \cup v) \geq c(B)$, and by definition of $K_v$, $c(B \cup v) \leq K$, hence $c(C_v \cap K_v) \leq K$. Next, for any $(C,D) \in F$ with $v \in C \cap D$, $C_v \subseteq C$; further, $C \cap D \cap K_v = \emptyset$, and, since $K_v \subseteq C_v \subseteq C$, $D \cap K_v = \emptyset$, i.e. $D \subseteq K_v$. Thus $(C_v,K_v)$ is the minimum of the sublattice $\{(C,D) \in F \mid v \in C \cap D\}$, proving (11). We omit the proof of
which is similar and proceed to (13), i.e. we show that \((C_v, \overline{C}_v)\) is the minimum of \(F^- := \{(C,D) \in F | (C,D) \notin (L_v, D_v)\} = \{(C,D) \in F | C \subseteq L_v \text{ or } D \subseteq L_v\}.\) First, \((C_v, \overline{C}_v) \in F^-\) since \(v \in D_v \cap \overline{C}_v\) and hence \(D_v \subseteq \overline{C}_v\). Next, take any \((C,D) \in F^-\). If \(C \subseteq L_v\), there exists \(w \in C \cap L_v\) and a chain \(B\) between \(w\) and \(v\), \(w \geq v\), of length \(c(B) > K\). Then \(v \in C\), hence \(C_v \subseteq C\). Also \(v \in D\), otherwise \(B \subseteq C \cap D\). But \(v \in D\) implies \(D \subseteq \overline{C}_v\). On the other hand, if \(D \subseteq \overline{C}_v\), then \(v \in \overline{D}\), hence \(D \subseteq \overline{C}_v\), and since \(C \cup \overline{D} = V\), \(v \in C\) and \(C_v \subseteq C\), concluding the proof of (13).

Using lemma 8.8 and (11) to (13), the graph \(H\) representing \(A\) is then characterized as follows: for any edge \(e\) with tail \(v\) and head \(w\): \(e \in E_b\) iff \((C_w \subseteq C_v\) and \(K_v \subseteq K_w)\), i.e.

\[
\begin{align*}
e \in E_b & \iff w \leq v, \\
e \in E_r & \iff w \leq v, \\
e \in E_{br} & \iff w \leq v \text{ and there exists a chain } B \text{ between } v \text{ and } w \text{ of length } c(B) > K, \\
e \in E_{rb} & \iff w \leq v.
\end{align*}
\]

As a result, if \(G = (V, E)\) is the comparability graph of \(V\) (\(ee \in E\) iff \(h(e) \leq t(e)\)), then \(H\) is obtained by making three copies of \(E\), yielding \(E_b\), \(E_r\) and \(E_{rb}\), and inserting an edge \(e \in E_{br}\) from \(v\) to \(w\) for all \(v, w \in V\), \(w \leq v\), such that there is a chain of length greater than \(K\) between \(v\) and \(w\).

Thanks to this simple structure of \(H\), we can partition the edges of the comparability graph \(G: E = E_k \cup E_{ek}\) where \((v,w) \in E_k\) iff there is a chain of length greater than \(K\) between \(v\) and \(w\), and restrict ourselves to those vectors \(\overline{A}_P \subseteq \{0, \pm 1\}^V\) associated to paths \(P = (v_1, \ldots, v_n)\), \(n \geq 1\), in \(G\) of the following type:

\[
\begin{align*}
\overline{A}_{Pv} & = 1, \ v \in \{v_1, v_n\}, \\
\overline{A}_{Pv} & = \overline{A}_{Pw} = 1 \text{ if } n > 1, \ 1 \leq i < n \text{ and } (v, w) = (v_i, v_{i+1}) \in E_k, \\
\overline{A}_{Pv} & = - \overline{A}_{Pw} \text{ if } n > 1, \ 1 \leq i < n \text{ and } (v, w) = (v_i, v_{i+1}) \in E_k, \\
\overline{A}_{Pv} & = 0 \text{ for } v \notin V(P).
\end{align*}
\]

Let \(\overline{A}\) be the matrix whose rows are all vectors of type (15).

