Symplectic Geometry and Nonlinear Optimal Feedback Control

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To my parents and

to my wife Nancy Jane
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Abstract

A global treatment of nonlinear optimal control requires, aside from an appropriate coordinate-free formalism, a fiberbundle setup respecting a wide class of topological properties of nonlinear control systems. The presentation of the presymplectic geometry in relation to the surrounding symplectic geometry illustrates the underlying structure between optimal control and suboptimal control. The intrinsic variational derivation of first order necessary conditions requires the abstract conception of control strategy which is a submanifold transversal to the fibers of the locally trivial state/time/control fiberbundle. The selective examination of structural properties such as transversality, integrability and complete integrability together with the approach to calculus of variation due to Carathéodory results in a complete set of necessary conditions for the optimal feedback control law. Since the nature of these conditions is constructive, it provides a strong tool for computing nonlinear examples of a polynomial type.

The introduction of homogeneous problems supports a global approach to the Hamilton-Jacobi-Theory. The Hamilton-Jacobi method of integration is extended to suboptimal control and thus is a keystone in the further development of a nonlinear robustness theory. The parallel derivation of the partial differential equation due to Hamilton-Jacobi and Bellman for optimal feedback and optimal pseudo-feedback control gives a canonical insight into the relationship between the optimal control problem and its inverse problem. The thorough understanding of complete, thin solutions closes the loop to the action of infinitesimal presymplectomorphisms.

For computational reasons, the so-called solution diffeomorphism from a specific Lagrange submanifold onto a given vector field is examined. As a special case there is the solution isomorphism which inherits under a Lyapunov stability assumption a Riemannian metric and transforms the suboptimal control problem into a nongeodesic spray on a double tangent bundle.

To sum up, the thorough research of the underlying geometry of nonlinear optimal control allows deep insight into the conditions which are necessary for the solution of the Synthesis problem. Since these conditions are constructive there are several integration methods which are based on them.
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Notation

Spaces

manifolds
atlas, charts
euclidean vectorspace
horizontal tangential subbundles
locally trivial fiberbundle
foliation
curves

Maps

relations
morphisms, functions
projections
inclusions

Objects

differential p-form on M
(r, s)-tensor on M
vector fields in TM

Groups

Lie-group
Cohomological group
Algebras

Lie-algebra to the Lie-group $G$  $g$

Cohomological algebra  $H(M, R)$

Operations

interior product  $\iota : TM \times \Lambda^p(M) \to \Lambda^{p-1}(M)$

exterior or wedge product  $\wedge : \Lambda^p(M) \times \Lambda^q(M) \to \Lambda^{p+q}(M)$

Lie-derivative  $\mathcal{L}_X T = S$, $T, S \in \tau^*_x(M), X \in TM$

exterior derivatives  $d, \bar{d}$

Control theoretic specifications

state space  $N, n$ - dimensional differentiable manifold

extended state space  $N \times \mathbb{R}$, $n+1$ - dimensional diff. manifold

control space  $B$, $m$ - dimensional diff. manifold

plant  $P (P \oplus 1)$, vectorfield in $\overline{HM}$ ($HM$)

initial- and endconditions  $B$, boundary relation

(optimal) control strategy  $(E_{opt}), E$

Signals

state, costate  $x, \lambda$ or $y, \mu$, time, cotime $t, \lambda_t$ or $t, \mu_t$

control, cocontrol  $u, \rho$ or $v, \sigma$

Problems

Optimal Control Problem  $(K, B, L, P)$

Hamiltonian Suboptimal Control Pr.  $(k, A, l, E)$

homogeneous problem  $(P, \omega; G)$
1. Optimal and Suboptimal Control

1.1 Optimal Control

In the first section of this chapter we give a very brief introduction to optimal control. An optimal control problem essentially consists of the computation of a control function which steers a plant while minimizing a functional. If the control depends not only on the time but on the state as well, then the solution is called an optimal feedback control. A plant may be thought of as any technical or physical dynamic system represented by a vector field on an appropriate state space. The functional is directly related to the desired optimization goal, i.e. like for example time-optimality, cost-optimality or energy-loss-optimality [Cesari, p. 197], [Alejksev, p. 242], [Athans, p. 254].

1.1.1 The problem statement of optimal control

There are three equivalent formulations of the optimal control problem due to Lagrange, Mayer and Bolza. In this work, we will deal with the Morse-type Bolza problem \((K, B, L, P)\) [Cesari, p. 313] and the Lagrange problem \((B, L, P)\). The four entries of the quadruple \((K, B, L, P)\) shall be defined from the right to the left in the sequence of their restrictive action to the problem. These problems require an \(n+1\)-dimensional state-manifold \((\text{configuration space}) \ N \times \mathbb{R}\) and an \(m\)-dimensional control-manifold \(B\).

**Definition 1.1 : Plant.** A vector field \(P \subset \pi \subset T M \subset TM\) on the total space of a local trivial fiber bundle \(m = (M, \pi, N \times \mathbb{R}, B)\) is called a plant if it is \(T \pi \mid_{\pi} -\)isomorphic to a vector field in \(T (N \times \mathbb{R})\).

If in time-dependent plants the time is emphasized as an independent variable the following notation will be used : \(P \oplus 1 \subset \pi \subset TM\), \(T \pi \mid_{\pi} (P \oplus 1) \subset T (N \times \mathbb{R})\). The assumption of local triviality of the fiberbundles is necessary with respect to the range of the controllability property of the plant [van der Schaft [90], p. 47].
Example 1.2: Linear and affine realization of a plant. On an open subset $U \subset M$ of the manifold $M$ an (abstract) plant $P \subset \mathcal{H}M|_{U} \subset TU$ may have a linear local representation $(U, \varphi (p) = (x, t, u))$ of the form

$$\dot{\mathbf{P}} : \quad \dot{x}(t) = A(t)x(t) + B(t)u, \quad (1.1.1)$$

$$x, \dot{x} \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, u \in \mathbb{R}^m, B \in \mathbb{R}^{n \times m}, \quad (1.1.2)$$

with local coordinates adapted to the submersion $\pi : M \to N \times \mathbb{R}$.

More often, or on a wider subset $U' \subset M$, there are plants $P \subset \mathcal{H}M|_{U'} \subset TU'$ with an affine representation $(U', \psi (p) = (x, t, u))$

$$\dot{\mathbf{P}} : \quad \dot{x}(t) = f(x(t), t) + g_j(x(t), t)u^j, \quad (1.1.3)$$

$$x, \dot{x} \in \mathbb{R}^n, u \in \mathbb{R}^m, f, g_j : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad (1.1.4)$$

[Isidori, p. 6], [Krener, p. 88], [Nejmeier, p. 148]. The abstract origin of the $u$-parametrized ordinary differential equation (1.1.3) could have been

$$\mathbf{P} = A \oplus B_j u^j, \quad \hat{A} = f, \hat{B}_j = g_j, \quad (1.1.5)$$

with $m$ abstract controls $u^j$. In other words, the concrete plant is determined up to an isomorphism $T\psi : TU' \to T(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m)$ by the abstract vector field $\mathbf{P} \subset \mathcal{H}M \subset \mathcal{TM}$.

The next entry in $(K, B, L, P)$ is a real valued smooth function $L \in \mathcal{F}(U)$ on an open subset $U$ of the manifold $M$. Its integral over a curve $c \subset U \subset M$

$$J(c) = \int_c L \quad (1.1.6)$$

defines a real-valued function $J$ on the space of curves

$$\mathcal{C} = \{c = im \gamma, \gamma : \mathbb{R} \to M \mid c \subset U \forall t \in [t_0, t_1]\} \quad (1.1.7)$$

For a treatment of nonsmooth curves see [Klötzer [83], p. 414], [Zezza, p. 5]. The evaluation of the so-called performance index $J \in \mathcal{F}(\mathcal{C})$ requires the pullback $\gamma^* : \Lambda^0(\mathcal{C}) \to \Lambda^0(\mathbb{R})$, or equivalently the related realizing mapping $\hat{\gamma} : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m, \hat{\gamma} = \varphi \circ \gamma$ together with the product

$$\hat{L} = L \circ \varphi^{-1} \quad (1.1.8)$$
\[ y^*J = \int_{t_0}^{t_1} y^* i^* L \, dt = \int_{t_0}^{t_1} \hat{L}(x(t), t, u(t)) \, dt, \]  

(1.1.9)

where \( y^*J \) is constant over \( \mathbb{R} \) and \( i^* \) is the pullback associated to the inclusion \( i : C \rightarrow M \). This evaluation is based on the synchronisation of the curve parameter of \( c = \text{im} \gamma(t) \) and the running time variable \( t \).

Boundary conditions will be introduced as a relation determined by a submanifold \( B \subset (N \times \mathbb{R}) \times (N \times \mathbb{R}) \). They interfere by the restriction of the performance index \( J \in \mathcal{J}(C) \) to the set of curves

\[ C_B = \{ c \in C \mid (c(t_0), c(t_1)) \in T^{-1}_{\text{zero}}(B) \}, \]  

(1.1.10)

where \( T^{-1}_{\text{zero}} \) denotes the diffeomorphism from \((N \times \mathbb{R}) \times (N \times \mathbb{R})\) onto the zero-section \( z_{\text{zero}} \times z_{\text{zero}}(M, \pi, N \times \mathbb{R}, B) \) in \( M \times M \), i.e. \( \pi(c(t_i)) = q_i \), \( q_i = (n_0, t_i) \in N \times \mathbb{R}, i = 0, 1 \). The next example deals with boundary relations \( B \) consisting of Cartesian products, i.e. \( B = B_0 \times B_1 \), \( B_i \subset N \times \mathbb{R}, i = 0, 1 \).

**Example 1.3 : Boundary relation.** Two specific boundary relations arise with boundary conditions of the form \( B = \{ (q_0, q_1) \} \) or \( B = \{ (n_0, n_1) \in N \times N \mid n_0 = n_1 \} \). In the first case \( C_B \) is the space of curves with prescribed initial state and initial time and fixed endstate and final time,

\[ B_0 = \{ q_0 \} = \{ (n_0, t_0) \}, \quad B_1 = \{ q_1 \} = \{ (n_1, t_1) \}. \]  

(1.1.11)

In the second case the elements of \( C_B \) are closed curves. For a detailed treatment of various boundary conditions in the context of necessary conditions see [Geering [75], p. 28].

The last entry of the quadruple \((K, B, L, P)\) is a function \( K \in \mathcal{J}(B) \) defined on the boundary relation \( B \) with weighting purposes.

**Example 1.4 : Boundary function.** If the boundary relation \( B \) is given by \( B_0 = \{ q_0 \} \), \( B_1 \subset N \times \mathbb{R} \) then the smooth function \( K_1 \in \mathcal{J}(B_1) \) is called the end-cost-term and may locally be represented by \( \hat{K}_1 = \hat{K}_1(x(t_1), t_1) \).

The synthesis of the given objects and spaces leads, together with the next property and the definition of a minimal curve, to the actual problem statement.

**Definition 1.5 : Side constraint.** A side constraint \( P \oplus 1 \) is strongly respected if
the projection \( m = \pi(c) \in \mathcal{M} \) of all curves \( c \in \mathcal{C} \) under consideration has first order contact with the vector field \( T\pi|_{\mathcal{H}M}(P \oplus 1) \subset T(N \times \mathbb{R}) \).

**Definition 1.6**: Minimal curve. The curve \( c \in \mathcal{C}_B \) in \( \mathcal{M} \) is called a minimal curve for the boundary relation \( \mathcal{B} \) with strongly respected side constraint \( P \oplus 1 \) if the restriction of \( J \) to \( \mathcal{C}_B \) has a minimum at \( c \).

For technical reasons, we choose the function \( L \in \mathcal{F}(U) \) with global convex level set. Thus the minimum of \( J \) at \( c \) can always be regarded as a global one.

**Definition 1.7**: Control strategy. A control strategy for a plant \( P \subset \mathcal{H}M \subset TM \) is an embedded submanifold \( E \) of \( \mathcal{M} \) of codimension \( m \) being transversal to the fibers of \( m = (M, \pi, N \times \mathbb{R}, \mathcal{B}) \) and thus diffeomorphic to the manifold \( N \times \mathbb{R} \).

There are several ways to generate this specific submanifold \( E \) of \( \mathcal{M} \).

**Definition 1.8**: Feedback controller. The mathematical realization of a feedback controller for a control strategy \( E \subset \mathcal{M} \) is a smooth section \( F : N \times \mathbb{R} \to \mathcal{M} \) such that \( F(N \times \mathbb{R}) \subset E \).

In local coordinates \( (U, \varphi(p) = (x, t, u)) \) this section is of the form

\[
\hat{F} : (x, t) \to (x, t, F(x, t)) ,
\]

(1.1.12)

where \( u = F(x, t) \) is the local technical realization of the feedback controller.

**Definition 1.9**: Steering. The mathematical realization of a steering for a control strategy \( E \subset \mathcal{M} \) is a smooth subimmersion \( S : N \times \mathbb{R} \to \mathcal{M} \) such that \( S(N \times \mathbb{R}) \subset E \).

The subimmersion \( S : N \times \mathbb{R} \to \mathcal{M} \) is decomposable into a submersion \( \sigma : N \times \mathbb{R} \to \mathbb{R} \) and an immersion \( \gamma : \mathbb{R} \to \mathcal{M} \), such that \( S = \gamma \circ \sigma \). In local coordinates \( (U, \varphi(p) = (x, t, u)) \) the subimmersion is pointwise of the form

\[
S(q) = p_t, \quad \hat{S} : (x, t) \to (x(t), t, S(t)) ,
\]

(1.1.13)

where \( u = S(t) \) is the local technical realization of the steering and \( q \in N \times \mathbb{R} \), \( p_t \in E \subset \mathcal{M} \).

In general, an automatization of a technical process is based on control strategies with a feedback controller realization.
Definition 1.10: Bolza-type optimal control (problem statement). In a Bolza-type optimal control problem there is given the quadruple \((K, B, L, P)\) and demanded a control strategy \(E \subseteq M\), for the plant \(P \subseteq \overline{HM} \subseteq TM\) such that the evaluation of the performance index on \(E\)

\[ J(c) = \pi^* K(B) + \int_{c} L , \quad c \in C_B, c \in E \quad (1.1.14) \]

attains a minimum. The submanifold \(E\) of \(M\) is then called an optimal control strategy \(E_{opt}\).

Definition 1.11: Closing-loop procedure. A closing-loop procedure is the product of three mappings, namely a section \(Z: N \times R \to M\), \(Z(N \times R) \subseteq E\) the plant \(P: M \to \overline{HM}\) and the projection \(T\overline{\pi}: TM \to TN \times R\), i.e.,

\[ Q = T\overline{\pi} \circ P \circ Z , \quad Q: N \times R \to TN \times R , \quad (1.1.15) \]

and its result \(Q(Z, P)\) is a vector field in \(TN \times R\).

If the control strategy \(E\) is realized by an extended state-feedback (time including) feedback controller \(F: N \times R \to M\), then the result of the closing-loop procedure is the closed-loop, extended state-feedback controlled vector field

\[ Q(F, P) = T\overline{\pi} \circ P \circ F , \quad Q \in X(N \times R) . \quad (1.1.16) \]

In local adapted coordinates \((U, \varphi(p) = (x, t, u))\), \(U \subseteq M\) there is the restriction of the plant \(P\) to the control strategy \(E\)

\[ \hat{P}|_E : \dot{x}(t) = f(x(t), t, u) , \quad (1.1.17) \]

where \(E\) is locally realized by a feedback control law \(\hat{F}: R^n \times R \to R^n \times R \times R^m\) such that

\[ \hat{P}|_{\hat{F}(R^n \times R)} : \dot{x}(t) = f(x(t), \hat{F}(x(t), t)) . \quad (1.1.18) \]

The closing-loop procedure ends with the application of the projection \(T\overline{\pi}\) which results in the local representation of the closed-loop, extended state-feedback controlled vector field

\[ \hat{Q} (\hat{F}, \hat{P}) \leftrightarrow \hat{Q}(F, P) : \dot{x}(t) = \hat{f}(x(t), t) . \quad (1.1.19) \]

If the control strategy \(E\) is realized by a steering \(S: N \times R \to M\), then the result
of the closing-loop procedure is the \textit{closed-loop, time-feedback controlled (steered) vector field}

\[ Q(S, P) = T\pi \circ P \circ S , \quad Q \subset T_z N \times R , \quad z = \pi \circ S(N \times R) . \quad (1.1.20) \]

In this case the closed-loop property is simply expressing the synchronisation of the plant time with the steering time. In local, adapted coordinates \((U, \varphi(p) = (x, t, u))\), \(U \subset M\) there is the plant \(P\) to the open subset of \(E\) realized by a steering \(\hat{S} : R^n \times R \to R^n \times R \times R^m, \hat{S} = \hat{y} \circ \hat{\sigma}, \hat{\sigma} : R^n \times R \to R\)

\[ \hat{p}|_{\hat{S}(R^n \times R)} : \quad \dot{x}(t) = f(x(t), t, S(t)) , \quad (1.1.21) \]

and via the projection \(T\pi\) we get to the \textit{closed-loop, time-feedback controlled (steered) vector field}

\[ \hat{Q}(\hat{S}, \hat{P}) \leftrightarrow \hat{Q}(S, \hat{f}) : \quad \dot{x}(t) = f(x(t), t) . \quad (1.1.22) \]

\textbf{Definition 1.12} : \textit{Optimal steering (solution).} The solution of a Bolza-type optimal steering control problem \((K, \{q_0\} \times B_1, L, P)\) is a subimmersion \(S_{opt} : N \times R \to M\) such that the image \(\text{im}(S_{opt}(n))\) of the integral curve of the closed-loop, optimally time-feedback controlled (steered) vector field \(Q(S_{opt}, P) \subset T_n N \times R\) fully determines a minimal curve for the boundary relation \(\{q_0\} \times B_1\) and the performance index \(J\) of equation \((1.1.14)\).

According to the uniqueness theorem of ordinary differential equations there is exactly one point \(q_0\) in the extended state-space \(N \times R\) which is associated to the point \(p_t \in E_{opt}\) via the flow \(\Phi : I \times N \times R \to N \times R\) generated by the vector field \(Q(S_{opt}, P) \oplus 1 \subset T(N \times R)\), i.e.

\[ q_0 = \Phi(t_0 - t, \pi|_E(p_t)) , \quad q_0 \in B_0 . \quad (1.1.23) \]

Hence, the image of the subimmersion \(S\) is a one-dimensional submanifold in \(E\) connecting the start point \(p_0\) with the running point \(p_t\). A pair \((p_0, S)\), with \(S(\pi(p_0)) = p_0\), is called a steering unit for a plant \(P \subset \overline{HM} \subset TM\). The image \(\text{im}(S(n))\) of the integral curve \(n\) of the vector field \(Q(S, P)\) passing through \(p_0 \in E\) is identical to the image of the subimmersion \(S\).

\textbf{Definition 1.13} : \textit{Optimal state-feedback control (solution)} The solution of a Bolza-type optimal feedback control problem \((K, B_0 \times B_1, L, P)\), \(B_0 \subset N \times R\) is a section \(F_{opt} : N \times R \to M\) such that the image \(\text{im}(F_{opt}(n))\) of integral
curves of the closed-loop, optimal, extended state-feedback controlled vector field $Q(F_{opt}, P) \subset TN \times R$ fully determines the set of minimal curves $c$ in $E_{opt} \subset M$ for the performance index $J$ of equation (1.1.14), obeying the boundary relation $B_0 \times B_1$.

Our main goal is the search of control strategies and their realization as state-feedback controllers by considering the hierarchy of necessary spaces, objects and maps involved in a nonlinear optimal feedback control problem.

**Summary of Section 1.1.1:**

The introduction of the problem statement of optimal control requires some basic definitions of control theoretical terms on an abstract, coordinate-free level.

**Definition 1.14**: Bolza-type optimal control problem (formal). The ordered quadruple $(K, B, L, P)$ consists from right to left, of a plant $P \subset HM \subset TM$, a smooth function $L \in \mathcal{F}(M)$ as an integrand for the performance index $J \in \mathcal{J}(C)$, a boundary relation $B \subset (N \times R) \times (N \times R)$ and a boundary cost-term $K \in \mathcal{F}(B)$.

Mathematical realization of an optimal steering: $S_{opt} : N \times R \to M$ (1.1.24)

Local technical realization: $u = S(t)$ (1.1.25)

Mathematical realization of an optimal state-feedback controller:

$$F_{opt} : N \times R \to M$$ (1.1.26)

Local realization

$$F|_V : V \to V \times B, F|_V = 1_V \times F_u|_V ,$$ (1.1.27)

where the map $F_u|_V : V \to B$ is the local restriction of the technical realization of the optimal state-feedback controller, i.e.,

$$F_u : N \times R \to B, u = F(x, t) .$$ (1.1.28)

The local realization (1.1.27) made implicit use of the local diffeomorphism $diff: \pi^{-1}(V) \to V \times B$, $V \subset N \times R$ inherited from the underlying local trivial fiberbundle. In general $n \geq m$, since the art of control theory is the influencing of a high amount of states $(x)$ by the lowest possible number of inputs (controls $u$) while performing a desired behaviour.
1.1.2 Necessary conditions for optimal steering

The variational calculus approach in optimal control theory is based on the integration of the side constraints (initial value problem with controls) into the minimizing functional using the Lagrange multiplier method [Bertsekas, p. 104].

Let there be introduced the $2(n+1)+m$-dimensional manifold $P$ as the total space of the cotangent vector-subbundle $v = (P, q, M, R^{n+1*})$, $P \subset T^*M$, with projection $q = \tau|_P$ being the restriction of the vectorbundle projection $\tau^* : T^*M \to M$ to the manifold $P$. In order to minimize $J \in \mathcal{J}(C)$ with strongly respected side constraint $P \oplus 1$, the function $L \in \mathcal{J}(M)$ has to be pulled back from $M$ to a Lagrangian function $L^e \in \mathcal{J}(TP)$ on the tangent manifold $TP$, i.e.,

$$L^e = \chi^*(L + \alpha (P \oplus 1)) - \bar{\beta}(X), \quad L^e \in \mathcal{J}(TP), L \in \mathcal{J}(M), (1.1.29)$$

where the projection $\chi : TP \to M$ is the product $\chi = q \circ \theta$ of the projections $\theta : TP \to P$ and $q : P \to M$. The one-form $\bar{\beta} \in \Lambda^1(P)$ is considered to be semi-basic and the restriction of the Liouville-form $\beta \in \Lambda^1(T^*M)$ to the manifold $P$ [Libermann, p.59]. The last summand in equation (1.1.29) is of the form

$$X \perp \bar{\beta} = \bar{\beta}(X), \quad \bar{\beta} = T^*\chi(\alpha), \quad X \subset TP, \alpha \in \Lambda^1(M), (1.1.30)$$

with $X$ being a general vector field. If the side constraint $P \oplus 1$ is strongly respected, then $\chi^*(\alpha (P \oplus 1)) - \bar{\beta}(X)$ is zero on $P$. The canonical symplectic 2-form $\Omega \in \Lambda^2(T^*M)$ on $T^*M$ induces via the canonical inclusion

$$\hat{q} : P \to T^*M, \quad q(V) \subset \tau^* \circ \hat{q}(V), \quad V \subset P \quad (1.1.31)$$

a presymplectic structure

$$\omega = \hat{q}^*(\Omega), \quad \Omega = d\beta, \quad \omega \in \Lambda^2(P) \quad (1.1.32)$$

on the coisotropic manifold $P$ which is equivalent to the one induced by

$$\omega = d\bar{\beta}, \quad \bar{\beta} \in \Lambda^1(P). \quad (1.1.33)$$

Hence, by construction, the total space $(T^*M, \Omega)$ of the cotangent bundle over the manifold $M$ is the minimal upper partner space of the presymplectic manifold $(P, \omega)$. The next conception is an extension of the control strategy $E$ in $M$ to a pseudo-control strategy $\tilde{E}$ in $P$. 

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Definition 1.15: The pseudo-control strategy. A pseudo-control strategy $\tilde{E}$ is a smooth embedded submanifold of codimension $m$ in $P$ which is transversal to the fibers $F$ of the locally trivial fiberbundle $(P, \partial, T^* (N \times R), F)$.

The submanifold $\tilde{E}$ is at least locally diffeomorphic to the symplectic manifold $(T^* (N \times R), \Omega_{N \times R})$. The kernel of a canonical presymplectic 2-form on the manifold $P$ is an integrable distribution which generates under some regularity assumptions a maximal stable foliation which is at least locally diffeomorphic to the locally trivial fiberbundle $(P, \partial, T^* (N \times R), F)$.

