ROBUSTNESS OF LINEAR SYSTEMS WITH POLYTOPIC PARAMETER UNCERTAINTIES

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1993
To my parents
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SUMMARY

In classical control engineering both the analysis and the design of control systems are based on a nominal model of the process. This model always reflects a mathematical approximation of the plant. For the control design no information on modeling errors, uncertain parameters or external disturbances is generally considered. The main purpose of the robust control approach is to make use of a priori knowledge of plant uncertainties in order to improve reliability of the control loop.

In the first part of this thesis, the robust D-stability of families of polynomials with polytopic parameter uncertainties is investigated in the frequency domain. Due to the polytopic structure of the uncertainties, the value set of the polynomial family is a parapolygon whose edges are the images of a subset of the edges of the parameter polytope. A procedure is presented for the explicit extraction of the edges of the value set, respectively of the associated exposed edges of the polytope. For s along \( \partial D \) frequency intervals can be found where the structure of the value set remains constant, i.e. where the building edges of the parameter polytope remain the same. Applying the zero exclusion principle, the essential condition for the D-stability of the polynomial family becomes the test of the D-stability of all exposed edges of the polytope which build the edges of the value set for s \( \in \partial D \).

These general results are applied to investigate the Hurwitz and the cone stability of the standard closed-loop with an interval plant and a fixed controller.

The special structure of the value set is used for the computation of the structured stability margin, a robustness measure indicating the largest size of the uncertainty domain such that D-stability of the
family of polynomials is preserved.

In the second part of this thesis, the value set of uncertain transfer functions is investigated. The numerator and the denominator polynomials are supposed to be products of polynomials with independent parameterization of the even and odd part. Then, the value sets of these specially parameterized polynomials are axis parallel boxes and the value set of the transfer function is obtained as a sequence of the mathematical operations product and ratio on axis parallel boxes. A geometrical procedure is presented for the direct computation of the boundary of the value set.

The robust design problem is addressed in the last part of this thesis. Using the Bode envelope of the uncertain system, procedures are presented for the design of robust dynamic compensators of lead/lag type in the frequency domain. The resulting compensator guarantees a minimal system performance in the sense of worst-case behavior. The design method is applied to an electromechanical system.
ZUSAMMENFASSUNG


Diese allgemeinen Ergebnisse werden zur Untersuchung der robusten Hurwitz- und Konusstabilität des Standardregelkreises mit Intervall-


Chapter 1

INTRODUCTION

The issue of uncertainty is omnipresent in control engineering. When working with real processes, this uncertainty can originate from different sources. Either the process is influenced by unknown disturbance signals, unknown initial conditions or uncertainty is inherent to the process itself such as time-varying parameters due to drifts, variable operating temperature, aging of components or even drastic changes of the process such as failures of components. A third possibility is that of uncertainties introduced in the mathematical model of the process. This model should reflect the essential behavior of the process with sufficient accuracy and uncertainties may describe unmodeled high frequency dynamics or express the lack of exact knowledge of system parameters. This thesis deals with the third kind of uncertainties, i.e. uncertain system parameters.

The main goal in control engineering is to design controllers that guarantee at least stability of the system in spite of these uncertainties. In the past years, two modern concepts have been established in order to deal with system uncertainties: the adaptive and the robust control approach. In the adaptive approach, the plant parameters are identified on-line, and detected changes lead to an automatic update of the controller. This concept is mainly used for slowly time-varying systems. In the robust control approach, a priori knowledge of the uncertainties in the system is taken into account; this knowledge is exploited to determine a fixed controller which stabilizes the system for all specified uncertainties.

The adaptive approach generally leads to better system performance,
but due to the more complex structure of the control loop, overall stability is hard to prove. The robust controller is more reliable since it has a simpler structure and is designed completely off-line.

The robust control approach can be partitioned into two main directions which essentially differ in the description of the uncertainty: parametric and non-parametric uncertainties, often denoted as structured and unstructured uncertainties. In the parametric approach, some system parameters are assumed to have a specific uncertainty structure, i.e. to belong to a specific set. In the non-parametric case, the uncertainty of the system is described by a norm which is a measure for the maximal deviation of the system from a nominal behavior. A widely used norm is the $\infty$-norm, which is treated with the $H_\infty$-approach [1]. This thesis deals with continuous-time linear systems with parametric uncertainties. The uncertain parameters are assumed to be time-invariant which means that they are fixed but uncertain in a specified uncertainty domain. Similarly to the classical control theory, the field of robust control with parametric uncertainties can be further divided into state space and frequency domain methods. In this thesis, the discussion is concentrated on frequency domain methods, with the systems described by transfer functions.

1.1 ROBUST CONTROL WITH PARAMETER UNCERTAINTIES

Research activities in this area were released in 1978 by a seminal paper of Kharitonov [2], where the Hurwitz stability of a family of polynomials with uncertain coefficients independently chosen in an interval was investigated. The surprising result was that it is necessary and sufficient to prove the Hurwitz stability of four special polynomials in order to prove Hurwitz stability for the whole family. Meanwhile, numerous contributions have been published dealing with stability of polynomial families. The common goal of these activities was to find minimal sets of necessary and sufficient conditions which guarantee stability of the whole polynomial family. Activities concentrated on two essential directions: i) the stability domains and ii) specific uncertainty structures
1.1. Robust control with parameter uncertainties

in parameter space. Discrete-time interval polynomials are investigated in [3], where it is shown that the Kharitonov result does not hold for Schur stability. In [4], the minimal set of conditions for the Schur stability of interval polynomials is presented. A discrete-time analogy to the Kharitonov theorem was obtained in [5, 6, 7] for specially parameterized families of polynomials. The stability of interval polynomials with respect to the left sector is discussed in [8, 9, 10]. Kharitonov-like results are obtained in [11, 12] for special stability domains bounded by rational functions. These results are generalized for convex stability domains in [13].

From a practical point of view, the uncertainties are related with physical parameters, who generally enter in a nonlinear way into the system description. In this context, two aspects are investigated: a) the uncertainty domain of the physical parameters and b) the relation between the uncertain parameters and the parameters of the system description.

The mostly used approach deals with physical parameters modeled by interval parameters, i.e. each uncertain parameter is located independently in an interval, which corresponds to a hyperrectangle in parameter space. The case of a diamond in parameter space was treated in [14], polytopical uncertainties in [15, 16], ellipsoidal uncertainty domains in [17] and general $\ell^p$-spheres in [18]. The relation between the uncertain interval parameters and the coefficients of the characteristic polynomial is discussed among others in [19, 20] for the linear case, in [21, 22, 23, 24] for multilinear dependencies and in [25] for nonlinear relations. An important case in this context is the closed-loop with an interval plant and a fixed controller. Caused by the controller polynomials, the coefficients of the characteristic polynomial become affine functions of the uncertain plant parameters which leads to polytopical uncertainty domains in coefficient space [26, 27, 28, 29].

In the last few years, the design problem gained increasing interest in research. The question of stabilizability of uncertain systems, i.e. the problem of existence of a single stabilizing controller for a family of systems, was discussed in [30, 31, 32]. Design methods were presented
in [33] for parameter space design, iterative procedures for the design of a robust stabilizer in [34, 35] and approaches based on numerical optimization techniques in [36, 37].

1.2 MOTIVATION AND GOAL OF THIS WORK

For the analysis of robust control systems most of the classical stability criteria such as Routh-Hurwitz and Nyquist criterion [38] can be extended to deal with uncertain systems. In the mathematical sense, an uncertain system is an infinite or finite set of functions and it is of central interest to find a minimal set of necessary and/or sufficient conditions to guarantee stability of the whole system. Therefore, the frequency domain approach turned out to be very promising. The key problem is the determination of the frequency domain representation of the uncertain system without extensive computations.

Due to the mathematical complexity, analysis and design of robust control systems generally cannot be done analytically. Numerical methods and, thanks to increasing computer power, also graphical methods have become of growing importance to provide supporting tools for a control engineer dealing with robust control systems.

The goal of this work is to investigate the robust stability of a special class of uncertain systems in the frequency domain and to present computationally tractable methods for the analysis and design of robust control systems.

1.3 OUTLINE

Chapter 2 introduces some notation and concepts needed for the mathematical description of uncertain systems in the parameter space and the frequency domain.

Chapter 3 investigates the stability of uncertain polynomials with respect to a general stability domain $D$. The main result is a procedure for the determination of the image of the uncertain polynomial in the frequency domain. The minimal set of conditions to be tested for the
1.3. Outline

$D$-stability of the uncertain polynomial is given. Based on this result, three important applications are presented: the Hurwitz and cone stability of the closed-loop and the computation of a robustness measure.

Chapter 4 deals with the frequency domain representation of uncertain transfer functions. A geometrical procedure is presented for the direct computation of the boundary of the image of the uncertain system for a fixed frequency.

In Chapter 5, the robust design problem is addressed. Two design methods from the literature are discussed and the classical design of dynamic compensators in the frequency domain is extended in order to design a robust compensator for an uncertain plant.

The robust positioning of an electromechanical system is studied in Chapter 6. The design is performed in the frequency domain applying the results of Chapter 5.
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Chapter 2

MATHEMATICAL DESCRIPTIONS OF UNCERTAIN SYSTEMS

In this chapter the mathematical descriptions of uncertain continuous-time systems are presented. Especially some definitions and set-ups are introduced that are used throughout this thesis.

2.1 NOTATION

\( \mathbb{R} \) The field of real numbers.

Polytope The convex hull of a set of points in \( \mathbb{R}^n \). The hull of the polytope is bounded by \( i \)-dimensional faces, \( i = 0, \ldots, n - 1 \). The 0-dimensional faces are called vertices and the 1-dimensional faces are the edges of the polytope. Every \( i \)-dimensional face of the polytope is itself the convex hull of its vertices [39].

Parpolytope Special polytope where two edges at a time are parallel.

Hyperrectangle Special parpolytope with all the edges parallel to the axis of the coordinate system.

Polygon A polytope in \( \mathbb{R}^2 \).

Parpolygon A parpolytope in \( \mathbb{R}^2 \)
2.2 UNCERTAIN POLYNOMIALS

An uncertain polynomial is described by

$$f(s, \gamma) = \sum_{i=0}^{n} a_i(\gamma)s^i$$  \hspace{1cm} (2.1)

where the coefficients $a_i$, $i = 0, \cdots, n$ of the polynomial depend on a set of uncertain parameters $\gamma \in \Gamma$ with $\Gamma$ a given uncertainty domain in the parameter space. For all $\gamma \in \Gamma$, (2.1) describes a continuum of polynomials, i.e. a family of polynomials. The uncertain parameters can be interpreted as a representation of uncertain physical parameters of the system. Generally they are assumed to be independent and to lie in an interval

$$\gamma_i \leq \gamma_i \leq \bar{\gamma}_i \quad i = 1, \cdots, \nu$$

Then, the domain $\Gamma$ is a $\nu$-dimensional hyperrectangle in parameter space.

The vector relation $\psi(\cdot)$

$$\gamma \in \Gamma \xrightarrow{\psi} a = \psi(\gamma) \ , \ a \in A$$

$$\Gamma \in \mathbb{R}^\nu \xrightarrow{\psi} A \in \mathbb{R}^{n+1}$$

(2.2)

defines a map of the domain $\Gamma$ in the parameter space $\mathbb{R}^\nu$ to a domain $A$ in the coefficient space $\mathbb{R}^{n+1}$. For special functions $\psi(\cdot)$ two interesting cases result:

(i) With the affine transformation

$$a = a_0 + T\gamma$$  \hspace{1cm} (2.3)

where $T$ is a real matrix of dimension $(n + 1) \times \nu$, with $n$ the degree of the uncertain polynomial $f(s, \Gamma)$ and $\nu$ the number of uncertain parameters $\gamma_i$. The coefficient vector $a$ is an affine function of the uncertain parameter vector $\gamma$. Therefore the hyperrectangle $\Gamma \in \mathbb{R}^\nu$ is mapped to a parpolytope $A \in \mathbb{R}^{n+1}$. 
(ii) For the special case \( T = I \) and \( a_0 = 0 \) we obtain

\[
a = \gamma
\]

with \( \nu = n + 1 \). Then, the uncertain parameters \( \gamma_i \) are directly the coefficients of the uncertain polynomial. This case is the important case of an interval polynomial

\[
f(s, \gamma) = \sum_{i=0}^{n} \gamma_{i+1}s^i
\]

where all coefficients lie independently from each other in an interval. Then, the domain \( \mathcal{A} \) is a hyperrectangle in coefficient space.

These two cases, where (ii) is an important special case of (i), are discussed in this thesis. Multiaffine dependence of \( a \) on \( \gamma \) is investigated in [40] and the references therein. For more general functions \( \psi \) analytical results can hardly be found in the literature.

2.3 UNCERTAIN CONTROL SYSTEMS

![Control system](image)

**Figure 2.1:** Control system.

The set-up of an uncertain control system with an uncertain plant and a fixed controller is shown in Fig. 2.1. The continuous-time SISO (single-input single-output) plant \( G_S \) is time-invariant but subject to parameter uncertainties. The transfer function of \( G_S \) is given as

\[
G_S(s, \gamma) = \frac{b(s, \gamma)}{a(s, \gamma)}
\]
where the polynomials $a(s, \gamma)$ and $b(s, \gamma)$ are uncertain polynomials as introduced in § 2.2. Then, $G_S(s)$ itself represents a family of plants for all $\gamma \in \Gamma$.

The fixed controller $G_R$ is described by the transfer function

$$G_R(s) = \frac{d(s)}{c(s)}$$

where $c(s)$ and $d(s)$ are polynomials with constant coefficients.

With this set-up, for each member of the plant family $G_S$ we obtain a possible realization of the control system in Fig. 2.1.

For the analysis of the robust stability of the closed-loop in Fig. 2.1 the characteristic polynomial

$$P(s, \gamma) = c(s)a(s, \gamma) + d(s)b(s, \gamma)$$  (2.4)

plays an important role. From § 2.2 it is clear that $P(s, \gamma)$ itself is a family of polynomials for $\gamma \in \Gamma$. To determine the stability of $P(s, \gamma)$ the stability of the whole family of polynomials, i.e. the stability of each member of the family, must be investigated.

### 2.4 D-STABILITY

The classical Hurwitz stability of continuous-time or Schur stability of discrete-time systems can be treated as special cases of the general D-stability. The left half plane respectively the unit disc are examples of D-stability domains.

**Definition 1** A stability domain $D$ is a union of open subsets of the complex plane symmetrical about the real axis. The domain $D$ can consist of several disjoint pieces whereas the border $\partial D$ of $D$ is not included.

Since we only consider polynomials with real coefficients, we assume the stability domain $D$ to be symmetric with respect to the real axis. Fig. 2.2 shows a possible D-domain in the complex plane consisting of several pieces and a hole inside $(D_3)$.
2.4. \textit{D-stability}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{d-stability-domain.png}
\caption{D-stability domain.}
\end{figure}

\textbf{Definition 2} \textit{The polynomial family $f(s, \Gamma)$ is robustly D-stable if and only if each polynomial $f(s, \gamma)$, $\gamma \in \Gamma$, has all its roots in the open domain $D$.}

If the uncertainty domain $\Gamma$ is connected, then, a $D$-stable polynomial family is root invariant which means that all polynomials of the family have the same root distribution in the different parts of $D$.

For the investigation of the root location the classical methods such as root locus [41] or frequency domain criteria can be applied.

\subsection*{2.4.1 The value set}

For the application of frequency domain methods the system representation in the complex plane is needed. Analog to the classical theory the image of a polynomial or a transfer function in the frequency domain is obtained by evaluating $s$ along the boundary $\partial D$ of the stability domain $D$. For a fixed frequency $s^*$ the image is a single point in the complex plane. In the case of uncertain polynomials or transfer functions $f(s, \Gamma)$ for every frequency a continuum of points is obtained. For this continuum the term \textit{value set} of an uncertain function $f(s, \Gamma)$ is introduced:

\textbf{Definition 3} \textit{The value set of $f(s^*, \Gamma)$ is the image of $f(s, \gamma)$ in the complex plane for a fixed frequency $s = s^*$ and all $\gamma \in \Gamma$.}
In § 3 the value set of special uncertain polynomials and in § 4 the value set of uncertain transfer functions are investigated.

2.4.2 Principle of the argument

In the frequency domain the stability of a system can be easily checked using the principle of the argument [38, 42]. Instead of determining the increase of argument for every member of the family of polynomials \( f(s, \Gamma) \), with \( s \) along \( \partial D \), the increase of argument of the whole value set is considered. To investigate the robust \( D \)-stability of \( f(s, \Gamma) \) the number of circles of the value set of \( f \) around the origin has to be determined [43]. This leads to the result given in [44] as generalization of the Nyquist criterion:

\[\text{Theorem 1 (Zero exclusion principle)}\]

The polynomial family \( f(s, \Gamma) \) is robustly \( D \)-stable if and only if

\[
\begin{align*}
\text{(i)} & \quad f(s, \gamma^*) \text{ is } D \text{-stable for some } \gamma^* \in \Gamma \\
\text{(ii)} & \quad 0 \not\in f(s, \Gamma) \quad \forall \ s \text{ along } \partial D
\end{align*}
\]

Condition (i) of the Theorem guarantees that one member of the polynomial family is \( D \)-stable, i.e. has the correct number of circles around the origin. Since the boundary \( \partial D \) is not part of the stability domain \( D \), no point \( s \in \partial D \) can occur as a root of \( f(s, \Gamma) \) and due to the continuity of the mapping, the origin must keep outside of the value set for all \( s \) along \( \partial D \) (condition (ii)).
Chapter 3

ROBUST $D$-STABILITY OF UNCERTAIN POLYNOMIALS

This chapter investigates the robust stability of a family of polynomials with respect to a stability domain $D$.

3.1 PROBLEM SET-UP

Consider the polynomial family of degree $n$

$$f(s, \Gamma) = \sum_{i=0}^{n} a_i(\gamma)s^i \quad (3.1)$$

and

$$\Gamma: \gamma_k \leq \gamma_k \leq \overline{\gamma}_k \quad k = 1, \ldots, \nu$$

where the coefficients $a_i$ depend in an affine way on a set of independent parameters $\gamma_k$. The relation between $a$ and $\gamma$ is given by

$$a = a_0 + T\gamma \quad (3.2)$$

with a full rank matrix $T \in \mathbb{R}^{(n+1)\times \nu}$. The uncertainty domain $\Gamma$ is a hyperrectangle in $\mathbb{R}^\nu$ and with the affine map (3.2)

$$\gamma \in \Gamma \in \mathbb{R}^\nu \rightarrow a \in A \in \mathbb{R}^{n+1}$$

the domain $A$ is a parpolytope in the coefficient space $\mathbb{R}^{n+1}$. 

13
Consider for example the polynomial family as the characteristic polynomial of a control system. Assuming the polynomials $a(s, \gamma)$ and $b(s, \gamma)$ of the plant in Fig. 2.1 to be interval polynomials, then with $c(s)$ and $d(s)$ fixed polynomials, the coefficients $p_i$ of the characteristic polynomial

$$P(s) = c(s)a(s, \gamma) + d(s)b(s, \gamma) = \sum_{i=0}^{n} p_i(\gamma)s^i$$

are affine functions of the parameters $\gamma$.

The goal is to find a test for the robust $D$-stability of the polynomial family $f(s, \Gamma)$. In other words, we have to find a criterion which ensures that all the roots of every member of the family $f(\cdot)$ are located in the open domain $D$. The central point is to avoid huge computation of the root space of the continuum of polynomials by using gridding methods over the family.

### 3.2 REVIEW OF PRELIMINARY RESULTS

In the sequel, two important results are presented that allowed a drastic reduction of the conditions for the stability test of uncertain polynomial families.

**Theorem 2 (Kharitonov theorem)**

The interval polynomial

$$f(s, \gamma) = \gamma_0 + \gamma_1 s + \gamma_2 s^2 + \cdots + \gamma_n s^n$$  \hspace{1cm} (3.3)

with \( \gamma_i \leq \gamma_i \leq \overline{\gamma}_i \) \hspace{1cm} \( i = 0, \cdots, n \)

is Hurwitz stable if and only if the four polynomials

$$f_1(s) = \underline{\gamma}_0 + \gamma_1 s + \gamma_2 s^2 + \gamma_3 s^3 + \gamma_4 s^4 + \cdots$$  \hspace{1cm} (3.4)

$$f_2(s) = \underline{\gamma}_0 + \gamma_1 s + \gamma_2 s^2 + \gamma_3 s^3 + \gamma_4 s^4 + \cdots$$  \hspace{1cm} (3.5)

$$f_3(s) = \overline{\gamma}_0 + \gamma_1 s + \gamma_2 s^2 + \gamma_3 s^3 + \gamma_4 s^4 + \cdots$$  \hspace{1cm} (3.6)
are Hurwitz stable.

Proof: The original proof can be found in [2]. Alternative proofs are presented in [45, 46, 47, 48].

This result is independent of the degree of the interval polynomial \( f(s, \gamma) \). The test of the four so-called Kharitonov polynomials is necessary and sufficient for the Hurwitz stability of the whole polynomial family. For low order polynomials \( f(s, \gamma) \) of degree \( n < 6 \) even less than four polynomials are necessary and sufficient to guarantee Hurwitz stability of the family [49].

The expressions of the four Kharitonov polynomials can be easily explained: If the polynomial \( f(s, \gamma) \) is decomposed into its even and odd part

\[
f(s) = h(s^2) + sg(s^2)
\]

then, the four Kharitonov polynomials correspond to

\[
f_1(s) = h(s^2) + sg(s^2)
\]

\[
f_2(s) = \overline{h}(s^2) + sg(s^2)
\]

\[
f_3(s) = \overline{h}(s^2) + s\overline{g}(s^2)
\]

\[
f_4(s) = h(s^2) + s\overline{g}(s^2)
\]

with

\[
h(s^2) = \gamma_0 + \gamma_2 s^2 + \gamma_4 s^4 + \cdots
\]

\[
\overline{h}(s^2) = \overline{\gamma}_0 + \overline{\gamma}_2 s^2 + \overline{\gamma}_4 s^4 + \cdots
\]

\[
g(s^2) = \gamma_1 + \gamma_3 s^2 + \gamma_5 s^4 + \cdots
\]

\[
\overline{g}(s^2) = \overline{\gamma}_1 + \overline{\gamma}_3 s^2 + \overline{\gamma}_5 s^4 + \cdots
\]
In [45], it was shown that the value set of an interval polynomial for every fixed frequency \( s = j\omega^* \) is an axis parallel box in the complex plane. The four vertices of the value set are just the images of the four Kharitonov polynomials.

Theorem 2 is often called strong Kharitonov theorem since the condition for the Hurwitz stability of the interval polynomial is independent of the degree \( n \). As seen before, the interval polynomial (3.3) is associated with an \( n \)-dimensional hyperrectangle in parameter space. The weak Kharitonov theorem [2] states that the Hurwitz stability of all \( 2^{n+1} \) vertices of the hyperrectangle is necessary and sufficient for the stability of the interval polynomial. The strong version of the theorem reduces therefore the number of vertices to be tested to simply four special ones, associated with the four Kharitonov polynomials, which means a drastic reduction of the necessary and sufficient stability conditions.

The Kharitonov theorem is restricted to the Hurwitz stability of interval polynomials, i.e. the Hurwitz stability of a hyperrectangle of polynomials. In the discrete-time case the Kharitonov theorem does not hold for interval polynomials. The conditions for Schur stability of interval polynomials are investigated in [4]. A discrete-time analogy to the weak Kharitonov theorem was derived in [5] and a result similar to the strong Kharitonov theorem was obtained in [6] for polynomial families with symmetric/antisymmetric parameterization.

The more general case of an uncertain polynomial is a polytope of polynomials where the edges are no longer axis parallel and the Kharitonov theorem is not valid. For this case an important result was presented in [15]:

**Theorem 3 (Edge theorem)**

Let \( \mathcal{A} \subset \mathbb{R}^n \) be a polytope of polynomials. Then, the boundary of the root space of \( \mathcal{A} \) is contained in the root space of all exposed edges of the polytope \( \mathcal{A} \).

The exposed edges in Theorem 3 are the one-dimensional faces of the polytope \( \mathcal{A} \). This theorem implies a reduction of the stability test of
a $n$-dimensional continuum to the multiple test of a one-dimensional continuum of polynomials.

For the test of the edge polynomials as convex combinations of two vertex polynomials there exist several methods [50, 51, 52, 53]. These tests are generally more complex than the stability test of vertex polynomials, which leads to the natural desire of similar results as the weak Kharitonov theorem, so-called vertex results, where the stability of an edge is guaranteed by the stability of the two associated vertex polynomials. Vertex results for special forms of edge polynomials and stability domains can be found in [11, 12, 20, 47, 54].

3.3 SOLUTION OF THE D-STABILITY PROBLEM

The problem of the robust $D$-stability of a polynomial family $f(s, \Gamma)$ is solved in the frequency domain using the zero exclusion principle (Theorem 1 on page 12). For this the value set of the polynomial family (3.1) for $s$ along the boundary $\partial D$ must be determined.

The polynomial family in (3.1) and (3.2), with $\Gamma$ a hyperrectangle in $\mathbb{R}^\nu$, is equivalent to a $\nu$-dimensional parpolytope $A$ in the coefficient space $\mathbb{R}^{n+1}$. For the value set of $f(s, \Gamma)$ follows:

**Lemma 1** Let $f(s, \Gamma)$ be a parpolytope of polynomials in the coefficient space. Then the value set of $f(s, \Gamma)$ with $s$ along $\partial D$ is a parpolygon. The edges of the value set are the images of a subset of the edges of the parpolytope of polynomials.

**Proof:** See Appendix A. ■

From Lemma 1, it follows that the preimages of the edges of the values set are a subset of the edges of $\Gamma$. However, the edges of the value set can be formed by different edges of the hyperrectangle $\Gamma$ during the movement of $s$ along $\partial D$ [45]:

**Definition 4** The exposed edges of $\Gamma$ with respect to $\partial D_i$ are all those edges of $\Gamma$ that are mapped to the edges of the value set for some $s \in \partial D_i$. 
The notation 
\[ f_A(s, \Gamma) \]

is introduced to indicate that \( f_A(s) \) is a polynomial family with a par-polytopic uncertainty domain in the coefficient space, i.e. that the coefficients \( a_i \) are an affine function of \( \gamma \in \Gamma \). With Lemma 1, Theorem 1 on page 12 can be modified and the test for the robust D-stability of the family of polynomials (3.1) follows [44]:

**Theorem 4** The polynomial family \( f_A(s, \Gamma) \) is robustly D-stable if and only if

1. \( f_A(s, \gamma^*) \) is D-stable for some \( \gamma^* \in \Gamma \)
2. \( 0 \not\in f_A(s_i, \Gamma) \) for some \( s_i \in \partial D_i \quad \forall i = 1, 2, \ldots, k \)
3. \( 0 \not\in f_A(s, \Gamma_{e}) \) \( \forall s \in \partial D_i \) where \( \Gamma_{e} \): exposed edges of \( \Gamma \).

With condition (i) of the theorem the D-stability of one polynomial of the family is tested for the correct increase of argument. This can be considered as the stability check of a "nominal" polynomial. For condition (ii) the value set is determined at one frequency \( s^* \) at each part \( \partial D_i \) of the boundary of the stability domain \( D \). If the origin is outside of the value set at one specific frequency, then, it could only enter into the value set for \( s \) along \( \partial D \) through one of the edges of the value set, hence condition (iii). Therefore, the edges of the value set, i.e. the exposed edges of \( \Gamma \), must be tested if they include the origin for some \( s \in \partial D \).

