Diss. ETH No. 11166

Stability and Stabilization of Time-Delay Systems

A dissertation submitted to the
SWISS FEDERAL INSTITUTE OF TECHNOLOGY ZURICH
for the degree of
Doctor of Technical Sciences

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Zurich 1995

To my parents and
to my wife Franziska

Acknowledgements

The author wishes to express his gratitude to Prof. Dr. H. P. Geering for proposing and supporting this project. Furthermore, I want to thank Prof. Dr. R. Longchamp of the Swiss Federal Institute of Technology (EPFL), Lausanne for accepting to be the co-examiner.

I am also indebted to my colleagues of the Measurement and Control Laboratory for their helpful discussions, especially to Urs Christen, Christian Roduner, Walter Schenker, Lorenz Schumann, and Martin Weilenmann. Moreover, I am grateful to Brigitte Rohrbach for her careful editing of the manuscript which helped improve the standard of my English.

Finally, a special thank to my wife Franziska for her patience and invaluable support.

Summary

The stability analysis of time-delay systems forms the centre of this work. Three tools are typically used to investigate the stability of such systems: Razumikhin theory, Lyapunov-Krasovskii theory, and (for linear time-delay systems) eigenvalue considerations. However, none of these basic concepts represents applicable stability tests in terms of the system matrices. Therefore, based on the three stability concepts mentioned some suitable algebraic stability tests are developed in this work. The stability tests obtained can be categorized into four groups, depending on how much information concerning the delays is required for these tests:

- *Delay-independent stability criteria*: The length of the delay need not be known for the application of these stability tests. The delays may be state-dependent and/or time variable. The only assumption needed is that the delays are continuous and bounded.
- *Stability criteria independent of constant delays*: In the second group it is assumed that the delays of the system are constant; no further information on the delays is necessary.
- *Stability criteria independent of a delay constant*: This type of stability criteria presumes that the delays are constant and that the ratios of size of the delays are known.
- *Delay-dependent stability criteria*: This group includes exact algebraic stability criteria depending on the delay and on the system constants and stability criteria which yield an upper bound of the admissible delay.

The need for delay-independent (and related) stability tests is obvious, since in practice the delays are difficult to estimate, especially those that are time variable and state dependent. While algebraic stability tests independent of delays are suitable to apply, exact algebraic stability conditions depending on the delay and the system constants are known only in some special cases. In this context a method is presented to achieve some extensions. The method permits the investigation of the stability of systems which are general enough to demonstrate the differences among the four types of stability tests. The stability of general, linear time-delay systems, however, can be checked exactly only by eigenvalue considerations. Unfortunately, the computation of

the eigenvalues is a cumbersome task, since the corresponding transcendental characteristic equation contains exponential terms which induce extreme gradients. An improved version of a well-known method for the computation of the eigenvalues is illustrated.

In connection with stability considerations the robustness of nondelayed systems against delays is studied as well. It turns out that the largest singular value of the control system can be used to analyse the robustness of the system against delays in the input. Furthermore, the H^∞-norm yields an estimate of the robustness against delays in the state. In order to analyse the robustness of time-delay systems against unstructured uncertainties, some suitable criteria are derived based on the three stability concepts.

Algebraic stability tests are also useful in connection with control methods for time-delay systems. A comparison between the assumptions and the possibilities of the various known control methods for time-delay systems shows that a combination of finite dimensional approximation techniques and optimal control theory is the most suitable approach for delayed systems. Nevertheless, the stability of the resulting closed-loop system cannot be guaranteed a priori. Hence, algebraic stability tests are necessary to check this property. The model of the Williams-Otto process is used to demonstrate that the three tools

- finite dimensional approximation
- optimal control
- algebraic stability criteria

in combination remarkably improve the behaviour of the control system. Besides, the stability of the closed-loop system is guaranteed even if the delay is not constant.

Zusammenfassung

Die Stabilitätsanalyse von totzeitbehafteten Systemen bildet den Schwerpunkt der vorliegenden Arbeit. Um die Stabilität von solchen Systemen abzuklären, stehen drei Werkzeuge zur Verfügung: Razumikhin-Theorie, Ljapunow-Krasovskii-Theorie und (für lineare Systeme) Eigenwertbetrachtungen. Diese grundlegenden Konzepte stellen aber bei Totzeitsystemen keine direkt anwendbaren Kriterien zur Untersuchung der Stabilität dar. Deshalb sind in dieser Arbeit mit Hilfe der allgemeinen Stabilitätskonzepte algebraische Stabilitätskriterien entwickelt worden. Die so gewonnenen Stabilitätstests lassen sich in vier Kategorien einteilen, abhängig davon, wieviel Information über die Grösse der Totzeit für den Stabilitätstest benötigt wird:

- *Totzeitunabhängige Stabilitätskriterien*: Die Information über die Grösse der Totzeiten wird nicht gebraucht. Die Totzeiten dürfen zeit- und zustandsabhängig sein. Es wird lediglich vorausgesetzt, dass die Totzeiten durch stetige und beschränkte Funktionen beschrieben werden können.
- *Stabilitätskriterien unabhängig von konstanten Totzeiten*: Bei der zweiten Gruppe von Stabilitätskriterien wird vorausgesetzt, dass die Totzeiten des Systems konstant sind. Weitere Informationen über die Totzeiten sind nicht erforderlich.
- *Stabilitätskriterien unabhängig von einer Totzeitkonstante*: Für diesen Typ von Kriterien wird vorausgesetzt, dass die Totzeiten konstant sind und dass die Grössenverhältnisse zwischen den verschiedenen Totzeiten bekannt sind.
- *Totzeitabhängige Stabilitätskriterien*: Zu dieser Gruppe gehören Kriterien, mit deren Hilfe man die Stabilität exakt untersuchen kann, sowie Kriterien, die eine obere Schranke für die Grösse der zulässigen Totzeit liefern.

Totzeitunabhängige Stabilitätstests zu entwickeln ist naheliegend, weil in der Praxis die Grösse der Totzeit oft schwierig abzuschätzen ist, vor allem, wenn diese zeit- und zustandsabhängig ist. Die meisten algebraischen Stabilitätsbedingungen, insbesondere die totzeitunabhängigen, sind selbst für MIMO-Systeme sehr gut anwendbar. Hingegen sind exakte algebraische Stabilitätskriterien in Abhängigkeit der Totzeiten und Systemkonstanten nur für sehr spezielle Fälle bekannt. Verallgemeinerungen gelingen durch eine neue Methode, die in diesem Rahmen vorgestellt wird. Diese Methode erlaubt es,

Systeme zu betrachten, die immerhin so allgemein sind, dass Unterschiede zwischen den vier verschiedenen Stabilitätstypen deutlich gemacht werden können. Um allgemeine, lineare Totzeitsysteme exakt auf ihre Stabilität hin zu untersuchen, sind wir auf Eigenwertbetrachtungen angewiesen. Die Berechnung der Eigenwerte ist aber insofern problematisch, als die numerische Lösung der entsprechenden transzendenten, charakteristischen Gleichung einen erheblichen Aufwand darstellt. Eine verbesserte Version der gängigen Methode zur Berechnung der Eigenwerte von Totzeitsystemen wird vorgestellt.

Im Zusammenhang mit der Stabilität wird auch die Robustheit von linearen, nicht totzeitbehafteten Systemen gegen Totzeiten betrachtet. Es zeigt sich, dass man mit Hilfe des grössten Singularwertes des Regelsystems auf einfache Weise die Robustheit des Regelsystems gegen Eingangstotzeiten analysieren kann. Ferner liefert die H^∞-Norm des Regelsystems eine Abschätzung der Robustheit des Regelsystems gegen Totzeiten im Zustand. Mittels der allgemeinen Stabilitätskonzepte lassen sich leicht anwendbare Kriterien zur Untersuchung der Robustheit eines Totzeitsystems gegen nichtlineare, zeitvariable, unstrukturierte Unsicherheiten entwickeln.

Algebraische Stabilitätskriterien sind auch in Verbindung mit Regelmethoden für Totzeitsysteme nützlich. Ein Vergleich der Möglichkeiten und Grenzen der verschiedenen, bekannten Zustandsregelmethoden für Totzeitsysteme zeigt, dass endlich dimensionale Approximationstechnik in Verbindung mit optimaler Regelung die geeignetste Regelmethode für Totzeitsysteme darstellt. Die Stabilität des resultierenden Regelsystems kann jedoch nicht a priori garantiert werden. Der Einsatz von algebraischen Stabilitätskriterien ist hier sinnvoll, insbesondere, wenn die Modellierung der Totzeiten Unsicherheiten beinhaltet. Am Beispiel des Williams-Otto-Prozesses wird illustriert, dass mittels der drei Werkzeuge

- endlich dimensionale Approximationstechnik
- optimale Regelung
- algebraische Stabilitätskriterien

auf einfache Weise das Verhalten des Regelsystems verbessert werden kann. Dabei ist die Stabilität des Gesamtsystems garantiert, selbst wenn in der Realität die Totzeiten nicht konstant sind.

Contents

Preliminaries

1 Introduction

In the early decades of this century, various theories of elasticity and of evolution were extensively tested. In both of these areas the need for explicit analytical reasoning became soon apparent. The mathematical description of certain processes within these topics led to an investigation of systems with delays.

One of the first descriptions of a system with retardation was given by Boltzman (1874), who studied retarded elasticity effects. His publication, however, did not point out clearly the need of the past states for a realistic modelling of retarded elasticity effects. In the early 1900's a controversy arose over the necessity of specifying the earlier history of a system in order to predict its future evolution. This view stood in contradiction with the Newtonian tradition which claimed that the knowledge of the present values of all relevant variables should suffice for prediction. Picard (1907) took the view that the past states are important for a realistic modelling. In his train of thought he analysed a system with essential hidden variables, not themselves accessible to observation. He claimed that the prediction of that system requires also the knowledge of the earlier values of the hidden variables. His paradigm for that situation was a pendulum clock whose descending weight is encased. As long as we cannot observe the present position of the weight and its rate of descent, a prediction of the future motion of the clock hand requires the knowledge of when the clock was last wound.

Systematic work with mathematical models on medicine and biology began with the epidemiological studies of Ross (1911). Ross was laying the foundation for the mathematical theory of epidemics in terms of differential equations. His results were extended and improved in the 1920's. The need for delays was emphasised both by Lotka (see Sharpe & Lotka, 1923), who discussed the discrete delays due to the incubation times in the Ross malaria epidemic model, and by Volterra (1927). Independently of each other, Lotka in the United States and Volterra in Italy began to concentrate their mathematical efforts on the problem of the variations and fluctuations in the numbers of individuals and species. From the very beginning of their ecological investigations, both Lotka and Volterra realized that, in order to achieve some degree of realism, delayed effects had to be explicitly taken into account.

Lotka's main previous interest had been in physical chemistry, with special emphasis on the oscillations of chemical reactions. He had also dealt with demographic problems and with evolutionary theory.

Volterra's previous interests were mostly in mechanics, including irreversible phenomena and elasticity. The latter had led him to develop the theory of functionals and integro-differential equations, for which he became well known [142], [143]. He also attempted to introduce a concept of energy function to study the asymptotic behaviour of the system in the distant future.

Minorsky (1942), in his study of ship stabilization and automatic steering, pointed out very clearly the importance of the delay considerations in the feedback mechanism. The great interest in control theory during those and later years has certainly contributed significantly to the rapid development of the theory of differential equations with dependence on the past state.

While it became clear a long time ago that retarded systems could be handled as infinite dimensional problems, the paper of Myshkis (1949) gave the first correct mathematical formulation of the initial value problem. Furthermore, in his book published in 1955, Myshkis introduced a general class of equations with delayed arguments and laid the foundation for a general theory of linear systems.

Subsequently, several books appeared which presented the then current knowledge on the subject and which greatly influenced later developments. In their monograph at the Rand Corporation, Bellman and Danskin (1953) pointed out the diverse applications of equations containing past information to other areas such as biology and economics. They also presented a well-organized theory of linear equations with constant coefficients and the beginnings of stability theory. A more extensive development of these ideas is contained in the book of Bellman and Cook (1963). Some important results were supplied also by Krasovskii, who studied stability and optimal control problems for time-delay systems [67]. Further important works have been written by El'sgol'ts (1966) and Hale (1977). In recent years several books have been published on this topic [40], [66], [83], [85], [131].

The above historical introduction shows that delays must be taken into account to describe or to control certain processes. Nowadays, one of the main goals of the development of automatic manufacturing processes is to reach a high production rate while maintaining a guaranteed quality level. This high production rate requires a high-speed variation of control variables. It is therefore necessary to include the consideration of delay effects within control methods. Delay effects occur not only in technology. They are equally observable in biology, chemistry, medicine, and economics. The most typical areas in which delays play an important role are transport, mixing, burning, evolution, bureaucracy, and economic fluctuations.

1.1 Classification of functional differential equations

Assume that $\tau_{max} = \text{const} \in [0, \infty)$, and let $x(t)$ be an n-dimensional variable describing the behaviour of a process in the time interval $t \in [t_0 - \tau_{max}, t_1]$. Most generally a functional differential equation (**FDE**) is formulated as follows. Let $T_1(t)$ and $T_2(t)$ be time-dependent sets of real numbers, defined for all $t \in [t_0, t_1]$. Let us assume that x is a continuous function in $[t_0, t_1]$. We shall use the convention that $\dot{x}(t)$ for $t \in [t_0, t_1]$ denotes the right-hand derivatives of x. For each $t \in [t_0, t_1]$, x_t is defined by $x_t(s) = x(t + s)$, where $s \in T_1(t)$. Analogously, $\dot{x}_t$ is defined by $\dot{x}_t(s) = \dot{x}(t + s)$, where $s \in T_2(t)$. We say that x satisfies an FDE in $[t_0, t_1]$ if for almost every $t \in [t_0, t_1]$ the following equality holds

$$\dot{x}(t) = f(t, x_t, \dot{x}_t, u(t)) \tag{1.1}$$

where the control u(t) is given for the whole time interval necessary. The equation above contains three types of differential equations.

i) An FDE is retarded or, as we say, a retarded functional differential equation (**RFDE**), if $T_1(t) \subset (-\infty, 0]$ and $T_2(t) = \varnothing$ for $t \in [t_0, t_1]$. Therefore the right-hand side of (1.1) does not depend on the derivative of x

$$\dot{x}(t) = f(t, x_t, u(t)) \,. \tag{1.2}$$

In other words, the rate of change of the state of an RFDE is determined by the inputs u(t), as well as the present and past states of the system. An RFDE is sometimes also designated as an hereditary differential equation or, in control theory, as a time-delay system.

ii) If the rate of change of the state depends on its own past values as well, the system can be governed by a neutral functional differential equation (**NFDE**). That is, we have $T_1(t) \subset (-\infty, 0]$ and $T_2(t) \subset (-\infty, 0]$ for $t \in [t_0, t_1]$. The following scalar linear system is an example of a neutral system

$$\dot{x}(t) = \dot{x}(t-1) + x(t) + u(t)$$

whereas the equation

$$\ddot{x}(t) = \dot{x}(t-1) + x(t-1) + u(t)$$

is of the retarded type (1.2), since the highest derivation is not delayed.

iii) An FDE is called an advanced functional differential equation (**AFDE**), if $T_1(t) \subset [0, \infty)$ and $T_2(t) = \varnothing$ for $t \in [t_0, t_1]$. An equation of the advanced type may represent a system in which the rate of change of a quantity depends on its present and future values of the quantity and of the input signal u(t).

Since in applications t usually represents time, the solution in the direction of an increasing t is required. One should note that an RFDE converts into an AFDE for $t < 0$, and vice versa, and an NFDE converts into another differential equation of a neutral type. However, in the following we deal mainly with RFDE, because in reality this type of system is encountered frequently.

In most applications (and in the above classification) the delays are usually bounded. Systems with infinite delay will not be considered here. Those aspects are treated comprehensively in [48].

If the set $T_1(t)$ is finite for every $t \in [t_0, t_1]$, a retarded FDE is called an FDE with lumped or discrete delays. Other names for this type of equations are retarded difference differential equations or simply difference differential equations or differential difference equations. An example of a system with a lumped delay is

$$\dot{x}(t) = f(x(t), x(t-\tau(t))) .$$

If the set $T_1(t)$ is a continuum, the FDE contains distributed delays. The following system has a distributed lag

$$\dot{x}(t) = \int_{t-h}^{t} g(x(s), t, s)ds .$$

Delays which are constant are called fixed point delays. Systems which have only multiple constant time lags can be classified further. Delays which are related by integers will be called commensurate delays. The linear commensurate time-delay system

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t-ih)$$

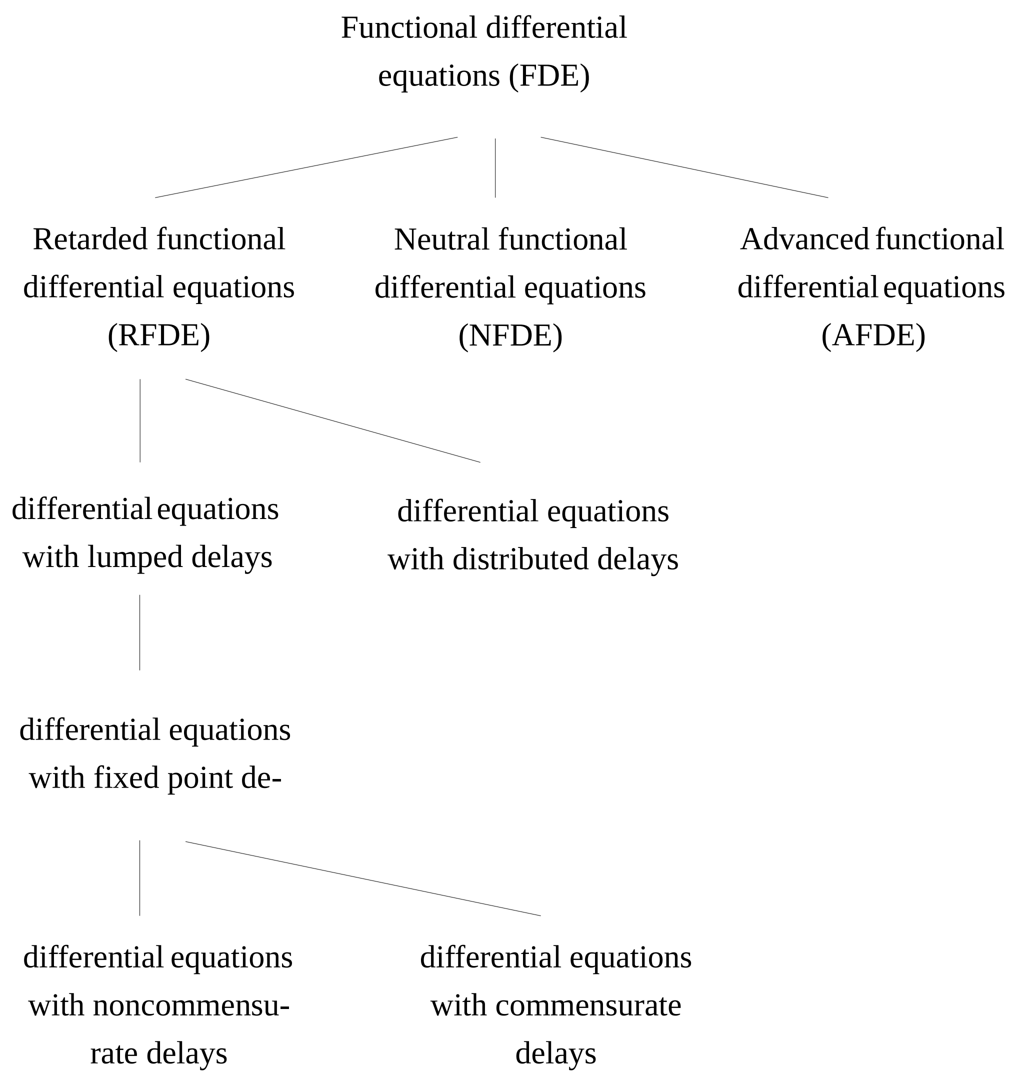

Fig. 1.1 Classification of FDEs and RFDEs

is frequently discussed in the literature. If the delays are not so related, the system is called a noncommensurate delay system. For example, the delays of the system

$$\dot{x}(t) = x(t) + x(t-1) + x(t-\pi)$$

are noncommensurate. A brief survey of the above mentioned expressions is given in Fig. 1.1.

1.2 Examples of systems with time-delays

The Williams-Otto process

The Williams-Otto process [153] has many characteristics of a typical chemical process and is therefore frequently discussed in the literature, especially in journals of chemical engineering. Here, common operations such as separation (through decanting and distillation) and reaction are involved. The system considered is a model of a refining plant. The schematic of the flowsheet for the Williams-Otto process is shown in Fig. 1.2.

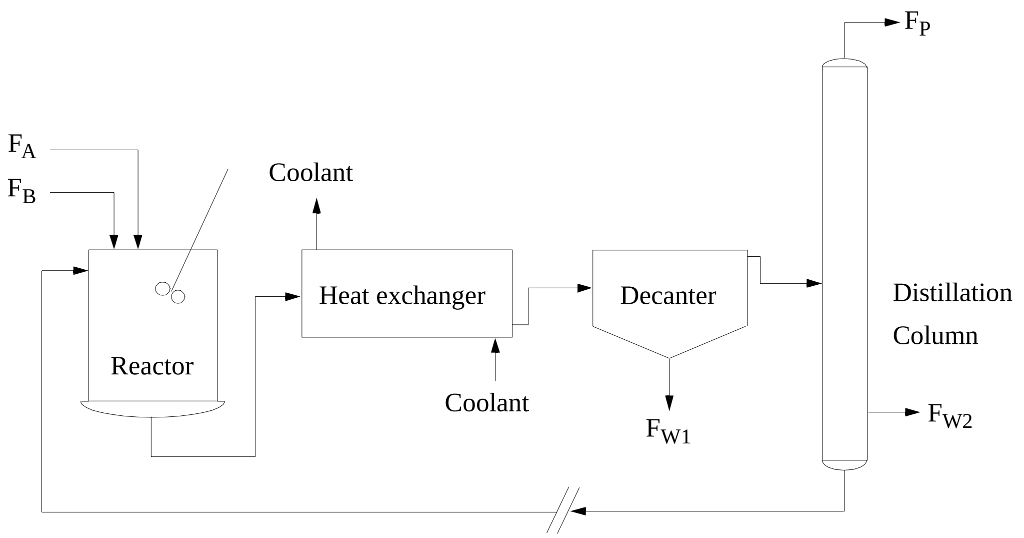

Fig. 1.2 Flowsheet of the Williams-Otto process

Upon entering the chemical reactor two kinds of raw materials take part in three chemical reactions which produce the desired product, along with some by-products. The feed rates of the raw materials are denoted by F_A and F_B. A heat exchanger is required to cool the reactants to a temperature at which an undesirable by-product (F_{W1}) will settle out of the reactant mixture. This settling takes place in the decanter. Subsequently the material enters a distillation column. The material contains the desired product, impurities, and a certain percentage of the raw material with some by-products of the chemical reaction. The valuable product (F_P) is removed in the overhead of the distillation column. At the bottom of the column the purge (F_{W2}) is led off, whereas the raw material with the by-products is recycled to the chemical reactor, where it is reprocessed. The recycle loop ensures that useful products will not be discarded.

The recycle loop represents a significant transport lag. In practical situations, it is not at all unusual for material to take ten minutes to travel from the chemical reactor through the cooler, the decanter, the distillation column, and the recycling to the reactor.

The differential equations governing this chemical process are nonlinear. However, for the determination of proper corrections of the feed rates F_A and F_B at the desired operating point, a corresponding linearized model is useful. For a recycle time of 10 minutes, the linearized and time-scaled (one time unit is 10 min) equations are [122]:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + Bu(t) \tag{1.3}$$

where

$$A_0 = \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.20 & -5.30 & -12.8 & 0 \\ 6.40 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11.0 & -3.96 \end{bmatrix}, \; A_1 = \begin{bmatrix} 1.92 & 0 & 0 & 0 \\ 0 & 1.92 & 0 & 0 \\ 0 & 0 & 1.87 & 0 \\ 0 & 0 & 0 & 0.724 \end{bmatrix} \tag{1.4}$$

and

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{1.5}$$

The dimensionless components x_1, x_2, x_3, and x_4 of the state vector x represents the deviations in the weight compositions of the raw materials A and B, of an intermediate product, and of the desired product, respectively, from their nominal values. The control inputs u_1 and u_2 are defined to be equal to $\delta F_A/6V_R$ and $\delta F_B/6V_R$, respectively, where V_R is the volume of the chemical reactor ($V_R = 92.8\,ft^3 \approx 2.628\,m^3$), and δF_A and δF_B are the deviations in the feed rates (in pounds per hour; 1 pound = 0.453592 kg) of the raw materials A and B, respectively, from their nominal values.

Wind tunnel

At the NASA Langley Research Center in Hampton, VA, a wind tunnel was constructed to achieve Reynolds numbers of one order of magnitude higher than those in existing tunnels. The desired test chamber temperatures are maintained at cryogenic levels by injection of liquid nitrogen into the airstream near the fan section of the tunnel. Fine control of the Mach number in the test chamber is effected through changes in the inlet guide vane angle setting in the fan section. Schematically, the tunnel can be depicted as in Fig. 1.3.

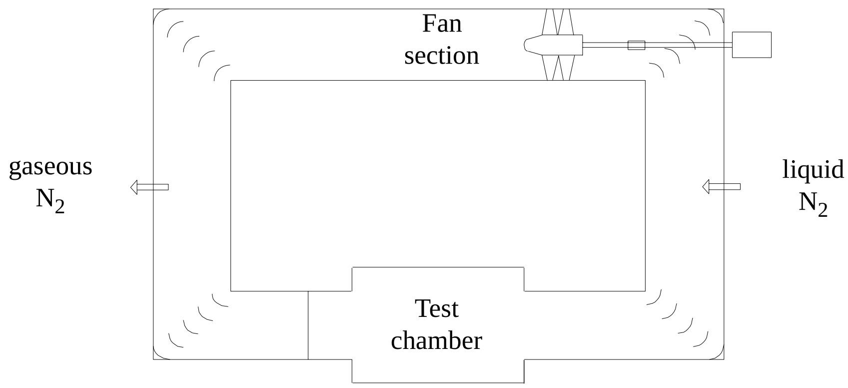

Fig. 1.3 Wind tunnel

Modelling this system based on the Navier-Stokes theory does not lead to useful equations for the design of a control law. A simple model for the Mach number control loop was proposed in [3]. In order to take into account the flow times through sections of the tunnel, a transport lag was included in the model. The proposed equations for this system are as follows

$$\dot{x}(t) = \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\omega^2 & -2\xi\omega \end{bmatrix} x(t) + \begin{bmatrix} 0 & ka & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t-0.33) + \begin{bmatrix} 0 \\ 0 \\ -\omega^2 \end{bmatrix} u(t) \qquad (1.6)$$

where $1/a = 1.964$ [$\sec^{-1}$], $\omega = 6.0$ [rad/sec], $\xi = 0.8$ [-], and $k = -0.0117$ [$\deg^{-1}$]. The state vector $x = [\,\delta M, \delta\theta, \delta\dot{\theta}\,]^T$ consists of the variation in Mach number δM, the variation in guide vane angle $\delta\theta$, and the variation in guide vane angle velocity $\delta\dot{\theta}$. The control $u(t)$ represents the guide vane angle actuator input.

Gasoline Engine

A modern engine test bench was developed at the Measurement and Control Laboratory of the Swiss Federal Institute of Technology (ETH) in Zurich. This test bench is used for various purposes, e.g. the emulation of the load dynamics of the drivetrain of the target vehicle for the engine under test, the development of system identification methods for SI engines, or the testing of multivariable model-based controllers for SI engines to control the air-to-fuel ratio and the speed.

In the work of Onder (1993), [112] an efficient method for the off-line identification of an engine model is presented. The general nonlinear, delayed model contains continuous-time and discrete-time subsystems. It turns out that delays have to be taken into account in order to describe this system in an appropriate way. The dynamic model can be partitioned into the following five subsystems:

- throttle actuator
- intake manifold
- torque generation and rotational inertia
- air-to-fuel ratio sensor
- wall-wetting dynamics.

In [112], the engine model is not given in the form of a linear delayed, state-space model. However, from the linear model described in [112] on page 137, together with the information about the delays presented on page 131, one immediately obtains a state-space model with seven different delays in the state and three delays in the control. Simulations show that for our purposes an appropriate single-delay system is a sufficiently good approximation to describe the behaviour of the engine. (All delays in the control are neglected, whereas the delays in the states are rounded to the maximal delay.) Following in this way the results of [112], the corresponding model for a six-cylinder 3.4-litre BMW engine with sequential injection working at idle speed,

$\alpha_{throttle} = 6°$	$\overset{*}{m}_{air\ flow} = 470$ g/min
$p_{manifold} = 0.45$ bar	$\alpha_{ignition} = 18°$
$n = 900$ rpm	$M_{load} = 38$ Nm
$\lambda = 1$	$T_{exhaust} = 693°K$

is

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + B_0 u(t) + B_d d(t) \tag{1.7}$$

where $\tau = 2/9$ [sec] and

$$A_0 = \begin{bmatrix} -3.00 \cdot 10^{1} & 0 & 0 & 0 & 0 \\ 2.62 \cdot 10^{-1} & -3.11 & -1.97 \cdot 10^{-3} & 0 & 0 \\ 0 & 2.05 \cdot 10^{2} & -4.96 \cdot 10^{-2} & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -8.60 \cdot 10^{-1} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1.61 \cdot 10^{2} & 1.17 \cdot 10^{3} & -6.68 \cdot 10^{-1} & 0 & 9.18 \cdot 10^{2} \\ -3.51 \cdot 10^{-1} & 4.83 & 2.76 \cdot 10^{-3} & 0 & -2.00 \\ 2.45 \cdot 10^{-2} & 0 & -1.33 \cdot 10^{-4} & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 30 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 7.89 \cdot 10^{2} & 1.66 \cdot 10^{1} \\ 0 & -1.72 & 0 \\ 0 & 1.20 \cdot 10^{-1} & 0 \end{bmatrix} \qquad B_d = \begin{bmatrix} 0 \\ 0 \\ -12.4 \\ 0 \\ 0 \end{bmatrix}.$$

The variables x(t), u(t), and d(t) represent the deviation from their nominal values (idle speed) with respect to the following physical meanings:

x_1	:	Throttle position [degree]
x_2	:	Intake manifold pressure [bar]
x_3	:	Engine speed [rpm]
x_4	:	Lambda signal [-]
x_5	:	State of the wall-wetting model [g/min]
u_1	:	Commanded throttle position [degree]
u_2	:	The base value of the metered fuel is multiplied by u_2 [-]
u_3	:	Difference between demanded spark angle to static calibration [degree]
d	:	External load torque [Nm].

Note that the load torque is considered as a disturbance. The corresponding model for a four-cylinder 1.8-litre BMW engine have recently been analysed. The model is of the form (1.7). Of course, the numerical values of the system matrices are different. At idle speed,

$\alpha_{\text{throttle}} = 10°$ $\qquad$ $\overset{*}{m}_{\text{air flow}} = 470$ g/min

$p_{\text{manifold}} = 0.48$ bar $\qquad$ $\alpha_{\text{ignition}} = 18°$

$n = 900$ rpm $\qquad$ $M_{\text{load}} = 20$ Nm

$\lambda = 1$ $\qquad$ $T_{\text{exhaust}} = 693°K$

the corresponding matrices are as follows:

$$A_0 = \begin{bmatrix} -3.50 \cdot 10^{1} & 0 & 0 & 0 & 0 \\ 1.68 \cdot 10^{-1} & -2.35 & -1.63 \cdot 10^{-3} & 0 & 0 \\ 0 & 1.23 \cdot 10^{2} & -6.36 \cdot 10^{-2} & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1.50 \cdot 10^{1} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2.87 \cdot 10^{1} & 1.83 \cdot 10^{3} & -8.87 \cdot 10^{-2} & 0 & 2.59 \cdot 10^{2} \\ -2.22 \cdot 10^{-1} & 3.97 & 2.22 \cdot 10^{-3} & 0 & -2.00 \\ 4.16 \cdot 10^{-1} & 0 & -3.47 \cdot 10^{-3} & 0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 35 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2.07 \cdot 10^{2} & 1.57 \cdot 10^{1} \\ 0 & -1.60 & 0 \\ 0 & 3.00 & 0 \end{bmatrix}, \qquad B_d = \begin{bmatrix} 0 \\ 0 \\ -16 \\ 0 \\ 0 \end{bmatrix}.$$

The delays of the models mainly represent retarded influences of some states on the torque generation and on the air-to-fuel ratio. The subsystem of the lambda sensor additionally contains a transport delay. The non-negligible influence of all these delays can be demonstrated with the models given above. The values of time constants for the corresponding delay-free systems are between 1 and 0.03 [sec]. The time-delay $\tau = 2/9$ [sec] is of the same order of magnitude.

The first four states can be measured. Some (or all) of these measured signals can be used to control the air-to-fuel ratio and the speed. The design of controllers which show a good disturbance rejection as well can be based on linear models. These controllers have been tested with success.

2 On the solution of time-delay systems

The initial value problem for RFDEs is briefly considered. Suppose that τ_{max} = const $\in [0, \infty)$. By $\boldsymbol{R}^n$ the n-dimensional vector space over the reals with the Euclidean norm $|\cdot|$ is denoted. $C([t_0 - \tau_{max}, t_0]; \boldsymbol{R}^n)$ is the space of continuous and bounded functions mapping the interval $[t_0 - \tau_{max}, t_0]$ into $\boldsymbol{R}^n$. For any $x \in C([t_0 - \tau_{max}, t_1]; \boldsymbol{R}^n)$, $t_1 > t_0$, x_t is defined as $x_t = x(t + \theta)$, $\theta \in [-\tau_{max}, 0]$. The initial value problem for the system

$$\dot{x}(t) = f(t, x_t, u(t)) \tag{2.1}$$

with a given control u(t) consists of determining a continuous solution x(t) of (2.1) for $t \geq t_0$ such that $x(t_0) = x_0$ and $x(t) = \varphi(t)$ for $t_0 - \tau_{max} \leq t < t_0$, where φ is a continuous function called the *initial function* (Fig. 2.1). It is often assumed that $\varphi(t_0) = x(t_0)$. For given initial values $(x_0, \varphi(t))$ the solution of equation (2.1) is often denoted as $x(x_0, \varphi, f)$ [43, p. 37]. If the function f in (2.1) is continuous and satisfies a local Lipschitz condition in φ and u, then the local existence and uniqueness of the solution can be proved as well as its continuous dependence on the initial data [40], [43].

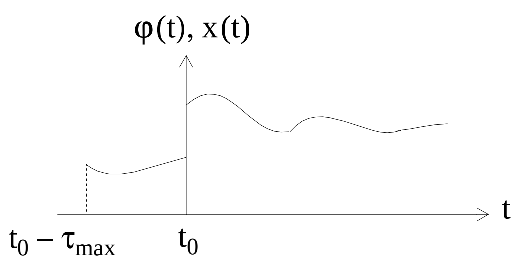

Fig. 2.1

The variable u(t) indicates the input to the system or the control variable. Usually, the state of a system at time t is defined as a collection of information which, together with the knowledge of the input, is sufficient to determine the output of the system. Therefore, the *state space* of continuous time-delay systems is an infinite-dimensional vector space $C([t_0 - \tau_{max}, t_1], \boldsymbol{R}^n)$, whereas the *phase space* of such a system is the space of the n-dimensional vector $x(t)$ at each instant time t.

2.1 Method of steps

In Part III simulation programs are applied to study the behaviour of some delayed control systems. One possibility to test these programs is by comparing the numerical with the exact solutions. The so-called method of steps (or method of successive integration) is a way to calculate explicit solutions [33].

The desired solution is found on successive intervals by solving ordinary differential equations without delays in each interval. To illustrate the method we consider the following delayed differential equation together with a given initial condition

$$\dot{x}(t) = f(t, x(t), x(t-\tau)) \qquad t \geq t_0 \tag{2.2}$$

$$x(t_0) = x_0 \qquad t = t_0 \tag{2.3}$$

$$x(t) = \varphi(t) \qquad t_0 - \tau \leq t < t_0\,. \tag{2.4}$$

For $t \in [t_0, t_0 + \tau]$ the above differential equation can be represented as an ordinary differential equation

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x(t-\tau)) && t_0 \leq t \leq t_0 + \tau \\ x(t_0) &= x_0 && t = t_0\,. \end{aligned}$$

Assuming the existence of a solution $x(t) = \varphi_1(t)$ on the entire segment $[t_0, t_0 + \tau]$, we obtain analogously for the next time interval

$$\dot{x}(t) = f(t, x(t), \varphi_1(t - \tau)) \qquad t_0 + \tau \le t \le t_0 + 2\tau$$

$$x(t_0 + t) = \varphi_1(t_0 + t) \qquad t = t_0 + \tau\,.$$

In this way, the method allows us to calculate step by step the solution on some finite segment.

Example 2.1: For the system

$$\dot{x}(t) = 6x(t - 1) \qquad t \ge 0$$

$$\varphi(t) = t \qquad -1 \le t \le 0$$

the solution on the time interval [0, 2] is

$$x(t) = 3(t - 1)^2 - 3 \qquad 0 \le t \le 1$$

$$x(t) = 6(t - 2)^3 - 18t + 21 \qquad 1 \le t \le 2.$$

The tested integration programs of the software package MatrixX yields very good results. A program which directly uses the method of steps is called Delsol [152].

For continuous φ and $f \in C^\infty$ the solution x(t) of equation (2.2) has a continuous derivative for $t_0 < t < t_0 + \tau$. Consequently the solution of equation (2.2) is twice differentiable for $t_0 + \tau < t < t_0 + 2\tau$, and so on. Therefore, the solution x(t) smooths out as t grows.

It is obvious that the method of successive integration can be extended to solve the initial value problem for systems with time-varying delays or for neutral systems. Furthermore, the method is always involved in existence and uniqueness proofs for FDEs [43]. However, the method is not suitable for the generation of stability or controllability criteria. Even for the simple system [33]

$$\dot{x}(t) = a \cdot x(t - 1) \qquad t \ge t_0$$

$$x(t) = c \qquad t_0 - \tau \le t < t_0$$

the solution

$$x(t) = c \sum_{i=1}^{N} a^i \frac{(t - t_0 - (i-1)^N)^i}{i!} \qquad N - 1 \le t \le N$$

is difficult to analyse concerning stability. In the next section, some other ways for representing the solution of a time-delay system are considered.

2.2 Fundamental matrix

The integral form of differential equation (2.1)

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x_s, u(s))ds$$

is sometimes quite useful to represent a solution of (2.1). We will use this form in Section 2.3 to establish a Comparison Theorem. Since an explicit solution for nonlinear time-delay systems can be given only in very special cases, we shall restrict our considerations in this section to the following linear differential equation

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - \tau_i) + B_0 u(t) \qquad t \ge t_0 \qquad (2.5)$$

$$x(t) = \varphi(t) \qquad t_0 - \tau_k \le t < t_0 \qquad (2.6)$$

where $0 < \tau_1 < \ldots < \tau_k$, $A_0, A_i \in \boldsymbol{R}^{n \times n}$, $x(t) \in \boldsymbol{R}^n$, $B_0 \in \boldsymbol{R}^{n \times m}$, $u(t) \in \boldsymbol{R}^m$. A widespread representation of the solution of (2.5) is given by

$$x(t) = e^{A_0 t} x_0 + \int_{t_0}^{t} e^{A_0(t-s)} \left\{ \sum_{i=1}^{k} A_i x(s - \tau_i) + B_0 u(s) \right\} ds \quad . \qquad (2.7)$$

Equation (2.7) is used in Section 3 to derive a simple algebraic stability condition. The first term on the right-hand side of equation (2.7) describes the influence of $x_0 = \varphi(t_0)$ on the system. An alternative characterization of the solution is obtained if the first term describes the influence of the entire φ on the system:

$$x(t) = \int_{t_0-\tau_k}^{t_0} \Phi(t, s)\varphi(s)ds + \int_{t_0}^{t} \Phi_u(t, s)B_0u(s)ds. \tag{2.8}$$

The fundamental matrices Φ and Φ_u are determined by

$$\begin{aligned} &\dot{\Phi}(t, s) = A_0\Phi(t, s) + \sum_{i=1}^{k} A_i\Phi(t-\tau_i, s) && t \geq t_0 \\ &\Phi(t, s) = I \cdot \delta(t-s) && t, s \in [t_0-\tau_k, t_0] \end{aligned}$$

$$\begin{aligned} &\dot{\Phi}_u(t, s) = A_0\Phi_u(t, s) + \sum_{i=1}^{k} A_i\Phi_u(t-\tau_i, s) && t \geq t_0 \\ &\Phi_u(t, t) = I && \\ &\Phi_u(t, t) = 0 && t < s \end{aligned}$$

where $I \in \boldsymbol{R}^{n \times n}$ is the identity matrix and δ is the Dirac function. An iterative method for the construction of Φ and Φ_u is given in [106, p. 93]. The example below illustrates an application of (2.7) together with the method of steps. Furthermore, the result obtained is used in Section 2.4.

A function is called absolutely continuous if it is continuous and maps bounded sets in C into bounded sets in $\boldsymbol{R}^q$. We shall denote by $L^2([a, b]; \boldsymbol{R}^q)$ the space of square integrable $\boldsymbol{R}^q$-valued functions on [a, b]. $W^{1,2}([a, b]; \boldsymbol{R}^q)$ is the space of absolutely continuous $\boldsymbol{R}^q$-valued functions on [a, b] with square integrable derivatives.

Example 2.2: Show that (2.5) admits a solution $x \in L^2([t_0-\tau_{max}, t_0]; \boldsymbol{R}^n) \cap W^{1,2}([t_0, t_1]; \boldsymbol{R}^n)$ for every input $u \in L^2([t_0, t_1]; \boldsymbol{R}^m)$ and every initial condition $\varphi \in L^2([t_0-\tau_{max}, t_0]; \boldsymbol{R}^n)$.

For $t \in [t_0, t_0+\tau_k]$, equation (2.7) can be rewritten as

$$x(t) = e^{A_0 t}x_0 + \int_0^t e^{A_0(t-s)}\left\{\sum_{i=1}^{k} A_i\varphi(s-\tau_i) + B_0u(s)\right\}ds \quad .$$

The right-hand side of the above equation implies that $x \in L^2$ for $t \in [t_0, t_0 + \tau_k]$. Again considering equation (2.5) we conclude that also $\dot{x} \in L^2$, which implies that $x \in W^{1,2}$. By analogy we may now proceed for the next time interval $t \in [t_0 + \tau_k, t_0 + 2\tau_k]$ and so on.

2.3 A Comparison Theorem

In order to establish a stability condition in Chapter 3, a Comparison Theorem is derived. As mentioned in Section 2.2, the solution of

$$
\begin{aligned}
\dot{x}(t) &= f(t, x_t) && t > t_0 && (2.9)\\
x(t_0) &= x_0 && t = t_0 && (2.10)\\
x(t) &= \varphi(t) && t_0 - \tau_{max} \le t < t_0\,. && (2.11)
\end{aligned}
$$

can be expressed as

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x_s)ds \quad .$$

For several reasons we will consider here the slightly more general integral equation

$$x(t) = g_0(t) + \int_{t_0}^{t} F(t, s, x_s)ds \quad . \tag{2.12}$$

The symbol $\mathfrak{I}^n = C([t_0 - \tau_{max}, t_0]; \boldsymbol{R}^n)$ denotes the space of continuous functions with domain $[t_0 - \tau_{max}, t_0]$ and range $\boldsymbol{R}^n$. J is defined by $J = [t_0, t_1]$. For any two elements $x, y \in \boldsymbol{R}^n$ we write $x \le y$ iff $x_i \le y_i$ for each $i = 1, ..., n$. Furthermore, we define the following expressions

$$\max[x, y] = z = (z_1, z_2, ..., z_n) \tag{2.13}$$

where $z_i = \max(x_i, y_i)$. For example $z(t) = \max[y(t), x(t)]$ implies that $x(t) \le z(t)$ and consequently $x(t + \theta) \le z(t + \theta)$ for $\theta \in [-\tau_{max}, 0]$, $t > t_0$. The last inequality between x and z is denoted by $x_t \le z_t = \max[y, x]_t$.

***Definition 2.1 (analogous to [76], Vol. 2, p. 36, Def. 6.9.2)**: Let $r(g_0, \varphi, F)$ be a solution of (2.12) defined on $[t_0, t_1]$. If any other solution $x(g_0, \varphi, F)$ of (2.12) defined on the same interval satisfies*

$$x(g_0, \varphi, F) \le r(g_0, \varphi, F)$$

then $r(g_0, \varphi, F)$ is said to be a maximal solution of (2.12).

***Definition 2.2 (analogous to [76], Vol. 1, p. 316, Def. 5.1.1)**: We shall say that the integral operator F is monotone nondecreasing if, for any ϕ, $\tilde{\phi} \in C([t_0 - \tau_{max}, t_1]; \mathbf{R}^n)$ such that for any $t_1 > t_0$*

$$\phi(t) \le \tilde{\phi}(t) \qquad\qquad t_0 - \tau_{max} \le t \le t_1$$

implies

$$\int_{t_0}^{t_1} F(t, s, \phi_s)ds \le \int_{t_0}^{t_1} F(t, s, \tilde{\phi}_s)ds \quad .$$

***Theorem 2.1 (Comparison Theorem)**: Let $F(t, s, x_s) \in C([J \times J \times \mathfrak{I}^n]; \mathbf{R}^n)$, be monotone nondecreasing in x_t for each (s, t) and*

$$x(t) \le g_0(t) + \int_{t_0}^{t} F(t, s, x_s)ds \qquad\qquad t_0 \le t \le t_1 \quad (2.14)$$

where $g_0 \in C([t_0, t_1]; \mathbf{R}^n)$, $x \in C([t_0 - \tau_{max}, t_1]; \mathbf{R}^n)$. Assume that $r(t)$ is a maximal solution of

$$m(t) = g_0(t) + \int_{t_0}^{t} F(t, s, m_s)ds \qquad\qquad (2.15)$$

existing on $[t_0, \infty)$. Then

$$x(t) \le r(t) \qquad\qquad \text{on } [t_0, t_1]. \quad (2.16)$$

Proof: Define

$$K(t, s, y_t) \; = \; F(t, s, \max[y, x]_t) \, . \qquad (2.17)$$

Equation (2.13) implies that $x_{t\check{s}} \leq \max[y, x]_t$ for any function y. From the monotonicity of F and (2.17), it therefore follows that

$$K(t, s, y_t) \geq F(t, s, x_t) \qquad \textit{for any function } y_t \, . \qquad (2.18)$$

Let $r^*(t)$ be the maximal solution of

$$m(t) \; = \; g_0(t) + \int_{t_0}^{t} K(t, s, m_s) ds$$

existing on $[t_0, t_1]$ such that

$$r^*(t) \; = \; g_0(t) + \int_{t_0}^{t} K(t, s, r^*_s) ds \, .$$

From (2.18) we conclude that

$$r^*(t) \geq g_0(t) + \int_{t_0}^{t} F(t, s, x_s) ds \, .$$

Applying (2.14) we obtain

$$r^*(t) \geq x(t) \, . \qquad (2.19)$$

To complete the proof we have to show that $r^*(t)$ is also a maximal solution of (2.15). It results from (2.19) and (2.13)

$$\max[r^*(t), x(t)] \; = \; r^*(t)$$

$$\max[r^*(t), x(t)]_t \; = \; r^*{}_t$$

and consequently, due to (2.17),

$$K(t, s, r^*{}_t) \; = \; F(t, s, r^*{}_t) \, .$$

Thus $r^*(t)$ is also the maximal solution of (2.15). Hence (2.19) proves the desired result (2.16). ❏

Corollary 2.1: *Let* $f(t, x_s) \in C([J \times \mathfrak{I}^n]; \boldsymbol{R}^n)$ *be monotone nondecreasing in* x_t *for each t and*

$$x(t) \le x_0 + \int_{t_0}^{t} f(s, x_s) ds$$

where $x_0 \in \boldsymbol{R}^n$, $x \in C([t_0 - \tau_{max}, t_1]; \boldsymbol{R}^n)$. *Assume that* $r(t)$ *is a maximal solution of*

$$m(t) = x_0 + \int_{t_0}^{t} f(s, m_s) ds \tag{2.20}$$

existing on $[t_0, \infty)$. *Then*

$$x(t) \le r(t) \qquad \text{on } [t_0, t_1]\,. \tag{2.21}$$

2.4 A transformation for time-delay systems

The transformation presented here converts a multiple delay system into a single delay system, leaving the trajectory invariant. Using this transformation, the stability and controllability criteria derived for single delay systems can at once be extended to multiple delay systems. The relation between the transformed and the original systems concerning system properties is analysed in Sections 3.3 and 4.1. In this section, both the construction of the transformation and sufficient conditions for the transformed system to have the same trajectory as the original system are studied. The time-delay system under consideration is described by a linear differential difference equation and a static output equation:

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^{k} A_ix(t-\tau_i) + B_0u(t) \qquad t \ge t_0 \qquad (2.22)$$

$$y(t) = C_0x(t) \qquad t \ge t_0 \qquad (2.23)$$

$$x(t) = \varphi(t) \qquad t_0 - \tau_k \le t \le t_0 \qquad (2.24)$$

where $0 < \tau_1 < \ldots < \tau_k$, $A_0, A_i \in \boldsymbol{R}^{n \times n}$, $x(t) \in \boldsymbol{R}^n$, $B_0 \in \boldsymbol{R}^{n \times m}$, $u(t) \in \boldsymbol{R}^m$, $C_0 \in \boldsymbol{R}^{p \times n}$, $y(t) \in \boldsymbol{R}^p$. We denote the natural numbers by $\boldsymbol{N}$, the rational numbers by $\boldsymbol{Q}$, and the positive real numbers by $\boldsymbol{R}^+$. Under the assumption that the system (2.22) is commensurate, the delays may be represented uniquely in the factorization

$$\tau_i = \frac{c_i}{d_i}v$$

where $c_i, d_i \in \boldsymbol{N}$ and $v \in \boldsymbol{R}^+ \setminus \boldsymbol{Q} \cup \{1\}$. In order to generate the transformation such that the number of state variables of the transformed system is minimal, we look for the maximal value of the delay constant τ^* which is derived from the equations

$$\begin{aligned} \tau_1 &= l_1\tau^* \\ \tau_2 &= l_2\tau^* \\ &\vdots \\ \tau_k &= l_k\tau^* \end{aligned} \qquad (2.25)$$

where $l_i \in \boldsymbol{N}$ and $\tau^* \in \boldsymbol{R}^+$. The tuple $\{l_1, ..., l_k, \tau^*\}$ determined by the equations (2.25) can be derived with the following Lemma.

