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Crossing Brownian motion in a soft Poissonian potential

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Crossing Brownian motion in a soft Poissonian potential

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Abstract

We consider $d$-dimensional crossing Brownian motion in a truncated soft Poissonian potential ($d \geq 2$): In a random medium with soft obstacles distributed according to a Poissonian law, we consider Brownian motion conditioned to reach a remote location. In this model there is a naturally defined random distance function, which measures the cost of the Brownian crossings in the presence of the Poissonian obstacles. This distance function satisfies a shape theorem, which says that asymptotically it behaves as a deterministic norm on $\mathbb{R}^d$.

In this thesis we introduce a critical exponent $\chi$ (depending on $d$), which measures finer asymptotics of this distance function. As a second critical exponent we introduce $\xi$ (depending on $d$), which describes the transverse fluctuations of the crossings. It is conjectured that $\chi$ and $\xi$ should satisfy the scaling identity, $2\xi - 1 = \chi$, in all dimensions $d \geq 2$. We provide here a rigorous version of the scaling identity, i.e., we prove lower and upper bounds on $\xi$ in terms of $\chi$.

Further, we provide numerical lower and upper bounds for $\xi$ and $\chi$. In particular, we are able to prove that the point-to-plane model behaves superdiffusively in dimension $d = 2$, resulting in $\xi \geq 3/5 (> 1/2)$, whereas for $d \geq 3$ we prove that the model behaves at least diffusively, $\xi \geq 1/2$.

In the final part of this thesis we compare, when we strengthen the potential, the Lyapounov norms to the time-constants of certain naturally associated random Riemannian metrics.
Wir betrachten $d$-dimensionale traversierende Brownsche Bewegungen in einem Poissonschen Potential ($d \geq 2$): D.h., in einem zufälligen Medium mit poisson-verteilten Hindernissen schauen wir alle Brownschen Pfade an, die ein entferntes Ziel in endlicher Durchlaufszeit erreichen. Für dieses Modell gibt es eine kanonisch definierte, zufällige Distanzfunktion, die die mittleren Kosten der traversierenden Brownschen Bewegungen im Poissonschen Potential misst. Es existiert nun eine deterministische Norm auf $\mathbb{R}^d$, die das asymptotische Verhalten dieser zufälligen Distanzfunktion beschreibt, wenn das Ziel der Pfade gegen unendlich strebt (Shape Theorem).

In dieser Doktorarbeit führen wir einen kritischen (dimensions-abhängigen) Exponenten $\chi$ ein, welcher das asymptotische Verhalten unserer Distanzfunktion für feinere Ordnungen beschreibt. Ein zweiter kritischer Exponent $\xi$, den wir hier betrachten wollen, misst die transversalen Schwankungen der traversierenden Pfade. In der physikalischen Literatur wurde die Behauptung aufgestellt, dass die beiden kritischen Exponenten eine Skalierungsgleichung erfüllen sollten: $2\xi - 1 = \chi$, in allen Dimensionen $d \geq 2$. In dieser Arbeit beweisen wir eine rigorose Version der Skalierungsgleichung, d.h., wir zeigen eine untere (bzw. obere) Schranke für $\xi$, die nur von $\chi$ abhängt.

Ferner beweisen wir numerische Schranken für $\xi$ und $\chi$. Insbesondere konnten wir zeigen, dass sich das "Punkt-zu-Ebene"-Modell in Dimension $d = 2$ superdiffusiv verhält, d.h., $\xi \geq 3/5 > 1/2$, während wir in höheren Dimensionen ($d \geq 3$) mindestens ein diffusives Verhalten haben, $\xi \geq 1/2$.

Im letzten Teil dieser Arbeit vergleichen wir die Lyapounov Normen mit den Zeit Konstanten, die wir erhalten, wenn wir die kanonisch definierte Riemannsche Metrik zum Poissonschen Potential betrachten.
Introduction and results

0. Motivation. In physics, chemistry and biology many phenomena can be modelled by spatial random processes, where the randomness lies both in the geometry of the space and in the behavior of the motion. A typical example is given by diffusions in a random medium. In such models a broad variety of questions appear. Questions of greatest interest involve the behavior in the large of particles moving in a random medium: "where do the particles prefer to stay in the long time behavior", "how much does it cost for a particle to travel large distances in the random medium", or questions related to growing interfaces and solids far from equilibrium. In the physics literature such models have been studied extensively (see, e.g., [51], [29]), but there are relatively few rigorous mathematical results obtained.

The aim of this thesis is to develop the mathematical investigation of crossing Brownian motion in a soft Poissonian potential: In a random medium with obstacles distributed according to a Poissonian law we consider Brownian motion starting at the origin conditioned to reach a remote location. Various properties of this model have been studied by Sznitman in several papers. A good overview can be found in the book [50]. The main object of this thesis is to define transverse and distance fluctuations for crossing Brownian motion. These fluctuations are measured in terms of two critical exponents $\xi$ and $\chi$. We give quantitative estimates for these exponents, and we provide relations between them (scaling identity).

We remark that there has been made some progress also in an other mathematical model for random media, namely in (first-passage) percolation theory (see Hammersley-Welsh [18], Alexander [3, 4], Kesten [26], Grimmett [17], Licea-Newman-Piza [30]). It turns out that in both models questions of similar flavor appear, but often the methods which work in one model do not apply to the other.

1. Definition of the model. We consider two different types of goals for the crossings, one is a point-to-point model, the other is a point-to-hyperplane model. The question which model should we choose is often essential, i.e., to prove certain statements one
benefits substantially from the geometry of the goal, and one cannot always translate the proof from one model to the other.

Let us start with the definition of the random medium. For $d \geq 1$ the space $\Omega$ denotes the set of locally finite, simple, pure point measures on $\mathbb{R}^d$. Let $\xi$ stand for the Poissonian law with fixed intensity $\nu > 0$ on $\Omega$. For $M > 0$, $\omega = \sum_i \delta_{x_i} \in \Omega$ and $x \in \mathbb{R}^d$ the truncated Poissonian potential is defined as

$$V(x, \omega) = \left( \sum_i W(x - x_i) \right) \wedge M - \left( \int_{\mathbb{R}^d} W(x - y) \omega(dy) \right) \wedge M,$$

(1)

where the shape function $W(\cdot) \geq 0$ is bounded, measurable, compactly supported and not a.e. equal to zero. Furthermore, we assume that

$$W(\cdot) \text{ is rotationally invariant.}$$

(2)

Our aim is to have a penalty to all random paths which experience the potential:

**Point-to-point model.** For $x \in \mathbb{R}^d$, we denote by $P_x$ the Wiener measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ starting at $x$, $Z$ denotes the canonical process on $C(\mathbb{R}_+, \mathbb{R}^d)$. For $\omega \in \Omega$, $\lambda \geq 0$ and $x, y \in \mathbb{R}^d$, the point-to-point crossing Brownian motion in the Poissonian potential from $x$ to $y$ on $C(\mathbb{R}_+, \mathbb{R}^d)$ is then defined by

$$d\hat{P}_x^{y} = \frac{1}{e_\lambda(x, y, \omega)} \exp \left\{ - \int_{0}^{H(y)} (\lambda + V)(Z_s, \omega) ds \right\} 1_{\{H(y) < \infty\}} dP_x,$$

(3)

where $H(y)$ is the entrance time of $Z.$ into the closed Euclidean ball $B(y, 1)$, and where $e_\lambda(x, y, \omega)$ is the normalizing constant. In a slightly formal way, we can interpret this crossing Brownian motion measure as a Brownian motion which feels a drift up to time $H(y)$, as described by the stochastic differential equation ($\beta$ is the $d$-dimensional Brownian motion):

$$dZ_s = d\beta_s + \frac{\nabla c_\lambda(\cdot, y, \omega)}{e_\lambda(\cdot, y, \omega)}(Z_s) ds \quad 0 \leq s \leq H(y),$$

$$Z_0 = x.$$

**Point-to-hyperplane model.** For $\theta \in [0, 2\pi)$, $L > 0$, let $\Lambda(\theta, L)$ be the half-space,

$$\Lambda(\theta, L) = \{ x \in \mathbb{R}^d; \langle x, \hat{x}(\theta) \rangle \geq L \},$$

(4)

where $\hat{x}(\theta) = (\cos \theta, \sin \theta, 0, \ldots, 0)$, and where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^d$. If $\theta = 0$, we write $\Lambda_L$ for $\Lambda(0, L).$ For $\lambda \geq 0$, $L > 0$, $\theta \in [0, 2\pi)$ and $\omega \in \Omega$, 


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the point-to-hyperplane crossing Brownian motion path measure on \( C(\mathbb{R}_+, \mathbb{R}^d) \) is then defined by

\[
d\tilde{P}_0^{\partial \Lambda(\theta, L)} = \frac{1}{e_\lambda(0, \partial \Lambda(\theta, L), \omega)} \exp \left\{ - \int_0^{H(\partial \Lambda(\theta, L))} (\lambda + V)(s, \omega) \, ds \right\} dP_0,
\]

where \( e_\lambda(0, \partial \Lambda(\theta, L), \omega) \) is the normalizing constant, and where \( H(\partial \Lambda(\theta, L)) = \inf\{ s > 0, \, Z_s \in \Lambda(\theta, L) \} \) is the entrance time of \( Z \) into the half-space \( \Lambda(\theta, L) \).

2. Properties and conjectures. Let us, for the moment, consider the point-to-point crossing Brownian motion model. Analogous statements to those of this paragraph also hold for the point-to-hyperplane model. The normalizing function \( u(\cdot) = e_\lambda(\cdot, 0, \omega) \) appears as the \((\lambda + V)\)-equilibrium potential of \( \bar{B}(0, 1) \), which satisfies in a weak sense the following second order elliptic equation (see Proposition 2.3.8 of [50] or Proposition 2.3 on p.86 below):

\[
\begin{cases}
-\frac{1}{2} \Delta u + (\lambda + V)u & = 0 \quad \text{in } \bar{B}(0, 1), \\
u & = 1 \quad \text{on } \partial B(0, 1), \\
u & = 0 \quad \text{at infinity (for typical } \omega). 
\end{cases}
\]

The exponential decay of the normalizing constant is (for typical \( \omega \)) described by a shape theorem (see Sznitman [50], Theorem 5.2.5):

**Theorem A (Sznitman)** For \( d \geq 1 \) and \( \lambda \geq 0 \) there exists a deterministic norm \( \alpha_\lambda(\cdot) \) on \( \mathbb{R}^d \) such that, on a set of full \( \mathbb{P} \)-measure, we have

\[
\lim_{x \to \infty} \frac{1}{|x|} \left| - \log e_\lambda(x, 0, \omega) - \alpha_\lambda(x) \right| = 0.
\]

The convergence takes place in \( L^1(\mathbb{P}) \) as well.

The norm \( \alpha_\lambda(\cdot) \) is called the Lyapounov coefficient or norm. Under (2), the Lyapounov norms are proportional to the Euclidean norm. If we symmetrize \( -\log e_\lambda(\cdot, \cdot, \omega) \), for \( \lambda \geq 0, \omega \in \Omega \) and \( x, y \in \mathbb{R}^d \),

\[
d_\lambda(x, y, \omega) = \max \left\{ - \inf_{\bar{B}(x, l)} e_\lambda(\cdot, y, \omega), - \inf_{\bar{B}(y, l)} e_\lambda(\cdot, x, \omega) \right\},
\]

then \( d_\lambda(\cdot, \cdot, \omega) \) is \( \mathbb{P} \)-a.s. a distance function on \( \mathbb{R}^d \), which induces the usual topology (see Sznitman [50], Lemma 5.2.1), moreover \( d_\lambda(\cdot, 0, \omega) \) satisfies the above shape theorem as well. This distance function measures the cost for crossing Brownian motion to reach
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its goal; observe that $d_{\lambda}(x, y, \omega)$ increases if we add additional points to the Poissonian cloud $\omega$. Let us mention that $d_{\lambda}(\cdot, \cdot, \omega)$ is in some sense comparable to the passage time $T(\cdot, \cdot, \omega)$ in first-passage percolation (see [26]); $T(\cdot, \cdot, \omega)$ also fulfills a shape theorem (see [26], Theorem 1.7). One difference between $d_{\lambda}(\cdot, \cdot, \omega)$ and $T(\cdot, \cdot, \omega)$ is that the distance function for the crossing Brownian motion model does not come as the length of minimizing geodesics (in fact, we need not have "flat triangles", see Sznitman [49], Example 1.1).

If we consider the set $B(t, \omega)$ of points in $\mathbb{R}^d$, which lie within $d_{\lambda}(\cdot, \cdot, \omega)$-distance $t$ of the origin, we get a growing interface in $\mathbb{R}^d$. Macroscopically, we know (via the shape theorem) that it is "close to" a ball with respect to the Lyapounov norm $\alpha_{\lambda}(\cdot)$. But, of course, the surface of $B(t, \omega)$ may in general be very rough, hence we speak from shape fluctuations or distance fluctuations. We want to learn more about this microscopical growth (for related models see Krug-Spohn [29]). The magnitude of these fluctuations is in physical literature usually described by a critical exponent $\chi$, depending on $d$.

If, for $d \geq 2$, $\xi$ is the critical exponent for the fluctuations of the Brownian crossings around the straight line from the origin to the goal $y$ (transverse fluctuations), it is conjectured that $\chi$ and $\xi$ should satisfy a scaling identity: namely, $\chi = 2\xi - 1$ for $d \geq 2$ (see Krug-Spohn [29]). Here, we provide an approach to this conjectured scaling identity. As a further step we provide numerical bounds for these critical exponents.