**Remark 1** If \( \partial D_i \) crosses the real axis, condition (ii) of the theorem can be simplified. A family of real polynomials \( f(s, \Gamma) \), with \( s_i \in \mathbb{R}, s_i \in \partial D_i \), is simply an interval on the real axis and it can be easily checked whether the origin is outside or inside the interval (Fig. 3.1).

To test the D-stability of an edge as a convex combination of two vertex polynomials \( f_i(s) \) and \( f_j(s) \) of the hyperrectangle \( \Gamma \) (at least one of them being D-stable), it is sufficient to check if the expression

\[ f_{ij}(s) = f_i(s) + \lambda[f_j(s) - f_i(s)] \quad \lambda \in (0, 1) \]
becomes zero along \( s \in \partial D \). Based on the concept of collinearity of pointers this zero-inclusion can be easily tested. The boundary \( \partial D \) of the stability domain \( D \) is parameterized as

\[
\partial D = \{ s = \varphi(\delta), \delta \in I_\varphi \} \tag{3.15}
\]

with \( \varphi(\cdot) \) a complex valued function and \( I_\varphi \) a given interval.

**Theorem 5** Let the vertex polynomials \( f_k(s) \) be parameterized on the boundary \( \partial D \) as

\[
f_k(\varphi(\delta)) = h_k(\varphi(\delta)) + g_k(\varphi(\delta))
\]

with \( \delta \in I_\varphi \) and

\[
f_k(\varphi(\delta)) = 0 \iff h_k(\varphi(\delta)) = g_k(\varphi(\delta)) = 0
\]

where \((h_k, g_k)\) is a linear independent decomposition of \( f_k \) along \( \partial D \). Then with

\[
z(\delta) = \det \begin{bmatrix} h_i(\varphi(\delta)) & h_j(\varphi(\delta)) \\ g_i(\varphi(\delta)) & g_j(\varphi(\delta)) \end{bmatrix}
\]

the edge \( f_i(s) \rightarrow f_j(s) \) does not contain the origin if and only if for all roots \( \delta^* \in I_\varphi \) of \( z(\delta) \) it holds

\[
h_i(\varphi(\delta^*))h_j(\varphi(\delta^*)) + g_i(\varphi(\delta^*))g_j(\varphi(\delta^*)) > 0 \tag{3.16}
\]
Proof: The edge \( f_i(s) \rightarrow f_j(s) \) contains the origin for a fixed \( s = s^* \) only if the associated pointers \( f_i(s^*) \) and \( f_j(s^*) \) are collinear and have a different orientation. In Fig 3.2 the situation is graphically clarified. Fig 3.2a shows an edge with arbitrary location in the complex plane. Collinearity of the two vertex pointers is shown in (b) and (c). But only in case (b) the edge contains the origin.

Testing the roots of \( z(\delta) \) equals testing the collinearity of the pointers and with (3.16), the orientation of the pointers is checked.

With Theorem 5 the investigation of the location of the origin on the edges is reduced to a test of polynomials where a positivity condition must be fulfilled for all roots \( \delta^* \in I_5 \) of \( z(\delta) \).

The simplest linear independent decomposition of the vertex polynomials \( f_i(s) \) in Theorem 5 is the decomposition into its real and imaginary part. However, depending on the parameterization of the boundary \( \partial D \) different decompositions can be more appropriate (see [6]).

Alternative tests for the \( D \)-stability of the edges can be found in [51, 52, 53].
3.3. Solution of the $D$-stability problem

3.3.1 Construction of the value set

The polynomial family $f(s, \Gamma)$ of (3.1) with (3.2) can be written in the affine form

$$f(s, \Gamma) = f_0(s) + \sum_{i=1}^{\nu} f_i(s) \gamma_i \quad \gamma_i \leq \gamma_i \leq \gamma_i$$

Then, the value set of $f(s, \Gamma)$, for $s = s^*$ fixed, is a parpolygon with generically $2\nu$ edges. With the pointers

$$l_i = \left[ \begin{array}{c} \text{Re} f_i(s) \\ \text{Im} f_i(s) \end{array} \right]_{s=s^*} \quad i = 1, 2, \ldots, \nu$$

the value set can now be constructed. The value set is characterized by an ordered list of vertices. Each vertex of the value set corresponds to a vertex of the parameter polytope $\Gamma$ which is given by a list of extrema of the associated uncertain parameters:

$$\gamma_i \in \{\gamma_i, \gamma_i\} \quad i = 1, 2, \ldots, \nu$$

This list, denoted as $e_k$, represents therefore a vertex polynomial.

Construction

All the pointers $l_i$ are given in the complex plane (Fig. 3.3). Assume the pointers are numbered with increasing argument:

$$\angle \tilde{f}_1 < \angle \tilde{f}_2 < \cdots < \angle \tilde{f}_{\nu}$$

where

$$\angle \tilde{f}_i = \angle f_i \pmod{\pi}$$

(see Fig. 3.4 on page 25).
Chapter 3. Robust D-stability

Figure 3.3: Pointers in the complex plane.

Step 1: Choice of the first vertex

Choose the vertex $\varepsilon_0 = [\gamma_i^*]$ of $\Gamma$ with

$$\gamma_i^* = \begin{cases} \gamma_i, & \text{if } \text{Im}f_i > 0 \lor (\text{Re}f_i > 0 \land \text{Im}f_i = 0) \\ \gamma_i^*, & \text{otherwise} \end{cases} \tag{3.17}$$

With this (arbitrary) setting, a leftmost vertex of the value set with the smallest imaginary part will be found. This vertex is denoted as $E_0$ in Fig. 3.4b. The sequence of the vertices of the value set is then determined anticlockwise.

Step 2: Determination of the sequence of vertices

Determine the vertices $\varepsilon_k$ recursively. The vertex $\varepsilon_k$ is obtained from $\varepsilon_{k-1}$ with:

$$e_k(i) = e_{k-1}(i) \quad i \neq i_k \quad k = 1, 2, \ldots, \nu \tag{3.18a}$$

$$e_k(i_k) = \int e_{k-1}(i_k) \tag{3.18b}$$

where the sign "$\int$" denotes the exchange of all the extrema of the associated parameter $\gamma_i$

$$\int \gamma_i = \gamma_i \quad \text{and} \quad \int \gamma_i = \gamma_i \tag{3.19}$$
3.3. Solution of the D-stability problem

Associated to every list $e_k$ there is a vertex of the value set denoted by $E_k$.

**Step 3: Closing the boundary**

The boundary of the value set is terminated by

$$e_{i+\nu} = \int e_i \quad i = 1, 2, \ldots, \nu$$  \hspace{1cm} (3.20)

**Remark 2**

1. Notice that $e_{\nu} = \int e_0$ and $e_{2\nu} = \int e_{\nu} = e_0$; thus the boundary is closed. The boundary of the value set is therefore determined by knowledge of the sequence of vertices from $e_0$ to $e_{\nu-1}$.

2. By construction, it is clear that the structure, i.e. the building vertices $e_k$, of the value set mainly depends on the relative position of the pointers $f_i$, $i = 1, \ldots, \nu$.

3. As long as the mutual position of the pointers stays the same, the vertices of the value set are images of the same vertex polynomials of $\Gamma$. Generally speaking, two pointer configurations are equal if and only if one can carry one configuration to the other without making any two pointers collinear.

4. For the determination of the vertices and the exposed edges, the length of the pointers is irrelevant. The vertex polynomials can be determined without knowledge of the actual bounds of the uncertain parameters.

For a clarification of the procedure see the following example.

**Example 1** Consider the pointer configuration shown in Fig. 3.4a. With (3.17) the list of the extrema of the first vertex is determined to

$$e_0 = [\gamma_1 \ \bar{\gamma}_2 \ \gamma_3 \ \gamma_4]'$$
and

\[ \angle f_3 < \angle f_1 < \angle f_2 < \angle f_4 \]

Skipping from one vertex to the other in anticlockwise direction around the boundary of the value set, the sequence of changing the extrema of \( \gamma_k \) is \( k = \{3, 1, 2, 4\} \). Therefore, the extreme value of the parameter \( \gamma_3 \) changes from vertex 0 to vertex 1

\[ e_1(3) = \int e_0(3) \Rightarrow e_1 = [ \gamma_1 \gamma_2 \gamma_3 \gamma_4 ]' \]

The associated vertices of the value set are \( E_0 \) and \( E_1 \) shown in Fig. 3.4b. The vertex polynomials are then

\[ f_{e_i}(s) = f_0(s) + [ f_1(s) f_2(s) f_3(s) f_4(s) ] e_i \]

and the first edge polynomial as convex combination of \( f_{e_0}(s) \) and \( f_{e_1}(s) \) follows by variation of \( \gamma_3 \).

The vertices of \( \Gamma \), associated with the value set vertices, are shown in Table 3.1 where "-" indicates the minimum and "+" the maximum value of the associated parameter \( \gamma_i \). Note that to identify the vertices of the value set, it suffices to know the vertices \( e_0 \) to \( e_3 \) which corresponds to half the boundary line. The rest is obtained by continuation of the sequence \( k = \{3, 1, 2, 4\} \) of changing the extrema of the uncertain parameters.

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Table 3.1: Vertices of the value set.

The edges of the value set are then given as a map of a convex combination between the neighboring vertices of \( \Gamma \), for example the first
edge polynomial is

\[ f_{ed_1}(s) = f_{e_0}(s) + \lambda(f_{e_1}(s) - f_{e_0}(s)) \quad \lambda \in [0, 1] \]

\[ \Delta \]

**Figure 3.4: Pointer configuration and value set.**

**Remark 3** If there are two or more collinear pointers in an interval on the border \( \partial D \) of the stability region, then, the associated extrema of the uncertain parameters will be changed in one step. The corresponding value set is degenerate, i.e. the number of vertices is smaller than \( 2\nu \). The collinear pointers can be combined together in the interval considered. Suppose

\[ \angle \tilde{f}_1 = \angle \tilde{f}_2 = \cdots = \angle \tilde{f}_\eta \]

then, this group can be replaced by

\[ f_1 \gamma_1 + f_2 \gamma_2 + \cdots + f_\eta \gamma_\eta = f_1 \left( \frac{f_2}{f_1} \gamma_2 + \cdots + \frac{f_\eta}{f_1} \gamma_\eta \right) = f_1 \Gamma_1 \]

The extreme bounds of the new parameter \( \Gamma_1 \) are

\[ \Gamma_1 \quad \text{and} \quad \bar{\Gamma}_1 = \int \Gamma_1 \]
where $\Gamma_1$ is reached with $\gamma_1$ and

$$
\gamma_k = \begin{cases} 
\gamma_k, & \text{if } \frac{f_k}{f_1} > 0 \\
\gamma_k, & \text{otherwise} 
\end{cases} \quad k = 2, \cdots, \eta
$$

Now, as long as the sign of $f_k/f_1$ for $k = 2, \cdots, \eta$ does not change, the pair $(f_1, \Gamma_1)$ can be used instead of $(f_1, \gamma_1)$. Every sign change causes a change in the vertex structure associated with the value set.

### 3.3.2 $\delta$-intervals with constant structure of the value set

As pointed out in Remark 2 on page 23, the structure of the value set remains constant for $s$ along the boundary $\partial D$ as long as no two pointers $f_i, f_j$ become collinear. If collinearity of two pointers occurs, then the building polynomials for the vertices of the value set change. With the given parameterization of the boundary $\partial D$ of the stability domain $D$ it is possible to determine those intervals where the structure of the value set remains constant. Then, the vertex polynomials can be determined once for each interval $\delta$. Let $\partial D$ be parameterized as follows

$$
\partial D = \{s = \varphi(\delta), \delta \in I_\delta\}, \quad I_\delta \text{ a given interval}
$$

(3.21)

The condition for collinearity of two pointers is

$$
z_{ij}(\delta) = \det \begin{bmatrix} h_i(\varphi(\delta)) & h_j(\varphi(\delta)) \\ g_i(\varphi(\delta)) & g_j(\varphi(\delta)) \end{bmatrix} = 0; \quad i, j = 1, \cdots, \nu; \quad i > j
$$

(3.22)

where $h_k, g_k$ is a linear decomposition of $f_i(s)$ along $\partial D$ as given in Theorem 5. The bounds of the intervals $I_\delta$ (Fig. 3.5) with constant structure of the value set are given by the set

$$
\{\delta_k\} = \{\delta \in I_\delta \mid z_{ij}(\delta) = 0; \quad i, j = 1, \cdots, \nu; \quad i > j\}
$$

(3.23)
3.3. Solution of the $D$-stability problem

This result plays an important role for the implementation of the $D$-stability test on a computer. With known intervals $I_k$ of $\delta$, the structure of the value set, i.e. the list of the building vertices given by the associated list of the extreme values of the uncertain parameters, has to be determined only once in each interval (§ 3.3.1). Since the structure of the value set is independent of the explicit bounds of the uncertain parameters, the list of vertices can be computed without knowledge of the exact uncertainty domain.

3.3.3 $s$-plane partitions

In § 3.3.2, intervals were found along $\partial D$ with constant structure of the value set. The boundaries between these intervals are characterized by the collinearity of any two pointers $f_i$ and $f_j$.

The resulting pointer configurations can be determined not only for $s$ along $\partial D$, but also directly for the whole $s$-plane.

Collinearity of pointers for some given $s$ can be expressed as

$$f_i(s) = kf_j(s)$$

with $k \in \mathbb{R}$. Without loss of generality we assume

$$\deg(f_i) > \deg(f_j).$$

Then, with a slight modification the known formulation

$$1 - k \frac{f_j(s)}{f_i(s)} = 0$$
of the root locus problem is obtained. The root loci for all pairs
\( i,j = 1, \ldots, \nu, i > j \) and \( k \in (-\infty, \infty) \) represent the boundaries of
the partitions in the \( s \)-plane with constant structure of the value set.
On these boundaries at least two associated pointers get collinear.
Therefore, for a given polynomial family, the partitions in the \( s \)-plane
can be obtained from the root loci of all combinations of two building polynomials \( f_i(s) \) and \( f_j(s) \). Then, the structure of the value set,
i.e. the vertex list, can be determined in each partition of the \( s \)-plane.
For any arbitrary stability domain \( D \), all occurring structures of the
value set and hence all the exposed edges of the parameter polytope
\( \Gamma \) which have to be tested for robust \( D \)-stability, can be obtained as
the union of all structures of partitions which are crossed by \( \partial D \) (some
multiple edges may have to be eliminated).

**Example 2** Consider the polynomial family

\[
f(s, \gamma) = f_0(s) + \sum_{i=1}^{\nu} \gamma_i s^{i-1}
\]

The building polynomials \( f_i(s) \) are \( 1, s, s^2, \ldots, s^{\nu-1} \) and the following
root loci result

\[
\frac{1}{s}, \frac{1}{s^2}, \frac{1}{s^3}, \ldots, \frac{1}{s^{\nu-1}}
\]

For \( \nu = 4 \), the partitions in the \( s \)-plane are shown in Fig. 3.6. The
associated structures of the value set are determined by the sequences
of the vertices and each vertex is described as a list of the extreme
values of the uncertain parameters. In Table 3.2 the vertex lists of
the four sectors in Fig. 3.6 are presented. Four essentially different
structures of the value set can occur, depending on the location of the
stability domain \( D \) in the \( s \)-plane.

**Remark 4** Due to symmetry, the value sets of the domains in the
lower half plane are the same as in the upper half (Fig. 3.6). However,
according to the construction rules on page 21, the sequence of the
vertices of the value set is opposite to the one obtained in the upper
half plane.
3.3. Solution of the $D$-stability problem

Aim

Figure 3.6: $s$-plane partitions.

Table 3.2: Different structures of the value set.
Example 3 Consider the polynomial family of degree 5 with symmetric/antisymmetric parameterization \([43]\]

\[
f(s, \gamma) = (\gamma_0 + \gamma_5)s^5 + (\gamma_1 + \gamma_4)s^4 + (\gamma_2 + \gamma_3)s^3
+ (\gamma_2 - \gamma_3)s^2 + (\gamma_1 - \gamma_4)s + (\gamma_0 - \gamma_5)
\]

with \(\gamma_i \in [\gamma_i, \gamma_i], i = 0, \ldots, 5\). The family can be rewritten in the usual representation

\[
f(s, \gamma) = (s^5 + 1)\gamma_0 + (s^4 + s)\gamma_1 + (s^3 + s^2)\gamma_2 + (s^3 - s^2)\gamma_3
+ (s^4 - s)\gamma_4 + (s^5 - 1)\gamma_5
\]

Now the partitions in the s-plane are determined by plotting all possible root loci (Fig. 3.7). It is obvious, that in comparison with Example 2 the partitioning of the s-plane is much more complex. \(\diamondsuit\)
3.4 TEST OF THE ROBUST $D$-STABILITY

With the results of the previous sections, the test of the robust $D$-stability of a family of polynomials is summarized. It consists of the following steps:

Step 1: Parameterization of the polynomial family along $\partial D$

There are two possibilities to parameterize the polynomial family $f(s, \Gamma)$ along the boundary $\partial D$ of the stability domain $D$:

1. Mapping of the stability domain $D$ to the left half of the $s$-plane and evaluation of $s$ along the imaginary axis. Then, the $D$-stability problem is transformed to the usual Hurwitz stability problem.

2. Direct parameterization of the polynomial family along the boundary $\partial D$ with $s = \varphi(\delta)$, $\delta \in I_\delta$.

The first case is only suitable for special domains $D$ where a closed transformation of the stability domain to the left half of the $s$-plane exists. An important example is the bilinear map of the unit circle to the left half plane such that the stability problem of discrete-time systems can be treated as Hurwitz stability problem of the transformed system [55].

Step 2: Affine representation of the polynomial family

The polynomial family is written as

$$f(\delta, \Gamma) = f_0(\delta) + \sum_{i=1}^{\nu} f_i(\delta) \gamma_i$$

with $\delta \in I_\delta$.

Step 3: Determination of the $\delta$-intervals

The intervals of $\delta$ along the boundary $\partial D$ with a constant structure of the value set are determined. This leads to the collinearity test of
all pairs of polynomials $f_i(s)$ and $f_j(s)$, $i > j$ and $i, j = 1, \ldots, \nu$. The roots of (3.22) give the bounds of the intervals of $\delta$.

**Remark 5** The nominal part $f_0(\delta)$ of the polynomial family simply causes a translation of the whole value set. The structure of the value set is therefore independent of $f_0(\delta)$. Naturally the nominal part must be taken into account for the stability test in *Step 5* to 7.

**Step 4: Construction of the value set**

The vertex and edge polynomials associated with the vertices and edges of the value set are determined for each $\delta$-interval at one frequency point (§ 3.3.1) and multiple edges are eliminated.

Now the preliminary work for the test of the robust $D$-stability of the polynomial family $f(s, \Gamma)$ was done and the stability test of Theorem 4 on page 18, which consists of three hierarchical parts, can be performed.

**Step 5: Test of the $D$-stability of a fixed polynomial**

One member of the polynomial family, often denoted as *nominal polynomial*, is chosen and its $D$-stability investigated, i.e. we check if all roots of this polynomial lie in $D$.

$$\Rightarrow$$ If Step 5 fails, then the polynomial family $f(s, \Gamma)$ is not robustly $D$-stable. Otherwise go to Step 6.

**Step 6: Test of the zero exclusion of the value set for some $s_i^* \in \partial D_i$**

For a fixed frequency $s_i^*$ on each part $\partial D_i$ of the boundary of the stability domain $D$ the value set is explicitly determined and tested if the origin is located outside.

If the boundary $\partial D_i$ crosses the real axis, the frequency point $s_i^*$ on the real axis should be chosen since the value set degenerates then to an interval on the real axis and the zero location can be easily tested (Remark 1 on page 18).
3.5. Applications

$$\Rightarrow$$ If Step 6 fails, then the polynomial family \( f(s, \Gamma) \) is not robustly D-stable. Otherwise go to Step 7.

**Step 7: Test of the D-stability of the exposed edges of \( \Gamma \)**

For all \( s \) along the boundary \( \partial D \) the exposed edges of the polytope \( \Gamma \) are tested if they do not contain the origin. In each interval \( I_s \) only the edges of the associated structure of the value set have to be tested.

$$\Rightarrow$$ If Step 7 holds, then the polynomial family \( f(s, \Gamma) \) is robustly D-stable.

Fig. 3.8 shows a flowchart of this procedure for the D-stability test.

The construction of the value set and the test of the robust D-stability of a family of polynomials can be easily implemented in a CACSD package on a computer to provide a useful tool for the analysis of uncertain systems. In [55], a toolbox implemented in MATLAB \(^1\) is described which allows the computation of the vertex lists of the value set and the test of the robust stability with respect to different stability domains \( D \).

3.5 APPLICATIONS

3.5.1 Robust Hurwitz stability of the closed-loop

In this section, the results of § 3.1 – § 3.4 are applied to investigate the robust Hurwitz stability of the closed-loop system depicted in Fig. 2.1.

The value set of the characteristic polynomial

The transfer function of the uncertain plant in Fig. 2.1 is given by

\[
G_\delta(s) = \frac{b(s)}{a(s)}
\]

\(^1\)MATLAB is a trademark of The MathWorks, Inc. [56]
Figure 3.8: D-stability test.
with \( a(\cdot) \) and \( b(\cdot) \) interval polynomials, i.e.

\[
\begin{align*}
a_i \leq a_i \leq \bar{a}_i & \quad i = 0, \ldots, n \\
b_j \leq b_j \leq \bar{b}_j & \quad j = 0, \ldots, m
\end{align*}
\]

and the fixed controller

\[
G_R(s) = \frac{d(s)}{c(s)}
\]

Then, the characteristic polynomial is

\[
P(s) = c(s)a(s) + d(s)b(s) \quad (3.24)
\]

which is a family of polynomials where the coefficients \( p_i \) of \( P(s) \) are affine functions of the interval coefficients \( a_i \) and \( b_j \).

The value set of \( P(s) \) could be determined for \( s \) along the imaginary axis by applying the construction rules of § 3.3.1. In what follows, the special structure of (3.24) will be utilized by a slightly different method to obtain general results in an easy manner. The value set of \( P(s) \) is composed by the simpler value sets of the interval polynomials \( a(s) \) and \( b(s) \). First, these simple value sets are determined and then, the influence of the controller polynomials \( c(s) \) and \( d(s) \) on those regions is investigated. Finally, the value set of the characteristic polynomial \( P(\cdot) \) is obtained by addition of the two resulting partial regions.

Each of the two polynomials \( a(s) \) and \( b(s) \) can be partitioned in an even and odd sub-polynomial:

\[
\begin{align*}
a(s) &= h_1(s^2) + sg_1(s^2) \\
b(s) &= h_2(s^2) + sg_2(s^2)
\end{align*}
\]

The value sets of \( a(s) \) and \( b(s) \), for \( s = j\omega^* \), are rectangles whose vertices are just the images of the four Kharitonov polynomials (see § 3.2). For these rectangles we introduce the term Kharitonov box. The box \( A \) of the polynomial \( a(s) \) is given by the four vertices

\[
a_1(s) = h_1(s^2) + sg_1(s^2)
\]
\[ a_2(s) = \overline{h}_1(s^2) + sg_1(s^2) \]
\[ a_3(s) = \overline{h}_1(s^2) + sg_1(s^2) \]
\[ a_4(s) = h_1(s^2) + sg_1(s^2) \]

and the vertices of the \( B \)-box are obtained in the same way with the polynomials \( h_2(s^2) \) and \( g_2(s^2) \) instead of \( h_1(s^2) \) and \( g_1(s^2) \).

It is obvious that for \( s = j\omega \), \( h_i(s^2) \) is the real part and \( \omega g_i(s^2) \) the imaginary part of \( a(s) \) and \( b(s) \) respectively. Therefore, there exist only two independent pointer directions for \( s = j\omega \) in the complex plane:

1 and \( j\omega \)

Starting from the Kharitonov boxes of \( a(s) \) and \( b(s) \) with the pointers 1 and \( j\omega \), the value sets of \( c(s)a(s) \) and \( d(s)b(s) \) can now be determined. For \( s = j\omega \) where \( \omega \in [0, \infty) \), the two boxes are distorted by \( c(j\omega) \) and \( d(j\omega) \). The resultant pointers are:

\[
\begin{align*}
H_1 &= 1 \cdot c(j\omega) \\
G_1 &= j\omega \cdot c(j\omega) \\
\tilde{A}\text{-box} \hspace{1cm} H_2 &= 1 \cdot d(j\omega) \\
G_2 &= j\omega \cdot d(j\omega) \hspace{1cm} \tilde{B}\text{-box}
\end{align*}
\]

As stated in § 3.3.1 for the construction of the structure of the value set, i.e. for the determination of the associated vertex polynomials, only the mutual position of the pointers is relevant. With \( \omega \in [0, \infty) \), four mainly different pointer configurations can occur (Fig. 3.9). It is important to notice that the configuration depends only on the difference angle between \( H_1 \) and \( H_2 \). This angle is just the phase of \( d(s)/c(s) \). The four configurations are therefore characterized by the phase of the controller transfer function \( d(s)/c(s) \):

\[
-\frac{\pi}{2} \leq \frac{d(s)}{c(s)} \bigg|_{s=j\omega} \leq -(\chi - 1)\frac{\pi}{2} \tag{3.25}
\]

where \( \chi = 1, \ldots, 4 \) correspond to the pointer configuration I,II, III or IV.
From the four pointer configurations, the vertex list and the exposed edges of the value set of the polynomial family $P(s)$ can be extracted using the procedure of § 3.3.1. In Table 3.3 the vertices of the four different structures of the value set are given by their lists of extrema of the associated parameters. Again "−" denotes the minimum and "+" the maximum value of the associated parameter. In Table 3.3 only half of the boundary of the value set is indicated since the rest is obtained by continued application of the operator "" ∫ ".

Table 3.3: The four different structures of the value set.
For the configuration I, the first vertex polynomial is
\[ p_1(s) = c(s)[h_1(s^2) + sg_1(s^2)] + d(s)[\bar{h}_2(s^2) + s\bar{g}_2(s^2)] \]
and the first edge goes from the vertex polynomial \( p_1(s) \) in the direction of
\[ c(s)[h_1(s) - h_1(s)] \]
to the vertex polynomial
\[ p_2(s) = c(s)[\bar{h}_1(s^2) + s\bar{g}_1(s^2)] + d(s)[\bar{h}_2(s^2) + s\bar{g}_2(s^2)] \]
The resulting value set for every pointer configuration I–IV is a par-polygon with 8 vertices. It is even more special since the edges \( k \) and \( k + 2, k = 1, \ldots, 6 \) are also orthogonal to each other.

The vertices and edges of the domain of the characteristic polynomial are uniquely determined by the corresponding extreme polynomials of \( a(s) \) and \( b(s) \) (or equivalently by the associated vertices of the parameter polytope \( \Gamma \)). From Table 3.3, Table 3.4 is directly obtained where the vertex polynomials of the plant are indicated explicitly. From Table 3.4 the eight exposed edges of each pointer configuration I–IV can be directly extracted.

Table 3.4: Vertex polynomials of the value set.

With the previous derivation, the critical exposed edges, i.e. those edges of \( \Gamma \) which have to be checked for the robust Hurwitz stability
3.5. Applications

Figure 3.10: Phase of a controller.

with a particular controller, can be determined. With no a priori knowledge about the controller, 32 edges must be checked. However, from the knowledge of the range of the controller phase, this result can be intensified since from (3.25), for each quadrant which contains the phase of the controller, one structure with 8 edges must be taken into account. Generally, for non-degenerate A- and B-boxes, there can occur only 8, 16, 24 or 32 exposed edges of the value set.

