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On the Definition of Nonlinear Stability for Numerical Methods

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Abstract

In this paper a new definition of nonlinear stability for the general nonlinear problem $F(u) = 0$ and the corresponding family of discretized problems $F_h(u_h) = 0$ is given. The notion of nonlinear stability introduced by Keller [Kel75] and later by López-Marcos and Sanz-Serna [LS88a] have the disadvantage that the Lipschitz constant of the derivative of $F_h(u_h)$ has to be known which, in many applications, is not practicable. The modification proposed here allows us to use linearized stability in a ball containing the solution $u_h$ to get nonlinear stability. The usual result remains true: nonlinear stability together with consistency implies convergence.

Keywords: nonlinear stability, linearized stability

Mathematics Subject Classification: 65H10, 65L20, 65M12, 65N12
1. Introduction and preliminaries. Let us consider the general nonlinear problem

$$F(u) = 0$$

(1)

where $F : D_F \subset X \to Y$ is a differentiable mapping between the Banach spaces $X, Y$ with domain $D_F$. We only consider isolated solutions $u^*$ of (1), i.e. the Fréchet derivative $d_u F : X \to Y$ shall be boundedly invertible.

A numerical method may be applied in order to solve equation (1). This leads in general to a family of equations

$$F_h(u_h) = 0$$

(2)

where the $F_h : D_{F_h} \subset X^h \to Y^h$ are mappings between finite dimensional Banach spaces. Here the domain $D_{F_h}$ is an open set and $F_h(\cdot)$ is continuous on $D_{F_h}$. The subscript $h$ indicates the dependence of the discretization on a small parameter such as the mesh size. Let us assume that $h$ takes values in a set $H$ of positive parameters with $\inf H = 0$ and $\sup H = h_0 < \infty$. In order to define convergence of the approximating solution $u_h^*$ of (2) to $u^*$, we require that there exist linear mappings

$$R^h_x : D_{R^h_x} \subset X \to X^h \quad \text{and} \quad R^h_y : D_{R^h_y} \subset Y \to Y^h$$

obeying the condition

$$\lim_{h \to 0} \| R^h_\nu (x) \| = \| x \|, \quad x \in D_{R^h_\nu}, \ \nu \in \{X, Y\}.\footnote{We suppress indices of norms: $\| z \|_\nu \equiv \| z \|$, $z \in \nu$, $\nu \in \{X, Y, X^b, Y^b\}$.}$$

Here the $D_{R^h_\nu}$ are required to be dense in $\nu$. For example, it may be reasonable to work with a function space $X$, say $L^2(\Omega)$, and with a restriction represented by a grid function $U(i,j) = u(ih,jh), (i,j) \in \Delta \subset \mathbb{Z}^2$. The domain of this restriction is a set containing all continuous functions $C(\Omega)$. But it is impossible to define such a restriction for all $u \in L^2(\Omega)$. Let us use henceforth the notation

$$[z]_h \equiv R^h_\nu (z), \quad \nu \in \{X, Y\}.$$

Convergence of the approximating solution $u_h^*$ to $u^*$ is now meant in the sense that

$$\lim_{h \to 0} \| u_h^* - [u^*]_h \| = 0.$$
Also the definition of consistency makes no difficulties. We introduce the local discretization error at a point \( u \in D_{F_h} \):

\[
\tau_h(u) = F_h([u]_h) - [F(u)]_h
\]

**Definition 1** The family \( \{F_h(\cdot)\} \) is said to be consistent of order \( p \) with \( F(\cdot) \) at \( u \) iff for some constant \( M \), independent of \( h \),

\[
\|\tau_h(u)\| \leq Mh^p. \tag{3}
\]

A further definition will be useful because our original problem (1) as well as the discretized problems (2) might have more than one isolated solution. Hence, the original solutions have to be related to the right discretized ones.