The predicted values (for models whose exponents should have the same values as in our case) are for $d = 2$: $\xi = 2/3$ (superdiffusive behavior) and $\chi = 1/3$ (see [21, 24, 22, 25]). For higher dimensions there are conflicting predictions for $\xi$ ranging from a lack of dependence on $d$ through their decreasing in $d$ while $\xi > 1/2$ to the possibility that above some critical dimension $\xi = 1/2$ (diffusive behavior) (for references to these conjectures see Licea-Newman-Piza [30], p.561). We remark that Johansson [23] has recently been able to prove that $\chi = 1/3$ for a model on $\mathbb{Z}^2$ close to directed site first-passage percolation, where the passage times of the sites are independent geometrically distributed. We prove that for our model we have (under suitable assumptions) in dimension $d = 2$ superdiffusive behavior and in higher dimensions at least diffusive behavior. As an upper bound for $\xi$ we get that in all dimensions $\xi \leq 3/4$.

We remark that if we consider a deterministic positive potential $V - \lambda > 0$, it is known that $\xi = 1/2$. So at least in dimension $d = 2$ ($\xi \geq 3/5$) we see that the transverse fluctuations due to the environment dominate the usual "thermal" fluctuations.

3. The results. Let us now restrict to the case $d \geq 2$. The results can be divided into three parts. In the first part we consider the point-to-point model. In the second part we
state superdiffusivity for the point-to-hyperplane model in dimension \(d = 2\). And in the final part we observe the behavior of our model for increasing strength of the Poissonian potential \(\lambda + V(\cdot, \omega)\).

**a) Point-to-point model.** We define the critical exponent \(\xi_1\) for transverse fluctuations as follows: Take \(y\) a non zero vector in \(\mathbb{R}^d\). Let \(l_y\) be the axis \(\{\alpha y; \alpha \in \mathbb{R}\}\). \(C(y, \gamma)\) is then the truncated cylinder with axis \(l_y\) and radius \(|y|\gamma\):

\[
C(y, \gamma) = \{ z \in \mathbb{R}^d; \text{dist}(l_y, z) \leq |y|\gamma \text{ and } -|y|\gamma \leq \langle z, y/|y| \rangle \leq |y| + |y|\gamma \}, \tag{9}
\]

where \(\text{dist}(\cdot, \cdot)\) is the Euclidean distance and \(\langle \cdot, \cdot \rangle\) denotes the usual scalar product on \(\mathbb{R}^d\). The truncation of the ends of the cylinder has a technical purpose for the calculations. \(A(y, \gamma)\) is the event that the Brownian crossings do not leave the cylinder \(C(y, \gamma)\), i.e.,

\[
A(y, \gamma) = \{ w \in C(\mathbb{R}_+, \mathbb{R}^d); Z_s(w) \in C(y, \gamma) \text{ for all } s \leq H(y) \}. \tag{10}
\]

We then define the critical exponent for transverse fluctuation as follows

\[
\xi_1 = \inf \left\{ \gamma \geq 0; \lim_{y \to \infty} \mathbb{E} \left[ \hat{P}_0^y [A(y, \gamma)] \right] = 1 \right\}. \tag{11}
\]

\(\xi_1\) measures the critical radius of the cylinder \(C(y, \cdot)\) such that it is very unlikely that the Brownian crossings exit from this cylinder. As for the distance fluctuations we introduce two critical exponents \(\chi^{(1)}\) and \(\chi^{(2)}\),

\[
\chi^{(1)} = \inf \left\{ \kappa \geq 0; \lim_{y \to \infty} \mathbb{P} \left[ |d_\lambda(0, y) - M_\lambda(0, y)| \leq |y|^{\kappa} \right] = 1 \right\}, \tag{12}
\]

where \(M_\lambda(0, y)\) is a median of \(d_\lambda(0, y)\), i.e., \(\mathbb{P} [d_\lambda(0, y) \geq M_\lambda(0, y)] \geq 1/2\) and \(\mathbb{P} [d_\lambda(0, y) \leq M_\lambda(0, y)] \geq 1/2\) (for the specific choice of the median, in the case where it is not unique, see p.55 below). In two parts of the proofs we require more uniformity for different directions and different distances, therefore we have to define a second critical exponent measuring distance fluctuations:

\[
\chi^{(2)} = \inf \left\{ \kappa \geq 0; \lim_{y \to \infty} \mathbb{P} \left[ \sup_{x \in B(0, r)} |d_\lambda(0, x) - M_\lambda(0, x)| \leq r^{\kappa} \right] = 1 \right\}. \tag{13}
\]

\(\chi^{(1)}\) measures the roughness of the boundary of \(B(t, \omega)\) for a fixed direction, whereas \(\chi^{(2)}\) measures the roughness of our growing interface uniformly in all directions. The natural conjecture is that \(\chi^{(1)} = \chi^{(2)}\), but unfortunately we are only able to prove the following:

**Proposition B (Proposition 0.1, p.43)** For all \(d \geq 2\) and \(\lambda \geq 0\),

\[
\chi^{(1)} \leq \chi^{(2)} \leq 1/2. \tag{14}
\]
This proposition is an easy consequence of Sznitman's exponential estimates on the fluctuations of $d_\lambda(\cdot, \cdot, \omega)$ around its mean (see Sznitman [49], Theorems 2.1, 2.5 and Corollary 3.4). Next we state one main result of this thesis:

**Theorem C (Theorems 0.2 and 0.3, p.43-44)** *For all $d \geq 2$ and $\lambda \geq 0$,*

$$\frac{\chi^{(1)} + \chi^{(2)}}{2} \leq \zeta_1 \leq \frac{\chi^{(2)} + 1}{2}. \quad (15)$$

This is an approach to the scaling identity. In particular, proving that $\chi^{(1)} = \chi^{(2)}$ would provide a rigorous version of the scaling identity. In first-passage percolation theory there is a similar upper bound (right-hand side of inequality (15)), but to my knowledge there is no similar lower bound proven in any other related mathematical model. In first-passage percolation on the lattice $\mathbb{Z}^d$ one serious obstacle to prove a lower bound of the type (15) is, that one faces a lot of difficulties caused by the non-rotational invariance of the model and the unknown character of the limiting shape of the growing surface of $B(t, \omega)$. The proof of the lower bound heuristically corresponds to the following picture (see Figure 1):

![Figure 1: Comparison of the lower and upper cylinder.](image-url)

We consider two parallel disjoint cylinders $C(y, \xi_1)$ and $C(y, \xi_1) + m$, with $m = (0, 3|y|^\xi_1 + c, 0, \ldots, 0)$. Since the crossing Brownian motion stays with high $\mathbb{P}$-probability in its cylinder, therefore $d_\lambda(m, m + y)$ and $d_\lambda(0, y)$ are almost independent, i.e., using distance fluctuations we can find cloud configurations $\omega$ for which the upper distance function $d_\lambda(m, m + y, \omega)$ is much smaller than the lower distance function $d_\lambda(0, y, \omega)$. If $\xi_1$ were too small, it would be favorable for the crossings from 0 to $y$ to walk to the upper cylinder, walk along that cylinder, and then go back to $y$ (see Figure 1): The cost of that
additional detour is of order $|y|^{2q_1-1}$ (connection to the upper cylinder) which may be compensated by the fact that the cloud in the upper cylinder, due to the distance fluctuations, can be more favorable for the crossings. This compared to the fact that $\xi_1$ is the critical cylinder radius turns out to yield the desired relation $1/2(\chi^{(1)} + \chi^{(1)}/\chi^{(2)}) \leq \xi_1$.

Whereas for the upper bound we explicitly calculate how much it costs walking to the boundary of the cylinder $C(y, \xi_1)$. We see that if we choose a cylinder $C(y, \gamma)$ with $2\gamma - 1 > \chi^{(2)}$ a detour walking to the boundary can not be compensated by the natural fluctuations of the distance functions $d_\lambda(\cdot, \cdot; \omega)$.

**Numerical bounds for transverse fluctuations:** We are able to provide an upper bound on transverse fluctuations of $3/4$. The lower bound (17) does not seem to be so interesting, because we expect at least diffusive behavior. But for the point-to-point model we are not able to provide any better lower bounds, and the technique to prove (17) is already involved (we will see that essentially the same method allows us to prove superdiffusivity for the point-to-hyperplane model in dimension $d = 2$).

**Theorem D** (Theorem 1.1, p.17, and Theorem 1.3, p.18)

\begin{align*}
\text{For } d \geq 2, & \quad \xi_1 \leq 3/4. \quad (16) \\
\text{If } d \geq 3 \text{ or } \lambda > 0, & \quad \xi_1 \geq 1/(d + 1). \quad (17)
\end{align*}

The upper bound in (16) is an easy consequence of Theorem C and Proposition B. Whereas the lower bound in (17) follows, if we combine the geometric construction used in Theorem C (see Figure 1) with the martingale technique used to prove Theorem F, formula (25), below.

It is more difficult to obtain lower bounds on the distance fluctuations. We therefore define the following critical exponent:

\[ \bar{\chi} = \sup \left\{ \kappa \geq 0, \limsup_{y \to \infty} |y|^{-2\kappa} \text{Var}(-\log e_\lambda(0, y)) > 0 \right\}. \quad (18) \]

$\bar{\chi}$ measures the distance fluctuations around its mean in the $L^2$-norm. The (unproven) conjecture is that $\bar{\chi}$ should be equal to $\chi^{(2)}$.

**Theorem E** (Theorem 1.2, p.18, Corollary 3.1, p.60)

\begin{align*}
\text{For } d \geq 2, & \quad \bar{\chi} \geq \chi^{(1)}. \quad (19) \\
\text{For } d = 2, & \quad \bar{\chi} \geq (1 - \xi_1)/2 \geq 1/8. \quad (20) \\
\text{For } d = 2, & \quad \max\{\bar{\chi}, \chi^{(2)}\} \geq 1/5. \quad (21)
\end{align*}
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Formula (21) is an easy consequence of Theorem C and (20). The proof of (20) is more involved, the strategy is qualitatively similar to a result in Newman-Piza [35]. The key technique is to use a martingale method and estimates on \(-\log c_{1}(0, y, \omega)\) if we locally (on compact sets) change the cloud configurations \(\omega\): Our main tool, the martingale technique, is a very common tool in this type of problems. It has been used in [52, 2, 35, 27, 49] and by many other authors. It enables us to write \(\text{Var}(-\log c_{1}(0, y)) \geq \sum_{k} \text{Var} \left( \mathbb{E} \left[ -\log c_{1}(0, y) | G_{k} \right] \right) \), where the \(G_{k}\)'s are \(\sigma\)-algebras generated by \(\omega\) on cubes \(C_{k} \subset \mathbb{R}^{d}\) with \(C_{k} (k \geq 1)\) chosen to be a paving of \(\mathbb{R}^{d}\). The next step is to estimate \(\text{Var} \left( \mathbb{E} \left[ -\log c_{1}(0, y) | G_{k} \right] \right) \) by \(c \mathbb{E} \left[ E_{k} \right]^{2}\), with \(E_{k}\) a random variable such that \(\mathbb{E} \left[ E_{k} \right] \geq c \mathbb{E} \left[ \tilde{P}_{0}^{y} [H(C_{k}) \leq H(y)] \right]\). The remaining steps are to reduce the above sum to the sum over all cubes \(C_{k}\) lying in the cylinder \(C(y, \gamma)\) \((\gamma > \xi_{1})\), and to get rid of the square in \(\mathbb{E} \left[ E_{k} \right]^{2}\) (this is done by the Cauchy-Schwarz inequality). Finally, we have to count all the cubes \(C_{k}\) in \(C(y, \gamma)\) that are visited during the crossings: this is a number which grows at least proportionally to \(|y|\) (thanks to the fact that \(\mathbb{E} \left[ \tilde{P}_{0}^{y} [A(y, \gamma)] \right] \to 1\) for \(|y| \to \infty\) \((\gamma > \xi_{1})\)).

b) Point-to-hyperplane model. Of course, the lower bound obtained in (17) is not satisfactory, because in dimension \(d = 2\) we expect superdiffusive behavior and in higher dimensions at least diffusive behavior. We can only prove results in that spirit for the point-to-hyperplane model, where we explicitly take advantage of the geometry of the goal. We define transverse fluctuations in that model as follows: For \(\theta \in [0, 2\pi)\), \(L > 0\) and \(\gamma > 0\), we consider \(l_{\theta}\) the line \(\{\alpha \hat{x}(\theta) \in \mathbb{R}^{d}; \alpha \in \mathbb{R}\}\) and the truncated cylinder with radius \(L^{\gamma}\) and symmetry axis \(l_{\theta}\),

\[
\tilde{C}(\hat{x}(\theta), L^{\gamma}) = \{z \in \Lambda(\theta, -L); \text{dist}(z, l_{\theta}) \leq L^{\gamma}\},
\]

where in view of (4): \(\Lambda(\theta, -L) = \{z \in \mathbb{R}^{d}; \langle z, \hat{x}(\theta) \rangle \geq -L\}\). Again this truncation serves technical purposes. \(A_{\theta}(L, \gamma)\) is the event that the perturbed Brownian paths starting at the origin with goal \(\partial \Lambda(\theta, L)\) do not leave the cylinder \(C(\hat{x}(\theta), L^{\gamma})\), i.e.,

\[
A_{\theta}(L, \gamma) = \{w \in C(\mathbb{R}_{+}, \mathbb{R}^{d}); Z_{s}(w) \in \tilde{C}(\hat{x}(\theta), L^{\gamma}) \text{ for all } s \leq H(\partial \Lambda(\theta, L))\}\).