**Proposition 8.9**

The polyhedral description of the family of all convex sets of length not greater than \(K\) is given by \(\{x \in \mathbb{R}_+^V \mid \overline{A}x \leq 1\}\) and the system \(x \geq 0, \overline{A}x \leq 1\) is homogeneous-tdi.

**Proof:**

Notice that \(H\) has no circuits. We assert that for all vectors \(x\) of a b-r-path in \(H\) there exists a row \(\overline{A}_P\) of \(\overline{A}\) with \(\overline{A}_P \geq x\). Assume that this assertion is false and let \(x\) be a componentwise maximal vector of a b-r-path \(P\) for which it is false. Let \(V(P) = (v_1, \ldots, v_{n+1})\) and \(E(P) = (e_1, \ldots, e_n)\), then \(x_{v_1} = x_{v_{n+1}} = 1\). Since \(x\) and \(P\) do not satisfy (15), one of the following cases must occur:
(i) \[ \exists 1 \leq i \leq n, x_{v_i} = 1, \quad x_{v_{i+1}} = -1, \quad (v_i, v_{i+1}) \in E_k, \]

(ii) \[ \exists 1 \leq i \leq n, x_{v_i} = -1, \quad x_{v_{i+1}} = 1, \quad (v_i, v_{i+1}) \in E_k, \]

(iii) \[ \exists 1 \leq i \leq n, x_{v_i} = -1, \quad x_{v_{i+1}} = -1. \]

In case (i), \( i+1 \leq n \) and there exists an index \( i+1 < k \leq n+1 \) with \( x_{v_k} = 1 \). Choose \( k \) minimal for this property. Then \( e' := (v_i, v_k) \in E_{br} \), since \( (v_i, v_{i+1}) \in E_k \). The vector \( x' \) of the \( b-r \)-path \( P' \) with \( E(P') = (e_1, \ldots, e_{i-1}, e', e_k, \ldots, e_n) \) satisfies \( x' \geq x, x' \neq x \); hence there exists a vector \( \bar{A}P' \) of a path \( P'' \) satisfying (15) with \( \bar{A}P'' \geq x' \geq x \), a contradiction.

In the case (ii) and (iii), \( 1 < i \) and there exists an index \( 1 < k < i \) with \( x_{v_k} = 1 \). Choose \( k \) maximal for this property. Then \( e' := (v_k, v_{i+1}) \in E_{br} \) in case (ii), \( e' \in E_b \) in case (iii) and the vector \( x' \) of the \( b-r \)-path \( P' \) with \( E(P') = (e_1, \ldots, e_{k-1}, e', e_{i+1}, \ldots, e_n) \) satisfies \( x' \geq x, x' \neq x \). Hence there exists a vector \( \bar{A}P' \) of a path \( P'' \) satisfying (15) with \( \bar{A}P'' \geq x' \geq x \), a contradiction.

Notice that in the antichain case \( (c := 1, K := 1) \), \( \bar{A} \) is - as expected - the incidence matrix of all chains. The case of unbounded convex sets \( (K \text{ large}) \) yields a result of Gröflin [1984].

**Dicuts**

Let \( G = (V, E) \) be an acyclic graph and \( F := \{ S \subseteq V \mid \delta(S) = \emptyset \} \). A column \( f_e \) of the incidence matrix \( A \in \{0, 1\}^{F \times E} \) of the dicuts of \( G \) is given by \( f_e(S) := 1 \) if \( e \in \delta(S) \) and \( f_e(S) := 0 \) otherwise.