Lemma 2.1: *The delay constant $\tau^* \in \boldsymbol{R}^+$ is maximal under consideration of the property $l_i \in \boldsymbol{N}$, iff relation (2.26) holds*

$$\tau^* = \frac{gcd\{c_i\}}{lcm\{d_i\}}v \ . \qquad (2.26)$$

Proof: First we have to check whether $\tau^* \in \boldsymbol{R}^+$ satisfies the equations (2.26). For any τ_i with $\breve{c} = gcd\{c_i\}$ and $\tilde{d} = lcm\{d_i\}$ we may write

$$\tau_i = \frac{c_i}{d_i} v = \frac{c_i \tilde{d} \breve{c}}{d_i \breve{c} \tilde{d}} v$$

$$l_i = \frac{c_i \tilde{d}}{\breve{c} d_i} \quad \Rightarrow \quad l_i \in \boldsymbol{N} .$$

Any combination $(\breve{c}', \tilde{d}')$ where

$$\breve{c}' > \breve{c} \qquad \tilde{d}' < \tilde{d}$$

destroys the property of l_i being an element of $\boldsymbol{N}$. ❑

With the following transformation the multiple delay system (2.22) – (2.24) is related to a single delay system with the same trajectory. Let us consider the maximal delay constant τ^*. Then the new state, control, and output vectors, $\bar{x}(t) \in \boldsymbol{R}^{l_k \cdot n}$, $\bar{u}(t) \in \boldsymbol{R}^{l_k \cdot m}$, and $\bar{y}(t) \in \boldsymbol{R}^{l_k \cdot p}$, respectively, are given by

$$\bar{x}_j(t) = x(t-(j-1)\tau^*)$$

$$\bar{x}(t) = \begin{bmatrix} \bar{x}_1(t) \\ : \\ \bar{x}_{l_k}(t) \end{bmatrix} \qquad t \geq t_0 \tag{2.27}$$

$$\bar{u}_j(t) = u(t-(j-1)\tau^*)$$

$$\bar{u}(t) = \begin{bmatrix} \bar{u}_1(t) \\ : \\ \bar{u}_{l_k}(t) \end{bmatrix} \qquad t \geq t_0 \tag{2.28}$$

$$\bar{y}_j(t) = y(t-(j-1)\tau^*),$$

$$\bar{y}(t) = \begin{bmatrix} \bar{y}_1(t) \\ : \\ \bar{y}_{l_k}(t) \end{bmatrix} \qquad t \geq t_0 \qquad (2.29)$$

where $j \in \{1, 2, \ldots, l_k\}$. Furthermore, we define $\bar{\varphi}(t) \in \boldsymbol{R}^{l_k \cdot n}$ and $\bar{f}(t) \in \boldsymbol{R}^{l_k \cdot n}$ to be

$$\bar{f}_j(t) = f(t-(j-1)\tau^*)$$

$$\bar{f}(t) = \begin{bmatrix} \bar{f}_1(t) \\ : \\ \bar{f}_{l_k}(t) \end{bmatrix} \qquad t \geq t_0 \qquad (2.30)$$

$$\bar{\varphi}_j(t) = \varphi(t-(j-1)\tau^*),$$

$$\bar{\varphi}(t) = \begin{bmatrix} \bar{\varphi}_1(t) \\ : \\ \bar{\varphi}_{l_k}(t) \end{bmatrix} \qquad t_0 - \tau_k \leq t \leq t_0 \qquad (2.31)$$

where the function f remains to be defined. The transformed system is then of the form

$$\dot{\bar{x}}(t) = \bar{A}_0\bar{x}(t) + \bar{A}_1\bar{x}(t-\tau_k) + \bar{B}_0\bar{u}(t) + \bar{f}(t) \qquad t \geq t_0 \qquad (2.32)$$

$$\bar{y}(t) = \bar{C}_0\bar{x}(t) \qquad t \geq t_0 \qquad (2.33)$$

$$\bar{x}(t) = \bar{\varphi}(t) \qquad t_0 - \tau_k \leq t \leq t_0 \qquad (2.34)$$

where the system matrices $\bar{A}_0, \bar{A}_1 \in \boldsymbol{R}^{l_k \cdot n \times l_k \cdot n}$, $\bar{B}_0 \in \boldsymbol{R}^{l_k \cdot n \times l_k \cdot m}$, and $\bar{C}_0 \in \boldsymbol{R}^{l_k \cdot p \times l_k \cdot n}$ have the following structures

$$\bar{A}_0 = \begin{bmatrix} A_0 & \cdot & \cdot & A_1 & \cdot & \cdot & A_{k-1} & & & & \\ & A_0 & \cdot & \cdot & A_1 & \cdot & \cdot & A_{k-1} & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & A_{k-1} \\ & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & & & \cdot & \cdot & \cdot & A_1 \\ & & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & & & \cdot & \cdot \\ \mathbf{0} & & & & & & & & & & A_0 \end{bmatrix} \tag{2.35}$$

$$\bar{A}_1 = \begin{bmatrix} A_k & & & & & & & & & & \mathbf{0} \\ \cdot & A_k & & & & & & & & & \\ \cdot & \cdot & \cdot & & & & & & & & \\ A_{k-1} & \cdot & \cdot & \cdot & & & & & & & \\ \cdot & A_{k-1} & \cdot & \cdot & \cdot & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\ A_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\ & A_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & A_1 & \cdot & \cdot & A_{k-1} & \cdot & \cdot & A_k \end{bmatrix} \tag{2.36}$$

$$\bar{B}_0 = \mathrm{diag}[B_0, \ldots, B_0] \tag{2.37}$$

$$\bar{C}_0 = \mathrm{diag}[C_0, \ldots, C_0]\ . \tag{2.38}$$

Because of (2.26) and (2.27), A_i (for $i \in \{1, 2, \ldots, k-1\}$) builds the l_i upper secondary block diagonal of $\bar{A}_0$ and the $l_k - l_i$ lower secondary block diagonal of $\bar{A}_1$. All other secondary block diagonals are zero. Furthermore, let us call the vector-valued function $\bar{f}(t)$ an initial control. This approach enables us to perform the transformation without restrictive assumptions on the original system. In the following, the symbol Δ is defined by $\Delta = \tau_k - \tau^*$.

Theorem 2.2: *The system (2.22) – (2.24) with multiple delays in state and with a given vector-valued initial function* $\varphi \in L^2([t_0 - \tau_k, t_0]; \boldsymbol{R}^n) \cap C^1([t_0 - \Delta, t_0]; \boldsymbol{R}^n)$ *can be represented by an equivalent single delay system (2.32) - (2.34) with the same trajectory if* $\bar{f}(t)$ *is chosen as follows*

$$\bar{f}_1(t) = 0 \qquad\qquad t \ge t_0$$

$$\bar{f}_2(t) = \begin{cases} f(t-\tau^*) & t_0 \le t < t_0 + \tau^* \\ 0 & t \ge t_0 + \tau^* \end{cases}$$

$$\vdots \tag{2.39}$$

$$\bar{f}_{l_k}(t) = \begin{cases} f(t-(l_k-1)\tau^*) & t_0 \le t < t_0 + \Delta \\ 0 & t \ge t_0 + \Delta \end{cases}$$

where

$$f(t) = \dot{\varphi}(t) - A_0\varphi(t) - \sum_{i=1}^{k} A_i\varphi(t-\tau_i) - B_0u(t) \tag{2.40}$$

for $t_0 - \Delta \le t < t_0$.

Proof: We first deal with the vector-valued initial function $\bar{\varphi}(t)$. Equation (2.31) requires the knowledge of $\varphi(t)$ on the interval $[t_0 - \tau_k - \Delta, t_0]$. Because of equation (2.24) the initial function $\varphi(t)$ is given only on $[t_0 - \tau_k, t_0]$. We extend $\varphi(t)$ to the interval $[t_0 - \tau_k - \Delta, t_0]$ such that $\varphi \in L^2([t_0 - \tau_k - \Delta, t_0]; \boldsymbol{R}^n) \cap C_1([t_0 - \Delta, t_0]; \boldsymbol{R}^n)$. Once $\varphi(t)$ is chosen on the prolonged interval, the function $\bar{\varphi} \in L^2([t_0 - \tau_k, t_0]; \boldsymbol{R}^{l_k \cdot n})$ is given by (2.31). For $\bar{u}(t)$ we proceed in an analogous way: We choose $u(t)$ arbitrarily on $[t_0 - \Delta, t_0]$ such that $u \in L^2([t_0 - \Delta, t_1]; \boldsymbol{R}^m)$. The control $\bar{u} \in L^2([t_0, t_1]; \boldsymbol{R}^{l_k \cdot m})$ is then determined by equation (2.28). Assuming the two functions $\bar{\varphi}(t)$ and $\bar{u}(t)$ to be fixed, the initial control function $\bar{f}(t)$ is fully determined. For $\bar{x}_1(t)$ with $t \ge t_0$

we have

$$\dot{\bar{x}}_1(t) = \dot{x}(t) = A_0x(t) + \sum_{i=1}^{k} A_ix(t-\tau_i) + B_0u(t) + f(t) \quad , \qquad (2.41)$$

$$f(t) = \dot{x}(t) - A_0x(t) - \sum_{i=1}^{k} A_ix(t-\tau_i) - B_0u(t) = 0 \quad . \qquad (2.42)$$

For $t \geq t_0$, f(t) vanishes. For the computation of $\bar{f}(t)$ we have to find f(t) on the interval $[t_0 - \Delta, t_0]$ using (2.40). Equations (2.30) and (2.42) yield (2.39). From the equations (2.31), (2.28), and (2.39) it follows that $\bar{\varphi} \in L^2([t_0 - \tau_k, t_0]; \boldsymbol{R}^{l_k \cdot n})$, $\bar{u} \in L^2([t_0, t_1]; \boldsymbol{R}^{l_k \cdot m})$, and $\bar{f} \in L^2([t_0, t_0 + \Delta]; \boldsymbol{R}^{l_k \cdot m})$. Therefore, it is obvious that the system (2.32) – (2.34) admits a unique solution $\bar{x} \in L^2([t_0 - \tau_k, t_0]; \boldsymbol{R}^{l_k \cdot n}) \cap W^{1,2}([t_0, t_1]; \boldsymbol{R}^{l_k \cdot n})$ [43]. ❑

Equation (2.39) shows that $\bar{f}(t)$ influences the transformed system only for $t \in [t_0, t_0 + \Delta]$. The initial control $\bar{f}(t)$ corrects the influence of the arbitrary extensions of φ(t) and u(t).

Example 2.3: The original system is given by

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-1) + A_2x(t-2) + A_3x(t-3) \\ &\quad + B_0u(t) && t \geq 0 \\ y(t) &= C_0x(t) && t \geq 0 \\ \varphi(t) &\equiv [1, -1]^T && 0 \geq t \geq -3 \\ u(t) &\equiv [2, -2]^T && t \geq 0 \end{aligned}$$

where

$$A_0 = \begin{bmatrix} -5 & 1 \\ -3 & -7 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \quad C_0 = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

The corresponding transformed system is described by

$$\dot{\bar{x}}(t) = \begin{bmatrix} A_0 & A_1 & A_2 \\ & A_0 & A_1 \\ & & A_0 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} A_3 & & \\ A_2 & A_3 & \\ A_1 & A_2 & A_3 \end{bmatrix} \bar{x}(t-3) +$$

$$+ \begin{bmatrix} B_0 & & \\ & B_0 & \\ & & B_0 \end{bmatrix} \bar{u}(t) + \bar{f}(t) \qquad (2.43)$$

$$\bar{y}(t) = \begin{bmatrix} C_0 & & \\ & C_0 & \\ & & C_0 \end{bmatrix} \bar{x}(t) \qquad (2.44)$$

The extension of φ(t) is chosen to be $\varphi(t) \equiv [0, 0]^T$ for all $t \in [-5, -3)$ and $u(t) \equiv [0, 0]^T$ for all $t \in [-2, 0)$. $\bar{\varphi}(t)$ and $\bar{u}(t)$ are calculated by (2.31) and (2.28). Equation (2.40) then yields

$$f(t) \equiv \begin{cases} [0,\ 0]^T & t \geq 0 \\ [7, -1]^T & -1 \leq t < 0 \\ [7, -2]^T & -2 \leq t < -1. \end{cases}$$

We then construct $\bar{f}(t)$ with (2.39)

$$\bar{f}_2(t) \equiv \begin{cases} [7, -1]^T & 0 \leq t < 1 \\ [0,\ 0]^T & t \geq 1 \end{cases}$$

$$\bar{f}_3(t) \equiv \begin{cases} [7, -2]^T & 0 \leq t < 1 \\ [7, -1]^T & 1 \leq t < 2 \\ [0,\ 0]^T & t \geq 2\ . \end{cases}$$

The solution $\bar{y}(t) = [\bar{y}_1(t), \bar{y}_2(t), \bar{y}_3(t)]^T$ to (2.43), (2.44) computed with the variable Kutta-Merson integration method is shown in Fig. 2.2.

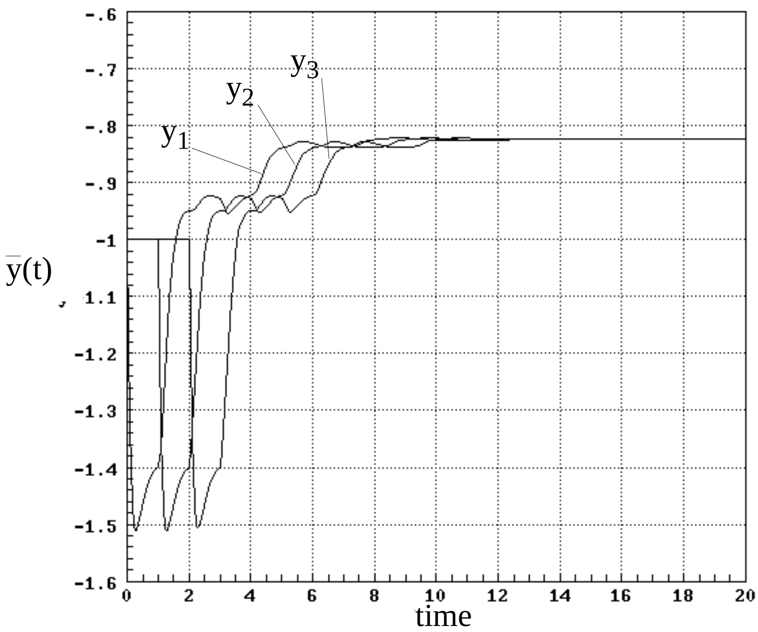

Fig. 2.2 Solution of the system (2.43) – (2.44)

Next, two cases are shown in which the introduction of an initial control $\bar{f}(t)$ can be neglected.

I) In the first case, we choose u(t) such that $u \in L^2([t_0 - \Delta, t_1]; \boldsymbol{R}^m)$. We assume A_k to be regular and $\varphi \in L^2([t_0 - \tau_k, t_0]; \boldsymbol{R}^n) \cap C^1([t_0 - \Delta, t_0]; \boldsymbol{R}^n)$. In order to achieve our goal, we calculate φ(t) on the extended interval $[t_0 - \tau_k - \Delta, t_0 - \Delta]$ by backward continuation. Under the above assumptions, the solution exists and is unique [43, Section 2.5]. We realize the backward continuation with the help of the inverse of the method of steps as follows.

Corollary 2.2: *If A_k is regular, $\varphi \in L^2([t_0 - \tau_k, t_0]; \boldsymbol{R}^n) \cap C^1([t_0 - \Delta, t_0]; \boldsymbol{R}^n)$, and the extension of φ(t) is chosen to be*

$$\varphi(t) = A_k^{-1}[\dot{\varphi}(t + \tau_k) - A_0\varphi(t + \tau_k) + - \sum_{i=1}^{k-1} A_i\varphi(t + \tau_k - \tau_i) - B_0u(t + \tau_k)] \tag{2.45}$$

for $t_0 - \Delta - j\tau^* \le t < t_0 - \Delta - (j-1)\tau^*$, $j \in \{2, \ldots, l_k\}$, *then* $\bar{f}(t)$ *vanishes.*

Proof: Consider $\bar{f}_j(t) = f(t-(j-1)\tau^*)$ where $t_0 \le t < t_0 + (j-1)\tau^*$. In order to show that $f(t-(j-1)\tau^*)$ vanishes, $f(t-(j-1)\tau^*)$ is expressed in terms of (2.40) where for $\varphi(t)$ relation (2.45) holds. ❑

The relation (2.45) is recursive, except if $\tau_{k-1} = \tau_1 = \tau^*$. Therefore, the solution is found on successive intervals. From the extended $\varphi(t)$, $\bar{\varphi}(t)$ follows immediately by equation (2.31). In the scalar case of (2.22) – (2.24), it is always possible to omit the initial control.

Example 2.4: For the system defined by

$$\dot{x}(t) = -x(t) + x(t-2) - x(t-3) + 2x(t-5) \qquad t \ge 0$$

$$\varphi(t) = -0.5t + 2 \qquad -5 \le t < 0$$

the extension of $\varphi(t)$ is calculated by applying (2.45)

$$\varphi(t) = -0.25t - 0.25 \qquad -7 \le t < -5$$

$$\varphi(t) = -0.125t - 1.125 \qquad -8 \le t < -7$$

$$\varphi(t) = -0.25t - 0.375 \qquad -9 \le t < -8.$$

The differential difference equation of the corresponding transformed system is described by

$$\dot{\bar{x}}(t) = \begin{bmatrix} -1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 & 2 \end{bmatrix} \bar{x}(t-5) \qquad t \ge 0 \quad (2.46)$$

$$\bar{x}(t) = \bar{\varphi}(t) \qquad -5 \le t < 0. \quad (2.47)$$

The construction of $\bar{\varphi}(t) \in \boldsymbol{R}^5$ and the solution of the corresponding transformed system is shown in Fig. 2.3.

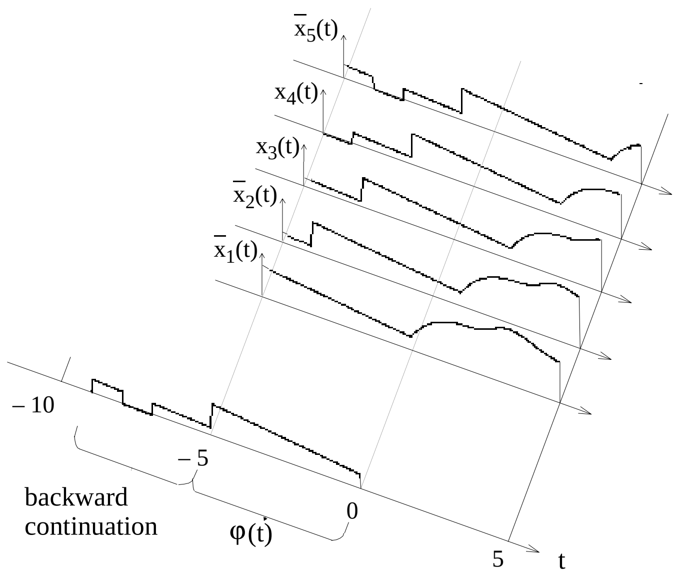

Fig. 2.3 $\bar{x}(t)$ *for all* $t \in [-5, 5]$ *for the system (2.46) – (2.47)*

II) In the second case, we choose $\varphi(t)$ on the extended interval $[t_0 - \tau_k - \Delta, t_0 - \tau_k]$ such that φ belongs to $L^2([t_0 - \tau_k - \Delta, t_0]; \boldsymbol{R}^n) \cap C^1([t_0 - \Delta, t_0]; \boldsymbol{R}^n)$.

Corollary 2.3: *If* $m = n$, B_0 *is regular,* $\varphi \in L^2([t_0 - \tau_k - \Delta, t_0]; \boldsymbol{R}^n) \cap C^1([t_0 - \Delta, t_0]; \boldsymbol{R}^n)$, *and the extension of* $u(t)$ *is chosen to be*

$$u(t) \;=\; B_0^{-1}\,[\,\dot{\varphi}(t) - A_0\varphi(t) - \sum_{i=1}^{k} A_i\varphi(t-\tau_i)\,] \quad , \tag{2.48}$$

$t_0 - \Delta \le t < t_0$, *then* $\bar{f}(t)$ *vanishes.*

Proof: The Corollary can simply be proved by replacing u(t) in (2.40) with the help of (2.48). ❑

Structural Properties

3 Stability

Roughly speaking, the stability of a system is its ability to resist any unknown small influences. Since in reality disturbances are always encountered, stability is an important property of any control system, delayed or nondelayed. The following Cauchy problem is considered:

$$\dot{x}(t) = f(t, x_t) \qquad t \geq t_0 \qquad (3.1)$$

$$x(t) = x_0 \qquad t = t_0 \qquad (3.2)$$

$$x(t) = \varphi(t) \qquad t_0 - \tau_{max} \leq t < t_0 . \qquad (3.3)$$

We assume in this and all further chapters that f is bounded and completely continuous (i.e., f is continuous and maps bounded sets in C into bounded sets in $\boldsymbol{R}^n$), and that it is regular such that for any initial state $\varphi \in C$ there exists a unique solution $x(x_0, \varphi)$. Furthermore, it is assumed that (x_0, φ) is bounded and the initial function φ is continuous.

Definition 3.1: *A constant function ϕ_e is called an equilibrium state if $f(t, \phi_e) = 0$ for all $t \geq t_0$.*

Even in the linear case of (3.1) the system may have more than one equilibrium state in general. But the stability analysis of any equilibrium ϕ may be reduced to the analysis of the zero equilibrium by the substitution $z(t) = x(t) - \phi_e$, $\psi(\theta) = \varphi(\theta) - \phi_e$. For z we obtain

$$\begin{array}{lcll} \dot{z}(t) & = & f(t, z_t + \phi_e) & t \geq t_0 \\ z(t_0) & = & 0 & t = t_0 \\ z(t) & = & \psi(t) & t_0 - \tau_{max} \leq t < t_0. \end{array}$$

Therefore, it is no restriction if we assume in the future that $f(t, 0) = 0$.

Definition 3.2: *The equilibrium state $\phi_e = 0$ is stable in the Lyapunov sense, if for any positive numbers t_0 and ε there exists a $\delta(\varepsilon, t_0) > 0$ such that every continuous solution of (3.1) which satisfies*

$$max\,|x(t)| \leq \delta(\varepsilon, t_0) \qquad t_0 \leq t \leq t_0 + \tau_{max}$$

will also satisfy

$$max\,|x(t)| \leq \varepsilon \qquad t_0 \leq t \leq \infty\,.$$

Definition 3.3: *The stable equilibrium state $\phi_e = 0$ is asymptotically stable if every continuous solution of (3.1) also satisfies $\lim\limits_{t \to \infty} x(t) \to 0$.*

In the definitions given above the number δ depends on both t_0 and ε. If a $\delta > 0$ can be found independent of t_0 the solution ϕ_e of (3.1) is designated as uniformly stable or uniformly asymptotically stable, respectively.

Definition 3.3 can be simplified if equation (3.1) is linear in x and $(x_0, \varphi) \in M^2 = \boldsymbol{R}^n \times L^2([t_0 - \tau_{max}, t_0]; \boldsymbol{R}^n)$. M^2 is a Hilbert space with the inner product $\langle (x_0, \varphi), (y_0, \psi) \rangle_{\boldsymbol{M}^2} = \langle x_0, y_0 \rangle_{\boldsymbol{R}^n} + \langle \varphi, \psi \rangle_{\boldsymbol{L}^2}$. Hence the induced norm is given by [39, p. 98]:

$$\|x_0, \varphi\|_{M^2} = \left[|x_0|^2 + \int_{t_0 - \tau_{max}}^{t_0} |\varphi(\theta)|^2 d\theta \right]^{1/2} .$$

The following conditions are equivalent and imply that the system is asymptotically stable if equation (3.1) is linear in x and $(x_0, \varphi) \in M^2$ [39, p. 99]

i) $\lim\limits_{t \to \infty} x(t) \to 0$

ii) $$\int_0^\infty |x(t)|^2 dt < \infty$$

iii) $$|x(t)| \le \beta e^{-\alpha t} \|(x_0, \varphi)\|_{M^2} \quad \alpha, \beta = \text{const} \in [0, \infty).$$

In the next section, we will introduce three basic tools to analyse the stability of retarded systems.

3.1 Stability concepts

In this section, the method of Lyapunov functionals and the method of Razumikhin are briefly introduced. Furthermore, we consider the characteristic equation to determine the stability of the solution for linear time-delay systems.

I) Lyapunov's direct method: From the stability of ordinary differential equations, the efficiency of Lyapunov's direct method (or second method) to analyse stability problems is well known. (Lyapunov's first method provided that an explicit solution of the considered differential equation is known.) Krasovskii [69] was the first who generalized this method to RFDE's. Since to each solution of an RFDE there is an integral curve in the space $\boldsymbol{R}^n \times C$ (see Sections 2.1 and 2.2), it is a natural generalisation to use Lyapunov functionals in this space instead of Lyapunov functions. These functionals are often called Lyapunov-Krasovskii functionals.

Suppose $V(t, x_t)$: $\boldsymbol{R} \times C \to \boldsymbol{R}$ is continuous function and $x(t)$ is the solution of (3.1). The function $\dot{V}(t, x_t)$ is the upper right-hand derivative of $V(t, x_t)$ along the solution of (3.1) (see, e.g., [66, p. 100], [43, p. 105]). We denote by $w_i(r)$ for $r \ge 0$ some continuous nondecreasing functions such that $w_i(0) = 0$ and $w_i(r) > 0$ for $r > 0$.

Theorem 3.1 [43, p. 105]: *Let there exist a continuous functional* $V(t, x_t)$: $\boldsymbol{R} \times C \rightarrow \boldsymbol{R}$ *such that*

$$w_1(|x|) \leq V(t, x_t) \leq w_2(|x_t|) \tag{3.4}$$

and

$$\dot{V}(t, x_t) \leq -w_3(|x|) \ . \tag{3.5}$$

Then the trivial solution of (3.1) is uniformly asymptotically stable.

It is an interesting fact that if the trivial solution of (3.1) is asymptotically stable, then there exists a continuous functional $V(t, x_t)$ satisfying (3.4) and (3.5) [69, Theorem 5.3].

II) Razumikhin's method: The idea of the Razumikhin-type theorem is to treat the stability problem with functions rather than with functionals. In the beginning of his research, Razumikhin (1958) considered the single delay system $\dot{x}(t) = f(x(t), x(t-\tau))$ and investigated the stability problem on the basis of first approximations. He demonstrated that the zero solution of this system is asymptotically stable if a positive-definite function V(t, x) has a negative-definite derivative along the solution of (3.1) with the additional condition $V(t-\tau, x(t-\tau)) < V(t, x(t))$. In the late seventies, Hale [43] presented a stronger version of the Razumikhin-type theorem. We read on page 126 in [43]: "A few moments of reflection in the proper direction indicate that it is unnecessary to require that $\dot{V}$ be nonpositive for all initial data in order to have stability. In fact, if a solution of an RFDE begins in a ball and is to leave this ball at some time t, then $|x_t| = |x(t)|$; that is $|x(t+\theta)| = |x(t)|$ for all $\theta \in [-\tau_{max}, 0]$. Consequently, one needs only consider initial data satisfying this latter property."

Theorem 3.2 [43, p. 127]: *Let there exist a continuous function* $V(t, x)$: $\boldsymbol{R} \times \boldsymbol{R}^n \rightarrow \boldsymbol{R}$ *such that*

$$w_1(|x|) \leq V(t, x) \leq w_2(|x|) \tag{3.6}$$

and

$$\dot{V}(t, x) \le -w_3(|x|) \ . \tag{3.7}$$

If

$$V(t + \theta, x(t + \theta)) \le w_4(V(t, x(t))) \tag{3.8}$$

for $\theta \in [-\tau_{max}, 0]$ *and* $w_1(s) \to \infty$ *as* $s \to \infty$*, then the trivial solution of (3.1) is uniformly asymptotically stable.*

III) Characteristic equation: The following linear autonomous system is a special case of equation (3.1), but it is important in control theory:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - \tau_i) + \int_{-h}^{0} A_{01}(\theta) x(t + \theta) d\theta \tag{3.9}$$

where $0 < \tau_1 < \ldots < \tau_k < \infty$, $h \in [0, \infty)$, A_0, $A_i \in \boldsymbol{R}^{n \times n}$. It is assumed that the elements of the function matrix $A_{01}(\theta)$ are continuous and bounded. The so-called characteristic equation for the linear hereditary differential equation (3.9) is given by

$$\det[\Delta(s)] = \det\left[sI - A_0 - \sum_{i=1}^{k} A_i e^{-s\tau_i} - \int_{-h}^{0} A_{01}(\theta) e^{s\theta} d\theta \right] . \tag{3.10}$$

The function $\det[\Delta(s)]$ is sometimes called the characteristic quasipolynomial. In the following, *Re*(s) designates the real part of s, and $\mathbb{C}$ is the set of complex numbers.

Theorem 3.3 [40, p. 54 and p. 132]*: System (3.9) is uniformly asymptotically stable iff*

$$Re(s) < 0 \tag{3.11}$$

for all $s \in \mathbb{C}$ *satisfying* $\det[\Delta(s)] = 0$.

Note that for *linear* neutral systems the ‘natural’ corresponding extension of Theorem 3.3 is not true in general. In [29] an example is given of a linear unstable neutral differential difference equation whose spectrum lies in the left half plane.

Calculating the exact value of the roots of the characteristic equation is possible only in very special cases. An approximate computation of the eigenvalues of the system (3.9), and some properties of the eigenvalues will be discussed in Section 3.4.

3.2 Stability tests

The tools to investigate the stability of RFDEs introduced in the last section are applied in the following to establish simple, algebraic stability conditions. The search for easily applicable stability tests has become a popular field of research over the last number of years. These stability criteria are classified into two categories. The stability criteria which do not need any information about the delay are called delay-independent criteria, while those which exploit information about the delays involved are called delay-dependent criteria. In [18], [49], [50], [57], [59], [60], [82], [145], [155] (to mention a few), a further classification is stated: the so-called stability criteria independent of delay, abbreviated to i.o.d. stability criteria. (These criteria are valid only for systems with constant lags.) The expression i.o.d. might appear to be equivalent to the term delay-independent, but it is not. For example in [57] the exact i.o.d. stability condition for the system $\dot{x}(t) = a_0 x(t) + a_1 x(t-h) + a_2 x(t-2h)$ is derived. However, this condition is not valid for the system $\dot{x}(t) = a_0 x(t) + a_1 x(t-h) + a_2 x(t-3h)$. The i.o.d. criteria discussed in Subsection 3.2.3 below do not need any information on the delay constant h. As soon as the system has further delays, the i.o.d. stability criteria depend on the ratio of the delays. This of course implies that the ratio of the delays has to be known exactly! (In the systems mentioned above the ratios of the delays are 1:2 and 1:3, respectively.) More accurate than the term “i.o.d.” would thus be the expression independent of a delay constant. We will use only the latter terminology. From now on, we say that a stability criterion is delay-independent or independent of delay if, for a system with multiple constant delays, the ratio of the delays does not have to be known for this criterion to be applicable. Furthermore, we distinguish whether it is assumed that the delays are constant or not. To sum up, we distinguish between the following four types of stability tests:

- delay-independent or independent of delay stability criteria (delays may be constant or variable)
- stability criteria independent of constant delays
- stability criteria independent of a delay constant
- delay-dependent stability criteria.

Of course, necessary and sufficient conditions for asymptotic stability criteria will be delay-dependent. Consequently, there is a gap between delay-independent criteria and exact ones. Several examples of this gap are visualized in Subsection 3.3.1. Note that exact stability conditions in terms of the system coefficients are known only for ‘simple’ systems, while delay-independent stability conditions are derived for much more general systems. Delay-independent stability criteria are very useful, since in reality it is difficult to estimate the delays, especially if those delays are time-varying and/or state-dependent.

Before starting the following subsections, some notation is introduced:

$\lambda_i(A)$	Eigenvalue of the matrix $A \in \boldsymbol{R}^{n \times n}$
$\lambda_{max}(A)$	Eigenvalues with the largest real part of the matrix A
$\lambda_{min}(A)$	Eigenvalues with the smallest real part of the matrix A
$Re(\cdot)$	Real part of $(\cdot)$
$Im(\cdot)$	Imaginary part of $(\cdot)$

$|x|.$ vector norm: $$|x|_1 = \sum_{i=1}^{n} |x_i|$$

$$|x|_2 = \left[\sum_{i=1}^{n} |x_i|^2 \right]^{1/2}$$

$$|x|_\infty = \max_i |x_i|$$

$\|A\|.$ matrix norm: $$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^T A)}$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

$\mu(A).$ matrix measure: $$\mu(A)_1 = \max_j \left[\mathrm{Re}(a_{jj}) + \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \right]$$

$$\mu(A)_2 = 0.5 \cdot \lambda_{max}(A^T + A)$$

$$\mu(A)_\infty = \max_i \left[\mathrm{Re}(a_{ii}) + \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right]$$

3.2.1 Stability tests: Independent of delays

For the system $\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau)$; $A_0, A_1 \in \boldsymbol{R}^{n \times n}$ with a constant delay τ, Mori *et al.* (1981) presented the well-known stability criterion $\mu(A_0). + \|A_1\|. < 0$. Cheres *et al.* (1989) [25] indicated that this stability condition for . = 2 is also valid if the delay is time-dependent. Wang *et al.* (1991) [145] showed that this type of criterion for . = 2 can also be formulated for systems with multiple constant delays. Indeed, the statement can be generalized for the following system

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{k} A_i x(t - \tau_i(t, x(t))) & t \geq t_0 \\ x(t) &= \varphi(t) & t_0 - \tau_{max} \leq t \leq t_0 \end{aligned} \tag{3.12}$$

where A_0, $A_1 \in \boldsymbol{R}^{n \times n}$. It is assumed that the delays are continuous and bounded, satisfying the inequality $0 < \tau_i(t, x(t)) < \tau_{max}$. The first stability condition is established using comparison techniques, while the other stability criteria are derived with the help of the Razumikhin concept.

Theorem 3.4: *System (3.12) is asymptotically stable independent of delays (a.s.i.d) if the inequality*

$$\mu(A_0)_{.} + \sum_{i=1}^{k} \|A_i\|_{.} < 0 \tag{3.13}$$

is valid for any . = 1, 2, ∞ .

Proof: The solution of (3.12) can be represented as

$$x(t) = e^{A_0 t} x_0 + \int_{t_0}^{t} e^{A_0(t-s)} \left[\sum_{i=1}^{k} A_i x(s - \tau_i(s, x(s))) \right] ds \qquad t \geq t_0$$

where $x_0 = \varphi(0)$. Taking the norm on both sides of the equation yields

$$|x(t)|_{.} \leq \|e^{A_0 t}\|_{.} |x_0|_{.} + \int_{t_0}^{t} \|e^{A_0(t-s)}\|_{.} \left[\sum_{i=1}^{k} \|A_i\|_{.} |x(s - \tau_i(s, x(s)))|_{.} \right] ds \quad .$$

Now we use the inequality [27]: $\|e^{At}\|_{.} \leq e^{\mu(A)t}$ for $t \geq t_0$ and define $v(t) = |x(t)|_{.}$; $v_0 = |x(t_0)|$.

$$\begin{aligned} v(t) &\leq e^{\mu(A_0)t} v_0 + \int_{t_0}^{t} e^{\mu(A_0)(t-s)} \left[\sum_{i=1}^{k} \|A_i\|_{.} v(s - \tau_i(s, x(s))) \right] ds \; ; & t \geq t_0 \\ v(t) &= |\varphi(t)|_{.} & t_0 - \tau_{max} \leq t \leq t_0 \end{aligned}$$

Corresponding to the above inequality, the following integral equation is considered

$$z(t) \;=\; e^{\mu(A_0)t}\,v_0 + \int_{t_0}^{t} e^{\mu(A_0)(t-s)} \left[\sum_{i=1}^{k} \|A_i\|_{\cdot}\, z(s-\tau_i(s, x(s)))\right] ds \;\; ; \qquad t \geq t_0$$

$$z(t) \;=\; |\varphi(t)|_{\cdot} \qquad\qquad t_0 - \tau_{max} \leq t \leq t_0\,.$$

The variable z(t) is the solution of the following scalar differential difference equation

$$\dot{z}(t) = \;\mu_{\cdot}(A_0)\, z(t) + \sum_{i=1}^{k} \|A_i\|_{\cdot}\, z(t-\tau_i(t, x(t))) \qquad t \geq t_0 \tag{3.14}$$

$$z(t) = |\varphi(t)|. \qquad\qquad t_0 - \tau_{max} \leq t \leq t_0\,. \tag{3.15}$$

Using the comparison theorem (see Section 2.3) we obtain

$$|x(t)|_{\cdot} \;=\; v(t) \leq z(t)\,.$$

Obviously, asymptotic stability of system (3.14) implies that of system (3.12). It is known (cf. [43, p. 129]; or [2]) that the solution of the scalar differential difference equation of the form

$$\dot{x}(t) \;=\; a_0 x(t) + \sum_{i=1}^{k} a_i x(t-\tau_i(t, x(t)))$$

is asymptotically stable for all bounded continuous functions $\tau_i(t, x(t))$ if the inequality

$$a_0 + \sum_{i=1}^{k} |a_i| < 0$$

holds. Applying this result to (3.14) and (3.15), Theorem 3.4 follows. ❑

Theorem 3.5: *System (3.12) is a.s.i.d. if the inequality*

$$\mu(PA_0)_2 + \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}\sum_{i=1}^{k} \|PA_i\|_2 < 0 \tag{3.16}$$

holds for some symmetric, positive-definite matrix $P \in \boldsymbol{R}^{n\times n}$.

Proof: The Lyapunov-Razumikhin function is chosen to be of the quadratic form

$$V(x) = x(t)^T P x(t) \tag{3.17}$$

where the symmetric matrix P is positive-definite. Using the properties of the numerical range of P (also called Rayleigh quotient), it is always possible to find a suitable $w_1(|x(t)|)$ and $w_2(|x(t)|)$ to satisfy condition (3.4) of Theorem 3.2. An appropriate choice of w_1 and w_2 is as follows:

$$w_1(|x(t)|) = \lambda_{min}(P) x(t)^T x(t) \le V(x(t))$$

$$w_2(|x(t)|) = \lambda_{max}(P) x(t)^T x(t) \ge V(x(t)) .$$

Determining the derivative of (3.17) along the trajectory of (3.12) yields

$$\dot{V} = x(t)^T [A_0^T P + PA_0] x(t) + 2x(t)^T P \sum_{i=1}^{k} A_i x(t - \tau_i(t, x(t)) . \tag{3.18}$$

Next, we have to satisfy inequality (3.8). The nondecreasing function w_4 is chosen such that $w_4(V(x(t))) = V(x(t))$. Based on condition (3.8) equation (3.17) implies that

$$|x(t - \tau_i(t, x(t)))| < \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} |x(t)| . \tag{3.19}$$

Thus, (3.18) together with (3.19) yields

$$\dot{V} < x(t)^T [A_0^T P + PA_0] x(t) + 2 \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \sum_{i=1}^{k} \|PA_i\|_2 |x(t)|^2 .$$

From the above inequality, it follows that $\dot{V}$ is negative along the trajectory of (3.12) if condition (3.4) is valid, and thus the proof is complete. ❑

Corollary 3.1 [25]: *The system (3.12) is a.s.i.d. if the inequality*

$$\mu(PA_0)_2 \cdot \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}(P)^3}} + \sum_{i=1}^{k} \|A_i\|_2 < 0 \tag{3.20}$$

is valid for some symmetric, positive-definite matrix $P \in \boldsymbol{R}^{n \times n}$.

Proof: Using the relation $\|P\| \cdot \|A_i\| \geq \|PA_i\|$ we immediately obtain Corollary 3.1 from Theorem 3.5. Corollary 3.1 was first presented by Cheres *et al*. (1989) for a system with a single time-varying delay. ❑

Note that Theorem 3.5 for $P = I$ produces the same stability condition as Theorem 3.4 for $. = 2$.

A remarkable stability criterion of the a.s.i.d. type was derived by Ameniya (1989) using the M-matrix technique. The reason for not stating that particular result here is that it is valid only for a special class of single-delay systems.

A conclusion of the above stability condition is that the asymptotic stability of a time-delay system is guaranteed, if the eigenvalues of the composite matrix $A_0 + A_0^T$ lie behind a certain boundary in the left half plane. The question now is whether or not it is sufficient to require that the eigenvalues of A_0 lie behind a certain boundary in the left half plane to guarantee asymptotic stability of the system. Such a stability criterion would be highly applicable, since a number of the controller design methods for linear nondelayed systems could be extended to delay systems by means of a pool-placing method.

3.2.2 Stability tests: Independent of constant delays

The concept of Lyapunov functionals is used to treat the stability problem for time-delay systems. If the delays were state- and/or time-dependent the Lyapunov's principle would yield stability criteria which would require the complete knowledge of $\dot{\tau}_i(t, x(t))$, as was demonstrated in [129]. However, for the system with *constant* delays described below, Lyapunov's principle is useful.

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^{k} A_i x(t-\tau_i) \tag{3.21}$$

whereby $A_0, A_i \in \boldsymbol{R}^{n \times n}$ and $0 < \tau_1 < \ldots < \tau_k < \infty$.

Theorem 3.6: *System (3.21) is asymptotically stable if there exist symmetric, positive-definite matrices* $P_0, P_i \in \boldsymbol{R}^{n \times n}$ *such that*

$$A_0^T P_0 + P_0 A_0 + \sum_{i=1}^{k} P_i + P_0 A_i P_i^{-1} A_i^T P_0 < 0 \ . \tag{3.22}$$

Proof: Let $V(x_t)$ be a Lyapunov functional given by

$$V(x_t) \ = \ x(t)^T P_0 x(t) + \sum_{i=1}^{k} \int_{t-\tau_i}^{t} x(s)^T P_i x(s) ds \ . \tag{3.23}$$

Then, according to Theorem 3.1 the sufficient stability conditions for (3.21) are

a) $w_1(|x(t)|) \le V(x_t) \le w_2(|x_t|)$

b) $\dot{V}(x_t) \le -w_3(|x_t|)$.

Approach (3.23) admits to fulfill a) since the functions w_1 and w_2 can be chosen as follows

$$w_1(|x(t)|) = \lambda_{min}(P_0)\,|x(t)|^2 \le V(x_t)$$

$$V(x_t) \le [\lambda_{max}(P_0) + \sum_{i=1}^{k} \tau_i \lambda_{max}(P_i)]|x(t)|^2 = w_2(|x_t|) .$$

To show b) the proof proceeds as follows. Calculating the derivative of V using (3.21) yields

$$\begin{aligned} \dot{V} \quad = \quad & x(t)^T\left\{ A_0^T P_0 + P_0 A_0 + \sum_{i=1}^{k} P_i \right\} x(t) + \sum_{i=1}^{k} x(t)^T P_0 A_i x(t-\tau_i) \quad + \\ & + \sum_{i=1}^{k} x(t-\tau_i)^T A_i^T P_0 x(t) - \sum_{i=1}^{k} x(t-\tau_i)^T P_i x(t-\tau_i) \ . \end{aligned}$$

The right-hand side of the above equation is expressed below as a quadratic form. In order to ensure asymptotic stability of the system (3.21) this quadratic form has to be negative-definite.

$$0 > v(t)^T M v(t) \tag{3.24}$$

where $v(t)^T = [x(t)^T, x(t-\tau_1)^T, \ldots, x(t-\tau_k)^T]$ and

$$M = \begin{bmatrix} A_0^T P_0 + P_0 A_0 + \sum_{i=1}^{k} P_i & P_0 A_1 & . \;\; . \;\; . & P_0 A_k \\ A_1^T P_0 & -P_1 & & 0 \\ \vdots & & \ddots & \\ A_k^T P_0 & 0 & & -P_k \end{bmatrix}$$

Inequality (3.24) is rewritten in the following, using a basic theorem for symmetric partitioned matrices. Kreindler and Jameson (1972) showed that for the matrices U_{11}, U_{12}, and U_{22} with appropriate dimensions, the condition

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{12}^T & U_{22} \end{bmatrix} < 0$$

is equivalent to

$$U_{11} < 0 \qquad \text{and}$$

$$U_{11} - U_{12}U_{22}^{-1}U_{12}^{T} < 0\ .$$

Applying this fact to condition (3.24), we obtain

$$A_0^T P_0 + P_0 A_0 + \sum_{i=1}^{k} P_i < 0 \qquad \text{and} \tag{3.25}$$

$$A_0^T P_0 + P_0 A_0 + \sum_{i=1}^{k} P_i + P_0 A_i P_i^{-1} A_i^T P_0 < 0\ . \tag{3.26}$$

Only the latter relation is relevant, since it includes condition (3.25). Thus, there exists a positive constant c_3 such that

$$\dot{V}(x_t) \le -c_3(|x_t|)$$

if condition (3.26) is satisfied. This completes the proof. ❑

Corollary 3.2 [105]: *Assume that $A_0 < 0$. If the symmetric, positive-definite matrices P_0 and Q associated with the Lyapunov equation*

$$A_0^T P_0 + P_0 A_0 = -(k+1) \cdot Q \tag{3.27}$$

satisfy the inequality

$$-Q + \sum_{i=1}^{k} P_0 A_i Q^{-1} A_i^T P_0 < 0 \tag{3.28}$$

then system (3.21) is asymptotically stable.