Corresponding to the introduction of the spaces of curves $C$ and $C_B$ we define the spaces of specified curves on a subset $V$ of $TP$ by

$$S = \{ s = im \sigma, \sigma : R \to TP \mid s \subset V \forall t \in [t_0, t_1] \} , \quad (1.1.34)$$

$$S_B = \{ s \in S \mid \tilde{\partial}(s(t_0), s(t_1)) \in \tilde{r}_0^{-1} (B) \} , \quad (1.1.35)$$

with the composition of projections $r = \pi \circ q$, $r : P \to N \times R$ and the diffeomorphism $\tilde{r}_0 (P \times P') \subseteq N \times R \times N' \times R$. The curve $s \in S_B$ is called a stationary curve for the boundary relation $B$ if the restriction of the function

$$J^e (s) = e^* K (B) + \int_s L^e , \quad e : TP \to N \times R, e = \pi \circ \chi \quad (1.1.36)$$

has an extremum at $s$, $\partial s \in B$. The variation of this expression is based on the introduction of an almost arbitrary flow $\Phi : R \times TP \to TP$, $(\Phi^e : TP \to TP)$ generated by a vector field

$$\frac{d}{de} \Phi^e (p) \bigg|_{e = 0} = V (\Phi^e (p)) , \quad \text{for all } p \in s \subset TP \text{ and } (1.1.37)$$

$$e_* V (p_0) = T_{e(p_0)} B , e_* V (p_1) = T_{e(p_1)} B , \quad V \in TTP . \quad (1.1.38)$$

This flow expresses the parametrization of a family of curves $s \in S_B$ all obeying to the same boundary relation. In other words, the introduction of the flow is equivalent to an embedding of the stationary curve into a family of curves being suboptimal with respect to the given performance index. The image of the function $J^e \in \mathcal{J} (S_B)$ at a stationary curve $s$ is an extremum of the function $\tilde{J}^e (s ; e) = J^e (\Phi^e (s))$ on $R \times TP$. The extremality property finds its infinitesimal expression in
\[
\frac{d}{d\varepsilon} \mathcal{J}^e(s; \varepsilon) \big|_{\varepsilon = 0} = dJ^e, \quad \frac{d}{d\varepsilon} \Phi^e(s) \big|_{\varepsilon = 0} = dJ^e. \quad V(s) = 0, \quad (1.1.39)
\]

where \( s \) is considered to be an extremal curve lying in an open connected subset \( V \) of the manifold \( TP \). If \( J^e \) attains a global minimum at \( s \) then the projected curve \( \chi(s) = c \in C_B \) is a minimal curve in \( M \).

**Theorem 1.16:** Necessary conditions of optimal control. The equation (1.1.39) defines up to isomorphisms a semi-basic one-form \( A \in \Lambda^1(P) \) and a semi-basic one-form \( a \in \Lambda^1(P) \) on \( P \), such that

\[
dJ^e(z) \cdot V(s) = e^* dK(B) \cdot V(\partial s) + \
+ \int_s (T^* \theta \{ [\hat{\beta} + A] + a \} + (T^* \chi)(d\alpha [P \oplus 1 - m]\}). V(s) = 0 \quad (1.1.40)
\]

holds true. The covector \( \alpha \) lies in the standard fiber of \( v = (P, q, M, \mathbb{R}^{n+1*}) \), i.e. in the Euclidean vector space \( \mathbb{R}^{n+1*} \). For a curve \( s \in S_B \) to be stationary it is necessary that

\[
\hat{\beta} = -A(\theta(s)), \quad m = P \oplus 1, \quad a(\theta(s)) = 0, \quad \beta_{\partial s} = r^* dK(B), \quad (1.1.41)
\]

for \( m = \chi(s) \).

**Proof:** In the preface of this theorem is given a description of the setup of the extended variational problem of minimizing the functional

\[
\mathcal{J}^e(s) = e^* K(B) + \int_s L^e. \quad (1.1.42)
\]

At the critical point of \( \mathcal{J}^e(z) \) over \( S_B \) there is formally the infinitesimal criterion

\[
dJ^e \cdot V(s) = 0, \quad (1.1.43)
\]

which has to be satisfied for all vector fields \( V \) in the sense of equations (1.1.37), and (1.1.38). The evaluation of this criterion is done by

\[
dJ^e = e^* dK + \int_s dL^e = 0. \quad (1.1.44)
\]

Its modification uses the *Poincare-Cartan-formula*

\[
L_X \eta = dX \perp \eta + X \perp d\eta \quad (1.1.45)
\]
for the second and third summand of the integrand in (1.1.44), i.e.
\[ d(T^* \chi(\alpha(P \oplus 1))) = T^* \chi d(\alpha(P \oplus 1)) = T^* \chi(L_{P \oplus 1} \alpha - (P \oplus 1) \perp d\alpha) \]
(1.1.46)
\[ d(\bar{\beta}(X)) = L_{\chi} \bar{\beta} - (P \oplus 1) \perp d\bar{\beta}, \]
(1.1.47)
on TP. The next step of the proof requires the restriction of the generalized transpor
tation theorem [Abraham, p. 471] to a specific initial condition, i.e.
\[ \frac{d}{dt} \int_{\Phi_t(p_0)} \eta = \int_{\Phi_t(p_0)} \left( \frac{\partial \eta}{\partial t} + L_{\chi} \eta \right), \quad \Phi_t(p_0) = s \]
(1.1.48)
for any time-dependent differential p-form \( \eta \in \Lambda^p(M) \) on any manifold M. The
application of the generalized transportation theorem is done by
\[ \int_s L_{\chi} \bar{\beta} = \frac{d}{dt} \int_s \bar{\beta} - \int_s \frac{\partial \bar{\beta}}{\partial t}, \quad \frac{\partial \bar{\beta}}{\partial t} = \hat{\beta}, \]
(1.1.49)
and results in the abstract boundary conditions
\[ \frac{d}{dt} \int_s \bar{\beta} = \bar{\beta}_{|s} = r^* dK(B). \]
(1.1.50)
In the last step of the proof we apply the exterior derivative with respect to base
manifolds, which is of the form
\[ dL(c) = \pi^* dL(\pi(c)). \]
(1.1.51)
By \( dL(\theta(s)) = r^* dL(e(s)) \) it follows that
\[ T^* \theta(A)(s) = T^* \chi(L_{P \oplus 1} \alpha) \perp dL(\chi(s)) = \]
(1.1.52)
\[ = -T^* \chi(\dot{\alpha}) = -T^* \theta(\dot{\bar{\beta}}), \]
(1.1.53)
\[ a(\theta(s)) = T^* q(L_{P \oplus 1} \alpha + dL(\chi(s))) - A(\theta(s)). \text{qed} \]
(1.1.54)
The projection \( \hat{\theta}(s) \) of a stationary curve lies under some existence assumptions
[Cesari, p. 308] in a connected extremal pseudo-strategy \( \hat{E} \). It is also the image of
the subimmersion \( \hat{S} : N \times \mathbb{R} \rightarrow P \), being the extremal pseudo-steering for the
problem \( (K, B, L, P) \). The necessary conditions of Theorem 1.16 are
Lagrangian. The dynamical part consists of two semi-basic forms and one vector
field.
Corollary 1.17: Necessary conditions of optimal steering (local). In local coordinates \((V, \Phi(p) = (x, t, u, \lambda, \lambda_1))\), \(V \subset P\) for a pseudo-steering \(\hat{S}: (x, t) \to (x(t), t, S(t), \lambda(t), \lambda_1(t))\) to be optimal in the sense of minimizing

\[
J(x_0, t_0; S(t)) = \hat{K}_1(x(t_1), t_1) + \int_{t_0}^{t_1} \hat{L}(x(t), t, S(t)) \, dt , \tag{1.1.55}
\]

with strongly respected constraint \(\dot{x}(t) = f(x(t), t, S(t))\), it is necessary to satisfy the system of ordinary differential equations and boundary condition given by

\[
\begin{align*}
\dot{x}(t) &= f(x(t), t, S(t)), \quad t = 1 , \\
\dot{\lambda}(t) &= -\frac{\partial \hat{L}}{\partial x}(x(t), t, S(t)) - \lambda(t) \frac{\partial f}{\partial x}(x(t), t, S(t)) , \\
\dot{\lambda}_1(t) &= -\frac{\partial \hat{L}}{\partial t}(x(t), t, S(t)) - \lambda(t) \frac{\partial f}{\partial t}(x(t), t, S(t)) , \\
0 &= -\frac{\partial \hat{L}}{\partial u}(x(t), t, S(t)) - \lambda(t) \frac{\partial f}{\partial u}(x(t), t, S(t)) ,
\end{align*}
\]

\[
\begin{align*}
x(t_0) &= x_0 , \quad t_0 = a , \\
\lambda(t_1) &= \frac{\partial \hat{K}_1}{\partial x}(x(t_1), t_1) , \quad \lambda_1(t_1) = \frac{\partial \hat{K}_1}{\partial t}(x(t_1), t_1) .
\end{align*}
\]

Proof: The variation of the Lagrangian function

\[
\hat{L}^e(x, t, \dot{x}, \dot{t}, \lambda, \lambda_1, u) = \hat{L}(x, t, u) + \lambda(f(x, t, u) - \dot{x}) + \lambda_1(1 - \dot{t}) \tag{1.1.62}
\]

with respect to all independent variables is written as

\[
\begin{align*}
d(L + \lambda(f - \dot{x}) + \lambda_1(1 - \dot{t})) &= \frac{\partial L}{\partial x} \, dx + \frac{\partial L}{\partial t} \, dt + \frac{\partial L}{\partial u} \, du + (f - \dot{x}) \, d\lambda + \\
&+ \lambda \left( \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial t} \, dt + \frac{\partial f}{\partial u} \, du \right) - \lambda \, d\dot{x} + (1 - \dot{t}) \, d\lambda_1 - \lambda_1 \, d\dot{t} .
\end{align*}
\]

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The application of a partial integration

\[
\int_{t_0}^{t_1} (-\lambda(t) \, dx(t)) \, dt = -\lambda(t) \, dx(t) \bigg|_{t_0}^{t_1} + \int_{t_0}^{t_1} (\dot{\lambda}(t) \, dx(t)) \, dt \tag{1.1.65}
\]

and the identification of vectors and covectors with identical basis leads directly to the equations

\[
\dot{A} = \frac{\partial L}{\partial x} \, dx + \lambda \frac{\partial f}{\partial x} \, dx , \quad \dot{A}_t = \frac{\partial L}{\partial t} \, dt + \lambda \frac{\partial f}{\partial t} \, dt , \quad \dot{a} = \frac{\partial L}{\partial u} \, du + \lambda \frac{\partial f}{\partial u} \, du , \tag{1.1.66}
\]

\[
\dot{\lambda}(t_1) = \frac{\partial \hat{K}_1}{\partial x} (x(t_1), t_1) , \quad \dot{\lambda}_t(t_1) = \frac{\partial \hat{K}_1}{\partial t} (x(t_1), t_1) . \tag{1.1.67}
\]

The next lemma and corollary are stated for representational reasons. Since the ordinary regularity assumption of Legendre-Clebsch is not satisfied, i.e.

\[
\text{rank } \frac{\partial^2 L}{\partial x^2} \neq 0 , \tag{1.1.69}
\]

in optimal control, the Legendre-(contact)-transformation cannot be applied to the necessary conditions of Lagrangian type. Nevertheless, there exists the possibility to use the symplectic and presymplectic structure on the corresponding manifold and a specific Hamiltonian function such that the necessary conditions are representable as a Hamiltonian vector field together with the vanishing of a semi-basic one-form.

**Lemma 1.18 : Hamiltonian embedding of a vector field.** Every complete vector field \( V \) on a manifold \( M \) can be smoothly embedded in a globally Hamiltonian vector field \( X \). The vector field \( X \) is assumed to be defined on an open connected compact subset of the cotangent bundle \( T^* M \) and generated by a Hamiltonian function \( H \in \mathcal{F}(T^* M) \), where \( H \) is the interior product \( H = W \cdot \beta \) of any vector field \( W \subset HT^* M \), \( T^* T^* M = \mathcal{V} \) with the Liouville one-form \( \beta \in \Lambda^1(T^* M) \). The fiber variable \( \zeta, \bar{\beta} = \zeta_i dz^i \) can locally, in canonical coordinates, be identified with the adjoint variable of the linearization of \( V \).

**Proof:** The Hamiltonian vector field \( X = \#dH \subset TT^* M \) is generated by
\[ dH = \mathcal{L}_X\beta - W \quad d\beta = -X \quad d\beta, \quad (1.1.70) \]

from which it follows that \( X|_{\text{HT}^*M} = W \) and

\[ X|_{\text{VT}^*M} \quad d\beta = -\mathcal{L}_W\beta. \quad (1.1.71) \]

We illustrate this assertion in local coordinates \( U \subset M, \phi(p) = (z) \) as follows:

\[ \dot{V} = g(z), \quad \dot{H} (z, \zeta) = \zeta_i g^i(z), \quad (1.1.72) \]

\[ \dot{z}^i(t) = \frac{\partial \dot{H}}{\partial \zeta_i}, \quad \dot{\zeta}_j(t) = \frac{\partial g^i}{\partial z^j}(z) \zeta_i, \quad (1.1.73) \]

where these last expressions determine a system of semicoupled first order ordinary differential equations. \textit{qed}

**Definition 1.19:** Partially Hamiltonian vectorfield. A vectorfield \( X \) on a presymplectic manifold \( (P, \omega) \) is said to be locally, partially Hamiltonian, or is called an infinitesimal automorphism of the presymplectic structure defined by \( \omega \in \Lambda^2(P) \), if its flow \( \Psi : R \times P \rightarrow P, \) \( (\Psi^t : P \rightarrow P) \) satisfies, for every \( t \in R, \)

\[ \Psi^t \ast \omega = \omega. \quad (1.1.75) \]

A partially Hamiltonian vector field \( X \) satisfies \( \mathcal{L}_X \omega = 0, \) where the Lie-derivative with respect to an embedded submanifold \( Z \) of \( P \) is given by

\[ \mathcal{L}_X \eta = z^* \mathcal{L}_{T_z(X)}\eta|_Z = z^* [T_z(X) \quad d\eta|_Z + d \{ T_z(X) \quad \eta|_Z \}]. \quad (1.1.76) \]

A vector field \( X \in TP \) is globally partially Hamiltonian if the \( \overline{d} \)-closed form \( X \quad \omega \) is \( \overline{d} \)-exact.

**Corollary 1.20:** Partially Hamiltonian embedding of a vector field. Every complete vector field \( P \oplus 1 \subset \text{HM} \subset TM \) on a manifold \( M \) can be smoothly embedded in a globally, partially Hamiltonian vector field \( X \). The vector field \( X \) is assumed to be defined on an open connected compact subset of the presymplectic manifold \( (P, \omega) \) and generated by a Hamiltonian function \( H \in \mathcal{F}(P) \) being the interior product \( H = O \quad \beta \) of any vector field \( O \subset HP, \) \( Tq^*O = P \) with the Liouville one-form \( \beta \in \Lambda^1(P) \). The fiber variables \( \xi, \xi_1, \xi_2, \xi_3, \xi_4 \in \mathbb{R}^{n+1} \)
\( \tilde{\beta} = \xi_i \, d\mathbf{x}^i + \xi_t \, dt, \) can locally be identified with the adjoint variable of the linearization of \( P. \)

**Proof**: Corresponding to the proof of Lemma 1.18, the partially Hamiltonian vector field \( X \in TP \) is generated by

\[
\tilde{d}H = \mathcal{L}_O \tilde{\beta} - O \quad \tilde{d}\tilde{\beta} = -X \quad \tilde{d}\tilde{\beta},
\]

which leads to the identifications \( X|_{HP} = O \) and

\[
X|_{VP} \quad \tilde{d}\tilde{\beta} = -\mathcal{L}_O \tilde{\beta}.
\]

In local coordinates \( \{ U, \phi(p) = (x, t, u) \} , U \subset M, \) these expressions result in

\[
\begin{align*}
\dot{P} &= f(x, t, u), & \dot{H}(x, t, u, \xi, \xi_t) &= \xi_i \, f^i(x, t, u) + \xi_t, \\
\dot{x}^i(t) &= \frac{\partial \dot{H}}{\partial \xi_i}, & \dot{x}^i(t) &= f^i(x, t, u), \quad i = 1, \\
\dot{\xi}_j(t) &= -\frac{\partial \dot{H}}{\partial x^j}, & \dot{\xi}_j(t) &= -\frac{\partial f^i}{\partial x^j}(x, t, u) \, \xi_i(t), \\
\dot{\xi}_t(t) &= -\frac{\partial \dot{H}}{\partial t}, & \dot{\xi}_t(t) &= -\frac{\partial f^i}{\partial t}(x, t, u) \, \xi_i(t),
\end{align*}
\]

where these last expressions determine a system of \( u \)-parametrized, semicoupled first order ordinary differential equations. \( \text{qed} \)

For representational reasons the Hamiltonian function \( H \in \mathcal{F}(P) \) is introduced by

\[
H = q^* L + T^* q \left( \alpha (P \oplus 1) \right). \tag{1.1.83}
\]

Since this Hamiltonian function is affine, we are now in the position to illustrate the more specific necessary conditions of optimal control in the sense of Bolza [Bolza, p. 519]. Although the the next assertion is stated as a theorem, its nature is less canonical than the one of Theorem 1.16.

**Theorem 1.21**: Hamiltonian necessary conditions of optimal control. The dynamical necessary conditions of the minimization problem \( (K, B, L, P) \)

\[
\min_{\mathcal{C}} J = \pi^* K(B) + \int_{\mathcal{C}} L \tag{1.1.84}
\]
subject to the strongly respected side constraints described by the plant $P$ are representable as a partially Hamiltonian vector field $\tilde{\bar{d}}H \subset TP$ on the total space of a locally trivial fiber bundle $(P, \partial, T^* (N \times \mathbb{R}), F)$, i.e., $\partial^* \Omega = \omega$

\[ T\partial^* (#\tilde{\bar{d}}H) = #dH|_{T^* (N \times \mathbb{R})} \wedge \Omega = -H|_{T^* (N \times \mathbb{R})} \]  

The smooth partially Hamiltonian vectorfield $\tilde{\bar{d}}H \in TP / \ker \omega$ on the presymplectic manifold $(P, \omega)$ has the exterior $\tilde{\bar{d}}$-derivative of the Poincaré-Cartan form $\chi$

\[ \chi = \tilde{\bar{\beta}} - H \, dt \]  

as an absolute integral invariant.

**Proof:** If we plug the Hamiltonian function given in (1.182) into the equation (1.176), then we can identify

\[ T\omega (#\tilde{\bar{d}}H) = P \oplus 1 \]  

and

\[ \tilde{\bar{d}}H|_{VP} \dashv \tilde{\bar{\beta}} = A \]  

where as the vanishing part of the exterior partially vanishing derivative of the Hamiltonian function is of the form

\[ dH - \tilde{d}H = a = 0 \]  

with $\tilde{\bar{d}}$ being the exterior partially disappearing differentiation. The other assertions are immediate consequences. $\boxed{}$

**Example 1.22:** In analytical mechanics [Santilli, p. 61] there are various approaches to the way systems with nonholonomic (nonintegrable) constraints can be treated by Pontryagin's maximum principle. The forces imposed by the constraints are regarded as controls. The converse point of view naturally leads to the connection between Differential Algebraic Systems (not in the sense of Kaplan-) and the presymplectic geometry [Léon, p. 399]. Overall, any physical force, exterior or potential, can be considered as a control [Cesari, p. 149].
The next corollary requires in an intrinsic version the locally trivial fiber-bundle $p = (M^0, \pi^0, N \times \mathbb{R} \times \mathbb{R}, B)$ and the cotangential vector-subbundle $v^0 = (P^0, q^0, M^0, \mathbb{R}^{n+2})$, $P^0 \subset T^*M^0$, with projection $q^0 = \tau^0|_{p^0}$ being the restriction of the vector-bundle projection $\tau^0 : T^*M^0 \to M^0$. However, the corollary is stated for comprehensive reasons, in its original coordinate specific language.

**Corollary 1.23 : Pontryagin's maximum principle.** In order that the pseudo-steering $\tilde{S}^0 : N \times \mathbb{R} \to P^0$ is optimal for the time-dependent problem $(K, B, L, P)$ it is necessary that there exists a nonnegative constant $\lambda_0 \geq 0$ and a pair $(\lambda(t), \lambda_t(t))$ such that:

a] $\tilde{S}^0 : (x, t) \to (x(t), t, u = S(t), \lambda_0, \lambda(t), \lambda_t(t))$ is a solution of the canonical system

\[
\dot{x}_0 = \frac{\partial \hat{H}_0}{\partial \lambda_0}, \quad \dot{x}_0(t) = \hat{L}(x(t), t, S(t)) \quad (1.1.91)
\]

\[
\dot{x} = \frac{\partial \hat{H}_0}{\partial \lambda}, \quad \dot{x}(t) = f(x(t), t, S(t)) \quad (1.1.92)
\]

\[
\dot{t} = \frac{\partial \hat{H}_0}{\partial \lambda_t}, \quad \dot{t} = 1 \quad (1.1.93)
\]

\[
\dot{\lambda}_0 = -\frac{\partial \hat{H}_0}{\partial x_0}, \quad \dot{\lambda}_0(t) = 0 \quad (1.1.94)
\]

\[
\dot{\lambda} = -\frac{\partial \hat{H}_0}{\partial x}, \quad \dot{\lambda}(t) = -\lambda_0(t) \frac{\partial \hat{L}}{\partial x}(x(t), t, S(t)) - \lambda(t) \frac{\partial f}{\partial x}(x(t), t, S(t)) \quad (1.1.95)
\]

\[
\dot{\lambda}_t = -\frac{\partial \hat{H}_0}{\partial t}, \quad \dot{\lambda}_t(t) = -\lambda_0(t) \frac{\partial \hat{L}}{\partial t}(x(t), t, S(t)) - \lambda(t) \frac{\partial f}{\partial t}(x(t), t, S(t)) \quad (1.1.96)
\]

satisfying the boundary conditions

\[
(x(t_0), t_0, \ldots) = (x_0, a, \ldots) \quad (1.1.97)
\]

\[
(\ldots, \lambda(t_1), \lambda_t(t_1)) = (\ldots, \frac{\partial \hat{K}}{\partial x}(x(t_1), t_1), \frac{\partial \hat{K}}{\partial t}(x(t_1), t_1)) \quad (1.1.98)
\]

where the Hamiltonian function $\hat{H}_0(x, t, \lambda, \lambda_t, \lambda_0, u)$ is given by
\[ \dot{H}_0 (x, t, \lambda, \lambda_t, \lambda_\circ, u) = \lambda_\circ \dot{L} (x, t, u) + \lambda f(x, t, u) + \lambda_t \]  

(1.1.99)

**b)** The function \( \dot{H}_0 (x(t), t, \lambda(t), \lambda_t(t), \lambda_\circ, u) \) has an absolute minimum as a function of \( u \) over \( V' \subseteq \mathbb{R}^m \) for \( t \) in \( [t_0, t_1] \) which shall be expressed, under the assumption of \( \dot{H}_0 \) being convex in the controls, by

\[ \theta = -\frac{\partial \dot{H}_0}{\partial u}, \quad \theta = -\lambda_\circ (t) \frac{\partial \dot{L}}{\partial u} (x(t), t, S(t)) - \lambda (t) \frac{\partial f}{\partial u} (x(t), t, S(t)). \]  

(1.1.100)

**Proof:** The proof is an immediate consequence of the proof of Theorem 1.16. In the regular case \( \lambda_\circ \) is normalized to 1. For a reference see [Athans, p. 305]. The Liouville-form in this fiberbundle setup is locally of the form

\[ \bar{\beta}_0 = \lambda_\circ dx_\circ + \lambda dx + \lambda_t dt, \quad \bar{\beta}_0 \in \Lambda^1 (P^0) \]  

(1.1.101)

Since a further functional analytic treatment would require the consideration of the fact that the functions \( f \) and \( \dot{L} \) belong to different functional classes, it is appropriate to leave the Mayer-type Pontryagin principle and to continue with the Morse-type Bolza problem of optimal control. The surjective submersion \( q^0 : P^0 \rightarrow M^0 \) projects the image of the optimal pseudo-steering onto the image of the optimal steering, or in other words,

\[ S^0_{opt} = q^0 \circ \tilde{S}^0_{opt} \quad \text{qed} \]  

(1.1.102)

**Summary of Section 1.1.2:**

The first order necessary conditions are deduced in a coordinate free framework. Their geometry is related to a Hamiltonian structure without the application of a Legendre-transformation. The Hamilton / Darboux machinery provides an efficient representational setup. The necessary conditions select extremal curves which strongly satisfy the given side constraint and boundary relation. A minimal curve is determinable under the convexity assumption on the Hamiltonian with respect to the control. The possible evaluation of a controller which drives the plant along a nonminimal extremal curve results in suboptimality of the first kind.
1.1.3 Standard evaluation of the necessary conditions to get a state-feedback controller

The composition of infinitely many optimal steerings resulting from problems which only differ by their initial conditions to one optimal state-feedback is called the Synthesis Problem [Athans, p. 361], [Boltjanskj, p. 326], [Cesari, p. 186] and is formally expressed by

\[(K, \{q_0\} \times B_1, L, P) \Rightarrow (K, B_0 \times B_1, L, P) \, , \, B_0 \subseteq N \times R \, . \quad (1.1.103)\]

In other words, the Synthesis problem deals with the glueing together process of optimal steering units, \{ (p_0, S_{opt}) \}, such that

(i) \[S_{opt} (N \times R) \subseteq E_{opt} \subseteq M , \quad (1.1.104)\]

(ii) \[\pi (p_0) \in B_0 , \, \pi (S(q_1)) \in B_1 . \quad (1.1.105)\]

In an abstract setup, the set of zeros of the form \[a \in \Lambda^1 (P) \] determines under regularity assumptions the submanifold \[S \subseteq P \] of codimension \[m \] in the presymplectic manifold \((P, \omega)\). The manifold \(S\) is at least locally diffeomorphic to the cotangent manifold \(T^* (N \times R)\) being the total space of the cotangential-bundle \(\tau = (T^* (N \times R), \tau^*_{N \times R} N \times R, R^{n+1})\). The symplectic manifolds \((S, \Omega_S)\), \((T^* (N \times R), \Omega_N)\) are maximal lower partner spaces [Weinstein [57], p. 333], [Schumann [94], p. 36] of \((P, \omega)\), i.e.,

\[1^* \omega = \Omega \, , \, 1 : T^* (N \times R) \rightarrow P . \quad (1.1.106)\]

In order to derive an optimal state-feedback control law \(F_{opt} : N \times R \rightarrow M \), for the optimal control strategy \(E_{opt} \supseteq F_{opt} (N \times R)\), the submanifold \(S\) has to be generated by a smooth, injective immersion \(\tilde{F} : T^* N \times R \rightarrow P\) from the contact manifold \((T^* N \times R, \tilde{\omega})\) into the presymplectic manifold \((P, \omega)\), such that \(\tilde{F} (T^* N \times R) \subseteq S\) [Section 2.1.1]. The second part of the necessary conditions, the control-parametrized, partially Hamiltonian vector field, is then restricted to the symplectic manifold \((S, \Omega_S)\). The integration or evaluation of the latter one is realized by the smooth injective immersion

\[P^+ : N \times R \rightarrow T^* (N \times R) \, , \, (\lambda, \lambda_t) = (P(x, t), P_1 (x, t)) \, . \quad (1.1.107)\]

The symbol \(P\) is now doubly engaged, but the potential of confusion is minimal, since the different meanings are far away from one another (abstract plant vs. con-
crete costate, depending on state and time). Finally, the product 
\( F^+ = F^* \circ \circ P^+ \) of the three mappings \( P^+: \mathbb{N} \times \mathbb{R} \to T^+ (\mathbb{N} \times \mathbb{R}) \),
\( \circ: T^+ (\mathbb{N} \times \mathbb{R}) \to T^* \mathbb{N} \times \mathbb{R} \) and \( F^*: T^* \mathbb{N} \times \mathbb{R} \to \mathbb{P} \) results in the desired optimal state-feedback control law \( F^+: \mathbb{N} \times \mathbb{R} \to \mathbb{P} \) for the Hamiltonian vector 
\( \# \tilde{a} \tilde{H} \in \mathbb{T} \mathbb{P} \). The projection \( q (F^+ (\mathbb{N} \times \mathbb{R})) \subset \mathbb{E}_{\text{opt}} \subset \mathbb{M} \) lies in the submanifold \( \mathbb{E}_{\text{opt}} \), i.e. the optimal state-feedback control law for the plant \( \mathbb{P} \) on \( \mathbb{M} \).