The second critical exponent is then defined via

\[
\xi^{(2)} = \inf \left\{ \gamma \geq 0; \limsup_{L \to \infty} \mathbb{E} \left[ \tilde{P}_{0}^{\partial\Lambda_{L}} [A_{0}(L, \gamma)] \right] = 1 \right\}.
\]

We remark that, thanks to (2), the direction does not play any role in our model. Therefore, we can restrict ourselves in the above definition to the case where the angle \(\theta = 0\). The main result in this part is the following theorem:
INTRODUCTION AND RESULTS

Theorem F (Theorems 0.1 and 0.2, p.64, Formula (0.10), p.63)

For $d = 2$ and $\lambda > 0$,  
\[ \xi^{(2)} \geq 3/5. \]  
(25)

For $d \geq 3$ or $\lambda = 0$,  
\[ \xi^{(2)} \geq 1/2. \]  
(26)

The results we obtain here have a similar flavor to questions which appear in the context of first-passage percolation on $\mathbb{Z}^d$ (see Licea-Newman-Piza [30]). However, in the first-passage percolation model the rotational invariance is lost. This turns out to be a serious problem, and it is only possible to prove superdiffusivity for "well-behaved" directions. Here in our model we obtain a lower bound which is uniform in all directions. It is also worth pointing out that the geometric construction (described below) becomes very nice and transparent in our model: At the heart of the proof of Theorem F lies the following

Figure 2: The two hyperplanes $\partial \Lambda(0, L)$ and $\partial \Lambda(\theta_L, L)$

picture: We consider two hyperplanes $\partial \Lambda(0, L)$ and $\partial \Lambda(\theta_L, L)$, where the choice of the angle $\theta_L = L^{-\varepsilon}$ is crucial (see Figure 2).

We prove formula (26) by contradiction: Assume $\xi^{(2)}$ were less than 1/2. Then we choose $\gamma = 1/2 - \varepsilon \in (\xi^{(2)} , 1/2)$ and define $\varepsilon = 1/2 + \varepsilon$. We now consider the concentration sets of the hitting points: $G_0(L) = \partial \Lambda(0, L) \cap \tilde{C}(\hat{x}(0), L^\gamma)$ and $G_{\theta_L}(L) = \partial \Lambda(\theta_L, L) \cap \tilde{C}(\hat{x}(\theta_L), L^\gamma)$ (For our choice of $\gamma$ we know that the $P_0^{\partial \Lambda_L}$-probability that $Z_{H(\partial \Lambda_L)}$ lies within $G_0(L)$ tends to 1 as $L \to \infty$). We observe that on the one hand $G_0(L)$ and $G_{\theta_L}(L)$ are well-separated (their distance is tending to infinity as $L \to \infty$), but on the other hand $G_0(L)$ is very close to $\partial \Lambda(\theta_L, L)$ and vice versa, i.e., the costs to travel from the
concentration sets to the other hyperplane are very low. This leads to a contradiction because we know that the concentration sets are very far apart.

To prove (25) for $d = 2$ and $\lambda > 0$, we apply the same geometric construction as above. We assume that $\xi^{(2)}$ were less than $3/5$. Then we choose $\gamma \in (\xi^{(2)}, 3/5)$ and $\epsilon = 1 - \gamma$. By this choice of the angle $\theta_L = L^{-\epsilon}$, the cylinders $\tilde{C}(\tilde{\omega}(0), L^\gamma)$ and $\tilde{C}(\tilde{\omega}(\theta_L), L^\gamma)$ are almost disjoint (their intersection is sublinear compared to $L$), therefore $-\log \epsilon_{\lambda}(0, \partial\Lambda(0, L))$ and $-\log \epsilon_{\lambda}(0, \partial\Lambda(\theta_L, L))$ are almost independent. Hence, we can adapt the martingale technique used to prove (17) to obtain a lower bound on $\text{Var}(\epsilon_{\lambda}(0, \partial\Lambda(0, L)) + \log \epsilon_{\lambda}(0, \partial\Lambda(\theta_L, L)))$. On the other hand because $G_0(L)$ is still close to $\partial\Lambda(\theta_L, L)$ and vice versa, we obtain an upper bound on the variance of the differences (here we use that we work with a truncated potential, hence we can calculate the upper bound in a deterministic fashion, using tubular estimates on Brownian motion). Comparing this upper bound with the lower bound leads to the desired contradiction. We remark that this general technique would also apply to the case $d \geq 3$, running the same arguments, one gets the following lower bound for $d \geq 3$: $\xi^{(2)} \geq 3/(d + 3)$, which is not interesting in view of (26).

c) **Strengthening of the potential.** Finally, we want to consider what is happening if we strengthen the potential. For this final part we may skip the assumption the $W$ is rotationally invariant, and we may also consider a soft Poissonian potential which is not truncated: We define, for $x \in \mathbb{R}^d$ ($d \geq 1$) and $\omega \in \Omega$,

$$V(x, \omega) = \sum_i W(x - x_i) = \int_{\mathbb{R}^d} W(x - y)\omega(dy), \quad \text{(27)}$$

where the shape function $W(\cdot) \geq 0$ is bounded, continuous, compactly supported and not a.e. equal to zero. The strengthening now comes via a parameter $\beta$ which will tend to infinity. The object of our main interest is then the following normalizing constant,

$$\epsilon_{\lambda, \beta}(x, y, \omega) = E_x \left[ \exp \left\{ -\int_0^{H(y)} \beta(\lambda + V)(Z_s, \omega)ds \right\}, H(y) < \infty \right]. \quad \text{(28)}$$

We remark that $\epsilon_{\lambda, \beta}(\cdot, 0, \omega)$ also fulfills the shape theorem (Theorem A) with Lyapounov coefficient $\alpha_{\lambda, \beta}(\cdot)$ (Of course these Lyapounov norms are a priori no longer rotationally invariant). Heuristically, for $\beta \gg 1$ there is a lot of penalty to Brownian paths which do not move along some optimal tube. Therefore in the zero temperature case ($\beta = \infty$) only the optimal paths survive. This leads to the idea to compare the crossing Brownian motion model to the following continuum first-passage percolation model on $\mathbb{R}^d$, obtained by considering the Riemannian distance between $x$ and $y$ with respect to the metrics
\[ ds^2 = 2(\lambda + V)dx^2: \]

\[ g_\lambda(x, y, \omega) = \inf_{\gamma \in \mathcal{P}(x, y, 1)} \left\{ \int_0^1 \sqrt{2(\lambda + V)(\gamma_s, \omega)}|\gamma_s|ds \right\}, \] (29)

where \( \mathcal{P}(x, y, 1) \) is the set of Lipschitz paths \( \gamma \) leading in time 1 from \( x \) to \( y \), i.e., \( \gamma_0 = x \) and \( \gamma_1 = y \). \( g_\lambda \) also fulfills a shape theorem:

**Theorem G (Theorem 0.2, p.79)** For \( d \geq 1 \) and \( \lambda > 0 \) there exists a deterministic norm \( \mu_\lambda(\cdot) \) on \( \mathbb{R}^d \) such that, on a set of full \( \mathbb{P} \)-measure, we have

\[ \lim_{|x| \to \infty} \frac{1}{|x|}|g_\lambda(x, 0, \omega) - \mu_\lambda(x)| = 0. \] (30)

The convergence also holds in \( L^1(\mathbb{P}) \).

Following the terminology of first-passage percolation on the lattice \( \mathbb{Z}^d \) (see Hammersley-Welsh [18], Kesten [26]) we call \( \mu_\lambda(\cdot) \) the time-constant. The main step in the proof of Theorem G is Kingman’s subadditive ergodic theorem, as in Theorem A (Sznitman) or as in analogous results in discrete first-passage percolation (see Kesten [26]). The second step then is to patch together the limits for different directions. This can be done by a maximal lemma (see also [50], Lemma 5.2.6). We remark that the shape theorem also holds in the case \( \lambda = 0 \), but then \( \mu_\lambda(\cdot) \) probably is no longer a norm (see discussion in [54], page 91 below). To avoid such difficulties we restrict ourselves to \( \lambda > 0 \). The connection between the two models comes in the following theorem, which is our main result in this part:

**Theorem H (Theorem 0.3, p.79)** For \( d \geq 1 \) and \( \lambda > 0 \) fixed,

\[ \lim_{\beta \to \infty} \frac{1}{\sqrt{\beta}}\alpha_{\lambda, \beta}(e) = \mu_\lambda(e) \quad e \in \mathbb{R}^d, \] (31)

where the above convergence is uniform on the unit sphere, and hence is uniform on every compact subset of \( \mathbb{R}^d \).

Let us point out that we obtain a much sharper lower bound on the difference between \( \beta^{-1/2} \log c_{\lambda, \beta}(x, 0, \omega) \) and \( g_\lambda(x, 0, \omega) \) than upper bound (compare Theorem 1.1, p.81, and Corollary 2.7, p.87). The main idea to prove the lower bound on \( \alpha_{\lambda, \beta} \) in terms of \( \mu_\lambda \) in (31) is to use the estimates of Agmon [1], which yield upper \( L^2 \)-bounds on certain solutions of second order elliptic equations (see (6)). In our special setting, \( c_{\lambda, \beta}(x, 0, \omega) \) is bounded from above by 1, the proof of Agmon’s results can substantially be simplified. From the \( L^2 \)-bounds we deduce the pointwise upper bounds on \( c_{\lambda, \beta}(\cdot, 0, \omega) \) using Harnack’s
inequalities. From this we easily deduce the lower bound on \( \alpha_{\lambda,\beta} \) in (31) using the shape theorems. To prove the upper bound on \( \alpha_{\lambda,\beta} \) in (31) we use the Cameron-Martin-Girsanov transformation to estimate expectations that the crossing Brownian motion moves along a well-chosen tube, i.e., for large \( \beta \) the particle can hardly reach its target unless it follows a nearly deterministic path that minimizes a certain functional. This inclusion shows that our results are closely related to Maupertuis’ least action principle which tells us to compare the \( L^2 \)- and \( L^1 \)-functionals
\[
L_2(\phi) = \int_0^t \beta(\lambda + V_\phi)(\phi_s, \omega) ds + \int_0^t 1/2|\dot{\phi}_s|^2 ds
\]
and
\[
L_1(\phi) = \int_0^t \sqrt{2\beta(\lambda + V_\phi)(\phi_s, \omega)}|\dot{\phi}_s| ds
\]
for \( \phi \in \mathcal{P}(x, 0, t) \).

Let us remark that for the discrete models (random walk in a random potential and first-passage percolation on \( \mathbb{Z}^d \)), Zerner [57] has proved a result which is qualitatively similar. However, a remarkable thing is that in the discrete setting the correct rescaling factor in (31) is \( \beta^{-1} \). This difference comes from the fact that in the continuous model Brownian motion has also the freedom to choose the “speed” at which it likes to travel through the obstacles. The proof of the discrete result is much simpler: On the one hand, one does not have to apply a technique like Agmon’s results because the first-passage geodesics have non-zero probability in the random walk context. On the other hand one can simply use Jensen’s inequality (because the normalization is \( \beta^{-1} \)).

In a simpler model, Sinai [41] has shown that the paths tend to stay “close” to the “natural geodesic”. One can wonder whether in the model of crossing Brownian motion (or also for random walks) one can define an environment depend “quasigeodesic” which plays a similar role.

4. Outline of this thesis. This thesis consists of four articles. The object of the first article Fluctuation results for Brownian motion in a Poissonian potential [53] (p.13-40) is to develop numerical bounds for \( \xi \) and \( \chi \) in the point-to-point crossing Brownian motion model. The second article Scaling identity for crossing Brownian motion in a Poissonian potential [55] (p.41-60) takes care of the scaling identity in the point-to-point model.

The third article Superdiffusive behavior of two-dimensional Brownian motion in a Poissonian potential [56] (p.61-76) proves superdiffusivity in the point-to-hyperplane model for \( d = 2 \), further it provides the diffusive lower bound on \( \xi \) for \( d \geq 3 \).

Finally, in Geodesics and crossing Brownian motion in a soft Poissonian potential [54] (p.77-94) we compare the Lyapounov norms to the time-constants when we strengthen the potential.
Fluctuation results for Brownian motion in a Poissonian potential

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Abstract

We consider Brownian motion in a truncated Poissonian potential conditioned to reach a remote location. If the Brownian motion starts in 0 and ends in the closed ball with center \( y \in \mathbb{R}^d \) and radius 1, then the transverse fluctuation is expected to be of order \(|y|^\xi\). We prove that \( \xi \leq 3/4 \) and \( \xi \geq 1/(d + 1) \), whereas for the lower bound we have to assume that the dimension \( d \geq 3 \) or that we have a potential with lower bound \( \lambda > 0 \). As a second result we prove, in dimension \( d = 2 \), that \( \chi \geq 1/8 \), where \( \chi \) is the critical exponent for the fluctuation for certain naturally defined random distance functions.

Résumé. — Nous considérons un mouvement Brownien dans un potentiel Poissonien tronqué atteignant un lieu éloigné. Si le mouvement Brownien démarre en 0 et termine dans la boule de centre \( y \in \mathbb{R}^d \) et de rayon 1, alors on attend que la fluctuation transversale soit d'ordre \(|y|^\xi\). Nous montrons que \( \xi \leq 3/4 \) et \( \xi \geq 1/(d + 1) \), où pour la borne inférieure nous devons admettre que la dimension \( d \) soit supérieure ou égale à 3 ou que le potentiel ait une borne inférieure \( \lambda > 0 \). Comme deuxième résultat nous montrons, en dimension \( d = 2 \) que \( \chi \geq 1/8 \), où \( \chi \) est l'exposant critique pour la fluctuation pour certaines fonctions de distance aléatoires définies naturellement.