Proof: If all P_i are replaced by Q in (3.22) and the term $A_0^T P_0 + P_0 A_0$ in (3.22) is replaced by $-(k+1)\cdot Q$ using (3.27), inequality (3.28) follows.❑

Corollary 3.3: *Assume that $\mu(A_0) < 0$, then the system (3.21) is asymptotically stable if the inequality*

$$\lambda_{max}\left[A_0 + A_0^T - 4k \cdot \sum_{i=1}^{k} A_i (A_0 + A_0^T)^{-1} A_i^T\right] < 0 \tag{3.29}$$

holds.

Proof: With the appropriate choices for P_0 and P_i:

$$P_0 = I \qquad \text{and} \qquad P_i = -\frac{1}{2k}(A_0^T + A_0) \tag{3.30}$$

Corollary 3.3 follows from Theorem 3.6. ❑

A discussion on the sharpness of the stability criteria presented follows later because first the influence of unstructured perturbations (Section 4.1) and certain methods to reduce the conservatism of the criteria (see Section 3.2.4) are introduced.

3.2.3 Stability tests: Independent of a delay constant

Given a delay differential system

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - ih) \tag{3.31}$$

with delays equal to integer multiples of a fixed delay constant $h \in [0, \infty)$. The characteristic equation of the system (3.31) is denoted in the following by $P(s, e^{-hs})$. From Theorem 3.3 it follows that the system is asymptotically stable independent of the delay constant h, iff

$$P(s, e^{-hs}) \neq 0, \qquad Re(s) \geq 0, \qquad \forall\, h \in [0, \infty)\,. \tag{3.32}$$

In this situation, two branches of development can be distinguished.

I) Kamen (1980) claimed that the condition (3.32) is equivalent to the two-variable criterion

$$P(s, z) \neq 0 , \qquad\qquad Re(s) \geq 0 , \qquad\qquad (3.33)$$

where $z = e^{j\omega}$, $\omega \in [0, 2\pi]$, $j = \sqrt{-1}$. Based on this work a number of papers were published. Jury and Mansour (1982) described a method to reduce the two-variable criterion to that of checking the positivity of a one-dimensional polynomial. They gave necessary and sufficient conditions of asymptotic stability independent of the delay constant h for several scalar systems. Lui and Mansour (1984) showed that under suitable conditions a system can be alternately stable; i.e. stable–unstable–stable ... as the delay h increases. Brierley *et al.* (1982) presented a stability criterion in terms of solutions of a complex Lyapunov matrix equation. They claimed that the system (3.31) is asymptotically stable independent of the delay constant h, iff for any positive-definite Hermitian matrix Q(z) the solution of the complex Lyapunov matrix equation

$$A(z)^T K(z) + K(z) A(z) = - Q(z) \quad \text{where } A(z) = \sum_{i=0}^{k} A_i z^i \qquad (3.34)$$

is also a positive-definite Hermitian matrix K(z) for all $\omega \in [0, 2\pi]$. Hmamed (1986) and Wang *et al.* (1991) [147] used this result of Brierley to derive a sufficient stability test for large-scale systems. (The stability of large-scale time-delay systems with uncertainties is discussed in Section 4.3.) Furthermore, Wang *et al.* (1991) [145] applied condition (3.34) to establish a sufficient stability criterion for the system (3.31):

$$\mu(A_0) + \mu\Big(\sum_{i=1}^{k} A_i z^i\Big) < 0 , \qquad \forall\ |z| = 1 .$$

An improved version of this condition was given by Hmamed (1991):

$$\mu\Big(A_0 + \sum_{i=1}^{k} A_i z^i\Big) < 0 \qquad \forall\ |z| = 1 . \qquad (3.35)$$

Using the properties of the matrix measure (see Appendix of [101]), it is easy to show that condition (3.35) is less restrictive than those given by Wang et al. (1991) [145] and Mori *et al.* (1981):

$$\mu(A_0 + \sum_{i=1}^{k} A_i z^i) \le \mu(A_0) + \mu(\sum_{i=1}^{k} A_i z^i) \le \mu(A_0) + \sum_{i=1}^{k} \|A_i\|_2 .$$

However, most of the authors mentioned above (and others) were not aware of the fact that Kamen (1983) corrected his own result: condition (3.32) is not equivalent to (3.33). The two-variable criterion (3.33) is equivalent to (3.32) plus the additional condition

$$P(0, z) \neq 0 , \qquad \forall\, |z| = 1 .$$

The consequences are that the stability criteria (3.33), (3.34), and those presented in [57], [82] are only sufficient rather than necessary and sufficient. Therefore, Boese (1989) made another attempt to find exact stability conditions independent of the delay constant h, in particular, for the system $\dot{x}(t) = a_0 x(t) + a_1 x(t-h) + a_2 x(t-2h)$. Astonishingly, he obtained the same necessary and sufficient condition as Kamen (1980) and Jury and Mansour (1982):

$$a_0 < \frac{a_1^2}{8a_2} + a_2 \qquad a_2 < 0 \tag{3.36}$$

or

$$a_0 + |a_1| + a_2 < 0 \qquad a_0 < 3a_2 . \tag{3.37}$$

Kamen (1983) showed that the system $\dot{x}(t) = -x(t) - x(t-h)$ is asymptotically stable independent of the delay constant h. Unfortunately, this system does not fulfill the conditions (3.36) and (3.37). Therefore, these conditions are only sufficient rather than necessary and sufficient, and hence the problem remains open.

II) Assume that the system (3.31) with h = 0 is asymptotically stable, then it is also asymptotically stable for sufficiently small values of h > 0 (cf. Sugiyama (1961)). Using this fact, Yoshizawa (1975), (cf. [40, p. 114]) showed that the system (3.31) is asymptotically stable independent of the delay constant h, iff the system

$$\dot{x}(t) = \left[A_0 + \sum_{i=1}^{k} A_i\right] x(t) \tag{3.38}$$

is asymptotically stable and the matrix

$$A_0 + \sum_{i=1}^{k} e^{j i \omega} A_i \tag{3.39}$$

has no nonzero eigenvalues on the imaginary axis for all $\omega \in [0, 2\pi]$. Several authors [119], [137], [138] proposed to simplify the calculation of the condition on the matrix (3.39) by a suitable auxiliary equation. Based on an attempt of Rekasius (1980), Thowsen in [137] showed that the system (3.31) is asymptotically stable independent of the delay-constant h, iff the system (3.38) is asymptotically stable and the matrix

$$sI - A_0 - \sum_{i=1}^{k} \left(\frac{1 - sT}{1 + sT}\right)^{2i} A_i \tag{3.40}$$

has no roots on the imaginary axis for some $T \geq 0$. Furthermore, MacDonald *et al.* (1985) used the ideas of Rekasius (1980) to show that the condition for the matrix (3.40) can also be formulated as follows: the matrix

$$sI - A_0 - \sum_{i=1}^{k} \left(\frac{1 - sT}{1 + sT}\right)^{i} A_i$$

has no roots on the imaginary axis for all real values of T. The substitution methods of Thowsen and Macdonald are sometimes designated as pseudo-delay techniques. It is important to note that these methods are not approximation methods. The substitution is not equivalent to Padé approximation techniques. The motivation for such an approach is that under such a change of variable from h to T, the characteristic equations are reduced to finite polynomial equations and hence possess only a finite number of solutions. Moreover, it is then possible to make use of results for delay-free systems such as the Routh-Hurwitz criteria. Furthermore, the pseudo-delay technique can be applied to also determine the range of h, where the system (3.31) is asymptotically stable. This is the topic of Subsection 3.3.2.

Olbrot in [109] has shown that if a delay grows to infinity, then either the delay differential system becomes unstable or, at least, some of its eigenvalues approach the imaginary axis. If the eigenvalues approach the imaginary axis, the system may become extremely sensitive to changes of other parameters, which means practical instability. Therefore, the case of h growing to infinity is always excluded here.

The stability criterion given by Yoshizawa and the related methods described in Subsection 3.2.3.II are the most interesting ones of this subsection, since they are easier to compute than the stability conditions mentioned in Subsection 3.2.3.I.

The stability criteria in this section presume that the delays are constant and that if $i > 1$ the ratios of the delays are exactly known. Especially the last assumption is often very restrictive from the practical point of view.

3.2.4 Stability tests: Delay-dependent

Exact stability conditions are delay-dependent. Unfortunately, exact algebraic stability conditions are known only for simple systems. Nevertheless, there exists a rich literature on this topic. Therefore, this subject is treated in a separate section (see Section 3.3). Here, only sufficient delay-dependent stability criteria are enumerated (which are valid for n-dimensional systems). The time-delay system considered is described by the following differential-difference equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau(t)) \tag{3.41}$$

where $0 < \tau(t) < \tau_{max}$ is continuous. From Sugujama (1961) we know that if the system $\dot{x}(t) = (A_0 + A_1)x(t)$ is asymptotically stable, then system (3.41) is so as well for sufficiently small values of τ_{max}. Su and Huang (1992) were able to give an estimate of the values τ_{max} such that the system (3.41) is asymptotically stable.

Theorem 3.7 [132]: *Suppose $A_0 + A_1$ is asymptotically stable. Then system (3.41) is asymptotically stable, if there exists a symmetric positive-definite matrix P such that the inequality*

$$\tau_{max} < \frac{-\mu(P(A_0 + A_1))_2}{\|A_1(A_0 + A_1)\|_2} \cdot \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}^3(P)}} \tag{3.42}$$

holds.

The stability criterion (3.42) is less conservative than the delay-independent stability criterion (3.4) when the delay is small [132].

Mori *et al.* (1989) presented another delay-dependent stability criterion. Their method has serious limitations since it requires to solve transcendental characteristic equations over a certain range. An attempt to simplify the calculation was made by Alastruey *et al.* (1992) using Taylor series approximations. However, that method guarantees the stability only over a small time interval. Furthermore, the calculation of the Euclidean norm of a matrix is required which necessitates a certain computational effort. Therefore, this method is not introduced here in detail.

3.2.5 Stability tests: An application of the transformation

In this Section, it is shown that the transformation introduced in Section 2.4 is a tool to extend certain stability criteria for single-delay systems to multiple-delay systems. The transformation converts the multiple-delay system

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t-\tau_i) \tag{3.43}$$

into the single delay system

$$\dot{\bar{x}}(t) = \bar{A}_0 \bar{x}(t) + \bar{A}_1 \bar{x}(t-\tau_k) + \bar{f}(t) \tag{3.44}$$

leaving the trajectory invariant. The matrices $\bar{A}_0$ and $\bar{A}_1$ are given by (2.35) and (2.36).

Theorem 3.8: *The eigenvalues of the original system (3.43) are a subset of the eigenvalues of the transformed system (3.44).*

Proof: In the sequel, the notation $\Gamma = sI - A_0 - A_k e^{-s\tau_k}$; $z^{l_i} = e^{\tau^* s \cdot l_i} = e^{\tau_i s}$ will be used. The characteristic equation of the system (3.44) is

$$\det[sI - \bar{A}_0 - \bar{A}_k e^{-s\tau_k}] = 0 \ . \tag{3.45}$$

Applying (2.35) and (2.36), the characteristic equation of the transformed system (3.44) is expressed as $\det(\Im) = 0$, where $\Im$ is defined by

$$\Im = \begin{bmatrix} \Gamma & \cdots & -A_i^{(1,\, l_i+1)} & & \\ \vdots & & & \ddots & \\ -A_i^{(l_k-l_i-1,\, 1)} z^{-l_k} & & \ddots & & -A_i^{(l_k-l_i,\, l_k)} \\ & \ddots & & \ddots & \\ & & -A_i^{(l_k,\, l_i)} z^{-l_k} & \cdots & \Gamma \end{bmatrix} . \tag{3.46}$$

$\Im \in \boldsymbol{R}^{l_k \cdot n \times l_k \cdot n}$ is a partitioned matrix with $n \times n$ submatrices. From (3.45), (2.35), and (2.35) it follows that the main diagonal of $\Im$ is built by Γ, while the outer diagonals are built of zeros, or $-A_i$, or $-A_i z^{-l_k}$. The symbols in parentheses indicate the place of the submatrices in $\Im$. In the following definition of the matrix T, these indices are displayed as subscripts to increase readability

$$T = \begin{bmatrix} I_{(1,1)} & Dz_{(1,2)} & \cdot\ \cdot & Dz^{l_i}_{(1,l_i+1)} & \cdot\ \cdot & Dz^{l_k-1}_{(1,l_k)} \\ & I_{(2,2)} & & & & \mathbf{0} \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ \mathbf{0} & & & & & I_{(l_k,l_k)} \end{bmatrix} . \qquad (3.47)$$

$I \in \boldsymbol{R}^{n \times n}$ is the identity matrix and D is defined by $D = -I$. The inverse T^{-1} of the matrix T is obtained from (3.47) by setting $D = I$. Using T for a similarity transformation, the proof is established as follows:

$$\det(T^{-1}\Im T) = \det\begin{bmatrix} sI - A_0 - \sum_{i=1}^{k} A_i e^{-s\tau_i} & 0 & \dots & 0 \\ \begin{bmatrix} V_1 \end{bmatrix} & & \begin{bmatrix} & V_2 & \end{bmatrix} & \end{bmatrix} \qquad (3.48)$$

$$= \det\left[sI - A_0 - \sum_{i=1}^{k} A_i e^{-s\tau_i}\right] \cdot \det(V_2) \ . \qquad ❑$$

Corollary 3.4: *If the transformed system (3.44) is asymptotically stable, then the original system (3.43) is asymptotically stable as well.*

Proof: Corollary 3.4 is a consequence of Theorem 3.8. The proof can also be performed independently of Theorem 3.8. Suppose the original system is unstable and the corresponding transformed system is asymptotically stable. Then it is not possible for the transformed system to have the same trajectory as the original system because the state of a stable system cannot increase. But the transformation assures trajectory invariance. From the contradiction the Corollary follows. ❑

The converse of Corollary 3.4 is false in general. To see this, we consider the system $\dot{x}(t) = -x(t) - x(t-1) + x(t-2)$. The dominant eigenvalue is $\lambda_{max} = -0.3651168....$ For the corresponding transformed system, we have $\lambda_{max} = 0.2963534....$

Stability criteria for single delay systems can immediately be extended with the help of the transformation to systems with multiple delays. This application of the transformation is illustrated by an example.

Example 3.1: Extend Theorem 3.7 for the following system with multiple constant delays

$$\dot{x}(t) = A_0 + \sum_{i=1}^{k} A_i x(t - ih) \ . \tag{3.49}$$

Corollary 3.4 and Theorem 3.7 imply that the system (3.49) is asymptotically stable if $\bar{A}_0 + \bar{A}_1 < 0$ and if there exists a symmetric, positive-definite matrix P such that the inequality

$$kh < \frac{-\mu(P(\bar{A}_0 + \bar{A}_1))_2}{\left\|\bar{A}_1(\bar{A}_0 + \bar{A}_1)\right\|_2} \cdot \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}^3(P)}}$$

holds, where $\bar{A}_0$ and $\bar{A}_1$ are given by (2.35) and (2.36).

Because the eigenvalues of the transformed and of the original system are not generally identical, it is expected that criteria extended with the help of the transformation have a certain loss of sharpness. Therefore, the transformation is only useful if this extension is not straightforward. This particular situation arises if one has to extend the stability conditions given by [99], [100], [101], [139], [132], and [1].

3.2.6 Stability tests: Reduction of the conservatism of the criteria

The stability criteria presented above can be remarkably improved by applying an appropriately chosen transformation matrix. Using transformed state vector techniques, it is easy to show that, for example, stability condition (3.4) of Theorem 3.4 can be rewritten as

$$\mu(T^{-1}A_0T)_{.} + \sum_{i=1}^{k} \left\| T^{-1}A_iT \right\|_{.} < 0 \ , \quad (. = 1, 2, \infty),$$

where T is a regular matrix. The determination of the matrix T which minimizes the left-hand side of the inequality leads to nonlinear transcendental conditions, in general. However, simple numerical methods are very effective to find an appropriate matrix T (e.g., method of steepest descent, threshold accepting, simulated annealing).

Example 3.2: The Williams-Otto process was introduced in Section 1.2. The homogeneous part of this system has the form

$$\dot{x}(t) = \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.20 & -5.30 & -12.8 & 0 \\ 6.40 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11.0 & -3.96 \end{bmatrix} x(t) + \begin{bmatrix} 1.92 & 0 & 0 & 0 \\ 0 & 1.92 & 0 & 0 \\ 0 & 0 & 1.87 & 0 \\ 0 & 0 & 0 & 0.724 \end{bmatrix} x(t-1) \ .$$

Without a similarity transformation, stability condition (3.4) fails for this system:

$$\mu(A_0)_2 + \|A_1\|_2 \ = \ 0.657 \not< \ 0 \ .$$

Using an appropriate diagonal matrix $T = \text{diag}[-0.37; 1.30; -0.52; 1.06]$, the asymptotic stability of the system is confirmed:

$$\mu(T^{-1}A_0T)_2 + \left\|T^{-1}A_1T\right\|_2 \ = \ -0.61 < 0 \ .$$

Moreover, the calculation above shows that the system is asymptotically stable even if the delay is not constant. The matrix T was computed with the help of the steepest descent method.

For systems with constant delays there is another possibility to improve the stability tests. The idea consists of omitting those elements of the matrices A_0, A_i which do not have any influence on the characteristic equation. This procedure yields the matrices $\tilde{A}_0$ and $\tilde{A}_i$ which are then used for the stability test instead of the matrices A_0, A_i. In the light of Theorem 3.3 the validity of this procedure is obvious.

3.3 Exact stability criteria

Necessary and sufficient algebraic stability conditions are known only for certain types of single-delay systems: for the scalar system $\dot{x}(t) = a_0x(t) + a_1x(t-h)$ (Subsection 3.3.1) and for the multivariable system $\dot{x}(t) = A_1x(t-h)$, where A_1 is a constant matrix (see Subsection 3.3.3). For a specific system with given time lags and with given values for all of the system parameters, the roots of a transcendental equation can be computed (see Section 3.4). This permits an analysis of the system's stability. In the τ-decomposition method (Section 3.3.2), the values of the delay for which the system is asymptotically stable is determined, while in the D-decomposition method, the time lag is held constant and the stability region in the parameter space is studied. In the subsection below, an improved D-decomposition method is presented.

3.3.1 Modified D-decomposition

Sufficient stability and instability criteria together with the D-decomposition method can be used to derive necessary and sufficient stability conditions for time-delay systems with constant delays. In this section, the linear system with multiple constant delays

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^{k} A_ix(t-\tau_i) \tag{3.50}$$

where $0 < \tau_1 < \ldots < \tau_k < \infty$ is studied. The characteristic equation of the system (3.50) can be written in the form

$$P(s, e^{-s\tau}) \;=\; \sum_{j=0}^{n} s^{n-j}\alpha_j(e^{-s\tau}) \tag{3.51}$$

where

$$\alpha_j(e^{-s\tau}) \;=\; \sum_{i_1 + \ldots + i_k \le j} a_j^{(i_1, \ldots, i_k)} e^{-s(\tau_1 i_1 + \ldots + \tau_k i_k)}$$

with $a_j^{(i_1, \ldots, i_k)} \in \boldsymbol{R}$, $a_0^{(0, \ldots, 0)} = 1$, and the sum is extended over all different sets of the positive integers for which $i_1 + \ldots + i_k \le j$. For numerical examples see [21] and [22]. Furthermore, let

$$p_{max} \;=\; \max_{i_1 + \ldots + i_k \le n} (\tau_1 i_1 + \ldots + \tau_k i_k) \quad . \tag{3.52}$$

The method developed here is based on the D-decomposition method [33, p. 49]. For a fixed value of the delay parameter h, the zeros of the characteristic equation (3.51) are continuous functions of its coefficients. In the usual D-decomposition method, the coefficient space is partitioned into different regions by means of hypersurfaces, the points of which are characterized by the corresponding characteristic equation (3.51) having at least one zero on the imaginary axis. (This procedure is called D-subdivision.) The points in the interior of each region correspond to a characteristic equation with the same number of zeros with positive real parts. The number of zeros with positive real parts can only change when a zero passes across the imaginary axis, i.e., when the point in the coefficient space passes across the boundary of the region.

Here, the goal is finding all of the regions in the coefficient space where the characteristic equation has no zero with positive real part. These regions κ are regions of asymptotic stability. In order to check how the number of roots with positive real parts changes when crossing the boundary, the differential $d\nu$ of the real part of the root (which crosses the imaginary axis) can be determined. The number of roots with positive real parts decreases (increases) if the sign of $d\nu$ is negative (positive). For a given characteristic equation

$$P(s, a_0^{(0, \ldots, 0)}, \ldots, a_j^{(i_1, \ldots, i_k)}, \ldots) \;=\; 0 \tag{3.53}$$

we obtain [33, p. 55]

$$d\nu = -Re\left(\frac{\sum_{j=0}^{n} \frac{\partial P}{\partial a_j^{(i_1, \ldots, i_k)}} da_j^{(i_1, \ldots, i_k)}}{\frac{\partial P}{\partial s}}\right) \quad . \tag{3.54}$$

However, equation (3.54) is not always well-suited for singling out asymptotically stable regions because, in general, there is an infinite number of boundaries which are given in parametric form. In order to simplify the search for the regions κ, sufficient stability criteria and the following instability criterion are used.

Theorem 3.9 [22]: *The system (3.50) with the characteristic equation (3.51) is not asymptotically stable if*

$$0 \geq \sum_{i_1 + \ldots + i_k \leq n} a_n^{(i_1, \ldots, i_k)} \tag{3.55}$$

or if

$$\begin{aligned} 0 \geq & \sum_{i_1 + \ldots + i_k \leq n-1} a_{n-1}^{(i_1, \ldots, i_k)} + \\ & - \sum_{1 \leq i_1 + \ldots + i_k \leq n} a_n^{(i_1, \ldots, i_k)}(\tau_1 i_1 + \ldots \tau_k i_k) + \\ & + p_{max} \sum_{i_1 + \ldots + i_k \leq n} a_n^{(i_1, \ldots, i_k)} \; . \end{aligned} \tag{3.56}$$

The search for the regions in the coefficient space corresponding to asymptotic stability of the dynamic system is particularly simple if the boundaries of the various regions do not intersect. In order to demonstrate the modified D-decomposition method, it is applied to two dynamical systems for which this particularly nice constellation arises.

I) In the first case, the system $\dot{x}(t) = a_0 x(t) + a_1 x(t-h)$ is considered. Its stability region in the coefficient space (a_0, a_1) has been established in [45] and in [43, p. 108 and 337], based on an extended Routh-Hurwitz criterion.

In [33, p. 56] the D-decomposition method was applied to study this stability problem. However, with the help of equation (3.54) only two boundaries were discussed there.

Theorem 3.10*: The time-delay system*

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-h) \tag{3.57}$$

is asymptotically stable, iff the following three conditions hold for some $y \in [0, \pi/h)$

$$a_0 + a_1 < 0 \tag{3.58}$$

$$a_0 = \frac{y \cdot \cos(yh)}{\sin(yh)} \tag{3.59}$$

$$a_1 > \frac{-y}{\sin(yh)} . \tag{3.60}$$

Proof: The characteristic equation of (3.57)

$$P(s, e^{-sh}) = s - a_0 - a_1 e^{-sh} \tag{3.61}$$

has a zero root if

$$0 = a_1 + a_2 . \tag{3.62}$$

Equation (3.62) defines the boundary curve c_1 depicted in Fig. 3.1. The other boundaries with purely imaginary roots of the form s = iy result from (3.61) in the following parametric form:

$$a_0 = \frac{y\cos(hy)}{\sin(hy)} \qquad a_1 = \frac{-y}{\sin(hy)} . \tag{3.63}$$

The infinite number of boundary curves defined by (3.63) are classified as follows

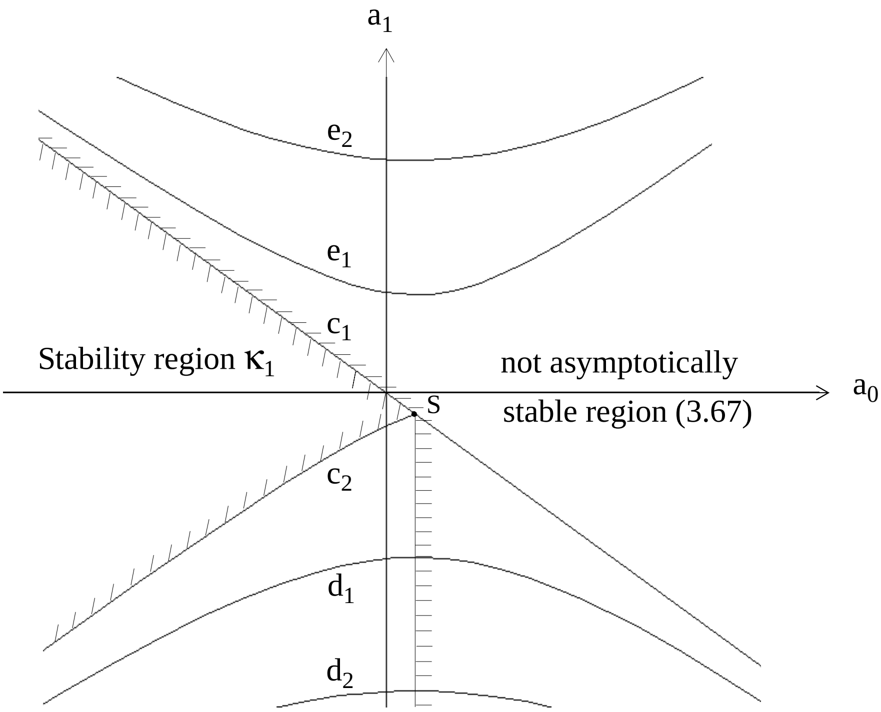

Fig. 3.1 Boundary curves c_1, c_2, d_1, d_2, e_1, e_2; Stability region κ_1; not asymptotically stable region (3.67)

$$c_2: \quad y \in [0, \pi/h,) \quad \text{or} \quad y \in (-\pi/h, 0] \tag{3.64}$$

$$d_j: \quad y \in (2j\cdot\pi/h, (2j+1)\cdot\pi/h) \quad \text{or} \quad y \in (-(2j+1)\cdot\pi/h, -2j\cdot\pi/h) \tag{3.65}$$

$$e_m: \quad y \in ((2m-1)\cdot\pi/h, 2m\cdot\pi/h) \quad \text{or} \quad y \in (-2m\cdot\pi/h, -(2m-1)\cdot\pi/h) \tag{3.66}$$

where j, m ∈ {1, 2, . . .}. The boundary curves c_1, c_2, d_1, d_2, e_1, e_2 are depicted in Fig. 3.1. The curve c_2 has a maximum for y = 0 and for the vector of coefficient $(a_0, a_1) = (1/h, -1/h)$. This point lies also on c_1 and is denoted in Fig. 3.1 as

$$S = \left(\frac{1}{h}, -\frac{1}{h}\right) .$$

The region κ_1 enclosed by c_1 and c_2 and lying to the left of the above-mentioned intersection points of c_1 and c_2 corresponds to asymptotic stability of the time-delay system (3.57) because it contains the family of systems $\dot{x}(t) = a_0 x(t)$ with $a_0 < 0$.

From (3.63) and (3.66) the following properties of all of the curves e_m can be derived: The curves e_m lie in the upper-half plane, $a_1 > 0$, and extend to $a_0 \to -\infty$ and $a_0 \to +\infty$. They do not intersect the straight line c_1 for finite values of a_0 and a_1. Furthermore, any two curves e_j, e_m, $m \neq j$, do not intersect. The obvious analog statement holds for the curves d_j in the lower-half plane, $a_1 < 0$. Of course, any two curves d_j and e_m do not intersect.

Now, Theorem 3.9 is used. According to (3.55), the system (3.56) is not asymptotically stable if

$$0 \le a_0 + a_1 \qquad \text{or} \qquad \frac{1}{h} \le a_0 \quad . \tag{3.67}$$

The instability region described by (3.67) is marked in Fig. 3.1. It follows that the region κ_1 is the only stability region because the region described by (3.67) intersects all of the other regions. ❑

II) In the second case, the time-delay system

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-h) + a_2 x(t-2h) \tag{3.68}$$

is considered. Delay-independent stability conditions were investigated by Jury and Mansour (1982), Liu and Mansour (1984), and Boese (1989). Buslowicz (1983) used this system to demonstrate his instability criterion. For some special cases of the system (3.68), the exact stability conditions were analysed by MacDonald (1989), by Stépán (1989), and by Walton and Marshall (1987). However, the exact delay-dependent algebraic stability conditions of the system (3.68) are not known.

Theorem 3.11: *The time-delay system (3.68) with $|a_2| < \pi/(2h)$ is asymptotically stable, iff the following three conditions hold for some $y \in [0, \pi/h)$*

$$a_0 + a_1 + a_2 < 0 \tag{3.69}$$

$$a_0 = \frac{y\cos(yh)}{\sin(yh)} + a_2 \tag{3.70}$$

$$a_1 > \frac{-y}{\sin(yh)} - 2a_2\cos(yh) \quad . \tag{3.71}$$

Proof: The characteristic equation of (3.68) is

$$0 = s - a_0 - a_1 e^{-sh} - a_2 e^{-2sh} \ . \tag{3.72}$$

The root s = 0 implies the boundary surface c_1 defined by

$$a_0 + a_1 + a_2 = 0 \ . \tag{3.73}$$

For purely imaginary roots s = iy the boundary surfaces are obtained in the following parametric form

$$a_0 = \frac{y\cos(yh)}{\sin(yh)} + a_2 \tag{3.74}$$

$$a_1 = \frac{-y}{\sin(yh)} - 2a_2\cos(yh) \quad . \tag{3.75}$$

The infinite number of boundary surfaces described by (3.74) and (3.75) are classified by (3.64), (3.65), and (3.66):

c_2: $y \in [0, \pi/h)$ or $y \in (-\pi/h, 0]$ (3.64)

d_j: $y \in (2j\cdot\pi/h, (2j+1)\cdot\pi/h)$ or $y \in (-(2j+1)\cdot\pi/h, -2j\cdot\pi/h)$ (3.65)

e_m: $y \in ((2m-1)\cdot\pi/h, 2m\cdot\pi/h)$ or $y \in (-2m\cdot\pi/h, -(2m-1)\cdot\pi/h)$ (3.66)

From now on, only nonnegative values of y will be considered since they define the complete surfaces. The region in the three-dimensional parameter space which is bounded by c_1 and c_2 and which contains the ray $(a_0, 0, 0)$ with $a_0 < 0$ is named κ_2.

This region κ_2 contains a stability region because its intersection with the plane $a_2 = 0$ is identical to the (two-dimensional) region κ_1 of Case 1. The whole region κ_2 is a stability region, provided it is not intersected by any of the surfaces d_j or e_m. However, for sufficiently large values of a_2 such intersections do occur.

We now derive the range for a_2 such that the surfaces d_j and e_m do not intersect the region κ_2 and hence do not intersect the surfaces c_1 and c_2.

Intersections with c_1: A condition for any surface d_j to intersect c_1 is obtained by combining (3.73) – (3.75) and by factoring the resulting equation. This leads to the intersection condition

$$(\cos(yh) - 1) \cdot (y - 2a_2 \sin(yh)) = 0 \tag{3.76}$$

where $y \in (2j \cdot \pi / h, (2j + 1) \cdot \pi / h)$. The first factor vanishes for $y = 2j\pi$. These values of y correspond to intersections of d_j with c_1 at infinity. The second factor cannot vanish provided

$$a_2 < \frac{3.894\ldots}{h} \ . \tag{3.77}$$

Hence, if (3.77) is satisfied, none of the surfaces d_j intersects c_1. Similarly, if the condition

$$a_2 > \frac{-2.301\ldots}{h} \tag{3.78}$$

is satisfied, none of the surfaces e_m intersects c_1.

Intersections with c_2: A condition for any surface d_j to intersect c_2 is obtained as follows. Let $y_0 \in [0, \pi / h)$ and $y_1 \in (2j \cdot \pi / h, (2j + 1) \cdot \pi / h)$. The intersection condition $a_0(y_0) = a_0(y_1)$ and equation (3.74) yield

$$\frac{y_1}{\sin(y_1 h)} \cdot \frac{\cos(y_1 h)}{\cos(y_0 h)} = \frac{y_0}{\sin(y_0 h)} \tag{3.79}$$

with $\cos(y_0 h) \neq 0$. The intersection condition $a_1(y_0) = a_1(y_1)$ and the equations (3.79) and (3.75) yield

$$0 = \left[1 - \frac{\cos(y_1 h)}{\cos(y_0 h)}\right] \cdot \left[\frac{y_1}{\sin(y_1 h)} - 2a_2 \cos(y_0 h)\right] \tag{3.80}$$

for $\cos(y_0 h) \neq 0$ (since values for y_0 such that $\cos(y_0 h) = 0$ do not produce any intersections of d_j and c_2). The first factor vanishes for $y = 2j\pi$. These values of y correspond to intersections of d_j and c_2 at infinity. The second factor cannot vanish if

$$\frac{\pi}{h} > |a_2| \ . \tag{3.81}$$

(This estimate is conservative.) Hence, if (3.81) is satisfied, none of the surfaces d_j intersects c_2. Similarly, if the condition

$$\frac{\pi}{2h} > |a_2| \tag{3.82}$$

is satisfied, none of the surfaces e_m intersects c_2. (Again, this is a conservative estimate.) Another (conservative) estimate shows that no pair of any of the surfaces defined by (3.65) and/or (3.66) intersects if (3.82) holds.

From (3.77), (3.78), (3.81), and (3.82) we obtain the sufficient condition (3.82) for the whole region κ_2 to be a stability region for the system (3.68). Now, Theorem 3.9 comes into play. According to Theorem 3.9, the system (3.68) is not asymptotically stable if

$$0 \le a_0 + a_1 + a_2 \quad \text{or} \quad \frac{1}{h} \le 2a_0 + a_1 \ . \tag{3.83}$$

In the coefficient space, equation (3.83) defines an instability region. Considering the above-mentioned properties of the surfaces d_j and e_m under the restriction (3.82) for a_2, equation (3.83) implies that the region κ_2 is the only stability region. ❑

The stability region κ_2 in the parameter space for the system (3.68) $\dot{x}(t) = a_0 x(t) + a_1 x(t-h) + a_2 x(t-2h)$ is depicted in Fig 3.2. Note that Theorem 3.11 is conservative with respect to the bound (3.82) assumed for a_2. Furthermore, Theorem 3.11 includes the result of Case 1.

Remark 3.1: The gap between delay-independent criteria and exact ones is sketched in Fig. 3.3. For the time-delay system

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-1) - 1 \cdot x(t-2) \tag{3.84}$$

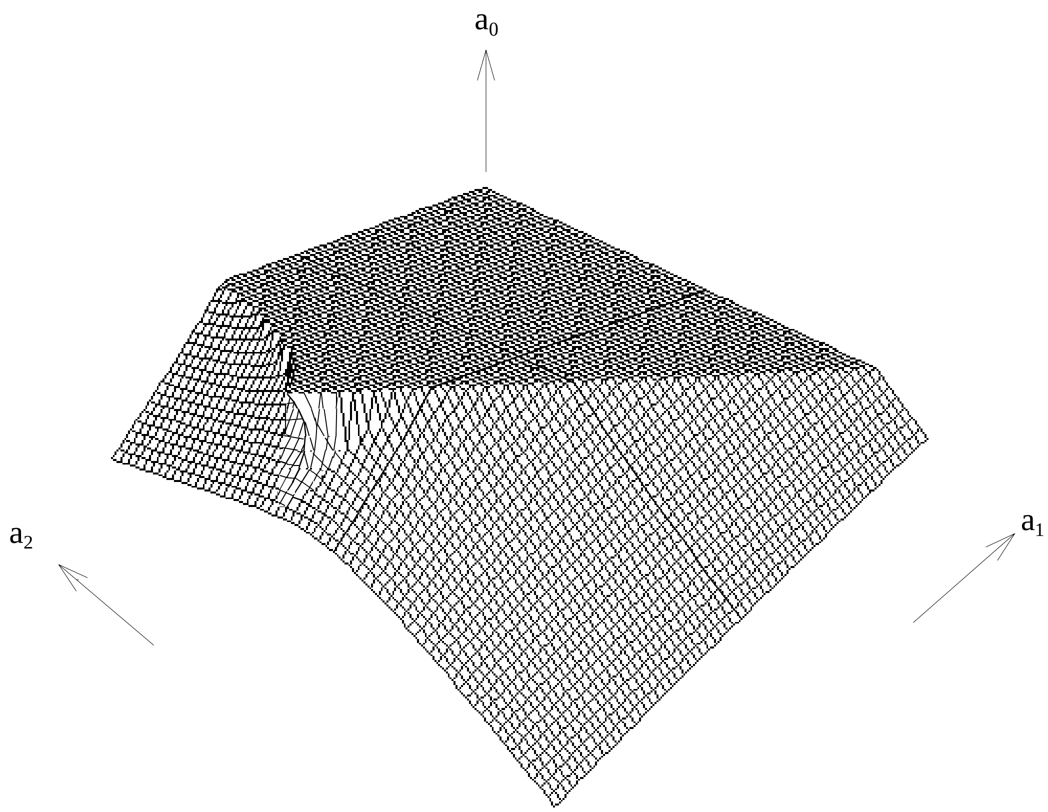

Fig. 3.2 Stability region for the system (3.68):
$\dot{x}(t) = a_0x(t) + a_1x(t-1) + a_2x(t-2)$.
The solid lines indicate where $a_1 = 0$ or $a_2 = 0$.

the exact conditions (3.69) – (3.71) are depicted. For the system (3.68) $\dot{x}(t) = a_0x(t) + a_1x(t-h) + a_2x(t-2h)$, the stability conditions independent of the delay constant h [16], [57], [82]:

$$a_0 < \frac{a_1^2}{8a_2} + a_2 \quad \text{for} \quad a_2 < 0 \tag{3.85}$$

or

$$a_0 + |a_1| + a_2 < 0 \quad \text{for} \quad a_0 < 3a_2 \tag{3.86}$$

are also shown in Fig. 3.3. Furthermore, for the related system

$$\dot{x}(t) = a_0x(t) + a_1x(t-\tau_1(t)) + a_2x(t-\tau_2(t)) \tag{3.87}$$

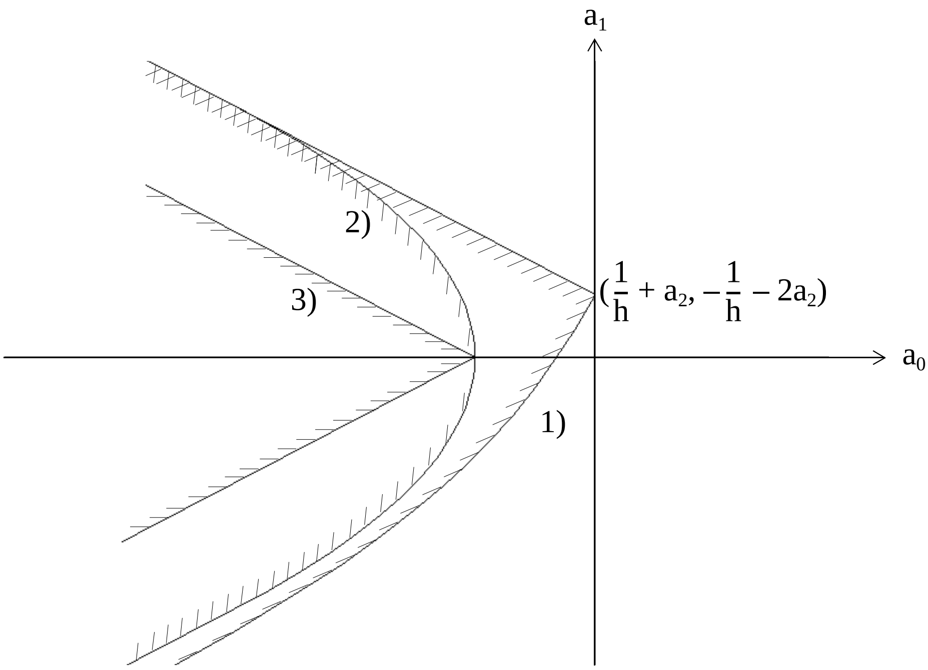

Fig. 3.3 Stability charts:
1) exact stability conditions (3.69) – (3.71) for the system
$$\dot{x}(t) = a_0x(t) + a_1x(t-1) - 1\cdot x(t-2)$$
2) stability conditions (3.85) – (3.86) independent of h for
$$\dot{x}(t) = a_0x(t) + a_1x(t-h) - 1\cdot x(t-2h)$$
3) delay-independent stability condition (3.88) for
$$\dot{x}(t) = a_0x(t) + a_1x(t-\tau_1(t)) - 1\cdot x(t-\tau_2(t))$$

where $\tau_1(t)$ and $\tau_2(t)$ are bounded continuous functions, the delay-independent stability condition given by Theorem 3.4

$$a_0 + |a_1| + |a_2| < 0 \tag{3.88}$$

is displayed in Fig. 3.3 as well.

Remark 3.2: For calculating the intersection of c_1 and c_2 in Case 2, equation (3.76) is used

$$(\cos(yh) - 1) \cdot (y - 2a_2\sin(yh)) = 0 \tag{3.76}$$

where $y \in [0, \pi/h)$. The first factor yields a solution for $y = 0$, and implies the intersection in form of a the straight line defined by

$$S = \left(\frac{1}{h} + a_2, -\frac{1}{h} - 2a_2, a_2\right) \tag{3.89}$$

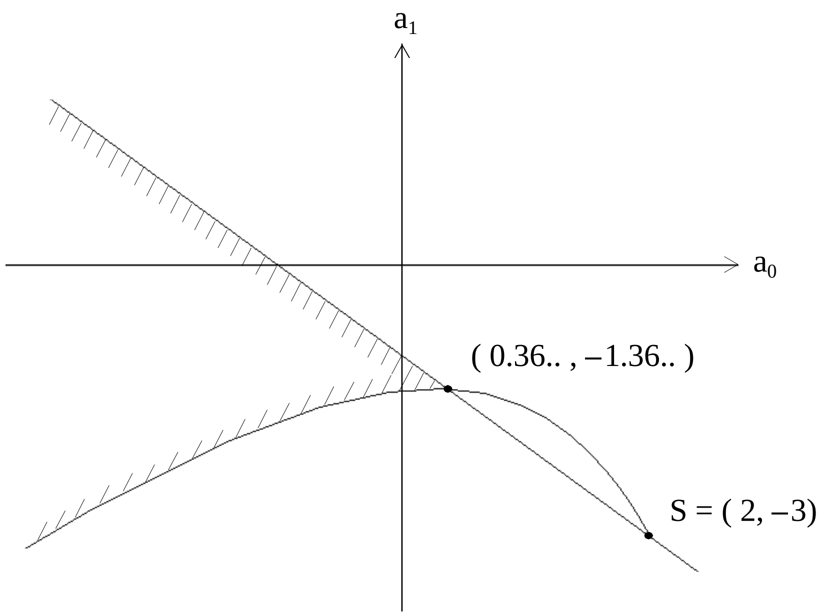

Fig. 3.4 Exact stability region for the system
$\dot{x}(t) = a_0x(t) + a_1x(t-1) + 1\cdot x(t-2)$

(cf. Figs. 3.2, 3.3, and 3.4). The second factor of (3.76) yields a solution iff

$$\frac{1}{2h} < a_2 \ . \tag{3.90}$$

If (3.90) holds, the intersection of c_1 and c_2 relevant for the stability region κ_2 is no longer given by (3.89), but is defined by the solution of the transcendental equation $y = 2a_2\sin(yh)$, and by (3.74), and (3.75). The part of the coefficient space which is bounded by c_1 and c_2 and for which the inequality $a_0 + a_1 + a_2 \geq 0$ holds corresponds to instability of the system due to (3.83). As an example, the stability region κ_2 of the system

$$\dot{x}(t) = a_0x(t) + a_1x(t-1) + 1\cdot x(t-2) \tag{3.91}$$

is displayed in Fig. 3.4. In contradistinction to the system of (3.84) (cf. Fig. 3.3) S of (3.89) does not give rise to the corner of the stability region of the system of (3.91) because (3.90) holds.

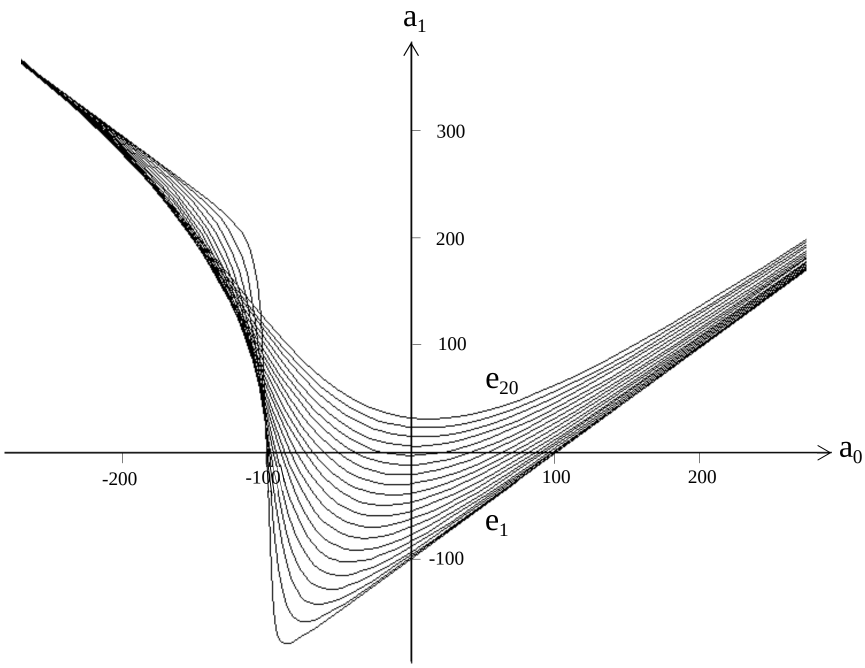

Fig. 3.5 *Boundary curves $e_1 \ldots e_{20}$ for the system $\dot{x}(t) = a_0x(t) + a_1x(t-1) - 100 \cdot x(t-2)$*

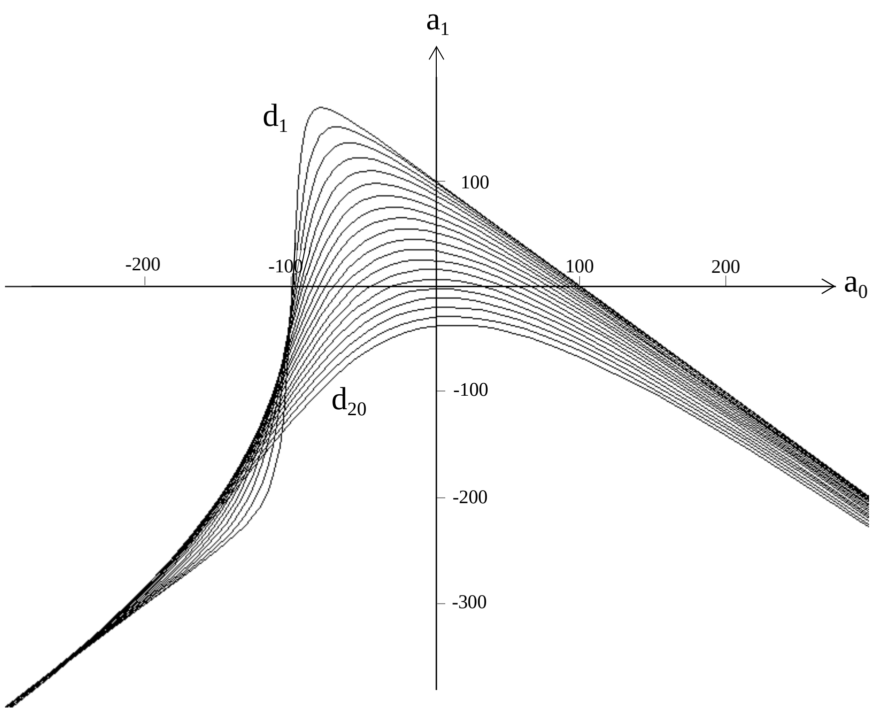

Fig. 3.6 *Boundary curves $d_1 \ldots d_{20}$ for the system $\dot{x}(t) = a_0x(t) + a_1x(t-1) - 100 \cdot x(t-2)$*

Remark 3.3: For the sake of completeness, some clues are given about the influences of the surfaces d_j and e_m on the stability region for the case (not covered by Theorem 3.11) in which (3.82) does not hold: As a_2 increases (decreases, respectively), more and more of the surfaces d_j and e_m intersect the surfaces c_1 and/or c_2. Therefore, an exact stability criterion for arbitrary values of a_2 would have to be formulated piecewise for intervals of values of a_2. For very large values of a_2 the various d_j's and e_m's are approximately described by (3.85) and (3.86).