For any \( v \in V \), let \( K_v := \{ w \in V \mid \exists \text{ a path in } G \text{ from } v \text{ to } w \} \). Clearly \( v \in S \in F \) implies \( K_v \subseteq S \). Then for any \( i \in E \):

\[
\begin{align*}
S_i^1 &= \cap \{ S \in F \mid i \in \delta(S) \} = K_{h(i)}; \\
S_i^2 &= \cup \{ S \in F \mid i \in \delta(S) \} = V \setminus \{ w \in V \mid \tau(i) \in K_w \}; \\
S_i^3 &= \cap \{ S \in F \mid S \nsubseteq S_i^2 \} = K_{t(i)}. 
\end{align*}
\]

Hence by lemma 8.8, for any edge \( e \) of \( H := (E, E_b \cup E_r \cup E_{br} \cup E_{rb}) \) with \( t(e) = i \) and \( h(e) = j \):

\[
\begin{align*}
e \in E_b & \iff S_i^1 \subseteq S_i^2 \iff h(j) \in K_{h(i)}, \\
e \in E_r & \iff S_i^2 \subseteq S_i^3 \iff t(j) \in K_{t(i)}, \\
e \in E_{br} & \iff S_i^3 \subseteq S_i^1 \iff t(j) \in K_{h(i)}, \\
e \in E_{rb} & \iff S_i^1 \subseteq S_i^3 \iff h(j) \in K_{t(i)}. 
\end{align*}
\]

Let \( \bar{A}' \) and \( \bar{R}' \) be the matrices whose rows are the vectors of all \( b-r \)-paths and \( b-r \)-circuits of \( H \), respectively. Then by theorem 8.7, \( \text{CONV}(A) = \{ x \in \mathbb{R}_q^E \mid \bar{A}'x \leq 1, \bar{R}'x \leq 0 \} \). In fact, this result has a simpler form.
In $G = (V,E)$, departing from the conventional (directed) paths and circuits, call $P$ an (elementary) path with node set $(v_1,...,v_{n+1})$ and edge set $(e_1,...,e_n)$, $n > 1$, if either $t(e_i) = v_i$ and $h(e_i) = v_{i+1}$, or $h(e_i) = v_i$ and $t(e_i) = v_{i+1}$, $1 \leq i \leq n$. Following standard terminology, $e_i$ is either forward or backward. If $v_1 = v_{n+1}$, call the path a circuit. The vector of a path $P$ (circuit $Q$) is the vector $x \in \mathbb{R}^E$ given by $x_e = 1$ if $e$ is forward in $P$, $x_e = -1$ if $e$ is backward in $P$, and $x_e := 0$ otherwise. For the remainder of this section, any path or circuit in $G$, and in $G$ only, is meant in the above sense.

Let $\bar{A}$ be the matrix whose rows are the vectors of all paths with first and last edge forward, and $\bar{R}$ the matrix whose rows are the vectors of all circuits.

**Proposition 8.10**

The polyhedral description of the family of all dicuts of $G$ is given by \{ $x \in \mathbb{R}^E_+ | \bar{A}x \leq 1$, $\bar{R}x \leq 0$ \} and the system $x \geq 0$, $\bar{A}x \leq 1$, $\bar{R}x \leq 0$ is homogeneous-tdi.

**Proof:**

Any vector $\bar{A}P$ of a path with edges $(e_1,...,e_n)$ in $G$ is the vector of a b-r-path in $H$ with node-set $(e_1,...,e_n)$: for $1 \leq i \leq n$, the four cases $(e_i,e_{i+1})$ being (forward (f), backward (b)), (b,f), (f,f), and (b,b) correspond to the cases $(e_i,e_{i+1}) \in E_b, E_r, E_{br}$ and $E_{rb}$. A similar claim holds for the vector of a circuit of $G$. Hence $\bar{A}$ and $\bar{R}$ are submatrices of $\bar{A}'$ and $\bar{R}'$.