In a real setup, in local coordinates, the application of the implicit function theorem to the equation

\[
0 = -\frac{\partial L}{\partial u} (x, t, u) - \lambda \frac{\partial f}{\partial u} (x, t, u) ,
\]

and its evaluation, namely, the determination of the solution \( u = \tilde{F} (x, t, \lambda) \), i.e. the pseudo-feedback control law of the system of partial differential equations

\[
0 = \frac{\partial^2 L}{\partial x^2} (x, \tilde{F}, t) + \frac{\partial^2 L}{\partial u^2} (x, \tilde{F}, t) \frac{\partial \tilde{F}}{\partial x} + \frac{\partial^2 f}{\partial x \partial u} (x, \tilde{F}, t) \lambda + \frac{\partial^2 f}{\partial u^2} (x, \tilde{F}, t) \frac{\partial \tilde{F}}{\partial \lambda} \lambda ,
\]

\[
0 = \frac{\partial^2 L}{\partial u^2} (x, \tilde{F}, t) \frac{\partial \tilde{F}}{\partial \lambda} + \frac{\partial^2 f}{\partial x \partial u} (x, \tilde{F}, t) \frac{\partial \tilde{F}}{\partial x} + \frac{\partial^2 f}{\partial u^2} (x, \tilde{F}, t) \frac{\partial \tilde{F}}{\partial u} \lambda ,
\]

\[
0 = \frac{\partial^2 L}{\partial t \partial u} (x, \tilde{F}, t) + \frac{\partial^2 L}{\partial u^2} (x, \tilde{F}, t) \frac{\partial \tilde{F}}{\partial t} + \frac{\partial^2 f}{\partial x \partial u} (x, \tilde{F}, t) \lambda + \frac{\partial^2 f}{\partial u^2} (x, \tilde{F}, t) \frac{\partial \tilde{F}}{\partial \lambda} \lambda ,
\]

is tedious if even possible. The second step is the evaluation of the system of ordinary differential equations (locally the restriction \( \# \tilde{a} \tilde{H} |_S \))

\[
\dot{x} (t) = \frac{\partial \tilde{H}}{\partial \lambda} (x (t), t, \lambda (t), \tilde{F} (x (t), t, \lambda (t))) , \frac{\partial \tilde{H}}{\partial \lambda} = 1 ,
\]

\[
\dot{\lambda} (t) = -\frac{\partial \tilde{H}}{\partial x} (x (t), t, \lambda (t), \tilde{F} (x (t), t, \lambda (t))) ,
\]

\[
\dot{\lambda} (t) = -\frac{\partial \tilde{H}}{\partial t} (x (t), t, \lambda (t), \tilde{F} (x (t), t, \lambda (t))) ,
\]

for all initial states under consideration \( (B_0 \subset \mathbb{N} \times \mathbb{R}) \). Together with the setup in equation (1.1.108), this leads immediately to the system of inhomogeneous quasi-linear partial differential equations
\[
\frac{\partial P}{\partial t} + \frac{\partial P}{\partial x} \tilde{f}(x, t, P(x, t)) = \tilde{g}(x, t, P(x, t)), \tag{1.1.115}
\]

where the two identifications

\[
\tilde{f}(x, t, P(x, t)) = f(x, t, \bar{F}[x, t, P(x, t)]), \tag{1.1.116}
\]

\[
\tilde{g}(x, t, P(x, t)) = (-\frac{\partial \mathcal{L}}{\partial x} - P(x, t) \frac{\partial f}{\partial x})(x, t, \bar{F}[x, t, P(x, t)]), \tag{1.1.117}
\]

for notational reasons assume to hold true. The boundary condition associated to the partial differential equation (1.1.114) is given by

\[
P(x(t_1), t_1) = \frac{\partial \tilde{\mathcal{H}}}{\partial x}(x(t_1), t_1). \tag{1.1.118}
\]

Finally, the technical realization of the optimal nonlinear state-feedback control law for the plant \( P \in \tilde{H}M \subset TM \) is locally of the form

\[
u = \tilde{F}(x, t, P(x, t)) = F(x, t), \tag{1.1.119}
\]

where the optimal pseudo-feedback control law \( \tilde{F}(T^*N \times R) \subset S \) for the Hamiltonian vector field \( ^\# \bar{d}H \big| S \) was given by the foot-point in \( P \), which is locally

\[
(x, t \mid u = \tilde{F}(x, t, P(x, t)) = F(x, t), P(x, t), P_t(x, t)). \tag{1.1.120}
\]

The second equality in formula (1.1.111) and the equation (1.1.113) may be integrated by quadrature. From technical point of view, the feedback control law is often required, in addition to its optimality, to have a certain amount of stabilizing potential with action on the qualitative behavior of the integral trajectories of the closed-loop plant \( Q \) [Section 3.2.2].

**Example 1.24 : LQ-regulator problem.** The Linear-Quadratic-regulator problem \( (x^T K x, (R^n \times R) \times B, \frac{1}{2} (x^T Q x + u^T R u), A x + B u) \) has within this framework the following two advantages with respect to the computational realization:

(I) The application of the implicit function theorem to the functional equation

\[
0 = \frac{\partial \tilde{H}}{\partial u}, \tilde{H} = \frac{1}{2} (x^T Q(t) x + u^T R(t) u) + \lambda (A x + B u). \tag{1.1.121}
\]
results in the pseudo-feedback-controller \( u(t) = -R^{-1}(t)B^T(t)\lambda \).

(II) The splitting \( \lambda(t) = P(x(t), t) = P(t)x(t) \) induces via separability the structure of an ordinary differential equation to the partial differential equation (1.1.114), i.e.

\[
P(t)x(t) + P(t)x(t) = -Q(t)x(t) - A^T(t)P(t)x(t), \quad (1.1.122)
\]

and thus

\[
\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) - Q(t) + P(t)B(t)R^{-1}(t)B^T(t)P(t),
\]

with end condition \( P(t_1) = K \). The \textit{LQ-controller} (regulator) is then of the form

\[
u = F(x, t) = -R^{-1}(t)B^T(t)P(t)x. \quad (1.1.123)
\]

Summary of Section 1.1.3:

The solution of the Synthesis Problem, is done in two steps. First, the rather tedious evaluation of the functional necessary conditions which results in the pseudo-feedback control law. Second, there is the elaborate evaluation of the Hamiltonian vector field in the sense of \textit{Liouville} by finding a map \( P^+ : N \times R \to T^*(N \times R) \). Finally, the two results have to be connected in series which leads to the desired optimal feedback control law realizing the optimal control strategy.
1.2 Suboptimal Control

1.2.1 Nonstandard Evaluation of the Necessary Conditions to Get a State-feedback Controller

In the present section there are exhibited, in order to solve the optimal feedback control problem, two computational effort-minimizing methods. Instead of restricting the vector field \( X = \#\bar{d}H \subset TP/\text{Ker } \omega \) to the submanifold \( S \) there is under some regularity assumptions, the possibility to embed the vector field \( X \) into a vector field \( Y \subset TP \) such that the projection \( \bar{\tau} : TP \to TP/\text{Ker } \omega \) applied to \( Y \) results in \( \bar{\tau}(Y) = X \) and the integral trajectories of \( Y \) remain in \( S \). This vector field contains under \( C^r \)-differentiability assumptions almost the same amount of information with respect to the construction of an optimal control strategy. The next lemma is related to the Theorem 1.16 and its proof contains an effective construction in local coordinates terms.

**Lemma 1.24** : The locally, differentially parametrized, partially Hamiltonian vector field \( Y \). The local representation of the necessary conditions derived in Theorem 1.16 is equivalent, under the regularity assumption of Legendre-Clebsch and the assumption that the controls belong at least to the function class \( C^1 \), equivalent to the local representation of the uniquely associated differentially parametrized, partially Hamiltonian vector field \( Y \subset TP \).

**Proof** : The so-called underlying differentially parametrized partially Hamiltonian vector field \( Y|_U = \left( \#\bar{d}H \oplus \bar{U} \right)|_U \) on \( P \) consists of \( \#\bar{d}H \subset TP / \text{Ker } \omega \) and the control vector field \( \bar{U} \subset \text{Ker } \omega \), which in local coordinates can be shown as

\[
\dot{x}(t) = f(x(t), t, u(t)) \quad (1.2.1)
\]

\[
\dot{1} = 1 \quad (1.2.2)
\]

\[
\dot{\lambda}(t) = -\frac{\partial L}{\partial x}(x(t), t, u(t)) - \lambda(t) \frac{\partial f}{\partial x}(x(t), t, u(t)), \quad (1.2.3)
\]

\[
\dot{\lambda}_t(t) = -\frac{\partial L}{\partial t}(x(t), t, u(t)) - \lambda(t) \frac{\partial f}{\partial t}(x(t), t, u(t)), \quad (1.2.4)
\]

and the smooth so-called control vector field \( \bar{U} \Rightarrow \bar{h}(x, t, \lambda, u), \bar{U} \subset \text{Ker } \omega, \text{Ker } \omega = \cup_{p \in P} \text{Ker } \omega(p) \),
\[ \dot{u} (t) = \tilde{h} (x (t), t, \lambda (t), u (t)) . \]  

The latter vector field is derived by the total time derivative of the m algebraic equations which originate from the local consideration of the vanishing of the one-form \( a \) in (1.26)

\[ 0 = \frac{\partial^2 L}{\partial u^2} f + \frac{\partial^2 L}{\partial u^2} \dot{u} + \frac{\partial^2 f}{\partial u^2} \dot{u} + \frac{\partial f}{\partial u} + \frac{\partial^2 f}{\partial u^2} \dot{u} + \frac{\partial^2 f}{\partial u^2} \dot{u} + \left\{ - \frac{\partial L}{\partial x} - \lambda \frac{\partial f}{\partial x} \right\} \frac{\partial f}{\partial u} \]  

We solve explicitly for \( \dot{u} (t) \) and get

\[ \dot{u} (t) = -\left( \frac{\partial^2 L}{\partial u^2} + \lambda \frac{\partial^2 f}{\partial u^2} \right)^{-1} \left( \frac{\partial^2 L}{\partial u^2} f + \frac{\partial^2 L}{\partial u^2} \dot{u} + \lambda \frac{\partial^2 f}{\partial u^2} \dot{u} + \frac{\partial^2 f}{\partial u^2} \dot{u} + \left\{ - \frac{\partial L}{\partial x} - \lambda \frac{\partial f}{\partial x} \right\} \frac{\partial f}{\partial u} \right) \]  

or in shorthand notation

\[ \dot{u} (t) = \tilde{h} (x (t), u (t), t, \lambda (t)) . \]  

The invertibility of the term

\[ \frac{\partial^2 L}{\partial u^2} + \frac{\partial^2 f}{\partial u^2} \lambda = \frac{\partial^2 \tilde{H}}{\partial u^2} \]  

is guaranteed either by the regularity assumption of Legendre-Clebsch, or by the positive-definiteness condition of the second order variation of optimal control. qed

There are two local methods for the derivation of an optimal feedback controller. Either one solves the coupled system of inhomogeneous, quasi-linear partial differential equations

\[ \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} f (x, t, F) = \tilde{h} (x, F, t, P) , \]  

\[ \frac{\partial P}{\partial t} + \frac{\partial P}{\partial x} f (x, t, F) = - \frac{\partial L}{\partial x} (x, t, F) - P (x, t) \frac{\partial f}{\partial x} (x, t, F) , \]  

where we set \( u (t) = F (x (t), t) \) and \( \lambda (t) = P (x (t), t) \) for the ordinary differential equations (1.2.3), (1.2.8), or the coupled system of inhomogeneous, quasi-linear partial differential equations
\[ \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} f(x, t, F) + \frac{\partial Q}{\partial u} \tilde{h}(x, F, t, Q) = - \left( \frac{\partial L}{\partial x} (x, t, F) + Q \frac{\partial f}{\partial x} (x, t, F) \right) \] (1.2.12)

\[ \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} f(x, t, F) = \tilde{h}(x, F, t, t), \quad (1.2.13) \]

with result \( u(t) = F(x(t), t) \) and \( \lambda(t) = Q(x(t), u(t), t) \). In addition to these approaches, which have to be chosen corresponding to the given problem \((K, B_0 \times B_1, L, P)\), and which are more efficient than the standard treatment described in Section 1.1.3, there is a net of theoretical consequences which shall be illustrated after the next example.

**Example 1.25**: *Polynomial nonlinear optimal control.* The optimal control problem \((K, B_0 \times B_1, L, P)\) is locally supposed to be of the following form:

Local realization of the plant \( P \):

\[ \dot{P} : x^f(t) = A^r_s(t) x^s(t) + \frac{1}{2} \tilde{A}^{q}_r(t) x^q(t) x^p(t) + B^r_m(t) u^m(t), \] (1.2.14)

Local realization of the integrand of the *performance index* \( J \):

\[ \hat{L} = \frac{1}{2} \Omega_{rs} x^r x^s + \frac{1}{6} \Pi_{ijkl} x^i x^j x^k + \frac{1}{8} \Sigma_{abcd} x^ax^bx^cx^d + \frac{1}{2} R_{mn} u^m u^n, \] (1.2.15)

with \( \Omega_{sr} = \Omega_{rs} > 0, \Sigma_{abcd} > 0, \Pi_{ijkl} > 0, \Pi_{ijk} = \Pi_{ikj} = \Pi_{ikl} = \Pi_{kj} \) and \( R_{mn} > 0, R_{nm} = R_{mn} \). The boundary relations are given in a (compact) connected open subset \( U \) of the extended state-space: \( B_0 \subseteq U \subseteq N \times R \), \( B_1 \subseteq B_0 \).

Local realization of the *end-cost-term*:

\[ \hat{K}_1 = \frac{1}{2} \tilde{\Omega}_{rs} (t_1) x^r x^s + \frac{1}{6} \tilde{\Pi}_{ijkl} (t_1) x^i x^j x^k. \] (1.2.16)

The system of quasi-linear partial differential equations (1.2.10) and (1.2.11) is in this case

\[ \dot{u}^l(t) = - R^{ln}(t) [ \dot{R}_{mn}(t) u^m(t) + \dot{B}_n^r(t) \lambda_r(t) - \Omega_{rs}(t) x^i(t) - \frac{1}{2} \Pi_{ij}(t) x^i x^j + \frac{1}{2} \Sigma_{abc} x^a x^b x^c - (A^r_s + \tilde{A}^q_r x^q(t)) B_n^s \lambda_r(t)] \] (1.2.17)

\[ \dot{\lambda}_r(t) = - \Omega_{rs} x^s - \frac{1}{2} \Pi_{rst} x^s x^t - \frac{1}{2} \Sigma_{rstv} x^s x^t x^v - A^r_s \lambda_s(t) - \tilde{A}^q_s x^t \lambda_s(t). \] (1.2.18)

If we set
\[
\lambda_j(t) = P_{ji}(t)x^i(t) + \frac{1}{2}Q_{jkl}(t)x^k(t)x^l(t), \quad (1.2.19)
\]
\[
u^m(t) = -N^m_r(t)x^r(t) - M^m_{pq}(t)x^p(t)x^q(t), \quad (1.2.20)
\]
and plug this into the previous equations we get a set of simple tensor differential equations. The end-cost-term fixes the end conditions of the adjoint equation whereas the control equation has free boundaries. For a solution see Example 2.19 in Section 2.1.3.

According to the interrelation between Lemma 1.18 and Corollary 1.19, illustrated thus far for representational reasons only, there is a specific Hamiltonian vector field \( Z \) on the cotangential-bundle \( T^*M \) such that
\[
Tq(Z) = Y, \quad \hat{q} : T^*M \to P, \quad T\hat{q} : TT^*M \to TP, \quad (1.2.21)
\]
\[
v = (P, q, M, \mathbb{R}^{n+1}), \quad \tau^* = (T^*M, \tau^*, M, \mathbb{R}^{n+1+m^*}). \quad (1.2.22)
\]
The Hamiltonian vector field \( Z \subset TT^*M \) is generated by the Hamiltonian function of the form
\[
F = \tau^*L + T^*\tau^*\left(\mu(P \oplus 1)\right) + \beta(\tilde{U}), \quad F \in \mathcal{F}(T^*M). \quad (1.2.23)
\]
The presymplectic manifold \( (P, \omega) \) is an invariant submanifold of the Hamiltonian vector field \( Z \in TT^*M \).
In local coordinates \( (V, \varphi(p) = (x, t, u, \lambda, \lambda_t, \rho)) \), \( V \subset T^*M \) the fundamental 2-form \( \Omega \in \Lambda^2(T^*M) \) is of the form
\[
\Omega = d\lambda \wedge dx + d\lambda_t \wedge dt + d\rho \wedge du \quad (1.2.24)
\]
and the Hamiltonian vector field \( Z = \#dF \subset TT^*M \) is given by
\[
\dot{x}(t) = f(x(t), t, u(t)) + \rho(t)\frac{\partial h}{\partial \lambda}(x(t), t, u(t), \lambda(t)), \quad (1.2.25)
\]
\[i = 1, \quad (1.2.26)
\]
\[
\dot{u}(t) = \tilde{h}(x(t), u(t), t, \lambda(t)), \quad (1.2.27)
\]
\[
\dot{\lambda}(t) = -\left(\frac{\partial \tilde{L}}{\partial x} + \lambda \frac{\partial f}{\partial x} + \rho \frac{\partial \tilde{h}}{\partial x}\right)(x(t), t, u(t), \lambda(t), \rho(t)), \quad (1.2.28)
\]
\[ \lambda_i(t) = - \left( \frac{\partial L}{\partial t} + \lambda \frac{\partial f}{\partial t} + \rho \frac{\partial h}{\partial t} \right) (x(t), t, u(t), \lambda(t), \rho(t)), \quad (1.2.29) \]

\[ \rho(t) = - \left( \frac{\partial L}{\partial u} + \lambda \frac{\partial f}{\partial u} + \rho \frac{\partial h}{\partial u} \right) (x(t), t, u(t), \lambda(t), \rho(t)). \quad (1.2.30) \]

The variable \( \rho \in \mathbb{R}^{m*} \) is called the cocontrol and has the property of being canonical. Essentially there are two cases to be distinguished. If the cocontrol is unequal to zero, the system of \( 2(n+1+m) \) first order ordinary differential equations determines suboptimal trajectories in the abstract phase space \( T^* M \). Furthermore, the set of zeros of the cocontrol on which its derivative vanishes is equal to the presymplectic manifold \( (P, \omega) \).

The equations (1.2.28) and (1.2.29) are in this case of the form

\[ \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \left[ f(x, t, F) + \rho \frac{\partial h}{\partial \lambda} (x, F, t, P) \right] = \tilde{h} (x, F, t, P), \quad (1.2.31) \]

\[ \frac{\partial P}{\partial t} + \frac{\partial P}{\partial x} f(x, t, F) = - \frac{\partial L}{\partial x} (x, t, F) - P(x, t) \frac{\partial f}{\partial x} (x, t, F) - \rho \frac{\partial h}{\partial x} (x, F, t, P), \quad (1.2.32) \]

where we set \( u(t) = F(x(t), t; \rho) \), \( \lambda(t) = P(x(t), t; \rho) \). If the cocontrol variable \( \rho \) vanishes identically over the whole time interval under consideration, we get locally the technical realization of the optimal state-feedback control law [see (1.2.10), (1.2.11)]. Another version arises if we set \( u(t) = F(x(t), t; \rho) \), \( \lambda(t) = Q(x(t), u(t), t; \rho) \), namely,

\[ \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \left[ f(x, t, F) + \rho \frac{\partial h}{\partial \lambda} (x, F, t, Q) \right] = \tilde{h} (x, F, t, Q), \quad (1.2.33) \]

\[ \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} f(x, t, F) + \frac{\partial Q}{\partial u} \tilde{h} (x, F, t, Q) = \left( \frac{\partial L}{\partial x} (x, t, F) + Q \frac{\partial f}{\partial x} (x, t, F) + \rho \frac{\partial h}{\partial x} (x, F, t, Q) \right). \quad (1.2.34) \]

**Lemma 1.26 : Vanishing of the cocontrol.** A necessary condition for a controller \( u(t) = F(x(t), t; \rho) \) to be optimal, is the vanishing of the time-derivative \( \dot{\rho}(t) \) of the cocontrol over the desired time interval, i.e.,
\[ \dot{\rho}(t) = 0 \quad \Rightarrow \quad \left( \frac{\partial L}{\partial u} + P_1 \frac{\partial f}{\partial u} + \rho \frac{\partial h}{\partial u} \right)(x, t, F(x, t; \rho), \rho) = 0 , \quad (1.2.35) \]

and equivalently
\[ \dot{\rho}(t) = 0 \quad \Rightarrow \quad \left( \frac{\partial L}{\partial u} + Q \frac{\partial f}{\partial u} + \rho \frac{\partial h}{\partial u} \right)(x, t, F(x, t; \rho), \rho) = 0 . \quad (1.2.36) \]

**Proof**: The proof of the this lemma is trivial but the importance of its assertions in the context of optimal control justifies its statement.

By the application of the implicit function theorem the constant cocontrol \( \rho \) may under regularity assumption of Legendre-Clebsch [Section 2.1.1] be determined as a function of the extended state, i.e., \( \rho = R(x, t) \). A special case asrises if we consider
\[ \rho = \left( \frac{\partial h}{\partial u} \right)^{-1} (\frac{\partial L}{\partial u} + \lambda \frac{\partial f}{\partial u})(x, t, u, \lambda) , \quad (1.2.37) \]

which leads via the results from above, \( u = F(x, t; \theta) \), \( \lambda = P(x, t; \theta) \), to \( \rho = R(x, t) = 0 \) in the optimal case.

Remaining is to plug the determined structure of the controller into the concrete plant and deduce some structural properties of the closed-loop system [Definition 1.11].

**Example 1.27**: The control vector field \( \tilde{U} \) of the previous example is locally determined by
\[ \tilde{h}^j(x, t, u, \lambda) = -R_{\text{ln}}(t) [ R_{mn}(t) um + B^r_n(t) \lambda_r - \Omega_{rs}(t) x^r - \frac{1}{2} \Pi_{ijs}(t) x^i x^j + \frac{1}{2} \Sigma_{abc}(t) x^a x^b x^c] - (A^i_s + \bar{A}^r_{qs} x^q) B^s_n \lambda_r ] \quad (1.2.38) \]

The equations (1.2.25)-(1.2.30) corresponding to the optimal control problem given in Example 1.25 are of the form
\[ x^f(t) = A^i_s x^s(t) + \frac{1}{2} \bar{A}^r_{qp} x^q(t) x^p(t) + B^r_m u^m(t) + \rho(t) (R_{\text{ln}}(B^r_n - (A^i_s + \bar{A}^r_{qs} x^q(t)) B^s_n)) \quad (1.2.39) \]
\[ u^1(t) = \tilde{h}^j(x(t), t, u(t), \lambda(t)) , \quad (1.2.40) \]
\[ \dot{\lambda}(t) = -\left( (\Omega_{rs}x^r + \frac{1}{2}\sum_{ijkl}x^i x^j + \frac{1}{2}\sum_{abcd}x^a x^b x^c) + \lambda (A^r_s + \overline{A}^r_{qs} x^d(t)) \right) \\
+ \rho \left\{ (\Omega_{rs} + \frac{1}{2}\sum_{ij} x^i x^j + \frac{1}{2}\sum_{abc} x^a x^b x^c) - \lambda \overline{\lambda} \overline{A}^r_{qs} x^d B^s_n \right\} \] 
\]

\[ \rho_n(t) = -R_{mn}(t) u^m(t) - B_n^r(t) \lambda_r(t) - R_{ln}(t) \dot{R}_{mn}(t) \rho_1(t), \] 

where the Hamiltonian function \( F \in \mathcal{F}(T^*M) \) has a local representation of the form

\[ \hat{F}(x, t, u, \lambda, \lambda_\nu, \rho) = \hat{L}(x, t, u) + \lambda f(x, t, u) + \lambda_1 + \rho \tilde{h}(x, t, u, \lambda), \] 

and where we omitted the equation for the cotime, since it always can be solved by quadrature and its integral does not directly interact in the pseudo-feedback control law. The necessary condition of Lemma 1.26 concerning the vanishing of the cocontrol is in matrix notation implicitly expressed by

\[ -u^T(t) R(t) - \lambda(t) B(t) - \rho(t) R^{-1}(t) \dot{R}(t) = 0, \] 

which leads via the identity \( \dot{R}^{-1} = -R^{-1} \dot{R} R^{-1} \) to the explicit determination of the cocontrol \( \rho \)

\[ \rho(t) = u^T(t) \dot{R}(t) + \lambda(t) B(t) R^{-1}(t) \dot{R}(t) \] 

We will see later in Section 3.1.2, that the optimal pseudo-feedback in this example is \( u = -R^{-1}(t) B^T(t) \lambda^T \) from which it follows that the constant cocontrol in equation (1.2.45) is identically zero.