In order to illustrate this case in Figs. 3.5 and 3.6, respectively, the intersections of the surfaces d_j and e_m with the plane $a_2 = -100$ are plotted for the time-delay system

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-1) - 100 \cdot x(t-2) \,. \tag{3.92}$$

3.3.2 τ-decomposition

In the τ-decomposition we are interested in the effects of the delay τ on the stability. The time-lag τ is allowed to vary while other parameters are kept fixed. The positive half of the τ-axis is first divided into intervals by boundary points at which purely imaginary roots of the characteristic equation exist. The points in the interior of each interval correspond to characteristic equations with the same number of zeros with positive real parts. This implies that within each interval the stability character of the system does not change. The number of zeros with positive real parts can only change when a zero passes across the imaginary axis, i.e., when the point in the interval passes across the boundary of the interval. In order to find the intervals in which the characteristic equation has no zero with positive real part, the direction of motion of the imaginary roots is calculated by differentiating the characteristic equation with respect to τ. This procedure is rather practical for singling out asymptotically stable intervals, since usually only a few boundary points have to be considered.

A number of suggestions [46], [83], [137], [138] have been made to improve the concept introduced above. In the remaining part of this section, an analytical method is introduced which is based on the results of Thowsen (1981) [137], [138] and Hertz *et al.* (1984) [46]. The following system is considered

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - ih) \ . \tag{3.93}$$

The characteristic equation of (3.93) is

$$\det\left[sI - A_0 - \sum_{i=1}^{k} A_i e^{-ihs}\right] = 0 \ . \tag{3.94}$$

Theorem 3.12 [138]: *The value $s = j\omega$ $(\omega \geq 0)$ is an imaginary root of (3.94) for some $h \geq 0$ iff $s = j\omega$ is also a root of*

$$det\left[sI - A_0 - \sum_{i=1}^{k} A_i\left(\frac{1 - sT}{1 + sT}\right)^{2i}\right] = 0 \tag{3.95}$$

for some nonnegative number T.

Corollary 3.5 [137]: *Let $0 \leq T < \infty$. Then e^{-sh} equals $(1 - Ts)^2/(1 + Ts)^2$ at $s = j\omega$ iff h and T are related by*

$$h = \frac{4}{\omega}arctan(\omega T) + 2\pi\frac{m}{\omega} \tag{3.96}$$

where $m = 0, \pm 1, \ldots$.

***Corollary 3.6 [138]**: If $s = j\omega$ is an imaginary root of (3.94) for $h = h_0$, then $s = j\omega$ is also an imaginary root of (3.94) for $h = h_0 + 2\pi m / \omega$.*

Theorem 3.12 and Corollaries 3.5 and 3.6 imply the following procedure to determine the intervals of delay values for which the system is asymptotically stable.

Step 1: The standard Routh (-Hurwitz) stability test reveals those values of T for which the polynomial (3.95) has imaginary roots. In the following, these particular values of T are designated by $\hat{T}$.

Step 2: Determine for each $\hat{T}$ the corresponding imaginary roots of (3.95) $s = \pm j\hat{\omega}$.

Step 3: Calculate for each $\hat{T}$ with the corresponding $\hat{\omega}$ the associated value of $\hat{h}$ by applying Corollary 3.6 under the restriction that $\hat{h} \geq 0$.

Step 4: Check the direction of motion of the imaginary roots $j\hat{\omega}$. The direction of the root loci is determined by the sign of the real part of (ds/dh) for the values of $s = j\hat{\omega}$ and the corresponding value of $\hat{h}$. If *Re*(ds/dh) = 0 at one of these points, and the multiplicity of the zero of *Re*(ds/dh) is odd, the root loci only touche the imaginary axis but do not cross it [46]. For this value of $\hat{h}$, the system is unstable. However, the two adjacent intervals for which the above value of $\hat{h}$ is a common point are both either asymptotically stable or unstable intervals. If the zero of *Re*(ds/dh) is of even multiplicity, the direction of the imaginary-axis crossing root is determined by the sign of the first derivative which is not zero.

Step 5: Analyse the stability of the system (3.93) for h = 0.

Step 6: The intervals of delay values for which the system is asymptotically stable follow from the results of Steps 4 and 5.

Example 3.3: Find all values of the delay constant h such that the system

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -x_1(t) - x_2(t - h)$$

with the characteristic equation

$$P(s, e^{-hs}) = s^2 + se^{-hs} + 1 = 0$$

is asymptotically stable. The auxiliary equation (3.95) becomes

$$T^2 s^4 + 2T(1 + T)s^3 + (1 + T^2)s^2 + 2(1 + T)s + 1 = 0$$

and the first column of the associated Routh's array (see, e.g., [34, p. 123]) is computed to be

$$\begin{array}{l} T^2 \\ 2T(1 + T) \\ T^2 - T + 1 \\ 2\dfrac{(T-1)^2(T+1)}{T^2 - T + 1} \\ 1\ . \end{array}$$

All elements in this column are positive for all $T \in (0, \infty)$ except for the penultimate one which is zero for $T = 1 = \hat{T}$. That $\hat{T}$ yields a pair of imaginary roots at $s = \pm j$. Since $\hat{T} = 1$ and $\hat{\omega} = 1$, the direction of the root loci has to be checked at $s = j$ and $h = \pi + 2\pi/|m|$

$$\frac{ds}{dh} = \frac{s^2 e^{-hs}}{2s + 1 + e^{-hs}(1 - sh)} \ .$$

For $s = j$, $h = (2|m| + 1)\pi$ we have

$$Re\left(\frac{ds}{dh}\right) = Re\left(j\frac{-1}{2 + h}\right) = 0 \ .$$

Therefore, we have to check the multiplicity of this zero

$$\left.\frac{ds^2}{dh^2}\right|_{s = j,\, h = (2|m| + 1)\pi} = \frac{-2(1 - j)}{(2 + h)^2} \ .$$

The multiplicity of the zero of *Re*(ds/dh) is 1 (odd), and hence the root locus only touches the imaginary axis but does not cross it. Thus, the system is asymptotically stable for all values of h except for

$$h = (2|m| + 1)\pi \ .$$

3.3.3 Stability of $\dot{x}(t) = A_1 x(t-\tau)$

Consider the linear delay-differential system

$$\dot{x}(t) = A_1 x(t-\tau) \ . \tag{3.97}$$

The stability problem of the system (3.97) has been studied by Barszcz and Olbrot (1979), Mori and Noldus (1984), and Buslowicz (1987).

The criterion for exponential stability with decay rate γ, $\gamma \geq 0$, of the system (3.97) given by Barszcz and Olbrot (1979) requires the solution of transcendental equations and is therefore not simple to apply.

The condition for asymptotic stability of the system (3.97) presented by Mori and Noldus (1984) is expressed in terms of the eigenvalue locations of the matrix A_1 in the complex plane.

Theorem 3.13 [99]: *System (3.97) is asymptotically stable iff all eigenvalues of A_1 lie in the open region Ω. This region is bounded by the parametrically defined curve*

$$b_1 = -y\sin(\tau y)$$
$$b_2 = y\cos(\tau y)$$

where $-\pi/2 < \tau y < \pi/2$ and is shown in Fig. 3.7

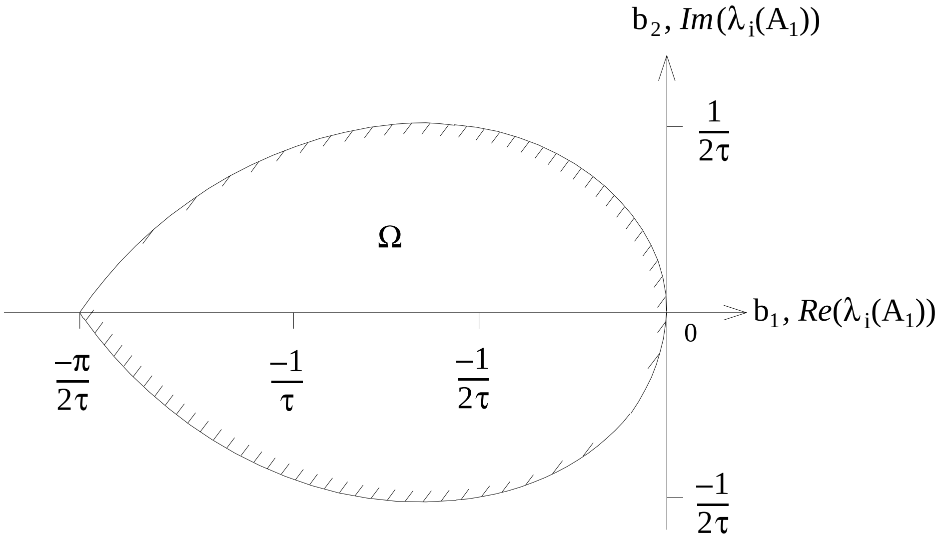

Fig. 3.7 Stability region Ω in the complex plane for the system $\dot{x}(t) = A_1 x(t-\tau)$

Based on the results of Mori and Noldus (1984), Buslowicz (1987) presented an analytical necessary and sufficient criterion for asymptotic stability of the system (3.97).

Theorem 3.14 [23]: *System (3.97) is asymptotically stable iff*

$$Re(\lambda_i(A_1)) < 0$$

and

$$\tau < \frac{arctan\left(\dfrac{Re(\lambda_i(A_1))}{Im(\lambda_i(A_1))}\right)}{|\lambda_i(A_1)|}$$

for all $i \in \{1, ..., n\}$.

3.4 Computation of the eigenvalues

The eigenvalues associated with the system

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - i \cdot h) \tag{3.98}$$

play an important role in many control problems such as stability (Theorem 3.3), controllability (Chapter 5), and feedback stabilization (Section 7.2). These eigenvalues coincide with the zeros of the characteristic equation of (3.98):

$$P(s, e^{-sh}) = \det[\Delta(s)] = \det\left[sI - A_0 - \sum_{i=1}^{k} A_i e^{-si \cdot h}\right] . \tag{3.99}$$

In general, equation (3.99) is transcendental, therefore numerical methods are applied to find a solution. Since this equation contains exponential functions, the solution is very sensitive with respect to s. A method which overcomes such problems was presented by Manitius *et al.* (1987) [92]. It is composed of several algorithms. The following is a brief summary of these algorithms. (They are explained in detail in the Subsections 3.4.2 through 3.4.6.)

- The algorithm of [20] enables us to compute the two-variable polynomial $P(s, e^{-sh})$ directly from the system matrices A_0, A_i (cf. Subsection 3.4.2).
- The eigenvalues of large modulus (modulus = absolute value of an imaginary number), which are distributed in some curvilinear strips, are estimated from the coefficients of the characteristic equation [13, Chapter 12] (cf. Subsection 3.4.3).
- The eigenvalues contained in some bounded region around the origin are approximately computed by an algorithm suggested by [73] (cf. Subsection 3.4.4).
- The roots estimated by the two above mentioned algorithms are used as initial guesses to start Newton's method for improvement. In

Subsection 3.4.5, a numerical procedure is proposed which is more efficient than the one suggested in [92].

- An algorithm proposed by Carpentier and Dos Santos (1982) [24] is used to verify that all eigenvalues of an analytic function in a given region have been found (cf. Subsection 3.4.6). In Subsection 3.4.1, upper bounds for the real and for the imaginary part of the eigenvalues are given. These bounds, together with the method of [24] enable us to check whether all eigenvalues with a positive real part have been found.

The method of [92] detailed in the following subsections is restricted to linear systems with commensurate delays of the form (3.98). However, an extension to systems with noncommensurate (and even neutral) delays is possible, albeit more cumbersome in coding. Since in practice the delays are always commensurate and noncommensurate time-delay systems can be approximated by (3.98), the method in the present form is useful. Before starting the introduction of the computation of the eigenvalues some properties of the eigenvalues are enumerated.

3.4.1 Properties of the eigenvalues

It is known that the eigenvalues of the system (3.98) have the following properties.

i) They are symmetric with respect to the real axis [40, p. 54].

ii) They are of finite multiplicity [40, p. 54] (see also Subsection 3.3.2 for properties of the root loci).

iii) The real parts of the eigenvalues are bounded above. For any constant ¢, the number of eigenvalues with real parts exceeding ¢ is finite

[40, p. 54]. This implies that the time-delay system (3.98) always has a finite number of eigenvalues with non-negative real parts.

In general, the set of eigenvalues is infinite and countable [40, p. 54]. In Subsection 3.4.3, an asymptotic description is given which reveals some qualitative information about the roots' location in the complex plane.

However, it is also possible that a time-delay system possesses only n eigenvalues. This case arises if the characteristic equation contains no 'delay terms' of the form e^{-sih} .

Example 3.4: The homogenous part of the wind tunnel model of Section 1.2 is an equation of the form

$$\dot{x}(t) = \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\omega^2 & -2\xi\omega \end{bmatrix} x(t) + \begin{bmatrix} 0 & ka & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t-0.33)$$

where 1/a = 1.964 [$\sec^{-1}$], ω = 6.0 [rad/sec], ξ = 0.8 [–], and k = – 0.0117 [$\deg^{-1}$]. This time-delay system has only three eigenvalues: $\lambda_1 = 0.5092$, $\lambda_{2/3} = -4.8 \pm 3.6 \cdot j$.

The following Corollary yields bounds on the real and the imaginary part of the eigenvalues of the system (3.98).

Corollary 3.7*: Every eigenvalue λ_i of the system (3.98) satisfies the following inequalities*

$$Re(\lambda_i) \le \|A_0\|_{\cdot} + \sum_{i=1}^{k} \|A_i\|_{\cdot} \tag{3.100}$$

$$|\mathrm{Im}(\lambda_i)| \le \|A_0\|_{\cdot} + \sum_{i=1}^{k} \|A_i\|_{\cdot} \quad \text{if} \quad Re(\lambda_i) \ge 0 \tag{3.101}$$

for any $\cdot = 1, 2,$ *and* ∞ .

Proof: Clearly, inequality (3.100) is satisfied for every λ_i with negative real part. In the following it is assumed that $Re(\lambda_i) \geq 0$. For every eigenvalue of the system (3.98) we may write

$$\lambda_i w = \left[A_0 + \sum_{i=1}^{k} A_i e^{-\lambda_i i \cdot h} \right] w$$

where w is some nonzero element of $\boldsymbol{R}^n$. Taking the norm on both sides of the equation yields

$$|\lambda_i||w| \leq \left[\|A_0\|_{\cdot} + \sum_{i=1}^{k} \|A_i\|_{\cdot} \left| e^{-\lambda_i i \cdot h} \right| \right] |w| \quad . \tag{3.102}$$

Since

$$\left| e^{-\lambda_i i \cdot h} \right| = e^{-Re(\lambda_i) i \cdot h}$$

and $Re(s) > 0$, we obtain from (3.102)

$$|\lambda_i| \leq \|A_0\|_{\cdot} + \sum_{i=1}^{k} \|A_i\|_{\cdot} \quad . \tag{3.103}$$

From (3.103) and inequality $Re(\lambda_i) \leq |\lambda_i|$ condition (3.100) follows. Furthermore, inequality $|Im(\lambda_i)| \leq |\lambda_i|$ together with inequality (3.103) implies (3.101). This completes the proof. ❑

Condition (3.100) can be improved by applying a transformation:

$$Re(\lambda_i) \leq \left\| T^{-1} A_0 T \right\|_{\cdot} + \sum_{i=1}^{k} \left\| T^{-1} A_i T \right\|_{\cdot} \quad .$$

An appropriate transformation matrix T can be found using numerical methods, e.g., the method of steepest descent. Those elements of the matrices A_0, A_i which do not arise in the characteristic equation can be omitted for the estimate.

The properties (i) – (iii) of the eigenvalues are also valid for linear hereditary systems of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - \tau_i) + \int_{-h}^{0} A_{01}(\theta) x(t + \theta) d\theta \quad .$$

The corresponding characteristic equation given by

$$\det\left[sI - A_0 - \sum_{i=1}^{k} A_i e^{-s\tau_i} - \int_{-h}^{0} A_{01}(\theta) e^{s\theta} d\theta \right] = 0 \tag{3.104}$$

can be rewritten in a form similar to (3.99) if the integral can be analytically solved. This is illustrated by the following examples.

Example 3.5: According to (3.104), the scalar system

$$\dot{x}(t) = a_0 x(t) + a_1 x(t - 0.5) + \int_{-1}^{0} k \cdot x(t + \theta) d\theta$$

has the characteristic function

$$g_1(s) = s - a_0 - a_1 e^{-0.5s} - k\frac{1 - e^{-s}}{s} = 0 \qquad \text{for } s \neq 0$$

$$a_0 + a_1 + k = 0 \qquad \text{for } s = 0 \,.$$

Let

$$g_2(s) = s^2 - [a_0 + a_1 e^{-0.5s}] \cdot s - k(1 - e^{-s}) = 0 \,.$$

The functions $g_1(s)$ and $g_2(s)$ have the same roots (except that $g_2(s)$ possesses an additional root at the origin). However, the form of $g_2(s)$ is similar to that of (3.99).

Example 3.6: The system

$$\dot{x}(t) = a_0 x(t) + a_1 x(t - 0.5) + \int_{-1}^{0} k \cdot \sin(\pi\theta) \cdot x(t + \theta) d\theta$$

is associated with the characteristic equation

$$s - a_0 - a_1 e^{-0.5s} - k\frac{1 + e^{-s}}{s^2 + \pi^2} = 0 \qquad \text{for } s \neq \pm \pi \cdot j$$

$$a_0 = 0,\ \pi - a_1 - \frac{k}{2} = 0 \qquad \text{for } s = \pm \pi \cdot j$$

which can be easily rewritten in the desired form (3.99).

3.4.2 The coefficients of the characteristic equation

The algorithm due to [20] and [92] enables us to compute the characteristic equation directly from the system matrices A_0, A_i. The numbers k and n are given by (3.98). The matrix $\Phi_{i,j} \in \boldsymbol{R}^{n \times n}$ and the numbers $\theta_{i,j} \in \boldsymbol{R}$ are defined by

$$\theta_{0,0} = 1$$

$$\theta_{i,j} = 0 \qquad \text{for } j < 0 \quad \text{or} \quad j > i \cdot k, \qquad i \geq 1$$

$$\Phi_{1,j} = 0 \cdot I \qquad \text{for } j < 0 \quad \text{or} \quad j > (i-1) \cdot k, \; i \geq 1$$

and by the recursive algorithm

$$\Phi_{1,0} = I$$

$$\theta_{i,j} = -\frac{1}{i} \operatorname{tr} \left\{ \sum_{r=0}^{k} A_r \Phi_{i,j-r} \right\} \qquad i = 1, ..., n, \qquad j = 0, ..., i \cdot k$$

$$\Phi_{i+1,j} = \sum_{r=0}^{k} A_r \Phi_{i,j-r} + \theta_{i,j} I \qquad i = 1, ..., n-1, \qquad j = 0, ..., i \cdot k \,.$$

The numbers $\theta_{i,j}$ are the coefficients of the characteristic equation of the system (3.98):

$$P(s, e^{-sh}) = s^n + \sum_{i=1}^{k} \theta_{1,j} e^{-sh \cdot j} s^{n-1} + \sum_{i=0}^{2k} \theta_{2,j} e^{-sh \cdot j} s^{n-2} + \qquad (3.105)$$
$$+ \sum_{i=0}^{nk} \theta_{n,j} e^{-sh \cdot j} \,.$$

3.4.3 Approximation of the poles with large modulus

Although it probably is impossible to express the roots of the characteristic equation by elementary operations and functions, asymptotic descriptions for roots with large moduli are available. Following [33, p. 32], [13, Chapter 12], [88], [92] we shall show a method for obtaining them.

Let $H \in \boldsymbol{R}^{(n+1)\times(k\cdot n+1)}$ denote the matrix of coefficients $\theta_{i,j}$ in the polynomial $P(s, e^{-sh})$, with $\theta_{0,0}$ in the upper left corner of H, and with j increasing in the horizontal direction, i in the vertical.

$$H = \begin{bmatrix} \theta_{0,0} & 0 & .. & 0 & 0 & .. & 0 & 0 & .. & 0 \\ \theta_{1,0} & \theta_{1,1} & .. & \theta_{1,k} & 0 & .. & 0 & 0 & .. & 0 \\ \theta_{2,0} & \theta_{2,1} & .. & \theta_{2,k} & \theta_{2,k+1} & .. & \theta_{2,2k} & 0 & .. & 0 \\ : & : & & : & : & & : & : & & : \\ \theta_{n,0} & \theta_{n,1} & .. & \theta_{n,k} & \theta_{n,k+1} & .. & \theta_{n,2k} & \theta_{n,2k+1} & .. & \theta_{n,kn} \end{bmatrix}$$

The nonzero elements of the matrix H are associated with a diagram, the so-called distribution diagram [13, p. 410]. Fig 3.8 shows the relationship of the distribution diagram to the characteristic equation:

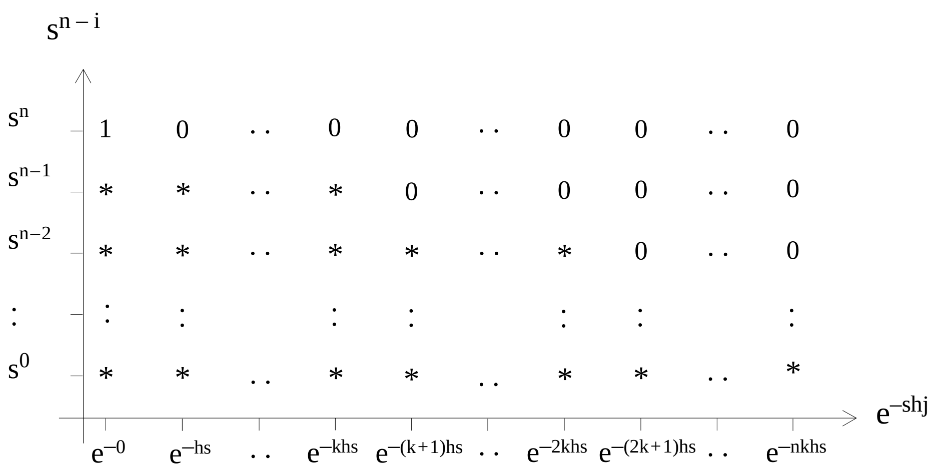

*Fig. 3.8 Coefficients $\theta_{i,j}$ stored as a quasi-triangular matrix. The symbol * denotes the (possibly) nonzero elements.*

From here on the distribution diagram is used in a normalized form, as depicted in Fig. 3.9.

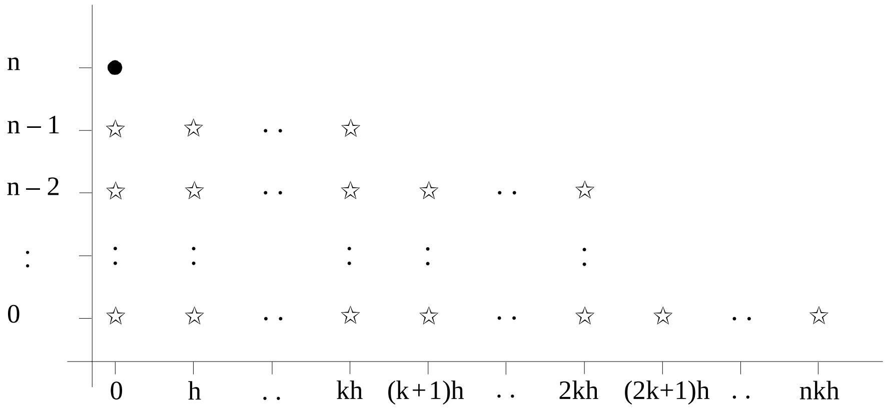

Fig. 3.9 The asterisk ☆ denotes a (possible) point in the normalized distribution diagram

Let $L_1, L_2, \ldots, L_v, \ldots, L_f$ denote an upward convex polygonal graph in the normalized distribution diagram, with the straight line L_v having a slope of $-m_v$, where $m_v > 0$. Furthermore, the polygon possesses the properties that no points lie above it, and the point (n, 0) is connected with the upper- and right-most point. An example of a polygon is displayed in Fig. 3.10.

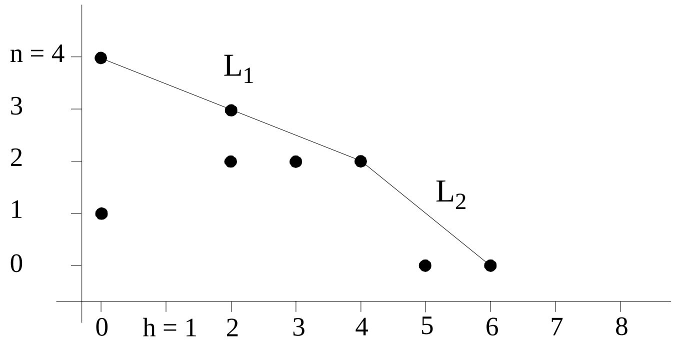

Fig. 3.10 An example of a normalized distribution diagram

For each L_v the polynomial

$$\phi_v(z) = \sum_{L_v} \theta_{i,j} z^{n-i} \tag{3.106}$$

is calculated. The sum is taken over those $\theta_{i,j}$ which are located on the segment L_v. A root z_i of the polynomial (3.106) is denoted by z_i. This root z_i corresponds to the eigenvalues s in the complex plane as follows [92], [13, p. 409]:

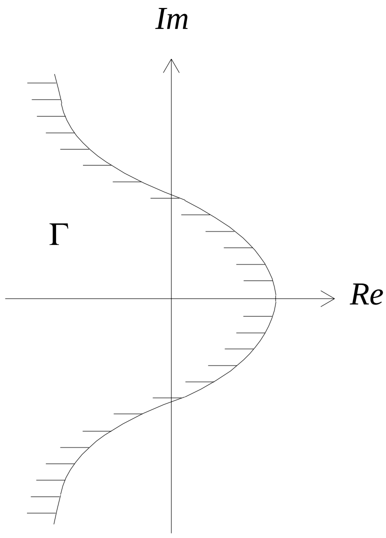

Fig. 3.11 Region Γ

$$Re(s) = m_v\left\{\ln|z_i| - \ln\left|2r\cdot m_v\pi + m_v\arg(z_i) - m_v\pi/2\right|\right\} + o(1) \quad (3.107)$$

$$Im(s) = m_v\left\{2r\pi + \arg(z_i) - \pi/2\right\} + o(1)\ . \quad (3.108)$$

where r = 1, 2, …; $\arg(z_i) \in [0, 2\pi]$; and the symbol $o(\cdot)$ indicates the error of magnitude. Equations (3.107) and (3.108) are an approximation of the eigenvalues with positive imaginary part. They describe a so-called chain of eigenvalues in the complex plane. If the polynomial (3.106) has several roots, there exist several chains. These chains are collected in a strip. For another m_v, we obtain another polynomial of the form (3.106) and hence, another strip with one or more chains. Furthermore, formula (3.107) and (3.108) show that the eigenvalues of (3.98) are asymptotically located on the curve [13, Theorem 12.8]

$$Re\left(s + \frac{\ln(s)}{h}\right) = \frac{\ln|z|}{h} \quad (3.109)$$

and are asymptotically separated from each other by the distance $2\pi/h$. The curve (3.109) bounds the region in the complex plane where the eigenvalues of a time-delay system are located in general [13, p. 100]. This region is denoted by Γ in Fig. 3.11.

Example 3.7: The scalar system

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-h) + a_2 x(t-2h) \qquad (3.110)$$

with the corresponding characteristic equation

$$P(s, e^{-sh}) = s - a_0 - a_1 e^{-sh} - a_2 e^{-s2h} = 0$$

yields the following matrix H

$$H = \begin{bmatrix} 1 & 0 & 0 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} .$$

This matrix implies the points (0, 0), (0, 1), (h, 0), and (2h, 0) in the normalized distribution diagram illustrated in Fig. 3.12.

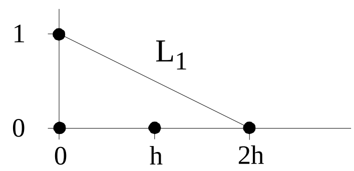

Fig. 3.12 Normalized distribution diagram for the system (3.110)

The only polygon connecting the points (0, 1) and (2h, 0) under the restriction of being convex is the straight line L_1 with the slope $-m_1 = -1/(2h)$. Therefore, the polynomial (3.106) is of the form

$$\phi_1(z) = z - a_2 .$$

Its root is $z_1 = a_2$. For this root, the equations (3.107) and (3.108) yield the following estimate of the eigenvalues:

$$Re(s) = 1/(2h)\left\{\ln|a_2| - \ln\left|\pi/h \cdot (r - 0.25)\right|\right\} + o(1)$$

$$Im(s) = 1/(2h)\left\{2r\pi - \pi/2\right\} + o(1)$$

where r = 1, 2, … .

Example 3.8: We consider the system

$$\ddot{x}(t) = -4\pi^2 x(t) - \pi^2 x(t-h) . \qquad (3.111)$$

Its characteristic equation is of the form

$$s^2 + 4\pi^2 + \pi^2 e^{-sh} = 0 \ .$$

The coefficients of the above equation determine the matrix H:

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 4\pi^2 & \pi^2 & 0 \end{bmatrix} .$$

The corresponding normalized distribution diagram is shown in Fig. 3.13.

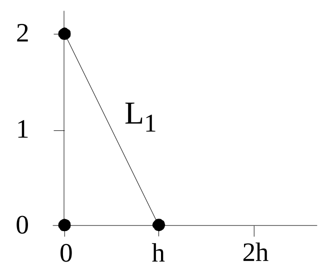

Fig. 3.13 Normalized distribution diagram for the system (3.111)

The only polygon connecting the points (0, 1) and (h, 0) under the restriction of being convex is the straight line L_1 with the slope $-m_1 = -2/h$. According to (3.106) the straight line L_1 produces the polynomial

$$\phi_1(z) = z^2 + \pi^2$$

with the roots $z_{1/2} = \pm\pi\cdot j$. This means that the eigenvalues with large moduli are located in one strip with two chains. The equations (3.107) and (3.108) yield the following estimates of the eigenvalues:

1st chain: $\quad Re(s) = 1/h\{\ln|h/(4\pi)|\} + o(1)$

$\quad Im(s) = 4\pi r/h + o(1)$

2nd chain: $\quad Re(s) = 2/h\{\ln|\pi| - \ln|2\pi(2r-1)/h|\} + o(1)$

$\quad Im(s) = 2\pi/h\{2r + 1\} + o(1)$

where r = 1, 2, … .

3.4.4 Approximation of the poles near the origin

The method of Kuhn [73] finds the roots of a function f(s) in a given rectangle Q in the complex plane. The rectangular region Q is partitioned into triangles such that a grid results as illustrated in Fig. 3.14.

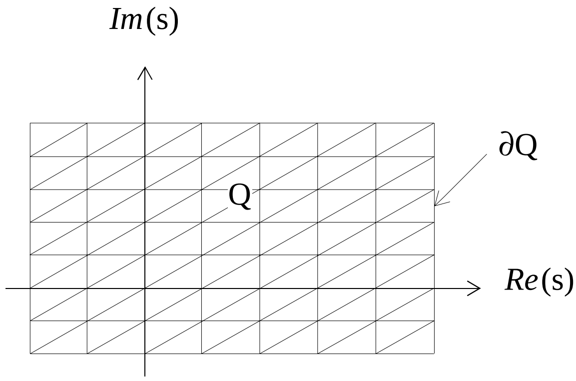

Fig. 3.14 Rectangle Q in the complex plane partitioned into triangles

Each mesh-point is weighted by the function $\ell(s)$:

$$\ell(s) = \begin{cases} 1 \text{ if } -\pi/3 \leq \arg(f(s)) \leq \pi/3 & \text{or if } \quad f(s) = 0 \\ 2 \text{ if } \quad \pi/3 < \arg(f(s)) \leq \pi & \\ 3 \text{ if } -\pi \quad < \arg(f(s)) < -€\pi/3 \ . & \end{cases} \tag{3.112}$$

Let $\{\ell(s_1), \ell(s_2), \ell(s_3)\}$ be a triple of distinct points. The triangle with vertices s_1, s_2, s_3 is said to be 'saturated' [92] or 'completely labelled' [73], if $\ell(s_1) = 1$, $\ell(s_2) = 2$, and $\ell(s_3) = 3$. Let the length of edge of a saturated triangle be smaller than ε. For all points lying inside of such an ε-small, saturated triangle it can be shown that $|f(s)| \leq 2\varepsilon/\sqrt{3}$ [73]. By subdividing a saturated triangle into smaller triangles and finding the saturated ones, a sequence of shrinking triangles can be constructed, the centres of which converge to a zero of the function f(s). Hence, the problem of finding the zeros of f(s) in a given bounded region can be attacked by constructing a subdivision of the region into triangles and finding the saturated ones.

In Fig. 3.15, the values of $\ell(s)$ with respect to the following characteristic equation

$$s^2 + 4\pi^2 + \pi^2 e^{-s\pi/4} = 0 \tag{3.113}$$

associated with the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -4\pi^2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -\pi^2 & 0 \end{bmatrix} x(t - \pi/4)$$

are shown. The plot reveals four eigenvalues. Furthermore, Fig. 3.15 illustrates how the algorithm can be computed efficiently. First, the boundary of Q is considered. The pairs of points of ∂Q at which the value of $\ell(s)$ changes are stored. From these points the algorithm follows the boundary of two zones into the inside of Q. In this way the algorithm proceeds along a boundary until either a saturated triangle has been found, or the boundary of Q is reached again. The search continues until all the boundaries have been traversed. In the next stage, the algorithm proceeds to a subdivision of rectangles, including saturated triangles, by decreasing the mesh size. This subdivision is performed several times until the desired accuracy of approximation is reached. The centres of the saturated rectangles (a rectamgle is called saturated if it contains a saturated triangle) then serve as starting points for a refinement via Newton's method (see next subsection). The roots of sufficiently large modulus are more quickly computed by the "asymptotic formulas" (3.107) and (3.108) introduced in the previous section.

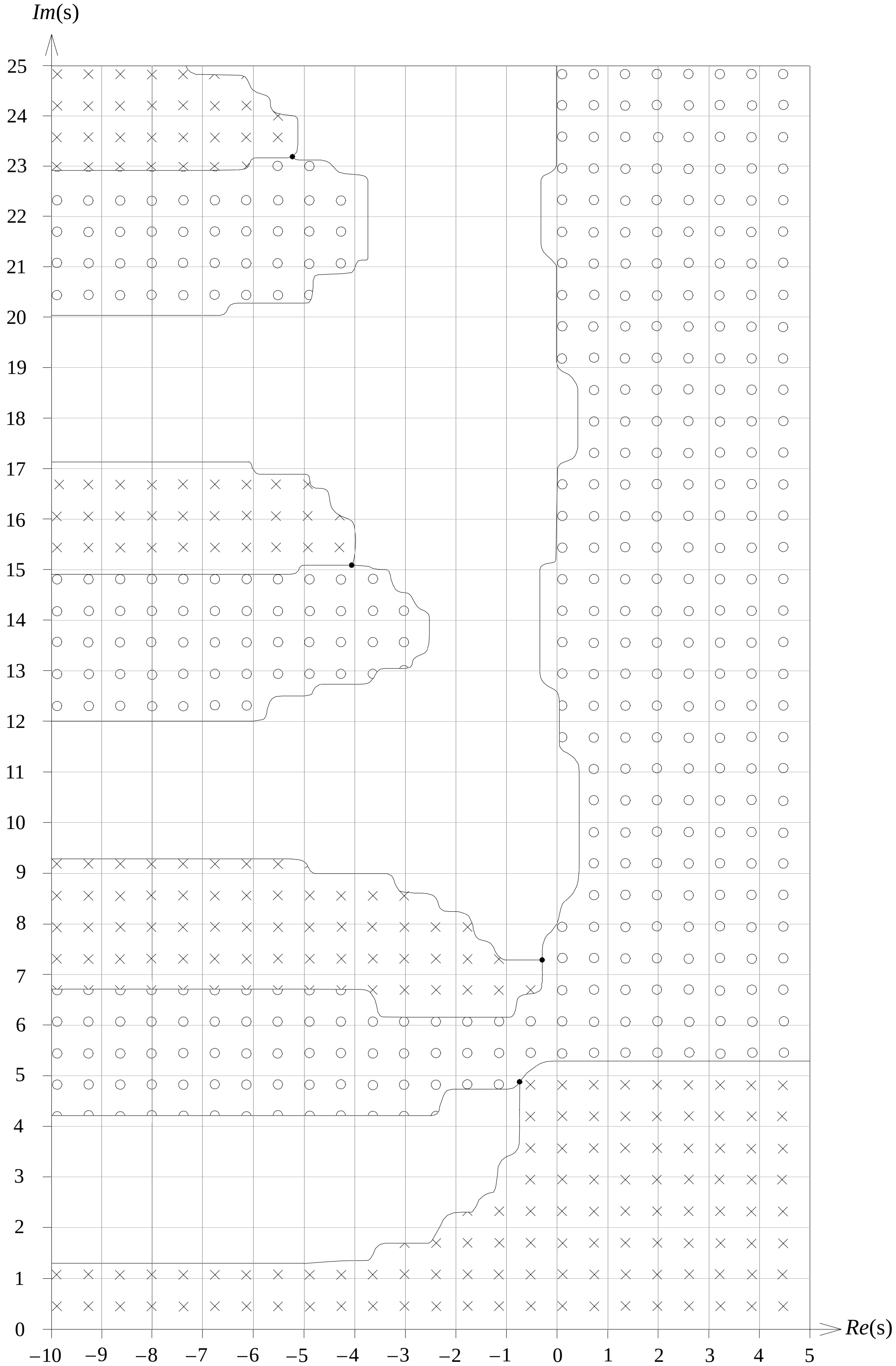

Fig. 3.15 l(s) computed for characteristic function (3.113)
×*-region: l(s) = 1;* ○*-region: l(s) = 2; empty spaces: l(s) = 3*

3.4.5 Refinement of the approximation

The approximation of the roots computed with the algorithms of Kuhn (Subsection 3.4.4) and Bellman and Cook (Subsection 3.4.3) are used as an initial guess to start Newton's method for improvement. This method is defined by the well-known iteration formula

$$\lambda_{i+1} = \lambda_i - \frac{\det[\Delta(\lambda_i)]}{\det[\Delta(\lambda_i)]'} \quad . \tag{3.114}$$

The formula requires the evaluation of the characteristic equation and its derivation $\det[\Delta(\lambda_i)]' = \frac{\partial}{\partial\lambda}\det[\Delta(\lambda)]$. In Subsection 3.4.2, a procedure was illustrated to determine the coefficients of the characteristic equation. Manitius *et al.* (1984) suggest to use this algorithm for obtaining $\det[\Delta(\lambda_i)]'$. However, there is a more efficient way to evaluate (3.114). Since λ_i is a given numerical value, there is no problem to compute

$$\det[\Delta(\lambda_i)] = \det\left[I\lambda_i - A_0 - \sum_{i=1}^{k} A_i e^{-\lambda_i h\cdot i}\right] \quad .$$

The term $\det[\Delta(\lambda_i)]'$ is evaluated in two steps. First, the matrices

$$\Delta(\lambda_i) = I\lambda_i - A_0 - \sum_{i=1}^{k} A_i e^{-\lambda_i h\cdot i} \tag{3.115}$$

and

$$\begin{bmatrix} \frac{\partial\Delta(\lambda_i)_{1,1}}{\partial\lambda_i} & \cdot\cdot & \frac{\partial\Delta(\lambda_i)_{1,n}}{\partial\lambda_i} \\ \vdots & & \vdots \\ \frac{\partial\Delta(\lambda_i)_{n,1}}{\partial\lambda_i} & \cdot\cdot & \frac{\partial\Delta(\lambda_i)_{n,n}}{\partial\lambda_i} \end{bmatrix} = I + \sum_{i=1}^{k} h\cdot i\, A_i e^{-\lambda_i h\cdot i} \tag{3.116}$$

are stored. Next, the following relation is used [14], [115, p. 18]:

$$\det[\Delta(\lambda_i)]' = \det\begin{bmatrix} \dfrac{\partial\Delta(\lambda_i)_{1,1}}{\partial\lambda_i} & \Delta(\lambda_i)_{1,2} & .. & \Delta(\lambda_i)_{1,n} \\ : & : & & : \\ \dfrac{\partial\Delta(\lambda_i)_{n,1}}{\partial\lambda_i} & \Delta(\lambda_i)_{n,1} & .. & \Delta(\lambda_i)_{n,n} \end{bmatrix} + \ldots +$$

$$+ \det\begin{bmatrix} \Delta(\lambda_i)_{1,1} & .. & \Delta(\lambda_i)_{1,n-1} & \dfrac{\partial\Delta(\lambda_i)_{1,n}}{\partial\lambda_i} \\ : & : & & : \\ \Delta(\lambda_i)_{n,1} & .. & \Delta(\lambda_i)_{n,n-1} & \dfrac{\partial\Delta(\lambda_i)_{n,n}}{\partial\lambda_i} \end{bmatrix} .$$

The right-hand side of the latter equation can be computed with the help of the stored matrices (3.115) and (3.116).

Note that λ_i is a multiple eigenvalue, if $\det[\Delta(\lambda_i)]' = 0$.

3.4.6 Test on the number of eigenvalues

The improved eigenvalues are arranged in order of increasing modulus in one list. The number of eigenvalues within a given disc D of prescribed radius ρ is then counted. Then the Carpentier-Dos Santos algorithm [24] is applied to verify that all eigenvalues have been found in the disc D. The algorithm consists of the computation of the value of a circulation integral over the disc D. For the numerical evaluation of this integral the circle is subdivided into M pieces as illustrated in Fig. 3.16.

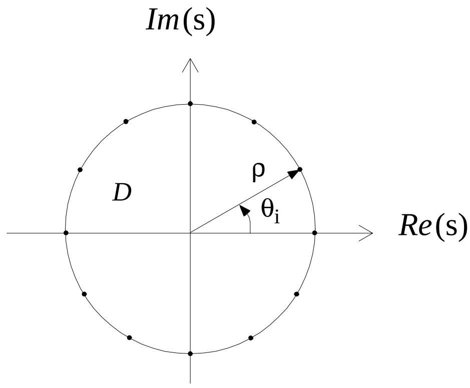

Fig. 3.16 Circle with mesh-points, Disc D, radius ρ*, and angle* θ_i *in the complex plane*

The function $\bar{\Delta}(\theta_i)$ is defined by

$$\bar{\Delta}(\theta_i) = \Delta(s = \rho e^{\theta_i})$$

Then, the number of eigenvalues C_0 within the given disc D is equal to [24]

$$C_0 \cong \frac{1}{2\pi j} \sum_{i=1}^{M} \ln \left[\bar{\Delta}(\theta_i) / \bar{\Delta}(\theta_i - 2\pi/M) \right] .$$

Numerical errors may occur if the roots are close to the boundary of the disc. Therefore, Carpentier and Dos Santos (1982) proposed the following test on the computed values of $\bar{\Delta}(\theta_i) / \bar{\Delta}(\theta_{i-1})$:

If for all i = 1, ..., M $\bar{\Delta}(\theta_i)$ is such that

(i) $\left| \arg\{ \bar{\Delta}(\theta_i) / \bar{\Delta}(\theta_{i-1}) \} \right| < 3\pi/4$

and

(ii) $1/6.1 < \left| \bar{\Delta}(\theta_i) / \bar{\Delta}(\theta_{i-1}) \right| < 6.1$

the computed value of C_0 is accepted.

If either of the conditions (i) or (ii) is not satisfied, the index C_0 is recalculated with M replaced by 2M. A good starting value of M is M = 128. In our experience this algorithm is very reliable.

3.4.7 Summary and examples

The various algorithms illustrated in the previous subsections are used to compute the eigenvalues directly from the system matrices A_0, A_i. The program realized with the help of the MatrixX software packages is structured as suggested by [92], with some modifications. The implementation of Newton's method has been considerably simplified (cf. Subsection 3.4.5). Furthermore, Corollary 3.7 is used to check whether all eigenvalues with nonnegative real part have been found.

First, Corollary 3.7 is used to find an upper bound of the real part of the roots in the right-hand side of the complex plane. Let us denote this value by ν. This value is used to define in the complex plane a quadratic region with the edges $e_1 = \nu + \nu \cdot j$, $e_2 = \nu - \nu \cdot j$, $e_3 = -\nu - \nu \cdot j$, and $e_4 = -\nu + \nu \cdot j$. According to Corollary 3.7, all eigenvalues with a nonnegative real part always lie inside of this square, designated in the following by Q_1. The algorithm of Kuhn is used to search the eigenvalues in the region Q_1. This estimate of the roots is improved by Newton's method. Next, the generalized Faddev algorithm (cf. Subsection 3.4.2) is applied to determine the coefficients of the characteristic equation from the system matrices A_0, A_1. The knowledge of the coefficients permits the computation of the eigenvalues with large moduli (cf. Subsection 3.4.3). Again, Newton's method is used for improving the numerical accuracy of the eigenvalues. Finally, the number of eigenvalues within a circle enclosing the square Q_1 is counted and tested by applying the procedure of [24]. If the test result is positive, we are sure that all eigenvalues with nonnegative real part have been found.

As an example we consider in the following the system (3.111)

$$\ddot{x}(t) = -4\pi^2 x(t) - \pi^2 x(t-h)$$

and its characteristic equation (3.112)

$$s^2 + 4\pi^2 + \pi^2 e^{-sh} = 0 \ .$$

Fig. 3.17 shows the first 22 eigenvalues for the constant delay $h = \pi/8$. It was checked that all roots had been found in the displayed region. The first ten eigenvalues are listed in Table 3.1. The calculations in example 3.6 reveal that the roots with a sufficiently large modulus are arranged in one strip consisting of two chains. In Fig. 3.17, these two chains are marked with the symbols ◆ and ✖. Since the delay $h = \pi / 8$ is small with respect to the system coefficients, the chains have a certain distance to the dominant eigenvalue. (The delay can even be neglected for stability considerations if the delay is sufficiently small [133]). However, if the value of the delay is of the same magnitude as the system coefficient, the eigenvalues of the chains are closer to the imaginary axis and the delay is of course relevant. (For an increasing delay the poles converge to the imaginary axis [109]). This is illustrated in Fig. 3.18 where the first 22 eigenvalues of the system (3.111) are depicted for $h = \pi$ are depicted. The numerical values of these poles are recorded in Table 3.2.

In Fig 3.19 the root loci of the first six eigenvalues for an increasing value of h are sketched. From the observation that the eigenvalues enter and leave the right-half plane, one may suppose that the system is alternately asymptotically stable (i.e., stable–unstable–stable–…, as the delay h increases). Indeed, applying the τ-decomposition method (cf. Subsection 3.3.2) we find that the system (3.111) is asymptotically stable iff

$$\frac{1}{\sqrt{3}} < h < \frac{2}{\sqrt{5}} \quad \text{or} \quad \frac{3}{\sqrt{3}} < h < \frac{4}{\sqrt{5}} \; .$$

As a further example, the eigenvalues of the model of the Williams-Otto process (cf. Section 1.2) of the form

$$\dot{x}(t) = \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.20 & -5.30 & -12.8 & 0 \\ 6.40 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11.0 & -3.96 \end{bmatrix} x(t) + \begin{bmatrix} 1.92 & 0 & 0 & 0 \\ 0 & 1.92 & 0 & 0 \\ 0 & 0 & 1.87 & 0 \\ 0 & 0 & 0 & 0.724 \end{bmatrix} x(t-1) \; .$$

are computed. According to the method described in Subsection 3.4.3, the eigenvalues with a large modulus are arranged in four chains collected in one strip (cf. Fig. 3.20).