0 INTRODUCTION

The theme of random motions in random potentials has attracted much interest recently. In the present work we want to consider Brownian motion in a truncated Poissonian potential conditioned to reach a remote location. Our purpose here is to study some fluctuation properties of certain distance functions.
Description of the model. Throughout this paper we look at Brownian motion in a truncated Poissonian potential. Let $\mathbb{P}$ stand for the Poisson law with fixed intensity $\nu > 0$ on the space $\Omega$ of simple pure point measures $\omega$ on $\mathbb{R}^d$. To the points $x_i$ of the Poissonian cloud $\omega = \sum_i \delta_{x_i} \in \Omega$ we want to attach soft obstacles: To model the soft obstacles we take a fixed shape function $W(\cdot) \geq 0$, which is assumed to be bounded, measurable, compactly supported, not a.s. equal to 0 and

$$W(\cdot) \text{ is rotationally invariant.}$$ (0.1)

By $a = a(W) > 0$ we denote the smallest possible $a \in \mathbb{R}^+$ such that $\text{supp}(W) \subset \bar{B}(0, a)$. For $M > 0$ (fixed truncation level), we define the truncated potential as follows:

$$V(x, \omega) = \left( \sum_i W(x - x_i) \right) \wedge M = \left( \int_{\mathbb{R}^d} W(x - y) \omega(dy) \right) \wedge M,$$ (0.2)

where $x \in \mathbb{R}^d$ and $\omega = \sum_i \delta_{x_i} \in \Omega$ is a simple pure locally finite point measure on $\mathbb{R}^d$.

In this medium, we look at Brownian motion. For $x \in \mathbb{R}^d$, $d \geq 2$, we denote by $P_x$ the Wiener measure on $C(\mathbb{R}_+; \mathbb{R}^d)$ starting from $x$; $Z = Z(w), w \in C(\mathbb{R}_+; \mathbb{R}^d)$, stands for the canonical process. For $x, y \in \mathbb{R}^d$, $\lambda \geq 0$, $\omega \in \Omega$, we define the following random variable

$$e_\lambda(x, y, \omega) = \mathbb{E}_x \left\{ \exp \left\{ - \int_0^{H(y)} (\lambda + V)(Z_s, \omega) ds \right\}, H(y) < \infty \right\},$$ (0.3)

where $H(y) = \inf\{ s \geq 0, Z_s \in \bar{B}(y, 1) \}$ is the entrance time of $Z$ into the closed ball $\bar{B}$ with center $y$ and radius 1. We will call $e_\lambda(x, y, \omega)$ the gauge function for $(x, y, \omega, \lambda)$, it plays the role of the normalizing constant for the path measure of the conditioned process. So the measure of the conditioned process is described by

$$\hat{P}_x^y(dw) = \frac{1}{e_\lambda(x, y, \omega)} \exp \left\{ \int_0^{H(y)} (\lambda + V)(Z_s, \omega) ds \right\} 1_{\{H(y) < \infty\}} P_x(dw).$$ (0.4)

We define

$$a_\lambda(x, y, \omega) = - \inf_{\hat{H}(u, t)} \log e_\lambda(\cdot, y, \omega).$$ (0.5)

$a_\lambda(x, y, \omega)$ is a nonnegative random variable that satisfies the triangle inequality. Thus,

$$d_\lambda(x, y, \omega) = \max(a_\lambda(x, y, \omega), a_\lambda(y, x, \omega)).$$ (0.6)
is a nonnegative random function that is symmetric and satisfies the triangle inequality [49]. We know that if \( d \geq 3 \) or \( \lambda > 0 \) or \( \omega \neq 0 \) (which is \( \mathbb{P} \)-a.s. the case) \( d_\lambda(\cdot, \cdot, \omega) \) is a distance function on \( \mathbb{R}^d \), which induces the usual topology.

From the results in [46], we know that there exist norms \( \alpha_\lambda(\cdot) \) on \( \mathbb{R}^d \) for which

\[
\mathbb{P}\text{-a.s.} \quad d_\lambda(0, y, \omega) \sim \alpha_\lambda(y) \quad \text{as} \quad y \to \infty, \tag{0.7}
\]

where in our case of rotationally invariant obstacles the quenched Lyapounov coefficients \( \alpha_\lambda(\cdot) \) are proportional to the Euclidean norm on \( \mathbb{R}^d \), this will imply that considerations on the Euclidean norm allow us to make statements on the quenched Lyapounov coefficients.

In this work we want to describe, how the Brownian paths are behaving when they feel the presence of the Poissonian distributed soft obstacles. As a first critical exponent, we look at the transverse fluctuation:

If we take a cylinder with axis passing through the origin and through our goal \( y \in \mathbb{R}^d \) and with radius \( |y|^\gamma, \gamma > 3/4 \), then we will show in Theorem 1.1 that \( \mathbb{P} \)-a.s. the \( \hat{P}^y_\gamma \)-probability of the event \( A(y, \gamma) \), that the path does not leave this cylinder, tends to 1 as \( |y| \to \infty \). On the other hand, if we take \( \gamma < 1/(d + 1) \), we are able to show (see Theorem 1.3) that for any sequence \((y_n)_n\) of goals tending to infinity, the \( \mathbb{E} \)-expectation of the random variable \( \hat{P}^y_\gamma[A(y_n, \gamma)] \) does not tend to 1. These two estimates give us a lower bound and an upper bound on the critical exponent \( \xi \), standing for the transverse fluctuation. Although this subdiffusive lower bound is far from the expected behavior of the paths, the proof is already mathematically involved. In dimension \( d = 2 \), we expect a superdiffusive behavior of the motion, we guess that the critical exponent \( \xi \) should equal 2/3 (This conjecture is based on the assumption that the behavior of this model should essentially be the same as in the first-passage percolation model (see below). We remark that in a closely related model, we have proved that \( \xi \geq 3/5 \) if \( d = 2 \) and \( \lambda > 0 \) (see Theorem 0.2 of [56])). Whereas in higher dimensions, \( \xi \) should be greater or equal to 1/2 (see Theorem 0.1 of [56] for the related model). But there are no rigorous proofs for these statements in the model considered here. In any case, the bounds which we derive here are a first approach to the expected behavior of the paths.

We also look at a second critical exponent \( \chi \), describing the asymptotic behavior of the variance of \(- \log e_\lambda(0, y)\) for \( |y| \to \infty \). The predicted asymptotic behavior for \( \text{Var}(\log e_\lambda(0, y)) \) is of the order \( |y|^{2\chi} \). We are able to give a nontrivial lower bound on \( \chi \) (see Theorem 1.2). For general \( d \) the following inequality is true:

\[
\chi \geq \frac{1 - (d - 1)\xi}{2}.
\]

This (together with Theorem 1.1) gives us a lower bound of 1/8 in dimension \( d = 2 \),
whereas in dimensions $d \geq 3$ (under the assumption $\xi = 1/2$), we do not get any new interesting features.

Physically, for fixed $\lambda$, $\omega$ and $y$, the gauge function $e_{\lambda}(\cdot,y,\omega)$ can be interpreted as the $\lambda + V(\cdot,\omega)$-equilibrium potential on $B(y,1)^c$ which formally satisfies

$$
\begin{aligned}
\frac{1}{2} \Delta u - (\lambda + V)u &= 0 \quad \text{on } B(y,1)^c, \\
u &= 1 \quad \text{on } B(y,1), \\
u &= 0 \quad \text{at infinity (for typical configurations)}. 
\end{aligned}
$$

(0.8)

We will see, that the model we study here, has lots of common properties with the models in first-passage percolation (see Kesten [26], [27], Newman-Piza [35], Licea-Newman-Piza [30]). The critical exponents, $\chi$ and $\xi$, for the longitudinal and transverse fluctuation are expected to depend on $d$, but nevertheless satisfy the scaling identity $\chi - 2\xi - 1$ for all $d$. But there is no proof for this scaling identity. In fact, loosely speaking, if $\chi'$ denotes the critical exponent for the fluctuation of the random distance function $d_{\lambda}(0,x,\omega)$ around the Lyapounov coefficient $\alpha_{\lambda}(x)$ ($\chi'$ is an exponent closely related to $\chi$), Theorem 1.1 tells on a heuristic level that $\chi' \geq 2\xi - 1$ (In view of Corollary 3.5 of [49] we see that in any dimension $d \geq 2$, $\chi' \leq 1/2$). Heuristic arguments tell also that $\chi \leq 2\xi - 1$ should be true.

In the next section we give precise statements of all the results and an overview on the results already known. In Sections 2, 3 and 4 all the statements are proved: In Section 2 we will prove the upper bound on $\xi$, here we use essentially the fact, that we can compare our random distance function $d_{\lambda}(\cdot,\cdot,\omega)$ to the Euclidean distance. In Section 3 we will prove the lower bound for $\chi$ and finally in Section 4 we give the proof of the subdiffusive lower bound for $\xi$, the main tool for these two bounds will be a martingale method similar to the methods used in the articles of Newman-Piza [35] and Licea-Newman-Piza [30].

1 SETTINGS AND RESULTS

We want to recall that in the whole article we only consider models with rotationally invariant obstacles. It follows that also our quenched Lyapounov coefficients $\alpha_{\lambda}(\cdot)$ are rotationally invariant. This means that $\alpha_{\lambda}(\cdot)$ is a norm on $\mathbb{R}^d$, which is proportional to the Euclidean norm.

We take $x$ a non zero vector in $\mathbb{R}^d$. We define the axis $L_x$ to be the line $\{ax \in \mathbb{R}^d; \alpha \in \mathbb{R}\}$ through $x$ and the origin. Take $r \geq 0$. We define $\bar{B}(x,r) = \{z \in \mathbb{R}^d; d(z,L_x) \leq r\}$ to be the cylinder with axis $L_x$ and radius $r$, where $d(\cdot,\cdot)$ is the Euclidean distance. For technical reasons we cut off the ends of the cylinders. For $x \neq 0$ and $1 \geq \gamma > 0$, let $S(x,\gamma)$
be the following slab $S(x, \gamma) = \{ z \in \mathbb{R}^d ; -|x|^\gamma \leq \langle z, \frac{y}{|y|} \rangle \leq |x| + |x|^\gamma \}$, then we define

$$Z(x, \gamma) = \tilde{Z}(x, |x|^\gamma) \cap S(x, \gamma).$$

(1.1)

For the boundary of $Z(x, \gamma)$ we use the notation $\partial Z(x, \gamma)$.

We are now able to define the event of our main interest. Let $A(x, \gamma)$ be the event that the path of the Brownian motion starting in 0 with goal $B(x, 1)$ does not leave the cylinder $Z(x, \gamma)$: For $x \neq 0$ and $\gamma > 0$,

$$A(x, \gamma) = \{ w \in C(\mathbb{R}_+, \mathbb{R}^d); w(0) = 0, H(x) < \infty \text{ and } Z_s(w) \in Z(x, \gamma) \text{ for all } s \leq H(x) \}.$$  

(1.2)

$\xi$ is then the following critical exponent:

$$\xi = \inf \left\{ \gamma > 0 ; \mathbb{P}-\text{a.s. } \lim_{|y| \to \infty} \tilde{P}_0^y[A(y, \gamma)] = 1 \right\}.$$  

(1.3)

We consider for $d \geq 2$ the model described in the introduction, where the obstacles are rotationally invariant and the Poissonian potential is truncated at the level $M > 0$.

**Theorem 1.1**

$$\xi \leq \frac{3}{4}.$$  

(1.4)

**Remark.** The above theorem gives us a superdiffusive upper bound on the transversal fluctuation of our Brownian motion in a truncated Poissonian potential. The proof of the theorem uses essentially the fact, that the obstacles are rotationally invariant and that the Poissonian potential is truncated at a fixed level. In Lemma 2.1 we will show that the random variable $\tilde{P}_0^y[A(y, \gamma)]$ is continuous in $y$, and therefore we know that $\liminf_{|y| \to \infty} \tilde{P}_0^y[A(y, \gamma)]$ is also a random variable. In the proof of Theorem 1.1, we show the fact that if $\gamma \in (3/4, 1)$, then, $\mathbb{P}$-a.s. for large $|y|$, $\tilde{P}_0^y[A(y, \gamma)] \geq 1 - c|y|^d \exp\{-c'|y|^{2\gamma-1}\}$, for suitable $c > 0$ and $c' > 0$, which of course goes to 1 as $|y|$ tends to infinity.

Once we know this statement, we are able to prove the divergence of the variance of $-\log e_\lambda(0, y, \omega)$ in dimensions $d = 2$. We define the critical exponent for longitudinal fluctuation

$$\chi = \sup \left\{ \kappa \geq 0 ; \exists C > 0 \text{ with } \text{Var}( -\log e_\lambda(0, y) ) \geq C|y|^{2\kappa} \text{ for all } |y| > 1 \right\}.$$  

(1.5)

We have the following theorem:
Theorem 1.2 For \( d \geq 2 \),
\[
\chi \geq \frac{1 - (d - 1)\xi}{2}.
\] (1.6)

In view of (1.4) for \( d = 2 \),
\[
\chi \geq 1/8.
\] (1.7)

Our next aim is to find a lower bound on the transverse fluctuation. For the start we get a subdiffusive lower bound on \( \xi_0 \) defined as
\[
\xi_0 = \inf \left\{ \gamma \geq 0 ; \limsup_{|y| \to \infty} \mathbb{E} \left[ \mathcal{P}^y[A(y, \gamma)] \right] = 1 \right\}.
\] (1.8)

Of course, \( \xi_0 \leq \xi \). We want to mention that in the definition of \( \xi_0 \) we could restrict ourselves to \( y \in \mathbb{R}^d \) of the form \( y = (|y|, 0, \ldots, 0) \). Indeed, in the case of rotationally invariant obstacles, the above expectation does not depend on the direction.