**Example 1.28 : Affine-quadratic optimal control.** The vector field \( Z \in T^*M \) of a Hamiltonian suboptimal control associated to an optimal control problem \((K, B, L_1 + L_2, P)\), with a plant \( P \subset \overline{HM} \subset TM \) with a local affine representation, is of the form

\[ \dot{x}(t) = \frac{\partial \hat{F}}{\partial \lambda}, \quad \dot{x}(t) = f(x(t), t) + g(x(t)) u(t) + \rho(t) \frac{\partial \tilde{h}}{\partial \lambda} \] 

\[ i = \frac{\partial \hat{F}}{\partial \lambda_i}, \quad i = 1, \] 

\[ \dot{\lambda}(t) = -\frac{\partial \hat{F}}{\partial x}, \quad \dot{\lambda}(t) = -\frac{\partial L_1}{\partial x} - \lambda(t) \frac{\partial f}{\partial x} - \lambda(t) \frac{\partial g}{\partial x} u - \rho(t) \frac{\partial \tilde{h}}{\partial x}, \]
\[
\dot{\lambda}_1(t) = -\frac{\partial F}{\partial \lambda_1}, \quad \dot{\lambda}_2(t) = -\frac{\partial (\hat{L}_1 + \hat{L}_2)}{\partial t} - \lambda(t) \frac{\partial f}{\partial t} - \rho(t) \frac{\partial h}{\partial t}, \quad (1.2.49)
\]

\[
\dot{u}(t) = \frac{\partial F}{\partial \rho}, \quad \dot{u}(t) = \tilde{h}(x(t), t, u(t), \lambda(t)) , \quad (1.2.50)
\]

\[
\rho(t) = -\frac{\partial L_2}{\partial u}, \quad \dot{\sigma}(t) = -\frac{\partial \hat{L}_2}{\partial u} - \lambda(t) \frac{\partial f}{\partial u} - \rho(t) \frac{\partial h}{\partial u} . \quad (1.2.51)
\]

In order to establish a connection between the Hamiltonian suboptimal control with an optimal control source, \( L_2 \) is assumed to have a Hessian matrix with full rank. Then we may choose \( L_2 \) for simplicity such that locally it is of the form

\[
\hat{L}_2(t, v(t)) = \frac{1}{2} u^T(t) R(t) u(t) . \quad (1.2.52)
\]

The control equation (1.2.50) is then explicitly of the form

\[
\dot{u}(t) = -R^{-1}(t) g^T(x(t)) \left[ -\frac{\partial \hat{L}_1}{\partial x} - \lambda(t) \frac{\partial f}{\partial x} - \lambda(t) \frac{\partial g}{\partial x} u(t) \right] . \quad (1.2.53)
\]

This result is stated as a theorem.

**Theorem 1.29 : Affine-quadratic optimal control.** The local representation of the control vector field of a Hamiltonian suboptimal control with a separated performance index with one part being a quadratic form in the control and a plant with affine representation, is again affine.

**Summary of Section 1.2.1 :**

The treatment of the evaluation of the necessary conditions in order to compute a feedback controller avoids the elaborate application of the implicit function theorem to the functional control equation. In addition, the cocontrol variable is canonical in the sense of Darboux and gives a local measure of the suboptimality of the Hamiltonian vector field \( Z \) which is associated to an optimal control problem \((K, B, L, P)\). If the cocontrol vanishes, the necessary conditions of optimal control appear. The evaluation of the desired feedback is done in wider circumstances depending on \( \rho \). In the last stage, there is \( u = F(x, t; [\rho = 0]) = F(x, t) \).

Finally, the impact of affinity and quadratic forms in the problem \((K, B, L, P)\) to the control vector field is illustrated in an example with adapted local coordinates.
1.2.2 The problem statement of suboptimal control

In this section we deal with (Hamiltonian) suboptimality. This situation arises if a given initial value problem \((A_0, E)\) is smoothly embedded into a variational problem \((k, A, l, E)\), in the sense of Lemma 1.18 in Section 1.1.2. The quadruple \((k, A, l, E)\) consists, from right to left, of a regular vector field \(E \in TM\), a function \(l \in \mathcal{F}(M)\), a boundary relation \(A \subset M \times M\) and a boundary cost-term \(k \in \mathcal{F}(A)\). The four entries of the quadruple \((k, A, l, E)\) shall be introduced in the sequence of their restrictive action to the problem.

The vector field \(E\) is closely related to the vector field \(P \subset \overline{HM} \subset TM\) and the vector field \(U \subset TM/\overline{HM}\). If for example the manifold \(M\) is the total space of a globally trivial fiber-bundle \(m = (M, \pi, N \times \mathbb{R}, B)\), \(M = N \times \mathbb{R} \times B\), then the following is possible

\[ E = P \oplus l \oplus U. \quad (1.2.54) \]

**Example 1.30**: One may consider a vector field \(E\) on \(M\) and a local chart \((U, \varphi(p)) = (x, t, u)\) on \(M\) with local coordinates adapted to the submersion \(\pi: M \to N \times \mathbb{R}\) such that \(E\) has a local representation of the form

\[ \dot{x}(t) = f(x(t), t, u(t)), \quad i = 1, \quad (1.2.55) \]
\[ \dot{u}(t) = h(x(t), t) + j_k(x(t), t) u_k, \quad k = 1, \ldots, m \quad (1.2.56) \]

In any case, the design strategy consists on the one hand, in the introduction of the technical plant \(P\), and on the other hand, in the structural setup with maximal appriory knowledge of the desired or possible algebraical structure of the controller to be implemented. This approach is differentially equivalent to the inverse problem of optimal control which is formally given by either

\[(K_1, \{q_0\} \times B_1, \mathcal{P}) (S) \to (L) \quad \text{or} \quad (K_1, B_0 \times B_1, \mathcal{P}) (F) \to (L) \quad (1.2.57)\]

The evaluation of the inhomogeneous partial differential equation

\[ \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} f(x, t, F(x, t)) = h(x, t) + j_k(x, t) F^k(x, t), \quad (1.2.58) \]

leads directly to the local feedback control law \(u = F(x, t)\) which is the state-feedback realization of a control strategy \(E \subset M\).
Example 1.31: An additional design goal could be established if the vector field \( E \) on \( M \) satisfies some stability properties. If the vector field \( E \) has compact support (i.e., if the closure of the set \( \{ p \in M | E(p) \neq 0 \} \) is compact, or especially if the manifold \( M \) is compact) then the integral curves of \( E \) have infinite lifetime [Abraham, p. 248]. In other words, if the original plant is not necessarily stable on the desired open subset of the manifold \( M \) for any control, then the equation (1.2.56) may be set up to generate a stabilizing control [Chapter 3, Section 6.2].

The next entry in the quadruple \((k, A, l, E)\) which is going to be discussed is the function \( l \in \mathcal{F}(M) \). This function is locally comparable with the function \( L \in \mathcal{F}(U) \) of an optimal control problem \((K, B, L, P)\). As mentioned above, the pullback of the function \( l \in \mathcal{F}(M) \) to \( T^*M \) is going to be added to the interior product of the general vector field \( X \in T^*M \) with the Liouville 1-form \( \beta \in \Lambda^1(T^*M) \),

\[
l = \zeta^*(l + \mu(E)) + \beta(X), \quad l \in \mathcal{F}(T^*M), (1.2.59)\]

where there is

\[
\zeta: T^*M \rightarrow M, \quad \theta_{T^*M}^*: T^*M \rightarrow T^*M, \quad \zeta = \tau^* \circ \theta_{T^*M}^*.
\]

Formally its integral over a curve \( b \subset V \subset T^*M \)

\[
j(b) = \int_b l \quad \text{(1.2.61)}
\]

defines a real-valued function \( j \) on the space of curves

\[
\mathcal{B} = \{ b = \text{im} \delta, \delta: \mathbb{R} \rightarrow TM | b \subset U \forall t \in [t_0, t_1] \}.
\]

The evaluation of the so-called performance index \( j \in \mathcal{F}(\mathcal{B}) \) is done via the pullback \( \delta^*: \Lambda^0(\mathcal{B}) \rightarrow \Lambda^0(\mathbb{R}) \)

\[
\delta^*j = \int_{t_0}^{t_1} \delta^*i^* l \, dt = \int_{t_0}^{t_1} l(x(t), t, u(t), \dot{x}(t), i, \dot{u}(t), \lambda(t), \lambda_i(t), \rho(t)) \, dt, \quad (1.2.63)
\]

where \( \gamma^*j \) is constant over \( \mathbb{R} \) and the pullback \( i^* \) is associated to the inclusion \( i: \mathcal{B} \rightarrow T^*M \). This evaluation is based on the synchronisation of the curve parameter \( b = \text{im} \delta(t) \) and the running time variable \( t \).
Definition 1.32: Minimal curve. The curve $b \in \mathcal{B}_A$ in $T^*M$ is called a minimal curve for the boundary relation $A$ if the restriction of $j$ to $\mathcal{B}_A$ has a minimum at $b$.

The boundary relation $A$ is in general supposed to be an appropriate submanifold of $A \subset M \times M$ of the cartesian product of the manifold $M$ with itself. The question of compatibility with the vector field $E$ is answered by the requirement of enough transversality.

Example 1.33: In order to introduce an initial value problem $(A_0, E)$, the boundary relation must be of the special form $A_0 \subset M$, an open submanifold transversal to the integral curves of the vector field $E$. In local coordinates there are two cases which shall be distinguished. The initial manifold is given by

$$
\mathcal{A}_0 = \{ x(t_0) = x_0(r), t_0 = a(r), u(t_0) = u_0(r) \}, \quad (1.2.64)
$$

with parametrization $r \in \mathbb{R}^{n+1+m}$ or there is just one initial point in $M$, i.e.,

$$
\mathcal{A}_0 = \{ x(t_0) = x_0, t_0 = a, u(t_0) = u_0 \}. \quad (1.2.65)
$$

Finally, the last entry in the quadruple $(k, A, l, E)$ is the smooth function $k \in \mathcal{F}(A)$ on the boundary relation $A$. In comparison with the optimal control problem, it has a similar purpose like the function $K \in \mathcal{F}(B)$ and is thus called a boundary value function.

The next corollary is stated in relation to the Lemma 1.18 of Section 1.1.2.

Corollary 1.34: Hamiltonian embedding. Every complete vector field $E \in TM$ on a manifold $M$ can be smoothly embedded in a globally Hamiltonian vector field $X$. The vector field $X$ is assumed to be defined on an open connected compact subset of the cotangent bundle $T^*M$ and generated by a Hamiltonian function $h \in \mathcal{F}(T^*M)$ being the interior product $h = W \lrcorner \beta$ of any vector field $W \in HT^*M$. $(\tau^*)_* W = E$ with the Liouville one-form $\beta \in \Lambda^1(T^*M)$. The fiber variable $\theta, (\xi, \xi^*_\theta, \chi) \in \mathbb{R}^{n+m}$, $\beta = \xi_i dx^i + \xi^*_i dt + \chi_k du^k$, can locally be identified with the adjoint variables of the linearization of $E$.

Proof: As in the proof of Lemma 1.18, we demonstrate the assertion in local, adapted coordinates $\{ U, \varphi(p) = (x, t, u) \}, U \subset M$,

$$
\tilde{E} = (f(x, t, u), l, h(x, t, u)), \quad (1.2.66)
$$
\[ \dot{h}(x, t, u, \xi, \xi_t) = \xi_x f^i(x, t, u) + \xi t + \chi_k h^k(x, t, u), \quad (1.2.67) \]

\[ x^i(t) = \frac{\partial h}{\partial \xi^i}, i = \frac{\partial h}{\partial \xi_t}, \dot{\xi}_j(t) = -\frac{\partial h}{\partial x^j}, \dot{\xi}_t(t) = -\frac{\partial h}{\partial t}, \quad (1.2.68) \]

\[ \dot{x}^i(t) = f^i(x(t), t, u(t)), i = 1, u^k(t) = h^k(x(t), t, u(t)), \quad (1.2.69) \]

\[ \dot{\xi}_j(t) = -\xi_i(t) \frac{\partial f^i}{\partial x^j}(x(t), t, u(t)) - \chi_k(t) \frac{\partial h^k}{\partial x^j}(x(t), t, u(t)), \quad (1.2.70) \]

\[ \dot{\chi}_i(t) = -\chi_i(t) \frac{\partial f^i}{\partial u^j}(x(t), t, u(t)) - \chi_k(t) \frac{\partial h^k}{\partial u^j}(x(t), t, u(t)), \quad (1.2.71) \]

where these last expressions (1.2.69), (1.2.70) and (1.2.71) determine a system of semicoupled first order ordinary differential equations. \textbf{qed}

**Theorem 1.35 : Hamiltonian embedding of suboptimal control.** To the suboptimal control problem \((k, A, l, E)\) is canonically associated a Hamiltonian vector field \(S \in T T^*M\) on \(T^*M\)

\[ S \perp \Omega = -dh, \quad \Omega \in \Lambda^2(T^*M) \quad (1.2.72) \]

generated by a Hamiltonian function \(h \in \mathcal{F}(T^*M)\) which is given by

\[ h = \tau^{**}(l + \mu(E)). \quad (1.2.73) \]

In local coordinates \((V, \theta(p) = (x, t, u, \lambda, \lambda_t, \rho))\), \(V \subset T^*M\) the Hamiltonian \(h\) is of the form

\[ \hat{h}(x, t, u, \lambda, \lambda_t, \rho) = \hat{f}(x, t, u) + \lambda f(x, t, u) + \lambda_t + \rho h(x, t, u). \quad (1.2.74) \]

The symplectic 2-form \(\Omega\) is a relative integral invariant of the Hamiltonian vector field \(S\).

**Proof:** The proof follows by construction given in Corollary 2.5. The symplectic 2-form \(\Omega\) is a relative integral invariant [Libermann, p. 395] of \(S\).

A control strategy of the vector field \(E \subset TM\) is an embedded submanifold \(E \subset M\) of codimension \(m\) described in Definition 1.7. The adapted version of this philosophy, that a vector field is driven or equivalently controlled by its foot points, requires in the case of suboptimal control the following definition:
Definition 1.36: Control/Costate/Cocontrol strategy. A control/costate/cocontrol strategy is an embedded submanifold $D$ of $T^*M$ of codimension $n+2m$ being transversal to the fibers of $(T^*M, r, N \times \mathbb{R}, F)$ and thus diffeomorphic to $N \times \mathbb{R}$.

A state feedback realization of this strategy is a section $F : N \times \mathbb{R} \to T^*M$ such that $F(N \times \mathbb{R}) \subset D$. A steering realization of this strategy is a subimmersion $S : N \times \mathbb{R} \to T^*M$ such that $S(N \times \mathbb{R}) \subset D$.

Definition 1.37: Suboptimal control problem. In a suboptimal control problem there is given the quadruple $(k, A, l, E)$ and demanded a control/costate/cocontrol strategy for the vector field $\#dh \in TT^*M$, such that the evaluation of the functional on $D$

$$j(b) = \tau^{**} k(A) + \int_{b} l , \quad b \in \mathcal{B}_A, b \subset D \quad (1.2.75)$$

attains a minimum.

Hence, the solution of a suboptimal open-loop control problem $(k, \{p_0\} \times A_1, l, E)$ is a subimmersion $S : N \times \mathbb{R} \to T^*M$ such that the vector field $\#dh$ restricted to $D \supset S(N \times \mathbb{R})$, $\#dh|_D$, fully determines a minimal curve for the boundary relation $\{p_0\} \times A_1$ and the performance index $j$ of equation (1.2.75).

The solution of a suboptimal closed-loop control problem $(k, A_0 \times A_1, l, E)$, $A_0 \subseteq M$ is a section $F : N \times \mathbb{R} \to T^*M$ such that the integral curves of the restricted plant $\#dh|_D$ determine the set of minimal curves $b$ in $D \subset T^*M$ for the performance index $j$ of equation (1.2.75) obeying to the boundary relation $A_0 \times A_1$.

Summary of Section 1.2.2:

Definition 1.38: Suboptimal control problem (formal). The ordered quadruple $(k, A, l, E)$ consists, from right to left, of a vector field $E \in TM$, a smooth function $l \in \mathcal{F}(M)$ as the first summand in the functional $j \in \mathcal{F}(\mathcal{B})$, a boundary relation $A \subseteq M \times M$ and a boundary value term $k \in \mathcal{F}(A)$.

Mathematical realization of a suboptimal steering:

$$s_{opt} : N \times \mathbb{R} \to T^*M \quad (1.2.76)$$
\[
\hat{\xi}_{\text{sub}F_{\text{opt}}} : (x, t) \rightarrow (x(t), t, u = s(t), \lambda(t), \lambda_1(t), \rho = r(t)) \quad (1.2.77)
\]

**Mathematical realization of a suboptimal feedback:**

\[
\hat{\xi}_{\text{sub}F_{\text{opt}}} : N \times \mathbb{R} \rightarrow T^* M . \quad (1.2.78)
\]

In technical reality, the control engineer has often a much better idea of the structure of the controller he is going to apply, than the precise mathematical model of the living plant. In addition, the rich structure of suboptimal control and its interaction with the special case, namely to the optimal control problem, opens various possibilities for efficient computational algorithms.
2. Optimal Feedback Control

2.1 Optimal state feedback control

So far we have derived the conditions of optimal control which are necessary regardless whatever the desired realization of the optimal control strategy \( E_{opt} \subset M \). In this chapter we introduce and illustrate the additional necessary conditions in order to get an optimal closed-loop state feedback control. In the sense of maximal applicability, the constructive part of these conditions shall be underlined such that the objects required can be used for the construction of the solution as well.

We assume the existence of solutions to the optimal steering problems \((K, \{ q_0 \} \times B_1, L, P)\), \( \forall q_0 \in B_0 \) in the differentially relaxed sense of Filippov [Cesari, p. 308] and the local controllability of the plant \( P \) on an open subset of \( N \).

**Definition 2.1 : Optimal feedback controllability.** A plant \( P \subset \tilde{H}M \subset TM \) in an optimal control problem \((K, B_0 \times B_1, L, P)\) is called locally optimal feedback controllable on an open subset \( U \subset M \) if the synthesis problem with \( B_0 \subset \pi(U) \) has a solution. It is called globally optimal feedback controllable if \( U \) lies dense in \( M \).

The result of the synthesis problem, i.e. the composition of optimal steering units \( \{ (p_0, S_{opt}) \} \), is an open subset of the optimal control strategy \( E_{opt} \subset M \). Since \( E_{opt} \) is required to be diffeomorphic to \( N \times \mathbb{R} \) there is always a possible realization of \( E_{opt} \) by an extended state-feedback controller, \( F_{opt} (N \times \mathbb{R}) \subset E_{opt} \).

In order to fully respect the controllability conception of control theory, the Definition 2.1 is modified.

**Definition 2.2 : Controllability in a set of optimal control problems.** A plant \( P \subset \tilde{H}M \subset TM \) in a set of optimal control problems \( \{ K_\alpha, B_{0\alpha} \times B_{1\alpha}, L, P \}_{\alpha \in \mathbb{Z}} \) is called optimal feedback controllable on an open subset \( U \subset E_{opt} \) if the synthesis problems have a solution [Geering [76], p. 120].

The sequel is based on Definition 2.1.
2.1.1 Necessary conditions for optimal state feedback control; transversality

The first necessary condition was already mentioned at the beginning of Section 1.1.2 in Chapter 1, where it was assumed that the 1-form $a \in \Lambda^1(P)$ is regular at its zeros in the sense of Legendre-Clebsch. Hence, this form determines a regular, submanifold $S$ of codimension $m$ in the presymplectic manifold $P$.

Definition 2.3 : Transversality. Let $M$, $F$, $P$ be differentiable manifolds. Let $F : M \to P$ and $G : F \to P$ be differentiable maps. We say that $F$ and $G$ are transverse if, for every pair $(p, q) \in M \times F$ satisfying $F(p) = G(q)$,

$$TF_p(T_qM) + TG_q(T_qN) = T_rP,$$

where $r = F(p) = G(q)$.

In optimal control theory we will concentrate our attention on the following. Let $F : M \to P$ be a differentiable map and let $F$ be a submanifold of $P$. The map $F$ is said to be transverse to the submanifold $F$ if it and the canonical injection $i : F \to P$ are transverse, i.e. if

$$TF_p(T_qM) + T_qN = T_qP$$

for every point $p$ in $M$ whose image $q=F(p)$ belongs to $F$.

A pseudo-strategy $\tilde{E} \subset P$ is the origin of an optimal control strategy $E_{opt} \subset M$, $q(\tilde{E}) = E_{opt}$ if the Hamiltonian $H$ attains a global minimum in $\tilde{E}$ with respect to the controls. For the next lemma we assume the submanifold $S$ to be realized by the section $\tilde{F} : T^*(N \times \mathbb{R}) \to P$, $\tilde{F}(T^*(N \times \mathbb{R})) \subset S$.

Lemma 2.4 : Transversality of $S$. In order that the submanifold $S$ in $P$ of codimension $m$ is the origin of an optimal control strategy $E_{opt}$ in $M$ it is necessary that it is an open submanifold of the optimal pseudo-strategy $\tilde{E}_{opt}$.

Proof : Since the surjective submersion $q$ restricted to the submanifold $S$ of $P$ is a projection along the fibers of the locally trivial fiberbundle $v = (P, q, M, \mathbb{R}^{n+1*})$, the transversality property of $S$ with respect to the fibers of the locally trivial fiberbundle $(P, \tilde{\nu}, T^*(N \times \mathbb{R}), F)$ becomes necessary if $E_{opt}$ is to be a control strategy. In local coordinates a point in $\tilde{E}$ is given by

$$(x, t, [u = \tilde{F}(x, t, \lambda)], \lambda, \lambda') \in \tilde{S} \subset \tilde{E} \quad \text{qed}$$
The next space can be regarded as the realization of the intersection of the conception of a strategy and of the Liouville type integration.

**Definition 2.5 : Lagrange submanifold.** A q-dimensional submanifold \( L \) of a 2q-dimensional symplectic manifold \( (\Sigma, \Omega) \) is called a Lagrangian submanifold if \( \Omega \) induces the zero form on \( L \) [Weinstein [57], p. 334].

A Lagrange submanifold is both isotropic and coisotropic. For a cotangent manifold \( T^*(N \times \mathbb{R}) = \Sigma \) one may distinguish the vertical subbundle \( \mathcal{V}T^*(N \times \mathbb{R}) \) consisting of tangent spaces of the fibers of the covector bundle \( (T^*(N \times \mathbb{R}), \tau_{N \times \mathbb{R}}^*, N \times \mathbb{R}, R^{n+1}) \). The vertical subbundle \( \mathcal{V}T^*(N \times \mathbb{R}) \) defines a foliation of \( T^*(N \times \mathbb{R}) \) whose leaves are Lagrangian submanifolds. For the purpose of control theory we are focusing on a type of Lagrange submanifold which is special by way of its generation.

**Lemma 2.6 : Realization of a specific class of Lagrange submanifolds.** The Lagrange submanifold \( L \subset T^*(N \times \mathbb{R}) \) is defined by a section \( P^+ : N \times \mathbb{R} \rightarrow T^*(N \times \mathbb{R}) \) if and only if \( \tau_{N \times \mathbb{R}}^* \big|_L \) is a diffeomorphism. A necessary condition for \( L \) to be defined by a section \( s \) is that \( L \) is transversal to the vertical foliation of \( \mathcal{V}T^*(N \times \mathbb{R}) \).

**Proof :** For a proof see [Vaisman, p. 8]. The generation of such a specific Lagrange submanifold is discussed at the end of this section.

**Lemma 2.7 : Transversality of \( P^+ (N \times \mathbb{R}) \).** In order that the injective immersion \( F^+ : N \times \mathbb{R} \rightarrow P, F^+ = \tilde{F}_{opt} \circ P^+ \) is a realization of the optimal pseudo-strategy \( \tilde{E}_{opt} \), it is necessary that the image \( P^+ (N \times \mathbb{R}) \) of the smooth section \( P^+ : N \times \mathbb{R} \rightarrow T^*(N \times \mathbb{R}) \) is a specific (Lemma 2.6) Lagrange submanifold \( L \) of \( (T^*(N \times \mathbb{R}), \Omega_{N \times \mathbb{R}}) \).