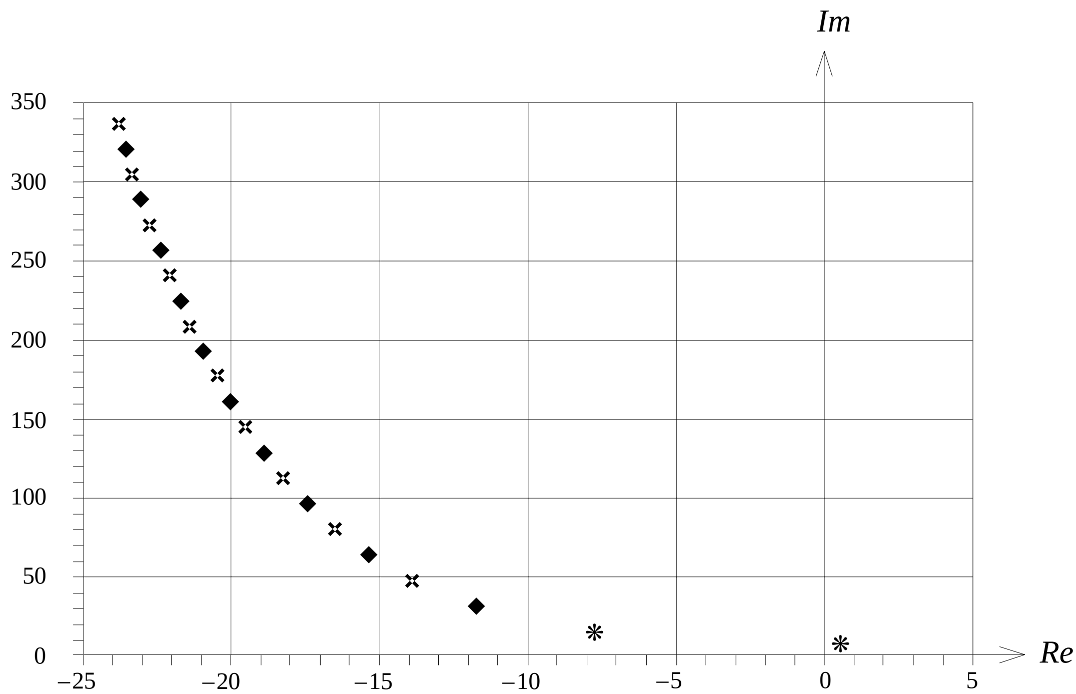

Fig. 3.17 The first 22 eigenvalues computed for the system

$$\ddot{x}(t) = -4\pi^2 x(t) - \pi^2 x(t - \pi/8) .$$

The roots, which were found by applying Kuhn's and Newton's method, are marked with the symbol ⁕. *The other roots were obtained by using the "asymptotic formulas" (3.107) and (3.107) together with the Newton method. The latter's eigenvalues are located in one strip containing two chains. The two chains are marked with the symbols* ◆ *and* ⌘ .

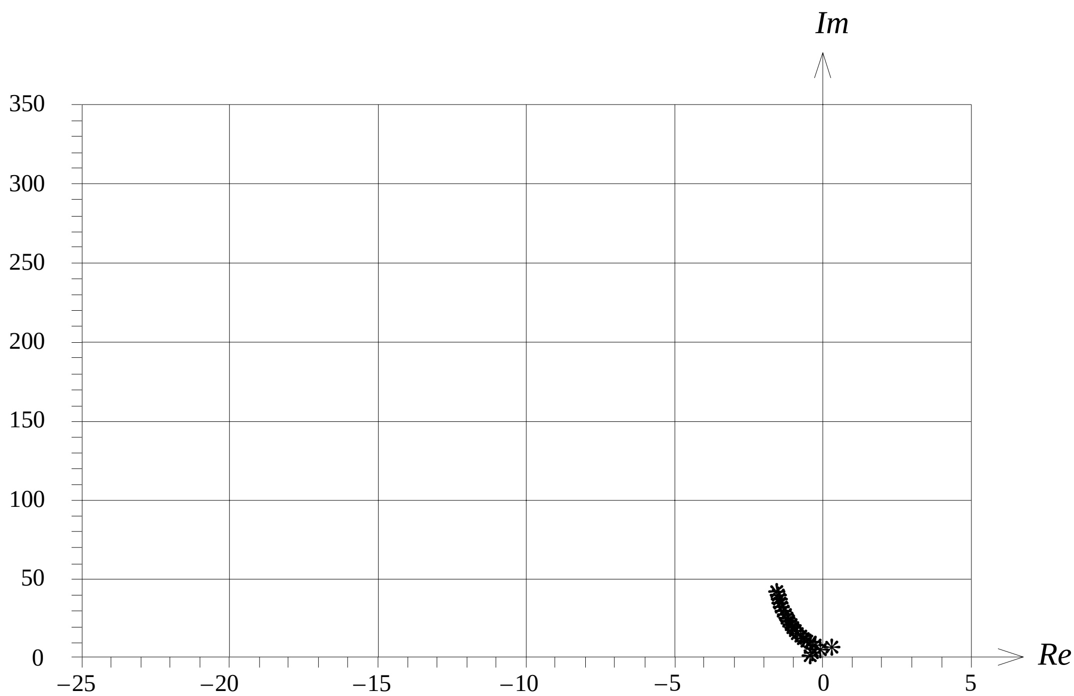

Fig. 3.18 The first 22 eigenvalues computed for the system

$$\ddot{x}(t) = -4\pi^2 x(t) - \pi^2 x(t - \pi).$$

Table 3.1 The first ten eigenvalues computed for the system
$\ddot{x}(t) = -4\pi^2 x(t) - \pi^2 x(t - \pi/8).$

k	λ_k	$\lvert\det\Delta(\lambda_k)\rvert$
1	$0.5125866951081 \pm 5.8612301778918 \cdot j$	$1.8 \cdot 10^{-13}$
2	$-7.7709153749941 \pm 12.7898921461502 \cdot j$	$2.7 \cdot 10^{-12}$
3	$-11.7909090749649 \pm 30.0267151620273 \cdot j$	$9.3 \cdot 10^{-12}$
4	$-13.9060327510865 \pm 46.4961588500000 \cdot j$	$5.4 \cdot 10^{-11}$
5	$-15.3787712955928 \pm 62.7650027658605 \cdot j$	$6.9 \cdot 10^{-11}$
6	$-16.5145897893815 \pm 78.9434946965780 \cdot j$	$1.9 \cdot 10^{-11}$
7	$-17.4407190325485 \pm 95.0721505091676 \cdot j$	$7.9 \cdot 10^{-11}$
8	$-18.2231767015346 \pm 111.1699761040171 \cdot j$	$1.8 \cdot 10^{-10}$
9	$-18.9008394703398 \pm 127.2472318439467 \cdot j$	$3.3 \cdot 10^{-10}$
10	$-19.4986017669266 \pm 143.3100003744767 \cdot j$	$1.8 \cdot 10^{-10}$

Table 3.2 The first ten eigenvalues computed for the system
$\ddot{x}(t) = -4\pi^2 x(t) - \pi^2 x(t - \pi).$

k	λ_k	$\lvert\det\Delta(\lambda_k)\rvert$
1	$0.29154380731165 \pm 6.39282352532310 \cdot j$	$1.3 \cdot 10^{-13}$
2	$-0.11751250169763 \pm 5.02636843241820 \cdot j$	$2.4 \cdot 10^{-13}$
3	$-0.28164636021213 \pm 7.94010300686203 \cdot j$	$1.8 \cdot 10^{-13}$
4	$-0.35964609615850 \pm 3.02267366287452 \cdot j$	$2.7 \cdot 10^{-13}$
5	$-0.43462577567080 \pm 1.00720873955413 \cdot j$	$4.8 \cdot 10^{-13}$
6	$-0.57490130595404 \pm 9.93907538142810 \cdot j$	$5.1 \cdot 10^{-13}$
7	$-0.75019152920180 \pm 11.94497509110028 \cdot j$	$9.5 \cdot 10^{-13}$
8	$-0.87923926724846 \pm 13.94983015985700 \cdot j$	$1.6 \cdot 10^{-12}$
9	$-0.98277788801663 \pm 15.95370323553014 \cdot j$	$2.1 \cdot 10^{-12}$
10	$-1.06985187323271 \pm 17.95686956898248 \cdot j$	$2.7 \cdot 10^{-12}$

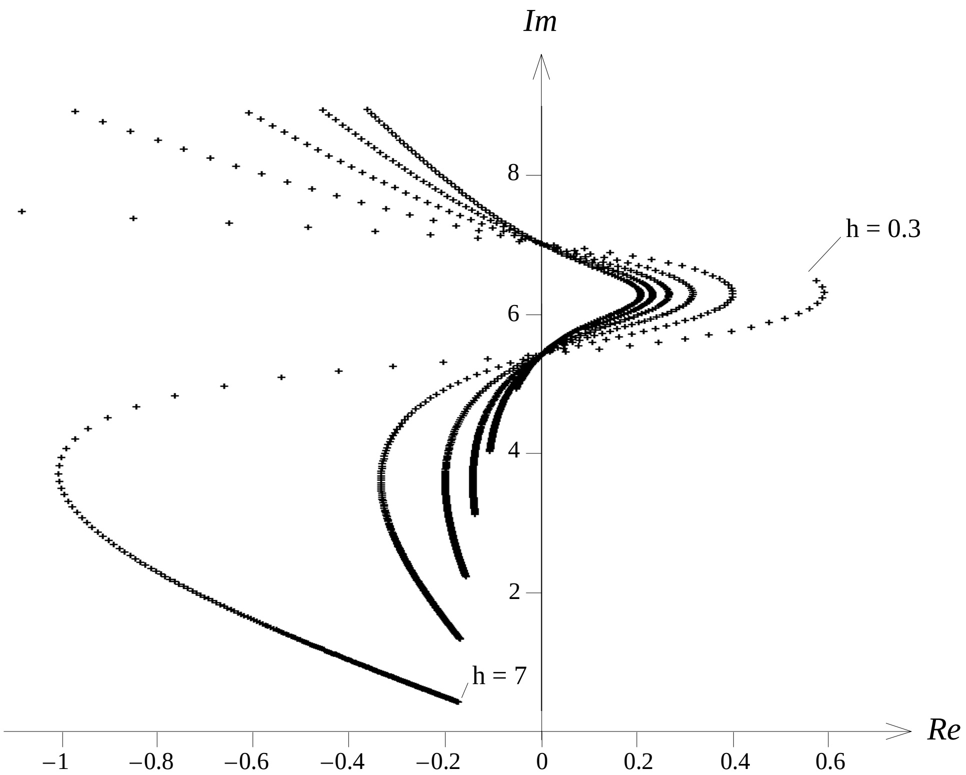

Fig. 3.19 Root loci of the first six eigenvalues of the system (3.111) for $h \in [0.3, 7]$

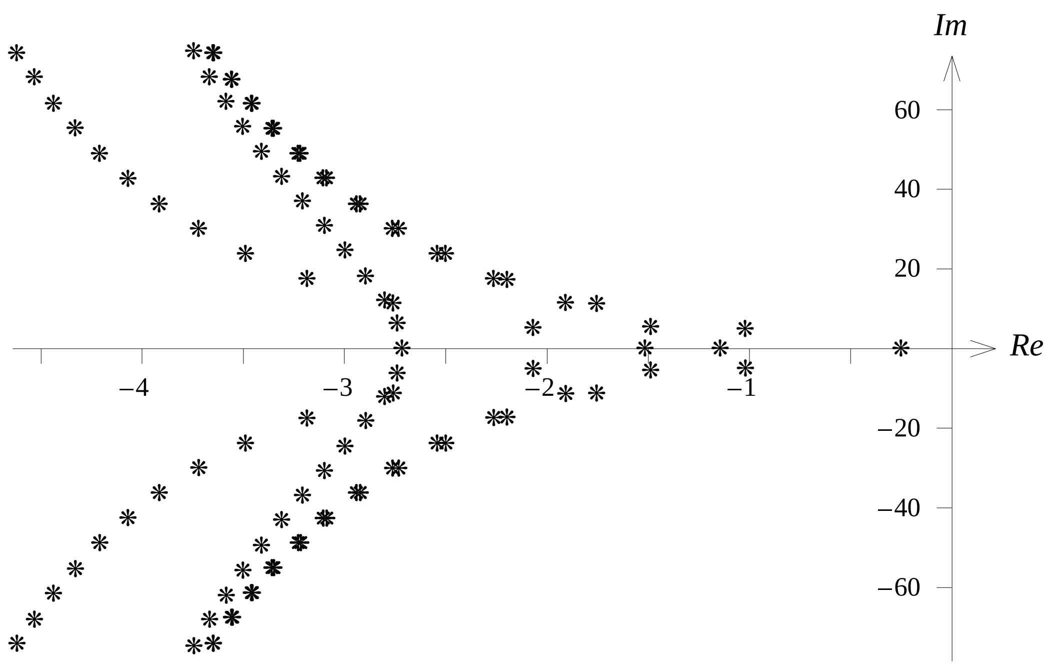

Fig. 3.20 Eigenvalues of the Williams-Otto process model

4 Robustness

In many cases, a system is considered as robust if its stability is unaffected by perturbations, as far as possible. These perturbations represent several types of modelling uncertainties. To begin with, we will treat delays as an extra perturbing input of a delay-free system. Frequency-domain concepts are mainly used to study this robustness problem. The frequency-domain representation of the system

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k_x} A_i x(t - i \cdot h) + B_0 u(t) + \sum_{i=1}^{k_u} B_i u(t - i \cdot h) \qquad (4.1)$$

$$y(t) = C_0 x(t) + \sum_{i=1}^{k_y} C_i x(t - i \cdot h) \qquad (4.2)$$

where $A_0, A_i \in \boldsymbol{R}^{n \times n}$, $B_0, B_i \in \boldsymbol{R}^{n \times m}$, and $C_0, C_i \in \boldsymbol{R}^{p \times n}$, expressed in terms of its system matrices, is

$$G(s, e^{-sh}) = [C_0 + \sum_{i=1}^{k_y} C_i e^{-sih}][sI - A_0 - \sum_{i=1}^{k_x} A_i e^{-sih}]^{-1}[B_0 + \sum_{i=1}^{k_u} B_i e^{-sih}]. \quad (4.3)$$

It is assumed that the plant $G(s, e^{-sh})$ is connected with the controller $K(s)$ as illustrated in Fig. 4.1.

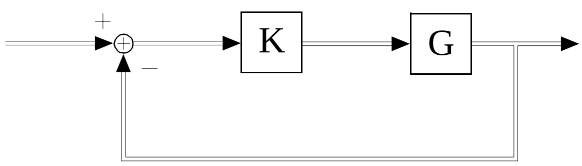

Fig. 4.1

In Section 4.1, the delays in the input/output are considered as a perturbation. As an introduction, the Nyquist criterion is briefly discussed for single-input single-output (SISO) systems. The relation between the phase margin of a delay-free system and its robustness against input/output delays is shown by using the Bode diagram. These considerations are then applied to analyse multi-input multi-output (MIMO) systems using singular values and the small-gain theorem. In Section 4.2, state delays are treated as a perturbation.

In the last two subsections, robustness bounds for unstructured uncertainties of time-delay systems are presented. With respect to modelling uncertainties of the delays these bounds are considered in Section 4.3. Robustness bounds for large-scale systems with time-dependent and state-dependent delays are investigated in Section 4.4.

4.1 Robustness against input/output delays

This section deals with control systems having delays in the input and/or in the output. First some graphical methods are discussed. The main graphical scheme for stability of systems with constant delays is the classical approach of the Nyquist criterion. The application of this method is suitable for checking the stability of SISO systems with input and/or output delays. The Bode diagram is then used to show the relation between the phase margin and the robustness of a nondelayed system against input and/or output delays. These considerations are extended for MIMO systems using standard tools in the frequency domain such as singular values, the small-gain theorem, and the complementary sensitivity function.

Let us start with the Nyquist criterion for SISO systems. The transfer function of the system shown in Fig. 4.1 is

$$T(s, e^{-sh}) = \frac{G(s, e^{-sh})K(s)}{1 + G(s, e^{-sh})K(s)} = \frac{G_0(s, e^{-sh})}{1 + G_0(s, e^{-sh})} \,. \tag{4.4}$$

It is assumed that

(i) all eigenvalues of the function $G_0(s, e^{-sh})$ have negative real part

(ii) $\lim\limits_{\omega \to \infty} G_0(j\omega, e^{-j\omega h}) = 0$.

Nyquist criterion, Theorem 4.1 [94, p. 55]: *The transfer function T(s) is asymptotically stable iff the frequency response of $G_0(j\omega, e^{-j\omega h})$ in the complex plane does not encircle the point $(-1, 0\cdot j)$ for $\omega \in [0, \infty]$.*

Theorem 4.1 is also valid if the control system G_0 contains delays in the state. However, condition (i) is particularly easy to check if the system only has delays in the control input or/and in the output, since the corresponding characteristic equation has no delay terms. Another possibility to study the stability of $T(s, e^{-sh})$ is to consider the characteristic equation of the closed-loop system. Since this characteristic equation contains delay terms, the Nyquist criterion is easier to apply.

Example 4.1: An illustrative example of the form

$$G_0(s, e^{-sh}) = \frac{b_0 + b_1 e^{-sh}}{s + a_0} \tag{4.5}$$

is considered. The Nyquist graph is given by

$$Re(G_0(j\omega, e^{-j\omega h})) = \frac{a_0 b_0 + b_1 a_0 \cos(\omega h) - b_1 \omega \sin(\omega h)}{a_0^2 + \omega^2}$$

$$Im(G_0(j\omega, e^{-j\omega h})) = -\frac{\omega b_0 + \omega b_1 \cos(\omega h) + a_0 b_1 \sin(\omega h)}{a_0^2 + \omega^2} \cdot j$$

The Nyquist curves for the following numerically given open-loop systems

$$G_0(s, e^{-sh}) = \frac{1 - 0.5e^{-sh}}{s + 1} \tag{4.6}$$

$$G_0(s, e^{-sh}) = \frac{e^{-sh}}{s} \tag{4.7}$$

are depicted in Figs. 4.3 and 4.4. The plots illustrate that both closed-loop systems are asymptotically stable. Assumption (i) is not fulfilled for (4.7), since this G_0 possesses a pole s = 0 on the imaginary axis. Nevertheless, the method yields a correct stability analysis (cf. Theorem 3.10). Indeed, Theorem 4.1 is also valid if G_0 has one or two poles at s = 0 [34, Subsections 4.4.3 and 4.4.5]. Finally, the example illustrates the typical spiralling of the Nyquist contours near the origin for open-loop systems with delays.

In the frequency domain there is no possibility to judge whether the delay term is associated with an input or an output retardation in the time-domain. Therefore, it is no restriction if in the following we consider only input delays.

Note that a SISO system may have delayed and non-delayed inputs simultaneously. As an example the signal flow diagram of the system $\dot{x}(t) = a_0x(t) + b_0u(t) + b_1u(t - h)$ related to the transfer function (4.5) is shown in Fig. 4.2.

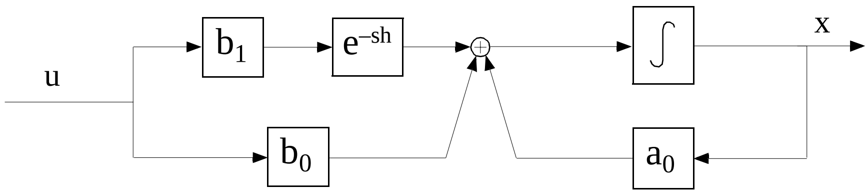

Fig. 4.2

If the input is purely delayed, which means that we have a delay-free system in cascade with a delay,

$$T(s, e^{-sh}) = \frac{G_0(s)e^{-sh}}{1 + G_0(s)e^{-sh}} \tag{4.8}$$

the checking of the stability of $T(s, e^{-sh})$ can be remarkably simplified.

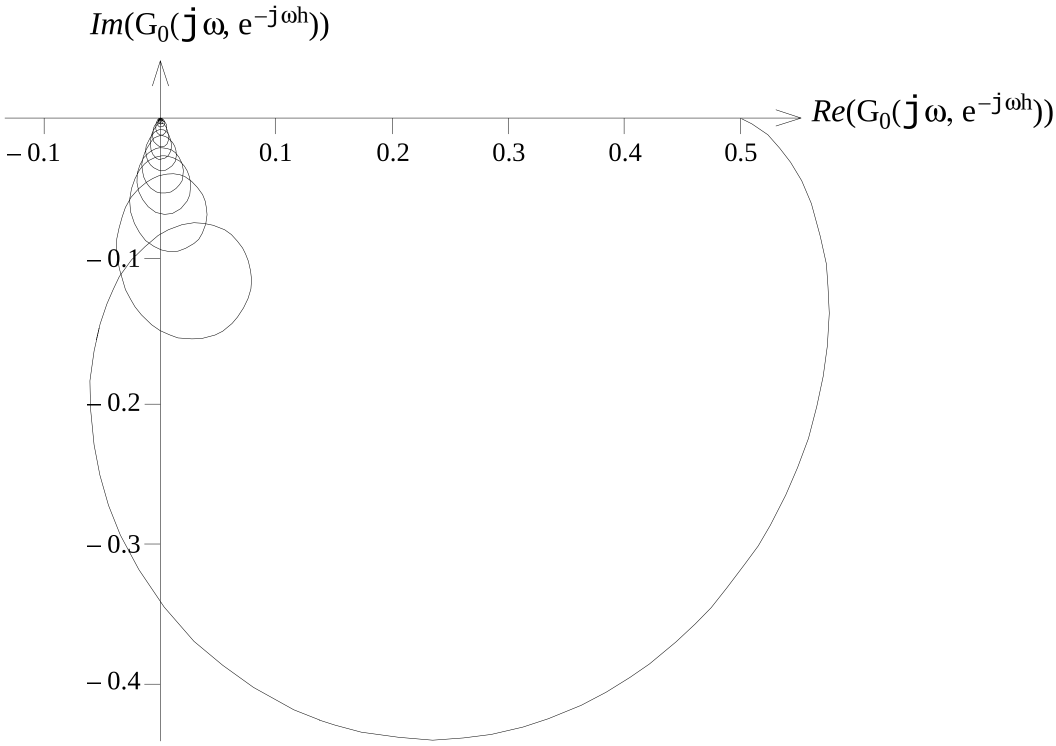

Fig. 4.3 Nyquist curve for (4.6)

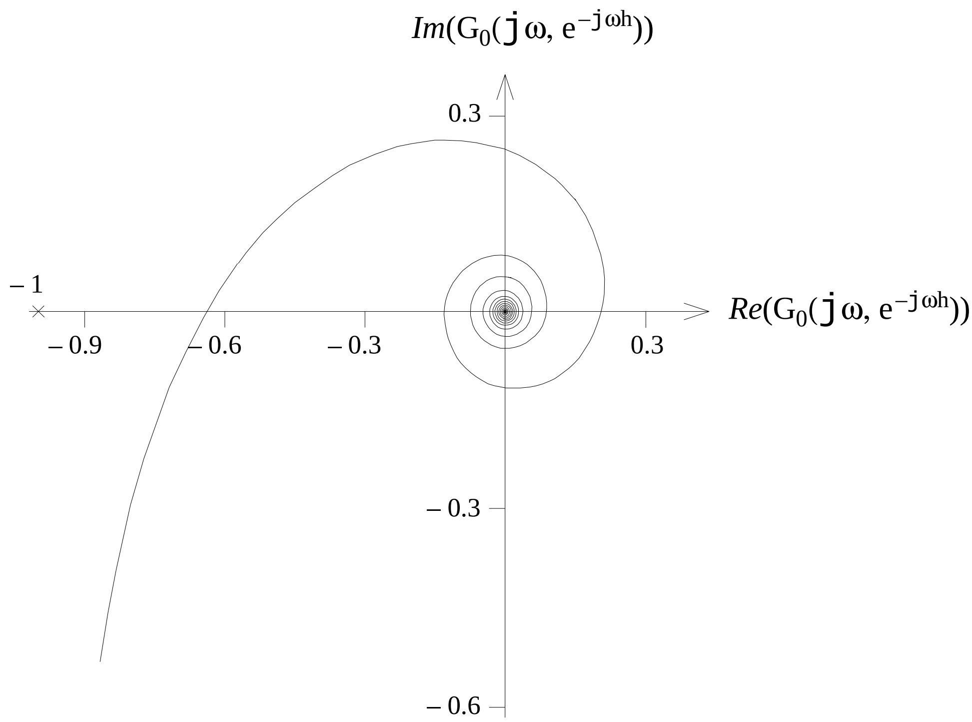

Fig. 4.4 Nyquist curve for (4.7)

This simplification is based on the following facts:

$$\left|G_0(j\omega)e^{-j\omega h}\right| = \left|G_0(j\omega)\right|$$

$$\arg\left|G_0(j\omega)e^{-j\omega h}\right| = \arg\left|G_0(j\omega)\right| - \omega h \ .$$

Using these facts several methods for testing the stability of (4.8) have been suggested. They are summarized in [94, Sections 4.4 – 4.6]. Here, the analysis is treated by superimposing the polar plot on the Bode diagram as suggested by [104]. We choose this concept since we extend it to the MIMO case later.

Let $\omega_{c,\,i}$ be a crossover frequency, which means that for this frequency $|G_0(j\omega_{c,\,i})| = 1$. The phase margin of the *nondelayed* system is defined by

$$\varphi = \min_i \left\{\pi + \arg(G_0(j\omega_{c,\,i}))\right\} = \pi + \arg(G_0(j\omega_{c,\,min})).$$

The system (4.8) is asymptotically stable if $\varphi - \omega_{c,\,min}h > 0$ or, in other words, the system (4.8) is asymptotically stable for all constant delays h satisfying the following inequality

$$h < h_{max} = \frac{\varphi}{\omega_{c,\,min}} \ . \tag{4.9}$$

The value h_{max} is usually designated as delay margin.

Example 4.2: The Bode diagram for the system

$$G_0(s) = \frac{16}{s^2 + 1.6s + 16} \tag{4.10}$$

is shown in Fig. 4.5. From condition (4.9) and the Bode diagram, we obtain the delay margin $h_{max} < 0.1$.

Nyquist criteria are also available for unstable MIMO systems. These extensions presume the solution of some transcendental equations and the computation of the eigenvalues of a delayed system (see, e.g., [75]). Furthermore,

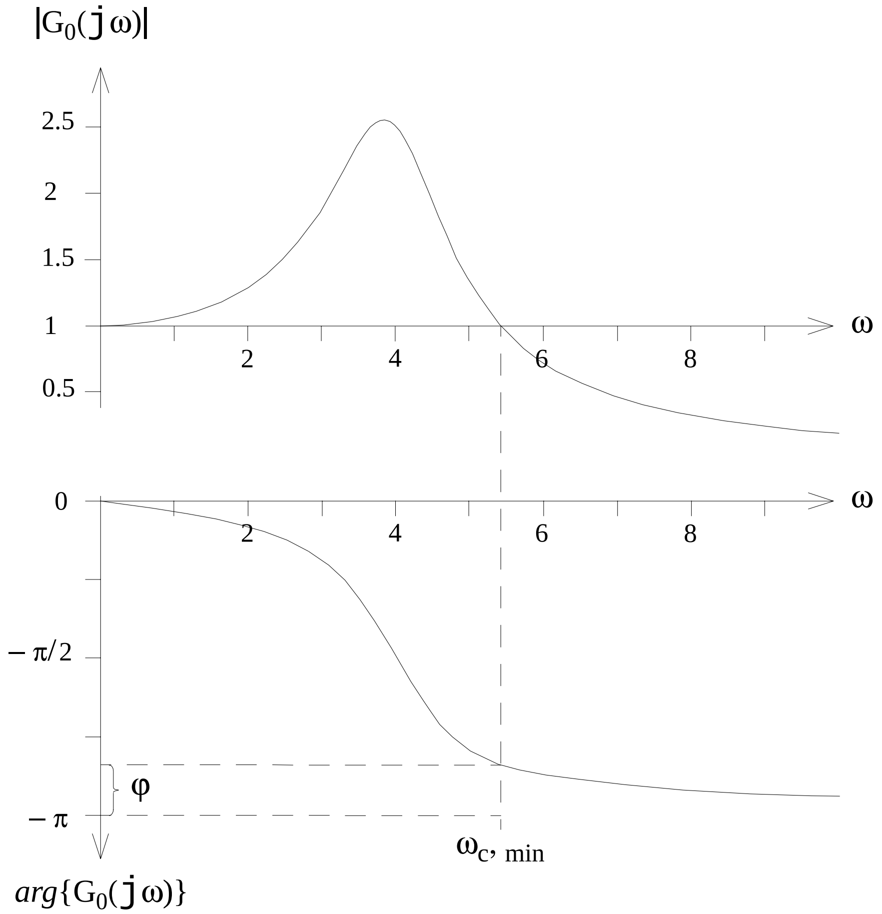

Fig. 4.5 Bode diagram for the system (4.10)

the method is only a stability test and does not yield robustness bounds for a nondelayed system. Therefore, we prefer to define the delay margin for MIMO systems and to investigate the relation between this delay margin and the complementary sensitivity function.

Definition 4.1: *The MIMO delay margin h_{max} is the largest constant retardation which can be tolerated independently in each input of a system such that it remains asymptotically stable.*

Since the input delays are considered as an uncertainty we may use the MIMO phase shift concept for linear delay-free systems [28, p. 52]. In Fig. 4.6, scalar multiplicative uncertainty is applied independently at each actuator or input of G. (This could also be done at the outputs of G.) This causes

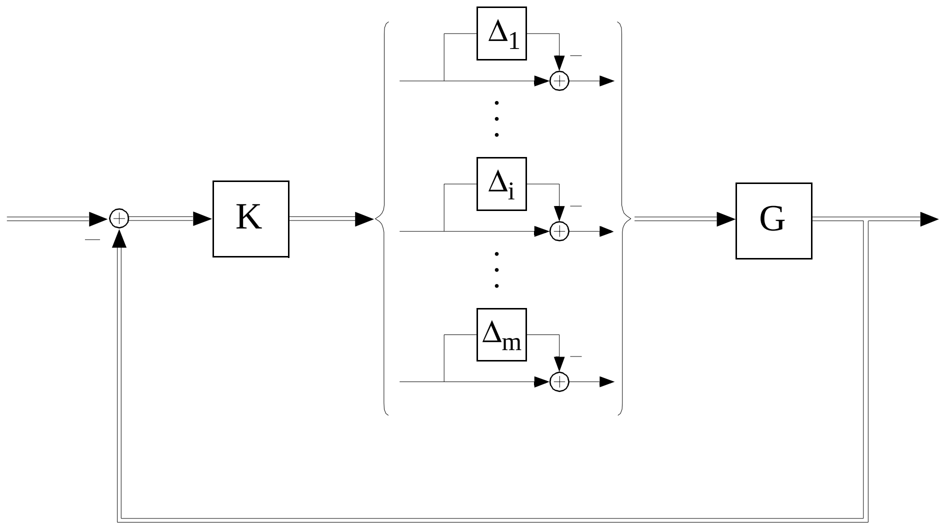

Fig. 4.6

the nominal gains in each channel, at the affected points in the loop, to be multiplied by $1 + \Delta_i$. When the scalar uncertainty blocks Δ_i are gathered into a single diagonal uncertainty matrix $\Delta = \mathrm{diag}(\Delta_1, ..., \Delta_m)$, the transfer matrix "seen" by Δ is the complementary sensitivity function at the input

$$T(s) = K(s)G(s)[I + K(s)G(s)]^{-1} . \tag{4.11}$$

A lower guaranteed bound of the delay margin can be computed using standard frequency-dependent singular values:

$$\bar{\sigma}(T(j\omega)) = \|T(j\omega)\|_2 = \sqrt{\lambda_{max}\{T(j\omega)^* T(j\omega)\}} .$$

Theorem 4.2: *The lower bound h* of the MIMO delay margin h_{max} for system (4.11) satisfies the inequality*

$$h^*\omega \leq 2 \cdot \arcsin\left(\frac{1}{2\bar{\sigma}(T(j\omega))}\right) . \tag{4.12}$$

Proof: The small-gain theorem [28, p. 47 and p. 53] says that the perturbed system shown in Fig. 4.11 is asymptotically stable if $\bar{\sigma}(\Delta)\,\bar{\sigma}(T) < 1$ for all frequencies $s = j\omega$. This implies the stability condition

$$\bar{\sigma}(\Delta) < \frac{1}{\bar{\sigma}(T)} = r(\omega) \,. \tag{4.13}$$

The function r(ω) is introduced for convenience. As mentioned above, each input of the plant is multiplied by $1 + \Delta_i$. The values of this factor are studied in the complex plane (cf. Fig. 4.7). The case in which the system is unperturbed, ($\Delta_i = 0$) is represented by the point $(1, 0 \cdot \mathtt{j})$. The circle with the radius r(ω) and the centre $(1, 0 \cdot \mathtt{j})$ covers all values of $1 + \Delta_i$ for which the perturbed system is asymptotically stable according to (4.13). When the Δ_i values are complex and satisfy the equality $1 + \Delta_i = e^{-\mathtt{j}h\omega}$, they correspond to delays at the plant inputs. These values of $1 + \Delta_i$ lie on the unit circle.

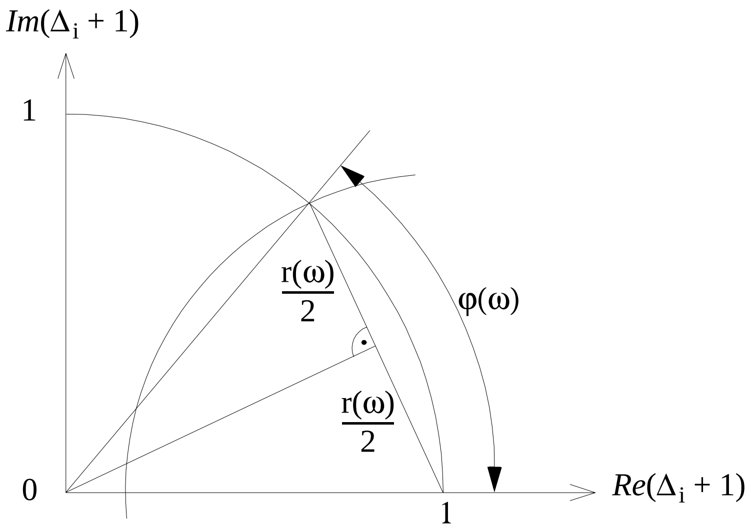

Fig. 4.7

The angle φ, defined in Fig. 4.7 by the points $(1, 0 \cdot \mathtt{j})$, $(0, 0 \cdot \mathtt{j})$, and the intersection point of the unit circle and the stability circle with radius r(ω) and the centre $(1, 0 \cdot \mathtt{j})$ is given by

$$\varphi(\omega) = 2 \cdot \arcsin(r(\omega)/2) \,. \tag{4.14}$$

This leads to the following condition

$$h\omega \le h^*\omega \le 2 \cdot \arcsin\left(\frac{r(\omega)}{2}\right) \tag{4.15}$$

which completes the proof. ❑

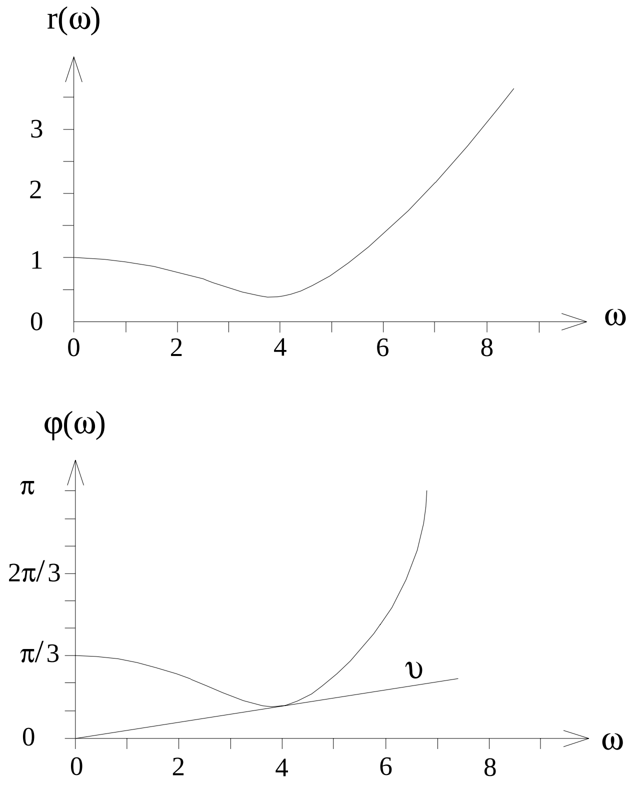

Fig. 4.8 The functions $r(\omega)$, $\varphi(\omega)$, and s are computed for the system (4.11), where $K(s)G(s) = G_0(s)$ is given by (4.10); here $h^* \approx h_{max} \approx 0.1$

Clearly, this test is conservative. The procedure of this robustness analysis is summarized in the following.

Step 1: Determine the function $r(\omega)$ from (4.13) and (4.11).

Step 2: Compute the function $\varphi(\omega)$ using (4.14).

Step 3: Draw the function $\varphi(\omega)$ in the phase diagram (cf. Fig. 4.8). Furthermore, plot the straight line υ which starts at the origin and is a tangent of the function $\varphi(\omega)$ (and never intersects it). Inequality (4.15) implies that the slope of s is the delay margin h^*.

The method confirms that a high bandwidth of the system (4.11) of the open-loop system reduces the delay margin.

4.2 Robustness against state delays

The robustness of a nondelayed system against state delays is considered. Again the small-gain theorem is used. Note that a closed-loop system is also of the form (4.3). Its stability is determined only by its dynamics denoted by

$$G(s)_{CL} = \left[sI - A_0 - \sum_{i=1}^{k} A_i e^{-sih} \right]^{-1} . \tag{4.16}$$

The dynamics of the corresponding delay-free system are denoted by

$$G(s)_{DF} = [sI - A_0]^{-1} . \tag{4.17}$$

The H_∞ norm of the delay-free system is denoted by $\gamma = \|G(j\omega)_{DF}\|_\infty = \max_\omega \bar{\sigma}(G(j\omega)_{DF})$. Some properties of singular values collected in the Lemma below are then used to establish Theorem 4.3.

Lemma 4.1 [115, p. 898]:

$$\textit{If } A^{-1} \textit{ exists, } \underline{\sigma}(A) = \frac{1}{\bar{\sigma}(A^{-1})} \tag{4.18}$$

$$\textit{If } A^{-1} \textit{ exists, } \bar{\sigma}(A) = \frac{1}{\underline{\sigma}(A^{-1})} \tag{4.19}$$

$$\bar{\sigma}(\alpha A) = |\alpha| \bar{\sigma}(A) \tag{4.20}$$

$$\underline{\sigma}(A) - \bar{\sigma}(E) \le \underline{\sigma}(A + E) . \tag{4.21}$$

Theorem 4.3: *System G_{CL} is asymptotically stable if the following inequality for the H_∞ norm γ of the delay-free system G_{DF} holds:*

$$\gamma < \frac{1}{1 + \sum_{i=1}^{k} \|A_i\|_2} . \tag{4.22}$$

Proof: Using the relations listed in Lemma 4.1, we may write

$$\bar{\sigma}(G(j\omega)_{CL}) = \bar{\sigma}\left(\left[j\omega I - A_0 - \sum_{i=1}^{k} A_i e^{-j\omega ih} \right]^{-1} \right)$$

$$\overline{\sigma}(G(j\omega)_{CL}) = \underline{\sigma}\left(\left[j\omega I - A_0 - \sum_{i=1}^{k} A_i e^{-j\omega ih}\right]\right)^{-1}$$

$$\frac{1}{\overline{\sigma}(G(j\omega)_{CL})} = \underline{\sigma}\left(\left[j\omega I - A_0 - \sum_{i=1}^{k} A_i e^{-j\omega ih}\right]\right)$$

$$\frac{1}{\overline{\sigma}(G(j\omega)_{CL})} \geq \underline{\sigma}([j\omega I - A_0]) - \overline{\sigma}\left(-\sum_{i=1}^{k} A_i e^{-j\omega ih}\right)$$

$$\frac{1}{\overline{\sigma}(G(j\omega)_{CL})} \geq \frac{1}{\overline{\sigma}([j\omega I - A_0]^{-1})} - \sum_{i=1}^{k} \|A_i\|_2 \quad .$$

From the small-gain theorem it follows that the inequalities

$$\frac{1}{\overline{\sigma}(G(j\omega)_{CL})} \geq \frac{1}{\overline{\sigma}([j\omega I - A_0]^{-1})} - \sum_{i=1}^{k} \|A_i\|_2 > 1 \tag{4.23}$$

must be satisfied for all frequencies ω to guarantee the stability of the system (4.16). From (4.23) we obtain

$$\frac{1}{\|G(j\omega)_{CL}\|_\infty} \geq \frac{1}{\gamma} - \sum_{i=1}^{k} \|A_i\|_2 > 1$$

which implies stability condition (4.22). ❑

Condition (4.22) can be rewritten as

$$\frac{1-\gamma}{\gamma} > \sum_{i=1}^{k} \|A_i\|_2 \quad .$$

A similar condition has been derived by Kojiama *et al*. (1993) using the Lyapunov method and the small-gain theorem. But their result is proven for a more restrictive assumption, namely: $\sum^{k}\|A_i\|_2 < 1$. Furthermore, those authors defined γ as the H_∞ norm of an optimally controlled time-delay system, while in our case γ is the H_∞ norm of a delay-free system.

Condition (4.22) can be remarkably improved, if all elements of the matrices A_0, A_i which are not involved in the corresponding characteristic equation are omitted (cf. Subsection 3.2.6).

4.3 Robustness bounds for unstructured uncertainties

The uncertain dynamical system

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t-\tau_i) + \Delta_0(t, x(t)) \tag{4.24}$$

is considered. The function $\Delta_0(t, x(t))$ is unknown and represents the system's nonlinear parametric perturbation with respect to the current state $x(t)$. It is assumed that $\Delta_0(t, x(t))$ is cone bounded, i. e.,

$$\|\Delta_0(t, x(t))\|_2 \le \beta \cdot \|x(t)\|_2 \ . \tag{4.25}$$

Here, a system is said to be robust if it is tolerant to changes within certain specific bounds of perturbation. As described in Section 3, we distinguish among several types of robust stability tests depending on the information of the delay involved in the test:

- delay-dependent stability criteria (delays may be constant or variable)
- stability criteria independent of constant delays
- stability criteria independent of a delay constant.

The proofs of the criteria presented in the following are straightforward. Since they are elaborated in Section 3 for single-delay systems, we will here only note the stability conditions.

Delay-independent: It is assumed that the delays τ_i of the system (4.24) are continuous and bounded, satisfying the inequality $\infty > \tau_i(t, x(t)) > 0$. Using the reasoning of Theorem 3.4, we obtain the following sufficient stability condition

$$\mu(A_0)_2 + \sum_{i=1}^{k} \|A_i\|_2 + \beta < 0 \tag{4.26}$$

while the stability test

$$\mu\Big(A_0 + \frac{1}{2}\sum_{i=1}^{k} A_i A_i^T\Big)_2 + \frac{1}{2} + \beta < 0 \tag{4.27}$$

is a special case of Theorem 4.4 established in the next subsection. An improvement of the stability condition using a transformed state vector technique of the form $z(t) = Tx(t)$ (cf. Subsection 3.2.6) is not likely, since the term β in (4.26) and (4.27) becomes $\|T^{-1}\|\cdot\|T\|\cdot\beta$.

Independent of constant delays: A convenient robustness test of this type follows from Corollary 3.3. The system (4.24) with constant delays τ_i is asymptotically stable if the inequality

$$\lambda_{max}\Big(A_0 + A_0^T + 2\beta I - 4k\sum_{i=1}^{k} A_i(A_0 + A_0^T + 2\beta I)^{-1} A_i^T\Big) < 0 \tag{4.28}$$

holds.

Independent of a delay constant: None of the various stability conditions listed in Subsection 3.2.3 have been analysed for perturbed systems of the form (4.24). However, the stability condition of Hmamed (1991) is suitable to be applied to this problem. The system (4.24) with constant delays of the form $\tau_i = i\cdot h$ is asymptotically stable if

$$\mu\Big(A_0 + \sum_{i=1}^{k} A_i z^i\Big)_2 + \beta < 0 \quad \forall\, |z| = 1 \tag{4.29}$$

where $z = e^{j\omega}$, $\omega \in [0, 2\pi]$. Checking the validity of (4.29) for all values $|z| = 1$ is generally a cumbersome task. However, for systems of low order the condition is applicable.

Example 4.3: The simple uncertain time-delay system

$$\dot{x}(t) = \begin{bmatrix} -3 & 1 \\ -1 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 & 0 \\ 0.75 & 1 \end{bmatrix} x(t-\tau) + \Delta(t, x(t)) \tag{4.30}$$

is considered to compare the various stability conditions. They yield the following bounds:

(4.26): $\|\Delta_0\|_2 < 1.71\ldots$

(4.27): $\|\Delta_0\|_2 < 2.34\ldots$

(4.28): $\|\Delta_0\|_2 < 2.36\ldots$

(4.29): $\|\Delta_0\|_2 < 2.41\ldots$

Condition (4.29) is less restrictive than the others for a constant delay but requires the greatest calculation effort. Condition (4.27) is remarkable since it is easy to apply and is valid also for systems with variable delays. However, a general comparison of the criteria is not available.

4.4 Robustness bounds for large-scale time-delay systems

In recent years, a number of stability criteria for large-scale systems with delays have been developed. Mori *et al.* (1981) derived a stability criterion using the comparison method. With the aid of the complex Lyapunov theorem, Suh and Bien (1982), Hmamed (1986), as well as Wang and Song (1989) obtained sufficient conditions for stability of large-scale systems. Furthermore, Lee *et al.* (1984) studied the stabilization problem of time-delay systems via generalized algebraic Riccati equations. Moreover, Wang *et al.* (1991) gave a stability criterion using the Lyapunov theorem.

We deal with the Razumikhin stability theorem for uncertain large-scale systems. The uncertainties may be linear, nonlinear, and/or time-varying. The result is presented in a scalar inequality which contains the matrices of each subsystem. While in the above-mentioned papers the stability conditions are derived for constant delays only, the result presented here is valid for arbitrary bounded continuous delays depending on time and state variables.