Example 4 Consider the phase of the controller

\[ \frac{d(s)}{c(s)} |_{s=j\omega} \]

given in Fig. 3.10. The phase lies in the sectors I, II and IV. According to this, there exist 3 × 8 exposed edges which build the boundary of the value set for \( \omega \) along the imaginary axis. If the phase crosses all four sectors, then all 32 exposed edges would have to be checked for robust stability.

\[ \Diamond \]
Remark 6

1. There is another, more graphical way to construct the value set of the polynomial family $P(s)$. This approach clarifies the foregoing more systematic procedure. Fig. 3.11a shows the Kharitonov boxes where $A_i, B_i, i = 1, \ldots, 4$ are the images of the four Kharitonov polynomials of the interval polynomials $a(s)$ or $b(s)$ respectively. In Fig. 3.11b these boxes were distorted by the controller. Adding the distorted boxes $\tilde{A}, \tilde{B}$ and building the convex hull, the value set of $P(s)$ in Fig. 3.12 results. There are 8 exposed edges and 24 edges inside the region. Which edges are outside depends only on the relative position of the regions $\tilde{A}, \tilde{B}$. If this position is unknown, all 32 potential exposed edges should be tested.

From Fig. 3.12 it is possible to extract all the exposed edges for every phase interval of the controller.

2. The same result of maximal 32 exposed edges to be tested for the robust Hurwitz stability of the characteristic polynomial was also obtained in [26] using a different approach in parameter space.

3. In [28, 29], it was shown that using a general first order controller, a necessary and sufficient condition for Hurwitz stability of the closed-loop is the stability of the loop for the 16 Kharitonov plants. This result is compatible with our result — the phase of such general first order controller has the range $(-\pi/2; \pi/2)$. Thus, we have just the structures I and IV to check.

In special cases, the number of plants to be tested can be further relaxed. However, the extreme point result presented in [29] cannot be obtained from the value set where 8 exposed edges would have to be tested for stability.

Stability test

After the determination of the exposed edges of the value set, the robust Hurwitz stability of the closed-loop in Fig. 2.1 can be checked
3.5. Applications

using Theorem 4. For Hurwitz stability the $D$-domain is just the left half of the $s$-plane. Since $\partial D$ crosses the real axis, condition (ii) of the Theorem can be used in the simplified form (see Remark 1 on page 18).

**Corollary 1** The closed-loop system (Fig. 2.1) is robustly Hurwitz stable if and only if

(i) the closed-loop with the nominal plant $b^*(s)/a^*(s)$ (or any other realization of the plant) is Hurwitz stable;

(ii) 

\[
(a_0c_0 + b_0d_0)(\alpha_0c_0 + \beta_0d_0) \geq 0 \quad \text{for} \quad c_0d_0 > 0 \tag{3.26a}
\]

\[
(\alpha_0c_0 + \beta_0d_0)(a_0c_0 + \beta_0d_0) > 0 \quad \text{for} \quad c_0d_0 < 0 \tag{3.26b}
\]

where $a_0$, $b_0$, $c_0$, $d_0$ are the constant terms of the polynomials $a(s)$, $b(s)$, $c(s)$ and $d(s)$.

(iii) none of the exposed edges of the value set contains the origin for any $\omega \in [0, \infty)$.

**Proof:** Only (ii) has to be proved. The origin must not be a root of the family $P(s)$ in (3.24). At the frequency $s = 0$ the value set of the characteristic polynomial degenerates to an interval on the real axis. Therefore, the condition for exclusion of the origin is

\[
\min(a_0c_0 + b_0d_0) \max(a_0c_0 + b_0d_0) > 0
\]

whereas for the minimum we choose

\[
b_0 = \begin{cases} b_0, & \text{if } d_0 \geq 0 \\ \bar{b}_0, & \text{otherwise} \end{cases} \quad a_0 = \begin{cases} a_0, & \text{if } c_0 \geq 0 \\ \bar{a}_0, & \text{otherwise} \end{cases}
\]

and vice versa for the maximum. Then (3.26) follows immediately.

Alternatively, the robust stability of a closed-loop can be checked using the Edge theorem (Theorem 3) applying the idea of exposed edges with respect to the stability domain $D$ and checking the robust stability of the exposed edges. Such a method is given in [50] for Hurwitz stability and in [51] for the Schur case. There, the number of edges to be tested for stability is much higher than the minimal necessary set obtained by the method described in this section (see § 3.6).
Chapter 3. Robust D-stability

Figure 3.11: Construction of the value set from the Kharitonov boxes.

Figure 3.12: Value set of $P(s)$. 
3.5. Applications

3.5.2 Robust cone stability of the closed-loop

In the previous section, the Hurwitz stability was discussed as an important special case of the general $D$-stability. In what follows, the robust stability of the closed-loop (Fig. 2.1) is investigated with respect to a cone in the left half of the $s$-plane (Fig. 3.13). The cone stability of an interval polynomial was already addressed in [8, 10, 57]. In this section, the problem is extended to affine parameter dependence and is also presented in a more general way.

![Figure 3.13: Cone stability domain.](image)

The conditions for testing the $D$-stability of a family of polynomials are given in Theorem 4. As the main point, one has to determine the exposed edges of the polynomial family with respect to $D$. Therefore, the following discussion concentrates on the determination of these exposed edges whereas the procedure described in § 3.3.1 for the determination of the value set is used.

The boundary $\partial D$ of the stability domain $D$ in Fig. 3.13 can be parameterized as

$$s = \delta e^{j\varphi_0}, \quad \delta \in [0, \infty)$$

(3.27)

where $\varphi_0$ is the opening angle of the cone.

$$f(\delta e^{j\varphi_0}) = \sum_{i=0}^{n} f_i s^i \bigg|_{s=\delta e^{j\varphi_0}} = f_0 + f_1 \delta e^{j\varphi_0} + \cdots + f_n \delta^n e^{jn\varphi_0}$$

(3.28)
Then, for an interval polynomial $f(s)$ and for a general angle $\varphi_0$ there exist $n + 1$ different directions of the pointers in the complex plane:

$$1, e^{j\varphi_0}, e^{j2\varphi_0}, \ldots, e^{jn\varphi_0}$$

These directions are independent of $\delta$ and therefore, the directions of the edges of the value set remain constant.

If $\varphi_0$ is a rational factor of $\pi$:

$$\varphi_0 = \frac{\rho}{\eta} \pi \quad \rho, \eta \text{ coprime integers} \quad (3.29)$$

then, for such special angles the number of vertices of the value set for high degrees $n$ degenerates because a period in the angles of the pointers occurs. Let

$$\angle s^k = k\varphi_0 = \frac{\rho}{\eta} k\pi \quad k = 0, \ldots, n.$$ 

From the first time when $\chi$

$$\chi = \frac{\rho}{\eta} k$$

becomes an integer for increasing $k$, no new pointer directions are obtained for higher $k$ and the number of vertices of the value set saturates (Fig. 3.14). Because of the coprimeness of $\rho$ and $\eta$, the minimal value of $k$ for this saturation is just $k = \eta$. Thus, one obtains maximally $\eta$ different pointer directions and therefore maximally $2\eta$ vertices. If $n \geq \eta - 1$, the number of vertices becomes independent of the degree of the polynomials.

According to the previous considerations, the determination of the vertices of the value set is now presented for a cone with $\varphi_0 = \frac{2}{3}\pi$. In [58], also the case with $\varphi_0 = \frac{3}{4}\pi$ is discussed.
Because of the periodic character of the trigonometric functions and \( \eta = 3 \), there exist only three different pointer directions with angles 0, \( \frac{2}{3}\pi \) and \( \frac{4}{3}\pi \):

\[
a(\delta e^{j\frac{2}{3}\pi}) = (a_0 + a_3\delta^3 + a_6\delta^6 + \cdots) \\
+ (a_1\delta + a_4\delta^4 + a_7\delta^7 + \cdots) e^{j\frac{2}{3}\pi} \\
+ (a_2\delta^2 + a_5\delta^5 + a_8\delta^8 + \cdots) e^{j\frac{4}{3}\pi}
\]

Following the construction rules of the value set (§ 3.3.1), each of the three groups of collinear pointers can be replaced by a new pointer and an associated parameter. From (3.32) it follows

\[
a(\delta e^{j\frac{2}{3}\pi}) = \Gamma_1 + \Gamma_2\delta e^{j\frac{2}{3}\pi} + \Gamma_3\delta^2 e^{j\frac{4}{3}\pi}
\]

The resulting pointers in the complex plane are then

\[1, \quad \delta e^{j\frac{2}{3}\pi}, \quad \delta^2 e^{j\frac{4}{3}\pi}\]
and the new parameters $\Gamma_i$ are

\begin{align*}
\Gamma_1 &= a_0 + a_3\delta^3 + a_6\delta^6 + \cdots \\
\Gamma_2 &= a_1 + a_4\delta^3 + a_7\delta^6 + \cdots \\
\Gamma_3 &= a_2 + a_5\delta^3 + a_8\delta^6 + \cdots
\end{align*}

(3.34) (3.35) (3.36)

The extremal values of $\Gamma_1$ are

\[
\Gamma_1 = [a_0 \ a_3 \ a_6 \cdots] \Delta = \int \Gamma_1
\]

(3.37)

with $\Delta = [1 \ \delta^3 \ \delta^6 \cdots]'$ and similarly for $\Gamma_2$ and $\Gamma_3$.

The polynomial $b(\delta e^{j\frac{2}{3}\pi})$ is decomposed in the same manner with the new parameters $\Gamma_4$, $\Gamma_5$ and $\Gamma_6$.

From these decompositions, it follows immediately that the directions of the edges of the value sets $A$ and $B$ of the two polynomials $a(s)$ respectively $b(s)$ are $0$, $\frac{2}{3}\pi$, and $\frac{4}{3}\pi$. For $n \geq m \geq 2$ the value sets of $a(s)$ and $b(s)$ are then parpolygons with six vertices (Fig. 3.15).

![Figure 3.15: Value set for $\varphi_0 = \frac{2}{3}\pi$.](image)

Next, the distortion of the value sets $A$ and $B$ caused by the controller polynomials $c(s)$ and $d(s)$ in the characteristic polynomial $P(s) = c(s)a(s) + d(s)b(s)$ has to be considered. The value sets $\tilde{A} = c(s)A$
and $\tilde{B} = d(s)B$ are determined by the pointers

\[
\begin{align*}
\alpha_1 &= c^* \\
\alpha_2 &= c^* \delta e^{j\frac{\pi}{3}} \\
\alpha_3 &= c^* \delta^2 e^{j\frac{\pi}{3}} \\
\beta_1 &= d^* \\
\beta_2 &= d^* \delta e^{j\frac{\pi}{3}} \\
\beta_3 &= d^* \delta^2 e^{j\frac{\pi}{3}}
\end{align*}
\]

with $c^* = c(\delta e^{j\frac{\pi}{3}})$ and $d^* = d(\delta e^{j\frac{\pi}{3}})$. For the mutual position of the two value sets $\tilde{A}$ and $\tilde{B}$ in the complex plane, there exist maximally six different sectors. The mutual position of the pointers remains the same in each of these sectors, and so the edges of the value set are images of the same original edges of the parameter polytope $\Gamma$. These six sectors are determined by the pointers $\alpha_i$ (see Fig. 3.16a). According to the location of the pointer $\beta_1$ in one of these six sectors there results a total of six different pointer configurations and therefore six different value set structures, each with twelve edges. For clarification, Fig. 3.16b shows the pointers of configuration I.

If $\delta$ varies from 0 to $\infty$, the resulting pointer configuration can be extracted directly from the phase diagram of the controller for $s = \delta e^{j\frac{\pi}{3}}$. 

Figure 3.16: Sectors of the pointers in the complex plane.
One has to determine the sectors crossed by the phase of the controller:

\[-\chi \frac{\pi}{3} \leq \left. \frac{d(s)}{c(s)} \right|_{s = \delta e^{i\frac{2\pi}{3}}} \leq -(\chi - 1) \frac{\pi}{3} \tag{3.40}\]

with \(\chi = 1, \cdots, 6\) according to the sectors I, II, \cdots, VI. The resulting list of vertices and exposed edges in Table 3.5 is analogously obtained to Table 3.3 on page 37. In each sector, the value set is bounded by the same set of twelve exposed edges. If the phase of the controller crosses all six sectors, then independent of the plant order, maximally \(6 \times 12 = 72\) exposed edges have to be tested.

\[
\begin{array}{c|ccc}
\alpha_1 & \Gamma_1 & \Gamma_2 & \Gamma_3 \\
\alpha_2 & \Gamma_3 & \Gamma_4 & \Gamma_5 \\
\alpha_3 & \Gamma_6 & \Gamma_7 & \Gamma_8 \\
\beta_1 & \Gamma_9 & \Gamma_{10} & \Gamma_{11} \\
\beta_2 & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\
\beta_3 & \Gamma_{15} & \Gamma_{16} & \Gamma_{17} \\
\end{array}
\]

Table 3.5: The six different structures of the value set.

Example 5 Let the characteristic polynomial

\[p(s) = c(s)a(s) + d(s)b(s)\]

be parameterized along the boundary \(\partial D\) with \(s = \delta e^{i\frac{2\pi}{3}}\). The phase of the controller is assumed to cross the sector

\[-\frac{\pi}{3} \leq \frac{d}{c} \leq 0\]
Then, the first vertex polynomial $p_1(s)$ in configuration I (Table 3.5) is obtained with

$$
\begin{bmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \cdots & \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \cdots
\end{bmatrix}
$$

The parameter $\Gamma_1$ varies from the first to the second vertex (along the direction of $\alpha_1$). Thus, the second vertex polynomial $p_2(s)$ is obtained with

$$
\begin{bmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \cdots & \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \cdots
\end{bmatrix}
$$

\[ \diamond \]

### 3.5.3 The structured stability margin

A measure for the robustness of an uncertain polynomial is the maximal size of the uncertainty domain for which the polynomial family remains robustly stable. This margin is especially interesting in the case where the family of polynomials is the characteristic polynomial of a closed-loop with a fixed controller and an uncertain plant (Fig. 2.1). Then, the stability margin gives the largest size of the uncertainties and therefore the largest family of plants with a given uncertainty structure that can be robustly stabilized by a specific controller. Using the properties of the value set derived in this chapter, the robustness measure can be easily computed in the frequency domain.

Let the polynomial family be described by

$$
f(s, \Gamma) = \sum_{i=1}^{n} f_i(s) \gamma_i
$$

with the uncertain interval parameters $\gamma_i \in \Gamma$ defined as

$$
| \gamma_i - \gamma_i^0 | \leq w_i \quad w_i \geq 0
$$

where $\gamma_i^0$ are the nominal values of the parameters and $w_i$ are nonnegative weights.
The nominal polynomial of the family (3.41) is given by

\[ f_0(s) = f(s, \gamma^0) = \sum_{i=1}^{n} f_i(s) \gamma_i^0 \]

Now we define the structured stability margin \( \rho \) for the polynomial family (3.41):

**Definition 5** The structured stability margin is the largest value \( \rho \) such that for an uncertainty domain \( \Gamma^* \) of the form

\[ \gamma \in \Gamma^* : |\gamma_i - \gamma_i^0| \leq \rho \omega_i \]

the polynomial family \( f(s, \Gamma^*) \) is robustly \( D \)-stable for all values \( \rho \) with \( 0 < \rho < \rho \).

The multiplication of the weights of the uncertainty intervals with \( \rho \) is equal to an expansion of the value set from the nominal point \( f_0(s) \) with the factor \( \rho \).

For the computation of the stability margin \( \rho \), we consider the value set for \( s \) along the boundary \( \partial D \) of the stability domain \( D \). From Theorem 4, it follows that \( D \)-stability of a polynomial family is lost if the origin gets on the boundary or inside of the value set for a specific frequency. As shown at the beginning of this chapter, the value set of the family (3.41) is a parpolygon whose edges are the images of a subset of the edges of the hyperrectangle \( \Gamma \). Let \( \rho_e \) denote the maximal expansion factor at the frequency \( s = s^* \), indicating how much the value set can be expanded from the nominal point \( f_0(s^*) \) before one of the edges includes the origin and, therefore, instability occurs (Fig. 3.17). The critical point \( M \) on the boundary of the value set — where the value set first touches the origin when expanded — is obtained as the intersection of the line \( f_0 \rightarrow O \) with the edge \( E_i \rightarrow E_{i+1} \). Then, the expansion factor is given by the ratio

\[ \rho_e(s^*) = \frac{|f_0(s^*)|}{|f_0(s^*) - M(s^*)|} \quad (3.43) \]
If the origin is on the boundary of the value set, then $M(s^*) = 0$ and with (3.43) follows $\rho_e = 1$. In the case where the origin is inside of the value set, we get

$$|f_0(s^*) - M(s^*)| > |f_0(s^*)|$$

and hence $\rho_e < 1$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{value_set_expansion.png}
\caption{Expansion of the value set.}
\label{fig:value_set_expansion}
\end{figure}

Now we state an important lemma:

**Lemma 2** Assume the nominal polynomial $f_0(s, \gamma^0)$ of the polynomial family $f(s, \Gamma)$, with $\gamma \in \Gamma$ defined in (3.42), to be $D$-stable. Then, the family $f(s, \Gamma)$ is robustly $D$-stable if and only if $\rho > 1$.

**Proof:** The proof uses the argument principle for the value set and the argumentation follows the proof of Theorem 4.

Since $f_0(s)$ is $D$-stable, at least one member of the family $f(s)$ has the correct increase of argument for $s$ along $\partial D$. 
At each frequency $s^*$, $\rho_e$ gives the expansion of the value set from the nominal point $f_0(s^*)$ for which the origin gets on the boundary of the expanded value set. If the origin is already inside or on the boundary, then, from (3.43) it follows that $\rho_e \leq 1$.

With Theorem 4, $D$-stability of the family $f(s)$ is guaranteed if the origin keeps outside of the value set sweeping $s$ along $\partial D$, i.e.

$$\bar{\rho} = \min_{s \text{ along } \partial D} \rho_e(s) > 1$$

hence the result.

The computation of $\rho_e$ for a specific frequency follows the steps:

1. Determine the vertex polynomials of the value set.

2. Determine the critical edge.

   The intersection $M$ of the line $f_0 \rightarrow O$ with the boundary of the value set is determined. Since two edges at a time are parallel, always an edge on each side of the nominal point is obtained. The critical edge is the one with the minimal distance of $M$ from the origin.

3. Compute the expansion factor $\rho_e$ with (3.43).

To simplify the search of the minimal $\rho_e(s^*)$ over all frequencies along the boundary $\partial D$ of the stability domain $D$, we analyze the value set in more detail.

As stated earlier, there exist frequency intervals with constant structure of the value set where the building edge polynomials remain the same. These intervals can be further partitioned: From Fig. 3.17 it is obvious that the critical edge remains the same as long as the critical point $M$ does not collapse with one of the associated vertices of the value set. Let $f_0(s^*)$ be the vector in the complex plane associated with the nominal polynomial $f_0(s)$ for the fixed frequency $s = s^*$ and $f_{e_i}(s^*)$ the vector of the vertex $E_i$ associated with the vertex polynomial $f_{e_i}(s)$. Then, the critical point $M$ collapses with the vertex $E_i$.
if the pointer $f_{e_i}(s^*)$ becomes collinear with $f_0(s^*)$. The condition for those frequencies $s^* \in \partial D$ where the critical edge changes is

$$Re f_0(s^*)Im f_{e_i}(s^*) - Im f_0(s^*)Re f_{e_i}(s^*) = 0 \quad i = 1, \ldots, \nu$$ (3.44)

where $2\nu$ is the number of vertices of the value set. Note that since the value set is a parpolygon, always two vertices at a time are collinear to $f_0(s^*)$.

For the $\delta$-parameterization of $\partial D$ (see Theorem 5), Equation (3.44) can be reformulated and the set $\{\zeta\}$ of the frequencies with collinearity is described by

$$\{\zeta\} = \{ \delta \mid r(\varphi(\delta)) = 0 ; \ i = 1, \ldots, \nu ; \ \delta \in I_\delta \}$$ (3.45)

with

$$r(\varphi(\delta)) = Re f_0(\varphi(\delta))Im f_{e_i}(\varphi(\delta)) - Im f_0(\varphi(\delta))Re f_{e_i}(\varphi(\delta))$$

Then, the bounds of all frequency intervals to be considered are given by

$$\{\mu_\delta\} = \{\xi_i\} \cup \{\zeta_k\}$$

where $\{\xi_i\}$ are the bounds of the intervals with constant structure of the value set (§ 3.3.2).

In each interval, the value set is determined for a fixed frequency and the critical edge is extracted. This edge remains the same over the whole interval and therefore, a search of the minimum of the expansion factor $\rho_e$ has to be performed along this edge for all frequencies of the respective interval.

Let the critical edge of the value set for a fixed frequency $\delta^*$ be described by

$$f_{ed}(\varphi(\delta^*)) = (1 - \lambda_1)f_{e_1}(\varphi(\delta^*)) + \lambda_1 f_{e_2}(\varphi(\delta^*)) \quad 0 \leq \lambda_1 \leq 1$$ (3.46)

and the line $f_0 \rightarrow O$ by

$$f(\varphi(\delta^*)) = \lambda_2 f_0(\varphi(\delta^*)) \quad 0 \leq \lambda_2 \leq 1$$ (3.47)
then, the critical point $M$ is given by the intersection of the two lines (3.46) and (3.47). We get

$$M = \hat{\lambda} f_0(\varphi(\delta*))$$

(3.48)

with (for notational simplicity, the arguments are skipped)

$$\hat{\lambda} = \frac{\text{Re} f_{e_2} \text{Im} f_{e_1} - \text{Im} f_{e_2} \text{Re} f_{e_1}}{\text{Re} f_0(\text{Im} f_{e_1} - \text{Im} f_{e_2}) - \text{Im} f_0(\text{Re} f_{e_1} - \text{Re} f_{e_2})}$$

Finally, the structured stability margin is obtained from

$$\rho = \min_{k=1,\ldots,n} \min_{M \in f_{ed_k}} \min_{\delta \in I_k} \frac{|f_0(\varphi(\delta))|}{|f_0(\varphi(\delta)) - M(\varphi(\delta))|}$$

(3.49)

with $f_{ed_k}$ the critical edge associated with the frequency interval $I_k$ with $k = 1, \ldots, n$. The inner minimum problem can be solved as a constraint optimization problem with standard routines. Then $\rho$ is the minimum taken from all intervals.

Discussion of $\rho_e(s)$

Let $s = \varphi(\delta)$ be parameterized along the boundary of the stability domain $D$. Then, the expansion factor $\rho_e(\delta)$ has the following properties:

- Discontinuity for $s = \sigma$, $\sigma \in \mathbb{R}$
  
  If the boundary of the stability domain crosses the real axis, then, the value set collapses at these frequencies to a simple interval on the real axis (see Remark 1 on page 18). In these cases the expansion factor $\rho_e$ has a discontinuity that must be computed separately. Note that this discontinuity is not necessarily the minimum of $\rho_e$ and therefore the stability margin $\rho$ (Fig. 3.18).

- Discontinuity for special frequencies $s^* = \varphi(\delta^*)$

  At special frequencies $s^*$ the value set can collapse to a single line segment, i.e. all the pointers $f_a(s^*)$ of the vertices become collinear. In this case we have two possibilities for $\rho_e$:
3.5. Applications

- If the pointer \( f_0(s^*) \) is collinear to \( f_{e_i}(s^*) \) then, \( \rho_e \) has a discontinuity at this frequency \( s^* \) and must be computed separately.

- If the pointer \( f_0(s^*) \) is not collinear to the pointers \( f_{e_i}(s^*) \), then \( \rho_e(s^*) = \infty \).

- Non-differentiable points.
  
The function \( \rho_e(\delta) \) is not differentiable at those frequencies where the critical edge of the value set changes.

**Example 6** Consider the closed-loop in Fig. 2.1 on page 9. Let the interval plant be described by

\[
G_S(s) = \frac{b_2 s^2 + b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}
\]

with the interval parameters

\[
\begin{align*}
    b_2 &\in [0.935, 0.965], & b_1 &\in [1.85, 2.15], & b_0 &\in [0.94, 1.06], \\
    a_3 &\in [1.035, 1.065], & a_2 &\in [2.7, 3.3], & a_1 &\in [1.7, 2.3], & a_0 &\in [0.05, 0.95],
\end{align*}
\]

and the (arbitrarily) designed controller

\[
G_R(s) = 0.5 \frac{(s + 0.5)(s + 4)}{(s + 0.1)(s + 5)}.
\]

The uncertain parameters of the plant can be written as

\[
|\gamma - \gamma^0| \leq w
\]

with the nominal values

\[
\gamma^0 = [0.95 \ 1.75 \ 0.9 \ 1.1 \ 3 \ 2 \ 0.5]'
\]

and the weights

\[
w = [0.015 \ 0.15 \ 0.06 \ 0.015 \ 0.3 \ 0.3 \ 0.45]'
\]
The structured stability margin $\overline{p}$ should be determined, i.e. the maximal value $\overline{p}$ that guarantees Hurwitz stability of the closed-loop.

It is easy to see that the nominal characteristic polynomial is Hurwitz stable. Inspecting the sectors of the phase of the controller, we find that there exist two different structures of the value set and that the change is at $\omega = 1.871$ rad/sec and hence

$$\{\zeta_k\} = \{1.871\}$$

Then, all the frequencies are computed where the critical point $M$ collapses with a vertex of the value set

$$\{\xi\} = \{0, 0.079, 0.104, 0.165, 0.204, 1.088, 1.168, 2.403, 2.434\}$$

Finally, the bounds of the intervals on the $j\omega$-axis are obtained as the union of the two sets:

$$\{\mu\} = \{0, 0.079, 0.104, 0.165, 0.204, 1.088, 1.168, 1.871, 2.403, 2.434\}$$

In each interval, the critical edge polynomial is computed and the minimal expansion factor $\rho_e$ is searched along this edge. In Fig. 3.18 the expansion factor $\rho_e$ is shown for $s = j\omega$.

The structured stability margin $\overline{p}$ is obtained as the minimum of $\rho_e$ over all intervals as

$$\overline{p} = 2.1468 \quad \text{at} \quad \omega_{\text{crit}} = 0.165 \text{ rad/sec}.$$

The controller $G_R(s)$ therefore guarantees the Hurwitz invariance of the closed-loop for all uncertainties $|\gamma - \gamma^0| \leq \rho \omega$ with $0 < \rho < \overline{p}$. ◊

### 3.6 SUMMARY

In this chapter, the robust stability of a family of polynomials was investigated with respect to a stability domain $D$. The coefficients of the polynomial family are an affine function of a set of $\nu$ uncertain interval
parameters which describe a $\nu$-dimensional hyperrectangle $\Gamma$ in the parameter space. The $D$-stability is investigated in the frequency domain where the value set of the family is determined. With the geometrical procedure presented in § 3.3.1, the structure of the value set as lists of the extreme values of the uncertain parameters can be explicitly determined. Depending on the parameterization of the boundary $\partial D$, intervals for $s$ along $\partial D$ can be found where the structure of the value set, i.e. the building exposed edges of the hyperrectangle $\Gamma$, remains the same. As a result this yields the critical edge polynomials of $\Gamma$ to be tested to guarantee the robust $D$-stability of the polynomial family (Theorem 4).

In comparison with the Edge theorem (Theorem 3), where all the edges of the polytope $A$ have to be tested for stability, the advantage of the presented $D$-stability test is the reduction of the number of edges to be tested from the set of all edges to the subset of exposed edges of $\Gamma$. This represents a substantial saving since the number of exposed edges grows linearly with the number of parameters $\nu$, while the number of
all edges of $\Gamma$ is exponential in $n$.