**Proof :** The mapping \( \Gamma : (T^*(N \times \mathbb{R}), \Omega_{N \times \mathbb{R}}) \rightarrow (S, \Omega_S) \) is according to Lemma 2.4 a symplectomorphism. Since, via the immersion \( i : L \rightarrow T^*(N \times \mathbb{R}) \) and \( \Gamma \), the pullback of the 2-form \( \Omega_S = \iota^* \omega \) onto the submanifold \( L \) vanishes identically, the image \( \Gamma (L) = L' \) of \( L \) is a Lagrange submanifold of \( (S, \Omega_S) \). The injective immersion \( F^+ \) is equivalently described by \( F^+ = \tilde{F}_{opt} \circ P^+ = i' \circ \Gamma \circ P^+ \) with the canonical inclusion \( i' : L' \rightarrow S \). The image \( P^+ (N \times \mathbb{R}) \) has to be a specific submanifold in the sense of Lemma 2.6 in \( T^*(N \times \mathbb{R}) \). The Lagrange property of this submanifold is determined in Section 2.1.4. qed
To sum up, in optimal control theory, the optimal control strategy \( E_{opt} = q(\tilde{E}_{opt}) \) is the projection of the optimal pseudo-strategy \( \tilde{E}_{opt} \) which itself is diffeomorphic to the cotangent manifold \( T^* (N \times \mathbb{R}) \). In this symplectic manifold lives the realizing Lagrange manifold \( L \) which is itself diffeomorphic to the extended state space \( N \times \mathbb{R} \).

After the introduction of this crucial property of transversality in its different appearances the question arises as to how to check the required properties. The transversality of the submanifold \( S \) in the sense of Definition 2.3 can be checked as follows.

Let us introduce the canonical injection \( i : S \to P \). Then the semi-basic 1-form \( a \in \Lambda^1 (P) \) vanishes on \( S \). If we consider the 2-form \( da \) on \( P \) then the identical vanishing of its pullback to \( S \) by \( i^\ast \) is significant for the required transversality property. In local coordinates this requirement is equivalent to the full rank property of the matrix

\[
\frac{\partial^2 H}{\partial u^2}.
\]  

The transversality of the submanifold \( L \) in the sense of Lemma 2.6 can be established as follows.

The generation of the realization of the class of specific Lagrange submanifolds [Lemma 2.6] \( L \) can be done, at least locally, by the exterior derivative of a \textit{monotonically increasing, bounded} function \( W \in F(U), U \subset N \times \mathbb{R} \) such that

\[
T^* \pi_p (\alpha) = d (W \circ s^{-1}) = i^\ast \beta_{N \times \mathbb{R}},
\]  

with \( \beta_{N \times \mathbb{R}} \in \Lambda^1 (T^* (N \times \mathbb{R})) \) being the \textit{Liouville} 1-form on \( T^* (N \times \mathbb{R}) \). In local adapted coordinates this is

\[
(\lambda, \lambda_t) = (P(x, t), P_t(x, t)) = \left( \frac{\partial \hat{W}}{\partial x} (x, t), \frac{\partial \hat{W}}{\partial t} (x, t) \right).
\]  

The fundamental property of the generated Lagrange submanifold is then evaluated by

\[
di^\ast \beta_{N \times \mathbb{R}} = i^\ast d\beta_{N \times \mathbb{R}} = i^\ast \Omega = 0
\]  

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and is locally of the form

\[(d\hat{W})^* [d\lambda \wedge dx + d\lambda_t \wedge dt] = \]

\[
\begin{align*}
\frac{\partial^2 \hat{W}}{\partial x^2} dx \wedge dx + \frac{\partial^2 \hat{W}}{\partial t \partial x} dt \wedge dx + \frac{\partial^2 \hat{W}}{\partial x \partial t} dx \wedge dt + \frac{\partial^2 \hat{W}}{\partial t^2} dt \wedge dt = 0 \, ,
\end{align*}
\]

(2.1.8)

where there is \(dW(N \times R) \subset L\). A function \(W\) exists globally if and only if the cohomology class \([i^* \beta_{N \times R}]\) in \(H^1(L, R)\) is zero.

Another type of generation of the specific class of Lagrange submanifolds mentioned in Lemma 2.6 is possible by the image \(\zeta(N \times R)\) of a closed Pfaffian form \(\zeta \in \Lambda^1(N \times R)\), \(d\zeta = 0\). Conversely, a submanifold \(L\) of \(T^* (N \times R)\), such that the restriction to \(L\) of the canonical projection \(\tau_{N \times R}^*: T^* (N \times R) \to N \times R\), \(\tau_{N \times R}^*|_L\) is a diffeomorphism from \(L\) onto \(N \times R\), is the image of a unique closed 1-form \(\zeta = (\tau_{N \times R}^*|_L)^{-1}\) on \(N \times R\). This 1-form is closed if and only if \(L\) is Lagrangian. The first kind of generation of the Lagrange submanifolds transversal to the vertical fibration of \(T^* (N \times R)\) is a special case of the latter.

The trajectories of the necessary conditions of Theorem 1.16 or equivalently of the Pontryagin minimum principle are composable via the synthesis problem only if the two transversality conditions are satisfied. In the following section we give some further methods for the verification of the required properties.

**Summary of Section 2.1.1:**

In the conception of optimal feedback controllability we are asking for the existence of an optimal control strategy. Since the latter is by definition diffeomorphic to the extended state manifold \(N \times R\), it can always be generated by an extended state feedback control law. The composition of optimal steering units into a field of integral curves with manifold property requires twice a geometric, structural property, namely the transversality. Once for the existence of a pseudo-strategy and a second time for the Lagrange submanifold carrying the previous one.
2.1.2 Necessary condition for optimal state-feedback control; integrability

In this section we introduce a different approach to additional necessary conditions for the solvability of the Synthesis problem

\[ (K, B_0 \times B_1, L, P) \quad (F) \]

\[ E_{opt} \iff \dddot{F} \]

We start on the level of integrability conditions for the integration of the Hamiltonian vector field \( \ddot{H} \) in the sense of Liouville.

**Definition 2.8 : Integrability.** Let \( (\Sigma, \Omega, H) \) be a Hamiltonian system with a \( 2q \)-dimensional symplectic manifold \( \Sigma \), a nondegenerate closed two-form \( \Omega \in \Lambda^2(\Sigma) \) and a Hamiltonian function \( H \in \mathcal{F}(\Sigma) \). Then the system is said to be integrable if it has \( q \) differentiable first integrals \( f_1, \ldots, f_q \in \mathcal{F}(\Sigma) \) defined on the whole manifold \( \Sigma \), whose differentials are linearly independent on a dense open subset of the manifold.

It turns out that this kind of integrability is a structural property of the problem \( (K, B_0 \times B_1, L, P) \) since it measures the number and range of integral invariants of the associated dynamical necessary conditions.

**Definition 2.9 : Verification of the integrability property.** On an open connected subset \( U \) of \( \Sigma \) the evaluation of the infinitesimal criterion

\[ \mathcal{L}_{\ddot{H}} f = 0 \]  

(2.1.11)

is equivalent with the determination of the function being constant under the Hamiltonian flow along an integral trajectory of the vector field \( \ddot{H} \).

**Proof :** An illustration of equation (2.1.11) in local coordinates \((U, \psi(p) = (z, \zeta)), U \subset \Sigma\)

\[ \frac{\partial \dot{H}}{\partial \zeta} (z, \zeta) \frac{\partial \dot{f}(z, \zeta)}{\partial z} - \frac{\partial \dot{H}}{\partial z} (z, \zeta) \frac{\partial \dot{f}(z, \zeta)}{\partial \zeta} = 0 \]  

(2.1.12)

results in a linear partial differential equation with nonlinear coefficients and a solution \( \dot{f}(z(t), \zeta(t)) = \text{const} \) along an integral curve \((z(t), \zeta(t))\), with a constant sometimes set equal to a part of the initial- or end-condition of the integral trajectory under consideration. Under enough regularity assumptions, the im-
plicit function theorem applies such that the image of the section in the cotangent bundle is a manifold,

\[ f_1, ..., q (z, \dot{z} (z)) = \text{const}_{1, ..., q} \Rightarrow \zeta = \dot{z} (z ; \text{const}_{1, ..., q}) \]  \hspace{1cm} (2.1.13)

For some further conceptions and concrete methods of integration see [Pommaret, p. 480]. In order to avoid the application of the implicit function theorem one directly integrates the equations

\[ \frac{\partial \dot{P}}{\partial t} + \frac{\partial \dot{P}}{\partial x} f (x, t, \dot{P} (x, t)) = \bar{g} (x, t, \dot{P} (x, t)) \] \hspace{1cm} (2.1.14)

\[ \frac{\partial \dot{P}_1}{\partial t} + \frac{\partial \dot{P}_1}{\partial x} f (x, t, \dot{P} (x, t)) = \bar{g}_1 (x, t, \dot{P} (x, t)) \] \hspace{1cm} (2.1.15)

with boundary condition given by the gradient of the end-cost-term \( \dot{K}_1 \) on the concrete submanifold \( \dot{B}_1 \), as they are given in Section 2.1.1. qed

Example 2.10 : A Casimir function \( C \in \mathcal{F} (P) \) which is constant along orbits of all possible Hamiltonian vector fields is a first integral [Olver, p. 380].

The space of first integrals of the Hamiltonian system \((\Sigma, \Omega, H)\) is a Lie subalgebra of \(C^\infty (\Sigma, \mathbb{R})\) for the Lie algebra structure defined by the Poisson bracket. For autonomous (time-independent) systems \((\Sigma^*, \Omega^*, H^*)\) the Hamiltonian \(H^*\) is a first integral. A first integral can always be regarded itself as a Hamilton function. Thus it generates a vector field \( \#df \) which is a symmetry vector field for the Hamiltonian vector field associated to the problem [Marle, p. 249], [Marsden [38], p. 122], [Vaisman, p. 8].

The next few theorems state the necessity of an integrability property in order to get a state-feedback controller.

Theorem 2.11 : Optimal state-feedback controllability (direct). A plant \( P \) in an optimal control problem \((K, B, L, P)\) is globally feedback controllable, only if for the partially Hamiltonian vector field \( Y \subset TP \) there exists \( n + 1 \) functionally independent first integrals \((f_1, ..., f_{n+1}) = \overline{G}, \overline{G} : P \rightarrow \mathbb{R}^{n+1} \) such that the submanifold \( \overline{G}^{-1} (c) \subset P, c \in \mathbb{R}^{n+1} \) is transversal to the vertical foliation \( VP \).

Proof : In order that the codimensional m submanifold \( \overline{G}^{-1} (c) \subset P \) is the carrier of the optimal pseudo-strategy it is necessary that it is diffeomorphic to \( M \). In lo-
cal adapted coordinates the partially Hamiltonian vector field $Y$ is given by

$$
\dot{x}(t) = f(x(t), t, u(t))
$$

(2.1.16)

$$
\dot{t} = 1,
$$

(2.1.17)

$$
\dot{\lambda}(t) = -\frac{\partial L}{\partial x}(x(t), t, u(t)) - \lambda(t) \frac{\partial f}{\partial x}(x(t), t, u(t)),
$$

(2.1.18)

$$
\dot{\lambda}_t(t) = -\frac{\partial L}{\partial t}(x(t), t, u(t)) - \lambda(t) \frac{\partial f}{\partial t}(x(t), t, u(t)),
$$

(2.1.19)

$$
\dot{u}(t) = \tilde{h}(x(t), t, \lambda(t), u(t)).
$$

(2.1.20)

The $n+1$ first integrals are formally

$$
\hat{f}_1, \ldots, \hat{f}_{n+1}(x, t, u, \lambda, \lambda_t) = \text{const}_{1...n+1},
$$

(2.1.21)

upon which, the application of the implicit function theorem, under the regularity assumption corresponding to the required transversality, would yield to the explicit expression

$$
\lambda = Q(x, t, u; \text{const}_{1...n+1}), \lambda_t = Q_t(x, t, u; \text{const}_{1...n+1}).
$$

(2.1.22)

The left-hand equation together with the setup $u(t) = F(x(t), t)$ plugged into equation (2.1.20) fully determines the technical realization of an optimal state-feedback controller. \textit{Qed}

**Theorem 2.12**: \textit{Optimal state-feedback controllability (indirect).} A plant $P$ in an optimal control problem $(K, B, L, P)$ is globally feedback controllable, only if for the underlying Hamiltonian system $(T^*M, \Omega, F)$ there exists $n+1+m$ functionally independent first integrals $(f_1, \ldots, f_n, f_{n+1}, \ldots, f_{n+1+m}) = G$, $G : T^*M \to \mathbb{R}^{n+1+m}$ such that the submanifold $G^{-1}(c) \subset T^*M, c \in \mathbb{R}^{n+1+m}$ is transversal to the vertical foliation $VT^*M$.

\textbf{Proof}: The existence of $m$ first integrals for $(T^*M, \Omega, F)$ is necessary for the generation of the invariant submanifold $P$ which is transversal to the vertical foliation. The remaining $n+1$ first integrals restricted to $P$ generate the integration of the Hamiltonian vector field $dF|_{\tilde{E}} = dH|_{\tilde{E}} \in T\tilde{E}$ in the sense of Liouville and the previous theorem (2.11). Since the requirement in this theorem is only part of all the necessary and sufficient conditions of optimal control, we can not expect to have conditioned an optimal control strategy.
In local, adapted coordinates the Hamiltonian vector field $dF$ is given by

$$\dot{x}(t) = f(x(t), t, u(t)) + \rho(t) \frac{\partial h}{\partial x}(x(t), t, \lambda(t), u(t)), \quad (2.1.23)$$

$$i = 1, \quad (2.1.24)$$

$$\dot{u}(t) = \tilde{h}(x(t), t, \lambda(t), u(t)) \quad (2.1.25)$$

$$\lambda(t) = -\frac{\partial L}{\partial x}(x(t), t, u(t)) - \lambda(t) \frac{\partial f}{\partial x} - \rho(t) \frac{\partial h}{\partial x}, \quad (2.1.26)$$

$$\lambda_i(t) = -\frac{\partial L}{\partial t}(x(t), t, u(t)) - \lambda(t) \frac{\partial f}{\partial t} - \rho(t) \frac{\partial h}{\partial t}, \quad (2.1.27)$$

$$\dot{\rho}(t) = -\frac{\partial L}{\partial u}(x(t), t, u(t)) - \lambda(t) \frac{\partial f}{\partial u} - \rho(t) \frac{\partial h}{\partial u}. \quad (2.1.28)$$

The $n+1+m$ first integrals are formally

$$\hat{f}_1, \ldots, \hat{f}_n, \hat{f}_{n+1}, \ldots, \hat{f}_{n+1+m}(x, t, u, \lambda, \lambda_i) = \text{const}_{1 \ldots n+1+m}, \quad (2.1.29)$$

upon which the application of the implicit function theorem, under the regularity assumption corresponding to the required transversality, would lead to the explicit expression

$$\lambda = Q(x, t, u; \text{const}_{1 \ldots n+1+m}), \quad \lambda_i = Q_i(x, t, u; \text{const}_{1 \ldots n+1+m}), \quad (2.1.30)$$

$$\rho = R(x, t, u; \text{const}_{1 \ldots n+1+m}). \quad (2.1.31)$$

The transition to the original optimal control problem is always possible by setting the the cocontrol identically zero. The rest of the proof proceeds in a similar fashion as the proof of the previous theorem (2.11). \texttt{qed}

**Theorem 2.13: Suboptimal feedback controllability.** A suboptimal control problem $(k, A, l, E)$ is suboptimal feedback controllable only if the associated Hamiltonian system $(T^*M, \Omega, h)$ has $n+1+m$ first integrals $(\hat{f}_1, \ldots, \hat{f}_{n+1+m}) = E$, $E : T^*M \to \mathbb{R}^{n+1+m}$ such that the submanifold $E^{-1}(c) \subset T^*M, c \in \mathbb{R}^{n+1+m}$ is transversal to the vertical foliation $VT^*M$. 

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Summary of Section 2.1.2:

The requirement of the structural property integrability is a necessary condition for the integration of Hamiltonian vector fields in the sense of Liouville. We introduced this property, demonstrated its verification and applied it to various optimal and suboptimal state-feedback control realizations of control strategies.
2.1.3 Necessary and sufficient condition for optimal state feedback control; Hilbert, Caratheodory

In a treatment of optimal control in the sense of Hilbert and Caratheodory one has to calibrate the function $J^c$ by adding the integrand of a path-independent integral to Lagrange function $L^c \in \mathcal{F}(TP)$, i.e.,

$$J_{\text{cara}}^c = J^c + (e^* W) (\partial s) , \quad J_{\text{cara}}^c = e^* K(B) + \int_s (L^c + \mathcal{L}_s e^* W)$$  \hspace{1cm} (2.1.32)

such that on all stationary curves $s \subset TP$ which projects $\chi : TP \rightarrow M$ onto minimal curves $c \subset M$ [Definition 1.6] the evaluation of the new performance index $J_{\text{cara}}^c \in \mathcal{F}(s)$ is minimal and equal to zero.

**Theorem 2.14 : Caratheodory's calibration applied to optimal state-feedback control.** For a control strategy $E$ in $M$ to be optimal with respect to a given problem $(K, B_0 \times B_1, L, P)$ it is necessary and sufficient that

(i) $q^* (L + \alpha (P \oplus 1)) = 0$, on $E_{\text{opt}}$, $q (E_{\text{opt}}) = E_{\text{opt}}$  \hspace{1cm} (2.1.33)

(ii) $\alpha = r^* dW$, \hspace{1cm} $W \in \mathcal{F}(N \times R)$, $\beta = T^* q (\alpha)$  \hspace{1cm} (2.1.34)

(iii) $L + \alpha (P \oplus 1)$ attains a minimum at $\tilde{E}_{\text{opt}}$  \hspace{1cm} (2.1.35)

(iv) $W (B_1) = K_1 (B_1)$  \hspace{1cm} (2.1.36)

**Proof :** The first and the last requirement result from the calibration which is according to Lemma 2.15 always possible. The second and third assertion are necessary since $J_{\text{cara}}^c \in \mathcal{F}(s)$ is set such that it takes a minimum at all stationary curves in $TP$ which projects onto minimal curves in $M$. qed

**Lemma 2.15 : Invariance of necessary condition of optimal control.** The subtraction of the term $L_s e^* W$, with $W \in \mathcal{F}(N \times R)$ and $e : TP \rightarrow N \times R$, from the Lagrange function $L^c \in \mathcal{F}(TP)$ associated to any given optimal control problem $(K, B, L, P)$ leaves the first order necessary conditions invariant.

**Proof :** In the coordinate-free derivation of the necessary conditions in Section 1.1.2 we ended up, among other results, with the evaluation of the term $\int_s dL^c$. In the present situation [equation (2.1.32)] there is

$$dL_s e^* W = e^* L_{Te(\hat{s})} dW = e^* (0) = 0 ,$$  \hspace{1cm} (2.1.37)
from which it follows that the first order necessary conditions for the two performance indexes $J^e$ and $J^{\text{cara}}$ remain the same. \textbf{qed}

In other words, since the Lagrange function $L^e$ in optimal control is degenerate in the meaning of the standard Legendre-Clebsch condition the function $W$ is basic, i.e. it is locally, in adapted coordinates, independent of the control, the costate and the cotime variables. For an illustration see the remark following the next corollary. In terms introduced earlier in this work, (iii) determines the optimal pseudo-strategy $\tilde{E}_{\text{opt}}$. (i) and (ii) require a specific Lagrange submanifold $L \subset T^* (N \times \mathbb{R})$ which carries the optimal pseudo-strategy in the sense of $\tilde{F} : T^* (N \times \mathbb{R}) \rightarrow P$, $\tilde{F} (T^* (N \times \mathbb{R})) \subset \tilde{E}_{\text{opt}}$.

Before we state Theorem 2.14 in local coordinates we repeat the calibration procedure of equation (2.1.32), adding to the Lagrange function $L^e \in \mathcal{F}(TP)$ the total time-derivative of an appropriate function [Hilbert]

$$J^{\text{cara}} = \dot{K}_1 (x(t_1), t_1) + \int_{t_0}^{t_1} \left[ \dot{L}(x(t), t, u) + \lambda (t) (f(x(t), t, u) - \dot{x}(t)) + \ight.$$  
$$+ \lambda(t) (1-t) \frac{\partial W}{\partial x} \dot{x}(t) + \frac{\partial W}{\partial t} \right] \ dt, \quad (2.1.38)$$

which is set up such that $J^{\text{cara}}$ is zero on any stationary curve.

Corollary 2.16: Caratheodory of optimal state-feedback control (local). In order that a control strategy $E$ in $M$ is optimal for a given problem $(K, B_0 \times B_1, L, P)$ it is necessary and sufficient that

(i) $\dot{L}(x, t, u) + \lambda f(x, t, u) + \lambda = 0$ on any minimal curve

(ii) $\lambda \dot{x} + \lambda \dot{t} = \dot{W}(x, t),$ \quad $\dot{W} \in \mathcal{F}(\mathbb{R}^n \times \mathbb{R}), \quad (2.1.40)$

(iii) $\dot{L}(x, t, u) + \lambda f(x, t, u) + \lambda$ attains a minimum at $\tilde{E}_{\text{opt}}$

(iv) $\dot{W}(x(t_0), t_0) = \dot{K}_0(\dot{B}_0), \dot{W}(x(t_1), t_1) = \dot{K}_1(\dot{B}_1)$

\textbf{Proof}: The proof of this corollary follows from that of Theorem 2.14 written in local coordinates. The first assertion is identical with the condition of the vanishing of the image of the Hamilton function $H \in \mathcal{F}(P)$ [equation (1.1.82)]. \textbf{qed}

In local coordinates Lemma 2.15 expresses the fact that the total time-derivative
of a function on the extended state-space is a null-Lagrangian for any optimal control problem \((K, B, L, P)\). This can be illustrated by

\[
L^{\text{cara}} = L^e + L^e W \rightarrow \dot{L}^{\text{cara}} = \frac{d\dot{W}}{dt},
\]

(2.1.43)

with the application of the local Euler-Lagrange operator

\[
\frac{d}{dt} \left( \frac{\partial L^{\text{cara}}}{\partial \dot{\gamma}} \right) - \frac{\partial L^{\text{cara}}}{\partial \gamma}, \quad \gamma = (x, t, u, \lambda, \lambda^e, \rho).
\]

(2.1.44)

In the case of

\[
\frac{d}{dt} \left( \frac{\partial L^{\text{cara}}}{\partial \dot{x}} \right) - \frac{\partial L^{\text{cara}}}{\partial x} = 0,
\]

(2.1.45)

there is explicitly

\[
\frac{d}{dt} \left( \frac{\partial \dot{W}}{\partial x} - \lambda \right) - \frac{\partial L}{\partial x} - \lambda \frac{\partial f}{\partial x} - \frac{\partial}{\partial x} \left( \frac{d\dot{W}}{dt} \right) = 0,
\]

(2.1.46)

from which results the so-called adjoint equation of the necessary condition of optimal control [Corollary 1.17].

Caratheodory's calibration works in the case of suboptimal as well. The corresponding function \(U \in \mathcal{F}(M)\) depends on the control variables

\[
ji = \zeta^* k^c(a) + \int_b^\alpha L^* \zeta^* U, \quad ji = j + \zeta^* U (\partial b).
\]

(2.1.47)

**Corollary 2.17 : Caratheodory of suboptimal control.** In order that a control costate/cotime/cocontrol strategy \(D\) in \(T^* M\) is optimal for a given problem \((k, A, l, E)\) it is necessary and sufficient that

(i) \(\tau^{**} (l + \mu (E)) = 0\), \(\beta = T^* \tau^* (\mu)\),

(2.1.48)

(ii) \(\mu = dU\), \(U \in \mathcal{F}(M)\),

(2.1.49)

(iii) \(\tau^{**} (l + \mu (E))\) attains a minimum at \(D\)

(2.1.50)

(iv) \(U (A_1) = k_1 (A_1)\)

(2.1.51)

**Proof :** For a proof we refer to [Bolza, p.132] and to Theorem 2.14.

The next definition is the key to a global treatment of the Hamilton-Jacobi method.
Definition 2.18: Homogeneous problem. A homogeneous problem is a triple $(P, \omega; G)$, where $(P, \omega)$ is a presymplectic manifold and $G$ is an embedded (connected) coisotropic submanifold of $(P, \omega)$.

In optimal open-loop control $(K, \{q_0\} \times B_1, L, P)$ the homogeneous problem $(P, \omega; G)$ is a result of the presymplectic total manifold of the locally trivial fiberbundle $v = (P, q, M, \mathbb{R}^{n+1})$ and the subset

$$G \subset (H)^{-1}(0), \quad G \subset P \quad (2.1.52)$$

such that $G \cap r^{-1}(B_1) = r^*dK_1$ and the one-dimensional fiber over the initial point has an intersection with $G$, i.e., $r^{-1}(q_0) \cap G \neq \{\emptyset\}$. Under some regularity assumptions the $G$ is a connected, codimensional one submanifold of $P$. In an optimal closed-loop problem $(K, B_0 \times B_1, L, P)$ the homogenous problem $(P, \omega; \tilde{G})$ is defined by the nonempty intersection

$$\tilde{G} \subset (H)^{-1}(0) \cap \tilde{E}, \quad (2.1.53)$$

where in addition there is $G \cap r^{-1}(B_1) = r^*dK_1$ and $r^{-1}(B_0) \cap G \neq \{\emptyset\}$. The definition of a homogeneous problem opens the door for the integration methods of Hamilton and Jacobi and for the necessary and sufficient conditions of optimal control in the sense of Caratheodory.

For the sake of completeness, we introduce the illustrative method of the characteristics. Since $\omega$ is closed, the distribution $\Delta$ defined by

$$\Delta = \{ V \in TG | V \perp \iota^*\omega|_G = 0 \} \quad (2.1.54)$$

is involutive, where $\iota : G \to P$. A maximal connected integral manifold of $\Delta$ is called a characteristic of $(P, \omega; G)$. Characteristics form the characteristic foliation of $(P, \omega; G)$. With respect to characteristics we refer to [Rund, p.48].