Consider an uncertain large-scale system with delays which is composed of N interconnected subsystems S_i, i = 1, 2, ..., N. Each subsystem is described by

$$S_i : \dot{X}_i(t) = A_i X_i(t) + \Delta_i(X_i(t), t) + \sum_{j=1}^{N} A_{ij} X_j(t - \tau_{ij}(X_j(t), t)) \tag{4.31}$$

where $X_i \in \boldsymbol{R}^{n_i}$ represents the state of the subsystem S_i. It is assumed that all of the delays $\tau_{ij}(X_j(t), t)$ are bounded and continuous functions. For briefness we shall write τ_{ij} instead of $\tau_{ij}(X_j(t), t)$. Furthermore, it is supposed that the nonlinear parametric uncertainties are bounded by the following inequalities:

$$\|\Delta_i(X_i(t))\|_2 \le \beta_i \|X_i(t)\|_2 \quad \text{where} \quad \beta_i \in [0, \infty) \ . \tag{4.32}$$

Theorem 4.4: *System (4.31) is asymptotically stable independent of delay, if*

$$\mu\Big(A_i + \frac{1}{2} \sum_{j=1}^{N} A_{ij} A_{ij}^T\Big)_2 + \beta_i + \frac{N}{2} < 0 \qquad i = 1, ..., N \ . \tag{4.33}$$

Proof: Given the assumption above, the Razumikhin theorem (cf. Section 3.1) can be applied to establish stability condition (4.33). The Lyapunov-Razumikhin function is chosen to be of the class of quadratic forms

$$V = \sum_{i=1}^{N} V_i = \sum_{i=1}^{N} X_i(t)^T X_i(t) \ . \tag{4.34}$$

Determining the derivative of (4.34) and using (4.31) and (4.32), we obtain

$$\begin{aligned} \sum_{i=1}^{N} \dot{V}_i \le \sum_{i=1}^{N} \{ & X_i(t)^T [A_i^T + A_i] X_i(t) + 2\beta_i \|X_i(t)\|_2^2 + \\ & + 2 \sum_{j=1}^{N} X_i(t)^T A_{ij} X_j(t - \tau_{ij}) \} \ . \end{aligned} \tag{4.35}$$

Using the fact that for any matrices U_1 and U_2 with appropriate dimensions [146], [156]

$$U_1^T U_2 + U_2^T U_1 \le U_1^T U_1 + U_2^T U_2$$

we obtain from (4.35)

$$\sum_{i=1}^{N} \dot{V}_i \le \sum_{i=1}^{N} \{X_i(t)^T[A_i^T + A_i + \sum_{j=1}^{N} A_{ij}A_{ij}^T]X_i(t) + 2\beta_i\|X_i(t)\|_2^2 + \\ + \sum_{j=1}^{N} X_j(t-\tau_{ij})^T X_j(t-\tau_{ij})\} \ . \tag{4.36}$$

According to Razumikhin, the inequality

$$\sum_{i=1}^{N} V_i(X_i(t)) \ge \sum_{i=1}^{N} V_i(X_i(\theta)) \qquad\qquad t-\tau_{max} \le \theta \le t$$

and hence

$$\sum_{j=1}^{N} X_j(t)^T X_j(t) \ge \sum_{j=1}^{N} X_j(t-\tau_{ij})^T X_j(t-\tau_{ij})$$

must be satisfied to ensure asymptotic stability. Thus, (4.36) yields

$$\sum_{i=1}^{N} \dot{V}_i < \sum_{i=1}^{N} \{X_i(t)^T[A_i^T + A_i + \sum_{j=1}^{N} A_{ij}A_{ij}^T]X_i(t) + 2\beta_i\|X_i(t)\|_2^2 + \\ + \sum_{j=1}^{N} X_j(t)^T X_j(t)\}. \tag{4.37}$$

Inequality (4.37) can be rewritten as

$$\sum_{i=1}^{N} \dot{V}_i < x(t)^T \begin{bmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_N \end{bmatrix} x(t) \tag{4.38}$$

where

$$x(t)^T = [X_1(t)^T, \ldots, X_N(t)^T]$$

and

$$M_i = A_i^T + A_i + \sum_{j=1}^{N} A_{ij} A_{ij}^T + I \cdot (2\beta_i + N) \ .$$

Here, $I \in \boldsymbol{R}^{n_i \times n_i}$ is the identity matrix. Since $\dot{V}_i$ has to be negative along the trajectories of (4.31) condition (4.33) follows from inequality (4.38). The proof is complete. ❑

Example 4.4: In order to illustrate the stability condition, we consider the large-scale time-delay system with linear uncertainties given in Wang *et al*. (1991):

$$\dot{X}_1(t) = \left(\begin{bmatrix} -5 & 1 \\ 2 & -7 \end{bmatrix} + \Delta_1 \right) X_1(t) + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X_1(t-\tau_{11}) +$$

$$+ \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.2 \end{bmatrix} X_2(t-\tau_{12}) + \begin{bmatrix} 0.2 & 0.3 \\ 0.2 & 0.3 \end{bmatrix} X_3(t-\tau_{13})$$

$$\dot{X}_2(t) = \left(\begin{bmatrix} -6 & 2 \\ 1 & -6 \end{bmatrix} + \Delta_2 \right) X_2(t) + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X_2(t-\tau_{22}) +$$

$$+ \begin{bmatrix} 0.3 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} X_1(t-\tau_{21}) + \begin{bmatrix} 0.4 & 0.2 \\ 0.4 & 0.2 \end{bmatrix} X_3(t-\tau_{23})$$

$$\dot{X}_3(t) = \left(\begin{bmatrix} -7 & 2 \\ 1 & -5 \end{bmatrix} + \Delta_3 \right) X_3(t) + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X_3(t-\tau_{31}) +$$

$$+ \begin{bmatrix} 0.4 & 0.3 \\ 0.4 & 0.3 \end{bmatrix} X_1(t-\tau_{32}) + \begin{bmatrix} 0.2 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} X_2(t-\tau_{33}) \ .$$

Applying condition (4.33) we calculate the robustness bounds for the above system such that the inequalities

$$\|\Delta_i\| \le -\mu(A_i + \frac{1}{2}\sum_{j=1}^{N} A_{ij} A_{ij}^T)_2 - \frac{N}{2} \tag{4.39}$$

hold. From (4.39) we find the following allowable bounds

$$\|\Delta_1\|_2 < 0.6336$$

$$\|\Delta_2\|_2 < 0.6700$$

$$\|\Delta_3\|_2 < 0.2850 \ .$$

The bounds given by Wang *et al.* (1991) are

$$\|\Delta_1\|_2 < 0.3117$$

$$\|\Delta_2\|_2 < 0.5833$$

$$\|\Delta_3\|_2 < 0.3117 \ .$$

Note that the method of Wang *et al.* (1991) has been proven for constant delays only.

Example 4.5: Applying Theorem 4.4, it can be shown that the large-scale system discussed by Hmamed (1986) and Wang and Song (1989) is also asymptotically stable independent of any continuous bounded time-varying and state-dependent delay. In contrast to the methods of Hmamed (1986) and Wang and Song (1989), no complex Lyapunov equation needs to be solved.

5 Controllability

Controllability is a fundamental structural attribute of any control system, dealing with the relationship between the input and the state of the system. More specifically, system controllability addresses the following question: Does a control u always exist which can transfer the initial state of the system to any desired state in a finite time?

The aim of this chapter is to give a survey of the various controllability and related stabilizability concepts of linear systems with time delays of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t-\tau_i) + Bu(t) \qquad t \geq t_0 \qquad (5.1)$$

$$x(t) = \varphi(t) \qquad t_0 - \tau_k \leq t \leq t_0 \qquad (5.2)$$

where A_0, $A_i \in \boldsymbol{R}^{n \times n}$, $B \in \boldsymbol{R}^{n \times m}$. Unless noted otherwise, it is assumed in the following that the delays are constant and commensurate, i.e., $\tau_i = i \cdot h$ with $h > 0$. In certain cases, controllability criteria for systems with noncommensurate or time-dependent delays are noted. It would be very useful to have controllability criteria independent of a delay constant or of time-varying delays, since the values of the delays are difficult to estimate. Unfortunately, such robust controllability criteria are known only in some special cases.

The literature on controllability of delay systems is quite rich. In early works, the research concentrated mainly on the reachability of the trajectory endpoint $x(t_1)$ for some final time t_1 (cf. Section 5.1). Next, some authors tried to examine reachability of arbitrary final states x_{t_1} in some function spaces (cf. Section 5.2). It soon appeared that this concept is much too strong to be useful in control theory since it typically requires rank(B) = n. It turns out that the concept of approximate controllability (cf. Section 5.3) is much less restrictive. However, in many systems not all of the components of the state are delayed. For such systems the requirement that all the components of the state must be approximately controllable in a function space might be too strong as well. This provides some motivation for a controllability concept, called F-approximate controllability which is weaker than the approximate controllability concept (cf. Section 5.4). The concepts of approximate, F-approximate, and function-space controllability are strongly related to spectral controllability (cf. Section 5.5). The latter mainly deals with eigenvalues considerations.

5.1 $\boldsymbol{R}^n$-controllability

In time-optimal control theory, it is assumed that starting from some initial state the target point can be reached in a finite time by using some admissible control. We consider here the target to be a point in the Euclidean space $\boldsymbol{R}^n$.

Definition 5.1 [26, p. 193]: *The linear control process (5.1) is $\boldsymbol{R}^n$-controllable (also denoted as Euclidean controllable or relatively controllable) if for every $\varphi \in C([t_0 - \tau_k, t_0], \boldsymbol{R}^n)$ there exist a finite time t_1 and a square integrable control u such that $x(t_1) = x_1 \in \boldsymbol{R}^n$.*

Definition 5.2: *The linear control process (5.1) is $\boldsymbol{R}^n$-null-controllable (also denoted as Euclidean null-controllable or relatively null-controllable) if for every $\varphi \in C$ there exist a finite time t_1 and a square integrable control u such that $x(t_1) = 0$.*

$\boldsymbol{R}^n$-null-controllability is sometimes designated as controllability to the origin. Since in the literature the latter expression is used for different types of controllability, e.g., compare [85, p. 134] and [150], we have not mentioned it in Definition 5.2. It may be somewhat surprising that $\boldsymbol{R}^n$-controllability and $\boldsymbol{R}^n$-null-controllability are defined separately, since $\boldsymbol{R}^n$-controllability implies $\boldsymbol{R}^n$-null-controllability. However, $\boldsymbol{R}^n$-null-controllability is not sufficient for $\boldsymbol{R}^n$-controllability. This fact is outlined below.

It is well known that the linear, nondelayed system

$$\dot{x}(t) = A_0 x(t)$$

has the property that for every $x_1 = x(t_1)$ there exists a vector x_0 such that the trajectory emanating from x_0 at time t_0 reaches x_1 at time t_1. A system with this property is called pointwise complete. Weiss (1967) conjectured that the system (5.1) with B = 0 is pointwise complete. Popov [117] and Zverkin [159] showed independently that this conjecture is false. There exist linear constant delay systems with the property that the trajectories associated with admissible initial functions all attain values in a subspace of $\boldsymbol{R}^n$ at some $t > t_1$. This feature is observed in Popov's example:

$$\dot{x}_1(t) = 2x_2(t) \qquad t \geq 0 = t_0 \tag{5.3}$$

$$\dot{x}_2(t) = -x_3(t) + x_1(t-1) \qquad t \geq 0 \tag{5.4}$$

$$\dot{x}_3(t) = 2x_2(t-1) \qquad t \geq 0\,. \tag{5.5}$$

Equation (5.4) yields

$$\ddot{x}_2(t) = -\dot{x}_3(t) + \dot{x}_1(t-1) \qquad t \geq 0$$

$$\ddot{x}_2(t) = -2x_2(t-1) + 2x_2(t-1) \qquad t \geq 1$$

$$\ddot{x}_2(t) = 0 \qquad t \geq 1\,. \tag{5.6}$$

Let c_1, c_2, and c_3 be integration constants. Equation (5.6) yields

$$x_2(t) = c_1 t + c_2 \tag{5.7}$$

for $t \geq 1$. Using (5.7) together with (5.3) we obtain

$$x_1(t) = c_1 t^2 + 2c_2 t + c_3 \tag{5.8}$$

and from (5.4) we find

$$x_3(t) = c_1 t^2 - 2c_1 t + 2c_2 t - 2c_2 + c_3\,. \tag{5.9}$$

Consequently,

$$\begin{bmatrix} 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \equiv 0 \tag{5.10}$$

for $t \geq 1$. Viewed geometrically all trajectories reach the plane P: $x_1(t) - 2x_2(t) - x_3(t) \equiv 0$ no later than $t \geq 1$ and remain on P for all future time. Such a system is called pointwise degenerated (as opposed to pointwise complete).

Definition 5.3 [135]: *The homogenous part of system (5.1) is pointwise degenerate if there exist some non-zero $\eta \in \boldsymbol{R}^n$ and some time t_1, $t_1 > t_0$ such that $\eta^T \cdot x(t_1) = 0$ for all initial conditions $\varphi \in C([t_0 - \tau_k, t_0], \boldsymbol{R}^n)$. (The complementary property is called pointwise completeness.)*

5.1.1 Pointwise completeness

Before discussing the connection between $\boldsymbol{R}^n$-controllability and pointwise degeneracy some interesting published results on pointwise degeneracy are briefly mentioned.

For single-delay systems, dimension 3 is the lowest dimension for which a system can be pointwise degenerated [117]. For systems with two delays, pointwise degeneracy may occur for n = 2 [159].

If a system is pointwise degenerate at time t_1, then it is also degenerate at any t_2 where $t_2 > t_1$ [117]. Furthermore, if n = 2 and k = 2 (k = 3) degeneracy can not occur before t = 2 (t = 3), where $t_0 = 0$ [5]. A corresponding extension of the latter statement is open as well as a general, easily verifiable condition for pointwise completeness. However, for a single-delay system of the form

$$\dot{x}(t) = A_0x(t) + A_1x(t-1) \tag{5.11}$$

the necessary and sufficient conditions are known [157]. The following notation is introduced to note the criterion. The matrices $\tilde{A}_j \in \boldsymbol{R}^{n(j+1) \times n(j+1)}$ and $E_j \in \boldsymbol{R}^{n \times n(j+1)}$ j = 0, 1, 2, ... are defined by

$$\tilde{A}_j = \begin{bmatrix} A_0 & & & 0 \\ A_1 & A_0 & & \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ 0 & & A_1 & A_0 \end{bmatrix}$$

$$E_j = [0, \ldots, 0, I]$$

where $\tilde{A}_0 = A_0$, $E_0 = I \in \boldsymbol{R}^{n \times n}$. The matrices $U_j \in \boldsymbol{R}^{n \times n}$, $Z_j \in \boldsymbol{R}^{n(j+1) \times n}$, and $F_j \in \boldsymbol{R}^{n(j+1) \times m}$ are defined in a recursive form:

$$U_0 = I$$

$$U_j = E_j \begin{bmatrix} I \\ e^{\tilde{A}_{j-1}} U_{j-1} \end{bmatrix}$$

$$Z_j = [U_0^T, \ldots, U_j^T]$$

$$F_j = Z_j B\,.$$

Theorem 5.1 [157]: *System (5.12) is pointwise complete at time $t_1 = j$, $j = 1, 2, \ldots$ iff the matrix*

$$M(j) = [E_{j-1}F_{j-1}, \ldots, E_{j-1}A_{j-1}^{nj-1}F_{j-1}, E_jZ_j] \tag{5.12}$$

has rank n.

Condition (5.12) is laborious to apply. The criteria stated below are easy to check but they are only sufficient.

Corollary 5.1 [19]: *System (5.11) is pointwise complete for all $t_1 \in [t_0, \infty)$ whenever $A_0A_1 = A_1A_0$.*

Corollary 5.2 [4]: *If there exist two n-dimensional column vectors a and b such that $A_1 = ab^T$, then the system (5.11) is pointwise complete.*

5.1.2 $\boldsymbol{R}^n$-controllability and $\boldsymbol{R}^n$-null-controllability

While $\boldsymbol{R}^n$-controllability is unaffected by whether the system is pointwise degenerate, $\boldsymbol{R}^n$-null-controllability is not. Since the null vector in $\boldsymbol{R}^n$ lies on the terminal manifold on which all trajectories of the (free) degenerated system end up, the available controls need only effect a transfer to the origin from any point on this manifold to guarantee $\boldsymbol{R}^n$-null-controllability. Conditions for $\boldsymbol{R}^n$-controllability are therefore sufficient for $\boldsymbol{R}^n$-null-controllability but not vice versa unless the system is pointwise complete. However, Gabasov and Kirillova [36, p. 61] presented an algebraic necessary and sufficient condition for $\boldsymbol{R}^n$-controllability. This criterion is valid for the system

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - \tau_i) + Bu(t) \tag{5.13}$$

where $0 < \tau_1 < \ldots < \tau_k < \infty$ are constant delays. First, the so-called determining equation is introduced

$$\begin{aligned} V_j(s) &= A_0 V_{j-1}(s) + \sum_{i=1}^{k} A_j V_{j-1}(s - \tau_i) \\ V_0(s) &= \begin{cases} B & s = 0 \\ 0 & s \neq 0 \end{cases} \end{aligned} \tag{5.14}$$

where $j = 0, 1, 2, \ldots, (n-1)$ and $s \in [t_0, t_1]$.

Theorem 5.2 [36, p. 61], [40, p. 259]: *System (5.13) is* $\boldsymbol{R}^n$*-controllable on* $[t_0, t_1]$ *iff*

$$rank[\{V_j(s)\}_{j,\,s}] = n \tag{5.15}$$

where $s \in \{t_0, t_1\}$, $j = 0, 1, ..., n-1$.

Theorem 5.3 [85, p. 137]: *System (5.13) is* $\boldsymbol{R}^n$*-null-controllable on* $[t_0, t_1]$ *if condition (5.15) is satisfied. Furthermore, this condition is also necessary if the system is pointwise complete.*

Example 5.1: The algebraic rank condition (5.15) is illustrated for the system

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) + Bu(t) \qquad\qquad t \geq 0 = t_0 \tag{5.16}$$

with n = 3. According to (5.14), V_0 is nonzero only for s = 0:

$$V_0(s = 0) = B$$

while V_1 is a nonzero matrix at s = 0 and s = h:

$$V_1(s = 0) = A_0V_0(0) + A_1V_0(-h)$$
$$V_1(0) = A_0B$$

$$V_1(s = h) = A_0V_0(h) + A_1V_0(0)$$
$$V_1(h) = A_1B\ .$$

Similarly, we obtain for V_2:

$$V_2(0) = A_0V_1(0) + A_1V_1(-h)$$
$$V_2(0) = A_0^2B$$

$$V_2(h) = A_0V_1(h) + A_1V_1(0)$$
$$V_2(h) = [A_0A_1 + A_1A_0]B$$

$$V_2(2h) = A_0V_1(2h) + A_1V_1(h)$$
$$V_2(2h) = A_1^2B\quad .$$

The nonzero values of V_i can be arranged in a scheme as shown in Table 5.1.

Table 5.1

	s = 0	s = h	s = 2h
$V_0(s)$	B		
$V_1(s)$	A_0B	A_1B	
$V_2(s)$	A_0^2B	$[A_0A_1 + A_1A_0]B$	A_1^2B

Now, Theorem 5.2 says that the system (5.16) is $\boldsymbol{R}^3$-controllable for $t_1 > 0$ iff

$$\text{rank}[B,\ A_0B,\ A_0^2B\,] = 3$$

and it is $\boldsymbol{R}^3$-controllable for $t_1 > h$ iff

$$\text{rank}[B,\ A_0B,\ A_0^2B\,,\ A_1B,\ [A_0A_1 + A_1A_0]B] = 3\ .$$

Furthermore, system (5.16) is $\boldsymbol{R}^3$-controllable for $t_1 > 2h$ iff

$$\text{rank}[B,\ A_0B,\ A_0^2B\,,\ A_1B,\ [A_0A_1 + A_1A_0]B,\ A_1^2B\,] = 3\ .$$

If the latter condition is not valid for a system of the form (5.16) where n = 3, then this system is not $\boldsymbol{R}^3$-controllable for any $t_1 > 0$.

Example 5.2: The system

$$\dot{x}(t) = A_0x(t) + A_1x(t - \tau_1) + A_2x(t - \tau_2) + Bu(t)$$

with n = 3 is considered. The matrices $V_1(s)$, $V_2(s)$, and $V_3(s)$ are shown in Table 5.2. It turns out that we have to distinguish between three cases, viz.: $\tau_1 < 0.5\tau_2$, $\tau_1 = 0.5\tau_2$, and $\tau_1 > 0.5\tau_2$.

This example illustrates that the criterion is easier to apply if the delays are commensurate ($\tau_i = i\cdot h$). The simplification is due to the fact that the variable s has to be considered only for the values 0, h, 2h, ... The simplified procedure for checking the $\boldsymbol{R}^3$-controllability of commensurate delay systems is summarized in the following corollary.

Table 5.2

$\tau_1 < 0.5\tau_2$:

	$s = 0$	$s = \tau_1$	$s = 2\tau_1$	$s = \tau_2$	$s = \tau_1 + \tau_2$	$s = 2\tau_2$
$V_0(s)$	B					
$V_1(s)$	A_0B	A_1B		A_2B		
$V_2(s)$	A_0^2B	$[A_0A_1 + A_1A_0]B$	A_1^2B	$[A_0A_2 + A_2A_0]B$	$[A_1A_2 + A_2A_1]B$	A_2^2B

$\tau_1 = 0.5\tau_2$:

	$s = 0$	$s = \tau_1$	$s = 2\tau_1 = \tau_2$	$s = \tau_1 + \tau_2 = 3\tau_1$	$s = 4\tau_1 = 2\tau_2$
$V_0(s)$	B				
$V_1(s)$	A_0B	A_1B	A_2B		
$V_2(s)$	A_0^2B	$[A_0A_1 + A_1A_0]B$	$[A_1^2 + A_0A_2 + A_2A_0]B$	$[A_1A_2 + A_2A_1]B$	A_2^2B

$\tau_1 > 0.5\tau_2$:

	$s = 0$	$s = \tau_1$	$s = \tau_2$	$s = 2\tau_1$	$s = \tau_1 + \tau_2$	$s = 2\tau_2$
$V_0(s)$	B					
$V_1(s)$	A_0B	A_1B	A_2B			
$V_2(s)$	A_0^2B	$[A_0A_1 + A_1A_0]B$	$[A_0A_2 + A_2A_0]B$	A_1^2B	$[A_1A_2 + A_2A_1]B$	A_2^2B

Corollary 5.3 *The system*

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - i \cdot h) + Bu(t) \qquad (5.17)$$

is $\mathbf{R}^n$*-controllable for* $t_1 > (j_f - 1) \cdot h$ *iff the following matrix Q has rank n:*

$$Q = [Q_{1,1}, \ldots, Q_{n,1}, Q_{1,2}, \ldots, Q_{n,2}, \ldots, Q_{n,j_f}] \qquad (5.18)$$

where

$$Q_{1,1} = B$$

$$Q_{r+1,j} = A_0 Q_{r,j} + \sum_{i=1}^{k} A_i Q_{r,j-i} \qquad (5.19)$$

$$r = 1, \ldots, n$$

$$j = 1, \ldots, j_f; \quad j_f \in [1, k(n-1)+1] \subset \mathbf{N}$$

and $Q_{r,j} = 0$ *for* $j \le 0$*, or* $r \le 0$*, or* $j > r + k - 1$*.*

Proof: For the system (5.17) the matrix $V_i(s)$ is nonzero for s = 0, h, 2h, …. Let be l = 0, 1, 2, …. We may write $V_i(l \cdot h) = Q_{i+1, l+1}$. Replacing i + 1 by r and l + 1 by j, the recursive definition of $Q_{i,j}$ (5.19) follows from (5.14) and the controllability matrix Q is obtained from (5.15). ❑

Corollary 5.3 is an extension of the controllability criterion of [85, p. 139] formulated for single-delay systems. However, in our notation the indices are defined differently such that the matrices $Q_{i,j}$ fit within the matrix scheme introduced in Examples 5.1 and 5.2 (cf. Tables 5.1 and 5.2). The general matrix scheme is shown in Table 5.3.

Table 5.3

	s = 0	s = h	..
$V_0(s)$	$Q_{1,1}$	0	..
$V_1(s)$	$Q_{2,1}$	$Q_{2,2}$	..
:	:	:	::

Furthermore, with this concept of indices, the exact lower bound for t_1 and its relation to the controllability matrix Q can be given. Theorem 5.2 shows that the delays affect the value of t_1 but not the algebraic rank condition (5.15). Consequently, a controllability criterion independent of any delay does not require the knowledge of the value τ_i for obtaining t_1. Therefore, $\boldsymbol{R}^n$-controllability on $[0, \tau_1]$ implies $\boldsymbol{R}^n$-controllability independent of constant delays since the only restriction for t_1 is $t_1 > 0$. Applying Theorem 5.2 for $\boldsymbol{R}^n$-controllability on $[0, \tau_1]$ yields the following result.

Corollary 5.4: *If the system*

$$\dot{x}(t) = A_0 x(t) + Bu(t)$$

is $\boldsymbol{R}^n$-controllable, then system (5.13) is $\boldsymbol{R}^n$-controllable independent of constant delays.

Proof: Choosing $j_f = 1$ in (5.19) Corollary 5.3 follows. ❑

In contrast to stability, $\boldsymbol{R}^n$-controllability can be checked by a linear delay-free system. Another controllability criterion based on the considera-tion of a delay-free system is given for the single-delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau(t)) + Bu(t) \,. \tag{5.20}$$

It is assumed that the delay is continuous and bounded such that $0 \le \tau(t) \le \tau_{max} < \infty$.

Corollary 5.5 [81]: *If the system*

$$\dot{x}(t) = (A_0 + A_1)x(t) + Bu(t)$$

is $\boldsymbol{R}^n$-controllable, then system (5.20) is $\boldsymbol{R}^n$-controllable for $t_1 > n \cdot \tau_{max}$.

5.2 Function-space controllability

The true state of the system (5.1) is an element of some function space. Thus the state at time t denoted by $x_t = x(t + \theta)$, $\theta \in [-\tau_k, 0]$ is a segment of the trajectory (cf. Chapter 2). This supplies some motivation for examining the question of controlling the difference-differential equation (5.1) from an initial function to a terminal function. In this context the space of square integrable $\boldsymbol{R}^q$-valued functions on [a, b] denoted by $L^2([a, b], \boldsymbol{R}^q)$ and the space of absolutely continuous $\boldsymbol{R}^q$-values functions on [a, b] with square integrable derivatives denoted by $W^{1,2}([a, b], \boldsymbol{R}^q)$ are useful. Indeed, if $\varphi \in W^{1,2}([t_0 - \tau_k, t_1], \boldsymbol{R}^n)$ and $u \in L^2([t_0, t_1], \boldsymbol{R}^m)$ then x(t) is absolutely continuous and by (5.1) it follows that $\dot{x} \in L^2([t_0, t], \boldsymbol{R}^n)$. Hence $x \in W^{1,2}([t_0, t_1], \boldsymbol{R}^n)$ so that $x_t \in W^{1,2}([t_0 - \tau_k, t_1], \boldsymbol{R}^n)$ for all $t \in [t_0, t_1]$. Therefore, L^2 as the class of admissible controllers and the Sobolev space $W^{1,2}$ as the state space are frequently used in the literature, e.g. in [9], [26, p. 197].

Definition 5.4 [26, p. 197]: *System (5.1) is called controllable (also denoted as $W^{1,2}$-controllable, complete controllable or controllable to all functions in $W^{1,2}$) if for every $\alpha, \varphi \in W^{1,2}$ there exist a finite time t_1 and a control $u \in L^2$ such that $x_{t_1} = \alpha_t$.*

Definition 5.5: *System (5.1) is called null-controllable (also denoted as exact null-controllable or controllable to zero function) if for every $\varphi \in W^{1,2}$ there exist a finite time t_1 and a control $u \in L^2$ such that $x_{t_1} = 0$.*

Function-space controllability criteria are available for the system

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - \tau_i) + Bu(t) \tag{5.13}$$

where the delays $0 < \tau_1 < \ldots < \tau_k < \infty$ are constant and $A_0, A_i \in \boldsymbol{R}^{n \times n}$, $B \in \boldsymbol{R}^{n \times m}$. It is easy to see that if rank[B] = n the system (5.13) is controllable. Banks *et al.* 1975 [9] showed that this condition is also necessary:

Theorem 5.4: *System (5.1) is controllable for any $t_1 > t_0 + \tau_1$ iff rank[B] = n.*

The full rank of B required by Theorem 5.4 is very strong since we must have as many control variables as state variables. There are very few practical situations where this condition will hold. However, weaker conditions are obtained if controllability to the zero function is required only.

Theorem 5.5 [9]: *System (5.13) is null-controllable for any $t_1 > t_0 + \tau_1$ iff*

$$BB^{\dagger} \sum_{i=1}^{k} A_i = \sum_{i=1}^{k} A_i \tag{5.21}$$

and

$$rank[B, A_0B, ..., A_0^{n-1}B] = n \tag{5.22}$$

where $B^{\dagger}$ denotes the Moore-Penrose generalized (or pseudo) inverse of B. (See [84, p. 32] for properties and [158, p. 156] for the computation of the pseudo inverse.)

The conditions for null-controllability are still very restrictive as Theorem 5.5 shows. These restrictive conditions for the system matrices arise since t_1 may be chosen from the interval $[t_0 + \tau_1, \infty)$ or, in other words, there are systems which are not null-controllable for some $t_1 > t_0 + \tau_1$ but for t_2 where $t_2 > t_1$. This fact is especially important for a final time chosen from the interval $[t_0, t_0 + n\tau_k]$. For a final time $t_1 \geq n\tau_k + t_0$, Banks *et al.* (cf. Corollary 5.1 in [9]) showed that function-space controllability remains invariant to the final time t_1. This means that if the system (5.13) is not controllable for any final time t_1 in the interval $[t_0, t_0 + n\tau_k]$, then it is also not controllable for any final time $t_1 > t_0 + n\tau_k$. (This is also true for $\boldsymbol{R}^n$-controllability as follows from Corollary 5.3; see also Example 5.1.) The dependence of function-space controllability on t_1 is illustrated by the two examples below.

Example 5.3: It follows from Theorem 5.5 that the system

$$\dot{x}(t) = A_1 x(t - \tau_1) + bu(t) \tag{5.23}$$

with $A_1 \in \boldsymbol{R}^{n \times n}$ $b \in \boldsymbol{R}^{n \times 1}$ is not null-controllable for any $t_1 > t_0 + \tau_1$. However, Gabasov and Kirillova (1977) [36, p. 84] showed that (5.23) is null-controllable for some finite time t_1 if $\text{rank}[B, A_1B, ..., A_1^{n-1}B] = n$. Equivalently, we could say that if (5.23) is $\boldsymbol{R}^n$-controllable, it is null-controllable for some finite time t_1.

Example 5.4: A further system where $\boldsymbol{R}^n$-controllability implies controllability to a zero function is the following one:

$$\dot{x}(t) = A_1x(t - \tau_1) + Bu(t)$$

where $A_1 \in \boldsymbol{R}^{n \times n}$, $B \in \boldsymbol{R}^{n \times m}$, and $\text{rank}[A_1] = n$. This system is null-controllable for a finite t_1 according to [36, p. 79], but it follows from Theorem 5.5 that this system is not null-controllable for every $t_1 > t_0 + \tau_1$.

These examples show that in some cases $\boldsymbol{R}^n$-controllability implies controllability to the zero function for some finite time t_1. However, this is not true in general as proven by a counterexample in [36, p. 84].

If $t_1 > n \cdot \tau_k$ then null-controllability is equivalent to spectral controllability [111]. Spectral controllability will be discussed in Section 5.5. In order to list all the important criteria for null-controllability the rank condition is stated here, as well. Recall that the matrix $\Delta(s)$ is defined in Section 3.1 as follows:

$$\Delta(s) = sI - A_0 - \sum_{i=1}^{k} A_i e^{-s\tau_i} \ . \tag{3.10}$$

The symbol Λ denotes the set of all eigenvalues of (5.13).

Theorem 5.6: *System (5.13) is null-controllable for any $t_1 > n \cdot \tau_k$ iff*

$$\text{rank}[\Delta(s), B] = n \qquad \forall\, s \in \Lambda \ .$$

5.3 Approximate controllability

The previous subsection showed that function-space controllability is very restrictive. However, in many practical situations it is sufficient to reach the prescribed final state x_{t_1} only approximately. A controllability concept which prescribes the final state only approximately is called approximate controllability. It is supposed that this type of reachability is less restrictive than function-space controllability. However to the author's knowledge, this conjecture has not been confirmed. The available exact algebraic rank condition for both controllability concepts are valid for different final times. Moreover, approximately null-controllable is equivalent to exact null-controllable for $t_1 > n \cdot \tau_k$ [111]. The following controllability definitions and criteria refer to the system

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t-\tau_i) + Bu(t) \qquad t \geq t_0 \qquad (5.24)$$

$$x(t_0) = x_0 \qquad t = t_0 \qquad (5.25)$$

$$x(t) = \varphi(t) \qquad t_0 - \tau_k \leq t < t_0 \qquad (5.26)$$

where $A_0, A_i \in \boldsymbol{R}^{n \times n}$, $B \in \boldsymbol{R}^{n \times m}$, $\tau_1 < \ldots < \tau_k < \infty$, $\varphi \in L^2([t_0 - \tau_k, t_0), \boldsymbol{R}^n)$, and $u \in L^2([t_0, t_1], \boldsymbol{R}^m)$. The product space $M^2 = \boldsymbol{R}^n \times L^2([-\tau_k, 0], \boldsymbol{R}^n)$ is the space of pairs $(x, x_t) = z$, $x \in \boldsymbol{R}^n$ $x_t \in L^2([-\tau_k, 0), \boldsymbol{R}^n)$ with the inner product

$$\langle z, y \rangle_{M^2} = x^T y + \int_{-\tau_k}^{0} x(t+\theta)^T x(t+\theta) d\theta .$$

This defines a norm [72, p. 129]

$$\|z\|_{M^2} = \sqrt{\langle z, z \rangle_{M^2}} = \sqrt{\|x\|^2_{\boldsymbol{R}^n} + \|x_t\|^2_{L^2}}$$

and a metric on M^2 given by

$$\|z - y\|_{M^2} = \sqrt{\langle z - y, z - y \rangle_{M^2}} .$$

Definition 5.6 [89]: *System (5.24) is M^2-approximately controllable if for any $z_0 = (x(t_0), x_{t_0}) \in M^2$ and any $\alpha \in M^2$ there exist a finite time t_1 and a control $u \in L^2$ such that $\|z_f - \alpha\|_{M^2} < \varepsilon$ for every $\varepsilon > 0$ where $(x(t_1), x_{t_1}) = z_f$.*

Definition 5.7 [89]: *System (5.24) is approximately null-controllable if for any $z_0 \in M^2$ there exist a finite time t_1 and a control $u \in L^2$ such that $\|z_f\|_{M^2} < \varepsilon$ for every $\varepsilon > 0$.*

The concept of the closure is usually applied to define M^2-approximate controllability [89], [111], and [40, p. 269]. We use the metric since it is closer to the engineering way of thinking. An algebraic characterization of this controllability concept was given by Manitius (1981):

Theorem 5.7 [89]: *System (5.24) is approximately controllable for any $t_1 > n \cdot \tau_k$ iff*

$$rank[\Delta(s), B] = n \qquad for\ all\ s \in \Lambda \tag{5.27}$$

and

$$rank[A_k, B] = n. \tag{5.28}$$

Theorem 5.8 [123]: *System (5.24) is approximately null-controllable for any $t_1 > n \cdot \tau_k$ iff*

$$rank[\Delta(s), B] = n \qquad for\ all\ s \in \Lambda\ . \tag{5.29}$$

Theorem 5.8 says that M^2-approximately null-controllability is equivalent to spectral controllability. A device for the verification of condition (5.27) or (5.29), respectively, is illustrated in Section 5.5 where spectral controllability is considered.

5.4 F-approximate controllability

Full state space controllability as illustrated in the last two sections has been studied by the approach of approximate controllability in M^2 and exact controllability in $W^{1,2}$. In each case, controllability in the full state space led to very restrictive conditions for the system matrices. This suggests that from the controllability point of view, the full state space is “too big” and that one therefore should search for a “smaller space” in which controllability would be characterized by less restrictive conditions, without losing the link with stabilizability and spectral controllability. The idea is that in many systems not all of the components of the state are delayed. For such systems the requirement that all the components of the state be approximately/exactly controllable might be too strong. In 1976, Manitius introduced the concept of F-approximate controllability which corresponds to controllability of a delayed system in a subspace of M^2. (Since this subspace is characterized by an operator denoted by F this reachability concept is called F-controllability. We shall not consider this operator here; for details see [123].) To this author’s knowledge, exact F-controllability is not discussed in the literature, except the dual problem: exact F-observability [107]. We shall not consider this concept here since the corresponding criterion is hard to apply. Before stating the available results on F-approximate controllability, some notation has to be defined. In this section, system (5.17) is considered:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{k} A_i x(t - i\cdot h) + Bu(t) && t \geq t_0 \qquad (5.17) \\ x(t_0) &= x_0 && t = t_0 \\ x(t) &= \varphi(t) && t_0 - \tau_k \leq t < t_0 \end{aligned}$$

where $A_i \in \boldsymbol{R}^{n\times n}$, $B \in \boldsymbol{R}^{n\times m}$, $\varphi \in L^2([t_0 - \tau_k, t_0), \boldsymbol{R}^n)$, and $u \in L^2([t_0, t_1], \boldsymbol{R}^m)$. Again the product space M^2 is used (cf. Section 5.3). The pairs $z_0 = (x_0, x_{t_0})$ and $z_f = (x(t_1), x_{t_1})$ are elements of M^2 and characterize the initial and final state, respectively.

Definition 5.8 [90]: *System (5.17) is F-approximately controllable if for any* $z_0 \in M^2$ *and any* $\alpha = (\alpha_0, \alpha_t) \in M^2$ *there exist a finite time* t_1 *and a control* $u \in L^2$ *such that*

$$\sqrt{\|x(t_1) - \alpha_0\|^2_{\mathbf{R}^n} + \sum_{i=1}^{k} \|A_i x_{t_1} - A_i \alpha_t\|^2_{L^2}} < \varepsilon$$

for every $\varepsilon > 0$.

Theorem 5.9 [123]: *System (5.17) is F-approximately controllable for any* $t_1 > n \cdot \tau_k$ *iff*

$$rank[\Delta(s), B] = n \tag{5.30}$$

and

$$rank \begin{bmatrix} A_0 - sI & A_1 & . & . & A_k & B \\ A_1 & . & . & A_k & 0 & 0 \\ . & & . & & . & . \\ . & . & & & . & . \\ A_k & & & 0 & 0 & 0 \end{bmatrix} = n + rank \begin{bmatrix} A_1 & . & . & A_k \\ . & & . & \\ . & . & & \\ A_k & & & 0 \end{bmatrix} \tag{5.31}$$

for all $s \in \Lambda$.

For systems with no delays, F-approximate controllability is equivalent to the standard Euclidean controllability. Moreover, if $\det[A_k] \neq 0$ F-approximate controllability is equivalent to approximate controllability [90]. Finally, from the fact that F-approximate controllability is weaker than approximate controllability and implies spectral controllability [90], Theorem 5.10 follows:

Theorem 5.10: *System (5.17) is F-approximately null-controllable for any* $t_1 > n \cdot \tau_k$ *if*

$$rank[\Delta(s), B] = n \qquad \text{for all } s \in \Lambda . \tag{5.32}$$

5.5 Spectral controllability

A linear system is spectrally controllable if there exists a controller which can move all of the eigenvalues of the system to any position in the complex plane. Moreover, (spectral) stabilizability implies the existence of a controller which can move all of the eigenvalues with positive real parts to the left side of the complex plane. In general, a time-delay system has infinitely many eigenvalues, but it possesses only a finite number of eigenvalues with positive real part. This caused some Russian authors, e.g. [68], to investigate the stabilizability of retarded system. Krasovskii considered the subspace of the state space which is spanned by the eigenvectors of the unstable eigenvalues. Since this subspace is finite dimensional, the verification of the stabilizability and the design of a stabilizing state-feedback controller is calculable. This method, based on the decomposition of the state space, belongs to the spectral decomposition theory (cf. [43] Section 7.1-7.4, [6], [7], [140]). A further result obtained with the help of this theory was the generalisation of the Popov-Belevitch-Hautus test for time-delay systems. This result was derived independently by Bhat and Koivo (1976) [7] and, for a more general system, by Pandolfi (1976) [114]. The definitions and theorems in the following refer to the difference-differential equation of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t-\tau_i) + Bu(t) \qquad (5.13)$$

where $A_0, A_i \in \boldsymbol{R}^{n \times n}$, $B \in \boldsymbol{R}^{n \times m}$, and $0 < \tau_1 < \ldots < \tau_k < \infty$.

Definition 5.9 [114]: *System (5.13) is spectrally controllable if there exists a controller of the form*

$$u(t) = K_0 x(t) + \int_{-\tau_k}^{0} K_1(\theta) x(t+\theta) d\theta \qquad (5.33)$$

such that all eigenvalues of the closed-loop system

$$\dot{x}(t) = [A_0 + K_0] x(t) + \sum_{i=1}^{k} A_i x(t-\tau_i) + \int_{-\tau_k}^{0} K_1(\theta) x(t+\theta) d\theta \qquad (5.34)$$

can be assigned.

Definition 5.10 [114]: *System (5.13) is ν-stabilizable (where ν is a finite real number) if there exists a controller of the form (5.33) such that all of the eigenvalues of (5.34) with $Re(\lambda) \geq \nu$ can be assigned.*

Theorem 5.11 [114]: *System (5.13) is stabilizable iff*

$$rank[\Delta(s), B] = n \qquad \text{for all } s \in \Lambda \text{ with } Re(s) \geq 0 \,. \tag{5.35}$$

Theorem 5.12 [114]: *System (5.13) is spectrally controllable iff*

$$rank[\Delta(s), B] = n \qquad \text{for all } s \in \Lambda \,. \tag{5.36}$$

Condition (5.36) can be formulated as an extended Kalman rank condition, which is a more common form of the criterion.

Theorem 5.13 [130]: *System (5.13) is spectrally controllable if*

$$rank\left[B, \left[A_0 + \sum_{i=1}^{k} A_i e^{-s\tau_i}\right] B, \ldots, \left[A_0 + \sum_{i=1}^{k} A_i e^{-s\tau_i}\right]^{n-1} B\right] = n \tag{5.37}$$

for all $s \in \Lambda$.

Note that condition (5.37) is only sufficient as has been shown by Spong and Tarn (1981). A generalisation of Theorem 5.11 for systems with additional delays in the control was given by Olbrot (1978). Condition (5.35) is strongly related to the existence of an optimal LQ-regulator for time-delay systems (cf. Chapter 6).

The verification of condition (5.35) can be performed numerically by computing the eigenvalues with positive real parts using the algorithm introduced in Section 3.4. The algorithm of Carpentier-Dos Santos (cf. Subsection 3.4.6) and Corollary 3.6 enable us to check whether all of the eigenvalues in the right half of the complex plane have been found. At first glance, (5.36) might require the computation of all the eigenvalues of (5.13).

This can actually be avoided by a device of Manitius and Triggiani (1978):

$$\operatorname{adj}[\Delta(s)]\cdot B \neq 0 \qquad \Rightarrow \qquad \operatorname{rank}[\Delta(s), B] = n$$

where $s \in \Lambda$ and adj means the matrix adjoint (cf. [84, p. 10]).

Example 5.5: The system

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{5.38}$$

is considered. For which values of the delay τ is the system (5.38) spectrally controllable or stabilizable? We have

$$\Delta(s) = \begin{bmatrix} s - e^{-s\tau} & 1 - 2e^{-s\tau} \\ -2 - e^{-s\tau} & s - 2 + 3e^{-s\tau} \end{bmatrix}$$

$$\operatorname{adj}[\Delta(s)] = \begin{bmatrix} s - 2 + 3e^{-s\tau} & -1 + 2e^{-s\tau} \\ 2 + e^{-s\tau} & s - e^{-s\tau} \end{bmatrix} .$$

Suppose

$$\operatorname{adj}[\Delta(s)] \cdot B = \begin{bmatrix} -1 + 2e^{-s\tau} \\ s - e^{-s\tau} \end{bmatrix} = 0 .$$

From the first row we have $e^{-s\tau} = 0.5$. Substituting this into the second row we have $s - 0.5 = 0$, hence s must be 0.5. Then the equality $e^{-s\tau} = 0.5$ is satisfied only for $\tau = -2\cdot\ln(0.5) \approx 1.386$. For that particular delay and for $s = 0.5$ the matrix

$$\left[\Delta(s = 0.5), B\right] = \begin{bmatrix} 0 & 0 & 0 \\ -2.5 & 1 & 1 \end{bmatrix}$$

has rank 1. Therefore, the system (5.38) is spectrally controllable as well as stabilizable for all $\tau > 0$ except for $\tau = -2\cdot\ln(0.5)$.

5.6 The dual problem: Observability

The concept of observability is concerned with the following problem. Given the system

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{k} A_i x(t - ih)\ \ Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

its input u, and output y over a finite time interval, determine the initial function. Since it is assumed that u and B are known and the solution of differential equation above can be superposed by the zero-initial state response and the zero-input response, the problem of system observability can be addressed when the control u is identically zero. This means that given a system and its zero-input response over a finite time interval, find the initial state. Thus, with no loss of generality we can assume that $u \equiv 0$ and study the observability of the system

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - ih) \qquad t \ge t_0 \tag{5.39}$$

$$x(t_0) = x_0 \qquad t = t_0$$

$$x(t) = \varphi(t) \qquad t_0 - \tau_k \le t < t_0$$

$$y(t) = Cx(t) \qquad t \ge t_0\,. \tag{5.40}$$

The observability of system described by (5.39) and (5.40) is dual to the controllability of the system

$$\dot{x}(t) = A_0^T x(t) + \sum_{i=1}^{k} A_i^T x(t - ih) + C^T u(t)\,. \tag{5.41}$$

This means that controllability of system (5.41) implies observability of system (5.39), (5.40) (cf. [40, Sections 8.1 and 8.4]). Consequently, there are several observability concepts.

An observability type is denoted as *strong* if the observation time is restricted to the length of the maximum delay $\tau_{max} = k \cdot h$ of the system. Furthermore, if a system is not observable on [0, n·k·h] it is not observable on any larger (or smaller) time interval. Below, the exact definitions together with the corresponding available criteria are enumerated.

Definition 5.11: *System (5.39), (5.40) is* $\boldsymbol{R}^n$*-observable if the initial point* $x(t_0)$ *can be uniquely determined from the observation* $y(t)$ *over a finite interval of time* $[t_0, t_1]$ *for any function* $\varphi \in L^2([t_0 - kh, t_0), \boldsymbol{R}^n)$.

Theorem 5.14: *System (5.39), (5.40) is* $\boldsymbol{R}^n$*-observable for* $t_1 > (j_f - 1)\cdot h$*, iff the following matrix Q has rank n:*

$$Q = [Q_{1,1}, \ldots, Q_{n,1}, Q_{1,2}, \ldots, Q_{n,2}, \ldots, Q_{n,j_f}] \tag{5.42}$$

where

$$Q_{1,1} = C^T$$

$$Q_{r+1,j} = A_0^T Q_{r,j} + \sum_{i=1}^{k} A_i^T Q_{r,j-i}$$

$$r = 1, \ldots, n$$

$$j = 1, \ldots, j_f; \quad j_f \in [1, k(n-1)+1] \subset \boldsymbol{N}$$

and $Q_{r,j} = 0$ *for* $j \le 0$*, or* $r \le 0$*, or* $j > r + k - 1$.

Definition 5.12: *System (5.39), (5.40) is called observable (or function space observable or* $W^{1,2}$*-observable) if every* $\varphi \in W^{1,2}([t_0 - kh, t_0], \boldsymbol{R}^n)$ *can be uniquely determined from the observation* $y(t)$ *over a finite time interval* $[t_0, t_1]$.

Theorem 5.15: *System (5.39), (5.40) is strongly observable (i.e., observable for* $t_1 > t_0 + kh$*) if* $rank[C] = n$.

A function-space observability criterion for time-varying systems has been suggested in [85, p. 145]. However, for a time-invariant system of the form (5.39), (5.40) this criterion is equivalent to the $\boldsymbol{R}^n$-observability rank test of Theorem 5.14. Therefore, these results are questionable.

Definition 5.13: *System (5.39), (5.40) is M^2-approximately observable if every initial state $z_0 = (x(t_0), x_{t_0}) \in M^2$ can be estimated by $\hat{z}_0 = (\hat{x}(t_0), \hat{x}_{t_0})$ from the observation $y(t)$ over a finite time interval $[t_0, t_1]$ such that $\|z_0 - \hat{z}_0\|_{M^2} < \varepsilon$ for every $\varepsilon > 0$.*

Theorem 5.16: *System (5.39), (5.40) is approximately observable for any $t_1 > n \cdot k \cdot h$, iff*

$$rank \begin{bmatrix} \Delta(s) \\ C \end{bmatrix} = n \tag{5.43}$$

for all $s \in \Lambda$ and

$$rank[A_k^T, C^T] = n \tag{5.44}$$

where Λ is the set of eigenvalues of (5.39).

Definition 5.14: *System (5.39), (5.40) is F-approximately observable if every initial state $z_0 = (x(t_0), x_{t_0}) \in M^2$ can be estimated by $\hat{z}_0 = (\hat{x}(t_0), \hat{x}_{t_0})$ from the observation $y(t)$ over a finite time interval $[t_0, t_1]$ such that*

$$\sqrt{\|x(t_0) - \hat{x}(t_0)\|^2_{\mathbf{R}^n} + \sum_{i=1}^{k} \|A_i x_{t_0} - A_i \hat{x}_{t_0}\|^2_{L^2}} < \varepsilon$$

for every $\varepsilon > 0$.