Next, for any vector $\bar{A}P'$ of a b-r-path $P'$ of $H$, there exists a vector $\bar{A}P$ of a path in $G$ and vectors $\bar{R}_k$, $k \in K$, of circuits $Q_k$ in $G$ such that $\bar{A}P$ and $\bar{R}_k$, $k \in K$, are rows of $\bar{A}$ and $\bar{R}$, and

\[
\bar{A}P' \leq \bar{A}P + \sum (\bar{R}_k | k \in K).
\]

Namely, let $(e_1,...,e_n)$ be the nodes of $P'$ in $H$. An edge of $P'$ joining $e_i$ to $e_{i+1}$, $1 \leq i < n$, means that there is a path $P_i$ in $G$ from $t(e_i)$ or $h(e_i)$ to $t(e_{i+1})$ or $h(e_{i+1})$. $P_i$ starts in $t(e_i)$ if $\bar{A}P_{e_i} = -1$ and in $h(e_i)$ if $\bar{A}P_{e_i} = 1$; it ends in $t(e_{i+1})$ if $\bar{A}P_{e_{i+1}} = 1$ and in $h(e_{i+1})$ if $\bar{A}P_{e_{i+1}} = -1$. There exists therefore a possibly non-elementary path $P''$ in $G$ with concatenated edge set $(e_1,E(P_1),e_2, E(P_2),...,E(P_{n-1}),e_n)$. $P''$ might traverse an edge $e$ more than once, say $d_e$ times in forward direction and $\bar{d}_e$ times in backward direction. Since all edges of the $P_i$'s are forward,

\[
\bar{A}P_{e} \leq d_e - \bar{d}_e \quad \text{for all } e \in E.
\]

$P''$ can be decomposed into an elementary path $P$, a collection of elementary circuits $Q_k$, $k \in K$, and non-elementary circuits consisting of one edge traversed forward and backward. The vectors $\bar{A}P$ of $P$ and $\bar{R}_k$ of $Q_k$, $k \in K$, satisfy $\bar{A}P_e + \sum (\bar{R}_ke | k \in K) = d_e - \bar{d}_e$ for all $e \in E$, establishing (18).
Similarly, any row of $\mathbf{R}'$ can be covered by a sum of rows of $\mathbf{R}$. Hence $\{x \in \mathbb{R}^p_+ | \mathbf{A}x \leq 1, \mathbf{R}x \leq 0\} = \{x \in \mathbb{R}^p_+ | \mathbf{A}'x \leq 1, \mathbf{R}'x \leq 0\}$ and the system $x \geq 0, \mathbf{A}x \leq 1, \mathbf{R}x \leq 0$ is homogeneous-tdi.

### 8.4 Approach via dicuts

In section 8.2, we have chosen the approach via intersections for finding the polyhedral description of lattice matrices. An approach via dicuts would have been possible too. In this section we compare the two approaches.

Let $A \in \{0,1\}^{p \times V}$ be a lattice matrix. The main ideas used in the intersection case can be summarized as follows:

(i) There exists a family of intersections $L$ on $V \cup J$ with incidence matrix $A_L$ such that $A$ is obtained from $A_L$ by positive deletion of $J$ (theorem 8.4).

(ii) Let $\mathbf{A}_L, \mathbf{R}_L$ be the matrices whose rows are the vectors of b-r-paths and b-r-circuits of the graph representing $L$. Then $\text{CONV}(A_L) = \{x \in \mathbb{R}^{v \cup J}_+ | \mathbf{A}_L x \leq 1, \mathbf{R}_L x \leq 0\}$ and $x \geq 0, \mathbf{A}_L x \leq 1, \mathbf{R}_L x \leq 0$ is homogeneous-tdi (theorem 5.4).

(iii) Let $\mathbf{A}, \mathbf{R}$ be the matrices obtained from $\mathbf{A}_L, \mathbf{R}_L$ by contraction of $J$, then $\text{CONV}(A) = \{x \in \mathbb{R}^V_+ | \mathbf{A}x \leq 1, \mathbf{R}x \leq 0\}$ and $x \geq 0, \mathbf{A}x \leq 1, \mathbf{R}x \leq 0$ is homogeneous-tdi (theorem 4.2).