Theorem 1.3 Assume \( d \geq 3 \) or \( \lambda > 0 \), then
\[
\xi_0 \geq \frac{1}{d + 1}.
\] (1.9)

Remarks. The first statement in Theorem 1.2 and the statement in Theorem 1.3 are also valid in a more general context; in fact in the two proofs we do not use that the shape function is rotationally invariant. We only require this assumption in the proof of Theorem 1.1 and as a consequence for the lower bound on \( \chi \) in dimension \( d = 2 \).

We sometimes use the terminology of first-passage percolation because the situation in the case of Brownian motion in a truncated Poissonian potential looks very similar. In first-passage percolation, to prove the statements one has often (unverified) assumptions on the curvature of the asymptotic shape, here we do not have these problems because in the case of rotationally invariant obstacles one knows that the asymptotic shape is a ball with positive radius. This fact indicates an advantage of our model.

To close this chapter we give some further general notations and state some already known results we will need later on. We usually denote positive constants by \( c_1, c_2, \ldots \) and \( \gamma_1, \gamma_2, \ldots \). These constants will only depend on the invariant parameters of our model, namely the dimension \( d \), the intensity \( \nu \), the shape function \( W \), the truncation level \( M \) and the parameter \( \lambda \). The constants which are used in the whole article are denoted by \( \gamma_i \), whereas \( c_i \) is only used for local calculations in the proofs.
If $U$ is an open subset of $\mathbb{R}^d$, we introduce the $(\lambda + V)$-Green function relative to $U$: Take $\omega \in \Omega$, $x, y \in \mathbb{R}^d$ then
\begin{equation}
    g_{\lambda, U}(x, y, \omega) = \int_0^\infty e^{-\lambda s} r_U(s, x, y, \omega) ds \in (0, \infty],
\end{equation}
where $r_U$, for a non void $U$, is known to be the kernel of the self-adjoint semigroup on $L^2(U, dx)$ generated by $-\frac{1}{2} \Delta + V$ with Dirichlet boundary conditions; for $\omega \in \Omega$, $x, y \in \mathbb{R}^d$ and $t > 0$ is
\begin{equation}
    r_U(t, x, y, \omega) = p(t, x, y) \cdot
    E_{x, y}^t \left[ \exp \left\{ -\int_0^t V(Z_s, \omega) ds \right\}, T_U > t \right],
\end{equation}
with $p(t, x, y)$ the Brownian transition density, $E_{x, y}^t$ the Brownian bridge in time $t$ from $x$ to $y$ and $T_U = \inf\{ s \geq 0 ; Z_s \notin U \}$ the exit time from $U$. When $U = \mathbb{R}^d$, we will drop the subscript $U$ from the notation.

Next we recall some properties of $e_{\lambda}(x, y, \omega)$ and $d_{\lambda}(x, y, \omega)$. For $e_{\lambda}(x, y, \omega)$ we have by a tubular estimate for Brownian motion the following nice lower bound (see for instance (1.35) in [49]):
\begin{equation}
    e_{\lambda}(x, y, \omega) \geq \gamma_1 \exp\{ -\gamma_2 |y - x| \},
\end{equation}
with $\gamma_1 \in (0, 1)$ and $\gamma_2 > 0$.

From [49] Proposition 1.3, we have a shape theorem: on a set of full $\mathbb{P}$-measure we know, for $\lambda \geq 0$, that
\begin{equation}
    \lim_{y \to \infty} \frac{1}{|y|} |d_{\lambda}(0, y, \omega) - \alpha_{\lambda}(y)| = 0,
\end{equation}
the convergence also holds in $L^1(\mathbb{P})$, and one can replace $d_{\lambda}(0, y, \omega)$ by $-\log e_{\lambda}(0, y, \omega)$, $-\log g_{\lambda}(0, y, \omega)$ or $\alpha_{\lambda}(0, y, \omega)$.

Next we introduce a paving of $\mathbb{R}^d$. For $q \in \mathbb{Z}^d$, we consider the cubes of size $l$ and center $q$
\begin{equation}
    C(q) = \left\{ z \in \mathbb{R}^d ; -\frac{l}{2} \leq z_i - lq^i < \frac{l}{2}, \; i = 1, \ldots, d \right\},
\end{equation}
with $l(d, \nu, a) \in (d(4 + 8a), \infty)$ fixed, but large enough, see for instance [48] or [49], and for $z \in \mathbb{R}^d$, $z^i$ denotes the $i$-th coordinate of $z$ for $i = 1, \ldots, d$. We also want to introduce a fixed ordering $q_1, q_2, \ldots$ of all $q \in \mathbb{Z}^d$. So we get also a ordering of our cubes: We define
\( C_k = C(q_k) \) for all \( k \in \mathbb{N} \). For \( y \in \mathbb{R}^d \) and \( w \in C(\mathbb{R}_+, \mathbb{R}^d) \) with \( H(y) < \infty \) we introduce the random lattice animal

\[
\mathcal{A}(w) = \{ k \in \mathbb{N} ; H_k \leq H(y) \},
\]

where \( H_k \) is the entrance time of \( Z \) into the closed cube \( C_k \). We know from [49] formula (1.31) that there exists a \( \gamma_3(d, \nu, W, M) \) small enough such that for \( x \in C(0) \) and \( y \in \mathbb{R}^d \):

\[
E_x \left[ \exp \left\{ \gamma_3 |A| - \int_0^{H(y)} V(Z_s, \omega) ds \right\} , H(y) < \infty \right] \leq 2^{N_0(\omega)/2},
\]

where \( N_0(\omega), \omega \in \Omega \), is a random variable with \( \mathbb{E} [2^{N_0(\omega)/2}] < \infty \), and \( |A| \) denotes the (random) number of cubes visited by the path. With the help of this exponential bound we get very important estimates on the expected value of any power of the number \( |A| \) of visited cubes.

Finally we quote the (for us) important part of Lemma 1.2 of Sznitman [49]. For \( |x - y| > 4 \) and \( \omega \in \Omega \)

\[
| - \log e_{13}(x, y, \omega) - d_{13}(x, y, \omega) | \leq \gamma_4(1 + F_{13}(x) + F_{13}(y)),
\]

where for \( x \in \mathbb{R}^d \) and \( \omega \in \Omega \)

\[
F_{13}(x) \leq \gamma_5, \quad \text{if } d \geq 3 \text{ or } \lambda > 0,
\]

\[
\leq \gamma_6 \left( 1 + \log^+ (\log \text{dist}(x, \text{supp} \omega)) \right), \quad \text{if } d = 2 \text{ and } \lambda = 0,
\]

provided \( \text{supp} \omega \) denotes the support of \( \omega \).

## 2 PROOF OF THE SUPERDIFFUSIVE UPPER BOUND ON \( \xi \)

First we start with a continuity result. We want to show that the limit in the definition of \( \xi \) is a well-defined random variable. Therefore we have to show that \( \tilde{P}_0^y[A(y, \gamma)] \) is continuous in \( y \).

**Lemma 2.1** For \( \lambda \geq 0 \) and \( \gamma > 0 \), the functions \( (y, \omega) \rightarrow e_{13}(0, y, \omega) \) and \( (y, \omega) \rightarrow \tilde{P}_0^y[A(y, \gamma)] \) are measurable in \( \omega \) and for all \( |y| > 1 \) continuous in \( y \).

We give the proof of this lemma in the appendix. Next we state a geometric lemma. If \( 0 \) and \( y \in \mathbb{R}^d \) are on the axis of the cylinder with radius \( |y|^\gamma \) we want to measure the cost of a detour to the boundary of the cylinder, as compared to the "direct way" from 0 to \( y \).
Lemma 2.2 We take $\gamma \in (0, 1]$. There exists a $\gamma_I \in (0, \infty)$ such that for all $y \in \mathbb{R}^d$ with $|y| > 1$ and $z \in \partial Z(y, \gamma)$ the following is true:

$$|0 - z| + |z - y| \geq |0 - y| + \gamma_I |y|^{2\gamma - 1}.$$  

(2.1)

Remark. The lemma will be important for us, because in the case of rotationally invariant obstacles, our quenched Lyapounov coefficients are proportional to the Euclidean norm, so in fact the above lemma is a claim for our quenched Lyapounov coefficients.

Proof. Take $y \in \mathbb{R}^d$ fixed. Without loss of generality we may assume that $y$ has the same direction as the first unit vector in $\mathbb{R}^d$. By $z_1$ we denote the first coordinate of the vector $z \in \partial Z(y, \gamma)$.

If $z_1 = -|y|^\gamma$ or if $z_1 = |y| + |y|^\gamma$, then

$$|0 - z| + |z - y| \geq |0 - y| + 2|y|^\gamma \geq |0 - y| + 2|y|^{2\gamma - 1}.$$  

If $z_1 \in (-|y|, |y| + |y|^\gamma)$, then we see, that if we embed an ellipsoid into the cylinder with focal points 0 and $y$ and tangent to the cylinder, that $|0 - z| + |z - y|$ is minimal for $z_1 = |y|/2$. Therefore

$$|0 - z| + |z - y| \geq 2\sqrt{\frac{|y|^2}{4} + |y|^{2\gamma}} = 2|y|\sqrt{\frac{1}{4} + |y|^{2\gamma-2}}.$$  

Now there exists a $c_1 > 0$ with

$$\frac{1}{4} + |y|^{2\gamma-2} \geq \frac{1}{4} + c_1 |y|^{2\gamma-2} + c_1^2 |y|^{4\gamma-4} = \left(\frac{1}{2} + c_1 |y|^{2\gamma-2}\right)^2.$$  

So we see that

$$|0 - z| + |z - y| \geq 2|y|\sqrt{\left(\frac{1}{2} + c_1 |y|^{2\gamma-2}\right)^2} = |y| + 2c_1 |y|^{2\gamma-1},$$  

which completes the proof, if we take $\gamma_I \in (0, \infty)$ suitable.

We recall the following result from [49]: Corollary 3.5 tells us that under assumption (0.1) in dimensions $d \geq 2$, $\mathbb{P}$-a.s., for large $|y|$,

$$|d_\lambda(0, y) - \alpha_\lambda(y)| \leq \gamma_8 (1 + |y|^{1/2} \log^2 |y|).$$  

(2.2)

Our aim is to formulate a similar result for the random distances $d_\lambda(0, z)$ and $d_\lambda(z, y)$ if $z$ is on the boundary of the cylinder $Z(y, \gamma)$. 

\qed
Lemma 2.3 Assume $(0.1)$. When $d \geq 2$, $\gamma \in (0,1]$, then $\mathbb{P}$-a.s., for large $|y|$ and $z \in \partial Z(y, \gamma)$, the following holds
\[
|d_\lambda(z, y) - \alpha_\lambda(y - z)| \leq \gamma_9 (1 + |y - z|^{1/2} \log^2 |y - z|) \tag{2.3}
\]
and
\[
|d_\lambda(0, z) - \alpha_\lambda(z)| \leq \gamma_{10} (1 + |z|^{1/2} \log^2 |z|). \tag{2.4}
\]

Remark. If $d \geq 3$ or $\lambda > 0$, one can improve the bounds of the above inequalities. But for our purposes we will not need any better bounds. The important thing in the proof will be that the distance of the two points is growing faster than the big holes in the Poissonian cloud.

Proof. First we want to prove (2.3) in the case $d \geq 3$ or $\lambda > 0$. We pick a fixed $y \in \mathbb{R}^d$, such that $|y - z| > 4$ for all $z \in \partial Z(y, \gamma)$. Take $z \in \partial Z(y, \gamma)$, then Theorem 2.1 from [49] tells us, that for $0 \leq u \leq c_1 |y - z|$ is
\[
\mathbb{P}\left[|d_\lambda(z, y) - D_\lambda(0, y - z)| > u \sqrt{|y - z|}\right] = \mathbb{P}\left[|d_\lambda(0, y - z) - D_\lambda(0, y - z)| > u \sqrt{|y - z|}\right] \leq c_2 \exp\{-c_3 u\}, \tag{2.5}
\]
where $D_\lambda(0, x) = \mathbb{E}[d_\lambda(0, x)]$.