Theorem 5.17: *System (5.39), (5.40) is F-approximately observable for any $t_1 > n \cdot k \cdot h$, iff*

$$rank \begin{bmatrix} \Delta(s) \\ C \end{bmatrix} = n \tag{5.45}$$

for all $s \in \Lambda$ and

$$rank \begin{bmatrix} A_0^T - sI & A_1^T & . & . & A_k^T & C^T \\ A_1^T & . & . & A_k^T & 0 & 0 \\ . & & . & & . & . \\ . & . & & & . & . \\ A_k^T & & & 0 & 0 & 0 \end{bmatrix} = n + rank \begin{bmatrix} A_1^T & . & . & A_k^T \\ . & & . & \\ . & . & & \\ A_k^T & & & 0 \end{bmatrix} . \tag{5.46}$$

Definition 5.15: *System (5.39), (5.40) is spectrally observable if all its eigenvalues are observable. An eigenvalue λ is observable when any corresponding eigensolution of the form $x(t) = x(t_0)e^{\lambda t}$, $x(t_0) \neq 0$ yields $y(t) \neq 0$ on $[0, \infty)$.*

Definition 5.16: *System (5.39), (5.40) is ν-detectable (where ν is finite and a real number), if all its eigenvalues with $Re(\lambda) \geq \nu$ are observable.*

Theorem 5.18: *System (5.39), (5.40) is spectrally observable iff*

$$rank \begin{bmatrix} \Delta(s) \\ C \end{bmatrix} = n \tag{5.47}$$

for all $s \in \Lambda$. System (5.39), (5.40) is ν-detectable iff (5.47) holds for all $s \in \Lambda$ with $Re(s) \geq \nu$.

Theorem 5.19: *System (5.39), (5.40) is spectrally observable if*

$$rank \begin{bmatrix} C \\ C\left[A_0 + \sum_{i=1}^{k} A_i e^{-shi}\right] \\ \vdots \\ C\left[A_0 + \sum_{i=1}^{k} A_i e^{-shi}\right]^{n-1} \end{bmatrix} = n\ .$$

for all $s \in \Lambda$.

Control

6 State-feedback control methods: A classification

This chapter is devoted to a brief overview of state-feedback control approaches for time-delay systems of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - \tau_i) + Bu(t) \ . \tag{6.1}$$

Various methods have been proposed to control this time-delay system. These methods can be classified into six groups, which are briefly described below:

I) The algebraic approach over a ring of polynomials
II) The spectrum decomposition method
III) The finite spectrum assignment technique
IV) Optimal control
V) Suboptimal control
VI) Finite dimensional approximations

I) *The algebraic approach over a ring of polynomials* [42], [61]: Using the delay operator $d^i x(t) := x(t - i \cdot h)$, the retarded system (6.1) can be rewritten as

$$\dot{x}(t) = A(d)x(t) + Bu(t) \tag{6.2}$$

where $A(d) = A_0 + A_1 d^1 + ... + A_k d^k$. System (6.2) is called a system over rings. It is of the form of an ordinary system, with the big difference that the elements of the matrix A(d) are polynomials in d. The necessary and sufficient conditions in terms of the matrices A(d) and B for stabilizability, controllability, and observability are known [42]. Perhaps one of the most interesting results of that theory is that a stabilizable delay system can always be stabilized by a finite-dimensional compensator [61]. However, up until now, this compensator has been determined approximately only [42], [61]. Furthermore, optimal control problems are not considered for systems over rings of the form (6.2). A survey of the results concerning systems over rings has been published in [42].

II) *The spectrum decomposition method [68], [114], [140], [43, Sections 7.1-7.4]*: In this method, the subspace C_+ of the state space C of (6.1) is considered. This subspace C_+ is spanned by the eigenfunctions of the unstable eigenvalues of (6.1). Since a time-delay system has only a finite number of eigenvalues with positive real part, the spectral projection of system (6.1) into C_+ is described by a finite dimensional system of the form $\dot{x}_\lambda(t) = A_\lambda x_\lambda(t) + B_\lambda u_\lambda(t)$ where A_λ and B_λ are real matrices. The stabilizability of the system (6.1) guarantees the existence of a controller $u_\lambda(t) = K_\lambda x_\lambda(t)$ such that the term $A_\lambda + B_\lambda K_\lambda$ is stable. The matrix K_λ can be found, e. g., by solving an algebraic Riccati equation of the form $A_\lambda^T P_\lambda + P_\lambda A_\lambda - P_\lambda B_\lambda R_\lambda^{-1} B_\lambda^T P_\lambda + Q_\lambda = 0$ where $Q_\lambda \geq 0$, $R_\lambda > 0$, and $K_\lambda = -R_\lambda^{-1} B_\lambda^T P_\lambda$. In the state space of the original system (6.1), the controller $u_\lambda(t) = K_\lambda x_\lambda(t)$ is given by

$$u(t) = K_0 x(t) + \int_{-k \cdot h}^{0} K_1(\theta) x(t+\theta) d\theta \ . \tag{6.3}$$

The matrices K_0 and $K_1(\theta)$ are determined by K_λ and the left and right eigenfunctions of the associated unstable eigenvalues (for details see [140]).

An advantage of this method is that all tools which are available for linear, nondelayed systems can be applied for time-delay systems as well. However, this requires the knowledge of the open-loop spectrum in the right-half complex plane as well as the calculation of the corresponding left and right eigenfunctions. Especially the latter calculations can be a cumbersome task, since they usually have to be performed analytically. Furthermore, the controller always influences only a finite number of eigenvalues of (6.1).

III) *The finite spectrum assignment technique [114], [148]*: The goal is the construction of a linear state feedback such that the corresponding closed-loop system has a finite number of eigenvalues located at an arbitrarily pre-assigned set of points in the complex plane. This method does not require a preliminary knowledge of the plant's spectrum. It requires only that n spectral points be assigned, while the others are automatically eliminated. Iff the system is spectrally controllable, it is finite spectrum assignable. However, to this author's knowledge, all spectrum assignment control laws are valid for SISO systems only. Furthermore, the resulting controller usually cannot be expressed in the form (6.3), since it requires additional terms. This increases the implementation effort of this controller.

IV) *Optimal control*: The study of optimal control for systems with delays has a history of over thirty years. In the early days, the so-called LQ-regulator was the main research topic (see, e.g., [67] and [122]). The LQ-regulator is the optimal regulator for the linear system (6.1) associated with the following quadratic cost function

$$J(u) = x(t)^T F x(t) + \int_{t_0}^{t_1} \{ x(t)^T Q x(t) + u(t)^T R u(t) \} dt$$

where $Q \geq 0$ and $R > 0$. The regulator is of the form

$$u(t) = K_0(t)x(t) + \int_{-k \cdot h}^{0} K_1(t, \theta)x(t + \theta)d\theta . \tag{6.4}$$

Subsequently, the existence of the optimal controller, its characterization by Riccati equations, and the existence of Riccati solutions for systems with delays in the state were studied, e.g., [30]. Generalizations of the delay structures in systems and costs were established in further research, e.g., [31], [52]. Furthermore, for a cost criterion with infinite horizon

$$J(u) = \int_{t_0}^{\infty} \{ x(t)^T Q x(t) + u(t)^T R u(t) \} dt \tag{6.5}$$

the stability of the optimally controlled system was established in [31]. Moreover, the resulting closed-loop system was found to have the same desirable sensitivity and robustness properties as finite-dimensional systems do. It has in fact been shown in [80] that the closed-loop system satisfies the circle condition. The calculation of the LQ-regulator for the system (6.2) associated with the cost criterion (6.5) involves the problem of solving partial differential equations, the so-called infinite-dimensional Riccati equation. Various efficient algorithms for this problem have been derived. Two suitable numerical methods are introduced in Chapter 8. In Chapter 7 the corresponding optimal control problem is considered.

A similar development can be observed for related optimization problems such as time-optimal control [26, Chapter 7], dynamic programming [85, Section 6.5], and the Hamilton-Jacobi-Bellman equation [8]. However, these developments are not subjects of this work.

V) *Suboptimal control [85, Chapter 7]*: The LQ-regulator problem is considered for a finite, preassigned time interval. Suboptimal control approaches for time-delay systems avoid the computation of the infinite-dimensional Riccati equation. The control laws obtained are of the form

$$u(t) \;=\; K_0 x(t) + g(t)$$

whereas the exact solution (LQ-regulator) is of the form (6.4)

$$u(t) \;=\; K_0(t)x(t) + \int_{-k \cdot h}^{0} K_1(t, \theta)x(t+\theta)d\theta \;.$$

Suboptimal control approaches involve solving nondelayed, linear optimization problems repeatedly, such that their solutions move closer to the optimal solution as the number of repetitions increases. One approach for determining suboptimal control for time-delay systems is based on the concept of optimal control sensitivity. In this technique, the control is expanded into a MacLaurin series in some parameters. The coefficients of the truncated series are computed from the optimization of some related nondelayed system. Another method is to treat the delay terms in the state as extra perturbing inputs, such that the problem is converted into a nondelayed problem. Suboptimal control methods are not considered here, since the corresponding control laws are valid only for a finite time.

VI) *Finite dimensional approximations*: The idea of this approach is to approximate time-delay systems by finite-dimensional systems. The approximation can be performed in the frequency and in the time domain. The relation between these two approaches has not been completely investigated. The approximation methods in the time-domain can be classified into two types: the semi-discretization technique and the full discretization technique. The first type consists of the replacement of the delay-differential equation by a linear ordinary differential equation (discretization of the space variable only). In the full discretization technique, a time-delay system is replaced by a discrete time system by a simultaneous discretization of the space and the time variables. In both methods, the dimension of the approximation system grows with the desired precision. During the last two decades the semi-discretization approach has mainly been considered and various efficient algorithms have been derived. A finite-dimensional approximation of a time-delay system allows to apply all control design tools to linear nondelayed systems or to discrete-time systems, respectively. However, these techniques lead to high-order systems. In Chapter 8, two approximation techniques of time-delay systems using state-space representations are considered.

Each approach has its advantages and disadvantages. However, combinations of the various approaches yield easily applicable controller design methods. For instance, finite-dimensional approximations of a time-delay system can be used to solve the infinite-dimensional Riccati equation for obtaining an approximation of the LQ-regulator. Moreover, stability criteria may be used to confirm the closed-loop stability. This procedure is outlined in Chapter 8. The other approaches are not considered further, since their applications are limited, as mentioned above.

7 Optimal Control: The optimal regulator

Consider a nonlinear retarded functional differential equation described by

$$\dot{x}(t) = f(x(t), x(t-h), \ldots, x(t-k\cdot h), u(t)) \qquad t \geq t_0 \qquad (7.1)$$

$$x(t) = x_0 \qquad t = t_0 \qquad (7.2)$$

$$x(t) = \varphi(t) \qquad t_0 - k\cdot h \leq t < t_0 \qquad (7.3)$$

where f is bounded and continuous. It is assumed that φ and u are continuous and bounded functions. Let the cost function be defined by

$$J(u) = F(x(t_1), t_1) + \int_{t_0}^{t_1} L(t, x(t), u(t))\,dt \qquad (7.4)$$

where $F(\cdot)$ is the final state penalty term and t_1 is a given final time. The function $L(\cdot)$ is supposed to reflect the cost of deviation from zero of the state variables and the control. The optimal regulator of a time-delay system can now be stated as follows. Find an optimal function $u^o(t)$, $t \in [t_0, t_1]$ which satisfies equation (7.1) for some given initial state (7.2), (7.3) and minimizes the performance index (7.4). The maximum principle is applied to perform this optimization. The extension of the maximum principle to time-delay systems has been developed by Kharatishvili (1967) (cf. also [85, Chapter 6]). As in the non-delayed case, a set of necessary conditions can be given. Let the Hamiltonian function be as follows

$$H(t, x(t), x(t-h), \ldots, x(t-k\cdot h), u(t), \lambda(t)) = -L(t, x(t), u(t)) +$$

$$+ \lambda(t)^T f(t, x(t), x(t-h), \ldots, x(t-k\cdot h), u(t)) \qquad (7.5)$$

where $\lambda(t) \in \boldsymbol{R}^n$ remains to be defined. If $u^o(t)$ is the optimal regulator and $x^o(t)$ the resulting optimal trajectory, then there exists a costate vector $\lambda^o(t)$ such that the following state equations holds:

state equations:

$$\dot{x}(t) = \nabla_\lambda H\big|_o \qquad t \geq t_0 \qquad (7.6)$$

$$x^o(t_0) = x_0 \qquad t = t_0 \qquad (7.7)$$

$$x^o(t) = \varphi(t) \qquad t_0 - k \cdot h \leq t < t_0 \qquad (7.8)$$

costate equations:

$$\dot{\lambda}^o(t) = -\nabla_x H\big|_o - \sum_{i=1}^{k} \nabla_x H(t + i \cdot h)\big|_o \qquad t_0 \leq t \leq t_1 - k \cdot h \qquad (7.9)$$

$$\dot{\lambda}^o(t) = -\nabla_x H\big|_o - \sum_{i=1}^{k-1} \nabla_x H(t + i \cdot h)\big|_o \qquad t_1 - k \cdot h < t \leq t_1 - (k-1) \cdot h \qquad (7.10)$$

$$\vdots$$

$$\dot{\lambda}^o(t) = -\nabla_x H\big|_o \qquad t_1 - h < t \leq t_1 \qquad (7.11)$$

$$\lambda^o(t_1) = \nabla_{x(t_1)} F(x(t_1), t_1)\big|_o \qquad t = t_1 \qquad (7.12)$$

maximization of the Hamiltonian:

$$0 = -\nabla_u H\big|_o \qquad t \geq t_0\,. \qquad (7.13)$$

A general solution of the two-point boundary-value (TPBV) problem (7.6) – (7.12) is not known. However, using the method of steps, an exact solution can be derived for linear time-delay systems and for an appropriate performance index. The method of steps reduces a delayed TPBV problem to a nondelayed TPBV problem (cf. also Section 2.1). It is obvious that this procedure can be applied only for a finite horizon, i.e., $t_1 < \infty$. The method is illustrated by an example.

Example 7.1: The optimal regulator for the system

$$\dot{x}(t) = x(t) + x(t-1) + u(t) \qquad t \geq 0 \qquad (7.14)$$

$$x(t) = c \qquad -1 \leq t \leq 0 \qquad (7.15)$$

minimizing the performance index

$$J = \frac{F}{2}x(2)^2 + \frac{1}{2}\int_0^2 u(t)^2 dt$$

is determined. The Hamiltonian function according to (7.5) is given by

$$H(x, \lambda, u) = -\frac{1}{2}u(t)^2 + \lambda(t)\cdot[x(t) + x(t-1) + u(t)] \, .$$

Condition (7.13) yields

$$\left.\frac{\partial H}{\partial u}\right|_o = 0 = -u^o(t) + \lambda^o(t) \, .$$

The above relation is used to eliminate u in the state equations (7.14) and in the differential equations of the adjoint system (7.9) – (7.12):

$$\dot{x}^o(t) = x^o(t) + x^o(t-1) + \lambda^o(t) \qquad 0 \le t \le 2 \qquad (7.16)$$

$$x^o(t) = c \qquad -1 \le t \le 0 \qquad (7.17)$$

$$\dot{\lambda}^o(t) = -\lambda^o(t) - \lambda^o(t+1) \qquad 0 < t \le 1 \qquad (7.18)$$

$$\dot{\lambda}^o(t) = -\lambda^o(t) \qquad 1 < t \le 2 \qquad (7.19)$$

$$\lambda^o(2) = F\cdot x^o(2) \qquad t = 2 \, . \qquad (7.20)$$

The method of steps (or method of successive integration) is now applied to evaluate $\lambda^o(t)$. In a first step, the differential equation (7.19) is integrated with respect to its boundary condition (7.20). We obtain

$$\lambda^o(t) = F\cdot x^o(2)e^{2-t} \qquad 1 \le t \le 2 \, . \qquad (7.21)$$

In the next step, solution (7.21) is used to solve (7.18):

$$\lambda^o(t) = \lambda^o(0)e^{-t} - F\cdot x^o(2)e^{1-t}\cdot t \qquad 0 \le t \le 1 \, . \qquad (7.22)$$

The constant of integration $\lambda^o(0)$ is chosen such that $\lambda^o(t)$ described by (7.21) and (7.22) is continuous at time t = 1. This continuity condition yield

$$\lambda^o(0) = F\cdot x^o(2)\cdot e[e + 1] \, . \qquad (7.23)$$

With the help of (7.22) and (7.23), $\lambda^o(t)$ is replaced in (7.16). The differential equation obtained is integrated with respect to its boundary condition (7.17). We obtain the optimal state trajectory for the interval of time $t \in [0, 1]$:

$$x^o(t) = -c + e^t\{2c + \frac{F}{2} x^o(2)e[0.5 + e]\} - e^{1-t}\frac{F}{2} x^o(2)\{0.5 + e - t\}. \quad (7.24)$$

Equation (7.24) represents the initial condition of (7.16) for $t \in [1, 2]$. The solution of (7.16) for $t \in [1, 2]$ is therefore

$$\begin{aligned} x^o(t) = e^t \cdot \Big\{ & 2c - 4e^{-1}c + \frac{F}{4}x^o(2)[e^{-1} - 1 - 2e + 2e^2] + \\ & + \left[2ce^{-1} + \frac{F}{2}x^o(2)[0.5 + e]\right] \cdot t \Big\} + c + \\ & + e^{-t} \cdot \frac{F}{4}x^o(2)[-e^2 + e^3 - e^2 \cdot t] \qquad 1 \le t \le 2 . \end{aligned}$$

The value of $x^o(2)$ can be determined by the latter equation. This value, together with the desired optimal regulator, is stated in the following equations:

$$u^o(t) = F \cdot x^o(2)e^{1-t}[e + 1 - t] \qquad 0 \le t \le 1$$

$$u^o(t) = F \cdot x^o(2)e^{2-t} \qquad 1 \le t \le 2$$

where

$$x^o(2) = \frac{-(1 + 2e^2)c}{-1 - \frac{3F}{4} + \frac{F}{2}e + \frac{F}{4}e^2 + \frac{F}{2}e^3 + \frac{F}{2}e^4} .$$

For the special case $c = 1$ and $F = 3$, this optimization problem was solved numerically in [10].

The application of the method of steps is only useful for an optimization involving linear, low-order control systems ($n \le 2$) associated with a quadratic performance index with short horizon. Otherwise, the effort of calculation becomes intolerable. Numerical methods are applied to solve more general optimization problems. Many of these methods use the Riccati equations for time-delay systems. These equations are discussed below. The considered system is of the form

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^{k} A_i x(t - i \cdot h) + Bu(t) \qquad t \geq t_0 \tag{7.25}$$

$$x(t_0) = x_0 \qquad t = t_0 \tag{7.26}$$

$$x(t) = \varphi(t) \qquad t_0 - kh \leq t < t_0 \tag{7.27}$$

where A_0, $A_i \in \boldsymbol{R}^{n \times n}$ and $B \in \boldsymbol{R}^{n \times m}$. The cost criterion which is to be minimized is of a quadratic form

$$J(u) = \int_{t_0}^{\infty} \{x(t)^T Q x(t) + u(t)^T R u(t)\} dt \tag{7.28}$$

where $Q \in \boldsymbol{R}^{n \times n}$ is a positive-semidefinite matrix and $R \in \boldsymbol{R}^{m \times m}$ is a positive-definite matrix. It is assumed that the system (7.25) is stabilizable, i.e.,

$$\operatorname{rank}\left[sI - A_0 - \sum_{i=1}^{k} A_i e^{-ihs}, B \right] = n$$

for all s being eigenvalues with nonnegative real parts of the system (7.25) (cf. Section 5.5, Theorem 5.11). Now, we can state the solution to the infinite-time optimal control problem: if the system (7.25) is stabilizable, then the optimal regulator (LQ-regulator) minimizing (7.28) is given by

$$u(t) = -R^{-1}B^T \left\{ P_0 x(t) + \int_{-kh}^{0} P_1(\theta) x(t + \theta) d\theta \right\} \tag{7.29}$$

provided the matrices $P_0 \in \boldsymbol{R}^{n \times n}$, $P_1(\theta) \in L^2([-kh, 0], \boldsymbol{R}^{n \times n})$, and $P_2(\theta, \xi) \in L^2([-kh, 0] \times [-kh, 0], \boldsymbol{R}^{n \times n})$ satisfy the following conditions [39], [40, p. 343], [52, p. 659]:

$$A_0^T P_0 + P_0 A_0 - P_0 B R^{-1} B^T P_0 + P_1(0) + P_1^T(0) + Q = 0 \tag{7.30}$$

$$-\frac{\partial P_1(\theta)}{\partial \theta} + [A_0^T - P_0 B R^{-1} B^T] P_1(\theta) + P_2(0, \theta) = 0 \tag{7.31}$$

$$\frac{\partial P_2(\xi, \theta)}{\partial \theta} + \frac{\partial P_2(\xi, \theta)}{\partial \xi} + P_1^T(\xi) B R^{-1} B^T P_1(\theta) = 0 \tag{7.32}$$

where $-kh \le \theta \le 0$ and $-kh \le \xi \le 0$. The matrix functions $P_1(\theta)$ and $P_2(\theta, \xi)$ obey the following boundary conditions:

$$P_1(-kh)^T = A_k^T P_0 \tag{7.33}$$

$$P_2(-kh, \theta) = A_k^T P_1(\theta) \ . \tag{7.34}$$

The matrix functions may be discontinuous in the form of “jumps”. These jumps are determined by [39, p. 104], [52, p. 659]:

$$P_1((-ih)^+)^T - P_1((-ih)^-)^T = A_i^T P_0$$

$$P_2((-ih)^+, \theta)^T - P_2((-ih)^-, \theta)^T = A_i^T P_1(\theta) \qquad i = 1, ..., k-1 \ .$$

Moreover, the matrices P_0 and $P_2(\theta, \xi)$ are symmetric

$$P_0 = P_0^T$$

$$P_2(\xi, \theta) = P_2(\theta, \xi)^T \ .$$

Given the conditions (7.30) – (7.34), the minimal value of (7.28) in terms of the initial function is [62, p. 1087], [52, p. 660]

$$J(u^o) = x_0^T P_0 x_0 + 2 x_0^T \int_{-kh}^{0} P_1(\theta)\varphi(\theta) d\theta + \int_{-kh}^{0}\int_{-kh}^{0} \varphi(\theta)^T P_2(\xi, \theta)\varphi(\theta) d\xi d\theta \ .$$

The asymptotic stability of the resulting closed-loop system (7.25), (7.29) has been established (cf., e.g., [31]).

Robustness properties of (7.25), (7.29) have been studied as well. Particularly the closed-loop system (7.25), (7.29) possesses simultaneously in each feedback control channel [80]

(i) $[0.5, \infty]$ gain margin

(ii) $\pm 60°$ phase margin

if $R > 0$ is diagonal and $Q > 0$. Note that the optimal regulator (7.29) consists of two parts. The first part of the regulator is similar to the regulator for the linear, nondelayed system. The integral part of (7.29) accounts for the delays of the system. An exact solution of the Riccati equations (7.30) – (7.34) is not known, even for very simple time-delay systems. However, this problem can be solved numerically. Two procedures are illustrated in the next chapter.

The Riccati method is also known for a quadratic performance index with finite horizon of the form (7.4). In that case the resulting optimal control law is time-dependent, whereas criterion (7.28) yields a state-feedback controller. Since the latter regulator is easier to implement, we consider here only criteria with infinite horizon.

8 Finite dimensional approximations

A finite dimensional approximation of a time-delay system can be performed in the frequency domain [32], [61], [94] or in the time domain. The approximation techniques in the time domain can be classified further: semi-discretization methods (discretization of the state only) and full-discretization methods (simultaneous discretization of the state and the time variables [30]). These methods can be used to solve the infinite-dimensional Riccati equation (7.30) – (7.34) for obtaining an approximation of the LQ-regulator (7.29). But none of these procedures can guarantee a priori the stability of the resulting closed-loop system, especially if the delays of the plant were estimated wrong. In case the controller is obtained with a semi-discretization method, the stability can easily be checked with the help of the algebraic stability tests presented in Subsection 3.2.6. Therefore, only semi-discretization methods are considered here. There is a rich literature on this topic, e.g., [10], [39], [56], [62], [77]. In the following, two suitable methods are illustrated: the averaging methods and the Legendre-Tau method. The averaging method excels by its feasibility, whereas the Legendre-Tau method has one of the highest convergence rates of the known approximation techniques.

8.1 The averaging approximation method

The averaging approximation method (frequently abbreviated as AVE-method) was invented by several Soviet authors in the early sixties and has been described in several publications (see, e.g., [70], [120]; further references and a detailed review can be found in the paper of Banks and Burns [10]). Krasovskii [70] and later Ross [122] used this method to compute the LQ-regulator for time-delay sytems. Next, the convergence and convergence rates of this approximation scheme (for a slightly different approximation of the initial state) were established in [39], [77], [93]. The approximation method is illustrated here in a popular and somewhat heuristic approach, since a detailed derivation would exceed the scope of this work.

I) *Approximation of the system*: The following system is considered:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - i \cdot h) + Bu(t) \qquad t \geq t_0 \tag{8.1}$$

$$x(t) = \varphi(t) \qquad t_0 - k \cdot h \leq t \leq t_0 . \tag{8.2}$$

Let N be a positive integer. The approximation starts by a division of the delay constant h into N equal subintervals with length h/N. Let

$$\begin{aligned} x_0^N(t) &= x(t) \\ x_1^N(t) &= x(t - h/N) \\ x_2^N(t) &= x(t - 2h/N) \\ &: \\ x_N^N(t) &= x(t - h) \\ &: \\ x_{kN}^N(t) &= x(t - kh) \end{aligned} \tag{8.3}$$

where $x_i^N(t) \in \boldsymbol{R}^n$ and $x^N(t) = [x_0^N(t)^T, \ldots, x_{kN}^N(t)^T]^T \in \boldsymbol{R}^{n(kN+1)}$. The derivative of $x_i^N(t)$ can be approximated by

$$\dot{x}_1^N(t) \cong \frac{x(t) - x(t - h/N)}{h/N} = \frac{N}{h}[x_0^N(t) - x_1^N(t)] .$$

In a similar fashion we obtain

$$\dot{x}_2^N(t) \cong \frac{x(t - h/N) - x(t - 2h/N)}{h/N} = \frac{N}{h}[x_1^N(t) - x_2^N(t)]$$

$$:$$

$$\dot{x}_N^N(t) \cong \frac{x(t - h + h/N) - x(t - h)}{h/N} = \frac{N}{h}[x_{N-1}^N(t) - x_N^N(t)]$$

$$:$$

$$\dot{x}_{kN}^N(t) \cong \frac{x(t - kh + h/N) - x(t - kh)}{h/N} = \frac{N}{h}[x_{kN-1}^N(t) - x_{kN}^N(t)] .$$

Hence, a finite-dimensional approximation of (8.1) can be formulated as follows:

$$\dot{x}^N(t) = A^N x^N(t) + B^N u(t) \tag{8.4}$$

where $A^N \in \boldsymbol{R}^{n(kN+1)\times n(kN+1)}$ and $B^N \in \boldsymbol{R}^{n(kN+1)\times m}$ are given by

$$A^N = \begin{bmatrix} A_0 & 0 & . & . & 0 & A_1 & . & . & A_k \\ \frac{N}{h}I & -\frac{N}{h}I & 0 & . & . & . & . & . & . \\ 0 & \frac{N}{h}I & -\frac{N}{h}I & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & \frac{N}{h}I & -\frac{N}{h}I \end{bmatrix} \qquad B^N = \begin{bmatrix} B \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} . \tag{8.5}$$

The symbol $I \in \boldsymbol{R}^{n\times n}$ denotes the identity matrix. From approach (8.3) it follows that the initial condition of (8.4) is given by

$$x_i^N(t_0) = \varphi(t - ih/N) . \tag{8.6}$$

For $N = 1$ and $k = 1$ the AVE-method is equivalent to a truncated Taylor series approach. This fact can be shown by an appropriate Taylor series expansion of x(t):

$$x(t) \approx x(t-h) + \frac{\dot{x}(t-h)}{1!}[t-(t-h)]^1 + \dots + \frac{x(t-h)^{(r)}}{r!}[t-(t-h)]^r \quad .$$

A first-order approximation yields

$$\frac{1}{h}[x(t) - x(t-h)] \approx \dot{x}(t-h) \approx \dot{x}_1^N(t) \quad .$$

Therefore, system (8.1) with $k = 1$ is approximated in the Taylor series approach by

$$\begin{aligned} \dot{x}_0^N(t) &= A^N x^N(t) + B^N u(t) \\ \dot{x}_1^N(t) &= \frac{1}{h}[x_0^N(t) - x_1^N(t)] \end{aligned} \tag{8.7}$$

which is just (8.4) for N = 1. For the sake of completeness it should be mentioned here that the Taylor series can also be used to approximate the delay term x(t – ih) [33, p. 22]:

$$x(t-ih) \approx x(t) - ih\dot{x}(t) + \ldots + (-1)^r \frac{(ih)^r}{(ih)!} x^{(r)}(t)$$

which leads to the following approximation of (8.1) and (8.2) [44]:

$$\begin{aligned} \dot{\tilde{x}}(t_0) &= \left[I + \sum_{i=1}^{k} ihA_i\right]^{-1} \left\{\sum_{i=0}^{k} A_i \tilde{x}(t) + Bu(t)\right\} \\ \tilde{x}(t_0) &= \varphi(t_0) \ . \end{aligned} \tag{8.8}$$

However, this approximation should be applied only if the delay is small [44]. But in this case it is more reasonable to approximate system (8.1) and its initial condition (8.2) by

$$\begin{aligned} \dot{\hat{x}}(t_0) &= \left\{A_0 + \sum_{i=1}^{k} A_i\right\} x(t) + Bu(t) \\ \hat{x}(t_0) &= \varphi(t_0) \end{aligned}$$

since this approach preserves stability properties if the delay is sufficiently small [133]. Note that the approximation schemes (8.7) and (8.8) do not offer an approximation parameter N. So let us return to the topic of this section: the AVE-method. What is the frequency domain representation of (8.4) in terms of the original system matrices? For simplicity a single-delay system is considered first:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t) \ . \tag{8.9}$$

The transfer function G(s) of (8.9) is given by

$$G(s) = [sI - A_0 - A_1 e^{-sh}]^{-1} B \ .$$

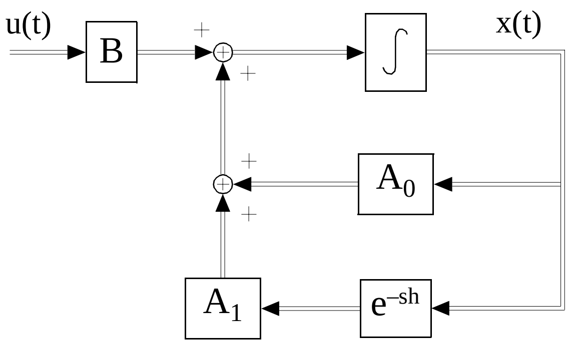

Fig. 8.1

An associated signal-flow diagram which involves the term e^{-hs} in one link is shown in Fig. 8.1. According to (8.4) the AVE-approximation of (8.9) is:

$$\dot{x}_0^N(t) = A_0 x_0^N(t) + A_1 x_N^N(t) + Bu(t) \tag{8.10}$$

$$\begin{aligned}
\dot{x}_1^N(t) &= \frac{N}{h}[x_0^N(t) - x_1^N(t)] \\
&: \\
\dot{x}_i^N(t) &= \frac{N}{h}[x_{i-1}^N(t) - x_i^N(t)] \\
&: \\
\dot{x}_N^N(t) &= \frac{N}{h}[x_{N-1}^N(t) - x_N^N(t)].
\end{aligned} \tag{8.11}$$

From (8.11) it follows that

$$x_i^N(s) = \{1 + s\frac{h}{N}\}^{-1} x_{i-1}^N(s)\,. \tag{8.12}$$

Relation (8.12) together with the Laplace transformation of (8.10) yields

$$x_0^N(s) = [sI - A_0 - A_1\{1 + s\frac{h}{N}\}^{-N}]^{-1} Bu(s)\,. \tag{8.13}$$

Equation (8.13) represents the AVE-approximation of (8.9). The term e^{-sh} is replaced by N successive links (see Fig. 8.2) with the rational fraction

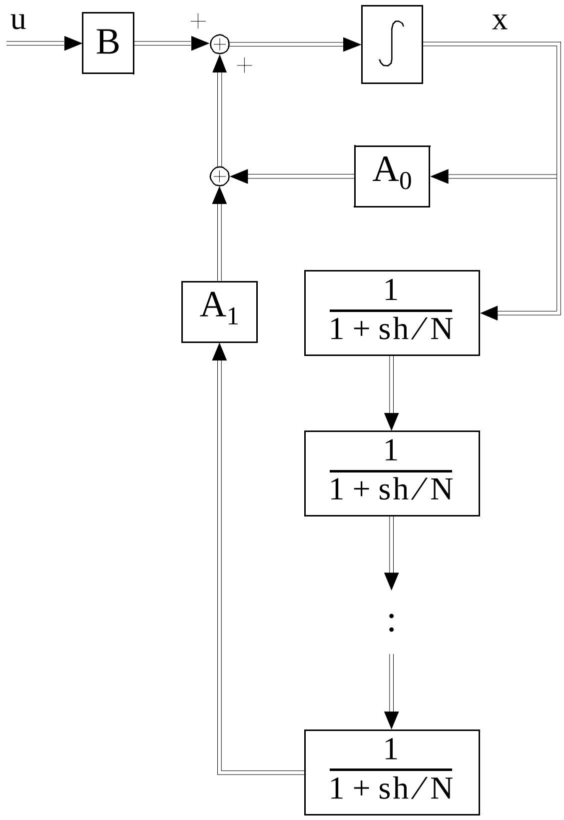

Fig. 8.2

transfer function $\{1 + s\frac{h}{N}\}^{-1}$. Since

$$e^{-sh} = \lim_{N\to\infty}\left\{1 + s\frac{h}{N}\right\}^{-N}$$

(cf. [10]) this approximation converges.

Gibsen (1983) conjectured that the approximating system (8.3) – (8.5) is exponentially stable for sufficiently large N if the underlying time-delay system is asymptotically stable. This stability preservation property of the averaging scheme was later proved by Salamon (1985).

II) *Optimal regulator on the infinite interval*: Let us consider the optimal regulator for the system

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t - i \cdot h) + Bu(t) \qquad t \geq t_0 \tag{8.14}$$

$$x(t) = \varphi(t) \qquad t_0 - kh \leq t \leq t_0 \tag{8.15}$$

minimizing the performance index

$$J(u) = \int_{t_0}^{\infty} \{ x(t)^T Q x(t) + u(t)^T R u(t) \} dt \tag{8.16}$$

where $Q \in \boldsymbol{R}^{n \times n}$ and $R \in \boldsymbol{R}^{m \times m}$ are symmetrical positive semi-definite and positive-definite matrices, respectively. Under the assumption that

$$\operatorname{rank}\left[sI - A_0 - \sum_{i=1}^{k} A_i e^{-ihs}, B \right] = n$$

and

$$\operatorname{rank}\begin{bmatrix} sI - A_0 - \sum_{i=1}^{k} A_i e^{-ihs} \\ Q^{1/2} \end{bmatrix} = n$$

for all s being eigenvalues with nonnegative real parts of the system (8.14), there exists an optimal regulator of the form

$$u(t) = -R^{-1} B^T \left\{ P_0 x(t) + \int_{-kh}^{0} P_1(\theta) x(t + \theta) d\theta \right\} \tag{8.17}$$

provided the matrices P_0, $P_1(\theta)$, and $P_2(\theta, \xi)$, satisfy the conditions (7.30) – (7.34). This optimization problem is treated here with the AVE-method. As stated above, in this method, system (8.14) is approximated by

$$\dot{x}^N(t) = A^N x^N(t) + B^N u(t) \tag{8.18}$$

where A^N and B^N are given by (8.5). From (8.3) it follows that the finite-dimensional averaging approximation of (8.16) is

$$J(u)^N = \int_{t_0}^{\infty} \{ x^N(t)^T Q^N x^N(t) + u(t)^T R u(t) \} dt \tag{8.19}$$

where $Q^N \in \boldsymbol{R}^{n(kN+1) \times n(kN+1)}$ is given by

$$Q^N = \begin{bmatrix} Q & 0 & . & . & 0 \\ 0 & 0 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & 0 \end{bmatrix} . \tag{8.20}$$

The solution of the optimization problem described by (8.18) and (8.19) is $u(t) = -R^{-1}(B^N)^T \Pi^N x^N(t)$ where the matrix $\Pi^N \in \boldsymbol{R}^{n(kN+1) \times n(kN+1)}$ satisfies the algebraic Riccati equation

$$(A^N)^T \Pi^N + \Pi^N A^N - \Pi^N B^N R^{-1} (B^N)^T \Pi^N + Q^N = 0 \ . \tag{8.21}$$

The matrix Π^N can be partitioned into $(N+1)^2$ submatrices. They are denoted by $\Pi^N_{r,j} \in \boldsymbol{R}^{n \times n}$ where $0 \le r \le Nk$ and $0 \le j \le Nk$:

$$\Pi^N = \begin{bmatrix} \boxed{\Pi^N_{0,0} \quad \Pi^N_{0,1} \quad . \quad . \quad \Pi^N_{0,Nk}} \\ \Pi^N_{1,0} \quad \Pi^N_{1,1} \quad . \quad . \quad \Pi^N_{1,Nk} \\ . \quad . \quad . \quad . \quad . \\ . \quad . \quad . \quad . \quad . \\ \Pi^N_{Nk,0} \quad . \quad . \quad . \quad \Pi^N_{Nk,Nk} \end{bmatrix} . \tag{8.22}$$

The relation $\Pi^N_{r,j} = (\Pi^N_{j,r})^T$ follows from the symmetry of Π^N. If one discretizes (7.31) and (7.32) and compares the algebraic relation thus obtained with (8.21) in terms of the submatrices $\Pi^N_{r,j}$, the two sets of equations are found to be in direct correspondence (for details see [122]). Let P^N_0 and $P^N_1(\theta)$ be an approximation of P_0 and $P_1(\theta)$. Then we have

$$P_0^N = \Pi_{0,0}^N \tag{8.23}$$

$$P_1^N(-jh/N) = N \cdot \Pi_{0,j+1}^N \tag{8.24}$$

for $0 \le j \le Nk - 1$. Furthermore, boundary condition (7.33) yields

$$P_1^N(-kh)^T = A_k^T \cdot \Pi_{0,0}^N \ . \tag{8.25}$$

The submatrices of Π^N, which are involved in (8.23) – (8.25), are located in the first row (first column, respectively,) of submatrices in Π^N. (This row is shaded in (8.22).) The other submatrices of Π^N are an approximation of $P_2(\xi, \theta)$. The corresponding relations are not stated here (cf. [122]), since the optimal control law (8.17) requires only P_0 and $P_1(\theta)$. The equations (8.23) – (8.25) determine almost completely an approximation of P_0 and $P_1(\theta)$. One question remains, however: How are the discontinuities of $P_1(\theta)$ represented in this approximation scheme? Recall that $P_1(\theta)$ may have discontinuities in the form of "jumps":

$$P_1((-ih)^+)^T - P_1((-ih)^-)^T = A_i^T P_0 \qquad i = 1, ..., k \ . \tag{8.26}$$

It turns out that (8.24) describes the upper value of $P_1(\theta)$ at the jumps:

$$P_1^N((-ih)^+) = N \cdot \Pi_{0,j=iN}^N \qquad i = 1, ..., k \ .$$

The lower value of $P_1(\theta)$ at a discontinuity denoted by $P_1^N(\theta^-)$ is given by (8.26):

$$P_1^N((-ih)^-)^T = (\Pi_{0,j=iN}^N)^T - A_i^T \Pi_{0,0}^N \ . \tag{8.27}$$

For those values of θ where $P_1(\theta)$ is continuous we have

$$P_1^N(\theta) = P_1^N(\theta^-) = P_1^N(\theta^+) \ .$$

The notation is illustrated in Fig. 8.3 for a scalar $P_1(\theta)$.

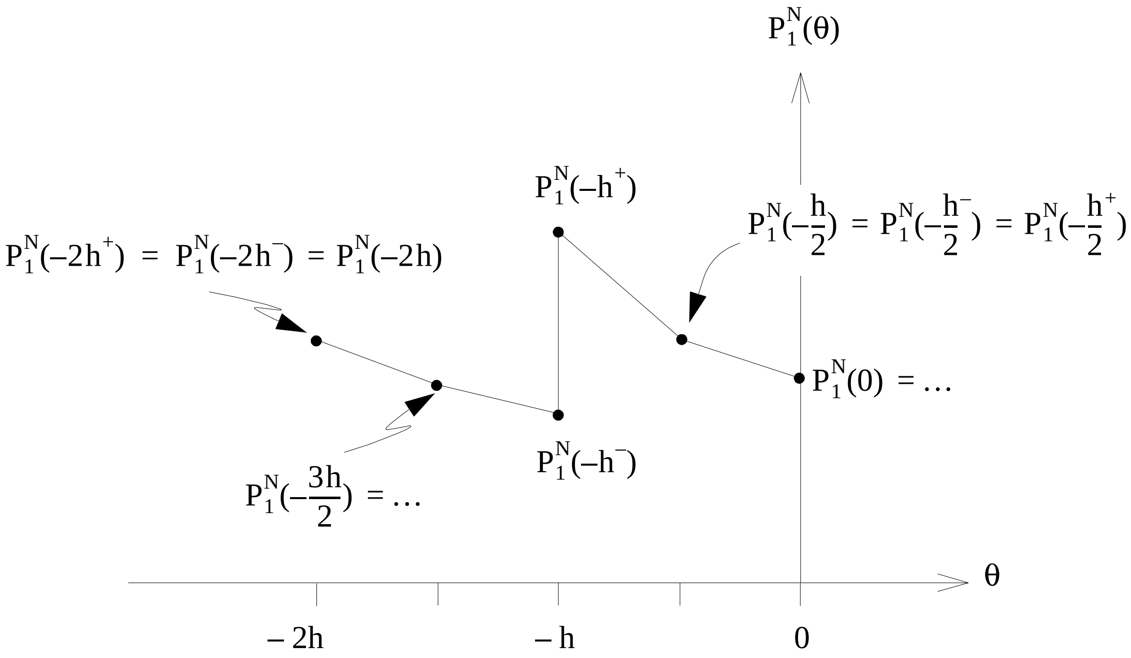

Fig. 8.3 Approximation of $P_1(\theta)$ by computation of some discrete values of $P_1^N(\theta)$ and splines.

Even if P_0 and $P_1(\theta)$ are exactly known, the LQ-regulator of the form (8.17)

$$u(t) = -R^{-1}B^T\left\{P_0x(t) + \int_{-kh}^{0} P_1(\theta)x(t+\theta)d\theta\right\}$$

is impossible to implement, since the integral term requires the storage of infinitely many values of x(t) and the integration has to be performed at each time. Thus, the integral is approximated by a trapezoidal integration, which can be expressed by a sum. Hence, the approximation of the LQ-regulator is given by

$$u^N(t) = -R^{-1}B^T\left\{P_0^Nx(t) + \sum_{j=1}^{Nk}\frac{P_1^N(-\frac{h}{N}(j-1)^-)x(t-\frac{h}{N}(j-1)) + P_1^N(-\frac{h}{N}j^+)x(t-\frac{h}{N}j)}{\frac{2N}{h}}\right\} . \tag{8.28}$$

The method is summarized in the following:

Step 1: Choose the integer $N \geq 1$.

Step 2: Compose the matrices $A^N \in \boldsymbol{R}^{n(kN+1) \times n(kN+1)}$ and $B^N \in \boldsymbol{R}^{n(kN+1) \times m}$ using (8.4).

Step 3: Define the matrix $Q \in \boldsymbol{R}^{n \times n}$ and $R \in \boldsymbol{R}^{m \times m}$ which determine the performance index (8.15).

Step 4: Compose the matrix Q^N (8.20).

Step 5: Solve the algebraic Riccati equation (8.21). The solution yields the matrix Π^N.

Step 6: The submatrices of Π^N according to (8.23) – (8.25) and (8.27) define the approximation (8.28) of the desired LQ-regulator for time-delay systems.

Usually, the procedure is executed several times for increasing values of N to verify the convergence of the solution (cf. the example 8.1 below).

The procedure involves the problem of the computation of a high-dimensional algebraic Riccati equation. The most widely available method for solution of the Riccati equation is the Laub-Schur algorithm [78]. This method is suitable for our purpose. More recently, a hybrid method has been suggested by Banks and Ito (1991). It possesses several computational advantages over the standard eigenvector based (Potter, Laub-Schur) techniques.

Example 8.1: The optimal regulator for the following scalar time-delay system

$$\dot{x}(t) = x(t) + 2x(t-1) + x(t-2) + u(t) \tag{8.29}$$

minimizing the cost function

$$J(u) = \int_0^\infty \{ x(t)^2 + u(t)^2 \} dt \tag{8.30}$$

is considered. For N = 2 we obtain from (8.5):

$$A^N = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 2 & -2 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix} \qquad B^N = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

Furthermore, equation (8.21) yields

$$Q^N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

From (8.30) it follows that $R = 1$. Hence, we are ready to compute the solution of the Riccati equation (8.21):

$$\Pi^N = \begin{bmatrix} 3.53 & 1.10 & 2.09 & 0.48 & 0.92 \\ 1.10 & 0.41 & 0.72 & 0.18 & 0.31 \\ 2.09 & 0.72 & 1.31 & 0.31 & 0.57 \\ 0.48 & 0.18 & 0.31 & 0.08 & 0.13 \\ 0.92 & 0.31 & 0.57 & 0.13 & 0.25 \end{bmatrix} .$$

According to (8.23) – (8.25) and (8.27) the above solution yields an approximation of P_0 and $P_1(\theta)$. The corresponding values are stored in Table 8.1. In order to illustrate the convergence of this procedure, the approximation of P_0 and $P_1(\theta)$ for $N = 1, 4, 8, 20$, and 50 has been computed as well. Only the values which are necessary for a comparison (with the solution for $N = 4$) are listed in Table 8.1. A further comparison of the solutions for $N = 1, 2, 4, 8, 20$, and 50 is illustrated graphically in Fig. 8.4.

As Table 8.1 shows, the approximation method describes $P_1(\theta)$ only at some discrete values of θ. In Fig. 8.4 the discrete approximation is connected via linear splines, which corresponds to the technically realizable form of the LQ-regulator (8.28).