For special combinations of the stability domain $D$ and of the affine parameterization of the uncertainty, it is possible to obtain even stronger results. As derived in § 3.5.1 for the Hurwitz stability of the closed-loop with an interval plant and a fixed controller, the number of exposed edges is independent of the number of uncertain parameters $\nu$, i.e. independent of the order of the plant. Depending only on the phase of the controller, four different structures of the value set can occur for $s$ along the imaginary axis and therefore maximal $4 \times 8 = 32$ exposed edges of the value set must be tested for $D$-stability. In § 3.5.2 the same closed-loop system was investigated with respect to a cone stability domain. For special opening angles $\varphi_0$ of the cone, it is also possible to obtain results independent on the number of uncertain parameters. The value set has been explicitly determined for a cone with $\varphi_0 = \frac{2}{3}\pi$. In this case, for $s$ along the boundary $\partial D$, only six different structures of the value set can occur and therefore at most $6 \times 12 = 72$ exposed edges have to be tested.

In § 3.5.3, the structured stability margin $\bar{p}$ was introduced. This margin is a measure of the robustness of the family of polynomials since it gives the largest size of the uncertainty domain for a specified uncertainty structure that still preserves $D$-stability of a polynomial. A procedure for the computation of this margin has been presented. The robustness measure $\bar{p}$ is especially interesting in the case where the family of polynomials is the characteristic polynomial of an uncertain control system. Then, $\bar{p}$ is a direct measure for the robustness of the chosen controller since it gives the maximal uncertainty domain (with a given structure) which can be stabilized by this controller.

The procedure for determination of the edges of the value set as well as the $D$-stability test can be easily implemented in computer programs such as MATLAB. This was successfully done using the structure of the flowchart in Fig. 3.8 for several $D$-stability domains such as the (shifted) left half plane, the unit circle, cones and combinations of these [55]. Appendix B presents a numerical example for the investigation of the robust $D$-stability of a family of polynomials with respect to a cone.
Chapter 4

COMPUTATION OF VALUE SETS OF UNCERTAIN TRANSFER FUNCTIONS

In this chapter, the value sets of special rational functions are investigated. Procedures for the direct computation of the boundary of the value sets are presented.

4.1 INTRODUCTION

As shown in § 3, the computation of value sets of uncertain polynomials or transfer functions plays a major role in the application of frequency domain methods for the analysis and design of robust control systems. The main problem is to find the boundary of the value set without having to compute it by simply gridding all the uncertain parameters. In this chapter, the computation of value sets is extended from the polynomial case (§ 3) to uncertain rational functions. A different approach as in the previous chapter is used.

In [59], a method to compute the value set of an uncertain polynomial is proposed. The resulting value set is obtained in an easy way for so-called tree decomposable polynomials by continued application of the operations “+” and “×” on value sets starting with the sets of the components1.

Further reduction of the computational burden is possible with results of [60] and [13]. Therein, the investigation is concentrated on the

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1W. Sienel is currently writing his Ph.D. thesis on this topic under the supervision of Prof. Ackermann at the DLR in Germany
boundary of the value sets. In [13], an argument condition is presented to obtain the essential points on the boundaries of the components to be mapped.

In this chapter, the primary interest is focused on uncertain plant transfer functions. In special cases where the numerator and denominator of the transfer function can be factored over polynomials with independent parameterization of the even and odd parts, the value set computation is decomposed into a sequence of products and ratios of simpler value sets. With the ideas of [13, 59, 60] used for the operations “×” and “/” and their combination we can get the resulting value set in a simple, straightforward manner.

4.2 PROBLEM SET-UP

Consider the three expressions

\[ f_1(s)f_2(s), \quad (4.1) \]
\[ \frac{f_1(s)}{f_2(s)}, \quad (4.2) \]
\[ \frac{f_1(s)f_2(s)}{f_3(s)f_4(s)}, \quad (4.3) \]

where the \( f_i(s) \), \( i = 1, \ldots, 4 \) are polynomials with mutually independent parameterizations of the even and odd parts.

For a fixed frequency \( s = j\omega\) , the value set of the expressions (4.1), (4.2) and (4.3) in the complex plane is investigated.

The well-known case of an interval plant is covered by (4.2), and (4.3) can be considered as a plant with multilinear parameter dependence.

4.2.1 The value set of the polynomials \( f_i \)

It is easy to show that the value set of a polynomial \( f_i(s) \) in (4.1) to (4.3) is an axis parallel box in the complex plane. The important case of an interval polynomial is a special case of \( f_i(s) \). The four vertices
of the box are then determined by the images of the four Kharitonov polynomials.

In the sequel, the notation

$$f(j\omega^*) = \alpha + j\beta$$

(4.4)

is used to describe the value set of a polynomial $f(s)$ for the fixed frequency $s = j\omega^*$ (Fig. 4.1).

The value sets of the expressions (4.1), (4.2) and (4.3) are formed by applying the operations “$\times$” and “$/$” on axis parallel boxes.

The generalization of this set-up leads to the following problem formulation. Let $V = A \circ B$, where “$\circ$” is a mathematical operator and $A$, $B$ and $V$ are value sets. The question of how to obtain the boundary $\partial V$ of the value set $V$ from the boundaries $\partial A$ and $\partial B$ of the two sets $A$ and $B$ is of major interest. The argument approach of [13] delivers a necessary condition for boundary points of $A$ and $B$ to be candidates to be mapped to the boundary of $V$. This condition, which was derived in [13] for the product of value sets, can be easily extended to the cases of (4.2) and (4.3).
4.3 PRODUCT OF TWO AXIS PARALLEL BOXES

Given are the boundaries $\partial A$ and $\partial B$ of two value sets $A$ and $B$. The boundary $\partial V$ of the value set

$$ V = AB $$

(4.5)

is to be determined.

As shown in [60], the boundary of $V$ is the envelope of the product of the boundaries of $A$ and $B$:

$$ \partial(AB) = \partial_0(\partial A \cdot \partial B) $$

(4.6)

where $\partial_0$ denotes the outer boundary.

To get this outer boundary of $V$ we would have to multiply each point of the boundary $\partial A$ with the whole boundary $\partial B$ and to finally take the envelope. In order to avoid this huge computational burden we use an explicit necessary condition for points on $\partial A$ and $\partial B$ [13]:

**Theorem 6** Let $\nu \rightarrow a(\nu), \, \nu \in I_a$ and $\mu \rightarrow b(\mu), \, \mu \in I_b$ be continuously differentiable positively (anticlockwise) oriented parameterizations of the boundaries $\partial A$ and $\partial B$ of the value sets $A$ and $B$. Then, the boundary $\partial V$ of $V = AB$ is contained in the set

$$ \left\{ a(\nu^*)b(\mu^*) : \arg \left( \frac{a(\nu)}{a'(\nu)} \right) \big|_{\nu=\nu^*} = \arg \left( \frac{b(\mu)}{b'\mu} \right) \big|_{\mu=\mu^*} \right\} $$

(4.7)

where "′" denotes the derivative with respect to the parameter $\nu$ respectively $\mu$.

Since Theorem 6 gives only necessary conditions, some points in the set described by (4.7) can be inside $V$. 
With Theorem 6, it is open how to find the associated pairs of $\nu^*, \mu^*$ in (4.7). As the value sets $A$ and $B$ in (4.1) are axis parallel boxes, the argument condition of Theorem 6 can be easily used in a geometric way to determine the segments on the boundary of each box which fulfill condition (4.7). Namely, there exist four distinct directions along the boundary which are the same in both value sets. Since only the relative position of the edges of the two value sets is relevant, all the arguments $\arg(a(\nu)/a'(\nu))$ of the edges can be determined relative to the first horizontal edge (Fig. 4.2).

4.3.1 Determination of the relative angles of an axis parallel box

The following procedure describes how to determine the relative arguments of (4.7) for an axis parallel box (see Fig. 4.2).

Procedure

1) The vertices of the value set are numbered in positive, i.e. counterclockwise, direction.

2) Start with the edge 1-2 and plot the angles of the two vertices.

3) At vertex 2 the direction of the edge 1-2 changes to the direction of the edge 2-3 by $90^\circ$. The vertex 2 itself is associated with a sector of angle $\varphi$ from $\varphi^* = \arg(\text{vertex } 2)$ to $\varphi^* - 90^\circ$.

4) Start from the end angle of vertex 2 and add the change of argument of the edge 2-3.

5) Add $-90^\circ$ to the angle of vertex 3.

6) Proceed as before until back to vertex 1.

According to the argument condition of Theorem 6 the sectors of angles of the two value sets $A$ and $B$ must be intersected to find those parts of the boundaries $\partial A$ and $\partial B$ which fulfill the argument equation. As result we get essentially four different cases for the product of two value sets $A$ and $B$: 
Figure 4.2: Relative arguments of the value set $A$.

i) Vertex of $A \times$ vertex of $B$

ii) Vertex of $A \times$ edge segment of $B$

iii) Edge segment of $A \times$ vertex of $B$

iv) Edge segment of $A \times$ edge segment of $B$

Case i) is simply a single point in the complex plane that is also obtained from the other cases. However, for recursive construction of value sets as in § 4.5, these points must also be considered. In case ii) and iii) all points of an edge segment are multiplied with a complex number. The result is again a straight line. The most interesting case iv) delivers a general boundary curve and is discussed in the sequel.

4.3.2 Product of two edge segments

The combination of the points on $\partial A$ and $\partial B$ which fulfill Theorem 6 is independent of the parameterization of the two boundaries of $A$ and $B$. But in the special case where the two boundaries consist of line segments, the affine parameterization of the segments leads to easy results:
Corollary 2 Suppose the boundaries of the value sets $A$ and $B$ are formed by line segments with positively oriented affine parameterization

$$a(\nu) = a_0 + a_1 \nu \quad \nu \in I_a \in \mathbb{R} \quad (4.8)$$

$$b(\mu) = b_0 + b_1 \mu \quad \mu \in I_b \in \mathbb{R} \quad (4.9)$$

where $a_i$ and $b_i$, $i = 1, 2$ are complex numbers. Then, for any two points on $\partial A$ and $\partial B$ which fulfill the argument condition (4.7) $\mu$ is an affine function of $\nu$:

$$\mu = k_1 \nu + k_0 \quad (4.10)$$

where $k_1$ and $k_0$ are real constants.

**Proof:** Let an edge of $a(\nu)$ be parameterized as

$$a(\nu) = a_0 + a_1 \nu.$$ 

Then, it follows

$$\frac{\partial a}{\partial \nu} = a_1$$

and so

$$\arg \left( \frac{a(\nu)}{a'(\nu)} \right) = \arg \left( \frac{a_0}{a_1} + \nu \right)$$

$$= \arctan \left( \frac{\text{Im} \left( \frac{a_0}{a_1} \right)}{\text{Re} \left( \frac{a_0}{a_1} \right) + \nu} \right)$$

Similarly for an edge of $b(\mu)$. With (4.7) we get

$$\frac{\text{Re} \left( \frac{a_0}{a_1} \right) + \nu}{\text{Im} \left( \frac{a_0}{a_1} \right)} = \frac{\text{Re} \left( \frac{b_0}{b_1} \right) + \mu}{\text{Im} \left( \frac{b_0}{b_1} \right)}$$

and hence immediately (4.10). \qed
Given the product of two line segments, Corollary 2 implies that only the pairs of points defined by (4.10) have to be multiplied with each other.

An important result for the recursive multiplication of value sets can be derived from Theorem 6:

**Corollary 3** Given the assumptions of Theorem 6, suppose there exists a real mapping \( g : \nu \rightarrow \mu \) determining pairs of points on the boundaries \( a(\nu) \) and \( b(\mu) \) of the value sets \( A \) and \( B \). Then, for each of these pairs the following equality holds

\[
\arg \left( \frac{a(\nu)}{\frac{\partial}{\partial \nu} a(\nu)} \right) = \arg \left( \frac{b(\mu)}{\frac{\partial}{\partial \mu} b(\mu)} \right) \bigg|_{\mu = g(\nu)} = \arg \left( \frac{a(\nu) b(\mu)}{\frac{\partial}{\partial \nu} [a(\nu) b(\mu)]} \right) \bigg|_{\mu = g(\nu)}
\]

**Proof:** Let

\[
\phi = \arg \left( \frac{a}{a'} \right) \bigg|_{\nu} = \arg \left( \frac{b}{b'} \right) \bigg|_{\mu}
\]

(4.11)

and use the relation \( g(\cdot) \) between the two parameterizations of \( a \) and \( b \). Then, we get

\[
a(\nu) b(\mu) = a(\nu) b(g(\nu)).
\]

With (4.7) and \( \mu = g(\nu) \) follows

\[
\frac{a(\nu) b(g(\nu))}{\frac{\partial}{\partial \nu} [a(\nu) b(g(\nu))] = \left[ \frac{\partial}{\partial \nu} a(\nu) \right] b(g(\nu)) + \left[ \frac{\partial}{\partial \mu} b(\mu) \right] \frac{\partial}{\partial \nu} g(\nu) a(\nu)
\]

\[
= \left[ \frac{a(\nu)}{\frac{\partial}{\partial \nu} a(\nu)} \frac{b(\mu)}{\frac{\partial}{\partial \mu} b(\mu)} \right] \left[ \frac{b(\mu)}{\frac{\partial}{\partial \mu} b(\mu)} + \frac{a(\nu)}{\frac{\partial}{\partial \nu} a(\nu)} \frac{\partial}{\partial \nu} g(\nu) \right]^{-1}
\]

and therefore

\[
\arg(\cdot) = \arg \left( \frac{a(\nu)}{\frac{\partial}{\partial \nu} a(\nu)} \frac{b(\mu)}{\frac{\partial}{\partial \mu} b(\mu)} \right) - \arg \left( \frac{b(\mu)}{\frac{\partial}{\partial \mu} b(\mu)} + \frac{a(\nu)}{\frac{\partial}{\partial \nu} a(\nu)} \frac{\partial}{\partial \nu} g(\nu) \right)
\]

\[
= 2\phi - \arg(p_1 + p_2 p_3)
\]
As $g(\cdot)$ is a real function $p_3$ is real. With (4.11) $p_1$ and $p_2$ and $p_2p_3$ are complex numbers with argument $\varphi$ and hence the result.

Corollary 3 implies that the relative arguments of the boundary of the value set $V = AB$ are the same ones as those of the associated points on $\partial A$ and $\partial B$. Consider for example the product $V = (AB)C$ where $A$, $B$ and $C$ are axis parallel boxes. With Corollary 3 only the relative arguments of the boundary of $(AB)$ must be considered for the product of $(AB)$ with $C$. Thus, the original arguments of the boxes $A$ and $B$ can be replaced by their intersection and further intersected with the relative arguments of $C$. This result will be used in § 4.5.

With the argument condition (4.7) the curves on the boundary $\partial V$ that result from the product of two edge segments of $A$ and $B$ can be determined explicitly. Let $A$ and $B$ be described as follows:

\[ A = \{ \alpha + j\beta, \quad \alpha \leq \alpha \leq \alpha, \quad \beta \leq \beta \leq \beta \} \quad (4.12) \]
\[ B = \{ a + jb, \quad a \leq a \leq \alpha, \quad b \leq b \leq \beta \} \quad (4.13) \]

Then, the edges of $A$ respectively $B$ are described by one varying parameter and the other fixed at one of his extreme values. So, one edge of $A$ is for example given by

\[ g_A = \alpha + j\beta^* \quad \text{with} \quad \beta^* \in \{\beta, \beta\}, \quad \alpha \leq \alpha \leq \alpha \quad (4.14) \]

For the product of two edge segments of $A$ and $B$ there exist four combinations of varying parameters in $(\alpha + j\beta)(a + jb)$:

\[ (\alpha, a), \quad (\alpha, b), \quad (\beta, a), \quad (\beta, b) \quad (4.15) \]

each of them with four different choices of the extreme values of the other parameters. For each of these cases the analytic expression of the product of the two edges can be determined.
(\alpha, a) \text{ variable} \\

The edge segments of A and B are given by

\[ g_A = \alpha + j\beta^* \]  \hspace{1cm} (4.16)

with \( \beta^* \in \{\overline{\beta}, \beta\} \), \( \alpha_1 \leq \alpha \leq \overline{\alpha}_1 \), \( [\alpha_1, \overline{\alpha}_1] \subseteq [\alpha, \overline{\alpha}] \)

\[ g_B = a + j b^* \]  \hspace{1cm} (4.17)

with \( b^* \in \{b, \overline{b}\} \), \( a_1 \leq a \leq \overline{a}_1 \), \( [a_1, \overline{a}_1] \subseteq [a, \overline{a}] \)

With (4.7) follows

\[ \frac{\partial}{\partial \alpha} g_A = 1, \quad \frac{\partial}{\partial a} g_B = 1 \]  \hspace{1cm} (4.18)

\[ \arg\left(\frac{\alpha + j\beta^*}{1}\right) = \arg\left(\frac{a + jb^*}{1}\right) \]  \hspace{1cm} (4.19)

\[ \Rightarrow \arctan\left(\frac{\beta^*}{\alpha}\right) = \arctan\left(\frac{b^*}{a}\right) \Rightarrow a = \frac{b^*}{\beta^*} \alpha \]  \hspace{1cm} (4.20)

and hence

\[ f = g_A \cdot g_B = (\alpha + j\beta^*) \left(\frac{b^*}{\beta^*}\alpha + jb^*\right) = \frac{b^*}{\beta^*}(\alpha + j\beta^*)^2 \]  \hspace{1cm} (4.21)

So, the potential contribution of the product of these two edge segments to the boundary of the value set \( V \) is a parabola.

(\alpha, b) \text{ variable} \\

The edges are described by (4.16) and

\[ g_B = a^* + j b \]  \hspace{1cm} (4.22)

with \( a^* \in \{a, \overline{a}\} \), \( b_1 \leq b \leq \overline{b}_1 \), \( [b_1, \overline{b}_1] \subseteq [b, \overline{b}] \)

With \( g'_A = 1 \) and \( g'_B = j \), where "\(^{'}\)" is the derivative with respect to the varying parameter, it follows

\[ \arg\left(\frac{\alpha + j\beta^*}{1}\right) = \arg\left(\frac{a^* + j b}{j}\right) \]  \hspace{1cm} (4.23)
4.3. Product of two axis parallel boxes

\[ \Rightarrow \ arctan \left( \frac{\beta^*}{\alpha} \right) = \ arctan \left( -\frac{a^*}{b} \right) \Rightarrow b = -\frac{a^*}{\beta^*} \alpha \]  

(4.24)

and therefore

\[ f = g_{AB} = (\alpha + j\beta^*) \left( a^* - j\frac{a^*}{\beta^*} \alpha \right) = -j\frac{a^*}{\beta^*} (\alpha + j\beta^*)^2 \]  

(4.25)

which is again a parabola.

The same procedure is used for (\( \beta, a \)) and (\( \beta, b \)). In Table 4.1, the four cases are summarized.

<table>
<thead>
<tr>
<th>Face</th>
<th>Relation</th>
<th>Curve in the value set</th>
<th>Type of curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha, a )</td>
<td>( a = \frac{b^<em>}{\beta^</em>} \alpha )</td>
<td>( f = \frac{b^<em>}{\beta^</em>} (\alpha + j\beta^*)^2 )</td>
<td>parabola</td>
</tr>
<tr>
<td>( \alpha, b )</td>
<td>( b = -\frac{a^<em>}{\beta^</em>} \alpha )</td>
<td>( f = -j\frac{a^<em>}{\beta^</em>} (\alpha + j\beta^*)^2 )</td>
<td>parabola</td>
</tr>
<tr>
<td>( \beta, a )</td>
<td>( a = -\frac{b^<em>}{\alpha^</em>} \beta )</td>
<td>( f = j\frac{b^<em>}{\alpha^</em>} (\alpha^* + j\beta)^2 )</td>
<td>parabola</td>
</tr>
<tr>
<td>( \beta, b )</td>
<td>( b = \frac{a^<em>}{\alpha^</em>} \beta )</td>
<td>( f = \frac{a^<em>}{\alpha^</em>} (\alpha^* + j\beta)^2 )</td>
<td>parabola</td>
</tr>
</tbody>
</table>

Table 4.1: Product of two edge segments.

Note that in general due to (4.7) the variable parameter is not varying along the whole edge of \( A \) respectively \( B \).

The results of Table 4.1 can be interpreted geometrically in the complex plane: The two varying parameters define a two-dimensional face (Fig. 4.3) and the relation between the varying parameters describes the slope of a line through the origin. The intersection of this line with the face gives those points on the edge segments of \( A \) and \( B \) with common relative arguments. The different combinations of the
Figure 4.3: Related variable parameters.

fixed extreme parameters (denoted with the "*"), define four such lines through the origin of such a face.

Remark 7 The relation (4.23) ⇒ (4.24) holds but not (4.23) ⇐ (4.24), since (4.23) has a period of $2\pi$ and (4.24) of $\pi$. The two cases depicted in Fig. 4.4 give the same result in (4.24), but with (4.23) only case (ii) is valid. So condition (4.23) is stronger than (4.24).

Figure 4.4: Two cases with the same relation.

4.3.3 Jacobian analysis

In this section a different approach, the so-called Jacobian method, is compared with the argument condition. It is shown that the same relations between the varying parameters (Table 4.1) can be obtained using Jacobian analysis.
4.3. Product of two axis parallel boxes

The expression

\[ f = (\alpha + j\beta)(a + jb) \]  \hspace{1cm} (4.26)

can be treated as the mapping of a four-dimensional hyperrectangle \((\alpha, \beta, a, b)\) into the complex plane.

For the discussion of the map it is necessary to distinguish between inner points and points on the surface of the hyperrectangle. After applying the map \(f\) on the hyperrectangle, all images of points of the box that can be locally varied in two independent directions in the complex plane are inner points of the value set (Fig. 4.5). On the other hand, all points that do not have locally two degrees of freedom for variation are potential points of the boundary of the value set. This fact can be expressed using the Jacobian matrix.

![Figure 4.5: Directions for variation.](image)

Let the hyperrectangle in the parameter space \(\mathbb{R}^4\) be described by the interval parameters \(\alpha, \beta, a, b\). Then, the expression

\[ f = (\alpha + j\beta)(a + jb) = \alpha a - \beta b + j(\beta a + \alpha b) \]  \hspace{1cm} (4.27)

defines the map of the hyperrectangle to the complex plane. The
Jacobian matrix $J$ of (4.27) is given by

$$J = \begin{bmatrix}
\frac{\partial (\text{Re} f)}{\partial (\alpha, \beta, a, b)} \\
\frac{\partial (\text{Im} f)}{\partial (\alpha, \beta, a, b)}
\end{bmatrix} = \begin{bmatrix}
a & -b & \alpha & -\beta \\
b & a & \beta & \alpha
\end{bmatrix}$$

Consider any inner point of the parameter hyperrectangle with local variation in all four directions $\alpha$, $\beta$, $a$ and $b$. This point can only be mapped on the boundary of the value set if $\text{rank}(J) < 2$. Investigating all $2 \times 2$-submatrices of $J$, at least one of the matrices

$$J_{12} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad J_{34} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

is always nonsingular, except at the origin of the parameter space, and hence $J$ has generally rank 2, i.e. there exist two independent directions in the complex plane. Hence, all inner points of the hyperrectangle are mapped inside of the value set.

For the analysis of the map of the 3-dimensional faces of the hyperrectangle, i.e. points with locally three degrees of freedom for variation, the column in $J$ corresponding to the fixed parameter can be removed (since there is no variation in that direction the corresponding partial derivatives in $J$ are zero) and the rank is determined. Since $J$ always contains the submatrix $J_{12}$ respectively $J_{34}$ we get $\text{rank}(J) = 2$ and hence only inner points of the value set.

The two- and one-dimensional faces are discussed in the sequel:

**Points on the 2-dimensional faces**

Consider an arbitrary point on a 2-dimensional face. This point can only be mapped to the boundary of the value set if the Jacobian $J$ drops in rank on this face. Again, the columns of the fixed parameters in $J$ are removed and the rank of $J$ is determined. Investigating all the $2 \times 2$-submatrices on the 2-dimensional faces for a drop of rank in $J$, the conditions in Table 4.2 are obtained. Points of the faces indicated in
4.3. Product of two axis parallel boxes

Table 4.2 which fulfill the condition for drop of rank are candidates for the boundary $\partial V$ of the value set $V = AB$. These faces are described by the varying parameters $(\alpha, a)$, $(\alpha, b)$, $(\beta, a)$ and $(\beta, b)$ and all other parameters fixed at their extreme values. Note that these parameter combinations are those with just one variable parameter in each factor of (4.27). Totally there exist 16 two-dimensional faces where $J$ can drop in rank.

<table>
<thead>
<tr>
<th>Face</th>
<th>Condition for drop of rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha, a$</td>
<td>$\beta^* a - \alpha^* b = 0$</td>
</tr>
<tr>
<td>$\alpha, b$</td>
<td>$\alpha^* a + \beta^* b = 0$</td>
</tr>
<tr>
<td>$\beta, a$</td>
<td>$\alpha^* a + \beta^* b = 0$</td>
</tr>
<tr>
<td>$\beta, b$</td>
<td>$\beta^* a - \alpha^* b = 0$</td>
</tr>
</tbody>
</table>

Table 4.2: Points on the 2-dimensional faces with rank($J$) = 1.

The conditions in Table 4.2 define a relation between the two varying parameters for points on the 2-dimensional faces where rank($J$) = 1. These relations are exactly the same ones as indicated in Table 4.1 on page 69 for the product of two edge segments and hence the same analytic expressions are obtained.

Points on the 1-dimensional faces

The Jacobian $J$ has always rank 1 on the one-dimensional faces, i.e. the edges of the hyperrectangle, which means that all these faces could be mapped to the boundary of the value set.

Comparison Jacobian method – Argument condition

Both methods, the Jacobian method and the argument condition of Theorem 6, give necessary but not sufficient boundary conditions. As
derived in the previous sections for the product of two value sets two cases must be discussed:

1) Product of an edge segment with a vertex (one-dimensional faces)
2) Product of two edge segments (two-dimensional faces)

For case 1) the argument condition delivers the relevant edges and furthermore, even the exact segments on these edges that potentially contribute to the boundary $\partial V$ of the value set $V = AB$. With the Jacobian method all the edges of the parameter hyperrectangle are potential candidates to be mapped to the boundary of the value set $V$ since this method cannot provide any specific information which one of all the edges of the hyperrectangle contribute to $\partial V$. In this case the Jacobian method is useless for further reduction of the computational burden.

In the second case the relations between the varying parameters of the two edges can be obtained from both methods. For every face defined by the two varying parameters the relation defines the slopes of four lines through the origin. These four slopes are given by all the combinations of the extreme values of the fixed parameters (see Fig. 4.6). The relevant line segments for multiplication on the associated two edges described by the varying parameters follow by the intersections of the lines with the face.

With these relations between the varying parameters of the face the analytical expression of the image of the product of the two edge segments in the resulting value set $V$ can be obtained.

From the Jacobian analysis there can also result some edges of $A$ and $B$ which fulfill the condition for drop of rank in $J$, but which do not fulfill the argument condition (4.7) (see Remark 7). To clarify this fact see the following example:

**Example 7** Given the two axis parallel boxes $A$ and $B$ with their vertices

\[
A : \alpha + j\beta : \begin{array}{cccc}
2.5 + 2i & 5 + 2i & 5 + 3.5i & 2.5 + 3.5i \\
B : \alpha + jb : \begin{array}{cccc}
2 + 0.5i & 3 + 0.5i & 3 + 1.5i & 2 + 1.5i 
\end{array}
\end{array}
\]
4.3. Product of two axis parallel boxes

Figure 4.6: Relations in a 2-dimensional face.

Figure 4.7: Parameterization of the value sets A and B.