The first step is to verify the lemma for a countable set of points because we want to use the Borel-Cantelli lemma. For every $n \in \mathbb{N}$, we take a finite covering of $\partial B(0, n)$ with balls $B(y, 1)$ such that $|y| = n$. We denote the set of the centers of these balls by $C_n$. We may and will choose $C_n$ such that $|C_n| \leq c_4 n^{d-1}$ (By $|A|$ we denote the number of points in $A$). For $\partial Z(y, \gamma)$ we do exactly the same and we denote the set of centers by $C_y$. We choose $C_y$ such that $|C_y| \leq c_5 |y|^{1+(d-1)\gamma}$. For a fixed $n \in \mathbb{N}$, $y \in C_n$ and $z \in C_y$ we define the following event
\[
A_{n, y, z} = \left\{ \omega \in \Omega ; \ |d_\lambda(z, y) - \alpha_\lambda(y - z)| \geq c_6 (1 + |y - z|^{1/2} \log |y - z|) \right\}.
\]
We will choose a suitable but fixed $c_6$ which is determined in (2.7) below. From Corollary 3.4 of [49] we know that, for $|y| \geq 1$,
\[
0 \leq D_\lambda(0, y) - \alpha_\lambda(y) \leq c_7 (1 + |y|^{1/2} \log |y|). \tag{2.6}
\]
So the triangle inequality and (2.6) imply for $n \in \mathbb{N}$, $y \in C_n$ and $z \in C_y$
\[
A_{n, y, z} \subset \left\{ \omega \in \Omega ; \ |d_\lambda(z, y) - D_\lambda(0, y - z)| \geq (c_6 - c_7) |y - z|^{1/2} \log |y - z| \right\}.
\]
Now we choose $c_6$. Let $c_6$ be large enough such that
\[ c_3 \gamma (c_6 - c_7) - d - (d - 1) \gamma \geq 2. \tag{2.7} \]
Then we choose $n_0$ such that $(c_6 - c_7) \log |y - z| \leq c_1 |y - z|$ for all $n \geq n_0$, $y \in C_n$ and $z \in C_y$. In view of (2.5), we get
\[
P[A_{n,y,z}] \leq c_3 \exp\{-c_3 (c_6 - c_7) \log |y - z|\}
\leq c_2 |y|^{-\alpha \gamma (c_6 - c_7)}
\leq c_2 n^{-d - (d - 1) \gamma - 2} \quad \text{for all } n \geq n_0.
\]
Therefore, the following sum is finite
\[
\sum_{n \geq 1} \sum_{y \in C_n} \sum_{x \in C_y} P[A_{n,y,z}] < \infty. \tag{2.8}
\]
The proof of (2.3) in the case $d \geq 3$ or $\lambda > 0$ follows with a Borel-Cantelli argument and the observation that $\sup_{|x - y| \leq 1} d_\lambda (x, y)$ is uniformly bounded by (1.12).

The proof of (2.3) in the case $d = 2$ and $\lambda = 0$ is almost the same, the only difference is that one has to use Theorem 2.5 of [49] instead of Theorem 2.1. It is at this point where the upper bound is weakened, i.e., here the power two in the logarithm is coming into the calculations. The proof of (2.4) goes analogously.

At this stage we can combine Lemma 2.1, Lemma 2.2 and Lemma 2.3 to find the upper bound on $\xi$. The idea is that with the help of the strong Markov property and the above lemmas one sees that the detour over the boundary of the cylinder costs too much.

**Proof of Theorem 1.1.** Take $\gamma \in (3/4, 1)$ fixed. We want to show that $\mathbb{P}$-a.s., for large $|y|$,\[
\hat{p}_y[A(y, \gamma)] \geq 1 - \gamma_1 |y|^d \exp\{-\gamma_1 |y|^{2\gamma - 1}\}. \tag{2.9}
\]
From Lemma 2.2 we know that for $|y| > 1$ and $z \in \partial Z(y, \gamma)$ the following holds
\[
|0 - z| + |z - y| \geq |0 - y| + \gamma_1 |y|^{2\gamma - 1}.
\]
We multiply the above inequality by $\alpha_\lambda (e_1)$. Thanks to the rotationally invariant obstacles, we get the following inequality for our quenched Lyapounov coefficients $\alpha_\lambda (\cdot)$:
\[
\alpha_\lambda (z) + \alpha_\lambda (z - y) \geq \alpha_\lambda (y) + \gamma_1 \alpha_\lambda (e_1) |y|^{2\gamma - 1}. \tag{2.10}
\]
Notice that if our path from 0 to \( y \) runs over the boundary of the cylinder \( Z(y, \gamma) \), we are allowed to add an extra term of order \( 2\gamma - 1 \) to the right-hand side of the above triangle inequality. In view of Lemma 2.3, (2.2) and (2.10), we get \( \mathbb{P} \)-a.s., for \( |y| \) large enough and any \( z \in \partial Z(y, \gamma) \),

\[
d_\lambda(0, z) + d_\lambda(z, y) \geq d_\lambda(0, y) + \gamma \alpha \lambda(e_1)|y|^{2\gamma - 1} - c_1(1 + |y|^{1/2} \log^2 |y|),
\]

for a suitable \( c_1 \). Here we see the importance of the order of the added term in (2.10). We want to have \( \gamma \) such that the correction term on the right-hand side of the above inequality stays positive. We see that \( |y|^{2\gamma - 1} \) tends faster to infinity than \( |y|^{1/2} \log^2 |y| \) whenever \( \gamma > 3/4 \). Thus, \( \mathbb{P} \)-a.s., for large \( |y| \) and \( z \in \partial Z(y, \gamma) \),

\[
d_\lambda(0, z) + d_\lambda(z, y) \geq d_\lambda(0, y) + c_2|y|^{2\gamma - 1}.
\]

(2.11)

As in the proof of Lemma 2.3 we take a finite covering of \( \partial Z(y, \gamma) \) with balls \( B(z, 1) \). We denote by \( C_y \) the set of the centers of these balls. We are able to choose \( C_y \) such that \( |C_y| \leq c_3\gamma^{1+(d-1)\gamma} \). With the help of the strong Markov property we find

\[
\hat{P}_0^y[A(y, \gamma)^c] \leq \sum_{z \in C_y} \hat{P}_0^y[H(z) \leq H(y)]
\]

\[
= \sum_{z \in C_y} \frac{1}{e_\lambda(0, y, \omega)} E_0 \left[ \exp \left\{ - \int_0^{H(y)} (\lambda + V)(Z_s, \omega) ds \right\}, H(z) \leq H(y) \leq \infty \right]
\]

\[
\leq \sum_{z \in C_y} \frac{1}{e_\lambda(0, y, \omega)} c_\lambda(0, z, \omega) \sup_{z' \in B(z, 1)} c_\lambda(z', y, \omega).
\]

First we want to treat the case \( d \geq 3 \) or \( \lambda > 0 \): We find with the help of (1.17) and (2.11), that \( \mathbb{P} \)-a.s., for large \( |y| \),

\[
\hat{P}_0^y[A(y, \gamma)^c] \leq \sum_{z \in C_y} \exp\{d_\lambda(0, y, \omega) - d_\lambda(0, z, \omega) - d_\lambda(z, y, \omega) + c_4\}
\]

\[
\leq \sum_{z \in C_y} c_5 \exp\{-c_2|y|^{2\gamma - 1}\}
\]

\[
\leq c_3 c_5 |y|^{1+(d-1)\gamma} \exp\{-c_2|y|^{2\gamma - 1}\}.
\]

That proves the theorem in the first case.

Now, the case when \( d = 2 \) and \( \lambda = 0 \) is a little bit more difficult, because (1.17) does not have that easy form as in the first case. If the Brownian motion is recurrent, we have to be sure, that there are obstacles in our space to get good results: As in (1.23) of [49] we see that \( \mathbb{P} \)-a.s., for large \( |y| \),

\[
\sup\{d(y, \text{supp}(\omega)), d(z, \text{supp}(\omega)), d(0, \text{supp}(\omega))\} \leq c_6 \log^{1/d} |y|.
\]
This follows from standard estimates on the Poissonian distributed cloud in \( \mathbb{R}^d \). Therefore we have \( \mathbb{P} \)-a.s., for all large \( |y| \), an upper bound on the function \( F_\lambda \) (see 1.17)

\[
\sup\{F_\lambda(0), F_\lambda(y), F_\lambda(z)\} \leq c_\ell \log \log |y|.
\]

So we find using (2.11) and (2.12) that \( \mathbb{P} \)-a.s., for large \( |y| \),

\[
\tilde{P}^y_0[A(y, \gamma)^c] \leq \sum_{z \in C_y} \exp\{d_\lambda(0, y) - d_\lambda(0, z) - d_\lambda(z, y) + c_9 + c_{10} \log \log |y|\} \\
\leq c_{11} |y|^{1+\gamma(\log \log |y|)}^{c_{10}} \exp\{-c_2 |y|^{2\gamma - 1}\}.
\]

This completes the proof of our theorem also in the second case.

\[\square\]

### 3 PROOF OF THE DIVERGENCE OF THE VARIANCE

We want to derive now the proof of Theorem 1.2. The proof has a very similar structure as an analogous power law result on the divergence of shape fluctuations in first-passage percolation. Our main tool will be the martingale technique used in the spirit of Wehr-Aizenman [52], Aizenman-Wehr [2], Newman-Piza [35] and many other authors.

**Proof of Theorem 1.2.** Take \( U \) a subset of \( \mathbb{R}^d \), we define the following \( \sigma \)-algebra,

\[
\mathcal{F}(U) = \sigma\{\omega(A); A \in B(\mathbb{R}^d) \text{ and } A \subseteq U\}.
\]

We introduce then the following filtration \((\mathcal{F}_k)_{k \geq 0}\) on \((\Omega, \mathcal{F}, \mathbb{P})\):

\[
\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_k = \mathcal{F}\left(\bigcup_{i=1}^{k} C_i\right), \quad k \geq 1,
\]

with \( C_i, i \geq 1 \), the cubes defined in Chapter 1. For a fixed \( y \in \mathbb{R}^d \), introduce the following non-negative martingale

\[
M_k = \mathbb{E}[-\log e_\lambda(0, y) \mid \mathcal{F}_k], \quad k \geq 0.
\]

In view of (1.12) \(-\log e_\lambda(0, y)\) is bounded above and \( M_k \) converges \( \mathbb{P} \)-a.s. and in \( L^p(\mathbb{P}) \), \( p \in [1, \infty) \), to \(-\log e_\lambda(0, y)\). By standard martingale identities we get

\[
\Var (-\log e_\lambda(0, y)) = \sum_{k \geq 1} \Var(\Delta M_k),
\]

where \( \Delta M_k = M_k - M_{k-1} \).
We denote by $G_k = F(C_k)$. So $F_k = G_1 \lor G_2 \lor \cdots \lor G_k$, with $G_1 \lor G_2$ the smallest $\sigma$-algebra containing $G_1$ and $G_2$. Because $G_k \subset F_k$ and $G_k \lor F_{k-1}$, we find

$$\text{Var} \left( -\log e_{\lambda}(0, y) \right) = \sum_{k \geq 1} \text{Var} \left( \Delta M_k \right) \geq \sum_{k \geq 1} \text{Var} \left( \mathbb{E}[\Delta M_k | G_k] \right) \geq \sum_{k \geq 1} \text{Var} \left( \mathbb{E}[\log e_{\lambda}(0, y) | G_k] \right).$$

Our next purpose is to apply similar considerations as Lemma 3 of [35]: If $\omega \in \Omega$ is a cloud configuration, we denote by $\tilde{\omega}_k$ the restriction of $\omega$ to $C_k^\delta$ and by $\omega_k$ the restriction of $\omega$ to $C_k$, so we can write $\omega = (\omega_k, \tilde{\omega}_k)$. We consider the following two disjoint events on $C_k$:

$$D_{0,k} = \{ \omega_k ; \omega_k(C_k) = 0 \},$$

this is the event that the cube $C_k$ receives no point of the cloud. Whereas

$$D_{1,k} = \{ \omega_k ; \omega_k(B(lq_k, 1)) \geq 1 \}$$

is the event that we have at least one point of the Poissonian cloud in the center (i.e., in the closed ball with center $lq_k$ and radius 1) of the cube $C_k$. We then define, for $\delta = 0$ or 1,

$$D^\delta_k = \{ \omega \in \Omega \text{ with } \omega_k \in D_{\delta,k} \}.$$

Of course, $D^0_k$ and $D^1_k$ are disjoint and $G_k$ measurable. We denote by $p = \mathbb{P}[D^0_k] > 0$ and by $q = \mathbb{P}[D^1_k] > 0$.

$$\mathbb{E}[ -\log e_{\lambda}(0, y, \omega) | G_k ] \cdot 1_{D^\delta_k}(\omega) = \mathbb{E}[ -\log e_{\lambda}(0, y, \omega) 1_{D^\delta_k}(\omega) | G_k ] \leq \mathbb{E} \left[ \sup_{\sigma_k^\delta \in D_{0,k}} -\log e_{\lambda}(0, y, (\sigma_k^\delta, \tilde{\omega}_k)) 1_{D^\delta_k}(\omega) \bigg| G_k \right] = \mathbb{E} \left[ \sup_{\sigma_k^\delta \in D_{0,k}} -\log e_{\lambda}(0, y, (\sigma_k^\delta, \tilde{\omega}_k)) \right] \cdot 1_{D^\delta_k}(\omega),$$

analogously one gets the following lower bound

$$\mathbb{E}[ -\log e_{\lambda}(0, y, \omega) | G_k ] \cdot 1_{D^\delta_k}(\omega) \geq \mathbb{E} \left[ \inf_{\sigma_k^\delta \in D_{1,k}} -\log e_{\lambda}(0, y, (\sigma_k^\delta, \tilde{\omega}_k)) \right] \cdot 1_{D^\delta_k}(\omega).$$
We remark that \( \sup_{\sigma_k^0 \in D_{0,k}} - \log e_{\lambda}(0, y, (\sigma_k^0, \cdot)) \) and \( \inf_{\sigma_k^1 \in D_{1,k}} - \log e_{\lambda}(0, y, (\sigma_k^1, \cdot)) \) are measurable. This can be seen by using an approximation of the cloud configuration by cloud configurations with rational coordinates. We define \( x_0 \) and \( x_1 \) as follows,

\[
x_0 = \frac{\mathbb{E} \left[ \mathbb{E} [- \log e_{\lambda}(0, y, \omega) | \mathcal{G}_k] \cdot 1_{D_k^0}(\omega) \right]}{\mathbb{P}[D_k^0]} \leq \mathbb{E} \left[ \sup_{\sigma_k^0 \in D_{0,k}} - \log e_{\lambda}(0, y, (\sigma_k^0, \hat{\omega}_k)) \right]
\]

and

\[
x_1 = \frac{\mathbb{E} \left[ \mathbb{E} [- \log e_{\lambda}(0, y, \omega) | \mathcal{G}_k] \cdot 1_{D_k^1}(\omega) \right]}{\mathbb{P}[D_k^1]} \geq \mathbb{E} \left[ \inf_{\sigma_k^1 \in D_{1,k}} - \log e_{\lambda}(0, y, (\sigma_k^1, \hat{\omega}_k)) \right].
\]