**Definition 2** We call the family \( \{F_h(\cdot)\} \) a proper discretization of \( F(\cdot) \) for the solution \( u^* \) iff there exists a radius \( \rho > 0 \), independent of \( h \), such that the following two conditions are satisfied for all relevant values of \( h \):

(i) There are solutions \( u^*_h \) of (2) such that \( B_\rho(u^*_h) \subset D_{F_h} \) and \( F_h(\cdot) \) restricted on \( B_\rho(u^*_h) \) is one-to-one.

(ii) \( [u^*_h]_h \in B_\rho(u^*_h) \)

2. **Classical definitions of stability.** There are several attempts to define nonlinear stability in the literature, see [Ste73], [Kel75], [LS88a] or [LS88b]. For a summary of these different notions see [LS88a]. They all require the existence of a stability constant \( S \), independent of \( h \), such that a stability inequality

\[
\|u_h - v_h\| \leq S\|F_h(u_h) - F_h(v_h)\| \tag{4}
\]

holds for \( u_h, v_h \) in some subset \( D_h \subset D_{F_h} \). Let us call \( D_h \) the **domain of stability**. It is exactly this domain of stability which distinguishes all the notions of nonlinear stability mentioned above. Clearly, if we choose \( D_h \) too large it may be difficult to find a suitable stability constant \( S \) satisfying (4). On the other hand, if we choose \( D_h \) too small, e.g. \( \text{diam}(D_h) = O(h^q) \), it may occur that we must impose a condition \( \|[u^*_h]_h - u^*_h\| = O(h^q) \), which itself guarantees convergence and our stability analysis becomes meaningless.
Let us select H. B. Keller’s definition of stability. He chooses $D_h = B_\rho(u_h^*)$, the open ball of radius $\rho$ about $u_h^*$. Here the stability threshold $\rho$ must be positive and independent of $h$. The problem now is how to determine $\rho$ and $S$. Let us assume that the linearized problem at $u_h^*$ is stable with stability constant $S_0$ independent of $h$, which is equivalent to the condition

$$\| [d_{u_h}^* F_h]^{-1} \| \leq S_0.$$  

If, in addition, the Fréchet derivative $d_{u_h}^* F_h$ has a Lipschitz constant $K_{\rho_0}$ independent of $h$ in some open ball $B_{\rho_0}(u_h^*)$, i.e.

$$\| d_{u_h}^* F_h - d_{v_h}^* F_h \| \leq K_{\rho_0} \| u_h - v_h \|, \quad u_h, v_h \in B_{\rho_0}(u_h^*), \quad (5)$$

then it can be shown (see [LS88b]) that we may choose an $S > S_0$ arbitrarily to get the condition

$$\rho = \min\{\rho_0, (S_0^{-1} - S^{-1})/K_{\rho_0}\}.$$  

We have transformed the problem of finding $\rho$ and $S$ into another problem, namely finding $S_0$ and $K_{\rho_0}$. It may be easy to handle the linearized problem, but to determine $K_{\rho_0}$ means in practice that we must estimate the norm of the second derivative $d_{u_h}^2 F_h$, which for a wide series of applications does not even exist.

Nevertheless, this definition of stability might be useful. An important lemma of Stetter gives us the connection to convergence.

**Lemma 1 ([Ste73])** Let $X^h, Y^h$ be finite dimensional Banach spaces and let $F_h : B_\rho(u_h^*) \subset X^h \to Y^h$ be a continuous mapping with $F_h(u_h^*) = 0$. If the bound (4) holds on $D_h = B_\rho(u_h^*)$ for some constant $S$, independent of $h$, then the inverse $F_h^{-1} : F_h(B_\rho(u_h^*)) \to B_\rho(u_h^*)$ is Lipschitz continuous with Lipschitz constant $S$ and

$$B_{\rho/S}(0) \subset F_h(B_\rho(u_h^*)).$$

Now we may state a convergence result.