Fig. 4.7 shows the two value sets with the vertices numbered in positive direction. Consider the edges

edge (1-2) : $\alpha$ varies from 2.5 to 5, $\beta^* = 2$
edge (7-8) : $a$ varies from 3 to 2, $b^* = 1.5$

Applying the Jacobian method, from Table 4.2 follows the relation

$$a = (b^*/\beta^*)\alpha$$

for a drop of rank in $J$ on the $(\alpha, a)$-face. All points of the $(\alpha, a)$-face located on the line through the origin with slope $b^*/\beta^*$ are potential candidates to be mapped on the boundary of $V$. As shown in Fig. 4.8 for the product of the edges (1-2)(7-8) intersection of the line with the face occurs.
On the other hand, from the procedure of § 4.3.1 on page 63 we get the following intervals of the relative arguments of the edges

\[ \arg(1-2) = [38.66^\circ, 21.80^\circ] \]
\[ \arg(7-8) = [-153.43^\circ, -143.13^\circ] \]

It is obvious that these two intervals have no intersections and therefore no points on those two edges fulfill (4.7) to be mapped on the boundary of \( V = AB \). The big difference in the angles of both intervals is due to the fact that the arguments of the edges are similar but the directions of the edges, expressed by the derivatives, are different:

\[ e'_{12} = 1, \quad e'_{78} = -1 \]

Hence, the argument condition which relies also on the orientation of the parameterization of the boundary is stronger than the Jacobian method. In the sense of Remark 7 the Jacobian delivers condition (4.24) but not the stronger (4.23).

![Figure 4.8: Face \((\alpha,a)\).](image)

The same argumentation holds for the division and the extension in § 4.4 and § 4.5. In both cases the argument condition is more restrictive than the Jacobian method.

### 4.3.4 Summary

The results for the product of two axis parallel boxes are summarized with the following corollary:
Corollary 4 The boundary of the value set $V = AB$ consists of line segments, i.e. the images of the product of a vertex of $A$ (resp. $B$) with an edge segment of $B$ (resp. $A$) and of parabolas i.e. images of the product of two edge segments.

From the discussion of the previous sections follows the procedure for the computation of the product of two axis parallel boxes $A$ and $B$:

Procedure

1. Determine the relative arguments of the two boundaries of the value sets $A$ and $B$ (§ 4.3.1).

2. With (4.7) the intersections of the relative arguments of $\partial A$ and $\partial B$ and the corresponding segments on the edges of $A$ and $B$ are determined (see Appendix C.1).

3. Compute the products of
   (a) Vertex $\times$ edge segment
   (b) Edge segment $\times$ edge segment

   In case (b), either Corollary 2 or Table 4.1 can be used to determine the related points on the edge segments which have to be multiplied.

A numerical example for the product of two axis parallel boxes is shown in Appendix C.1.

4.4 DIVISION OF TWO AXIS PARALLEL BOXES

The case of an uncertain plant (4.2) is discussed in this section where the polynomials $f_i(s)$ have mutually independent parameterizations of the even and odd parts. For any fixed frequency $s = j\omega^*$ the resulting value set is the quotient of two axis parallel boxes $B$ and $A$. This case covers the important case of an interval plant where the numerator and denominator of the transfer function are interval polynomials.
The procedure for the analysis of the boundary of the value set $V = B/A$ is essentially the same as in § 4.3. The result of [13] in Theorem 6 can be easily extended to the case of division of two value sets.

**Theorem 7** Let $\nu \to a(\nu)$, $\nu \in I_a$ and $\mu \to b(\mu)$, $\mu \in I_b$ be continuously differentiable positively oriented parameterizations of the boundaries of the value sets $A$ and $B$. Then, the boundary $\partial V$ of $V = B/A$ is contained in the set

$$\left\{ \frac{b(\mu^*)}{a(\nu^*)} : \arg \left( \frac{b(\mu)}{b'(\mu)} \right) \big|_{\mu = \mu^*} = \arg \left( \frac{-a(\nu)}{a'(\nu)} \right) \big|_{\nu = \nu^*} \right\}$$

where $"^*"$ indicates the derivative with respect to the parameter $\mu$ respectively $\nu$.

**Proof:** The division can be considered as multiplication:

$$\frac{b(\mu)}{a(\nu)} = b(\mu) \frac{1}{a(\nu)}$$

and

$$\frac{\partial}{\partial \nu} \left( \frac{1}{a(\nu)} \right) = -\frac{a'(\nu)}{a^2(\nu)}$$

Then, from Theorem 6 follows

$$\arg \left( \frac{b(\mu)}{b'(\mu)} \right) = \arg \left( \frac{1}{a(\nu) a'(\nu)} \right) = \arg \left( \frac{-a(\nu)}{a'(\nu)} \right)$$

Theorem 7 shows that the boundary of $V = B/A$ is only formed by images of the division of boundary points of $B$ and $A$.

In the special case where the boundaries of the value sets $A$ and $B$ consist of line segments a similar relation between the two parameterizations as in Corollary 2 for the multiplication can be obtained from Theorem 7.
4.4. Division of two axis parallel boxes

Corollary 5 Let the value sets $A$ and $B$ be bounded by line segments with positively oriented affine parameterization

$$a(\nu) = a_0 + a_1 \nu \quad \nu \in I_a \quad (4.29)$$

$$b(\mu) = b_0 + b_1 \mu \quad \mu \in I_b \quad (4.30)$$

where $a_i, b_i, i = 1, 2$ are complex numbers. Then, for any two points on $\partial A$ and $\partial B$ which fulfill the argument condition (4.28) $\mu$ is an affine function of $\nu$:

$$\mu = k_1 \nu + k_0 \quad (4.31)$$

where $k_1$ and $k_0$ are real constants.

Proof: The proof follows the same lines as for Corollary 2.

For the recursive computation of value sets the next result is similar to Corollary 3 and shows that the relative arguments for associated points on $\partial A$ and $\partial B$ are the same as those of the boundary points $\partial V$ of $V = B/A$.

Corollary 6 Given the assumptions of Theorem 7, suppose there exists a real mapping $g : \nu \to \mu$ determining pairs of points on the boundaries $a(\nu)$ and $b(\mu)$ of the value sets $A$ and $B$. Then, for each of these pairs the following equality holds:

$$\arg \left( \left. \frac{b(\mu)}{\frac{\partial}{\partial \mu} b(\mu)} \right|_{\mu=g(\nu)} \right) = \arg \left( \left. \frac{a(\nu)}{-\frac{\partial}{\partial \nu} a(\nu)} \right|_{\nu=g(\nu)} \right) = \arg \left( \left. \frac{b(\mu)/a(\nu)}{-\frac{\partial}{\partial \nu} [b(\mu)/a(\nu)]} \right|_{\mu=g(\nu)} \right)$$

Proof: The proof follows the same steps as in Corollary 3.

With Theorem 7 the procedure for determination of the edge segments of the two boundaries of $B$ and $A$ which must be divided by each other is exactly the same except the argument condition (4.28) where the negative sign causes an additional phase of $\pi$ for the relative angles of the value set $A$. 
As result one gets essentially the same four cases to discuss for the division of two value sets $B$ and $A$ as for the product on page 64. Case i), i.e. a vertex of $B$ divided by a vertex of $A$, is simply a single point that is also obtained from the other cases and is therefore not separately discussed here. In case iii) all points of an edge segment of $B$ are divided by a complex number. The result is again a straight line. The interesting cases are ii) and iv) which are investigated in the sequel.

4.4.1 Vertex of $B$ divided by an edge segment of $A$

The division of a vertex of $B$ with a line segment of $A$ can be considered as the product of a complex number with an inverted line segment which is an arc. Let $w = x + jy$ be a point on the line segment of $A$ and $z = u + jv$ the corresponding point on the inverted set $1/A$. Then, $z$ and $w$ are related by

$$z = \frac{1}{w} = \frac{1}{x + jy} = \frac{x - jy}{x^2 + y^2} = u + jv$$

with

$$u = \frac{x}{x^2 + y^2} ; \quad v = \frac{-y}{x^2 + y^2}$$

Therefore, a straight line

$$2\alpha x + 2\beta y + 1 = 0 \quad (4.32)$$

is mapped into

$$(u + \alpha)^2 + (v - \beta)^2 = \alpha^2 + \beta^2 \quad (4.33)$$

It is obvious that (4.33) describes a circle in the complex plane with center $-\alpha + j\beta$ and radius $r = \sqrt{\alpha^2 + \beta^2}$ which is going through the origin. For a line segment on the boundary of $A$ an arc is obtained whose end points are the inverted end points of the segment (see Fig 4.9).

Remark 8 In the case where the origin is on the boundary or inside of the box $A$ the inverted value set is no longer a closed domain as
the origin is mapped to infinity. If the origin is inside of $A$, then, the boundary of the value set is the inner boundary around a hole in the complex plane (Fig. 4.10).

### 4.4.2 Ratio of two edge segments

The analytic expression for the curve on the boundary of $V = B/A$ that results from the division of two edge segments can be directly derived. The procedure is exactly the same as in § 4.3 on page 62 for the product of two line segments.

The same combinations of varying parameters for the division of two edge segments are obtained as for the multiplication in (4.15) on page 67. Again, for each of these four cases the analytic expression of the curve in the value set $V$ can be determined.

**(a, a) variable**

Given the edge segments of $A$ and $B$ by (4.16) and (4.17). With (4.28) follows

$$
\frac{\partial}{\partial \alpha} g_A = 1, \quad \frac{\partial}{\partial a} g_B = 1
$$

$$
\arg \left( \frac{a + j b^*}{1} \right) = \arg \left( \frac{\alpha + j \beta^*}{1} \right)
$$

$$
\Rightarrow \quad \arctan \left( \frac{b^*}{a} \right) = \arctan \left( \frac{\beta^*}{\alpha} \right) \Rightarrow \quad a = \frac{b^*}{\beta^*} \alpha
$$

and so

$$
f = \frac{g_B}{g_A} = \frac{b^* \alpha + j b^*}{\alpha + j \beta^*} = \frac{b^*}{\beta^*}
$$

which is a single point in the complex plane defined by the ratio of the fixed parameters.

The same procedure is used for $(\alpha, b)$, $(\beta, a)$ and $(\beta, b)$, see Table 4.3. It follows that for the division of two edge segments in all cases a single point in the value set is obtained which is located either on the boundary or inside of $V = B/A$. So, for the computation of the boundary of $V$ the division of two line segments can be ignored.
Chapter 4. Computation of value sets

Figure 4.9: Inversion of an axis parallel box.

Figure 4.10: Inversion of an axis parallel box with the origin inside.
4.4. Division of two axis parallel boxes

<table>
<thead>
<tr>
<th>Face</th>
<th>Relation</th>
<th>Curve $\hat{f}$</th>
<th>Type of curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha, a$</td>
<td>$a = \frac{b^<em>}{\beta^</em>} \alpha$</td>
<td>$\hat{f} = \frac{b^<em>}{\beta^</em>}$</td>
<td>point</td>
</tr>
<tr>
<td>$\alpha, b$</td>
<td>$b = -\frac{a^<em>}{\beta^</em>} \alpha$</td>
<td>$\hat{f} = -j \frac{a^<em>}{\beta^</em>}$</td>
<td>point</td>
</tr>
<tr>
<td>$\beta, a$</td>
<td>$a = -\frac{b^<em>}{\alpha^</em>} \beta$</td>
<td>$\hat{f} = j \frac{b^<em>}{\alpha^</em>}$</td>
<td>point</td>
</tr>
<tr>
<td>$\beta, b$</td>
<td>$b = \frac{a^<em>}{\alpha^</em>} \beta$</td>
<td>$\hat{f} = \frac{a^<em>}{\alpha^</em>}$</td>
<td>point</td>
</tr>
</tbody>
</table>

Table 4.3: Division of two line segments.

4.4.3 Summary

The results for the division of two axis parallel boxes are summarized and the computation procedure is presented.

Corollary 7 The boundary of the value set $V = B/A$ consists of line segments, i.e. the images of the division of an edge segment of $B$ with a vertex of $A$ and of arcs, i.e. the images of the division of a vertex of $B$ with an edge segment of $A$.

Procedure

1. Determine the relative arguments of the two value sets (§ 4.3.1). Due to the negative sign in (4.28) an additional phase of $-\pi$ is added to the relative arguments of the value set in the denominator.

2. With (4.28), the intersections of the relative arguments of the two value sets and the corresponding segments on the edges of $A$ and $B$ are determined (see Appendix C.2).
3. Compute the ratios of

(a) Vertex divided by an edge segment
(b) Edge segment divided by a vertex

This procedure is exemplified in Appendix C.2.

4.5 EXTENSION

The elementary operations "\( \times \)" and "\( \div \)" on axis parallel boxes discussed in § 4.3 and § 4.4 can be combined. The boundary of the value set \( f(j\omega) \) in the complex plane is determined using the same approach as in the sections before. For notational simplicity the axis parallel boxes will be \( A_i \) and \( B_j \) and the result set denoted as \( V \):

\[
V = \frac{B_1 B_2 \cdots B_m}{A_1 A_2 \cdots A_n} \quad (4.35)
\]

The following result for a sequence of products and ratios of axis parallel boxes can be derived from Theorem 6 and Theorem 7:

**Theorem 8** Let the boundaries \( a_i(\nu_i), \nu_i \in I_{ai}, b_j(\mu_j), \mu_j \in I_{bj} \) of the value sets \( A_i, B_j \) be positively oriented parameterized. Only such points on \( \partial A_i \) and \( \partial B_j \) can form the boundary \( \partial V \) of the value set \( V \) for which the following argument condition holds:

\[
\arg \left( \frac{b_j(\mu_j)}{b_j(\mu_j)} \right) = \arg \left( \frac{a_i(\nu_i)}{a_i(\nu_i)} \right) \quad \forall \ i = 1, \ldots, n, \ \forall \ j = 1, \ldots, m \quad (4.36)
\]

where "\( \prime \)" denotes the derivative to the appropriate parameter \( \mu_i, \nu_j \).

**Proof:** With Theorem 6 and Corollary 3 for the product of value sets the condition for the associated boundary points is
arg\left( \frac{b_j(\mu_j)}{b_j'(\mu_j)} \right) = arg\left( \frac{b_k(\mu_k)}{b_k'(\mu_k)} \right) = arg\left( \frac{b_j(\mu_j)b_k(\mu_k)}{(b_j(\mu_j)b_k(\mu_k)')} \right) \\
\forall \ j, k = 1, \ldots, m,

and therefore

arg\left( \frac{\prod_{j=1}^{n} b_j(\mu_j)}{(\prod_{j=1}^{n} b_j'(\mu_j))'} \right) = arg\left( \frac{b_k(\mu_k)}{b_k'(\mu_k)} \right) \forall \ k = 1, \ldots, m. \quad (4.37)

Similarly for the product of the value sets $A_i$ we get

arg\left( \frac{\prod_{i=1}^{n} a_i(\nu_i)}{(\prod_{i=1}^{n} a_i'(\nu_i))'} \right) = arg\left( \frac{a_k(\nu_k)}{a_k'(\nu_k)} \right) \forall \ k = 1, \ldots, n. \quad (4.38)

For the division of value sets with Theorem 7 and Corollary 6 follows

arg\left( \frac{\prod_{j=1}^{n} b_j(\mu_j)}{(\prod_{j=1}^{n} b_j'(\mu_j))'} \right) = arg\left( -\frac{\prod_{i=1}^{n} a_i(\nu_i)}{(\prod_{i=1}^{n} a_i'(\nu_i))'} \right) \quad (4.39)

and with (4.37) and (4.38) the result (4.36) is obtained.

From (4.36) follows that the relative angles of the value sets $A_i$ and $B_j$ must be intersected to find the common intersections of all value sets. The value sets in the denominator of (4.35) get an additional phase of $\pi$ due to the negative sign in (4.36). From the intersections of arguments, the associated points on the edges of the value sets $A_i$ and $B_j$ can be determined. By inspection, further reduction of the computational burden is possible since the argument condition (4.36) delivers also combinations of edge segments that are mapped to single points in the value set $V$ (see § 4.4).

Remark 9 The extension to the case of value sets $A_i$ and $B_j$ of general parameter polytopes is straightforward. All essential Theorems 6, 7
and 8 and Corollaries 2, 3, 5 and 6 do not take advantage of the property of axis parallelism of the value sets $A_i$ and $B_j$. For the computation of the relative angles the direction of the edges (not anymore a multiple of $\pi/2$) as well as the different opening sectors of the vertices must be considered. The analytic calculation of the boundary curves has to be slightly modified.

4.5.1 Extended case for $n = m = 2$

In the sequel the extended case is discussed for $n = m = 2$. This case can be interpreted as a cascade of two interval plants. Using the results of § 4.3 and § 4.4 we determine the boundary of the value set

$$V = \frac{B_1 B_2}{A_1 A_2} = \frac{a_1 + jb_1}{a_1 + j\beta_1} \frac{a_2 + jb_2}{a_2 + j\beta_2}$$

(4.40)

where the parameters vary along the edges of the boxes $A_i$ and $B_j$. Then, with § 4.4 only one- and two-dimensional faces of the 8-dimensional parameter hyperrectangle $(\alpha_i, \beta_i, a_i, b_i)$, $i = 1, 2$ must be discussed. A one-dimensional face is defined by just one parameter varying and all the others fixed at their extreme values. In this case the resulting curve in the value set is either a straight line or an arc depending on the location of the variable parameter in (4.40).

For the two-dimensional faces with § 4.4.2 only those combinations of two varying parameters must be considered where both variable parameters are located either in the numerator or in the denominator of (4.40). For all such combinations of two varying parameters with (4.36) the curves in Table 4.4 result. All other combinations of two variable parameters are mapped to single points of the value set. The procedure for the calculation of the relation between the two varying parameters and the resulting analytic expression $\hat{f}$ in the value set is exactly the same as in § 4.3 and § 4.4 and is not repeated here.
### Table 4.4: Two-dimensional faces.

<table>
<thead>
<tr>
<th>Face</th>
<th>Relation</th>
<th>Curve $\hat{f}$</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1, \alpha_2$</td>
<td>$\alpha_2 = \frac{\beta_2^*}{\beta_1} \alpha_1$</td>
<td>$\frac{\beta_1^* (a_1^* + j b_1^<em>) (a_2^</em> + j b_2^<em>)}{\beta_2^</em> (a_1 + j \beta_1)^2}$</td>
<td>1</td>
</tr>
<tr>
<td>$\alpha_1, \beta_2$</td>
<td>$\beta_2 = \frac{-\alpha_2^*}{\alpha_1}$</td>
<td>$\frac{j \beta_1^* (a_1^* + j b_1^<em>) (a_2^</em> + j b_2^<em>)}{\alpha_2^</em> (a_1^* + j \beta_1)^2}$</td>
<td>1</td>
</tr>
<tr>
<td>$\beta_1, \alpha_2$</td>
<td>$\alpha_2 = \frac{-\beta_2^*}{\alpha_1} \beta_1$</td>
<td>$-\frac{j \alpha_1^* (a_1^* + j b_1^<em>) (a_2^</em> + j b_2^<em>)}{\beta_2^</em> (a_1^* + j \beta_1)^2}$</td>
<td>1</td>
</tr>
<tr>
<td>$\beta_1, \beta_2$</td>
<td>$\beta_2 = \frac{a_2^<em>}{a_1^</em>} \beta_1$</td>
<td>$\frac{\alpha_1^* (a_1^* + j b_1^<em>) (a_2^</em> + j b_2^<em>)}{\alpha_2^</em> (a_1^* + j \beta_1)^2}$</td>
<td>1</td>
</tr>
<tr>
<td>$a_1, a_2$</td>
<td>$a_2 = \frac{b_2}{b_1} a_1$</td>
<td>$\frac{b_2^* (a_1 + j b_1^<em>)^2}{b_1^</em> (a_1^* + j \beta_1^<em>)(a_2^</em> + j \beta_2^*)}$</td>
<td>2</td>
</tr>
<tr>
<td>$a_1, b_2$</td>
<td>$b_2 = \frac{-a_2^*}{b_1} a_1$</td>
<td>$-\frac{j a_2^* (a_1 + j b_1^<em>)^2}{b_1^</em> (a_1^* + j \beta_1^<em>)(a_2^</em> + j \beta_2^*)}$</td>
<td>2</td>
</tr>
<tr>
<td>$b_1, a_2$</td>
<td>$a_2 = \frac{-b_2}{a_1^*} b_1$</td>
<td>$\frac{j b_2^* (a_1^* + j b_1^<em>)^2}{a_1^</em> (a_1^* + j \beta_1^<em>)(a_2^</em> + j \beta_2^*)}$</td>
<td>2</td>
</tr>
<tr>
<td>$b_1, b_2$</td>
<td>$b_2 = \frac{a_2^<em>}{a_1^</em>} b_1$</td>
<td>$\frac{a_2^* (a_1^* + j b_1^<em>)^2}{a_1^</em> (a_1^* + j \beta_1^<em>)(a_2^</em> + j \beta_2^*)}$</td>
<td>2</td>
</tr>
</tbody>
</table>

Type of curve: 1: Rational function 2: Parabola
The computation of the value set of \( V = (B_1 B_2)/(A_1 A_2) \) goes through the following steps:

1. Computation of the arguments \( \arg(a/a') \) of the boundaries of the four value sets.

2. Determination of the intersection of the relative arguments of the four value sets with (4.36).

3. Extraction of the line segments on each boundary of \( A_1, A_2, B_1 \) and \( B_2 \).

4. Computation of the boundary \( \partial V \) as the image of the subset of the one-dimensional (vertex \( \times \) edge segment) and two-dimensional (edge segment \( \times \) edge segment) faces. Note, that these products can be located either in the numerator or in the denominator of (4.40).

A numerical example can be found in Appendix C.3.

### 4.6 SUMMARY

This chapter has presented a procedure for the computation of value sets of uncertain transfer functions where the numerator and denominator are uncertain polynomials or products of polynomials with mutually independent parameterization of the even and odd parts. Interval plants or cascades of interval plants are special cases of this set-up. The value sets of these polynomials are axis parallel boxes in the complex plane. Therefore, the value set of the uncertain transfer function can be determined as a sequence of the operations multiplication and division on axis parallel boxes. It was shown that the boundary of the image set in the complex plane is obtained from the mapping of boundary points of the operands and, thus, no inner points of the simpler value sets have to be considered. The boundary of the image set is determined using an argument condition for the points on the boundaries of the operands to be candidates for being mapped on the
4.6. Summary

boundary of the image set. Since this condition is only necessary but not sufficient some points which fulfill the argument condition can be mapped to the inside of the image set.

The described procedure can be easily implemented on a computer program such as MATLAB\(^2\). The boundaries of the value sets and their relative arguments can be treated as lists. Then, the extraction of the relevant points on the boundaries of the simple value sets is performed as the intersection of lists. In the final step, the exact coordinates of the boundary points of the operands have to be extracted from these lists in order to compute the boundary of the value set (Appendix C).

The presented procedure for the computation of value sets of uncertain transfer functions provides a strong basis for the investigation of uncertain control systems. This procedure can be easily embedded in CACSD tools for the interactive analysis and design of controllers in the frequency domain [61].

\(^2\)The examples in Appendix C were computed with our toolbox "value_set" written in MATLAB
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Chapter 5

DESIGN OF ROBUST CONTROLLERS FOR UNCERTAIN PLANTS

The design of a robust controller for an uncertain system is addressed in this chapter. The simultaneous stabilization problem and some design algorithms are discussed. In the second part, the classical design of dynamic compensators of lead/lag type in the frequency domain is extended to uncertain systems.

5.1 INTRODUCTION

In the last few years, numerous contributions to the analysis of uncertain control systems with emphasis on stability problems have been published. Tests have been derived to investigate if a given controller can robustly stabilize an uncertain plant. However, the synthesis problem of designing a fixed robust controller has been recognized as a very challenging problem. Until now, only few results on the design of robust controllers were published. Especially algebraic algorithms for a systematic design process are missing. The reason may be that the simultaneous stabilization problem is very complex from the mathematical point of view. Further on, the conditions for robust stability are in general computationally intractable when addressing the synthesis problem.

The discussion of the synthesis problem concentrates on two subproblems: (1) the existence of a robust controller for a given uncertain plant and (2) the design of the controller.
5.2 THE SIMULTANEOUS STABILIZATION PROBLEM

The synthesis problem is centered around the question of existence of a robust controller which is formulated in a general form as the so-called Simultaneous Stabilization Problem: Given a specific family of plants $F(s)$ described by a finite or infinite set of transfer functions. Under what conditions does there exist a fixed dynamic compensator $C(s)$ which simultaneously stabilizes the family of plants?

- Vidyasagar and Viswanadham [30] proved that two $p \times m$ plants $G_0(s)$ and $G_1(s)$, where $p$ is the number of inputs and $m$ the number of outputs, can generically be stabilized simultaneously if $\max(p, m) > 1$. In the single-input-single-output case ($p = m = 1$), the simultaneous stabilizability of $G_0$ and $G_1$ is no longer a generic property. As an extension, the authors proved that the problem of simultaneous stabilization of $l + 1$ plants can be reduced to the simultaneous stabilization of $l$ plants using a stable compensator.

- Ghosh and Byrnes [31] found a sufficient condition for the generic simultaneous stabilizability of a $r$-tuple of $p \times m$ proper plant transfer functions $G_1(s), \ldots, G_r(s)$ with a dynamic compensator $C(s)$:

**Theorem 9** The generic $r$-tuple of $p \times m$ plants is simultaneously stabilizable by a compensator $C(s)$ of degree $\leq q$ if

$$\max(m, p) \geq r$$

and $q$ satisfies the inequality

$$q[\max(m, p) + 1 - r] \geq \sum_{i=1}^{r} n_i - \max(m, p)$$

where $n_i$ is the McMillan degree of $G_i(s)$.

For the case of two plants ($r = 2$) the condition given by the theorem yields $\max(p, m) > 1$ and hence the result of Vidyasagar [30].
Note that (5.1) is also a sufficient condition for generic simultaneous pole assignability of a set of $r$ strictly proper plants where the compensator $C(s)$ is of degree $q$ with $q$ satisfying (5.2) [31].

- The problem of simultaneous stabilization of a family of interval plants was first investigated by Ghosh [32] where he gave a sufficient condition for an interval plant to be stabilized by the compensator $C(s)$. Four special stable polynomials guarantee the stability of the whole uncertain system. These four polynomials correspond to the four Kharitonov polynomials obtained by overbounding the coefficients of the characteristic polynomial to get an interval polynomial (hence the sufficiency of this condition).

If the interval plant is stabilized using a constant feedback gain $K$, then, it is necessary and sufficient to simultaneously stabilize eight special plants. These eight plants are a subset of the sixteen Kharitonov plants resulting from all combinations of the Kharitonov polynomials in the numerator and the denominator of the interval plant.

- Barmish and Wei [34] investigated the family of plants

$$P(s, q) = \frac{\sum_{i=0}^{n(q)} b_i(q) s^{m(q)-i}}{s^{n(q)} + \sum_{i=1}^{n(q)} a_i(q) s^{n(q)-i}}$$

(5.3)

with $q \in Q$ and $Q$ an indexing set. This family of plants describes a finite or infinite set and it also covers the case of an interval plant. It was proved that under a set of assumptions such as strictly properness, minimum phase, one sign high frequency gain of the plant, the family $P(s, q)$ is simultaneously stabilizable and a strictly proper and stable compensator $C(s)$ can be constructed. An iterative design algorithm for $C(s)$ was given (see § 5.3.2).

- Barmish et al. [29] treated the problem of simultaneous stabilization of an interval plant using a first-order compensator. A necessary and sufficient condition for the stabilizability of the whole family is the stabilizability of the sixteen Kharitonov plants. Depending on the root distribution of the compensator, simultaneous stabilization of eight or
twelve Kharitonov plants can be sufficient to guarantee stability of loop.