To simplify the notation we introduce:

\[
E_{k}(0, y, \hat{\omega}_k) = \inf_{\sigma_k^1 \in D_{1,k}} - \log e_{\lambda}(0, y, (\sigma_k^1, \hat{\omega}_k)) - \sup_{\sigma_k^0 \in D_{0,k}} - \log e_{\lambda}(0, y, (\sigma_k^0, \hat{\omega}_k)).
\]

In view of the above bounds on \( x_0 \) and \( x_1 \), we see that \( x_1 - x_0 \geq \mathbb{E} [E_{k}] \geq 0 \) and therefore

\[
(x_1 - x_0)^2 \geq \mathbb{E} [E_{k}]^2. \tag{3.4}
\]

Using Lemma 3 of [35], we have the following estimate

\[
\text{Var} \left( \mathbb{E} [- \log e_{\lambda}(0, y) | \mathcal{G}_k] \right) \geq \frac{\mathbb{P}[D_k^0] \mathbb{P}[D_k^1]}{\mathbb{P}[D_k^0] + \mathbb{P}[D_k^1]} (x_1 - x_0)^2,
\]

therefore, together with (3.3) and (3.4), this is yielding us a lower bound for the variance of \( - \log e_{\lambda}(0, y) \)

\[
\text{Var} (- \log e_{\lambda}(0, y)) \geq \frac{pq}{p + q} \sum_{k \geq 1} \mathbb{E} [E_{k}]^2. \tag{3.5}
\]

Take \( \gamma > \xi \) and define \( \mathcal{E}_y = \{ k \in \mathbb{N} \text{ with } C_k \cap Z(y, \gamma) \neq \emptyset \} \), to be all the cubes, that intersect the cylinder \( Z(y, \gamma) \) with radius \( |y|^\gamma \). We know that \( |\mathcal{E}_y| \leq c_1 |y|^{1+(\ell-1)\gamma} \) for a suitable \( c_1 \subset (0, \infty) \). With the help of Cauchy-Schwarz inequality, we get

\[
\text{Var} (- \log e_{\lambda}(0, y)) \geq \frac{pq}{p + q} \sum_{k \in \mathcal{E}_y} \mathbb{E} [E_{k}]^2
\]

\[
\geq \frac{pq}{p + q} \frac{1}{|\mathcal{E}_y|} \left( \sum_{k \in \mathcal{E}_y} \mathbb{E} [E_{k}] \right)^2
\]

\[
\geq c_2 |y|^{-1-(\ell-1)\gamma} \left( \sum_{k \in \mathcal{E}_y} \mathbb{E} [E_{k}] \right)^2.
\]
If we are able to prove that there exists a $c_3 \in (0, \infty)$ with
\[
\liminf_{|y| \to \infty} \frac{1}{|y|} \sum_{k \in \mathcal{K}_y} E[E_k] \geq c_3,
\]
then $\chi \geq (1 - (d - 1)\gamma)/2$ for all $\gamma > \zeta$, so the claim of Theorem 1.2 follows for $d \geq 2$, whereas for the bound $1/8$ for $d = 2$ one simply inserts the bound for $\xi$ given in Theorem 1.1. It remains to prove (3.6).

We use the following notation to prove the rest of the theorem. We will therefore prove two technical lemmas, the proof of the two lemmas will essentially be the same as the proof of formulas (2.10), (2.11) and (2.13) of [49]: For $k \geq 1$, take $\sigma_k^{\delta} \in D_k$, with $\delta = 0$ or 1, $\omega \in \Omega$ and $x \in \mathbb{R}^d$. We define the potential
\[
V_k^{\delta}(x, \hat{\omega}_k) = M \wedge \left( \int_{C_k} W(x - y)\sigma_k^{\delta}(dy) + \int_{C_k} W(x - y)\hat{\omega}_k(dy) \right).
\]

If $\tilde{C}_k$ is the closed $\alpha$-neighborhood of $C_k$, then notice that
\[
V_k^1 = V_k^0 \text{ on } \tilde{C}_k^c \quad \text{and} \quad V_k^1 \geq V_k^0 \text{ on } \tilde{C}_k,
\]
and there exists a $c_4 > 0$ and a domain $G \subset C_k$ (depending on $\sigma_k^1 \in D_{1,k}$) such that $G$ has positive Lebesgue measure and the difference $V_k^1 - V_k^0 > c_4$ on $G$. Define $\tilde{H}_k - II_{\tilde{C}_k}$ to be the entrance time of $Z$ into $\tilde{C}_k$, and $H_k = H_{\tilde{C}_k}$ to be the entrance time of $Z$ into the closed cube $\tilde{C}_k$. So we denote the path measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ generated by $V_k^{\delta}$ with start in $x \in \mathbb{R}^d$ and goal in $y \in \mathbb{R}^d$ as follows
\[
d\hat{P}_x^{k,\delta} = d\hat{P}_x^{k,\delta} = \frac{1}{e_\lambda(x, y, \sigma_k^{\delta}, \hat{\omega}_k)} \exp \left\{ - \int_0^{H(y)} (\lambda + V_k^{\delta})(Z_s) ds \right\} 1_{(H(y) < \infty)} dP_x,
\]
to avoid overloaded notations we will drop the brackets for the cloud configuration in the gauge function.

**Lemma 3.1** With the above notations the following statements are true:

\[
\frac{e\lambda(0, y, \sigma_k^1, \hat{\omega}_k)}{e\lambda(0, y, \sigma_k^0, \hat{\omega}_k)} - 1 + \hat{P}_0^{k,0} \left[ \tilde{H}_k \leq H(y), \frac{e\lambda(Z_{\tilde{H}_k}; y, \sigma_k^1, \hat{\omega}_k)}{e\lambda(Z_{\tilde{H}_k}; y, \sigma_k^0, \hat{\omega}_k)} - 1 \right]
\]

and
\[
\frac{e\lambda(0, y, \sigma_k^0, \hat{\omega}_k)}{e\lambda(0, y, \sigma_k^1, \hat{\omega}_k)} = 1 + \hat{P}_0^{k,1} \left[ \tilde{H}_k \leq H(y), \frac{e\lambda(Z_{\tilde{H}_k}; y, \sigma_k^0, \hat{\omega}_k)}{e\lambda(Z_{\tilde{H}_k}; y, \sigma_k^1, \hat{\omega}_k)} - 1 \right].
\]
Proof of the lemma. With the help of the strong Markov property follows

\[
\frac{e_\lambda(0, y, \sigma^1, \hat{\omega}_k)}{e_\lambda(0, y, \sigma^0, \hat{\omega}_k)} = \hat{P}^{k,0}_0 [\hat{H}_k > H(y)] + \hat{P}^{k,0}_0 [\hat{H}_k \leq H(y), e_\lambda(Z_{\hat{H}_k}, y, \sigma^1, \hat{\omega}_k)] = 1 + \hat{P}^{k,0}_0 [\hat{H}_k \leq H(y), e_\lambda(Z_{\hat{H}_k}, y, \sigma^0, \hat{\omega}_k) - 1].
\]

Whereas for the second claim of the lemma one has to exchange the role of \(\sigma^1\) and \(\sigma^0\).

This finishes the proof of Lemma 3.1.

The second lemma tells us, how to handle the fraction on the right-hand side of the equation in the above lemma. For \(k \geq 1\), take \(\sigma^k \in D_{\delta,k}\), with \(\delta = 0\) or \(1\), \(\omega \in \Omega\) and \(y \in \mathbb{R}^d\), we denote by \(g^\delta_{\lambda,k} (\cdot, \cdot)\) the \((\lambda + V_k^\delta)\)-Green function on \(U = B(y, 1)^c\), that is, for \(x, z \in \mathbb{R}^d\),

\[
g^\delta_{\lambda,k} (x, z) = \int_0^\infty e^{-\lambda s} p(s, x, z) E_{x,z}^\delta \left[ \exp \left\{ - \int_0^d V_k^\delta(Z_s) ds \right\} , s < H(y) \right] ds,
\]

see also formula (1.10).

**Lemma 3.2** With the above notations the following two statements are true:

\[
e_\lambda(x, y, \sigma^0, \hat{\omega}_k) - e_\lambda(x, y, \sigma^1, \hat{\omega}_k) = \int_{C_k} g^\delta_{\lambda,k} (x, z) (V_k^1 - V_k^0) (z) e_\lambda(z, y, \sigma^0, \hat{\omega}_k) dz
\]

and

\[
e_\lambda(x, y, \sigma^0, \hat{\omega}_k) - e_\lambda(x, y, \sigma^1, \hat{\omega}_k) = \int_{C_k} g^\delta_{\lambda,k} (x, z) (V_k^1 - V_k^0) (z) e_\lambda(z, y, \sigma^1, \hat{\omega}_k) dz.
\]

Proof of the lemma. By a classical differentiation and integration argument one has for \(w \in C(\mathbb{R}_+, \mathbb{R}^d)\), with \(H(y) < \infty\),

\[
\exp \left\{ - \int_0^{H(y)} (V_k^0 - V_k^1)(Z_s) ds \right\} = 1 + \int_0^{H(y)} (V_k^1 - V_k^0)(Z_s) \exp \left\{ - \int_s^{H(y)} (V_k^0 - V_k^1)(Z_u) du \right\} ds.
\]

To prove the first claim, we multiply both sides by \(\exp \left\{ - \int_0^{H(y)} (\lambda + V_k^1)(Z_s) ds \right\}\) and take the integration with respect to \(1_{\{H(y) < \infty\}} P_x\). Whereas for the second claim one has to exchange the role of \(\sigma^0\) and \(\sigma^1\). This finishes the proof of Lemma 3.2.
Now we are able to prove (3.6). Take $\sigma_k^0 \in D_{0,k}$. (In fact $D_{0,k}$ contains only one element.) Then
\[
E_k(0, y, \omega_k) = \inf_{\sigma_k^1 \in D_{1,k}} - \log \left( \frac{e_{\lambda}(0, y, \sigma_k^1, \omega_k)}{e_{\lambda}(0, y, \sigma_k^0, \omega_k)} \right) \geq 0.
\]
(3.10)

We want to find a good lower bound on the term on the right-hand side of the above equation. We take a fixed $\sigma_k^1 \in D_{1,k}$. Then by Lemma 3.1 and Lemma 3.2 we find
\[
- \log \left( \frac{e_{\lambda}(0, y, \sigma_k^1, \omega_k)}{e_{\lambda}(0, y, \sigma_k^0, \omega_k)} \right) = - \log \left( 1 + \tilde{E}_0^{k,0} \left[ \bar{H}_k \leq H(y), \frac{e_{\lambda}(Z_{\bar{H}_k}, y, \sigma_k^1, \omega_k)}{e_{\lambda}(Z_{\bar{H}_k}, y, \sigma_k^0, \omega_k)} - 1 \right] \right)
\]
\[
= - \log \left( 1 - \tilde{E}_0^{k,0} \left[ \bar{H}_k \leq H(y), \int_{\bar{C}_k} \frac{g_{\lambda,k}^{y,1}(Z_{\bar{H}_k}, z)(V_k^1 - V_k^0)(z)}{e_{\lambda}(Z_{\bar{H}_k}, y, \sigma_k^0, \omega_k)} dz \right] \right).
\]

Now, we have to distinguish whether $\bar{C}_k$ is a neighboring box of our goal or not: Choose $R$ minimal, such that $\bar{C}_k \subset B(lq_k, R)$. We say $\bar{C}_k$ is a neighboring box of $y \in \mathbb{R}^d$ if $y$ is contained in the closure of $B(lq_k, R + 2)$. Define
\[
N_y = \left\{ k \in \mathbb{N} ; \bar{C}_k \text{ is a neighboring box of } y \right\}.
\]

Of course the number of points contained in $N_y$ is bounded by a constant only depending on $\alpha, l$ and $d$.