**Theorem 1** Let the family $\{F_h(\cdot)\}$ be consistent of order $p > 0$ with $F(\cdot)$ at $u^*$ and stable on the stability domain $B_\rho(u_h^*)$ with stability constant $S$. If,

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1In fact he chooses $D_h = B_\rho([u^*]_h)$, but this difference is only of technical nature.
in addition, it is a proper discretization for \( F(\cdot) \) at \( u^* \) (i.e. \([u^*]_h \in B_h(u^*_h)\)), then
\[
\|u^*_h - [u^*]_h\| \leq MSh^p, \quad h \leq \left(\frac{\rho}{MS}\right)^{1/p}.
\]
The order of convergence is not smaller than the order of consistency.

Proof. Since by (3)
\[
\|\tau_h(u^*)\| = \|F_h([u^*]_h) - [F(u^*)]_h\| = \|F_h([u^*]_h)\| \leq Mh^p,
\]
we see that \( h \leq \left(\frac{\rho}{MS}\right)^{1/p} \) implies \( F_h([u^*]_h) \in B_{p/S}(0) \). Thus, Lemma 1 applies and the result follows by the stability inequality (4).

From the proof of Theorem 1 we learn the following: the inverse \( F_h^{-1} \) is not required to be Lipschitz continuous on the whole range \( F_h(B_h(u^*_h)) \) to imply convergence for the numerical method. It would suffice for \( F_h^{-1} \) to be Lipschitz continuous on \( B_{p/S}(0) \) only. This is exactly the point where we want to modify the definition of nonlinear stability.

3. Stability by range. Since we only consider isolated solutions \( u^* \) of (1), the Fréchet derivative \( d_u F : X \to Y \) is boundedly invertible. If \( F(\cdot) \) is continuously differentiable\(^{\dagger}\), then, by the inverse mapping theorem (see e.g. [Rud73]), there exists a ball \( B_h(u^*) \) such that \( F : B_h(u^*) \to F(B_h(u^*)) \) is an isomorphism. This allows us to define an inverse mapping \( F_h^{-1} \) on \( F(B_h(u^*)) \). It is the inverse mapping we want to compute, therefore we think stability in some sense has to be defined for this inverse mapping.

Definition 3 Let \( F_h(u^*_h) = 0 \) and let \( R_h \subset Y^k \) be an open set with \( 0 \in R_h \). The family \( \{ F_h(\cdot) \} \) is said to be stable on the stability range \( R_h \) with stability constant \( S \), independent of \( h \), iff for all relevant \( h > 0 \)
\( (i) \quad F_h^{-1} : R_h \to X_h \) can be uniquely defined with \( u^*_h \in F_h^{-1}(R_h) \),
(\(ii\) \( F_h^{-1}(\cdot) \) is Lipschitz continuous on \( R_h \) with Lipschitz constant \( S \).

Again, we get the convergence result.

\(^{\dagger}\)Recall that in general \( F(\cdot) \) need not be continuously differentiable since it is only defined on a subset \( D_F \), but we will see that this requirement (at least the continuity) is essential for the family \( \{ F_h(\cdot) \} \).

5
Theorem 2 Let the family \( \{F_h(\cdot)\} \) be consistent of order \( p > 0 \) with \( F(\cdot) \) at \( u^* \) and stable on the stability range \( B_r(0) \) with stability constant \( S \). If, in addition, it is a proper discretization for \( F(\cdot) \) at \( u^* \) with \( \rho \geq S \rho \), then

\[
\| u_h^* - [u^*]_h \| \leq MSh^p, \quad h \leq \left( \frac{r}{M} \right)^{1/p}.
\]

Proof. As in the proof of Theorem 1 inequality (3) yields

\[
\| \tau_h(u^*) \| = \| F_h([u^*]_h) \| \leq Mh^p.
\]