Summary of Section 2.1.3:

The necessary and sufficient conditions of optimal state-feedback control are stated in the sense of Hilbert and Caratheodory. In order to connect them with a global approach to the integration method of Hamilton and Jacobi, we introduced and discussed the conception of homogeneous systems.
2.1.4 Complete integrability

In this section we climb onto the next level of integrability - the complete integrability of a Hamiltonian system \((\Sigma, \Omega, H)\). This property is crucial, because if the associated Hamiltonian system \((T^*M, \Omega, F)\) of an optimal control problem \((K, \mathcal{B}, L, P)\) is in addition to its integrability completely integrable, then its analytical and/or numerical integration is reducible to half of the dimensions.

**Definition 2.19 : Complete Integrability.** Let \((\Sigma, \Omega, H)\) be a Hamiltonian system with a 2q-dimensional symplectic manifold \(\Sigma\) a nondegenerate, closed two-form \(\Omega \in \Lambda^2(\Sigma)\) and a Hamiltonian function \(H \in \mathcal{F}(\Sigma)\). Then the system is said to be completely integrable if it has \(q\) differentiable first integrals \(f_1, \ldots, f_q \in \mathcal{F}(\Sigma)\) defined on the whole manifold \(\Sigma\), which are pairwise in involution, \(\{f_i, f_j\} = 0\), and whose differentials are linearly independent on a dense open subset of the manifold.

We quote a strong version of Darboux's theorem. Let \((\Sigma, \Omega)\) be a 2q-dimensional symplectic manifold with \(q\) functions \(f_1, \ldots, f_q \in \mathcal{F}(\Sigma)\) in involution. Then about each point of \(\Sigma\) there is a neighborhood \(U\) in which it is possible to find \(q\) other functions \(g_1, \ldots, g_q \in \mathcal{F}(\Sigma)\) which are in involution and have the property that \(\Omega|_U = df_i \wedge dg^i\). (2.1.55)

From this it follows that \(\{f_j, g^k\} = \delta^k_j\). The \(2q\) functions \(f_k, g_k \in \mathcal{F}(\Sigma)\) provide a symplectic chart in \(U\), sometimes referred to as a Heisenberg coordinate system. The differentiable generalized distribution generated by the vector fields \(\#df_1, \ldots, \#df_q\) is invariant under the family of diffeomorphism generated by these vector fields. Sussmann's theorem [Sussmann [1], p. 171] shows that this generalized distribution is completely integrable, and therefore defines a generalized foliation of \(\Sigma\). Through each point \(p\) of \(\Sigma\) passes a unique leaf of this generalized foliation, whose dimension is equal to the rank of the generalized distribution at that point. Each trajectory of \(\#dH\), or any one of the vector fields \(\#df_1, \ldots, \#df_q\), is wholly contained in one of these leaves. The restriction of this generalized foliation to the dense open subset \(U\) on which \(\#df_1, \ldots, \#df_q\) are linearly independent is a Lagrange foliation of \(U\), whose leaves are closed Lagrange submanifolds of \(U\) defined by equations of the form \(f_k = \text{const}_k, 0 \leq k \leq q\).

For constructional reasons we state the following theorem which appeared early
in the history of the theory of dynamical systems and their integration methods.

**Theorem 2.20 : Completion.** Let \((\Sigma, \Omega)\) be a symplectic manifold and \(p\) a point in \(\Sigma\). If \(f_a, a = 1, \ldots, h\) is a system of functionally independent functions in involution on a neighborhood \(U\) of \(p\), then \(h \leq n\) and if \(h < n\), the system can be completed by \(f_{h+1}, \ldots, f_n\) to a system of \(n\) functionally independent functions in involution on a certain smaller neighborhood \(V\), where \(p \in V \subset U\).

**Proof:** The functions \(f_a\) in question are functionally independent if and only if \(df_a \in \Lambda^1(\Sigma)\) are linearly independent. Let \(f_a, a = 1, \ldots, h\) be the given system in involution, and let us look at the local foliation \(\{\Phi_U\}\) defined by \(f_a = \text{const.}\). Let \(#df_a \in T\Sigma\) be the symplectic gradients of \(f_a\) which are linearly independent vector fields at every point \(p\) in \(U\). If \(Y \in T\Sigma\) is a tangent vector field of \(\{\Phi_U\}\), then the \(a\) equalities ensue

\[
\Omega(#df_a, Y) = 0, \quad \text{(2.1.56)}
\]

from which it follows that \(#df_a\) span the skeworthogonal distribution \(\phi\Delta\). On the other hand, we have

\[
\Omega(#df_a, #df_b) = L_{#df_a}f_b = df_b(#df_a) = 0, \quad \text{(2.1.57)}
\]

from which \(#df_a \in T\Phi_U\). It follows that \(\{\Phi_U\}\) is a coisotropic foliation of codimension \(h \leq n\). The foliation generated by the distribution \(\phi\Delta\) is a subfoliation of \(\{\Phi_U\}\). If \(h < n\), there will be at least one more first integral \(g\) of \(\phi\Delta\) defined on a neighborhood \(V, p \in V \subset U\), independent of the first integrals \(f_a\). This function satisfies

\[
dg(#df_a) = #df_a(g) = \Omega(#df_a, #dg) = 0. \quad \text{(2.1.58)}
\]

Hence \(f_1, \ldots, f_h, f_{h+1} = g\) is again a larger system of functionally independent functions in involution. The same procedure can be repeated until we arrive at \(f_n\).

This theorem goes back to the work of Jacobi and Caratheodory. qed

In order to see the topological and computational consequences of the complete integrability property we refer to Liouville's theorem [Libermann, p.167].

In order to avoid the application of the implicit function theorem to the evaluation of the vanishing semi-basic 1-form \(a \in \Lambda^1(P)\) [Section 1.1.2] for the verification of the integrability property, we have to state the following correspondence.
Lemma 2.21: Reduction invariance of complete integrability. If a completely integrable projectable Hamiltonian system \((\Sigma, \Omega, F)\) has a complete involutive set of first integrals \(\{f_1, \ldots, f_m, f_{n+m}, f_{n+m+1}\}\), then the restriction of \(n+1\) of these first integrals to the embedded submanifold \(P\) of codimension \(m\) in \(\Sigma\) constitutes a complete involutive set of first integrals for the projected Hamiltonian system \((P, \omega, H)\).

**Proof:** On the \(2n + 2 + m\)-dimensional submanifold \(P\) of \(\Sigma\) there are \(m\) first integrals with constant value and the remaining integrals have linearly independent differentials spanning a rank \(2n + 2 + m\) tensor. The restriction of these functions to \(P\) leaves the involutivity invariant, i.e. with the canonical injection \(t : P \rightarrow \Sigma\) there is

\[
\{t^* f, t^* g\} = t^* \{f, g\} = 0 . \quad \text{qed} \quad (2.1.59)
\]

This lifting property is already applied in Theorem 2.12.

Theorem 2.22: Optimal state-feedback controllability (direct). A plant \(P\) in an optimal control problem \((K, B, L, P)\) is globally feedback controllable, if and only if for the partially Hamiltonian vector field there exists \(n + 1 + m\) functionally independent first integrals \(\{f_1, \ldots, f_m, f_{n+1}\} = \overline{G}, \overline{G} : P \rightarrow \mathbb{R}^{n+1}\) constituting a complete involutive set, and if \(\overline{G}^{-1}(c) \subset P\) is a submanifold transversal to the vertical foliation generated by \(VP\).

**Proof:** In order that the codimensional \(n+1\) submanifold \(\overline{G}^{-1}(c) \subset P\) is the carrier of the optimal pseudo-strategy it is necessary and sufficient that it is diffeomorphic to \(M\). The intersection \(\overline{G}^{-1}(c) \cap \hat{E}_{opt} = L'\) is a Lagrange submanifold of \((S, \Omega_s)\). The \(n+1\) first integrals are formally

\[
\hat{f}_1, \ldots, \hat{f}_{n+1}(x, t, u, \lambda, \lambda_t) = \text{const}_{1 \ldots n+1}, \quad (2.1.60)
\]

upon which, under the regularity assumption corresponding to the required transversality, the application of the implicit function theorem would lead to the explicit expression

\[
\lambda = Q(x, t, u; \text{const}_{1 \ldots n+1}) , \quad \lambda_t = Q_t(x, t, u; \text{const}_{1 \ldots n+1}). \quad (2.1.61)
\]

The left-hand equation together with the setup \(u(t) = F(x(t), t)\) plugged into equation (2.1.20) fully determines the technical realization of an optimal state-feedback controller. **qed**
Example 2.23: *LQ-regulator.* Since the Lie algebra \( \text{sp}(2q,\mathbb{R}) \) of the linear symplectic transformation group \( \text{Sp}(2q,\mathbb{R}) \) is simple and of rank \( n \) it follows that a linear Hamiltonian system on \( \mathbb{R}^{2q} \) has \( n \) linearly independent quadratic first integrals. The underlying Hamiltonian function \( F \) of a time-dependent linear plant
\[
\dot{x}(t) = A(t)x(t) + B(t)u(t)
\]
is a quadratic function
\[
\hat{F} = \frac{1}{2} \{ x^TQx + u^TRu + \lambda K\lambda^T + \rho L\rho^T \} + \lambda Ax + \lambda Bu + \lambda C\rho^T + u^TMx + u^TN\rho^T + \rho Sx.
\]
\[(2.1.62)\]
\[(2.1.63)\]
Thus the underlying Hamiltonian system of the *LQ-problem*
\[
\begin{align*}
\dot{x}(t) &= \frac{\partial \hat{F}}{\partial \lambda}; \quad \dot{x} = Ax + K\lambda^T + Bu + C\rho^T, \\
\dot{u}(t) &= \frac{\partial \hat{F}}{\partial \rho}; \quad \dot{u} = L\rho^T + C^T\lambda^T + N^Tu + Sx, \\
\dot{\lambda}(t) &= -\frac{\partial \hat{F}}{\partial x}; \quad \dot{\lambda}^T = -Qx - A^T\lambda^T - M^Tu - S^T\rho^T, \\
\dot{\rho}(t) &= -\frac{\partial \hat{F}}{\partial u}; \quad \dot{\rho}^T = -Ru - B^T\lambda^T - Mx - N\rho^T,
\end{align*}
\]
\[(2.1.64)\]
\[(2.1.65)\]
\[(2.1.66)\]
\[(2.1.67)\]
where the matrices \( C, N \) and \( S \) are given by
\[
C^T = R^{-1}B^TA^T; \quad N^T = -R^{-1}\hat{R}; \quad S = R^{-1}B^TQ,
\]
\[(2.1.68)\]
omitting the dynamic of the cotime.

Since in the case of vanishing matrices \( K \) and \( M \), for an arbitrary matrix \( L \) the integrability structure of the problem remains invariant, every *LQ-problem* has, under the assumption of controllability of the plant, a regular feedback control law. Solving (4.19) as a linear inhomogeneous partial differential equation and using the standard setting \( \lambda(t) = P(t)x(t) \) the linearity of the regulator appears immediately. For another approach to the linear case we refer to [Nakamura, p. 2].

Example 2.24: Let us consider again the optimal control problem \((K, \alpha, L, P)\) given in Example 1.25. The adjoint equation, as representation of a part of the Hamiltonian vector field of optimal control, takes the form
\[ \dot{\lambda}_r(t) = -\Omega_{rs} x^s - \frac{1}{2} \Pi_{rst} x^s x^t - \frac{1}{2} \Sigma_{rstv} x^s x^t \dot{x}^v - A^s_t \dot{\lambda}_s(t) - \overline{A}^s_r x^t \dot{\lambda}_s(t), \]  

(2.1.69)

with the end condition

\[ \lambda_r(t_1) = \overline{\Omega}_{rs}(t_1) x^s + \overline{\Pi}_{rjk}(t_1) x^k. \]  

(2.1.70)

The minimum of the Hamiltonian function with respect to the control is attained in the interior of the control manifold B, i.e.

\[ \theta = \frac{\partial \mathcal{H}}{\partial u} \rightarrow \theta = R_{mn} u^n + B^r_m \dot{\lambda}_r. \]  

(2.1.71)

In order to solve the equation (2.1.69), we suggest the costate variable to be of the form

\[ \lambda_j(t) = P_{ji}(t) x^i(t) + \frac{1}{2} Q_{jkl}(t) x^k(t) x^l(t), \]  

(2.1.72)

with both tensors being of full rank. If in the next step this is plugged into the equation (2.1.71), then by comparative argument of the order, we realize one way of determining \( P_{ji}(t) \) and \( Q_{jkl}(t) \), which become identical to the solutions of the two differential equations and the algebraic Lyapunov-type equation

\[ \dot{P} + PA + A^T P + \Omega - PB R^{-1} B^T P = 0, \]  

(2.1.73)

\[ \dot{Q} + PA + 2A^T P + AQ + 2QA + \Pi - PB R^{-1} B^T Q - 2QB R^{-1} B^T P = 0, \]  

(2.1.74)

\[ Q (\overline{A} - BR^{-1} B^T Q) + A^{-1} Q + \Sigma = 0, \]  

(2.1.75)

with the end conditions

\[ P(t_1) = \overline{\Omega} \quad Q(t_1) = \overline{\Pi}. \]  

(2.1.76)

For notational reasons, the quantities of order 3 are not printed in bold face.

**Summary of Section 2.1.4:**

Complete integrability, from a historical viewpoint, is a structural property which is necessary for the computation of the trajectories of a dynamic system in Hamiltonian layout. It unifies the property of transversality to the vertical foliation and the integrability introduced in an earlier section.
3. Hamilton-Jacobi Integration Technic

3.1 Global approach to Hamilton-Jacobi integration theory

A Hamilton-Jacobi partial differential equation is regarded here as a geometric object which is naturally visualized by a homogeneous problem \((P, \omega; G)\), whereas a parametrized Hamilton-Jacobi partial differential equation has \((P, \omega; G)\) as an associated homogeneous problem. Four different types of solutions occur - the standard, the thin, the complete and the thin complete solution.

3.1.1 Standard solution to the global setup

The presymplectic manifold \((P, \omega)\) is the total space of a cotangent vector sub-bundle of \(T^*M\) [Chapter 1, Section 1.1.2]. The homogeneous problem \((P, \omega; G)\) specifies functions \(s^{(\alpha)} \in \mathcal{F}(U_\alpha), U_\alpha \subset M, U_\alpha \in \mathcal{A}_M\), which generate a Lagrange submanifold \(L\) in \(T^*M\). These functions are assumed to satisfy the compatibility condition \(v^{(\alpha)} = v^{(\beta)}\) on \(U_\alpha \cap U_\beta\).

**Definition 3.1**: Standard solution. A set of functions \(\{v^{(\alpha)}| v^{(\alpha)} \in \mathcal{F}(U_\alpha)\}\), where \(U_\alpha\) is an open subset of \(M\), is called a solution of the Hamilton-Jacobi partial differential equations associated with the homogeneous problem \((P, \omega; G)\) if it generates an isotropic submanifold \(I\) of the cotangential bundle \((T^*M, \Omega)\) such that

\[
\overline{d}v^{(\alpha)}(U_\alpha) \subseteq G.
\]

The isotropic submanifolds of \(T^*M\) generated in this way are sections of the projections \(\tau^*_M\) so that these solutions exist locally only if the projection of \(G\) by \(\tau^*_M\) is sufficiently regular [Lions [79], p. 23]. We will treat this problem in the next section. If \((T^*U, T^*\phi(p) = (z, \bar{z}, t, \xi, \xi_{\bar{z}}, \xi_t))\) are canonical coordinates of \(T^*M\), then the isotropic submanifold \(I\) [Weinstein [50], p. 4] generated by the set of compatible functions \(\{v^{(\alpha)}| v^{(\alpha)} \in \mathcal{F}(U_\alpha)\}\) is locally defined by the equations...
\begin{equation}
\xi^{(\alpha)} = \frac{\partial s^{(\alpha)}}{\partial z} ; \xi_t^{(\alpha)} = \frac{\partial s^{(\alpha)}}{\partial t} .
\end{equation}

When \( G \) is locally represented by the equations
\begin{equation}
H(z, \ddot{z}, t, \xi, \xi_t) = 0 ,
\end{equation}
then the Hamilton-Jacobi partial differential equations gives rise to the equation
\begin{equation}
H\left( z, \ddot{z}, t, \frac{\partial s^{(\alpha)}}{\partial z}, \frac{\partial s^{(\alpha)}}{\partial t} \right) = 0 ,
\end{equation}

In the case where the manifold \( P^+ \) is \( 2(n+1+m) \) dimensional, the exterior derivation \( d \) in equation (5.1) becomes identical with the ordinary one \( (d) \) and the isotropic submanifold \( I \) is Lagrangian in the sense of Chapter 2, Section 2.1.1.

**Example 3.2**: By applying the Hamilton-Jacobi theory to an optimal open-loop control problem \( (K, \{ p_0 \} \times B_1, L, P) \) it is possible to derive first-order necessary and sufficient conditions in a way which, to our knowledge, has been overlooked so far. In what follows, we assume the existence of a globally defined Lagrange-function \( L \in \mathcal{F}(M) \) with compact support on \( M \) and a function \( K_1 \in \mathcal{F}(B_1) \) on the set of all endpoints of integral trajectories in \( M \). Let us consider again the evaluated integral
\begin{equation}
\gamma^* v = \gamma^* K_1 + \int_{\gamma^* t^* L} d\tau , \quad [t_0, t_1] \subset \mathbb{R} ,
\end{equation}
with \( \gamma : \mathbb{R} \rightarrow M ; \hat{\gamma}(t) = (x(t), t, u(t)) , t : \mathcal{C} \rightarrow M , c = im \gamma \in \mathcal{C} \) and the pullback \( \gamma^* : \Lambda^0(M) \equiv \mathcal{F}(M) \rightarrow \Lambda^0(\mathbb{R}) \), represented in local coordinates by
\begin{equation}
\hat{\gamma}(x(t), u(t), t) = \hat{K}_1(x(t_1), t_1) + \int_{t_1}^{t_1} \hat{L}(x(\tau), u(\tau), \tau) d\tau .
\end{equation}

Instead of applying methods from variational calculus due to Euler and Lagrange [Bolza, p. 580], the function \( v \in \mathcal{F}(U) \) shall be minimized with respect to the free parameters, i.e. the controls. The total time derivative of the equation (3.1.5), or equivalently the Lie-derivative of \( v \in \mathcal{F}(U) \) in the direction tangent to the integral curve \( c = im \gamma , c \subset M \) expressed in coordinate-free language is
\[ L_c v(c_0) = -L(c_0) \]  

(3.1.7)

and pulled back to \( \mathbb{R} \) by \( \gamma^* \), it is in local coordinates of the form

\[
\frac{\partial \tilde{\psi}}{\partial x} \dot{x} + \frac{\partial \tilde{\psi}}{\partial u} \dot{u} + \frac{\partial \tilde{\psi}}{\partial t} \dot{t} = -\hat{L}(x(t), u(t), t) + \frac{\partial \tilde{\psi}}{\partial u} = 0 \quad \forall t \in [t_0, t].
\]  

(3.1.8)

The equation (3.1.3) describes a codimensional, globally defined submanifold \((H)^{-1}(0)\) and is thus the Hamilton-Jacobi partial differential equation of open-loop optimal control. A solution of the collection of Hamilton-Jacobi partial differential equations associated with the homogeneous problem \((P, \omega, G)\) is a set of functions \( \{ v^{(\alpha)} \mid v^{(\alpha)} \in \mathcal{F}(U_\alpha) \} \), \( U_\alpha \subseteq M \) such that

\[
F[\tilde{\psi}^{(\alpha)}(U)] \subseteq G.
\]  

(3.1.9)

The exterior, partially disappearing derivative [Section 1.1.2] of the equation (3.1.7) gives

\[
\partial [L_c v(c_0) + L(c_0)] = 0,
\]  

(3.1.10)

which is evaluated in local coordinates by

\[
\frac{\partial^2 \tilde{\psi}}{\partial u \partial x} \dot{x} + \frac{\partial f}{\partial u} \frac{\partial \tilde{\psi}}{\partial x} + \frac{\partial^2 \tilde{\psi}}{\partial u^2} \dot{u} + \frac{\partial^2 \tilde{\psi}}{\partial u \partial t} \dot{t} = -\frac{\partial \hat{L}}{\partial u}, \quad \frac{\partial^2 \tilde{\psi}}{\partial u^2} = 0, \quad \frac{\partial^2 \tilde{\psi}}{\partial u \partial x} = 0.
\]  

(3.1.11)

Finally, this results in the expression

\[
\frac{\partial \hat{H}}{\partial u} = \frac{\partial^2 \tilde{\psi}}{\partial u \partial t} \dot{t} = -\frac{\partial \hat{L}}{\partial u} - \frac{\partial f}{\partial u} \frac{\partial \tilde{\psi}}{\partial x} = 0.
\]  

(3.1.12)

In a Lagrangian derivation of necessary conditions for extremality the last equation follows naturally by considering the affine Pontryagin-function to be

\[
\hat{H}(x(t), u(t), \lambda(t), t) = \hat{L}(x(t), u(t), t) + \lambda(t)f(x(t), u(t), t).
\]  

(3.1.13)

In formula (3.1.12) appears the local representation of the one-form \( a \in \Lambda^1(P) \) in Theorem 1.16, Section 1.1.2 of Chapter 1.

**Example 3.3**: The nonvanishing Hamiltonian function \( F \in \mathcal{F}(T^*M) \) introduced in Section 1.2.1 generates a Hamiltonian vector field which is associated to an optimal control problem \((K, B, L, P)\). Its inverse function \( F^{-1} : \mathbb{R} \rightarrow T^*M \) determines under regularity assumptions submanifolds of \( T^*M \) of codimension
one and thus a homogeneous problem in a symplectic manifold. A standard solution has to satisfy

\[ d\nu(U) \subseteq F^{-1}(I) \], for an interval \( I \subset \mathbb{R} \). \hspace{1cm} (3.1.14)

Though, the cocontrol variable may be locally identified with the partial derivative of the function \( \hat{s} \) with respect to \( u \), i.e.

\[ \rho = \frac{\partial \hat{s}}{\partial u} \]. \hspace{1cm} (3.1.15)

According to the conception demonstrated in Chapter 2, the Hamilton-Jacobi integration method can be comfortably applied to suboptimal control.

**Summary of Section 3.1.1**:

The standard solution to a homogeneous problem is defined and illustrated. This conception allows a derivation of the necessary conditions of open-loop control in the sense of Caratheodory and has a great impact on developing effective computational algorithms and canonical representations. The standard solution also closes the theoretical loop in theory to the Hamiltonian vector field associated to the optimal control problem as it was introduced in Section 1.2.1.
3.1.2 The thin solution to the global setup

A thin solution shall be defined as a smooth, specific section in the cotangent bundle $T^* (N \times R), \tau_{N \times R}, N \times R, R^{n+1}$. The manifold $T^* (N \times R)$ is an embedded $\tau (T^* N) \subset T^* M$ submanifold of the total space $T^* M$ of the cotangential bundle over $M$.