5.3 DISCUSSION OF DESIGN METHODS

5.3.1 Method of Evans and Xianya (1985)

A numerical procedure for the design of a robust controller \( G_R(s) \) stabilizing an interval plant \( G(s) \) was proposed in [36]:

1. A controller is designed for the nominal plant using any classical design procedure such as LQG, pole placement etc.

2. The coefficients of the controller are adjusted in order to render the Hurwitz invariance of the characteristic polynomial, denoted by \( P(s) \), for all uncertainties. Let

\[
G_R(s) = \frac{d(s)}{c(s)} \quad \text{and} \quad v = [d \ c]
\]

where \( c \) and \( d \) are the coefficient vectors of the controller polynomials \( c(s) \) respectively \( d(s) \). Further let \( v_0 \) be the coefficient vector of the nominal controller. Then, the constrained optimization problem can be formulated as:

\[
\min_{\hat{v}} \|v - v_0\|_2^2
\]

such that

\[
\det H_{j,k}(v) > 0 \quad j = 1, \ldots, n-1 \quad k = 1, \ldots, 4
\]

where \( H_{j,k}(\cdot) \) are the leading principle minors of the Hurwitz matrices of the four Kharitonov polynomials \( \tilde{P}_k(s) \), \( k = 1, \ldots, 4 \) that result from overbounding \( P(s) \) to an interval polynomial \( \tilde{P}(s) \). By inspection, the set of constraints can be reduced to the critical inequalities, depending on the extreme values of the plant coefficients.

The proposed cost function shall guarantee that the behavior of the robust controller remains similar to that of the nominal one.
The constraints of the optimization problem give only a sufficient condition for the Hurwitz invariance of the closed-loop [32]. Moreover, the overbounding of the characteristic polynomial to an interval polynomial can be rather conservative since the coefficients of the characteristic polynomial of a closed-loop with an interval plant are affine functions of the coefficients of the plant.

The constraints can be replaced by the necessary and sufficient conditions for the Hurwitz invariance of the closed-loop. As shown in § 3.5.1, the Hurwitz stability of at most 32 edge polynomials, depending on the phase of the controller, must be guaranteed. Therefore, one has to find the minimal Hurwitz determinants along each of the 32 edges and their positivity guarantees positivity of all Hurwitz determinants along the 32 edges and hence Hurwitz invariance of the closed-loop.

The constrained optimization problem can be reformulated as:

\[
\min_{\lambda} \|u - u_0\|_2^2
\]

such that

\[
\min_{\lambda} \{\det H_{j,k}(u, \lambda)\} > 0
\]

\[0 \leq \lambda \leq 1\]

with

\[j = 1, \ldots, n - 1 \quad k = 1, \ldots, 32\]

where \(H_{j,k}(\cdot), j = 1, \ldots, n - 1\) are the leading principle minors of \(H_k(u, \lambda)\) and \(H_k(\cdot)\) is the Hurwitz matrix along the \(k\)-th edge built as a convex combination of the Hurwitz matrices \(H(p_1), H(p_2)\) of the associated vertex polynomials \(p_1(s)\) and \(p_2(s)\):

\[
H_k = (1 - \lambda)H(p_1) + \lambda H(p_2) \quad \lambda \in (0, 1).
\]

This design method proposed by Evans and Xianya delivers, if it exists, a specific controller which guarantees the Hurwitz invariance of the closed-loop. In the presence of large plant uncertainties, i.e. the members of the plant family can heavily deviate from the nominal
plant, the uncertain system remains stable but can show a great variety in the behavior. This is due to the fact that no performance specifications are considered in this design procedure. Moreover, the cost function does not reflect any specifications for the whole family of plants. It simply guarantees that the behavior of the loop with the nominal plant and the robust controller does not substantially deviate from the nominal design.

Although this method is quite basic and does not deliver satisfactory system behavior it was discussed here since it represents one of the approaches in actual research for robust controller design: With numerical optimization the controller is determined minimizing a suitable robustness measure\(^1\). A key problem is the optimization algorithm since, due to the constraints for robust stability and robust performance which result in generally nonlinear and often non-differentiable functions, the optimization problem becomes non-smooth. Most of the standard optimization techniques such as gradient methods get into troubles and show bad convergence properties.

### 5.3.2 Method of Barmish and Wei (1986)

An iterative design procedure was presented by Barmish and Wei [34] for the simultaneous stabilization of a single input-single output family and in [62] extended to MIMO systems. The same procedure was used in [35] where beside the parameter uncertainties also high-order norm bounded uncertainty was considered. The iterative design steps are essentially the same ones as in [34].

Let the family of plants be given as:

\[
P(s, q) = \frac{N_p(s, q)}{D_p(s, q)} = \frac{\sum_{i=0}^{m(q)} b_i(q) s^{m(q)-i}}{s^n(q) + \sum_{i=1}^{n(q)} a_i(q) s^{n(q)-i}}
\]

\(^1\)See the Ph.D. thesis of G. Peretti, Automatic Control Laboratory, ETH Zürich, to be published at the beginning of 1994
Assume that $P(s, q)$ is:

- strictly proper
- has bounded degree, i.e. $n(q) < \infty \ \forall q \in Q$
- continuous coefficients
- has a compact index set, i.e. $Q$ is a compact subset of $\mathbb{R}^p$
- has a closed degree set
- minimum phase
- has a constant sign of the high frequency gain over $Q$.

Then, the family $P(s, q), q \in Q$ is simultaneously stabilizable and a strictly proper and stable compensator can be constructed iteratively. Consider a controller of the form

$$A^c = A^0 M \ll S \ K$$

The design algorithm consists of the following steps:

**Step 1:**

$N_C(s)$ is chosen to be any Hurwitz polynomial with $\delta_k > 0$ and degree

$$k \geq \max_{q \in Q}\{n(q) - m(q)\} - 1$$

**Step 2:** (Initialization)

Define

$$\Delta_0(s, q) = N_C(s)N_p(s, q)$$

$$D_{C,0} = 0$$

**Step 3:** (Inductive Step)

For given $\Delta_i(s, q)$ and $D_{C,i}(s)$, select $\epsilon_i > 0$ such that

$$\Delta_{i+1}(s, q) = \Delta_i(s, q) + \epsilon_i s^i D_p(s, q)$$
is Hurwitz invariant and

\[ D_{C,i+1}(s) = D_{C,i}(s) + \epsilon_i s^i \]

is Hurwitz. The assumptions on the plant guarantee the success of the procedure for \( i = 0, 1, \ldots, l \).

**Step 4: (Termination)**

The denominator of the controller is then

\[ D_C(s) = D_{C,l+1}(s) \]

The main problem of this design procedure is the inductive step (Step 3) where a numerical search for \( \epsilon_i \) has to be performed in order to make the partial denominator \( \Delta_{i+1}(s, q) \) Hurwitz invariant. With an interval plant, \( \Delta_{i+1}(s, q) \) is a family of polynomials where the coefficients are affine functions of the interval parameters of the plant. The numerical search of \( \epsilon_i \) over the whole polynomial family \( \Delta \) is quite complex. The authors in [34] propose to overbound the polynomial family \( \Delta_{i+1}(s, q) \) to an interval polynomial \( \tilde{\Delta} \) and then, to use the Kharitonov theorem to guarantee Hurwitz invariance. Therefore, \( \epsilon_i \) is determined making the four Kharitonov polynomials of \( \tilde{\Delta}_{i+1}(s, q) \) Hurwitz stable. Naturally, this step brings conservatism into the design method since the Kharitonov theorem gives only a sufficient condition for the Hurwitz invariance of the polynomial family \( \Delta_{i+1}(s, q) \).

This design method can merely guarantee the Hurwitz invariance of the closed-loop. The zeros of the controller are chosen freely and the denominator is numerically adjusted for Hurwitz stability of the characteristic polynomial. The resulting closed-loop behavior is thereby rather accidental since not even a nominal behavior is considered in the design steps.

This method shows a key problem with the algebraic design of robust controllers: The mathematical complexity of the stability conditions. On one hand, these conditions for simultaneous stability of a finite or infinite set of systems are generally in a mathematical form useless for
5.4. Design of robust dynamic compensators

the design. A “pass/fail” answer is often obtained from the stability tests and it is difficult to decide how to adjust the controller parameters to improve stability. On the other hand, the number of conditions that has to be simultaneously satisfied can be quite large. To deal with these problems, two approaches are typically used: (1) the introduction of a more conservative description of the uncertainties, such as interval parameters. This yields less and easier conditions for robust stability but also leads to conservative results for the controller design. (2) The controller parameters are determined by numerical search methods.

5.4 DESIGN OF ROBUST DYNAMIC COMPENSATORS

5.4.1 Preliminary

In the field of linear control system design, an important method for the approximative design of dynamic compensators uses the frequency domain representation of the open-loop system. The Bode plot of the plant is very useful for the design of dynamic compensators in the case where the requirements on the closed-loop system behavior are specified for the open-loop system in the frequency domain (such as cross over frequency, gain and phase margin etc.). In that case a lead or lag compensator — or a cascade of them — can be designed in an iterative way using the magnitude and phase plot of the open-loop system. If the plant is subject to parameter uncertainties, the frequency domain representation is no longer a simple curve but now at each frequency a continuum of points results. Hence, the Nyquist curve of the system changes to a band, the so-called Nyquist band (Definition 6 on page 100).

In § 4, the computation of value sets of uncertain transfer functions was discussed. These results can be directly applied for obtaining the frequency domain representation of uncertain systems. Based on the frequency plot of the uncertain system, procedures for the design of robust lead/lag compensators are presented. Similar procedures for classical systems can be used, but the design specifications must be discussed and partly redefined for robust control system design.
Consider the control system in Fig. 2.1 on page 9 where the uncertain plant is described by

\[ G(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0} \]  

(5.4)

with

\[ a_i(\gamma), \ b_j(\gamma) \quad \gamma \in \Gamma \quad i = 0, \cdots, n - 1 \quad j = 0, \cdots, m \]

and \(\Gamma\) the uncertainty region.

The goal is to design a robust controller in the frequency domain with a fixed transfer function \(G_R(s)\) fulfilling a number of specifications on the behavior of the closed-loop for every member \(G(s)\) of the family of plants (5.4).

### 5.4.2 Frequency domain representations of the open-loop transfer function

**The Nyquist plot**

For the design of a controller in the frequency domain the open-loop representation

\[ G_0(s) = G_R(s)G(s) \]

of the control system in Fig. 2.1 is needed. For every fixed frequency \(s = s^*\), the image of \(G_0(s)\) in the complex plane is the value set (§ 4). For \(s = j\omega\), \(\omega\) varying, the *Nyquist band* is obtained.

**Definition 6** The *Nyquist band of an uncertain system* is the envelope of the value sets for \(s = j\omega\) with \(\omega \in [0, \infty)\).

The main problem is to find the boundary of the value set of the uncertain system. As shown in § 4 for special structures of the uncertainty, this problem can be solved without huge computational burden. For the special case of an interval plant \(G(s)\), the boundary of the value set is non-convex and consists of line segments and arcs (§ 4.4).
At any fixed frequency $s = j\omega^*$, the transfer function $G_R(s)$ of the controller is a complex number. The effect of the controller on the value set is a rotation according to the controller phase and an expansion caused by the controller magnitude.

Since there is no way to directly compute the Nyquist band of $G_0(j\omega)$ one has to compute the value set at different frequencies and finally take the envelope of these subsets. But for the design problem one can often concentrate on a finite number of different frequencies near the interesting points for the design, i.e. the critical point or the cross over frequency. At the end of the design process, the design specifications can be checked by computing the Nyquist band using a fine gridding of the frequency $\omega$.

**Remark 10** Problems with the value set occur if the interval plant has poles on the imaginary axis since the value set at that frequency $s = j\omega^*$ is no longer a closed domain (Fig. 4.10 on page 82). This is similar to the the classical case – linear systems without uncertainties – where the Nyquist curve goes to infinity at those frequencies of pure imaginary open-loop poles. These cases are difficult to handle with the Nyquist plot and are not discussed here.

**The Bode plot**

From the Nyquist band, the Bode envelope of the uncertain system is obtained by extracting the extremal gains and phases of the value sets at each frequency (Fig. 5.2).

Note that the extremal curves of the amplitude and the phase band in the Bode plot do not correspond to each other, since points of the value set with extremal gain are in general not identical to those with extremal phase. Fig. 5.3 illustrates that the Bode plot at each frequency is an overbound of the exact value set.

As a consequence, the design of a controller with the Bode plot is more conservative than the design with the exact Nyquist band. However, the advantage of the Bode plot is the explicit frequency information present in the magnitude and phase plot.
Figure 5.1: Nyquist band of an uncertain system (high frequency part at the bottom).
5.4. Design of dynamic compensators

5.4.3 Design specifications

Some of the controller design specifications in the frequency domain must be revised in the sense of worst case conditions for the uncertain system.

Gain margin and phase margin

The gain margin $GM$ is the minimal gain increase of the open-loop which causes some of the poles of the closed-loop system to lie on the stability boundary. This gain can be easily extracted from the Nyquist diagram. In the case of an uncertain plant, the gain margin can be defined as follows:

Definition 7 The gain margin $GM$ of an uncertain system is the smallest gain margin of all members of the family of systems.

The phase margin $PM$ is defined as the smallest phase increase which,
applied to a system at the cross over frequency, causes instability of the system.

**Definition 8** The phase margin \( PM \) of an uncertain system is the minimal phase margin of all members of the family of systems.

Fig. 5.4 shows the gain and phase margin of an uncertain system in the Nyquist plot.

**Cross over frequency \( \omega_D \)**

The cross over frequency \( \omega_D \) of a linear system is the frequency where the gain of the open-loop becomes one, i.e. 0dB. It is an important measure for the dynamic quality of the control loop. The value of \( \omega_D \) determines the transient behavior of the system and especially the damping ratio. The higher \( \omega_D \) is, the faster the control loop can react to changes in the reference signals or to disturbances. On the other hand, the value of \( \omega_D \) should not be too high in order to suppress high frequency noise.
5.4. Design of dynamic compensators

Figure 5.4: Gain margin GM and phase margin PM of an uncertain system.

For an uncertain system there exists no discrete cross over frequency $\omega_D$. In general the value sets of different frequencies are overlapping such that there results a range of frequencies for $\omega_D$. For example in Fig. 5.5 all the value sets in the frequency range $0.673 \text{ sec}^{-1} \leq \omega \leq 0.942 \text{ sec}^{-1}$ have points with gain 0dB. The two value sets shown in Fig. 5.5 are the two extremal ones with only one member of the family having 0dB. For each frequency in the interval $\omega_D$ there exists a subset of systems inside of the family having zero gain.

The size of the frequency interval $\omega_D$ depends on the size of the uncertainty and on the slope of the magnitude plot at the cross over. The natural way to reduce the width of the frequency interval would be to add poles to the system with the aim of increasing the slope of the magnitude plot. But the secondary effect would be an increase in the phase brought by the poles. Either the upper or the lower bound of the interval $\omega_D$ can be specified since the other bound will be automatically fixed from the design. From practical experiences, it turns out that in most cases it is impossible to design a compensator guaranteeing simultaneously a minimal phase margin $PM$ and the lower bound, denoted as $\omega_D$. If for example a lead is used to increase the phase at $\omega_D$ to obtain the specified $PM$, then, the upper bound, denoted as $\omega_D$, is automatically shifted towards higher frequencies due to the increase
Figure 5.5: Range of the cross over frequency $\omega_D$ of an uncertain system.

of the magnitude caused by the lead. Moreover, the necessary phase increase at the new $\omega_D$ grows rapidly. The discussion in this chapter concentrates on the specification of the upper bound $\omega_D$.

Steady state error

A common specification for linear controller design is the maximal steady state error $e(\infty) = r(\infty) - y(\infty)$ for a certain kind of input signals $r(t)$ such as steps or ramps. Depending on the difference between the type of the open-loop system and the type of the input signal, the steady state error becomes zero, constant or infinite. If $e(\infty)$ is constant, then, the necessary gain of the open-loop and hence the gain of the controller can be derived.

For an uncertain system there can be some difficulties with the type of the system. Consider the transfer function in (5.4). A pole at zero
can belong to two different cases:

i) Structural pole at zero: \( a_0 \equiv 0 \)

ii) Change of system type: \( 0 \in [a_0, \bar{a}_0] \)

In the first case, the type of the system is fixed and causes no problem. In the second case the system is generally of type zero except in those cases, where the coefficient \( a_0 \) of the denominator of the plant becomes exactly zero. However, in the sense of worst case this kind of system can be treated as being of type zero.

**Time domain specifications for the closed-loop**

Specifications in the time domain for the closed-loop system such as the damping ratio or the maximal overshoot of the step response can be also given for uncertain systems as in the classical case. These values are then simply interpreted as worst case specifications for the uncertain system. For the design procedure in the frequency domain these closed-loop specifications must be transformed in the usual way to open-loop specifications [38, 63].

**5.4.4 Design procedures**

In classical linear control system design a very common type of dynamic compensator is the lead or lag structure

\[
G_R(s) = K_c \frac{s + z}{s + p}
\]

with \( z, p > 0 \) and

\[
\begin{align*}
    z < p & \rightarrow \text{lead} \\
    z > p & \rightarrow \text{lag}
\end{align*}
\]

or a cascade of these elementary compensators. These compensators can be designed in an easy way using the Bode plot of the open-loop system.
The maximal phase shift of the controller is obtained at the frequency

\[ \omega_\mu = \sqrt{p z} \]

In the logarithmic representation of the \( \omega \)-axis in the Bode plot, \( \omega_\mu \) lies exactly in the middle of the two frequencies of the controller. Then, with \( \alpha = p/z \), we get

\[ z = \frac{\omega_\mu}{\sqrt{\alpha}} \]

\[ p = \omega_\mu \sqrt{\alpha} = z \alpha \]

The maximal phase is

\[ \phi_{\text{max}} = \arctan \sqrt{\alpha} - \arctan \frac{1}{\sqrt{\alpha}} \]

and hence

\[ \alpha = \frac{1 + \sin \phi_{\text{max}}}{1 - \sin \phi_{\text{max}}} \]  

and for the magnitude at the frequency \( \omega_\mu \) we get

\[ |G(j\omega_\mu)| = -10 \log \alpha \]

For the design of dynamic compensators, the most common specifications on the performance of the closed-loop are

- Range for the upper bound of the cross over frequencies:
  \[ \omega_{\text{min}} \leq \omega_D \leq \omega_{\text{max}} \]

- Minimal phase margin \( PM \)

- Maximal steady state error \( e(\infty) \leq \epsilon \).

If \( e(\infty) \) is constant then it follows \( K_{\text{comp}} = K_{\text{tot}}/K_S \) where \( K_S \) is the minimal gain of the plant, \( K_{\text{tot}} \) the minimal necessary gain of the open-loop to fulfill the steady state requirements and \( K_{\text{comp}} \) the necessary gain of the compensator.
In the sequel, the design procedures for proportional controller, lag, lead and lead-lag compensators are presented using the Bode plot of the uncertain system. Depending on the required specifications these compensators may have to be cascaded. Note that these design methods are valid for any kind of uncertain system for which a Bode envelope of the open-loop is available (§ 6). As mentioned earlier for special structures of the uncertainties, the Bode envelope can be easily computed from the value sets (§ 4).

**Proportional controller**

The proportional controller is used to guarantee the required steady state error of the closed-loop. In the frequency domain, the additional gain of the proportional controller causes a change of the interval of cross over frequencies $\omega_D$. Thereby, the interval is not simply shifted to higher or lower frequencies but, depending on the slope of the magnitude plot, the width of $\omega_D$ can be changed.

**Lag compensator**

The lag compensator is used to increase the gain at low frequencies. As a secondary undesired effect the phase decreases.

- **Step 1**: Determine the minimal necessary phase increase $\phi_c$ to guarantee the phase margin $PM$.

  \[ \phi_c = PM + \Delta \phi_{LAG} \]

  where $\Delta \phi_{LAG}$ is approximatively 5° for the secondary decrease of phase at the cross over frequencies.

  Draw a line at $\varphi = -180^\circ + \phi_c$ in the phase plot and determine the intersection of this line with the minimal phase curve and extract the frequency $\omega_1$.

- **Step 2**: The frequency $\omega_1$ will be the upper bound $\omega_D$. Compare $\omega_1$ with the specified range for $\omega_D$: 
Figure 5.6: Design of a lag compensator.

\[ \omega_1 < \omega_{\text{min}} \quad \text{an increase of phase is necessary;} \]
\[ \text{choose a new structure for the controller.} \]
\[ \omega_1 > \omega_{\text{max}} \quad \text{enlarge PM until } \omega_{\text{min}} \leq \omega_1 \leq \omega_{\text{max}} \]

- **Step 3**: Determine the additional gain \( K_c [\text{dB}] \) for the lag compensator at \( \omega_1 \) on the maximal magnitude curve. The frequency \( \omega_2 \) at the gain \( K_c [\text{dB}] \) on the minimal magnitude curve becomes the lower bound of the cross over frequencies.

- **Step 4**: A lag with gain \( K_c \) and with a phase decrease of 5° at the frequency \( \omega_1 \) is designed.

\[
\alpha [\text{dB}] = K_{\text{comp}} [\text{dB}] - K_c [\text{dB}]
\]

\[
z = 0.1 \omega_1 \quad p = \frac{z}{\alpha}
\]
and the transfer function of the lag compensator follows

\[ G_{\text{LAG}} = K_c \frac{s + z}{s + p} \]

- \textbf{Step 5:} Check if all the design specifications are fulfilled:
  - If \( \Delta \phi_{\text{LAG}} > 5^\circ \) at \( \omega_1 \), then decrease \( z \)
  - Compute the Nyquist band of the plant with the lag
  - Simulate the control loop

Note that in general, the resulting phase margin \( PM \) is slightly bigger than the required one. This is due to the conservatism of the design in the Bode plot.

\textbf{Lead compensator}

The lead compensator is used to guarantee the required phase margin \( PM \) at the cross over frequency \( \omega_D \).

- \textbf{Step 1:} From the specification of the steady state behavior of the system determine the gain \( K_{\text{comp}} = K_c \) of the compensator. Draw a new "0dB"-line in the magnitude plot. The intersection of this line with the maximal magnitude curve gives the frequency \( \omega_1 \).

- \textbf{Step 2:} Determine the necessary phase increase at the frequency \( \omega_1 \) with the \textit{minimal} phase curve:

\[ \phi_m = PM - 180 - \phi_1 + \Delta \phi \]

with \( \phi_1 = \phi(\omega_1) \) and \( \Delta \phi \) being an additional phase increase for counteracting the shift of the interval \( \omega_D \) caused by the gain of the lead compensator. Compute \( \alpha \) for \( \phi_m \) with (5.5).

- \textbf{Step 3:} Increase the magnitude plot with \( 10 \log \alpha \) after \( \omega_1 \) and determine the new upper bound \( \omega_2 \) of the cross over frequencies.
• **Step 4:** Determine $\phi_2 = \phi(\omega_2)$ with the minimal phase curve. If $\phi_2 + \phi_m > PM - 180^\circ$ then go to Step 5 else go back to Step 2 and increase $\Delta \phi$.

• **Step 5:** Determine the lead compensator with maximal phase shift at $\omega_2$:

$$z = \frac{\omega_2}{\sqrt{\alpha}} \quad p = \alpha z$$

The transfer function of the lead compensator is then

$$G_{\text{LEAD}} = K_c \alpha \frac{s + z}{s + p}$$

Note that we assume the phase band to monotonically decrease in the interval $\omega_D$. The lead is then designed for the maximal phase increase to occur at the upper bound $\omega_D$. Since the phase at $\omega_D$ is generally higher than the one at $\omega_D$, the phase increase at $\omega_D$ should suffice to guarantee the minimal phase margin $PM$ over the whole interval $\omega_D$.  

---

**Figure 5.7:** Design of a lead compensator.
Lead-lag compensator

The lead-lag compensator is a cascade of the lead, used to guarantee the required phase margin \( PM \) at a specified cross over frequency \( \omega_D \), and the lag which ensures the gain at low frequencies for the stationary behavior.

![Diagram of lead-lag compensator](image)

**Figure 5.8: Design of a lead-lag compensator.**

- **Step 1:** Choose the upper bound \( \omega_D \) for the cross over frequency interval \( \omega_D \).

- **Step 2:** Determine the necessary phase increase \( \phi_m \) at \( \omega_D \) with the minimal phase curve:

\[
\phi_m = PM - 180^\circ - \phi_1 + \Delta\phi_{LAG}
\]

with \( \Delta\phi_{LAG} \approx 5^\circ \) for the decrease of phase caused by the lag at \( \omega_D \). Compute \( \alpha_{LEAD} \) with (5.5) for \( \phi_m \).
• **Step 3**: Increase the magnitude plot with $10 \log \alpha_{\text{lead}}$ near $\omega_D$.

• **Step 4**: Draw the new "0dB"-line through the intersection of $\omega_D$ with the maximal magnitude curve. Extract the gain $K_c$ and design the lead:

$$z_1 = \frac{\omega_D}{\sqrt{\alpha_{\text{lead}}}} \quad p_1 = z_1 \alpha_{\text{lead}}$$

• **Step 5**: From the specification of the steady state behavior follows

$$\alpha_{\text{lag}} = \frac{K_{\text{tot}}}{K_S K_c}$$

• **Step 6**: A lag with phase decrease of less than $5^\circ$ at the frequency $\omega_D$ is designed.

$$z_2 = 0.1 \omega_D \quad p_2 = \frac{z_2}{\alpha_{\text{lag}}}$$

The transfer function of the lead-lag compensator is therefore

$$G(s) = K_c \ \alpha_{\text{lead}} \ \frac{(s + z_1)(s + z_2)}{(s + p_1)(s + p_2)}$$

### 5.5 SUMMARY

The design of a dynamic compensator in the frequency domain has to be performed iteratively in general. To support the design process, computer based tools are of great importance in the case of uncertain systems because of the complexity of the frequency representations. Starting with the given plant description, the value sets of the uncertain transfer function have to be computed using a fine gridding of the frequency $\omega$. Then, the associated Bode envelope is extracted from these value sets. During the design process there is no need to re-compute the value sets. The design of the compensators can be performed with the Bode envelope of the uncertain plant and the effect
of the compensator is directly studied with the magnitude and phase band in the Bode plot. Finally, the exact Nyquist band (i.e. the value sets of the open-loop) is computed to check the design specifications. Since the Bode envelope is an overbound of the value sets at each frequency the design process can be conservative. Therefore, the resulting gain and phase margin of the compensated system are higher than the minimal required. In the case of cascaded complex compensators it can be worth of re-computing the Bode envelope after having added the first part of the compensator to the plant. Then, a new accurate Bode envelope can be derived from the exact Nyquist band and therefore some conservatism can be eliminated from the following design steps.

Examples for the presented design procedures can be found in [61]. In the next chapter the design of a robust controller for an electromechanical system is discussed.
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Chapter 6

ROBUST POSITIONING OF AN ELECTROMECHANICAL SYSTEM

In this chapter, the design of a dynamic compensator for the robust position control of an uncertain electromechanical system is presented. The purpose is to apply some of the theoretic results exposed in the previous chapters and to discuss some implementation problems.

6.1 SYSTEM DESCRIPTION

The considered uncertain system is a flexible servo system from the Automatic Control Laboratory (Fig. 6.1). It consists of two servos and two fly wheels whose rotational inertia can be varied by adding several rings. Each fly wheel is rigidly connected with the respective motor and the two wheels are coupled with a spring. In this application, simply one of the servos is used to control the movement of the second fly wheel (Fig. 6.2).

The positions of the two fly wheels given by the angles $\phi_1$ and $\phi_2$ are measured with incremental encoders and the velocities $\dot{\phi}_1$ and $\dot{\phi}_2$ with a tacho meter. All measurements from the model are available as continuous signals.