Thus, \( \tau_h(u^*) \) is in \( B_r(0) \) if only \( h \leq (r/M)^{1/p} \). \( F^{-1}_h(\cdot) \) is Lipschitz continuous on \( B_r(0) \) with Lipschitz constant \( S \). This implies \( F^{-1}_h(B_r(0)) \subset B_{Sr}(u^*_h) \subset B_p(u^*_h) \). Since the discretization is proper we conclude \( [u^*]_h \in F^{-1}_h(B_r(0)) \) and \( [u^*]_h = F^{-1}_h(\tau_h(u^*)) \). The result follows now by the Lipschitz continuity of \( F^{-1}_h(\cdot) \).

\[\]

4. Nonlinear stability by linearized stability. We want to have a practicable tool that allows us to verify the assumptions of Theorem 2. It would be helpful if we could work with the linearized problems

\[
L_h(u_h) \equiv F_h(v_h) + d_{v_h} F_h(u_h - v_h) = 0 \quad (6)
\]

for suitable \( v_h \in X_h \). Clearly, the linear family \( \{L_h(\cdot)\} \) is stable in all above senses iff there is a constant \( S \), independent of \( h \), such that

\[
\|[d_{v_h} F_h]^{-1}\| \leq S. \quad (7)
\]

We will now state our main theorem.

Theorem 3 Let the family \( \{F_h(\cdot)\} \) satisfy the following conditions for all relevant values of \( h \):

(a) \( \{F_h(\cdot)\} \) is a proper discretization for \( F(\cdot) \) at \( u^* \) on \( B_p(u^*_h) \).

(b) \( F_h(\cdot) \) is continuously differentiable on \( B_p(u^*_h) \).

(c) The linearized families \( \{L_h(\cdot)\} \) of (6) are stable with stability constant \( S \) uniformly for \( v_h \in B_p(u^*) \).

(d) The constants \( \rho \) and \( S \) are independent of \( h \).
Then the family $\{F_h(\cdot)\}$ is stable on the stability range $B_{h/S}(0)$ with stability constant $S$.

**Corollary 1** Let the assumptions of Theorem 3 hold. If, in addition, $\{F_h(\cdot)\}$ is consistent of order $p$ with $F(\cdot)$ at $u^*$ then

$$\|u_h^* - [u^*]_h\| = O(h^p).$$

The order of convergence is not smaller than the order of consistency.

To proof our main theorem, it suffices to proof the following, more general lemma.

**Lemma 2** Let $X$, $Y$ be arbitrary Banach spaces and let the mapping $f : B_p(u^*) \subset X \to Y$ satisfy the conditions

(a) $f(u^*) = 0$

(b) $f(\cdot)$ is continuously differentiable and one-to-one on $B_p(u^*)$.

(c) $d_u f : X \to Y$ is a linear isomorphism and $\|d_u f^{-1}\| \leq S$, uniformly for $u \in B_p(u^*)$.

Then the following holds.

(i) $f : B_p(u^*) \to V \equiv f(B_p(u^*))$ is a diffeomorphism. In particular $f^{-1}(\cdot)$ can be uniquely defined on $V$.

(ii) $B_{p/S}(0) \subset V$

(iii) $f^{-1}$ is Lipschitz continuous with Lipschitz constant $S$ on $B_{p/S}(0)$.

**Proof.** Because $f(\cdot)$ on $B_p(u^*)$ is one-to-one, we get (i) from the inverse mapping theorem. For the rest we may assume that $Y \setminus V$ is nonempty, for otherwise (ii) is always satisfied and (iii) follows from Taylor’s formula

$$f^{-1}(y) - f^{-1}(y') = \int_0^1 d_{y+(1-t)y'} f^{-1}(y-y') dt$$

(see e.g. [Die69, Theorem 8.14.3]) together with hypothesis (c). Thus, let $y \in Y \setminus V$ and $r_1 = \|y\| + 1$, i.e. $B_{r_1}(0) \setminus V \neq \emptyset$. By the inverse mapping theorem we know that there is a $0 < r_0 < r_1$ with $B_{r_0}(0) \subset V$. We define

$$r = \sup \{s \in [r_0, r_1] : B_s(0) \subset V\}.$$
Thus, $B_r(0)$ is the biggest ball around zero, which lies entirely in $V$.