**Definition 3.4**: Thin solution. A set of functions $\{ V^{(\alpha)}| V^{(\alpha)} \in \mathcal{F}(U_\alpha) \}$, where $U$ is an open connected subset of $N \times R$, is called a thin solution of the system of Hamilton-Jacobi partial differential equations associated with the homogeneous problem $(P, \omega; \tilde{G})$, if it generates a Lagrangian submanifold $L$ of the symplectic manifold $(T^* (N \times R), \Omega_N)$, such that

$$t [dV^{(\alpha)}(U)] \subseteq \tilde{G}. \quad (3.1.16)$$

If $(T^* U, T^* \phi (p, \theta)) = (z, t, \xi, \xi_t)$ are canonical coordinates of $T^* (N \times R)$, then the Lagrangian submanifold $L$ [Weinstein [56], p. 3] generated by the set of compatible functions $\{ V^{(\alpha)}| V^{(\alpha)} \in \mathcal{F}(U_\alpha) \}$ is defined by the equations

$$\xi^{(\alpha)} = \frac{\partial \hat{V}^{(\alpha)}}{\partial z}; \xi_t^{(\alpha)} = \frac{\partial \hat{V}^{(\alpha)}}{\partial t}. \quad (3.1.17)$$

When $\tilde{G}$ is locally, implicitly represented by the equation

$$\tilde{H} (z, t, \xi, \xi_t) = 0 \iff \tilde{G} \subset (H)^{-1} (0) \cap \tilde{E} \quad (3.1.18)$$

The Hamilton-Jacobi equations give rise to the equation

$$\tilde{H} \left( z, t, \frac{\partial \hat{V}^{(\alpha)}}{\partial z}, \frac{\partial \hat{V}^{(\alpha)}}{\partial t} \right) = 0. \quad (3.1.19)$$

**Example 3.5**: The Hamilton-Jacobi method of integration appears in optimal closed-loop control as well. Let there be the function $L \in \mathcal{F}(M)$ considered on an open subset $U$ of $M$ such that for an open connected subset $V$ of $N \times R$ there is $\pi (U) \subset V$. Furthermore, there is the mathematical realization of an optimal feedback control law, i.e. the section $F : N \times R \to M$ and $F (V) \supset U$. Then there are two functions $R, V \in \mathcal{F}(N \times R)$ is defined by

$$R = K_0 (B_0) + F^* \int_c L|_{U}, \quad V = K_1 (B_1) + F^* \int_{\tilde{c}} L|_{U}, \quad c = \zeta + \tilde{c}. \quad (3.1.20)$$
which are called the cost function and the cost-to-go function [Athans, p.88], [Edelen, p. 381] respectively. In the following we compare two slightly different appearances of the Hamilton-Jacobi method in optimal control. The difference is the key stone in the inverse problem of optimal control [Kalman, p. 52], [Jameson, p. 1]. This fact becomes even more clear by Example 3.12, formula (3.1.83). In local coordinates the evaluation of the cost-to-go function on an interval \((t, t_1] \subset \mathbb{R}\) is given by

\[
\hat{V}(x(t), t) = \tilde{K}_1(x(t_1), t_1) + \int_t^{t_1} \hat{L}(x(\tau), F\begin{\{x(\tau), \tau\}\}, \tau) \, d\tau ,
\]

(3.1.22)

\[
\vec{c} = \text{im} \, \gamma(\, t, t_1 \, ) , \, t_0 < t \leq t_1 ,
\]

(3.1.23)

on \(G \cap \tilde{E} \_\text{opt}\). Since the knowledge of the optimal control strategy \(E_{opt}\) is either of a priori type or dependent on the optimal pseudo strategy \(\tilde{E}_{opt} \subset P\) we consider in the forward problem the optimal pseudo-feedback control law \(\tilde{F} : T^*N \times \mathbb{R} \to M\). Then, on an open subset \(U' \subset \tilde{F}(\tau_{opt}^*(V') \times I)\) there is

\[
V = K_1(B_1) + \tilde{F}^* \int_{\tilde{G}} L|_{U'} ,
\]

(3.1.24)

on \(\tilde{G} \subset G \cap \tilde{E}_{opt}\) with local representation

\[
\hat{V}(x(t), t) = \hat{K}_1(x(t_1), t_1) + \int_t^{t_1} [\hat{L}(x(\tau), \tilde{F}\{x(\tau), \tau, \lambda(\tau)\}, \tau) \, d\tau .
\]

(3.1.25)

The cost-to-go function defined in formula (3.1.20) is a thin solution of the homogeneous problem \((P, \omega; G)\), where the submanifold \(G\) is, according to Section 2.1.3, determined by the surjective submersion

\[
H : U \to \mathbb{R} ,
\]

(3.1.26)

for an open subset \(U \subset P\). This fact is expressed, under the assumption of connectedness of \(U\), by the Lie-derivative of the equality (3.1.20) in the direction of the minimal curve, i.e.

\[
\mathcal{L}_{\pi_*(\delta)} V = -\tilde{F}^* L \iff H|_{\delta} = 0 .
\]

(3.1.27)
The thin solution of this problem is determined locally, in adapted coordinates \((U, \phi(p) = (x, t, \tilde{F}(x, t, \lambda)))\), by the highly nonlinear partially differential equation

\[
-\frac{\partial \tilde{V}}{\partial t} = \dot{L}(x, \tilde{F}(x, t, \frac{\partial \tilde{V}}{\partial x}), t) + \frac{\partial \tilde{V}}{\partial x} f(x, \tilde{F}\{x, t, \frac{\partial \tilde{V}}{\partial x}\}, t) \tag{3.1.28}
\]

The equation (3.1.28) is based on the identification in the sense of Caratheodory

\[
\lambda(t) = \frac{\partial \tilde{V}}{\partial x} (x(t), t), \quad \lambda_t(t) = \frac{\partial \tilde{V}}{\partial t} (x(t), t). \tag{3.1.29}
\]

In the case of an \(H\)-minimizing pseudo-feedback control law the equation (3.1.28) is called the Hamilton-Jacobi-Bellmann partial differential equation of nonlinear optimal control.

The Hamilton-Jacobi equation for the inverse problem is easily determined on the manifold \(G \cap E\). The exterior partially disappearing derivative of this equation gives

\[
\tilde{d} [L \, V = -F^*L] = 0, \tag{3.1.30}
\]

which is evaluated in local coordinates

\[
\frac{\partial^2 \tilde{V}}{\partial x \partial t} = -\frac{\partial \tilde{L}}{\partial x} - \frac{\partial \tilde{L}}{\partial u} \frac{\partial F}{\partial x} - \frac{\partial^2 \tilde{V}}{\partial x^2} f - \frac{\partial \tilde{V}}{\partial x} \frac{\partial f}{\partial x} - \frac{\partial \tilde{V}}{\partial x} \frac{\partial f}{\partial u} \frac{\partial F}{\partial x}, \tag{3.1.31}
\]

where a similar expression of the one-form \(a \in \Lambda^1(P)\) appears. For a stationary curve, we may split equation (3.1.31) into

\[
\frac{\partial^2 \tilde{V}}{\partial x^2} \dot{x} + \frac{\partial^2 \tilde{V}}{\partial x \partial t} = -\frac{\partial \tilde{L}}{\partial x} - \frac{\partial \tilde{V}}{\partial x} \frac{\partial f}{\partial x}, \text{ and } \left[\frac{\partial \tilde{L}}{\partial u} + \frac{\partial \tilde{V}}{\partial x} \frac{\partial f}{\partial x}\right] \frac{\partial F}{\partial x} = 0. \tag{3.1.32}
\]

With the parallel treatment of two thin solutions corresponding to equivalent optimal control problems, we try to illustrate the conception of thin solutions with maximal clarity, especially since it turns out that a global analysis of optimal control requires a clear understanding of the thin solutions.

**Theorem 3.6**: Relationship standard solution \(\iff\) thin solution. If the standard solution \(\{v^{(\alpha)} | v^{(\alpha)} \in F(V_\alpha)\}\) of a homogeneous problem \((P, \omega; G)\) is defined on an open subset of \(M\) such that \(V_\alpha \subset F(U_\alpha)\), then the thin solution

\[69\]
\{ V^{(\alpha)} | V^{(\alpha)} \in \mathcal{F}(U_\alpha) \} \text{ is composable by } 
V^{(\alpha)} = v^{(\alpha)} \circ F \iff v^{(\alpha)}|_{E_{\text{opt}}} = V^{(\alpha)} \tag{3.1.33} 

where there are the maps \( v^{(\alpha)} : V_\alpha \to \mathbb{R} \), \( F : N \times \mathbb{R} \to M \) and \( V^{(\alpha)} : U_\alpha \to \mathbb{R} \).

The most general disposition of homogeneous problems includes, from the point of view of partial differential equation of Hamilton-Jacobi type, the initial- and/or end conditions. It provides thus a setup for optimal control with vector-valued integrals. Furthermore, the symmetry vector fields generated by the first integrals may be considered in this context as characteristic vector fields to this kind of optimal control problem.

**Example 3.7** : *LQ-regulator and canonical transformation.* There is a degenerate case of a thin solution which shall be illustrated by a suboptimal control problem \((k, A_0 \times A_1, l, E)\) with a linear/quadratic, time-dependent local representation given by

\[
x(t_0) = x_0, \dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{3.1.34}
\]

\[
\lambda(t_1) = \lambda_1, \dot{\lambda}^T(t) = -Q(t)x(t) - A^T(t)\lambda^T(t) - M^T(t)p^T(t), \tag{3.1.35}
\]

\[
u(t_0) = u_0, \dot{u}(t) = M(t)x(t) + N(t)u(t), \tag{3.1.36}
\]

\[
\rho(t_0) \neq 0, \quad \dot{\rho}^T(t) = -R(t)u(t) - N^T(t)p^T(t) - B^T(t)\lambda^T(t), \tag{3.1.37}
\]

\[x \in \mathbb{R}^n, \lambda \in \mathbb{R}^{*n}, u \in \mathbb{R}^m, \rho \in \mathbb{R}^{*m}, \lambda_1^T = K(t_1)x(t_1), R = R^T, \tag{3.1.38}
\]

\[A, Q, K \in \mathbb{R}^{n \times n}, N, M, R \in \mathbb{R}^{m \times m}, B \in \mathbb{R}^{n \times m}, Q = Q^T, K = K^T, \tag{3.1.39}
\]

with the Hamiltonian function \( \hat{h} \in \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^{*n} \times \mathbb{R}^m \times \mathbb{R}^{*m} \times \mathbb{R}) \),

\[
\hat{h} = \frac{1}{2}x^T(t)Q(t)x(t) + \frac{1}{2}u^T(t)R(t)u(t) + \lambda(t)[A(t)x(t) + B(t)u(t)] + \rho(t)[M(t)x(t) + N(t)u(t)]. \tag{3.1.40}
\]

Our goal is the computation of a suboptimal feedback cocontroller such that after a given time period the Hamiltonian equations are driven into the necessary conditions associated to a familiar optimal control problem \((K, B, L, P)\). Hence, the gradient of the boundary function \( k \) on \( A_1 \) in the direction of the fiber \( B \) has to be zero, i.e. locally there is
\[ \rho (t_1) = \frac{\partial k}{\partial u} (x(t_1), t_1, u(t_1)) = 0. \]  

(3.1.41)

The smooth solution

\[ \hat{s} = \frac{1}{2} y^T P_1 (t) y + v^T P_2 (t) y + \frac{1}{2} v^T P_4 (t) v + \lambda x + \rho u + \mu y + \sigma v, \]  

(3.1.42)

\[ P \in \mathbb{R}^{m \times m}, \quad P^T = P \]  

(3.1.43)

of the Hamilton-Jacobi-equation

\[ \hat{h}(x, \frac{\partial s}{\partial x}, u, \frac{\partial s}{\partial u}, t) + \frac{\partial s}{\partial t} = \hat{h}^+(y, \frac{\partial s}{\partial y}, v, \frac{\partial s}{\partial v}, t, \frac{\partial s}{\partial t}) \]  

(3.1.44)

generates a canonical transformation

\[ [\lambda dx + \rho du] - [\mu dy + \sigma dv] = d\hat{s}. \]  

(3.1.45)

The corresponding time-dependent, block-symplectic transformation \( T \) (fiber-morphism, \( (T^* M, \tau^*, \wedge^* M, R^{n+m^*}) \)),

\[
\begin{bmatrix}
    y \\
    \nu \\
    \mu \\
    \sigma \\
\end{bmatrix} =
\begin{bmatrix}
    I & 0 & 0 & 0 \\
    0 & I & 0 & 0 \\
    -P_1 (t) & -P_2 (t) & I & 0 \\
    -P_2^T (t) & -P_4 (t) & 0 & I \\
\end{bmatrix}
\begin{bmatrix}
    x \\
    u \\
    \lambda \\
    \rho \\
\end{bmatrix}
= \begin{bmatrix}
    I & 0 & 0 & 0 \\
    0 & I & 0 & 0 \\
    P_1 (t) & P_2 (t) & I & 0 \\
    P_2^T (t) & P_4^T (t) & 0 & I \\
\end{bmatrix}
\begin{bmatrix}
    y \\
    \nu \\
    \mu \\
    \sigma \\
\end{bmatrix}
\]  

(3.1.46)

modifies the equations (3.1.34)-(3.1.37) into

\[ y(t_0) = x_0, \mu(t_1) = \lambda_1, v(t_0) = u_0, \sigma(t_0) = \rho(t_0), \sigma(t_1) = 0 \]  

(3.1.47)

\[ \dot{y}(t) = A(t) y(t) + B(t) v(t), \]  

(3.1.48)

\[ \dot{v}(t) = M(t) v(t) + N(t) v(t), \]  

(3.1.49)

\[ \dot{\mu}(t) = -[\dot{P}_1 + P_1 A + A^T P + Q + P_2 M + M^T P_2^T] (t) y(t) + \]  

\[ -[\dot{P}_2 + P_1 B + P_2 N + A^T P_2 + M^T P_4] (t) v(t) + \]  

\[ -M^T (t) \sigma^T (t) - A^T (t) \mu^T (t) \]  

(3.1.50)
\[ \dot{\sigma}^T(t) = -[\dot{P}_2^T + P_2^T A + P_4 M + N^T P_2^T + B P_1] y(t) - [\dot{P}_4 + P_4 N + N^T P_4 + R + P_2 B + B^T P_2^T] \nu(t) - N^T(t) \sigma^T(t) - B^T(t) \mu^T(t) \]  

(3.1.51)

If the matrix \( P_2 \) is set to be zero, in other words if there is possibly a clean feedback, then the following two aspects can be demonstrated. First there is the vanishing cocontrol, \( \dot{\sigma}(t_1) = \sigma(t_1) = 0 \), and thus \( P_4 = 0 \). As a result the optimal closed-loop control law plugged into equation (3.1.50) yields

\[ \mu^T(t_1) = -[\dot{P}_1 + P_1 A + A^T P_1 + Q - P_1 B R^{-1} B^T P_1] y(t_1) - A^T \mu^T(t_1) \]

(3.1.52)

Second, under the assumption that the equality \( M^T P_4 = -P_1 B \) holds true for all \( t \) in the given interval, the last two equations of the transformed system of Hamiltonian equations are of the form

\[ \mu^T(t) = -Q(t) y(t) - M^T(t) \sigma^T(t) - A^T(t) \mu^T(t) , \]

(3.1.53)

\[ \sigma^T(t) = -R(t) \nu(t) - N^T(t) \sigma^T(t) - B^T(t) \mu^T(t) . \]

(3.1.54)

**Summary of Section 3.1.2 :**

The thin solution of the homogeneous problem \( (P, \omega; \tilde{G}) \) is examined. In optimal control the solution is realized by a set of cost-to-go functions satisfying a compatibility assumption. The relationship between the thin solution and the standard solution is examined. It is the integral version of the interdependence of the forward optimal control problem and its inverse.
3.1.3 Complete solution

Since function families [Courant II, p. 7] cannot describe first-order partial differential equations of higher complexity, we have to introduce the complete integrals [Courant II, p. 19]. We will extract the complete integral in the case of optimal control in Example 3.12. First, let us consider a locally trivial fiberbundle \( a = (A, \pi, N' \times R, B') \) and a fiberbundle morphism \((F, G)\) onto \( m = (M, \pi, N \times R, B)\), where there are the maps \( F : A \to M \) and \( G : N' \times R \to N \times R \). On the cartesian product \( A \times M \), with projections \( pr_1 : A \times M \to A \) and \( pr_2 : A \times M \to M \), there is an isomorphism between the cotangent bundles, \( T^* (A \times M) \cong T^* A \times T^* M \). A complete solution is a specific section in the cotangent bundle over the cartesian product \( A \times M \) and it is defined as follows.

**Definition 3.8 : Complete solution.** A set of functions \( \{ s^{(\alpha)} | s^{(\alpha)} \in J(V_\alpha) \} \), where \( V_\alpha \) are open connected subsets of \( A \times M \), is called a complete solution of the Hamilton-Jacobi partial differential equation associated with the homogeneous problem \((P, \omega; G)\) if it generates isotropic submanifolds \( J^{(\alpha)} \) of the cotangential bundle \((T^*(A \times M), \Omega_{A \times M})\), such that

\[
pr_2 (\tilde{d} s^{(\alpha)} (V_\alpha) ) \subseteq G .
\]

(3.1.55)

For any point \((a, p) \in A \times M\) each function \( v^{(\alpha)} (a, \cdot) \) is a standard solution [Definition 5.1]. A complete solution consists of generating functions of a family of left presymplectic reduction relations \( L^{(\alpha)} \) in \( J^{(\alpha)} \subset T^* A \times T^* M \)

\[
\text{graph } L^{(\alpha)} = \{ ((a, \epsilon), (p, \theta)) \in J^{(\alpha)} | (\epsilon, \theta) = \tilde{d} v (p, q) , \tau_{A \times M}^* ((p, \theta), (q, \theta)) = (p, q) \} .
\]

(3.1.56)

This setup has a direct effect on the integration of the Hamilton-Jacobi partial differential equation associated with the homogeneous problem \((P, \omega, G)\). A special case arises if the \( \tilde{d} \)-exterior derivative is replaced by the usual \( d \)-exterior derivative. Then the isotropic submanifolds \( J^{(\alpha)} \) are of dimension \( 2(n+1+m) \) and thus Lagrangian. In local canonical coordinates \((T^* V, T^* \psi (a, p, \epsilon, \theta)) = (z, \bar{z}, \xi, \bar{\xi}, t, \xi, a, \bar{a}, b, \bar{b}, s, b_s)\) the isotropic submanifolds \( J^{(\alpha)} \) of \( T^* A \times T^* M \) generated by the set of functions \( \{ s^{(\alpha)} | s^{(\alpha)} \in J(V) \} \) are defined by the equations

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When $G^{-1} t (G)$ is represented by the equations

$$H (z, \bar{z}, t, \xi, \bar{\xi}, \xi_t) = 0$$

then the Hamilton-Jacobi equation gives rise to the equation

$$H \left( z, \bar{z}, t, \frac{\partial \delta^{(\alpha)}}{\partial z}, \frac{\partial \delta^{(\alpha)}}{\partial \bar{z}}, \frac{\partial \delta^{(\alpha)}}{\partial t} \right) = 0 .$$

**Example 3.9** : Suboptimal control of the third kind. Under some regularity assumptions, the Hamiltonian function $h : T^* M \to \mathbb{R}$ generates as a surjective submersion a homogeneous problem $(T^* M, \Omega; Q)$. The corresponding Hamilton-Jacobi partial differential expressing locally the submanifold $Q$ of $T^* M$ is of the form

$$L_E s(a, \ldots) = -l , \quad l \in \mathcal{F}(M) ,$$

and in local coordinates it is linear and inhomogeneous, i.e.,

$$\frac{\partial \delta}{\partial x} f(x, u, t) + \frac{\partial \delta}{\partial u} h(x, u, t) + \frac{\partial \delta}{\partial t} = -l(x, u, t) .$$

The complete solution depends on $n+m+1$ constants

$$\hat{s} = \hat{s}(a, w, s, x, u, t) ,$$

and at the final time the solution equals the boundary function, i.e.

$$\hat{s}(a, w, s, x(t_1), u(t_1), t_1) = \hat{k}(x(t_1), u(t_1), t_1) .$$

**Example 3.10** : Suboptimal control of the second kind. Under some regularity assumptions, the Hamiltonian function $F : T^* M \to \mathbb{R}$ generates as a surjective submersion a homogeneous problem $(T^* M, \Omega; Q)$. Locally there is the Hamilton-Jacobi type equation

$$L_{\xi} s(a, \ldots) = -L , \quad L \in \mathcal{F}(M) ,$$

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which is explicitly of the form

\[
\frac{\partial \hat{s}}{\partial x} \left[ f(x, u, t) + \frac{\partial \hat{h}}{\partial u} \frac{\partial \hat{s}}{\partial t} \right] + \frac{\partial \hat{s}}{\partial u} \hat{h}(x, \frac{\partial \hat{s}}{\partial x}, u, t) + \frac{\partial \hat{s}}{\partial t} = -\hat{\mathcal{L}}(x, u, t) .
\] (3.1.66)

This partial differential equation is modified into

\[
\frac{\partial \hat{s}}{\partial x} \left[ f(x, u, t) + \rho \frac{\partial \hat{h}}{\partial u} \right] + \rho \hat{h}(x, \frac{\partial \hat{s}}{\partial x}, u, t) + \frac{\partial \hat{s}}{\partial t} = -\hat{\mathcal{L}}(x, u, t) ,
\] (3.1.67)

such that the first term is "quadratic" in \( \frac{\partial \hat{s}}{\partial x} \), whereas the second term is linear in this argument [Section 1.2.1]. The complete solution is of the form

\[
\hat{s} = \hat{s}(a, w, s, x, u, t; \rho) ,
\] (3.1.68)

and if in the limit where the cocontrol variable tends to zero

\[
\lambda = \left. \frac{\partial \hat{s}}{\partial x} \right|_{\rho = 0} ,
\] (3.1.69)

then there is the equality with the \textit{Lagrange multiplier of optimal control}, i.e. with the costate. Hence the embedding of the necessary conditions associated with an optimal control problem into the flow of a Hamiltonian vector field and thus determining integral curves with suboptimality of the second kind, results in an efficient algorithm while avoiding the elaborate application of the implicit function theorem to the control equation and the almost impossible evaluation of the \textit{Hamilton-Jacobi-Bellmann} partial differential equation.

In order to introduce the geometric structure for the understanding of partial differential equations which arise in optimal control, we give the definition of the \textit{complete thin solution}.

**Definition 3.11**: \textit{Complete thin solution}. A set of functions \( \{ S^{(\alpha)} \} \), where \( \mathcal{U}_\alpha \) are open connected subsets of \( (\mathbb{N} \times \mathbb{R}) \times (\mathbb{N} \times \mathbb{R}) \), is called a complete thin solution of the Hamilton-Jacobi partial differential equation associated with the homogeneous problem \( (P, \omega; \tilde{G}) \) if it generates Lagrangian submanifolds \( \bar{L}^{(\alpha)} \) of the cotangential bundle \( (T^* \mathcal{U}, \Omega_{\mathcal{U}}) \), such that

\[
pr_2(\{ dS^{(\alpha)} (\mathcal{U}) \}) \subseteq \tilde{G} ,
\] (3.1.70)
with the regular embedding \( 1 : T^* \left( (N' \times R) \times (N \times R) \right) \to T^* (A \times M) \).

**Example 3.12 : Optimal control.** Let us consider an open connected subset \( \mathcal{U} \) of the Cartesian product \( (N \times R) \times (N \times R) \). The two parametric group of diffeomorphisms on \( A \) generated by the "curve" \( k \in \mathfrak{X} \), \( \mathfrak{X} = \{ k = im \kappa \mid \kappa : R \times R \to \mathcal{U} \} \) splits globally since \( \kappa : R \times R \to \mathcal{U} ; (t, \tau) \to (\eta (t), \eta (\tau)) \). Furthermore, we introduce a section-variational problem

\[
S = K_1 (B_1) + F^* \int_{c_1} L + F^* \int_{c_2} \tilde{L} ,
\]

\[
\bar{c} = \bar{c}_1 + \bar{c}_2 , \quad (3.1.71)
\]

\[
\partial \bar{c}_1 = (N \times R, \bar{c}) , \quad \partial \bar{c}_2 = (\bar{c}, B_1) \quad (3.1.72)
\]

of which the function \( S \in \mathcal{F}(\mathcal{U}) \) is defined on \( \mathcal{U} \) where \( \tilde{L} \in \mathcal{F}(U) \), \( U \subset M \), \( L \in \mathcal{F}(U') \) and \( U' \subset A \). In local, adapted coordinates of \( A \times M \), \((U' \times U, \phi (a, p) = (x (t), t, F (x (t), t), x (\tau), \tau, F (x (\tau), \tau))) \) this is

\[
S (x (t), t, x (\tau), \tau) = \int_{t_1}^{\tau} L (x (s), F \{ x (s), s \}, s) ds +
\]

\[
+ \int_{t}^{\tau} \tilde{L} (x (r), F \{ x (r), r \}, r) dr . \quad (3.1.73)
\]

The "curve" \( k \) is generated by two vector fields \( S_1, S_2 \in TA \) given by

\[
S_1 = (\kappa (t, \cdot), D\kappa (t, \cdot) \frac{\partial}{\partial t}) , \quad S_2 = (\kappa (\cdot, \tau), D\kappa (\cdot, \tau) \frac{\partial}{\partial \tau}) , \quad (3.1.75)
\]

with vanishing Lie-bracket \([S_1, S_2] = 0\). The two Lie-derivatives of \( S \in \mathcal{F}(\mathcal{U}) \) with respect two \( S_1 \) and \( S_2 \) are

\[
\mathcal{L}_{S_1} S (\cdot, p) = -F^* L , \quad \mathcal{L}_{S_2} S (a, \cdot) = F^* (L - \tilde{L}) , \quad (3.1.76)
\]

and in local coordinates, evaluated on \( R \)

\[
\frac{\partial \tilde{s}}{\partial x} f (x (t), F \{ x (t), t \}, t) + \frac{\partial \tilde{s}}{\partial t} = -L (x (t), F \{ x (t), t \}, t) , \quad (3.1.77)
\]

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\[ \frac{\partial \xi}{\partial x} x'(\tau) + \frac{\partial \xi}{\partial \tau} = L(x(\tau), F\{x(\tau), \tau\}, \tau) - \overline{L}(x(\tau), F\{x(\tau), \tau\}, \tau) \]

(3.1.78)

There are several cases which are worth being investigated. In optimal closed-loop control the identity \( L = \overline{L} \) holds. We examine the exterior partially vanishing derivative of the right hand side of the equation (3.1.76)

\[ \xi (L, S) = 0, \]

(3.1.79)

which is in the neighborhood \( V \) of a point \( p \) in \( A \)

\[ \frac{\partial^2 \xi}{\partial x^2} x'(\tau) + \frac{\partial \xi}{\partial x} \frac{\partial f}{\partial x} + \frac{\partial \xi}{\partial x} \frac{\partial f}{\partial u} \frac{\partial F}{\partial x} + \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial \tau} = 0. \]

(3.1.80)

If we naturally locally identify

\[ \frac{\partial \xi}{\partial x}(x(t), t, x(\tau), \tau) = \lambda(\tau), \]

(3.1.81)

it reduces the scalar partial differential equation of second order (3.1.80) to

\[ \frac{\partial^2 \xi}{\partial x^2} x'(\tau) + \frac{\partial \xi}{\partial x} \frac{\partial f}{\partial x} - \frac{\partial \xi}{\partial x} \frac{\partial f}{\partial x} \frac{\partial F}{\partial x}, \]

(3.1.82)

a system of \( n \) linear ordinary differential equations

\[ \lambda^T(\tau) = -\left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \right) \lambda^T(\tau). \]

(3.1.83)

This system has one distinguished equilibrium point \( (\lambda' = 0, \lambda = 0) \). We thus illustrate our understanding of a complete integral \((S(a, x) : \text{standard})\) in optimal control. Namely, the constants \( a \) are parametrized by time \( t \) and identified with the state of the plant, and the variables \( x \) are future states lying on the trajectory generated by the plant \( P \) which is driven by an optimal controller. In order to close the gap between this new interpretation of a complete solution and the classical generation of solutions for the two-point-boundary-value-problem of optimal control we give a brief introduction of the infinitesimal symplectic transformation \( T \) on \( T^* \mathbb{R}^n \equiv \mathbb{R}^n \times \mathbb{R}^n \) for a linear system and quadratic Hamiltonian. The original linear Hamiltonian equations of the pseudo-closed system are of the form
\( x(t_0) = x_0, \lambda^T(t_1) = K(t_1)x(t_1) \) \hspace{1em} (3.1.84)
\( x(t) = A(t)x(t) - B(t)R^{-1}(t)B^T(t)\lambda^T(t) \), \hspace{1em} (3.1.85)
\( \dot{\lambda}^T(t) = -Q(t)x(t) - A^T(t)\lambda^T(t) \). \hspace{1em} (3.1.86)

The pushforward of the infinitesimal automorphism \( T \) in \( T^*\mathbb{R}^n \), generated by the function \( \hat{S} \in \mathcal{F}(\mathbb{R}^n \times \mathbb{R}) \),

\[ \hat{S} = \frac{1}{2}x(t)P(t)x(t), \] \hspace{1em} (3.1.87)

is an isomorphism in \( T(T^*\mathbb{R}^n) \), so that by \( x(\tau) \Rightarrow y(t), \lambda(\tau) = \mu(t) \) and

\[ \hat{T} : \begin{bmatrix} y(t) \\ \mu(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ -P(t) & I \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}, \quad \hat{T}^{-1} : \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ P(t) & I \end{bmatrix} \begin{bmatrix} y(t) \\ \mu(t) \end{bmatrix} \] \hspace{1em} (3.1.88)

the transformed linear Hamiltonian system is of the form

\[ \dot{y}(t) = [A(t) - B(t)R^{-1}(t)B^T(t)P(t)]y(t) - [B(t)R^{-1}(t)B^T(t)]\mu^T(t), \] \hspace{1em} (3.1.89)
\[ \mu^T(t) = [-P - PA - Q + PBR^{-1}B^TP]y(t) \] \hspace{1em} (3.1.90)

By the linear version of formula (3.1.83) we conclude the vanishing of the Riccati-type ordinary matrix differential equation

\[ -\dot{P}(t) - A(t)P(t) - PA(t) - Q(t) + P(t)B(t)R^{-1}(t)B^T(t)P = 0, \] \hspace{1em} (3.1.91)
\[ P(t_1) = K(t_1). \]

An extension of this last part to a higher degree is given in the Example 3.13. For a general nonlinear problem we conclude that the cost-to-go function does neither generate a decoupling transformation [Hildebrandt, p. 67] nor does it map the system locally to canonical equilibrium points [Lichnerovicz [33], p. 287].