Let us now consider the case of dicuts. To obtain the polyhedral description of lattice matrices, three steps again are necessary, similar to the steps above. Clearly, step (ii) consists now in the polyhedral description of the dicuts. The main differences in step (i) and (iii) between intersections and dicuts are the switched role of the operations deletion and contraction. The three steps of above read then:

(i) There exists a family of dicuts $D$ in an acyclic graph $G = (V^*, V \cup J)$ with incidence matrix $A_D$ such that $A$ is obtained from $A_D$ by contraction of $J$ (theorem 8.5).

(ii) Let $\mathbf{A}_D, \mathbf{R}_D$ be the matrices whose rows are the vectors of paths and circuits (with forward and backward edges) of $G$. Then $\text{CONV}(A_D) = \{x \in \mathbb{R}^{V \cup J}_+ | \mathbf{A}_D x \leq 1, \mathbf{R}_D x \leq 0\}$ and $x \geq 0, \mathbf{A}_D x \leq 1, \mathbf{R}_D x \leq 0$ is homogeneous-tdi (proposition 8.10).

(iii) Let $\mathbf{A}, \mathbf{R}$ be the matrices obtained from $\mathbf{A}_D, \mathbf{R}_D$ by negative deletion of $J$, then $\text{CONV}(A) = \{x \in \mathbb{R}^V_+ | \mathbf{A}x \leq 1, \mathbf{R}x \leq 0\}$ and $x \geq 0, \mathbf{A}x \leq 1, \mathbf{R}x \leq 0$ is homogeneous-tdi (theorem 4.3 and proposition 4.11).

Notice that in the intersection case no property (like homogeneous-tdi) of the system $x \geq 0, \mathbf{A}_L x \leq 1, \mathbf{R}_L x \leq 0$ is used to get the polyhedral description in (iii). In contrast, the polyhe-
dral description of $A$ can only be obtained in (iii') because $x \geq 0$, $\bar{A}Dx \leq 1$, $\bar{R}Dx \leq 0$ is homogeneous.

In our approach via intersections, a graph $H$ representing the lattice matrix $A$ has been constructed (see lemma 8.8) such that vectors of b-r-paths and circuits (defined on the node set of $H$) are the rows of $\bar{A}$ and $\bar{R}$ in (iii). Similarly, for the dicut approach, we could construct another graph representing $A$, say $H^D$, such that vectors associated to the edge set of paths and circuits in $H^D$ are the rows of $\bar{A}$ and $\bar{R}$ in (iii'). Not very surprisingly, both $H$ and $H^D$ are essentially given by the elements $S^i_1$, $S^3_v$, $v \in V$, and the order-relation between $S^i_v$, $S^j_w$, $i, j \in \{1, 3\}$; $v, w \in V$.

Assume that the construction of $H$ and $H^D$ as well as the derivation of $\bar{A}$ and $\bar{R}$ from $H$ and $H^D$ are known. A natural question arises if one or the other approach is better for finding the polyhedral description of a given lattice matrix $A$. Actually the main difference stems from the fact that once we are interested in the node set of the paths and circuits (intersection approach) and once in the edge sets of them (dicut approach). Consequently, the context from which $A$ arises will be determinant for the choice of an approach. If $A$ is for instance the incidence matrix of a family defined on the node set of a graph (like convex sets of bounded length), a node-oriented approach (intersection case) seems more appropriate than an edge-oriented approach (dicut case).

In the proof of the polarity relation between subclasses of lattice polyhedra and switching paths polyhedra (theorem 7.1), the approach via intersection has been adopted. Surely, the approach via dicuts presented above would have been an alternate tool for the proof but we did not carry it out.
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