We may now choose $\epsilon > 0$ arbitrarily. $B_{r+\epsilon}(0)$ is no more a subset of $V$. We may therefore choose $y \in (Y \setminus V) \cap B_{r+\epsilon}(0)$. Let

$$T_y = \{ty : 0 \leq t < 1\} \quad \text{and} \quad t^* = \sup\{\tau : T_{\tau y} \subset V\}.$$  

If we could walk on $T_y$ beginning at zero, we would leave $V$ for the first time at the point $y^* = t^*y$. Clearly, $y^* \notin V$ by the definition of $t^*$, but $y^* \in \partial V = \overline{V \setminus V}$ and by (9) we have

$$r \leq \|y^*\| \leq r + \epsilon.$$  

If we choose a sequence $\{y_k\} \subset T_{y^*} \subset V$ with limit $y^*$, then the sequence $\{v_k = f^{-1}(y_k)\}$ is well defined in $B_p(u^*)$. Since $T_{y^*}$ is convex, we may apply Taylor’s formula to get

$$v_k - v_m = \int_0^1 d_{ty} + (1-t)y_m f^{-1}(y_k - y_m) dt, \quad k, m \geq 1.$$  

and together with hypothesis (c)

$$\|v_k - v_m\| \leq S\|y_k - y_m\|, \quad k, m \geq 1.$$  

By this, both $\{y_k\}$ and $\{v_k\}$ are Cauchy sequences. Let $v^* = \lim_k v_k$. It is impossible for $v^*$ to be an element of $B_p(u^*)$, for else, by the continuity of $f(\cdot)$, $y^* = f(x^*) \notin V$, which is a contradiction. Hence, $v^* \in \partial B_p(u^*)$, i.e. $\|v^* - u^*\| = \rho$. We may choose $y_m = 0$ in (11) and pass to the limit; then we have, recalling (10),

$$\rho = \|v^* - u^*\| \leq S\|y^*\| \leq S(r + \epsilon).$$

Since $\epsilon$ can be chosen arbitrarily, (ii) is proven. $B_r(0)$ is convex; thus again, we can apply Taylor’s formula and verify (iii) by hypothesis (c). This completes our proof.

5. **Numerical examples.** Let us look at the nonlinear problem

$$-u_{xx} + g(u) - f = 0, \quad x \in [0, \pi]$$

$$u(0) = u(\pi) = 0,$$  

(12)
where \( g : \mathbb{R} \to \mathbb{R} \) is a contraction (i.e. \( g(\cdot) \) is Lipschitz continuous with Lipschitz constant \( L < 1 \)) and \( f \in L^2(0, \pi) \). Here \( F(u) = Au + g(u) - f \), where \( A \) is an unbounded self-adjoint linear operator with dense domain \( D_F \) in \( L^2(0, \pi) \) and spectrum \( \sigma(A) = \{k^2 : k \geq 1\} \). Problem (12) has a unique solution \( u^* \in L^2(0, \pi) \) (see [Bra91]). If \( f \) is continuous, i.e. \( f \in C_0^0(0, \pi) \), then \( u^* \in C^2(0, \pi) \). In the following we will always assume that \( f \) is continuous.

Let us discretize (12) with central differences on the grid \( \{ih : i = 0, \ldots, n + 1, h = \pi/(n+1)\} \). We get a system

\[
F_h(u_h) = A_h u_h + g(u_h) - f_h = 0
\]

with \( u_h, f_h \in \mathbb{R}^n \) and

\[
A_h = \frac{1}{h^2} \begin{bmatrix}
  2 & -1 & & & & \\
  -1 & 2 & -1 & & & \\
  & \ddots & \ddots & \ddots & & \\
  & & -1 & 2 & -1 & \\
  & & & -1 & 2 & \end{bmatrix} \in \mathbb{R}^{n \times n}.
\]