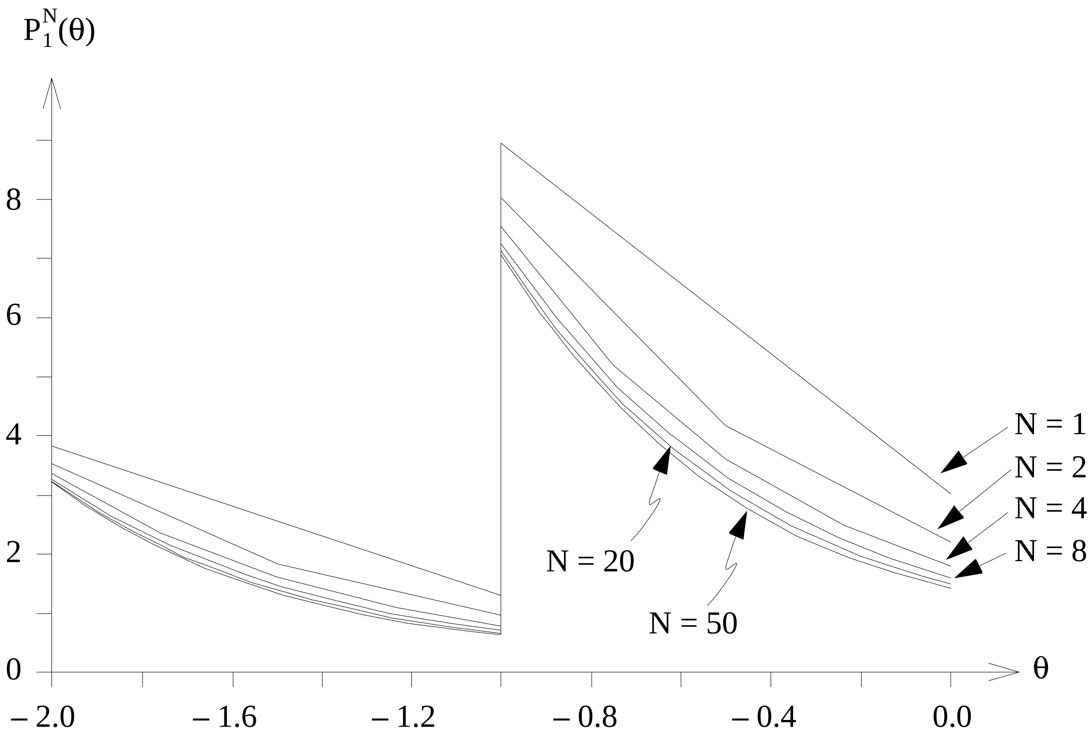

Fig. 8.4 $P_1^N(\theta)$ *computed for N = 1, 2, 4, 8, 20, 50 subject to (8.29) and (8.30)*

Table 8.1 P_0^N *and* $P_1^N(\theta)$ *computed for example 8.1*

	N = 1	N = 2	N = 4	N = 8	N = 20	N = 50
P_0^N	3.83	3.53	3.37	3.28	3.23	3.21
$P_1^N(0)$	3.01	2.21	1.80	1.60	1.49	1.45
$P_1^N(-0.25)$	–	–	2.54	2.29	2.13	2.05
$P_1^N(-0.50)$	–	4.18	3.62	3.33	3.13	3.10
$P_1^N(-0.75)$	–	–	5.21	4.89	4.70	4.62
$P_1^N(-1.00)^-$	8.94	8.03	7.53	7.27	7.12	7.07
$P_1^N(-1.00)^+$	1.28	0.96	0.79	0.71	0.66	0.65
$P_1^N(-1.25)$	–	–	1.13	1.02	0.95	0.91
$P_1^N(-1.50)$	–	1.83	1.61	1.48	1.40	1.36
$P_1^N(-1.75)$	–	–	2.32	2.20	2.12	2.08
$P_1^N(-2.00)$	3.83	3.53	3.37	3.28	3.23	3.21

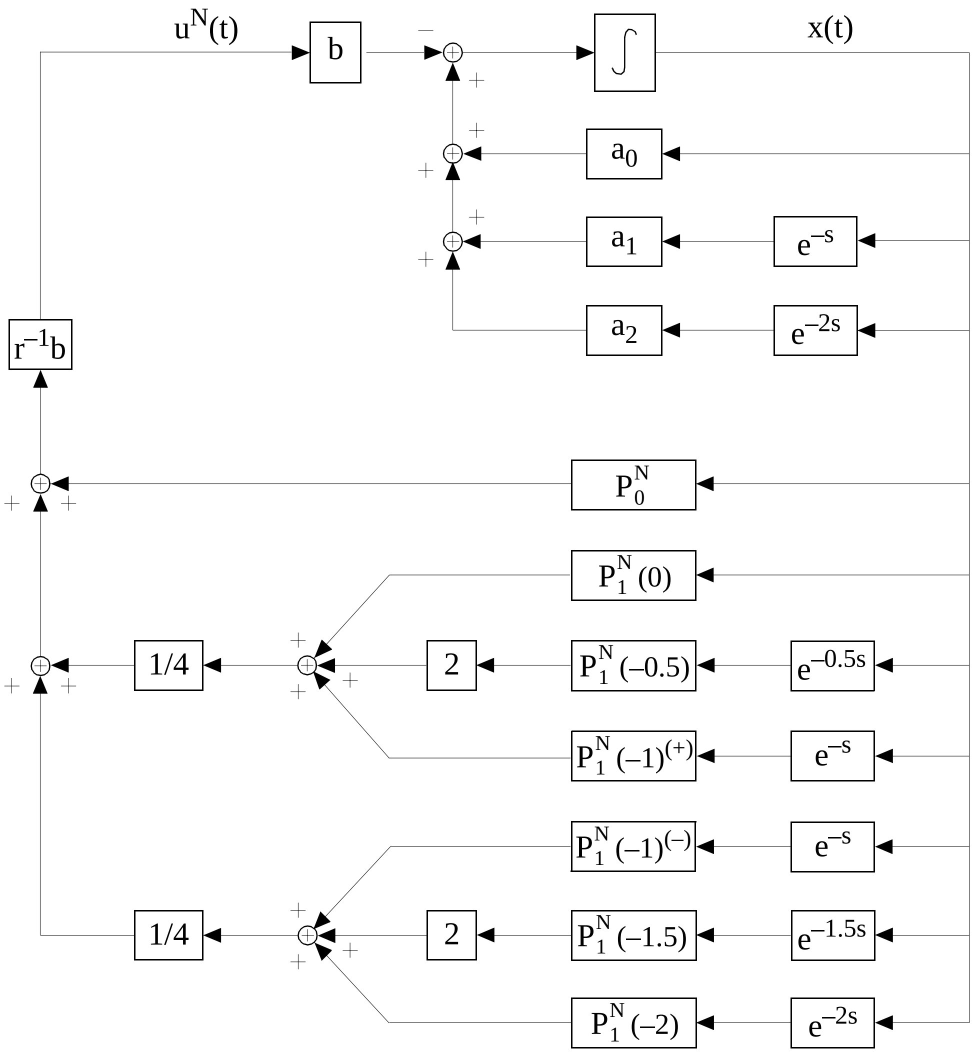

Fig. 8.5 Signal-flow diagram of the approximated (N = 2) LQ-regulator

A signal-flow diagram of the approximated regulator is sketched in Fig. 8.5. This scheme can easily be extended for multidimensional systems. An alternative scheme of the implementation of the LQ-regulator is shown in the next section.

Using Theorem 3.4 (cf. Section 3.2), it can easily be shown that for the approximation degree N = 1 the resulting closed-loop system is asymptotically stable. For briefness, checking the stability of the closed-loop system is demonstrated only in context with the Legendre-Tau approximation method (cf. next section).

8.2 The Legendre-Tau method

While the AVE-method excels by its simplicity and its feasibility, the Legendre-Tau method developed in [53] – [56] is well known for its fast convergence rate [53]. The Legendre-Tau method has one of the highest convergence rates of the known approximation schemes. The eigenvalues of the time-delay system with positive real part or near the imaginary axis are approximated with high precision [54]. Furthermore, if the original time-delay system is asymptotically stable and if the approximation degree N is sufficiently large, the approximating system is asymptotically stable as well [54]. However, the theory of this approximation scheme and its application for designing an optimal state-feedback controller is complete only for single delay systems. An extension for systems with two delays is considered in [55]. But that work is not a consequent extension of the original Legendre-Tau method (cf. [55, p. 1384] for details). Nevertheless, it reveals that the effort of notation would be enormous if the Legendre-Tau technique were to be applied for multiple delay systems. However, for single delay systems the Legendre-Tau method is suitable. In the second part of this section, this method is applied to compute an approximation of the LQ-regulator. Since the stability of the resulting closed-loop system cannot be confirmed a priori with this procedure or any semi-dicretization method, algebraic stability tests (presented in Section 3.2) are applied to check the stability. In the LQ-regulator approach, it is assumed that the delays are exactly known. However, in reality it is difficult to estimate the value of a delay. Furthermore, in practice the delays are frequently not constant and vary within a certain range. Again, algebraic stability tests can be used to investigate the robustness of a system (controlled by an approximated LQ-regulator) against uncertain delays. An illustrative example is given at the end of this section.

In the Legendre-Tau method the state x_t of a linear time-delay system is approximated by Legendre polynomials. The approximation degree N is equivalent to the degree of the Legendre polynomials involved. In order to briefly illustrate the basic ideas of the Legendre-Tau approximation technique some well-known properties of the Legendre polynomials are reviewed (cf. [55]). The Legendre polynomial $p_i(t)$ of degree i is defined by

$$p_i(t) = \frac{1}{2^i \cdot i!} \cdot \frac{d^i}{dt^i}[(t^2 - 1)^i] \qquad -1 \le t \le 1 .$$

It is important to note that $p_i(t)$ is defined only for $t \in [-1, 1]$. For instance the first six Legendre polynomials are:

$$p_0(t) = 1$$

$$p_1(t) = t$$

$$p_2(t) = \frac{1}{2}(3t^2 - 1)$$

$$p_3(t) = \frac{1}{2}(5t^3 - 3t)$$

$$p_4(t) = \frac{1}{8}(35t^4 - 30t^2 + 3)$$

$$p_5(t) = \frac{1}{8}(63t^5 - 70t^3 + 15t)$$

$$p_6(t) = \frac{1}{16}(231t^6 - 315t^4 + 105t^2 - 5) .$$

The Legendre expansion of the function $f \in ([-1, 1]; \boldsymbol{R}^1)$ is

$$f(t) = \sum_{i=0}^{\infty} \alpha_i \cdot p_i(t) \tag{8.31}$$

where the Legendre coefficients α_i are given by

$$\alpha_i = \frac{2i + 1}{2} \cdot \int_{-1}^{1} f(t) \cdot p_i(t)dt . \tag{8.32}$$

Moreover, for all i = 0, 1, ...

$$p_i(\pm 1) = (\pm 1)^i . \tag{8.33}$$

Example 8.2: The function $f(t) = \sin(t)$ is approximated for the time interval $[0, \pi]$ using a truncated Legendre series. Applying the transformation $\sigma = \frac{2t}{\pi} - 1$ the function f can be considered in the appropriate interval:

$$f(\sigma) = \sin[\frac{\pi}{2}(\sigma + 1)] \qquad -1 \le \sigma \le 1 .$$

From (8.32) we obtain

$$\alpha_0 = \frac{2}{\pi}$$

$$\alpha_1 = 0$$

$$\alpha_2 = \frac{10}{\pi}\left\{1 - \frac{12}{\pi^2}\right\} .$$

Thus, f can be approximated by

$$f(\sigma) \cong \alpha_0 p_0(\sigma) + \alpha_1 p_1(\sigma) + \alpha_2 p_2(\sigma) \qquad -1 \le \sigma \le 1$$

which can be rewritten as

$$f(t) \cong \frac{2}{\pi} + \frac{10}{\pi}\left\{1 - \frac{12}{\pi^2}\right\} \cdot \left[\frac{6}{\pi^2}\left\{t - \frac{\pi}{2}\right\}^2 - \frac{1}{2}\right] \qquad 0 \le t \le \pi .$$

The l_1-error of the approximation is smaller than 0.06.

I) *Approximation of the system*: A system with a single and constant delay is considered:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t) \qquad t \ge t_0 \qquad (8.34)$$

$$x(t) = \varphi(t) \qquad t_0 - h \le t \le t_0 \qquad (8.35)$$

where $A_0, A_1 \in \boldsymbol{R}^{n \times n}$, $B \in \boldsymbol{R}^{n \times m}$, and $h > 0$. In the Legendre-Tau method the state $x_t = x(t + \theta)$, $-h \le \theta \le 0$ is described by a truncated Legendre series. Since we intend to approximate x_t for any $t \ge 0$ the coefficients (8.32) of the corresponding Legendre polynomial are time-varying. Let x_t^N be an approximation of x_t. Then we have

$$x_t^N = x^N(t+\theta) = \sum_{i=0}^{N} \alpha_i(t) \cdot p_i\left(\frac{2\theta + h}{h}\right) \qquad \theta \in [-h, 0] \tag{8.36}$$

where p_i denotes the i^{th} Legendre polynomial and $\alpha_i(t) \in \boldsymbol{R}^n$ its coefficient vector. The fraction $(2\theta + h)/h$ maps every $\theta \in [-h, 0]$ into $[-1, 1]$. From (8.36) it follows that

$$x^N(t) = \sum_{i=0}^{N} \alpha_i(t) p_i\left(\frac{2\theta + h}{h}\right)\Bigg|_{\theta = 0} = \sum_{i=0}^{N} \alpha_i(t) p_i(1) = \sum_{i=0}^{N} \alpha_i(t) \ . \tag{8.37}$$

The coefficients $\alpha_i(t)$ are determined by the following two relations:

$$\frac{\partial x^N(t+\theta)}{\partial t} = \frac{\partial x^N(t+\theta)}{\partial \theta} \tag{8.38}$$

$$\dot{x}^N(t) = A_0 x^N(t) + A_1 x^N(t-h) + Bu(t) \ . \tag{8.39}$$

Since

$$\frac{\partial x(t+\theta)^N}{\partial t} = \sum_{i=0}^{N} \dot{\alpha}_i(t) \cdot p_i\left(\frac{2\theta + h}{h}\right)$$

$$\frac{\partial x(t+\theta)^N}{\partial \theta} = \frac{2}{h} \cdot \sum_{i=0}^{N-1} (2i+1) \sum_{\substack{j = i+1 \\ j+i = \text{odd}}}^{N} \alpha_j(t) \cdot p_i\left(\frac{2\theta + h}{h}\right)$$

(cf. [54, p. 741]) condition (8.38) yields

$$\begin{bmatrix} \dot{\alpha}_0(t) \\ \vdots \\ \dot{\alpha}_{N-1}(t) \end{bmatrix} = \frac{2}{h} S^N \otimes I_n \cdot \begin{bmatrix} \alpha_0(t) \\ \vdots \\ \alpha_N(t) \end{bmatrix}$$

where $I \in \boldsymbol{R}^{n \times n}$ is the identity matrix and $S^N \in \boldsymbol{R}^{nN \times n(N+1)}$ is defined by

$$S^N = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & . & . & 1 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 & . & . & 0 & 3 \\ 0 & 0 & 0 & 5 & 0 & 5 & . & . & 5 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & 2N-3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & 0 & 2N-1 \end{bmatrix} .$$

Condition (8.39) is used to determine $\alpha^N(t)$. This approach is called the Tau-method or spectral method (cf. [41, pp. 11ff] for a general description of the spectral methods and their applications).

$$\sum_{i=0}^{N} \dot{\alpha}_i(t) = A_0 \sum_{i=0}^{N} \alpha_i(t) \cdot p_i(1) + A_1 \sum_{i=0}^{N} \alpha_i(t) \cdot p_i(-1) + Bu(t)$$

$$\dot{\alpha}_N(t) = -\sum_{i=0}^{N-1} \dot{\alpha}_i(t) + A_0 \sum_{i=0}^{N} \alpha_i(t) + A_1 \sum_{i=0}^{N} \alpha_i(t)(-1)^i + Bu(t)$$

$$\dot{\alpha}_N(t) = -\left\{ \sum_{i=0}^{N-1} \frac{4i+2}{h} \sum_{\substack{j = i+1 \\ j+i = odd}}^{N} \alpha_j(t) \right\} +$$

$$+ A_0 \sum_{i=0}^{N} \alpha_i(t) + A_1 \sum_{i=0}^{N} \alpha_i(t)(-1)^i + Bu(t) \ .$$

Now, we have n differential equations for every $\alpha_i(t)$. Defining the vector $\alpha(t) \in \boldsymbol{R}^{n(N+1)}$

$$\alpha(t) = \begin{bmatrix} \alpha_0(t) \\ : \\ \alpha_N(t) \end{bmatrix}$$

these differential equations can be collected in a state-space representation:

$$\dot{\alpha}(t) = A^N \alpha(t) + B^N u(t) \ . \tag{8.40}$$

The matrices $A^N \in \boldsymbol{R}^{n(N+1) \times n(N+1)}$ and $B^N \in \boldsymbol{R}^{n(N+1) \times m}$ are given by

$$A^N = \begin{bmatrix} & \frac{2}{h} S^N \otimes I & \\ D_1 & \dots & D_N \end{bmatrix} \qquad\qquad B^N = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B \end{bmatrix} \tag{8.41}$$

where $D_i = -\frac{i(i+1)}{2} + A_0 + (-1)^i A_1$. The initial condition for the system (8.40) is defined by

$$\alpha(0) = \begin{bmatrix} \alpha_0(0) \\ \vdots \\ \alpha_N(0) \end{bmatrix} \tag{8.42}$$

where

$$\alpha_i(0) = \frac{2i+1}{h} \cdot \int_{-h}^{0} \varphi(\theta) \cdot p_i\left(\frac{2\theta + h}{h}\right) d\theta \qquad 0 \le i \le N-1$$

$$\alpha_N(0) = \varphi(0) - \sum_{i=0}^{N-1} \alpha_i(0) .$$

Actually, we are interested in approximating $x(t)$ by $x^N(t)$. According to (8.37), the relation between $x^N(t)$ and $\alpha(t)$ is:

$$x^N(t) = \sum_{i=0}^{N} \alpha_i(t) .$$

This relation can be connected with (8.40) using a state transformation:

$$\xi(t) = \Omega \cdot \alpha(t) = \begin{bmatrix} I & & 0 \\ & \ddots & \\ I & \dots & I \end{bmatrix} \alpha(t) = \begin{bmatrix} \alpha_0(t) \\ \vdots \\ \alpha_{N-1}(t) \\ x^N(t) \end{bmatrix} .$$

Hence, system (8.40) and its initial state (8.42) can be rewritten as

$$\dot{\xi}(t) = \Omega A^N \Omega^{-1} \xi(t) + B^N u(t) \tag{8.43}$$

$$\xi(0) = \begin{bmatrix} \alpha_0(0) \\ \vdots \\ \alpha_{N-1}(0) \\ \varphi^N(0) \end{bmatrix}. \tag{8.44}$$

The eigenvalues of (8.43) are related with diagonal (or all-pass) Padé approximations [32]. In [54 p. 198] it was demonstrated that λ is an eigenvalue of A^N iff λ satisfies $\det[\Delta^N(\lambda)] = 0$ where

$$\Delta^N(\lambda) = \lambda I - A_0 - A_1 \cdot \text{Padé}(e^{-\lambda h}) = \lambda I - A_0 - A_1 \cdot \frac{\sum_{i=0}^{N} \binom{N}{i} / \binom{2N}{i} \cdot (-\lambda h)^i}{\sum_{i=0}^{N} \binom{N}{i} / \binom{2N}{i} \cdot (\lambda h)^i} .$$

II) *Approximation of the LQ-regulator*: The optimal regulator for the system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t) \qquad t \ge t_0 \tag{8.45}$$

$$x(t) = \varphi(t) \qquad t_0 - h \le t \le t_0 \tag{8.46}$$

subject to the performance index

$$J(u) = \int_{t_0}^{\infty} \{ x(t)^T Q x(t) + u(t)^T R u(t) \} dt \tag{8.47}$$

where $Q \ge 0$, $Q \in \boldsymbol{R}^{n \times n}$ and $R > 0$, $R \in \boldsymbol{R}^{m \times m}$ is considered. It is assumed that

$$\operatorname{rank} [sI - A_0 - A_1 e^{-sh}, B] = n \tag{8.48}$$

and

$$\operatorname{rank} \begin{bmatrix} sI - A_0 - A_1 e^{-sh} \\ Q^{1/2} \end{bmatrix} = n \tag{8.49}$$

for all s being eigenvalues of the system (8.45) with nonnegative real parts. Given these assumptions, there exits an optimal regulator of the form

$$u(t) = -R^{-1}B^T\left\{P_0x(t) + \int_{-h}^{0} P_1(\theta)x(t+\theta)d\theta\right\} . \qquad (8.50)$$

In order to tackle this optimization problem with the Legendre-Tau method, we approximate the performance index (8.47) as follows

$$J(u) = \int_{t_0}^{\infty}\{x^N(t)^TQx^N(t) + u(t)^TRu(t)\}dt \quad . \qquad (8.51)$$

Equivalently, we may write

$$J(u) = \int_{t_0}^{\infty}\{\xi(t)^TQ^N\xi(t) + u(t)^TRu(t)\}dt \qquad (8.52)$$

where $Q^N \in \boldsymbol{R}^{n(N+1)\times n(N+1)}$ is defined by

$$Q^N = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \qquad (8.53)$$

and $\xi(t) \in \boldsymbol{R}^{n(N+1)}$ is determined by (8.43) and (8.44). It is well known that the solution of the optimization problem (8.43), (8.52) is

$$u^N(t) = -R^{-1}(B^N)^T\Pi^N\xi(t) \qquad (8.54)$$

where $\Pi^N \in \boldsymbol{R}^{n(N+1)\times n(N+1)}$ satisfies the following Riccati equation:

$$0 = (\Omega A^N\Omega^{-1})^T\Pi^N + \Pi^N(\Omega A^N\Omega^{-1}) - \Pi^NB^NR^{-1}(B^N)^T\Pi^N + Q^N . \quad (8.55)$$

Rewriting (8.54) as

$$u^N(t) = -R^{-1}(B^N)^T \left[\begin{array}{c|c} \Pi_{11}^N & \Pi_{10}^N \\ \hline \Pi_{01}^N & \Pi_{00}^N \end{array}\right] \left[\begin{array}{c} \xi_1(t) \\ \vdots \\ \xi_{N-1}(t) \\ \hline x^N(t) \end{array}\right] \tag{8.56}$$

it becomes obvious that $\Pi_{00}^N \in \boldsymbol{R}^{n \times n}$ is the desired approximation of P_0:

$$\Pi_{00}^N = P_0^N \ . \tag{8.57}$$

Π^N also yields an approximation of $P_1(\theta)$, denoted by $P_1^N(\theta)$. The proof of the following procedure appeared in [56]. Defining the matrix Λ

$$\Lambda = \mathrm{diag}(h, \ldots, \frac{h}{2i+1}, \ldots, \frac{h}{2N-1}, 1) \otimes I \tag{8.58}$$

the following similarity transformation can be performed

$$\Gamma^N = (\Lambda^N)^{-1}\Pi^N\Lambda^N \ . \tag{8.59}$$

The matrix Γ^N contains the Legendre coefficients of the Legendre polynomials defining $P_1^N(\theta)$. The elements of the matrices $P_1^N(\theta)$ are therefore determined by a truncated Legendre series

$$P_1^N(\theta)_{i,j} = \sum_{l=0}^{n-1} \Gamma^N_{i+n\cdot l,\,j} \cdot p_l\left(\frac{2\theta+h}{h}\right) \tag{8.60}$$

where $1 \le i \le n$ and $1 \le j \le n$. The optimal regulator (8.50) can now be approximated by

$$u^N(t) = -R^{-1}B^T\left\{P_0^N x(t) + \int_{-h}^{0} P_1^N(\theta)x(t+\theta)d\theta\right\} \ . \tag{8.61}$$

Since the integral term of (8.61) is not technically realizable, it is approximated by a trapezoidal integration such that the controller is of the form

$$u^N(t) = \sum_{i=0}^{N} K_i x(t - i\frac{h}{N}) .$$

Strictly speaking, the matrix function $P_1^N(\theta)$ is computed for $\theta = 0, \frac{h}{N}, ..., h$ and hence the LQ-regulator is approximated by

$$u^N(t) = -R^{-1}B^T\left[P_0^N x(t) + \sum_{j=1}^{N} \frac{P_1^N(-\frac{h}{N}(j-1))x(t-\frac{h}{N}(j-1)) + P_1^N(-\frac{h}{N}j)x(t-\frac{h}{N}j)}{\frac{2N}{h}}\right] . \tag{8.62}$$

In order to increase the readability of the description of the method, the procedure is summarized in the following:

Step 1: Let $N \geq 1$ be an integer.
Step 2: Compose the matrices $A^N \in \boldsymbol{R}^{n(N+1) \times n(N+1)}$ and $B^N \in \boldsymbol{R}^{n(N+1) \times m}$ according to (8.41).
Step 3: Define the matrix $Q \in \boldsymbol{R}^{n \times n}$ and $R \in \boldsymbol{R}^{m \times m}$ which determine the performance index (8.51).
Step 4: Compose the matrix Q^N (8.53).
Step 5: Solve the algebraic Riccati equation (8.55). The solution yields the matrix Π^N.
Step 6: According to (8.56) and (8.57) the submatrix Π_{00}^N of Π^N is equivalent to P_0^N.
Step 7: Compute $P_0^N(\theta)$ for $\theta = 0, \frac{h}{N}, ..., h$ using (8.60).
Step 8: Implement the approximation of the regulator (8.62).

Example 8.3: The Williams-Otto process is considered. It consists of a chemical reactor, a cooler, a decanter, and a distillation column (cf. Section 2.1). This refining plant can be described by a single delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + Bu(t) \tag{8.63}$$

where

$$A_0 = \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.20 & -5.30 & -12.8 & 0 \\ 6.40 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11.0 & -3.96 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1.92 & 0 & 0 & 0 \\ 0 & 1.92 & 0 & 0 \\ 0 & 0 & 1.87 & 0 \\ 0 & 0 & 0 & 0.724 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The dimensionless state variables represent the deviations in the weight composition of two raw materials, of an intermediate product, and of the desired product, respectively, from their nominal values. The spectrum of the system is shown in Fig. 3.20. It reveals that this system is asymptotically stable. In Subsection 3.2.6, it was shown that system (8.63) is asymptotically stable even if the delay is not constant (but bounded and continuous). However, the system shows a sluggish time response for an initial disturbance. Fig. 8.6 shows a typical time response of this system. The initial disturbance was chosen as follows

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_2(t) \\ x_2(t) \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \forall\, t \in [-1, 0]. \tag{8.64}$$

Since the natural response of the system is unacceptable, an LQ-regulator is applied to diminish deviations of the state variable. An appropriate regulator is obtained for the following choice of the matrices Q and R:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{8.65}$$

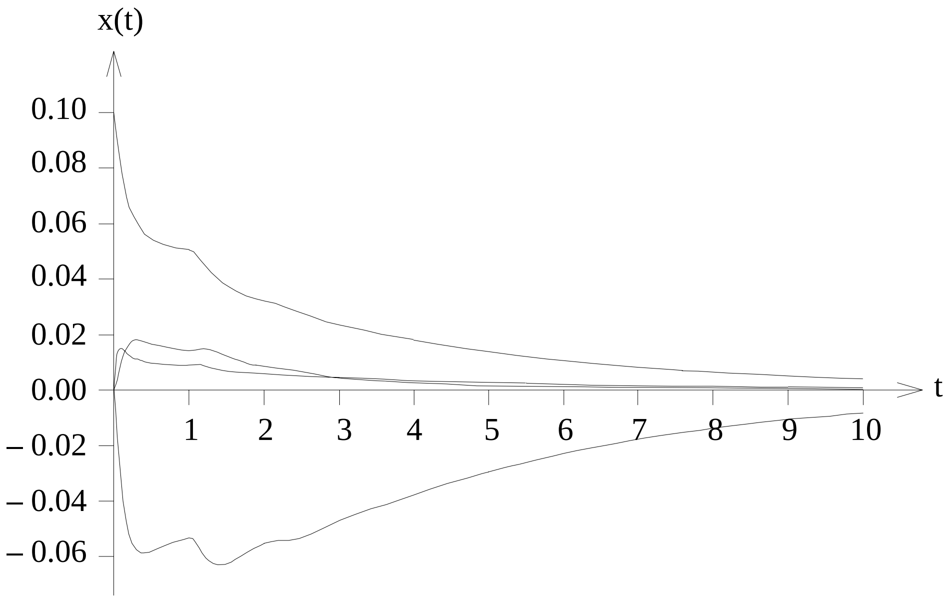

Fig. 8.6 Uncontrolled time response of system (8.63) for the initial disturbance (8.64)

Recall that system (8.63) is spectrally controllable iff

$$\mathrm{rank}[sI - A_0 - A_1 e^{-sh}, B] = n = 4 \qquad\qquad \forall\ s \in \mathcal{C}\ .$$

The matrix $[sI - A_0 - A_1 e^{-sh}, B]$ for this example is of the form

$$\begin{bmatrix} * & * & 0 & 0 & 1 & 0 \\ * & * & * & 0 & 0 & 1 \\ c_1 & * & * & * & 0 & 0 \\ 0 & c_2 & * & * & 0 & 0 \end{bmatrix} . \tag{8.66}$$

The asterisks * denote functions of s. The symbols c_1, c_2 represent constants. Since the rank of the matrix (8.66) is always equal 4, system (8.63) is spectrally controllable. Obviously, the assumptions (8.48) and (8.49) which guarantee the existence of the LQ-regulator are satisfied. For convenience the approximation parameter N is chosen to be N = 1. (The approximation of the LQ-regulator for N = 3 will be briefly given at the end of this example.) Hence, we obtain from (8.41):

$$A^N = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ -3.01 & -1.01 & 0 & 0 & -8.85 & -1.01 & 0 & 0 \\ -3.20 & -3.38 & -12.80 & 0 & -3.20 & -9.22 & -12.80 & 0 \\ 6.40 & 0.35 & -30.63 & -1.04 & 6.40 & 0.35 & -36.37 & -1.04 \\ 0 & 0.83 & 11.00 & -3.20 & 0 & 0.83 & 11.00 & -6.68 \end{bmatrix}$$

$$(B^N)^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

The matrix Q^N is given by (8.53) and (8.65). Next, the solution of the Riccati equation (8.55) is computed. From the resulting Π^N we obtain by (8.57)

$$P_0^N = \begin{bmatrix} 1.01 & -0.23 & 0.64 & 1.44 \\ -0.23 & 0.94 & 0.00 & 0.86 \\ 0.64 & 0.00 & 0.95 & 2.84 \\ 1.44 & 0.86 & 2.84 & 10.81 \end{bmatrix} .$$

Furthermore, with the help of (8.58), (8.59), and (8.60), the matrix Π^N yields an approximation of $P_1(\theta)$ for $\theta = 0$ and $\theta = -1$:

$$P_1^N(0) = P_1^N(-1) = \begin{bmatrix} 1.06 & -0.38 & 0.78 & 0.75 \\ -0.17 & 0.66 & 0.13 & 0.43 \\ 0.40 & -0.01 & 0.59 & 0.77 \\ 0.80 & 0.69 & 1.66 & 2.49 \end{bmatrix} .$$

Since N is chosen to be equal 1, $P_1(\theta)$ is approximated by constant functions and hence $P_1^N(0) = P_1^N(-1)$. The resulting closed-loop system is given by

$$\dot{x}(t) = \tilde{A}_0 x(t) + \tilde{A}_1 x(t-1) \tag{8.67}$$

where

$$\tilde{A}_0 = A_0 - BR^{-1}B^T\left\{P_0^N + \frac{1}{2}P_1^N(0)\right\}$$

$$\tilde{A}_1 = A_1 - \frac{1}{2}BR^{-1}B^TP_1^N(-1) \ .$$

The stability of system (8.67) cannot be guaranteed a priori by the Legendre-Tau method or any semi-discretization method like the AVE-method. However, the algebraic stability tests presented in Section 3.2 can be applied to confirm the stability of the closed-loop system. In particular, the stability condition of Theorem 3.4

$$\mu(\tilde{A}_0)_2 + \|\tilde{A}_1\|_2 = -0.51 < 0 \tag{8.68}$$

is satisfied. Thus, the stability could be successfully analysed without having to compute the eigenvalues of the closed-loop system. Similarly, we may establish the stability of the closed-loop system in case the delay of the plant (8.63) was estimated wrong or if the delay is not constant. The corresponding differential equation of the closed-loop system is of the form

$$\dot{x}(t) = \hat{A}_0x(t) + \hat{A}_1x(t - \tau(t, x(t))) + \hat{A}_2x(t-1)$$

where

$$\hat{A}_0 = A_0 - BR^{-1}B^T\left\{P_0^N + \frac{1}{2}P_1^N(0)\right\}$$

$$\hat{A}_1 = A_1$$

$$\hat{A}_2 = -\frac{1}{2}BR^{-1}B^TP_1^N(-1) \ .$$

The stability condition of Theorem 3.4 can be satisfied as follows

$$\mu(T^{-1}\hat{A}_0T)_2 + \|T^{-1}\hat{A}_1T\|_2 + \|T^{-1}\hat{A}_2T\|_2 = -0.89 < 0$$

where the matrix $T = \mathrm{diag}(1, 1, 0.65, 1)$. The diagonal matrix T is used to reduce the conservatism of the criterion (cf. Subsection 3.2.6). In this example the excellent stability property of the resulting closed-loop system

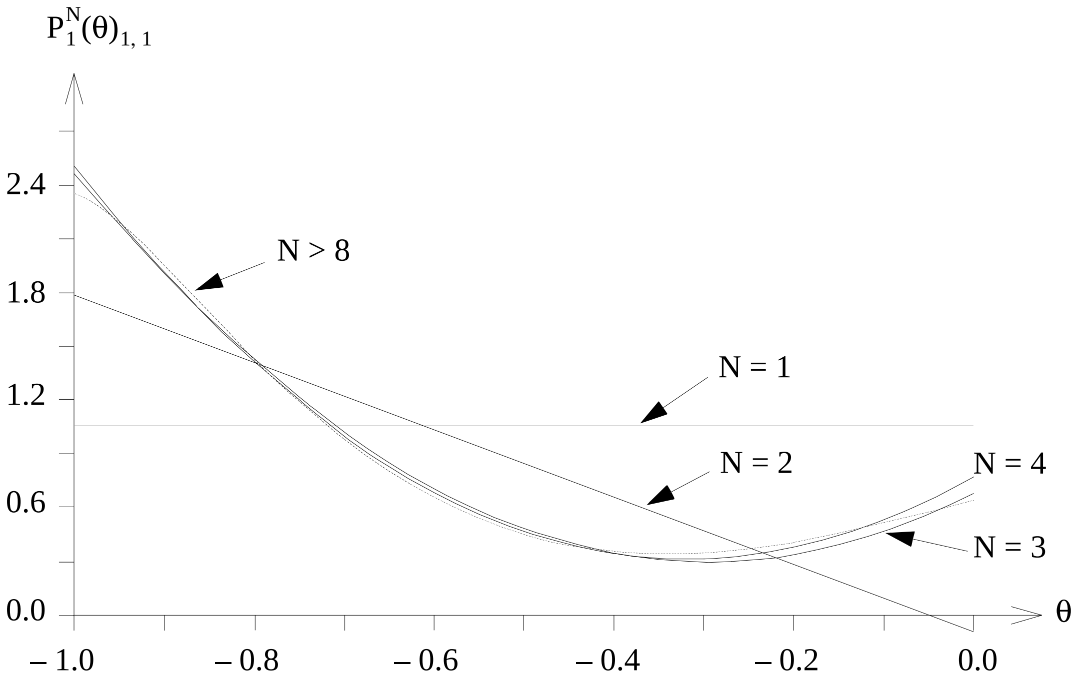

Fig. 8.7 $P_1^N(\theta)_{1,1}$ shown for the system (8.29) subject to the performance index (8.30) for N = 1, 2, 3, 4, and N > 8.

could also be established if the AVE-method (cf. Section 8.1) were applied to compute P_0^N, $P_1^N(0)$, and $P_1^N(-1)$.

N was chosen to be equal one in order to illustrate the method in a simple way. But what is a reasonable choice for N? The question can be answered in a practical way by studying the convergence. In Fig. 8.7 the approximations of the first element of $P_1(\theta)$ denoted by $P_1^N(\theta)_{1,1}$ are shown for $N = 1, 2, 3, 4$, and for $N > 8$. There is no visible difference among the several plots of $P_1^N(\theta)_{1,1}$ with $N > 8$ (dashed plotted in Fig. 8.7). (The convergence behaviour of $P_1^N(\theta)_{1,1}$ is representative for other entries of $P_1^N(\theta)$.) The corresponding program, realized with the help of the MatrixX software package, is able to compute solutions up to $N = 12$. Fig. 8.7 reveals that $N = 3$ is a reasonable choice. For this choice of N we obtain the following approximations of P_0 and $P_1(\theta)$:

$$P_0^N = \begin{bmatrix} 1.25 & -0.28 & 0.76 & 1.65 \\ -0.28 & 1.09 & -0.02 & 0.97 \\ 0.76 & -0.02 & 1.07 & 3.10 \\ 1.65 & 0.97 & 3.10 & 11.49 \end{bmatrix}$$

$$P_1^N(0) = \begin{bmatrix} 0.69 & 0.05 & 0.00 & -0.14 \\ -0.28 & 0.58 & -0.09 & 0.09 \\ 0.47 & -0.01 & 0.46 & 0.46 \\ 0.79 & 0.47 & 1.14 & 1.55 \end{bmatrix}$$

$$P_1^N(-1/3) = \begin{bmatrix} 0.31 & -0.15 & 0.19 & 0.21 \\ -0.07 & 0.08 & -0.08 & 0.15 \\ 0.00 & -0.05 & -0.07 & -0.06 \\ -0.10 & 0.20 & -0.11 & -0.07 \end{bmatrix}$$

$$P_1^N(-2/3) = \begin{bmatrix} 0.90 & -0.43 & 0.80 & 0.80 \\ -0.10 & 0.49 & 0.16 & 0.40 \\ 0.24 & 0.01 & 0.40 & 0.55 \\ 0.50 & 0.59 & 1.19 & 1.85 \end{bmatrix}$$

$$P_1^N(-1) = \begin{bmatrix} 2.46 & -0.78 & 1.81 & 1.65 \\ -0.37 & 1.81 & 0.15 & 0.87 \\ 1.18 & -0.13 & 1.86 & 2.31 \\ 2.58 & 1.64 & 5.04 & 7.31 \end{bmatrix}$$

The resulting closed-loop system is of the form

$$\dot{x}(t) = \tilde{A}_0 x(t) + \tilde{A}_1 x(t - 1/3) + \tilde{A}_2 x(t - 2/3) + \tilde{A}_3 x(t - 1) \qquad (8.69)$$

where

$$\tilde{A}_0 = A_0 - BR^{-1}B^T\left\{P_0^N + \frac{1}{6}P_1^N(0)\right\}$$

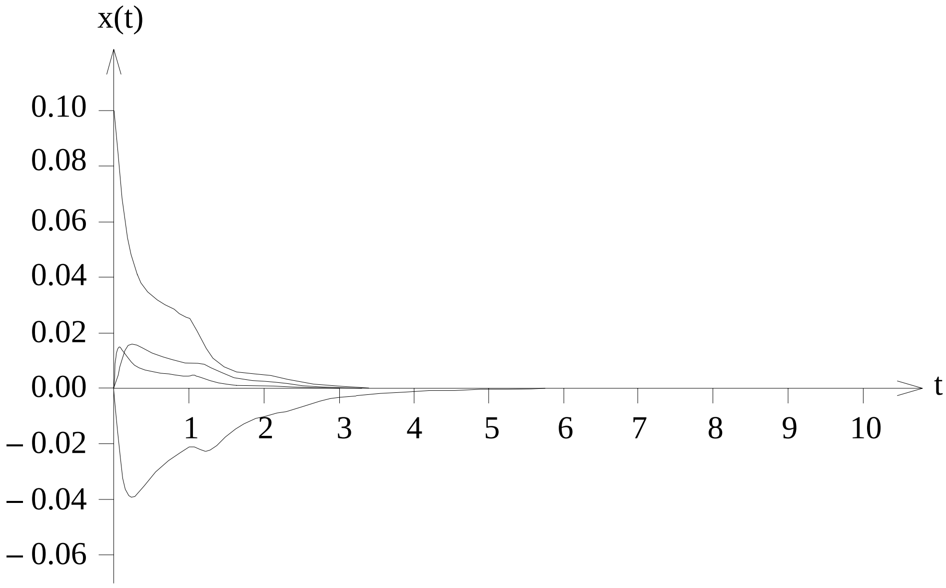

Fig. 8.8 Controlled Williams-Otto process described by (8.69) (N = 3)

$$\tilde{A}_1 = -\frac{1}{3}BR^{-1}B^TP_1^N(-1/3)$$

$$\tilde{A}_2 = -\frac{1}{3}BR^{-1}B^TP_1^N(-2/3)$$

$$\tilde{A}_3 = A_1 - \frac{1}{6}BR^{-1}B^TP_1^N(-1)\ .$$

The asymptotic stability of (8.69) can be shown using again Theorem 3.4:

$$\mu(T^{-1}\hat{A}_0T)_2 + \sum_{i=1}^{3}\left\|T^{-1}\hat{A}_iT\right\|_2 = -0.81 < 0 \tag{8.70}$$

where the diagonal matrix T = diag(1, 1, 0.65, 1). Similarly, the stability of the closed-loop system can be established even in case the delay of the plant was estimated wrong or if it is state-dependent and time-varying.

An implementation scheme of the LQ-regulator is shown in Fig. 8.9. The simulation of the closed-loop system (8.69) associated with the initial condition (8.64) is shown in Fig. 8.8. This simulation illustrates the considerable improvement of the performance, while the stability and the robustness against a modelling error of the delay is guaranteed.

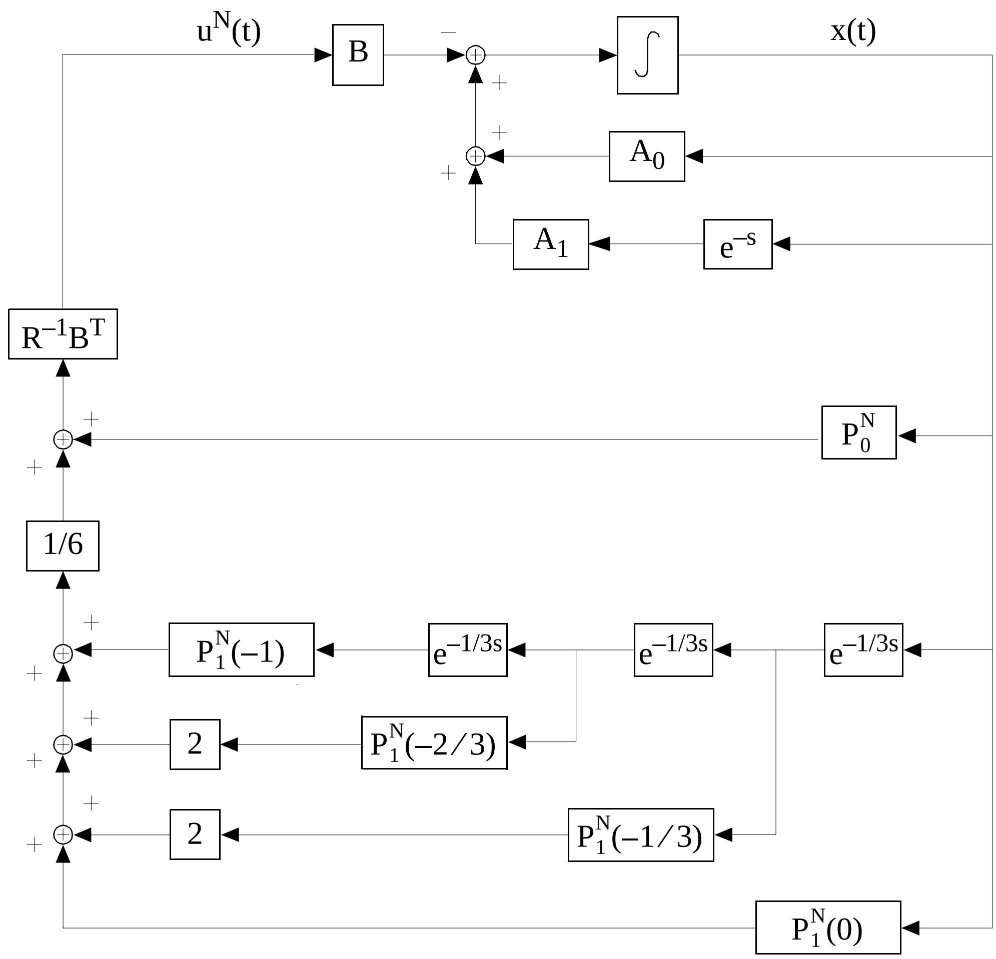

Fig. 8.9 Approximation of the LQ-regulator (N = 3)

Conclusions

The main part of this work deals with stability criteria for time-delay systems. As in the nondelayed case the stability of linear time-delay systems is determined by its eigenvalues. These eigenvalues coincide with the zeros of a characteristic equation, which in general is transcendental. An improved version of a well-known method for the computation of the eigenvalues of time-delay systems is illustrated.

Another way to check the stability of retarded systems is by means of algebraic stability criteria. Several easily applicable algebraic stability tests are developed in this work. Further applications of these stability tests are as follows.

- Some of the algebraic stability criteria are valid even if the delays are unknown and variable. The development of these criteria is quite useful since in practice it is very difficult to estimate the value of the delays especially if they are time-varying and state-dependent. In this context also, robustness criteria are developed (robustness against input, output, and state delays and robustness bounds for unstructured uncertainties).
- The simple stability tests in combination with an instability criterion can be used to derive exact, delay-dependent, algebraic stability tests. The new method is demonstrated by extending a known exact stability condition. The author is convinced that further generalisations can be achieved due to this procedure.
- Algebraic stability tests are useful in connection with control methods for time-delay systems as well. It is demonstrated that the three tools finite dimensional approximation, optimal control, and algebraic stability criteria in combination remarkably improve the behaviour of the control system. The stability of the closed-loop system is guaranteed even if the delay is not constant.

Further research should reveal whether these stability criteria can be used to develop powerful control methods using LMI techniques.

Notation

FDE	Functional differential equations
RFDE	Retarded functional differential equations
NFDE	Neutral functional differential equations
AFDE	Advanced functional differential equations
gcd	Greatest common divisor
lcm	Lowest common multiple
$\boldsymbol{R}^n$	n-dimensional Euclidean space
$\boldsymbol{R}^+$	Set of all positive real numbers
$\boldsymbol{Q}$	Set of all rational numbers
$\boldsymbol{N}$	Set of all natural numbers; $\boldsymbol{N} = \{1, 2, 3, \dots\}$
N	$N \in \boldsymbol{N}$
$\mathbb{C}$	Set of the complex numbers
s	$s \in \mathbb{C}$
j	$j = \sqrt{-1}$
L^2	Space of square integrable functions
$W^{1,2}$	Space of continuous functions with square integrable derivatives
C	Space of continuous functions
C^1	Space of continuous functions with continuous derivatives
M^2	$\boldsymbol{R}^n \times L^2$
x(t)	Phase space at time t, $x(t) \in \boldsymbol{R}^n$
x_t	State vector of an RFDE, $x_t := x(t + \theta), -\tau_{max} \leq \theta \leq 0$
t_0	Initial time
x_0	$x_0 = x(t_0)$
φ(t)	Initial function, $\varphi(t) \in \boldsymbol{R}^n$ (cf. Chapter 2)
u(t)	Control signal, $u(t) \in \boldsymbol{R}^m$
y(t)	Output signal, $y(t) \in \boldsymbol{R}^p$

τ_{max}	Maximal time-retardation of an RFDE: $\tau_{max} > \tau(t, x(t))$
τ_i	Delay constant of a noncommensurate time-delay system of the form: $\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t-\tau_i) + Bu(t)$ where $0 < \tau_1 < \ldots < \tau_k$
h	Delay constant of a commensurate time-delay system of the form: $\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t-ih) + Bu(t)$
$\lambda_i(A)$	Eigenvalue of the matrix $A \in \boldsymbol{R}^{n \times n}$
$\lambda_{max}(A)$	Eigenvalue with the largest real part of the matrix A
$\lambda_{min}(A)$	Eigenvalue with the smallest real part of the matrix A
$Re(\cdot)$	Real part of $(\cdot)$
$Im(\cdot)$	Imaginary part of $(\cdot)$
$\lvert x\rvert$	Absolute value of $x \in \boldsymbol{R}^n$: $\lvert x\rvert = \sqrt{x^T x}$
$\lvert x\rvert_\cdot$	Vector norm: $\lvert x\rvert_1 = \sum_{i=1}^{n} \lvert x_i\rvert$ $\lvert x\rvert_2 = \left[\sum_{i=1}^{n} \lvert x_i\rvert^2\right]^{1/2}$ $\lvert x\rvert_\infty = \max_i \lvert x_i\rvert$
$\lVert A\rVert_\cdot$	Matrix norms: $\lVert A\rVert_1 = \max_j \sum_{i=1}^{n} \lvert a_{ij}\rvert$ $\lVert A\rVert_2 = \sqrt{\lambda_{max}(A^T A)}$ $\lVert A\rVert_\infty = \max_i \sum_{i=1}^{n} \lvert a_{ij}\rvert$
$\mu(A)_\cdot$	Matrix measure: $\mu(A)_1 = \max_j \left[Re(a_{jj}) + \sum_{i=1, i \neq j}^{n} \lvert a_{ij}\rvert\right]$ $\mu(A)_2 = 0.5 \cdot \lambda_{max}(A^T + A)$ $\mu(A)_\infty = \max_i \left[Re(a_{ii}) + \sum_{j=1, j \neq i}^{n} \lvert a_{ij}\rvert\right]$

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