Analysis and design of the system are performed in MATLAB, the data acquisition and control are implemented in Modula 2 [64] on a Compaq Deskpro 386/33 with Burr Brown interface. The 12 bit A/D and D/A converters of the interface have a range of $\pm 10$ Volts.
6.1.1 Mathematical model

A diagram of the system is shown in Fig. 6.2. Physical modeling yields the following state-space description of the electromechanical system [65].

\[
\begin{bmatrix}
\dot{x} \\
y
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-\frac{f}{J_1} & -\left(\frac{\kappa^2}{J_1 R} + \frac{\xi}{J_1}\right) & \frac{f}{J_1} & 0 \\
0 & 0 & 0 & 1 \\
\frac{f}{J_2} & 0 & -\frac{f}{J_2} & -\frac{\xi}{J_2}
\end{bmatrix}
\begin{bmatrix}
x \\
x \\
0 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
\frac{\kappa}{J_1 R} \\
0 \\
\end{bmatrix} u
\]

\[y = \begin{bmatrix} 0 & 0 & \alpha & 0 \end{bmatrix} x\]

with the state vector \(x = [\phi_1 \dot{\phi}_1 \phi_2 \dot{\phi}_2]^T\) and \(u\) being the input voltage of the servo.
6.1. System description

The physical parameters of the model are:

\[
\begin{align*}
R &= 6.2 \ \Omega \quad \text{Motor resistance} \\
\kappa &= 0.030 \ \text{Nm/A} \quad \text{Motor constant} \\
f &= 0.0158 \ \text{Nm/rad} \quad \text{Spring constant} \\
\alpha &= \frac{1}{\pi} \ \text{V/rad} \quad \text{Sensor constant} \\
c_1 &= 3 \cdot 10^{-4} \ \text{Nm sec/rad} \quad \text{Damping coefficient} \\
c_2 &= 1 \cdot 10^{-6} \ \text{Nm sec/rad} \quad \text{Damping coefficient}
\end{align*}
\]

The damping coefficients \( c_1 \) and \( c_2 \) were identified by numerically shaping the frequency response with \( J_1 = J_2 = 3.6 \cdot 10^{-4} \). The inductivity \( L \) of the motor is neglected.

The inertias of the two fly wheels can be varied by adding small rings. The minimal inertia is \( J = 3.6 \cdot 10^{-4} \ \text{kg m}^2 \) and 3 rings with each \( J = 3.42 \cdot 10^{-5} \ \text{kg m}^2 \) and one with \( J = 1.97 \cdot 10^{-4} \ \text{kg m}^2 \) can be added. Therefore, we can treat the inertias as interval parameters with

\[
\begin{align*}
J_1 \in [3.6 \cdot 10^{-4}, 6.5 \cdot 10^{-4}] \ \text{kg m}^2 \\
J_2 \in [3.6 \cdot 10^{-4}, 6.5 \cdot 10^{-4}] \ \text{kg m}^2
\end{align*}
\]

Remark 11 Throughout this chapter, the term nominal will be used referring to the plant with the inertias \( J_1 = J_2 = 4.28 \cdot 10^{-4} \) which is the inertia of the fly wheel with 2 small rings.
Our goal is to design a controller in the frequency domain for the positioning of the second fly wheel for all possible combinations of \(J_1\) and \(J_2\).

### 6.2 FREQUENCY RESPONSE

In order to apply the theory of § 4 and § 5.4, we have to compute the Bode envelope of the uncertain plant. From the state-space description the transfer function is easily obtained:

\[
G(s) = \frac{y}{u} = \frac{\alpha_f \kappa}{s(a_4s^3 + a_3s^2 + a_2s + a_1)}
\]

with

\[
\begin{align*}
    a_4 &= RJ_1J_2 \\
    a_3 &= Rc_2J_1 + J_2(Rc_1 + \kappa^2) \\
    a_2 &= Rc_1c_2 + c_2\kappa^2 + Rf(J_1 + J_2) \\
    a_1 &= Rf(c_1 + c_2) + f\kappa^2
\end{align*}
\]

The numerator as well as the coefficient \(a_1\) are independent of \(J_1\) and \(J_2\). The coefficients \(a_3\) and \(a_2\) depend linearly and \(a_4\) multilinearly on the two uncertain physical parameters \(J_1\) and \(J_2\). Therefore, we cannot directly apply the results of § 4 for the computation of the value set and the Bode envelope of the transfer function. One possible solution is to overbound \(G(s)\) to an interval transfer function with independent uncertain coefficients. This leads to

\[
\tilde{G}(s) = \frac{b_0}{s(\tilde{a}_4s^3 + \tilde{a}_3s^2 + \tilde{a}_2s + \tilde{a}_1)}
\]

with

\[
\begin{align*}
    b_0 &= 1.5088 \cdot 10^{-4} \\
    a_1 &= 0.4371 \cdot 10^{-4} \\
    \tilde{a}_2 &\in [0.0705 \cdot 10^{-3}, 0.1274 \cdot 10^{-3}] \\
    \tilde{a}_3 &\in [0.0996 \cdot 10^{-5}, 0.1798 \cdot 10^{-5}] \\
    \tilde{a}_4 &\in [0.0804 \cdot 10^{-5}, 0.2619 \cdot 10^{-5}]
\end{align*}
\]
The value sets of \( \tilde{G}(s) \) are computed using the procedure in § 4. Due to the overbounding of the coefficients, pure imaginary roots and even roots in the right half plane are introduced that do not exist in \( G(s) \). In particular the pure imaginary roots cause problems for the value set computation since for some frequencies the value set contains the origin. Hence, the gain and phase is not defined anymore at these frequencies. Fig. 6.3 shows the Bode envelope of \( G(s) \) (computed with parameter gridding) and of \( \tilde{G}(s) \). The maximal gain of \( \tilde{G} \) is infinite in the frequency interval \([5.11, 6.65] \text{ sec}^{-1}\).

Comparing the Bode plots of \( G \) and \( \tilde{G} \) the conservatism introduced by overbounding the coefficients of the transfer function is significant in the frequency range of the two complex roots of the system.

\[ \text{--- } \tilde{G}(j\omega) \quad \text{-- } G(j\omega) \]

**Figure 6.3: Bode envelope of \( G \) and \( \tilde{G} \).**
For the design of dynamic compensators in the frequency domain the complex roots near the imaginary axis generally cause troubles, i.e. the resonance in the magnitude plot and the steep descent of the phase. Since the cascaded compensators of lead/lag type have only real poles and zeros there is hardly a chance to find a suitable controller for this kind of plants. To overcome this problem we use a trick: With an underlying control loop we first modify the dynamics of the plant so to move the roots from the imaginary axis towards the left of the complex plane. Note that with the underlying control loop the uncertain parameters $J_1$ and $J_2$ are smeared all over the coefficients of the transfer function of the inner loop. Hence, the transfer function can no longer be treated as an interval plant without introducing too much conservatism. Thus, the results of § 4 and § 5.4.2 to compute the Bode plot of the uncertain system are not applicable. So, we are forced to use parameter gridding over $J_1$ and $J_2$ instead.

6.2.1 Underlying control loop

From Fig. 6.4 it is easy to see that the roots of the plant are close to the imaginary axis. With an underlying control loop designed for the nominal plant we try to shift the roots to the left to get a more suitable Bode plot of the system. A design problem is the sensitivity of the poles to parameter variations which can be drastically increased when closing the loop. It is important that the order of the underlying controller is as small as possible and also no pole-zero cancellations may be used in the design since cancellations are only valid with the nominal plant.

The poles of the electromechanical system with the nominal inertias $J_1$ and $J_2$ are located at

$$s_1 = 0 \quad s_2 = -0.523 \quad s_{3,4} = -0.26 \pm 8.57i$$

To shift these roots several classical concepts could be applied and shall be shortly discussed from a practical point of view:

1. **PD-controller**: From root locus considerations, the simplest possibility would be the use of three PD-controllers in cascade
6.2. Frequency response

with the plant to shift the complex roots away from the imaginary axis into the left half plane. However, this type of controller with only zeros and no poles is not realizable.

The realizable form of the PD-controller \( sT_D/(1 + sT_D/N) \) with \( N \gg 1 \) is useless since, due to the three poles of the controller, two complex parts of the root locus always tend to the right half plane.

2. Dynamic output feedback: With an algebraic design of a dynamic output feedback the poles of the closed-loop could be arbitrarily placed using a controller of at least third order. The number of system poles is at least doubled and the new zeros of the controller are arbitrarily placed, maybe even in the right half plane. The variation domain of the poles in function of \( J_1 \) and \( J_2 \) can be quite large and at least two closed-loop poles are always close to the imaginary axis.

3. Observer-based state-feedback controller: Suppose only the input \( u \) and the output \( y \) of the system are measured. Then, the four roots of the system could be placed with observer-based state-feedback. The observer is designed using an identical model of the plant and, due to the separation principle, the dynamics of the observer and of the state-feedback controller can be chosen separately. In the input-output transfer function of the closed-loop the poles of the observer are canceled out by zeros: the dynamics of the observer disappears. In the presence of parameter uncertainties in the plant description this concept gets into troubles since the model of the plant and of the observer can heavily deviate. The consequence is that the poles of the observer may not be canceled out in the closed-loop transfer function. Thus, we get a function of order \( 2n \) for a \( n \) th order plant. In the frequency domain this high number of poles and zeros gives a quite complex Bode plot useless for our design procedures.

4. State-feedback controller: At our laboratory model all four state variables are measurable. The four poles of the system
can be arbitrarily placed using a state-feedback controller. This concept will be applied in the sequel.

State-feedback controller

In order to obtain a shape of the Bode plot of the uncertain system which is suitable for the design of a compensator in the frequency domain, we should be able to arbitrarily place the four system poles. This requires full state-feedback. With the following choice of the nominal poles

\[ p_1 = -2 \quad p_2 = -12 \quad p_{3,4} = -5 \pm 4i \]

the necessary feedback gain \( K \) is

\[ K = \begin{bmatrix} 11.6163 & 2.0329 & -9.2536 & -0.1713 \end{bmatrix} \]

Fig. 6.4 shows the root space of the plant and the underlying control loop for all possible values of the interval parameters \( J_1 \) and \( J_2 \). The Bode envelope of the inner loop is then computed through a parameter gridding of \( J_1 \) and \( J_2 \). It is shown in Fig. 6.5.

6.3 ROBUST POSITION CONTROL

After this preliminary work, a robust controller for the positioning of \( \phi_2 \) can be designed guaranteeing a minimal performance.

With the frequency plot of the "new" plant, i.e. of the inner loop in Fig. 6.6, we can apply the procedures presented in § 5.4 for the design of a dynamic compensator.

6.3.1 Design specifications

For the design of the robust controller several hardware limits have to be respected:

- The controller is to be implemented on a digital computer in discrete form. The achievable sampling rate depends on the Burr Brown
6.3. Robust position control

Figure 6.4: Root space.

Figure 6.5: Frequency responses.
interf...as well as on the number of operations implemented in the controller program.

- The A/D and D/A converter of the Burr Brown interface have a voltage range of maximally ±10 Volts. All signals of the electromechanical plant are adapted to this range. However, the hardware limitation influences the controller output from the PC since the output can saturate, thus introducing non-linearity in the control loop.

- The laboratory model suffers from friction effects from the servo and the bearings. We do not compensate the friction in the control loop. So, it is important that no pure integrator is present in the loop since, due to steady state errors caused by the friction, the integrator and therefore the controller output would never tend to zero.

The design of the robust controller is performed assuming a linear model of the plant. Because of the saturation of the controller output the cross over frequency \( \omega_D \) of the open-loop should not be chosen too high since otherwise the controller would react too fast to sudden changes in the reference signal. We choose the following upper bound for \( \omega_D \)

\[
\bar{\omega}_D = 3 \text{ sec}^{-1}
\]

while the lower bound follows from the design (see § 5.4). A satisfactory settling time of the step response requires a phase margin

\[
PM \geq 65^\circ
\]

The gain of the closed-loop is adjusted using a prefilter.

6.3.2 Dynamic compensator

From the Bode plot in Fig. 6.5 it is obvious that a lead compensator is necessary to guarantee the required minimal phase margin \( PM \) at \( \bar{\omega}_D \). The phase increase is

\[
\phi_m = 29.2^\circ
\]
6.3. Robust position control

![Control scheme](image)

Figure 6.6: Control scheme.

and $\alpha_{\text{lead}} = 2.905$. The gain $K_c$ yielding a gain of 0dB for the maximal magnitude plot at $\omega_D$ is

$$K_c = 8.59$$

and the transfer function of the lead compensator is

$$G_R(s) = 8.59 \cdot 2.905 \cdot \frac{s + 1.7601}{s + 5.1134}$$

Finally, the design specifications are checked by computing the Bode plot of the compensated system (Fig. 6.5). We obtain

$$68.5^\circ \leq PM \leq 103.1^\circ$$

$$2.5 \text{ sec}^{-1} \leq \omega_D \leq 3 \text{ sec}^{-1}$$

and from the simulation of the step responses (Fig. 6.7) a maximal settling time $T_{s,1\%} = 3.3$ sec.

The prefilter in the forward path of the system is chosen so to make the steady state gain of the closed-loop equal to one. The resulting gain of the filter is $v = 1.864$.

6.3.3 Implementation

The control system was simulated with SIMULAB\(^1\) and then successfully implemented on the computer with a sampling time of $T = 0.02$ sec.

---

\(^1\)SIMULAB, lately changed to SIMULINK, is a graphical environment based on MATLAB for simulating dynamic control systems [66]
The program for the state-feedback controller and the lead compensator was written in Modula 2. The state-feedback was implemented in quasi-continuous form, i.e. neglecting the influence of the sampling rate, and the lead was discretized using the Tustin approximation. Fig. 6.7 shows the step responses of the simulated loop and the measured responses of the controlled electromechanical system for some combinations of the inertias $J_1$ and $J_2$. It is obvious that the process reacts slower to the step than in simulation and that the maximal settling time $T_{s,1\%} = 3.8$ sec is a little longer. The reason for that are (1) the uncompensated stick-slip effects of the servo and the bearings and (2) the saturation of the controller output that occurs for a short time after the step of the reference signal.

6.3.4 Comparison robust design – nominal design

The goal of the procedures for the frequency domain design of dynamic compensators presented in § 5.4 is to achieve a minimal system performance for the whole uncertain system. To clarify this result the robust design is compared with a so-called nominal design where the lead compensator is designed for the nominal plant (see Remark 11). This nominal controller is then used to control the uncertain system.

The nominal design uses the frequency response of the underlying control loop with the nominal plant (Fig. 6.8). The design specifications are the same as for the robust design. Then, the phase increase at the cross over frequency $\omega_D = 3$ sec$^{-1}$ is $\phi_m = 4.55^\circ$ and the necessary gain is $K_c = 13.64$. The transfer function of the lead (§ 5.4.4) is

$$G_{R,n} = 13.64 \cdot 1.1723 \cdot \frac{s + 2.771}{s + 3.248}$$

and the gain of prefilter is $v = 1.544$. The Bode envelope of the open loop with the lead controller $G_{R,n}$ is shown in Fig. 6.8 with the phase margin and the cross over frequency in the following intervals

$$45.4^\circ \leq PM \leq 80^\circ$$

$$2.65 \text{ sec}^{-1} \leq \omega_D \leq 3 \text{ sec}^{-1}$$
Figure 6.7: *Step responses of the controlled system.*
The step responses of the uncertain system with the lead $G_{R,n}$ (nominal design) are depicted in Fig. 6.7(bottom). In comparison with Fig. 6.7(top) the maximal overshoot is approximatively the same with both design methods. However, in the nominal case the maximal amplitude depends on the parameter variations which is not the case with the robust design. The maximal settling time with the nominal design $T_s,1\% = 4.6$ sec is larger than with the robust design. The specified phase margin $PM$ is not guaranteed with the nominally designed lead compensator and therefore this design does not fulfill the required robust performance.

6.4 SUMMARY

In this chapter the design of a dynamic compensator for an uncertain electromechanical system has been presented. First, the dynamics of
the plant was changed with a state-feedback controller to get a suitable frequency response of the plant for the frequency domain design of the robust controller. A lead compensator could guarantee a minimal phase margin of the uncertain system for all possible combinations of the inertias $J_1$ and $J_2$ of the fly wheels.

The problem with the frequency response of the system is symptomatic for controller design in the frequency domain since complex-conjugate roots near the imaginary axis are difficult to handle with compensators with only real poles and zeros.
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Chapter 7

CONCLUSIONS AND DIRECTIONS FOR FUTURE RESEARCH

7.1 CONCLUSIONS

This thesis contributes to the analysis and design of uncertain control systems with polytopic uncertainties.

The first part investigated in the frequency domain the $D$-stability of families of polynomials whose coefficients are affine functions of the uncertain interval parameters. Using the zero exclusion principle $D$-stability of the polynomial family was guaranteed if the origin stays outside the value set for all $s$ along the boundary $\partial D$ of the stability domain $D$. The value set of the polynomial family $f(s)$ for $s$ along $\partial D$ was determined. For any fixed frequency the value set is a par-polygon whose edges are the images of a subset of the edges of the hyperrectangle in parameter space. A procedure was derived for the explicit determination of the exposed edges building the boundary of the value set. Further on, it was possible to determine intervals of the frequency along the boundary $\partial D$ of the stability domain $D$ where the essential structure of the value set, i.e. the building edges of the parameter hyperrectangle, remains the same. The necessary and sufficient test of the $D$-stability consisted of three necessary conditions where the essential one is the test of the zero inclusion of the exposed edges for $s$ along $\partial D$.

The number of exposed edges (i.e. the number of edges to be tested for $D$-stability) was drastically reduced with respect to the number of
edges required by the Edge theorem [15].
An important aspect of the presented $D$-stability test is its computational tractability. The stability test can be easily implemented in a software environment such as MATLAB. Then, the test can be performed either pure numerically by computing the $D$-stability of the exposed edges, or graphically by plotting the explicit value sets for $s$ along $\partial D$ and checking the location of the origin on the plot.
A special case of the investigated class of uncertain polynomials is the characteristic polynomial of a standard loop with an interval plant and a fixed controller. The value sets can be used to determine the Hurwitz invariance of the polynomial. It was shown that the value set of the characteristic polynomial is generally a parpolygon with eight edges. For $s$ along the imaginary axis, depending on the phase of the controller, only four essentially different structures of the value set can occur and therefore at most $4 \times 8 = 32$ exposed edges of the parameter box can be mapped to the edges of the value set. This result is independent of the order of the plant.
Similar results could be obtained when considering a conic sector with special cone angles.
Using the special structure of the boundary of the value set a procedure for the efficient computation of an important robustness measure was presented. For a specific uncertainty structure the structured stability margin $\rho$ gives the largest size of the uncertainty domain which still preserves robust $D$-stability of the family of polynomials. If one member of the polynomial family is $D$-stable, then, the condition $\rho > 1$ is necessary and sufficient for $D$-stability of the whole family of polynomials.
This robustness measure applied to the standard closed-loop with a fixed controller gives a measure for the richness of the class of uncertain plants that can be stabilized by the fixed controller.

In the second part of this thesis the value set of uncertain transfer functions was investigated where the value sets of the numerator and denominator of the transfer functions are products of axis parallel boxes. The interval plants are a special case of this set-up. Geometrical procedures for the direct computation of the boundary of the product
and ratio of axis parallel boxes were presented. An argument condition gave a necessary condition for boundary points of the operands to be mapped on the boundary of the resulting value set. A cascade of the procedures for product and ratio of axis parallel boxes allowed us to compute value sets of more complex transfer functions.

The value set of uncertain transfer functions is of central interest for the analysis and design of robust control systems in the frequency domain. The presented procedures allow to directly determine the boundary of the value set without having to resort to parameter gridding.

The third part of this thesis addressed the robust design problem. Some results on the stabilizability of families of plants by a single fixed controller were discussed. These results are important from a system theoretic point of view but do not deliver conditions which can be used for design problems. For example in [29] it was shown that a necessary and sufficient condition for the simultaneous stabilizability of an interval plant using a first-order compensator is the simultaneous stabilizability of sixteen Kharitonov plants. This result is useful for the analysis of a closed-loop with a fixed first-order compensator. However, it does not address the problem of designing a single compensator which simultaneously stabilizes 16 plants. Moreover, the question if a specific interval plant is stabilizable at all using a first order compensator – the problem of existence of the robust controller – remains unsolved.

In classical control engineering the design of dynamic compensators in the frequency domain is a widely used method. This method was extended to uncertain systems. The classical design specifications had to be newly defined in the sense of worst case requirements for the uncertain control loop. From the Nyquist plot of the uncertain system the Bode envelope was easily obtained. Procedures for the design of robust lead, lag and lead-lag compensators have been presented using the Bode plot of the uncertain system.

The design of dynamic compensators for uncertain systems in the frequency domain suffers from restrictions similar to those known in the classical case. The method is applicable especially for plants with low
pass characteristics. In these cases the design delivers a dynamic compensator which not only guarantees pure robust stability but also a minimal performance of the overall system. In this sense, this design method presented in this thesis is a first step towards the practical design of robust controllers.

7.2 DIRECTIONS FOR FUTURE RESEARCH

One of the main problems in the analysis of systems with parametric uncertainties is the relation between uncertain physical parameters and the coefficients of the appropriate system description. In actual research strict assumptions on the uncertainty domains of the system parameters are often made in order to get mathematically tractable results. The models of real processes often have to be overparameterized for the description to fit into the mathematical framework. Quite frequently such models turn out to be much too restrictive. An important direction for future research is to find more realistic models dealing with uncertain physical parameters.

The field of robust controller design for systems with parameter uncertainties is in its beginning. In particular, algorithms for systematic controller design are missing completely. Design of useful controllers should consider not only robust stability of the closed-loop but also specifications for robust performance. It could prove to be impossible to find an analytic solution for the controller. Research may focus on efficient numeric algorithms.

A last aspect is on system identification. In the field of robust control, assumptions on uncertain parameters have been made without ensuring that this kind of uncertainties could be obtained from the identification of process parameters. In the last few years, this problem has returned to mind and first steps in this direction have been done [67]. New identification algorithms should be able to provide interval estimates of uncertain parameters.
Appendix A

PROOF OF LEMMA 1

The proof of Lemma 1 on page 17 is given in this appendix. For convenience the lemma is restated here:

**Lemma 1** Let \( f(s, \Gamma) \) be a parpolytope of polynomials in the coefficient space. Then the value set of \( f(s, \Gamma) \) with \( s \) along \( \partial D \) is a parpolygon. The edges of the value set are the images of a subset of the edges of the parpolytope of polynomials.

**Proof:** The proof is presented in two steps:

**Step 1:** The map \( \Gamma \to A \)

Let the domain \( \Gamma \) of the uncertain parameters be a \( \nu \)-dimensional hyperrectangle in parameter space. Associated with each uncertain parameter \( \gamma_k \) there is a set of collinear edges described by

\[
\{g_k\} : \gamma_k = [\gamma_1^* \cdots \gamma_{k-1}^* \gamma_k \gamma_{k+1}^* \cdots \gamma_\nu^*]' \quad (A.1)
\]

with

\[
\gamma_k \leq \gamma_k \leq \bar{\gamma}_k
\]

and all the other parameters fixed at one of their extreme values.

The coefficients \( a \) of the polynomial family \( f(\cdot) \) of order \( n \) are affine functions of the uncertain parameters \( \gamma \in \Gamma \). Thus, the domain \( \Gamma \) is mapped by the affine rule

\[
a = a_0 + T\gamma \quad (A.2)
\]

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to the range $\mathcal{A}$ in coefficient space.
With no loss of generality we assume the matrix $T \in \mathbb{R}^{(n+1) \times \nu}$ to have full rank $\nu$. Further we assume

$$-1 \leq \gamma_i \leq 1 \quad i = 1, \ldots, \nu$$

what can always be achieved by appropriate choice of $T$ and $a_0$.
Now, consider an inner point $\gamma \in \Gamma$ with locally $\nu$ degrees of freedom for variation. This point can only be mapped to the boundary of $\mathcal{A}$ if the Jacobian matrix $J$ of the expression (A.2) drops in rank. In this case the image of $\gamma$ in the coefficient space would have locally less degrees of freedom for variation. For $J$ we get

$$J = \frac{\partial a}{\partial \gamma} = \frac{\partial (a_0 + T\gamma)}{\partial \gamma} = T$$

and hence, $J$ is a constant matrix with rank $\nu$. With this property of $J$, every point of $\Gamma$ is mapped to a point in $\mathcal{A}$ with the same degrees of freedom for variation. Thus, every inner point of $\Gamma$ is mapped inside of $\mathcal{A}$ and every $\mu$-dimensional face of $\Gamma$, with $0 \leq \mu \leq \nu$, is mapped to a $\mu$-dimensional face of $\mathcal{A}$. Especially the 1-dimensional faces of $\Gamma$, i.e. the edges, are mapped to edges of $\mathcal{A}$. Due to the linear map (A.2), collinearity is an invariant property and hence collinearity of edges is preserved. Thus, $\mathcal{A}$ is a $\nu$-dimensional parapolytope in $\mathbb{R}^{n+1}$ whose edges are the images of the edges of $\Gamma$.

The image of the set of collinear edges $\{g_k\}$ in (A.1) is described by

$$\{g_k\}_A : a_k = a_0 + T\gamma_k = a_0 + T\tilde{\gamma}_k + \underline{t}_k \gamma_k \quad (A.3)$$

with $\gamma_k$ as in (A.1) and

$$\tilde{\gamma}_k = [\gamma_1^* \ldots \gamma_{k-1}^* \ 0 \ \gamma_{k+1}^* \ldots \ \gamma_{\nu}^*]$$

and $\underline{t}_k$ the $k$-th column of the matrix $T$. 
Step 2: The value set of $f(s, \Gamma)$

The family of polynomials $f(s, \Gamma)$ can be written as

$$f(s, a) = s'a = s'(a_0 + T\gamma)$$

with $a \in A$ and $s' = [1 \ s \ s^2 \ \cdots \ s^n]$. Then, for a fixed frequency $s^* \in \partial D$ we get the value set

$$
\begin{bmatrix}
\text{Re}f(s^*, a) \\
\text{Im}f(s^*, a)
\end{bmatrix} = \tilde{s}'(a_0 + T\gamma)
$$

with

$$\tilde{s}' = \begin{bmatrix} 1 & \text{Res} & \text{Res}^2 & \cdots & \text{Res}^n \\ 0 & \text{Im}s & \text{Im}s^2 & \cdots & \text{Im}s^n \end{bmatrix}_{s=s^*}
$$

a constant matrix. As image of the set $\{g_k\}_A$ in the complex plane we get

$$
\begin{bmatrix}
\text{Re}f(s^*, a_k) \\
\text{Im}f(s^*, a_k)
\end{bmatrix} = \tilde{s}'(a_0 + T\gamma_k + \bar{t}_k\gamma_k) = f_0 + f_k \gamma_k
$$

with

$$
\begin{align*}
f_0 &= \tilde{s}'(a_0 + T\gamma_k) \\
f_k &= \tilde{s}'\bar{t}_k
\end{align*}
$$

For each member of the set $\{g_k\}_A$, $f_0$ is a constant vector and thus, $f(s^*, a_k)$ is a set of collinear line segments (Fig. A.1). The direction of this set is given by the vector $f_k$.

The two extremal edges of $\{g_k\}_A$ which contribute to the boundary of the value set can be easily extracted. Let $f_\perp$ be orthonormal to the vector $f_k$. Then, the distance $d$ of the edges from the origin in the direction $f_\perp$ is given as the scalar product

$$d = <f_0, f_\perp> = \tilde{s}'(a_0 + T\gamma_k)f_\perp$$

(A.4)
The two extremal edges of the image of the set \( \{g_k\}_A \) are found as those combinations of the extreme values \{-1, 1\} in \( \hat{y}_k \) which deliver the minimal and maximal distance \( d \) in (A.4).