Let us use the notation \( U \equiv u_h \). Then by \( g(U) \) we denote the vector \( [g(U_1), \ldots, g(U_n)]^T \in \mathbb{R}^n \). The boundary conditions \( U_0 = U_{n+1} = 0 \) imply that we can work with the reduced grid \( \{ih : i = 1, \ldots, n\} \). The restriction from \( L^2(0, \pi) \) on this reduced grid is surely defined on the dense set \( C_0^0(0, \pi) \). We write \( f_h = [f]_h \equiv [f(h), f(2h), \ldots, f(nh)]^T \). Let us use the discrete \( L^2 \)-norm on grid functions

\[
\|U\| = (\sum_{i=1}^n U_i^2 h^2)^{1/2}.
\]

We first show that the family \( \{F_h(\cdot)\} \) is a proper discretization for the solution \( u^* \) of (12) with \( \rho = \infty \). As one can easily check, this can be done by showing that the system (13) has a unique solution \( U^* \in \mathbb{R}^n \). It is well-known that \( A_h \) has eigenvalues \( \lambda_k = 4 \sin^2(kh/2)/h^2 \), \( k = 1, \ldots, n \). This implies

\[
\|A_h^{-1}\| = \frac{h^2}{4 \sin^2(h/2)}.
\]

From now on we only admit values of \( h \) so small that a condition

\[
L \frac{h^2}{4 \sin^2(h/2)} \leq L' < 1
\]
can be satisfied. System (13) has a unique solution iff the function $G_h : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$G_h(U) = -A_h^{-1}(g(U) - f_h)$$

has a single fixed-point. Since $g(\cdot)$ is a contraction, we have

$$\|G_h(U) - G_h(V)\| \leq \|A_h^{-1}\|\|g(U) - g(V)\| \leq L\|A_h^{-1}\|\|U - V\| \leq L'\|U - V\|,$$

hence $G_h(\cdot)$ is a contraction, too. Now the result follows by the contraction theorem.

Next we verify that the linearizations of (13) are uniformly stable with stability constant $S = \frac{L'}{L(1 - L')}$

provided that $g(\cdot)$ is continuously differentiable. From (13) we derive

$$dU F_h = A_h + \text{diag}(g'(U)) = A_h[I + A_h^{-1}\text{diag}(g'(U))].$$

Thus, by the Banach lemma [Nas76, p. 333]

$$\|dU F_h^{-1}\| \leq \frac{\|A_h^{-1}\|}{1 - \|g'(U)\|_\infty\|A_h^{-1}\|} \leq \frac{L'}{L(1 - L')}.$$

Now all assumptions of Theorem 3 hold for $\rho = \infty$. We conclude that the family $\{F_h(\cdot)\}$ is stable on the stability range $\mathbb{R}^n$ with stability constant $S = L'/(L(1 - L'))$.

It is a well-known property of the central difference discretization that it is consistent of order $p = 1$ provided the solutions $u^*$ lie in $C^3(0, \pi)$, which is true if both $f(\cdot)$ and $g(\cdot)$ are continuously differentiable. The solutions $U^*$ of (13) therefore converge to $u^*$ with order not smaller than $p = 1$.

We mention that we have never imposed a condition on the Lipschitz continuity of the derivative $dU F_h$ as in (5). This is the main advantage of our analysis.

\textsuperscript{1}If $f(\cdot)$ and $g(\cdot)$ are twice continuously differentiable then $u^* \in C^4(0, \pi)$ which implies consistency of order $p = 2$.
We conclude this paper with a remark on partial differential equations. Exactly the same kind of analysis as we have made for equation (12) can be done on related two-dimensional problems such as
\[
\begin{align*}
    u_{tt} - u_{xx} + g(u) - f &= 0 \\
    u(0, t) = u(\pi, t) &= 0 \\
    u &= \text{2}\pi\text{-periodic}.
\end{align*}
\]
For more details on these problems see [Bra91].

References