For a suboptimal embedding of a sensitivity analysis it is convenient to consider a Lagrange-function of the form \( \tilde{L} = L + \epsilon L \) in equation (3.1.78).

**Example 3.13**: Let us demonstrate Example 1.25 in presence of the given Hamilton-Jacobi method. The cost-to-go function \( \hat{S} \in \mathcal{F}(\mathbb{N} \times \mathbb{R}) \) is of the form

\[ \hat{S} = \frac{1}{2}P_{ij}(t)x^i\dot{x}^j + \frac{1}{6}Q_{ijk}(t)x^i\dot{x}^j\dot{x}^k, \] \hspace{1em} (3.1.92)
which is the generator of a symplectoautomorphism $D$ on $T^* (N \times \mathbb{R})$ for the time-dependent nonlinear pseudoclosed system written in Darboux's coordinates

$$x^i(t) = A^i_j(t) x^j(t) + \frac{1}{2} \tilde{A}^i_{kl} x^k(t) x^l(t) - B^i_n R^{nl} B^m_l \lambda_m(t) , \quad (3.1.93)$$

$$\dot{\lambda}_i(t) = -\Omega_{ik} x^k - \frac{1}{2} \Pi_{jkl} x^j x^k x^l - \frac{1}{2} \sum_{jkln} x^j x^k x^l x^m - A_i^j \dot{\lambda}_j - \tilde{A}^i_{kl} x^l \lambda_k . \quad (3.1.94)$$

The infinitesimal symplectoautomorphism $D$ maps the set of extended states and costates at time $t \{ \delta(t) \} (q_0) \subset T^* (N \times \mathbb{R})$ with origin $q_0$ (i.e. initial condition) to the set of states and costates at time $\tau, \| t - \tau \| < \varepsilon$, denoted by $\{ \delta(\tau) \} (q_0) \subset T^* (N \times \mathbb{R})$, as realized by the transformation

$$y(t) = \Phi (x(t), t, \lambda(t), \lambda_t(t)) , \quad t = t , \quad (3.1.95)$$

$$\mu(t) = \Psi (x(t), t, \lambda(t), \lambda_t(t)) , \quad \mu_t(t) = \Psi_t (x(t), t, \lambda(t), \lambda_t(t)) \quad (3.1.96)$$

where we identified $y(t) = x(t), \mu(t) = \lambda(t)$ and $\mu_t(t) = \lambda_t(t)$. In a first order approximation we have

$$y(t) = x(t), \mu(t) = -\frac{\partial S}{\partial x} + \lambda(t) \quad \mu_t(t) = -\frac{\partial S}{\partial t} + \lambda_t(t) . \quad (3.1.97)$$

The infinitesimal pushforward $D_* : TT^* (N \times \mathbb{R}) \to TT^* (N \times \mathbb{R})$ is of the form

$$\begin{bmatrix} \dot{y}(t) \\ \dot{\mu}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0^T & 1 & 0 \\ T & b & I \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} , \quad T = -\frac{\partial^2 S}{\partial x^2} , \quad d = -\frac{\partial^2 S}{\partial t^2} , \quad b = -\frac{\partial^2 S}{\partial t \partial x} . \quad (3.1.98)$$

The image of the nonlinear pseudo-closed system (3.1.93),(3.1.94) is given here for notational reasons in the same manner as in the previous example, such that a comparison is possible.

$$\dot{y}(t) = (A - BR^{-1}B^T P) y + \frac{1}{2} (\tilde{A}(t) - BR^{-1}B^T Q) yy - BR^{-1}B^T \mu T , \quad (3.1.99)$$
\[ \mu(t) = -\left( \dot{P} + A^TP + PA + \Omega - PBR^{-1}B^TP \right)y - \]
\[ -\frac{1}{2} \left( \dot{Q} + P\dot{A} + 2\bar{A}P + A^TQ + 2QA + \Pi - PBR^{-1}B^TQ - 2QBR^{-1}B^TP \right)y - \]
\[ -\frac{1}{2} \left( \dot{Q\bar{A}} + \bar{A}Q + \Sigma - QBR^{-1}B^TQ \right)yy - (A^T - PBR^{-1}B^T)\mu^T - \]
\[ -\frac{1}{2} \left( \bar{A}^T(t) - QBR^{-1}B^T \right)y\mu^T \] (3.1.100)

In optimal control the nonregularity of the system (3.1.93),(3.1.94) including the equation for \( \dot{\lambda}_q(t) \) and \( \dot{f} \) is a result of the construction of adjointness. Hence this part of the Hamiltonian system is coupled unidirectional and may be integrated by quadrature. For simplicity of the illustration we omitted the introduction of its orbit under the action of \( D \).

Any generalizations to a higher polynomial degree than two can thus be expected to have a solution of the Synthesis Problem by the repeated application of our construction.

**Summary of Section 3.1.3:**

In this section we introduced the thin and the complete thin solution of the Hamilton-Jacobi equations associated with some homogeneous problems. These solutions are interesting in context of the cut system, since they generate the required presymplectic morphisms.
3.2 Solution diffeomorphism

3.2.1 Introduction

Let us consider the cotangential bundle $T^*M$ and the Hamiltonian flow generated by the Hamiltonian vector field $\#dh \in TT^*M$. It represents some necessary conditions of the suboptimal control problem $(k, A, l, E)$. The suboptimal feedback control law is a section $F$ of the locally trivial fiberbundle $(T^*M, r, N \times R, F)$, such that its image $F(N \times R)$ lies in the Lagrange submanifold $L_M$ of $(T^*M, \Omega)$ associated to problem $(k, A, l, E)$. In this paragraph we are looking for diffeomorphisms from this submanifold $L_M$ in the cotangential bundle onto any smooth section in the tangential bundle with the following purpose. The realization of the section $c$ as well as the determination of the Lagrange manifold $L_M$ is elaborate. Thus, with the method of the solution diffeomorphism we decouple the solving process and gain some efficient algorithm and some further insight into the differential geometric structure of optimal and suboptimal control.

A smooth section in the tangential bundle $\zeta : M \to TM$ shall have as a realization the vector field $E \in TM$ with the following properties. There are many other vector fields which may be considered but about the one from the problem $(k, A, l, E)$ we have maximal apriori knowledge.

**Definition 3.14 : Solution diffeomorphism.** A solution diffeomorphism is a bijective differentiable mapping from a Lagrange submanifold $L_M$ of a Hamiltonian system $(T^*M, \Omega, H)$ onto the image of a smooth section in the tangent bundle TM.

**Theorem 3.15 : Existence of solution diffeomorphism.** A solution diffeomorphism always exists.

**Proof :** Any specific Lagrange submanifold $L_M$ is diffeomorphic to the base manifold $M$. So is the image of $\zeta (M)$. Since suboptimal feedback controllability requires among others the existence of the Lagrange submanifold this realization conception leaves the structural propery invariant. **qed**

A global suboptimal feedback control law is then a section $\tilde{c} : N \times R \to TM$ such that
\[ L_M \supset c(N \times \mathbb{R}) \supset g \circ \tilde{c}(N \times \mathbb{R}) . \]  

A solution diffeomorphism may be surrounded by an immersion \( i : L_M \to TM \) with \( i (L_M) = \zeta(M) \), by a submersion \( s : T^*M \to \zeta(M) \) with \( s^{-1}(\zeta(M)) = L_M \) or by a diffeomorphism \( G : T^*M \to TM \). In the sequence we are going to focus our attention to the last case.

**Definition 3.16**: Abstract second order ordinary differential equation. A second order ordinary differential equation on \( M \) is a vector field \( X \) on \( TM \) such that the associated morphism \( T\tau \circ X \) is the identity mapping of \( TM \).

The fibration \( \tau : TM \to M \) admits a special transverse field \( W \in TTM/VTM \), which corresponds to the identity mapping of \( TM \), and is comparable with the Liouville form. A vector field \( X \) on \( TM \) defines an abstract second order ordinary differential equation if and only if the transverse field \( \tilde{X} = \tilde{\rho}(X) \) is equal to \( W \), where \( \tilde{\rho} : TTM \to TTM/VTM \).

**Lemma 3.17**: Origin of an abstract second order ordinary differential equation. Let \( \tau^* : T^*M \to M \) be a surjective submersion and \( Y \) a vector field on \( T^*M \). If the morphism \( G = T\tau^* \circ s \), based on the section \( s : M \to TM \), \( s(M) = Y \), is a diffeomorphism from \( T^*M \) onto \( TM \), then the transform of \( Y \) by \( G \) defines an abstract second order ordinary differential equation on \( M \).

**Proof**: The transform \( V = TG(Y) \) of \( Y \) is the vector field

\[ V = TG \circ Y \circ G^{-1} , \quad V \in TTM \]  

(3.2.2)

Thus, there is

\[ T\tau \circ V = T\tau \circ TG \circ Y \circ G^{-1} = T\tau^* \circ Y \circ G^{-1} = G \circ G^{-1} = id_{TM} \]  

qed

**Lemma 3.18**: Generation of an abstract second order ordinary differential equation. The Hamiltonian function \( H \in \mathcal{F}(T^*M) \) is under some regularity assumptions the generating function of the diffeomorphism \( G : T^*M \to TM \) in the sense that \( G = T\tau^* \circ \#dh \). This diffeomorphism leaves the Hamiltonian property invariant, i.e.,

\[ E^* L_{\#dh}d\beta = 0 . \]  

(3.2.3)

with \( E = G^{-1} \) being the inverse-morphism \( E : TM \to T^*M \) and \( \beta \in \Lambda^2(T^*M) \).

**Proof**: The Hamiltonian function \( h \) transforms under the diffeomorphism \( G \) to a
function $b$, which generates via the closed 2-form
\[ \Pi = E^* d\beta, \quad \Pi \in \Lambda^2 (TM) \] (3.2.4)
a vector field $^#db \in TTM$. In other words, there is
\[ TG (^#dH) \perp E^* d\beta = -E^* (dH) = -db, \quad b \in \mathcal{F}(TM), \] (3.2.5)
By Lemma 6.4 the vector field $^#db$ on TM determines an abstract second order ordinary differential equation. The image of the smooth section $\zeta(M)$ is a Lagrangian submanifold of the Hamiltonian system $(TM, \Pi, H)$. qed

**Example 3.19**: *Legendre morphism*. If the abstract second order ordinary differential equation is a result from a regular Lagrange problem then there is
\[ L = E^* (Z \perp dH - H), \quad b = W \perp dL - L, \] (3.2.6)
with $Z$ being the Liouville vector field on $T^*M$ and $L$ the Lagrange function on $TM$. In local coordinates the morphism $G$ is described by
\[ \dot{y} = \frac{\partial \hat{H}}{\partial \xi} (y, \xi) \iff \xi = \hat{E} (y, \dot{y}). \] (3.2.7)
The transformed closed 2-form $\Pi$ is then
\[ \hat{\Pi} = d\hat{E} (y, \dot{y}) \wedge dy = (\frac{\partial \hat{E}}{\partial y} dy + \frac{\partial \hat{E}}{\partial \dot{y}} d\dot{y}) \wedge dy, \] (3.2.8)
the new Hamilton function $\hat{b} (y, \dot{y}) = \hat{H} (y, \hat{E} (y, \dot{y}))$ and the second order ordinary differential equation
\[ \hat{A} (y, \dot{y}) = \dot{y}, \quad \hat{B} (y, \dot{y}) = \left( \frac{\partial^2 \hat{L}}{\partial y^2} \right)^{-1} \left( - \frac{\partial^2 \hat{L}}{\partial y \partial \dot{y}} + \frac{\partial \hat{L}}{\partial \dot{y}} \right) = \dot{y}, \] (3.2.9)
since there is locally the formula
\[ \hat{A} (y, \dot{y}) \frac{\partial}{\partial y} + \hat{B} (y, \dot{y}) \frac{\partial}{\partial \dot{y}} \perp \frac{\partial^2 \hat{L}}{\partial y^2} d\dot{y} \wedge dy + \frac{\partial^2 \hat{L}}{\partial y \partial \dot{y}} d\dot{y} \wedge dy = -d\hat{b}. \] (3.2.10)
Because of regularity reasons, the Legendre-morphism does not apply to neither optimal nor suboptimal control problems. In other words, there is no canonical associated abstract second order ordinary differential equation to the Hamiltonian
necessary conditions of optimal or suboptimal control being the result of a familiar Lagrange problem.

However, an interesting situation arises if the realization of the surrounding morphism $G: T^* M \to TM$ can be done with a twice covariantly tensor $\Xi \in \tau_2 (M)$ on $M$. In local coordinates $(U, \phi (p) = (y))$, $U \subset M$ there is

$$y^p = \Xi^{pq} (y) \xi_q, \quad \xi_q = \Xi_{qr} (y) y^r, \quad \Xi^{pq} \Xi_{qr} = \delta^p_r$$  \hspace{1cm} (3.2.11)

The pushforward $TG: T(T^* M) \to TTM$ maps the Hamiltonian vector field $^\#dH$ on $T^* M$ onto an abstract second order ordinary differential equation $V$ on $TM$.

In a local index notation it follows from the right side of (3.2.11)

$$\xi_q = \frac{\partial \Xi_{qr} (y)}{\partial y^j} y^j y^r + \Xi_{qr} (y) y^r,$$  \hspace{1cm} (3.2.12)

and by symmetrization and inversion we get

$$y^s + \frac{1}{2} \Xi^{sq} \left( \frac{\partial \Xi_{qr}}{\partial y^j} + \frac{\partial \Xi_{qs}}{\partial y^j} \right) y^j y^r - \Xi^{sq} \xi_q = 0.$$  \hspace{1cm} (3.2.13)

Finally, the second order ordinary differential equation appears as a nongeodesic spray

$$y^i + \Gamma^i_{kl} y^k y^l + \hat{N}^i = 0$$  \hspace{1cm} (3.2.14)

$$\Gamma^i_{kl} = \frac{1}{2} \Xi^{ij} \left[ \frac{\partial \Xi_{jl}}{\partial y^k} + \frac{\partial \Xi_{jk}}{\partial y^l} - \frac{\partial \Xi_{kl}}{\partial y^j} \right],$$

$$\hat{N} = \Xi^{ij} \left[ \frac{\partial H}{\partial y^j} (y, \Xi y) + \frac{1}{2} \frac{\partial \Xi_{kl}}{\partial y^j} y^k y^l \right].$$  \hspace{1cm} (3.2.15)

In the other direction we redefine the Pontryagin function $H \in \mathcal{F}(T^* M)$ given in local coordinates by

$$\hat{H} (y, \xi) = \frac{1}{2} \Xi^{pq} (y) \xi_p \xi_q + \hat{L} (y)$$  \hspace{1cm} (3.2.16)

being the generator of the Hamiltonian vector field $^\#dH \subset T(T^* M)$

$$y^p (t) = \Xi^{pq} (y) \xi_q$$  \hspace{1cm} (3.2.17)
If we plug the total time derivative of the equation (3.2.17) into (3.2.18) we get

$$y^p(t) - \Gamma^{prs} \xi_r(t) \xi_s(t) - \frac{1}{2} \Xi^{pq} \frac{\partial L}{\partial y^q} = 0$$

(3.2.19)

with the symbol $\Gamma^{prs}$ being of the form

$$\Gamma^{prs} = \frac{1}{2} \left\{ \Xi^{qr} \frac{\partial \Xi^{ps}}{\partial y^q} + \Xi^{qs} \frac{\partial \Xi^{pr}}{\partial y^q} - \Xi^{pq} \frac{\partial \Xi^{rs}}{\partial y^q} \right\}.$$  

(3.2.20)

**Example 3.20**: Solution isomorphism. The realization of the solution diffeomorphism is done by the twice covariantly tensor $\Xi \in \tau_2(M)$ on $M$ which may be locally $(U, \varphi(p) = (x, t, u))$, $U \subset M$ expressed in its diagonal form

$$\Xi = \begin{bmatrix} G & 0 & 0 \\ 0^T & g & 0^T \\ 0 & 0 & D \end{bmatrix} (x, t, u).$$  

(3.2.21)

**Summary of Section 3.2.1**:  

The purpose of the solution diffeomorphism is its belonging to a sufficiently short algorithm solving any optimal or suboptimal control problem with two sets of quasilinear partial differential equations. The solution isomorphism provides a mechanism to make use of a maximal amount of apriori knowledge about the expected solution [Section 1.2.2] in order to minimize lengthy computation. The required integrability conditions for the existence of a feedback solution remain the same [Section 2.1.4].
3.2.2 The Metric and Stability

If optimal and suboptimal control laws have in addition to their extremizing properties some stabilizing action, then there is a higher interest of their implementation to a technical plant. In this section we introduce what is meant by the stability of a critical point of the plant \( P \in \mathcal{H}M \subset TM \). In addition, we illustrate the relationship between the solution isomorphism of the previous section and this structural property.

**Definition 3.21 : Critical point.** An equilibrium point (or critical point) of a differentiable vector field \( E \) on a manifold \( M \) is any point \( p \) where \( E \) vanishes.

The maximal integral curve of a vector field \( E \) which passes through such a point is defined for all \( t \) in \( \mathbb{R} \), and is simply the constant map \( t \rightarrow c \).

Let \( \Phi : \mathbb{R} \times M \rightarrow M \) be the flow of the vector field \( E \). An equilibrium point \( p \) of \( E \) is \( \omega \)-stable (resp. \( \alpha \)-stable) in the sense of Lyapunov if, for every neighbourhood \( U \) of \( p \), there exists another neighbourhood \( V \) of \( p \), \( V \subset U \), such that for every point \( q \) in \( V \), \( \Phi(t, q) \) is defined for all \( t \geq 0 \) (resp. \( t \leq 0 \)), and lies in \( U \). The point \( p \) is stable if it is simultaneously \( \omega \)-stable and \( \alpha \)-stable. The mathematical meaning of a control law is that of an embedded submanifold \( K \) in \( M \) of codimension \( m = (M, \pi, \mathbb{N} \times \mathbb{R}, B) \) and diffeomorphic to the zero-section of \( m \).

**Definition 3.22 : Stabilizing control.** A control law is said to be stabilizing for a plant \( P \in \mathcal{L} \subset TM \) at a critical point \( p \in \mathbb{N} \times \mathbb{R} \) of the vector field \( T\pi(P) \in T(\mathbb{N} \times \mathbb{R}) \) if for every neighbourhood \( U \) of \( p \), there exists another neighbourhood \( V \) of \( p \), \( V \subset U \), such that for every point \( q \) in \( \pi^{-1}(V) \cap K \), the flow \( \Psi(t, q) \) of the vector field \( P|_K \) is defined for all \( t \geq 0 \), and lies in \( \pi^{-1}(U) \cap K \) (compare with [Sontag, p.150]).

In general, the critical points of the vector field \( P|_K \in TK \) do not coincide with the equilibrium points of \( T\pi(P) \). Hence, the conception of stabilizing control leaves the question about stability properties of the controlled plant open. The controlled plant \( P|_K \) is stable at a critical point \( r \) of \( P|_K \) in \( K \) if \( r \) is a \( \omega \)-stable equilibrium.

The control law \( K \subset M \) may either be realized by a smooth regular section, i.e. by a closed-loop state-feedback control law \( F : \mathbb{N} \times \mathbb{R} \rightarrow M \) or by the
i.e. by the open-loop control law $S : N \times \mathbb{R} \to M$

According to a theorem due to Lejeune and Dirichlet, a point $p$ of $T^* M$ is an equilibriumpoint of a Hamiltonian system $(T^* M, \Omega, H),$ i.e. a critical point of the Hamiltonian vector field $^*dH \in T T^* M,$ if and only if $dH \in \Lambda^1 (T^* M)$ vanishes at $p.$ If, in addition, the Hessian $D^2 H (p)$ is positive-definite or negative-definite, then the point $p$ is stable in the sense of Lyapunov.

**Theorem 3.23 : Suboptimal Open-loop stabilizability.** The cost-to-go function $W \in \mathcal{F}(M)$ of the suboptimal control problem $(k, A, l, E)$ is a Lyapunov function for the vector $E \in TM$ in a neighbourhood $U$ of the critical point $p \in M$ under the assumption of the existence of a solution isomorphism, if and only if the tensor $S \in \tau_2 (M)$ is negative definite.

**Proof :** The cost-to-go function $W \in \mathcal{F}(M),$ i.e. the standard solution of example of the previous paragraph,

$$W = k_1 (A_1) + \int_c l \quad (3.2.22)$$

is monotone increasing on a minimal curve and has positive values in an open neighbourhood $U$ of $p.$ The second requirement of Lyapunov is given by the evaluation of

$$L_E W = -S^* L . \quad (3.2.23)$$

In local coordinates $\{ U, \varphi (p) = (x, t, u) \}$ this is

$$\frac{\partial W}{\partial x} f(x(t), t, u(t)) + \frac{\partial W}{\partial t} + \frac{\partial W}{\partial u} h(x(t), t, u(t)) = -L (x(t), t, u(t)) . \quad (3.2.24)$$

The tensor $\Xi \in \tau_2 (M)$ has a natural decomposition into a symmetric and a skew-symmetric part

$$S = \frac{1}{2} \{ \Xi + \Xi^T \} , \quad A = \frac{1}{2} \{ \Xi - \Xi^T \} . \quad (3.2.25)$$

In the next formula the first equality follows by the definition of the solution isomorphism while the second is immediate

$$L_E W = \Xi (E, E) = S (E, E) . \quad (3.2.26)$$

Hence, it is necessary and sufficient that the symmetric part of the tensor $\Xi,$ real-
izing the solution isomorphism, is negative-definite such that the cost-to-go function becomes Lyapunov function for the vector field $E$ on an open neighborhood $U$ of the critical point $p E$ in $M$. qed

**Theorem 3.24 : Suboptimal Closed-loop stability.** The cost-to-go function $W \in \mathcal{F}(M)$ of the suboptimal control problem $(k, A, l, E)$ is a Lyapunov function for the vector $Z \in TM$ in a neighborhood $U$ of a point $p \in M$ if and only if the tensor $S \in S_{\tau}(M)$ is negative definite.

**Proof :** According to [Section 3.1.2] we have in local coordinates \( \{ U, \varphi(p) = (y, t, v) \} \)

$$\mathcal{L}_Z W = -\eta^* L , \quad (3.2.27)$$

$$\frac{\partial \hat{W}}{\partial y} f + \frac{\partial \hat{W}}{\partial t} + \frac{\partial \hat{W}}{\partial v} h = -\hat{L}(y(t), t, v(t)) . \quad (3.2.28)$$

In the next formula the first equality follows by the definition of the solution isomorphism while the second is immediate

$$\mathcal{L}_Z W = \Xi(Z, Z) = S(Z, Z) . \quad (3.2.29)$$

Hence, it is necessary and sufficient that the symmetric part of the tensor $\Xi$, realizing the solution isomorphism, is negative definite such that the cost-to-go function becomes Lyapunov function for the vector field $Z$ on a neighborhood $U$ of a point $p$ in $M$. qed

**Summary of Section 3.2.2 :**

The existence properties of Filipov are in fact some growth conditions for the function $L$ and the plant $P$. In the terms of this section, this conditions guarantee the stability of the closed-loop, optimal state-feedback controlled vector field $Q$. 

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