We have shown that the set \( \{g_k\}_A \) is the image of the set \( \{g_k\} \) of collinear edges of the hyperrectangle \( \Gamma \). Further on, the set \( \{g_k\}_A \) is mapped to a set of collinear lines in the complex plane and two extreme lines of this set build a part of the boundary of the value set. Moreover, this is true for all \( k = 1, 2, \ldots, \nu \) and hence the lemma. \( \blacksquare \)
Appendix B

ROBUST D-STABILITY OF A FAMILY OF POLYNOMIALS: AN EXAMPLE

Consider the standard control loop in Fig. 2.1 on page 9 with the interval plant

\[ G_S(s) = \frac{b(s)}{a(s)} = \frac{s + b_0}{s^3 + a_2s^2 + a_1s + a_0} \]

with \( b_0 \in [3, 8] \quad a_2 \in [12, 18] \quad a_1 \in [61, 69] \quad a_0 \in [135, 180] \)

and an arbitrarily designed controller

\[ G_R(s) = \frac{d(s)}{c(s)} = 2 \cdot \frac{s + 5}{s + 8} \]

The stability domain for the closed-loop poles is supposed to be a cone in the left half plane with \( \varphi = \frac{2}{3} \pi \) (Fig. 3.13).

We want to check whether the closed-loop is robustly stable with respect to the cone, i.e. if all roots of the characteristic polynomial are located within the cone. This is done by applying the stability test of § 3.4. The value set is determined using directly the results of § 3.5.2.

The boundary of the stability domain is parameterized with

\[ s = \delta e^{i \frac{2}{3} \pi} \quad \delta \in [0, \infty) \]

and the polynomials \( a(s) \) and \( b(s) \) can be reformulated as

\[ a(\delta e^{i \frac{2}{3} \pi}) = \Gamma_1 + \Gamma_2 \delta e^{i \frac{2}{3} \pi} + \Gamma_3 \delta^2 e^{i \frac{2}{3} \pi} \]
Appendix B. Robust $D$-stability: an example

\[ b(\delta e^{j\frac{3\pi}{2}}) = \Gamma_4 + \Gamma_5 \delta e^{j\frac{3\pi}{2}} + \Gamma_6 \delta^2 e^{j\frac{3\pi}{2}} \]

with the generalized parameters

\[
\begin{align*}
\Gamma_1 &= a_0 + \delta^3 \\
\Gamma_2 &= a_1 \\
\Gamma_3 &= a_2 \\
\Gamma_4 &= b_0 \\
\Gamma_5 &= 1 \\
\Gamma_6 &= 0
\end{align*}
\]

The minima of all parameters $\Gamma_i$, $i = 1, \ldots, 6$ are given by the associated minima of the uncertain parameters $a_i$ respectively $b_0$.

From the phase condition (3.40), we find that the phase of the controller $G_R$ lies only in the sector VI. Hence, there exists a single structure of the value set for all $s$ along the boundary of the cone. The 12 vertices of the value set can be extracted from Table 3.5 for sector VI. Note that in this example the value set is degenerate, i.e. does not have the maximal number of 12 edges, since there is no variation in the direction of $\beta_2 (\Gamma_5)$ and $\beta_3 (\Gamma_6)$. The value set has only 8 vertices and 8 exposed edges (see Table B.1).

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>$\Gamma_1$</th>
<th>$\alpha_2$</th>
<th>$\Gamma_2$</th>
<th>$\alpha_3$</th>
<th>$\Gamma_3$</th>
<th>$\beta_1$</th>
<th>$\Gamma_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>- + + +</td>
<td>- - - +</td>
<td>+ + + +</td>
<td>- - - +</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 2 3 4 5 6 7 8 9 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table B.1: Vertices of the value set.

The associated vertex polynomials can be extracted from the list of the vertices in Table B.1, given by the extreme values of the generalized parameters $\Gamma_i$:

\[
\begin{align*}
P_1(\delta e^{j\frac{3\pi}{2}}) &= d^*(b_0 + \delta e^{j\frac{3\pi}{2}}) + c^*(a_0 + \delta^3 + a_1 \delta e^{j\frac{3\pi}{2}} + a_2 \delta^2 e^{j\frac{3\pi}{2}}) \\
P_2(\delta e^{j\frac{3\pi}{2}}) &= d^*(b_0 + \delta e^{j\frac{3\pi}{2}}) + c^*(a_0 + \delta^3 + a_1 \delta e^{j\frac{3\pi}{2}} + a_2 \delta^2 e^{j\frac{3\pi}{2}}) \\
P_3(\delta e^{j\frac{3\pi}{2}}) &= d^*(b_0 + \delta e^{j\frac{3\pi}{2}}) + c^*(a_0 + \delta^3 + a_1 \delta e^{j\frac{3\pi}{2}} + a_2 \delta^2 e^{j\frac{3\pi}{2}}) \\
P_4(\delta e^{j\frac{3\pi}{2}}) &= d^*(b_0 + \delta e^{j\frac{3\pi}{2}}) + c^*(a_0 + \delta^3 + a_1 \delta e^{j\frac{3\pi}{2}} + a_2 \delta^2 e^{j\frac{3\pi}{2}})
\end{align*}
\]
\[ P_5(\delta e^{j\frac{3}{4}\pi}) = d^*(b_0 + \delta e^{j\frac{3}{4}\pi}) + c^*(\alpha_0 + \delta^3 + \alpha_1 \delta e^{j\frac{3}{4}\pi} + \alpha_2 \delta^2 e^{j\frac{3}{4}\pi}) \]
\[ P_6(\delta e^{j\frac{3}{4}\pi}) = d^*(b_0 + \delta e^{j\frac{3}{4}\pi}) + c^*(\alpha_0 + \delta^3 + \alpha_1 \delta e^{j\frac{3}{4}\pi} + \alpha_2 \delta^2 e^{j\frac{3}{4}\pi}) \]
\[ P_7(\delta e^{j\frac{3}{4}\pi}) = d^*(b_0 + \delta e^{j\frac{3}{4}\pi}) + c^*(\alpha_0 + \delta^3 + \alpha_1 \delta e^{j\frac{3}{4}\pi} + \alpha_2 \delta^2 e^{j\frac{3}{4}\pi}) \]
\[ P_8(\delta e^{j\frac{3}{4}\pi}) = d^*(b_0 + \delta e^{j\frac{3}{4}\pi}) + c^*(\alpha_0 + \delta^3 + \alpha_1 \delta e^{j\frac{3}{4}\pi} + \alpha_2 \delta^2 e^{j\frac{3}{4}\pi}) \]

with \( c^* = c(\delta e^{j\frac{3}{4}\pi}) \) and \( d^* = d(\delta e^{j\frac{3}{4}\pi}) \). The exposed edges of the value set are given as convex combinations of two neighboring vertices:

\[ p_1(\delta) = P_1(\delta) + \lambda(P_2(\delta) - P_1(\delta)) \quad \lambda \in (0, 1) \]
\[ p_2(\delta) = P_2(\delta) + \lambda(P_3(\delta) - P_2(\delta)) \]
\[ \vdots \]
\[ p_8(\delta) = P_8(\delta) + \lambda(P_1(\delta) - P_8(\delta)) \]

After this preliminary work, the stability test (Theorem 4 on page 18) can be performed:

- **Step 1**: Test of the cone stability of one member of the polynomial family \( P(s) \). We choose the plant with all uncertain parameters centered in their intervals:

\[ G_{S0}(s) = \frac{s + 5.5}{s^3 + 15s^2 + 65s + 157.5} \]

The nominal characteristic polynomial is

\[ P_0(s) = s^4 + 23s^3 + 187s^2 + 698.5s + 1315 \]

with the roots

\[ s_1 = -9.75 \quad s_2 = -8.21 \quad s_{3,4} = -2.52 \pm j3.17 \]

The roots are all located in the cone stability domain from which follows that \( P_0(s) \) is cone stable.
Appendix B. Robust $D$-stability: an example

- **Step 2:** The value set of the family $P(s)$ is determined for one frequency $s$ on the boundary of the cone and the location of the origin has to be investigated. Since the boundary of the cone crosses the real axis, we choose $\delta = 0$, i.e. $s = 0$ (see Remark 1 on page 18). Then, the value set of $P(s)$ is simply an interval on the real axis where the two end points are given by:

$$V = [1110, 1520]$$

Since the origin is outside this interval, condition (ii) of Theorem 4 is fulfilled.

- **Step 3:** The cone stability of all exposed edges of the value set has to be tested for $s = \delta e^{j\frac{\pi}{3}}, \delta \in (0, \infty)$. We use the collinearity test of Theorem 5. It follows that the edges $p_1$, $p_3$ and $p_4$ do contain the origin for certain values of $\delta$ and are therefore that they are not cone stable. This implies that the whole family of polynomials $P(s)$ is not robustly cone stable.

From the exact root locations of the edges, we note that a conjugate-complex root along the edges $p_1$, $p_3$ and $p_4$ intersects the boundary of the cone (Fig. B.3). Hence, the origin crosses the value set through these edges. According to the collinearity test the edge $p_2$ does not contain the origin for any $\delta$. However, the whole edge $p_2$ is unstable since a conjugate-complex root pair always lies outside the cone.

Fig. B.1 shows the value set of the characteristic polynomial $P(\delta)$ for $\delta$ along the boundary of the cone. It is easy to see that the value set does contain the origin and therefore $P(s)$ is not robustly cone stable. Fig. B.2 shows the roots of the edges of the value set which corresponds to the boundary of the root space of $P(s)$ (Edge Theorem) and it is obvious that there exist roots of $P(s)$ outside of the cone (see also Fig. B.3).
Nyquist curves of the vertices 1, ⋯, 8

Value set at $s = \delta e^{i\frac{3\pi}{4}}$

Figure B.1: Value set of $P(s)$ for $s$ along $\partial D$.

Figure B.2: Boundary of the root space of $P(s)$. 
Figure B.3: Critical part of the root space of $P(s)$ (see Fig. A.2).
Appendix C

COMPUTATION OF VALUE SETS: EXAMPLES

The examples in this appendix illustrate the procedures described in Chapter 4.

C.1 PRODUCT OF TWO AXIS PARALLEL BOXES

Given are two axis parallel boxes by their vertices in the complex plane. Compute the boundary $\partial V$ of the value set $V = AB$.

$$A : \begin{array}{cccc} 2 + 3.6i & 6 + 3.6i & 6 + 7.6i & 2 + 7.6i \\ B : \begin{array}{cccc} -1.2 + 1.2i & 1.2 + 1.2i & 1.2 + 3.2i & -1.2 + 3.2i \end{array} \end{array}$$

Table C.1 shows the relative angles $\arg(a/a')$ and $\arg(b/b')$ along the boundaries of $A$ respectively $B$. Next, the intersections of the intervals of relative angles between the two value sets are extracted. Table C.2 lists the pairs of edges of the two value sets which fulfill the argument condition (4.7), i.e. where the angle intervals intersect. From the angles of the intersections we can find on each edge of $A$ and $B$ the corresponding line segments or vertex points to be multiplied for obtaining a candidate part of the boundary of the resulting value set. Table C.2 does not list products of two vertices, as the resulting points are also given from the edge segments.

The absolute arguments $\arg(a)$ of the end points of the edges can be extracted from the third column of Table C.2. The angles of the points
Table C.1: Relative arguments \( \arg(a/a') \) of the boundaries of \( A \) and \( B \).

Table C.2: Intersections of the relative arguments of \( \partial A \) and \( \partial B \) (for computational reasons the arguments were shifted to positive angles).
on the edges are given by

\[ \phi_A = \psi + (k - 1) \cdot 90^\circ \quad k = 1, \cdots, 4 \]

respectively

\[ \phi_B = \psi + (k - 5) \cdot 90^\circ \quad k = 5, \cdots, 8 \]

where \( \psi \) are the angles out of the third column of Table C.2 and \( k \) is the first index number indicating an edge (e.g. for edge 3-4 it is \( k = 3 \)). Now, for each argument \( \phi_A \) respectively \( \phi_B \) the points on the corresponding edges can be determined. Consider the edge given by the two end points \( E_1 \) and \( E_2 \) (Fig. C.1). Then, the point \( P \) with the given argument \( \phi \) on this edge is obtained by

\[ P = x + jy \]

with

\[ y = x \tan \phi \]

and hence

\[ P = x(1 + j \tan \phi) \]

As \( P \) must lie on the line segment between \( E_1 \) and \( E_2 \), it follows

\[ x(1 + j \tan \phi) = E_1 + \lambda(E_2 - E_1) \quad \lambda \in [0, 1] \]

and after some calculations

\[ x = \frac{\text{Im}E_1 \cdot \text{Re}E_2 - \text{Re}E_1 \cdot \text{Im}E_2}{(\text{Re}E_2 - \text{Re}E_1) \tan \phi - \text{Im}E_2 + \text{Im}E_1} \]

The result is a list of segments of the two boundaries \( \partial A \) and \( \partial B \), which have to be multiplied pairwise to obtain the boundary \( \partial V \) (Table C.3). The product of two line segments can be replaced using directly the expression given in the third column in Table 4.1. The final value set \( V \) is shown in Fig. C.2. Note that the additional curves inside of \( V \) are due to the fact that (4.7) gives only a necessary but not sufficient condition for the product of two points from the boundaries of \( A \) and \( B \) to be boundary points of \( AB \).
Figure C.1: *Determination of a point with given argument* $\phi$ *on an edge.*

<table>
<thead>
<tr>
<th>Value set A</th>
<th>Value set B</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.00 + 3.60i</td>
<td>0.67 + 1.20i</td>
</tr>
<tr>
<td>3.60 + 3.60i</td>
<td>1.20 + 1.20i</td>
</tr>
<tr>
<td>6.00 + 3.60i</td>
<td>1.20 + 1.20i</td>
</tr>
<tr>
<td>6.00 + 6.00i</td>
<td>1.20 + 1.20i</td>
</tr>
<tr>
<td>6.00 + 6.00i</td>
<td>1.20 + 1.20i</td>
</tr>
<tr>
<td>6.00 + 3.60i</td>
<td>1.20 + 1.20i</td>
</tr>
<tr>
<td>6.00 + 7.60i</td>
<td>1.20 + 3.20i</td>
</tr>
<tr>
<td>6.00 + 7.60i</td>
<td>1.20 + 3.20i</td>
</tr>
<tr>
<td>2.85 + 7.60i</td>
<td>1.20 + 3.20i</td>
</tr>
<tr>
<td>2.85 + 7.60i</td>
<td>1.20 + 3.20i</td>
</tr>
<tr>
<td>6.00 + 7.60i</td>
<td>-1.20 + 3.20i</td>
</tr>
<tr>
<td>2.00 + 7.60i</td>
<td>1.20 + 3.20i</td>
</tr>
<tr>
<td>2.00 + 7.60i</td>
<td>1.20 + 3.20i</td>
</tr>
<tr>
<td>2.00 + 3.60i</td>
<td>-1.20 + 1.20i</td>
</tr>
</tbody>
</table>

Table C.3: *Coordinates of the end points describing the associated line segments on $\partial A$ and $\partial B$ to be multiplied.*
FIGURE C.2: Value set $V = AB$.

C.2 DIVISION OF TWO AXIS PARALLEL BOXES

Given are two axis parallel boxes by their corners in the complex plane.

$A : 3 + 2i$ $6 + 2i$ $6 + 4i$ $3 + 4i$

$B : -3 - i$ $-1 - i$ $-1 + 4i$ $-3 + 4i$

Determine the boundary of the value set $V = B/A$.

The relative angles according to the argument condition (4.28) are given in Table C.4 and the corresponding segments of the boundaries of $A$ and $B$ are shown in Table C.5. The intersections of two edge segments are not listed. From the relative angles $\psi$ of the third column in Table C.5 the absolute angles at the end points of the segments on the edges are obtained by

$$\phi_B = \psi + (k - 1) \cdot 90^\circ \quad k = 1, \ldots, 4$$

$$\phi_A = \psi + (k - 3) \cdot 90^\circ \quad k = 5, \ldots, 8$$
where \( k \) is the first index number indicating an edge. With the arguments \( \phi \) of the end points the coordinates on the edges are computed (Table C.6). Finally, Fig. C.3 shows the boundary of the value set \( V \).

<table>
<thead>
<tr>
<th>Edge</th>
<th>Angles [°]</th>
<th>Edge</th>
<th>Angles [°]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 2</td>
<td>-161.57</td>
<td>5 - 6</td>
<td>-146.31</td>
</tr>
<tr>
<td>2 - 2</td>
<td>-135.00</td>
<td>6 - 6</td>
<td>-161.57</td>
</tr>
<tr>
<td>2 - 3</td>
<td>-225.00</td>
<td>6 - 7</td>
<td>-251.57</td>
</tr>
<tr>
<td>3 - 3</td>
<td>14.04</td>
<td>7 - 7</td>
<td>-236.31</td>
</tr>
<tr>
<td>3 - 4</td>
<td>-75.96</td>
<td>7 - 8</td>
<td>-326.31</td>
</tr>
<tr>
<td>4 - 4</td>
<td>-53.13</td>
<td>8 - 8</td>
<td>-306.87</td>
</tr>
<tr>
<td>4 - 1</td>
<td>-143.13</td>
<td>8 - 5</td>
<td>-36.87</td>
</tr>
<tr>
<td>1 - 1</td>
<td>-71.56</td>
<td>5 - 5</td>
<td>-56.31</td>
</tr>
</tbody>
</table>

Table C.4: Relative arguments along the boundaries of \( A \) and \( B \).

<table>
<thead>
<tr>
<th>Edge resp. Vertex</th>
<th>Edge resp. Vertex</th>
<th>Intersection [°]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 2</td>
<td>5 - 5</td>
<td>213.69 225.00</td>
</tr>
<tr>
<td>2 - 2</td>
<td>5 - 6</td>
<td>213.69 198.43</td>
</tr>
<tr>
<td>2 - 3</td>
<td>6 - 6</td>
<td>135.00 108.43</td>
</tr>
<tr>
<td>2 - 3</td>
<td>7 - 7</td>
<td>123.69 33.69</td>
</tr>
<tr>
<td>2 - 3</td>
<td>8 - 8</td>
<td>53.13 14.04</td>
</tr>
<tr>
<td>3 - 3</td>
<td>8 - 5</td>
<td>303.69 323.13</td>
</tr>
<tr>
<td>3 - 4</td>
<td>5 - 5</td>
<td>284.04 303.69</td>
</tr>
<tr>
<td>4 - 4</td>
<td>8 - 5</td>
<td>306.87 303.69</td>
</tr>
<tr>
<td>4 - 1</td>
<td>5 - 5</td>
<td>216.87 288.43</td>
</tr>
<tr>
<td>1 - 1</td>
<td>5 - 6</td>
<td>213.69 198.43</td>
</tr>
</tbody>
</table>

Table C.5: Intersections of the relative arguments of \( \partial A \) and \( \partial B \).
C.2. Division of two axis parallel boxes

<table>
<thead>
<tr>
<th>Value set B</th>
<th>Value set A</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.50 - 1.00i</td>
<td>3.00 + 2.00i</td>
</tr>
<tr>
<td>-1.00 - 1.00i</td>
<td>6.00 + 2.00i</td>
</tr>
<tr>
<td>-1.00 - 1.00i</td>
<td>6.00 + 2.00i</td>
</tr>
<tr>
<td>-1.00 - 0.67i</td>
<td>6.00 + 4.00i</td>
</tr>
<tr>
<td>-1.00 + 0.75i</td>
<td>3.00 + 4.00i</td>
</tr>
<tr>
<td>-1.00 + 4.00i</td>
<td>3.00 + 4.00i</td>
</tr>
<tr>
<td>-1.00 + 4.00i</td>
<td>3.00 + 2.00i</td>
</tr>
<tr>
<td>-3.00 + 4.00i</td>
<td>3.00 + 2.00i</td>
</tr>
<tr>
<td>-3.00 - 1.00i</td>
<td>3.00 + 2.00i</td>
</tr>
</tbody>
</table>

Table C.6: Coordinates of the end points of the associated line segments on $\partial A$ and $\partial B$.

Figure C.3: Value set $V = BA^{-1}$. 
Appendix C. Computation of value sets: examples

C.3 EXTENSION

Given are four axis parallel boxes by their vertices in the complex plane.

\[
\begin{array}{cccc}
A_1: & -1 - 4i & 1 - 4i & 1 - 2i & -1 - 2i \\
A_2: & -3 + i & -1 + i & -1 + 4i & -3 + 4i \\
B_1: & 2 + 3i & 5 + 3i & 5 + 7i & 2 + 7i \\
B_2: & 1 + 0.5i & 3 + 0.5i & 3 + 3i & 1 + 3i
\end{array}
\]

Compute the value set \( V = (B_1 B_2)(A_1 A_2)^{-1} \). \( V \) can be determined as a sequence of the two operations product and division

\[
V = B_1 A_1^{-1} \cdot B_2 A_2^{-1}
\]  
(C.1)

The relative arguments of the boundaries of the four boxes are given in Table C.7 and C.8. Table C.9 gives the relative arguments of the intersections of all four boundaries and the indices of the belonging edges (respectively vertices) of the four value sets \( A_1 \)–\( B_2 \). Note that the vertices of \( B_1 \) and \( B_2 \) are numbered from 1 to 4 and those of \( A_1 \) and \( A_2 \) from 5 to 8. A partial list of the line segments on the four boundaries which build the boundary of \( V \) according to (C.1) are shown in Table C.10. (For complete numerical results of this example, see [68])

<table>
<thead>
<tr>
<th>Value set ( B_1 )</th>
<th>Value set ( A_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edge</td>
<td>Angles [°]</td>
</tr>
<tr>
<td>1 - 2</td>
<td>56.31</td>
</tr>
<tr>
<td>2 - 2</td>
<td>30.96</td>
</tr>
<tr>
<td>2 - 3</td>
<td>-59.04</td>
</tr>
<tr>
<td>3 - 3</td>
<td>-35.54</td>
</tr>
<tr>
<td>3 - 4</td>
<td>-125.54</td>
</tr>
<tr>
<td>4 - 4</td>
<td>-105.95</td>
</tr>
<tr>
<td>4 - 1</td>
<td>-195.95</td>
</tr>
<tr>
<td>1 - 1</td>
<td>-213.69</td>
</tr>
</tbody>
</table>

Table C.7: Relative arguments of the boundaries of \( B_1 \) and \( A_1 \).
### Table C.8: Relative arguments of the boundaries of $B_2$ and $A_2$.

<table>
<thead>
<tr>
<th>Edge</th>
<th>Angles [$\degree$]</th>
<th>Edge</th>
<th>Angles [$\degree$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 2</td>
<td>26.57 9.46</td>
<td>5 - 6</td>
<td>-18.43 -45.00</td>
</tr>
<tr>
<td>2 - 2</td>
<td>9.46 -80.54</td>
<td>6 - 6</td>
<td>-45.00 -135.00</td>
</tr>
<tr>
<td>2 - 3</td>
<td>-80.54 -45.00</td>
<td>6 - 7</td>
<td>-135.00 -165.96</td>
</tr>
<tr>
<td>3 - 3</td>
<td>-45.00 -135.00</td>
<td>7 - 7</td>
<td>-165.96 -255.96</td>
</tr>
<tr>
<td>3 - 4</td>
<td>-135.00 -108.43</td>
<td>7 - 8</td>
<td>-255.96 -233.13</td>
</tr>
<tr>
<td>4 - 4</td>
<td>-108.43 -198.43</td>
<td>8 - 8</td>
<td>-233.13 -323.13</td>
</tr>
<tr>
<td>4 - 1</td>
<td>-198.43 -243.43</td>
<td>8 - 5</td>
<td>-323.13 -288.43</td>
</tr>
<tr>
<td>1 - 1</td>
<td>-243.43 -333.43</td>
<td>5 - 5</td>
<td>-288.43 -18.43</td>
</tr>
</tbody>
</table>

### Table C.9: Intersections of the relative arguments on $\partial B_1$, $\partial A_1$, $\partial B_2$ and $\partial A_2$.

<table>
<thead>
<tr>
<th>$B_1$</th>
<th>$A_1$</th>
<th>$B_2$</th>
<th>$A_2$</th>
<th>Intersections [$\degree$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 2</td>
<td>6 - 6</td>
<td>1 - 1</td>
<td>8 - 8</td>
<td>56.31 36.87</td>
</tr>
<tr>
<td>1 - 2</td>
<td>6 - 6</td>
<td>1 - 1</td>
<td>8 - 5</td>
<td>56.31 36.87</td>
</tr>
<tr>
<td>1 - 2</td>
<td>6 - 6</td>
<td>1 - 1</td>
<td>5 - 5</td>
<td>56.31 30.96</td>
</tr>
<tr>
<td>2 - 2</td>
<td>6 - 6</td>
<td>1 - 2</td>
<td>5 - 5</td>
<td>14.04 26.57</td>
</tr>
<tr>
<td>2 - 2</td>
<td>6 - 7</td>
<td>1 - 2</td>
<td>5 - 5</td>
<td>14.04 26.57</td>
</tr>
<tr>
<td>2 - 2</td>
<td>7 - 7</td>
<td>1 - 2</td>
<td>5 - 5</td>
<td>9.46 26.57</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Table C.10: Coordinates of the end points of the line segments on $\partial A_1$, $\partial B_1$, $\partial A_2$ and $\partial B_2$.

Figure C.4: Value set $V = (B_1B_2)(A_1A_2)^{-1}$. 

<table>
<thead>
<tr>
<th>Value set $B_1$</th>
<th>Value set $A_1$</th>
<th>Value set $B_2$</th>
<th>Value set $A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 + 3i$</td>
<td>$4 + 3i$</td>
<td>$1 + 0.5i$</td>
<td>$-3 + 4i$</td>
</tr>
<tr>
<td>$2 + 3i$</td>
<td>$4 + 3i$</td>
<td>$2 + 0.5i$</td>
<td>$-3 + 2i$</td>
</tr>
<tr>
<td>$2 + 3i$</td>
<td>$5 + 3i$</td>
<td>$1 + 0.5i$</td>
<td>$-3 + i$</td>
</tr>
<tr>
<td>$5 + 3i$</td>
<td>$5 + 3i$</td>
<td>$2 + 0.5i$</td>
<td>$-3 + i$</td>
</tr>
<tr>
<td>$5 + 3i$</td>
<td>$5 + 3i$</td>
<td>$3 + 0.5i$</td>
<td>$-3 + i$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
BIBLIOGRAPHY


[61] W. Truöl, “Computer aided design of robust dynamic compensators in the frequency domain”, Technical Report 92-08, Auto-
matic Control Laboratory, Swiss Federal Institute of Technology (ETH), Zurich, Switzerland, 1992.


CURRICULUM VITAE

I was born on October 17, 1963 in Luzern, Switzerland, as son of Gisela and Horst Truöl-Nick. I attended primary school for six years in Ebikon and the gymnasium for seven years in Luzern where I obtained the Matura Certificate Type C (natural sciences and mathematics) in May 1983. In autumn 1983 I started my studies with the faculty of Electrical Engineering at the Swiss Federal Institute of Technology (ETH) in Zurich where I graduated on March 1, 1988 as an electrical engineer.

In May 1988 I joined the Automatic Control Laboratory at the ETH as a teaching and research assistant.

At the same time I began with my post-graduate studies in Automatic Control at the ETH which I finished with the certificates and in July 1990 with a post-graduate thesis.

Since about 1990 I worked on my Ph.D. thesis.