First case, $k \in N_y$.
\[
- \log \left( \frac{e_{\lambda}(0, y, \sigma_k^1, \omega_k)}{e_{\lambda}(0, y, \sigma_k^0, \omega_k)} \right) \geq 0.
\]

Second case, $k \notin N_y$. From Harnack's inequality (see for instance [46] after (1.28)), we get, for $k \geq 1$,
\[
\inf_{z \in \bar{C}_k} \frac{e_{\lambda}(z, y)}{\sup_{z \in \bar{C}_k} e_{\lambda}(z, y)} \leq c_6(d, \lambda, M, \alpha).
\]
(3.11)

Thus, because $(V_k^1 - V_k^0)(z) \geq 0$ we see that
\[
\tilde{E}_0^{k,0} \left[ \bar{H}_k \leq H(y), \int_{\bar{C}_k} \frac{g_{\lambda,k}^{y,1}(Z_{\bar{H}_k}, z)(V_k^1 - V_k^0)(z)}{e_{\lambda}(Z_{\bar{H}_k}, y, \sigma_k^0, \omega_k)} dz \right]
\]
\[
\geq c_5 \tilde{E}_0^{k,0} \left[ \bar{H}_k \leq H(y), \int_{\bar{C}_k} \frac{g_{\lambda,k}^{y,1}(Z_{\bar{H}_k}, z)(V_k^1 - V_k^0)(z)}{e_{\lambda}(Z_{\bar{H}_k}, y, \sigma_k^0, \omega_k)} dz \right]
\]
\[
\geq c_5 \tilde{E}_0^{k,0} \left[ \bar{H}_k \leq H(y), \int_{\bar{C}_k} \frac{g_{\lambda,k}^{y,1}(Z_{\bar{H}_k}, z)(V_k^1 - V_k^0)(z)}{e_{\lambda}(Z_{\bar{H}_k}, y, \sigma_k^0, \omega_k)} dz \right].
\]
On \( k \notin N_y \), there exists a \( c_6 > 0 \), independent of \( k \), such that for all \( x \in \partial \tilde{C}_k \) and all \( \sigma_1^k \in D_{1,k} \)

\[
\int_{C_k} g_{\lambda,k}^{y^1}(x, z)(V_k^1 - V_k^0)(z)dz \geq \int_{C_k} g_{\lambda+M,B(y,1)}(x, z, \omega = 0)(V_k^1 - V_k^0)(z)dz \\
\geq \int_{C_k} g_{\lambda+M,B(y,1)}(x, z, \omega = 0)(V_k^1 - V_k^0)(z)dz \geq c_6.
\]

Therefore, we find

\[
\hat{E}_0^{k,0} \left[ \hat{H}_k \leq H(y), \int_{C_k} g_{\lambda,k}^{y^1}(Z, \sigma_k^0, \omega_k)(V_k^1 - V_k^0)(z)dz e_{\lambda}(z, y, \sigma_k^0, \omega_k)dz \right] \\
\geq c_7 \hat{P}_0^{k,0} \left[ \hat{H}_k \leq H(y) \right].
\]

If we insert this result into formula (3.10), we get

\[
E_k(0, y, \hat{\omega}_k) = \inf_{\sigma_k^0 \in D_{1,k}} \log \left( e_{\lambda}(0, y, \sigma_k^0, \hat{\omega}_k) \right) \\
\geq \inf_{\sigma_k^0 \in D_{1,k}} - \log \left( 1 - c_7 \hat{P}_0^{k,0} \left[ \hat{H}_k \leq H(y) \right] \right) \\
\geq c_7 \hat{P}_0^{k,0} \left[ \hat{H}_k \leq H(y) \right] \geq c_7 \hat{P}_0^{y} \left[ \hat{H}_k \leq H(y) \right] \\
\geq c_7 \hat{P}_0^{y} \left[ H_k \leq H(y) \right],
\]

where we have used the Lemma 3.3 which follows below. Thus, we find the following lower bound

\[
E_k(0, y, \hat{\omega}_k) \geq \begin{cases} 
0 & \text{if } k \in N_y \\
1 & \text{if } k \notin N_y
\end{cases}
(3.12)
\]

If \( A(y, \gamma) \) again denotes the event that the path of the Brownian motion starting in 0 with goal \( B(y, 1) \) does not leave the cylinder \( Z(y, r) \), we know, because we have taken \( \gamma > \xi \), that \( \mathbb{P}\text{-a.s. } \lim_{|y| \to \infty} \hat{P}_0[A(y, \gamma)] = 1 \).

Therefore,

\[
\sum_{k \in E_y} \mathbb{E}[E_k] \geq \sum_{k \in E_y \setminus N_y} \mathbb{E}[E_k] \\
\geq c_7 \mathbb{E} \left[ \sum_{k \in E_y \setminus N_y} \hat{P}_0^{y} \left[ H_k \leq H(y) \right] \right] \\
\geq c_7 \mathbb{E} \left[ \hat{P}_0^{y} \left[ \sum_{k \in E_y \setminus N_y} 1_{H_k \leq H(y)} \cdot \mathbb{1}(A(y, \gamma)) \right] \right] \\
\geq c_7 \mathbb{E} \left[ \hat{P}_0^{y}[A(y, \gamma)] \frac{|y|}{2\sqrt{d}} \right] \quad \text{for all large } |y|.
\]
So the claim (3.6) follows by the lemma of Fatou, this completes the proof of our theorem.

\[\square\]

**Lemma 3.3** Take \(y \in \mathbb{R}^d, k \geq 1, \omega \in \Omega, \sigma_k^0 \in D_{0,k} \) and \(\sigma_k^1 \in D_{1,k}.\) Then we have

\[\hat{P}_0^{k,0} \left[ \hat{H}_k \leq H(y) \right] \geq \hat{P}_0^{k,1} \left[ \hat{H}_k \leq H(y) \right] \tag{3.13}\]

and

\[\hat{P}_0^{k,0} \left[ \hat{H}_k \leq H(y) \right] \geq \hat{P}_0^{y} \left[ \hat{H}_k \leq H(y) \right]. \tag{3.14}\]

**Proof.** We know by (3.8) that \(V_k^0(\hat{\omega}_k) \leq V_k^1(\hat{\omega}_k)\) and also \(V_k^0(\hat{\omega}_k) \leq V(\omega).\) Therefore it suffices to prove the first claim, the second one then follows analogously. Let us define for \(\delta = 0, 1, w \in C(\mathbb{R}_+, \mathbb{R}^d),\)

\[f(\hat{\omega}_k, \sigma_k^\delta, w) = \exp \left\{ - \int_{0}^{H(y)} (\lambda + V_k^\delta)(Z_s, \omega) ds \right\} \cdot 1_{\{H(y) < \infty\}},\]

and observe that

\[f(\hat{\omega}_k, \sigma_k^1, w) \cdot 1_{\{\hat{H}_k > H(y)\}} = f(\hat{\omega}_k, \sigma_k^0, w) \cdot 1_{\{\hat{H}_k > H(y)\}} \tag{3.15}\]

and

\[f(\hat{\omega}_k, \sigma_k^1, w) \cdot 1_{\{\hat{H}_k \leq H(y)\}} \leq f(\hat{\omega}_k, \sigma_k^0, w) \cdot 1_{\{\hat{H}_k \leq H(y)\}}. \tag{3.16}\]

Therefore by (3.15) and (3.16) we get

\[\hat{P}_0^{k,1} \left[ \hat{H}_k \leq H(y) \right] = \frac{E_0 \left[ f(\hat{\omega}_k, \sigma_k^1, w) \cdot 1_{\{\hat{H}_k \leq H(y)\}} \right]}{E_0 \left[ f(\hat{\omega}_k, \sigma_k^0, w) \cdot \left(1_{\{\hat{H}_k \leq H(y)\}} + 1_{\{\hat{H}_k > H(y)\}} \right) \right]}
\leq \frac{E_0 \left[ f(\hat{\omega}_k, \sigma_k^0, w) \cdot 1_{\{\hat{H}_k \leq H(y)\}} \right]}{E_0 \left[ f(\hat{\omega}_k, \sigma_k^0, w) \cdot 1_{\{\hat{H}_k \leq H(y)\}} \right] + E_0 \left[ f(\hat{\omega}_k, \sigma_k^1, w) \cdot 1_{\{\hat{H}_k > H(y)\}} \right]}
= \hat{P}_0^{k,0} \left[ H_k \leq H(y) \right].\]

This completes the proof of the lemma.

\[\square\]
4 PROOF OF THE SUBDIFFUSIVE LOWER BOUND OF $\xi_0$

The strategy of the proof of Theorem 1.3 is an extension of the arguments used in the proof of Theorem 1.2. For a similar result in first-passage percolation we want to refer to Section 3 of [30]. The idea of the proof is to compare $-\log e_\lambda(y, y)$ for the two different starting points 0 and $m$ and the two different endpoints $y$ and $y + m$. One easily gets an upper bound on the variance of the difference of these two “passage times”: If the distance between 0 and $m$ is of order $|y|^\gamma$ then $\text{Var}(-\log e_\lambda(0, y) + \log e_\lambda(m, y + m)) = O(|y|^{2\gamma})$.

On the other hand, we will get, by the methods presented in the preceding chapter, a lower bound of the form $c|y|^{1-(d-1)}$. So if we choose $\gamma$, the power of the cylinder radius, smaller than $1/(d + 1)$ we get a contradiction, to the two bounds mentioned above, this leads us to the claim of Theorem 1.3.

**Proof of Theorem 1.3.** Assume that for a fixed $\gamma \in [0, 1]$, there exists a sequence $(y_n)_n \subset \mathbb{R}^d$ with $|y_n| \to \infty$ such that

$$
\mathbb{E} \left[ \hat{P}_{y_n}^0 \left[ A(y_n, \gamma) \right] \right] \to 1 \quad \text{as } n \to \infty. \quad (4.1)
$$

In fact, by rotation invariance of the obstacles, we can and will choose the sequence $(y_n)_n$ such that $y_n = (|y_n|, 0, \ldots, 0)$ for all $n$. We want to show that under these assumptions $\gamma \geq 1/(d + 1)$.

Define, for $\omega \in \Omega$, $n \geq 1$, the difference of an “upper” and a “lower” passage time as follows

$$
\delta \log e_n(\omega) = -\log e_\lambda(0, y_n, \omega) - (-\log e_\lambda(m_n, y_n + m_n, \omega)),
$$

where we choose $m_n \in \mathbb{R}^d$ as follows: $m_n = (0, |m_n|, 0, \ldots, 0)$ points into the direction of the second coordinate axis, and $|m_n|$ is minimal such that the two cylinders $Z(y_n, |y_n|^\gamma + \sqrt{d}l)$ and $Z(y_n, |y_n|^\gamma + \sqrt{dl}) + m_n$ are disjoint. We remark that $|m_n| \leq c_1|y_n|^{\gamma}$ for all large $n$ because of our choice of the sequence $(y_n)_n$. Using the strong Markov property, we see that

$$
e_\lambda(0, y_n) \geq e_\lambda(0, m_n) \inf_{z \in B(m_n, 1)} e_\lambda(z, y_n + m_n) \inf_{z \in B(y_n + m_n, 1)} e_\lambda(z, y_n).
$$

By Harnack’s inequality (see (3.11)), one gets

$$
-\log \left( \frac{e_\lambda(0, y_n)}{e_\lambda(m_n, y_n + m_n)} \right) \leq c_2 - \log e_\lambda(0, m_n) - \log e_\lambda(y_n + m_n, y_n),
$$

analogously, by exchanging the role of the passage times,

$$
-\log \left( \frac{e_\lambda(0, y_n)}{e_\lambda(m_n, y_n + m_n)} \right) \geq -c_2 + \log e_\lambda(m_n, 0) - \log e_\lambda(y_n, y_n + m_n).
$$
Therefore the following estimate on $|\delta \log e_n|$ holds,

$$
- \log \left( \frac{e_\lambda(0, y_n)}{e_\lambda(m_n, y_n + m_n)} \right) \leq c_2 - \log e_\lambda(0, m_n) - \log e_\lambda(y_n + m_n, y_n) - \log e_\lambda(m_n, 0) - \log e_\lambda(y_n, y_n + m_n).
$$

In view of (1.12), we find a suitable constant $c_3 \in (0, \infty)$ such that for all large $n$

$$
\text{Var}(\delta \log e_n) \leq c_3 |m_n|^2 \leq c_1^2 c_3 |y_n|^{2\gamma}.
$$

(4.2)

Hence, we have found the desired upper bound on the variance of the difference of these two passage times. During the rest of the proof we try to show that one gets the following lower bound on the same variance:

$$
\text{Var}(\delta \log e_n) \geq c_4 |y_n|^{1-(d-1)\gamma}.
$$

(4.3)

If we have verified those two bounds, we see that $2\gamma \geq 1 - (d-1)\gamma$ from which our claim, $\gamma \geq 1/(d+1)$, follows. It remains to find the lower bound (4.3) on the variance of the difference of the two passage times.

With the same notations as in the previous chapters, we get by martingale identities as before

$$
\text{Var}(\delta \log e_n) \geq \sum_{k \geq 1} \text{Var}(\mathbb{E}[\delta \log e_n | G_k]).
$$

(4.4)

So we are again interested in a lower bound for $\text{Var}(\mathbb{E}[\delta \log e_n | G_k])$. We introduce the same events $D_{0,k}, D_{1,k}, D^0_k$ and $D^1_k$ on our cubes $C_k$ as in Chapter 3. We also keep the notation for the decomposition of $\omega = (\omega_k, \hat{\omega}_k) \in \Omega$ on the cube $C_k$ and on its complement.

In view of Lemma 3 in [35], we find

$$
\text{Var}(\mathbb{E}[\delta \log e_n | G_k]) \geq \frac{pq}{p+q} (x_1 - x_0)^2,
$$

where $p = \mathbb{P}[D^0_k] > 0$, $q = \mathbb{P}[D^1_k] > 0$ and

$$
x_\delta = \mathbb{E} \left[ \mathbb{E}[\delta \log e_n | G_k] 1_{D^\delta_k} \right] \mathbb{P} [D^\delta_k],
$$

for $\delta = 0$ or 1.

We have to find a "good" lower bound on $x_1 - x_0$. For $\omega \in \Omega$, we define the random variable

$$
G_k(\hat{\omega}_k) = \inf_{\sigma^1_k \in D_{1,k}} \delta \log e_n(\sigma^1_k, \hat{\omega}_k) - \sup_{\sigma^0_k \in D_{0,k}} \delta \log e_n(\sigma^0_k, \hat{\omega}_k).
$$

The expression $G_k(\hat{\omega}_k)$ is a lower bound on the difference of the two passage times.

Therefore, we have

$$
\text{Var}(\delta \log e_n) \geq c_4 |y_n|^{1-(d-1)\gamma}.
$$

(4.3)
As in the preceding chapter, we see that $x_1 - x_0 \geq \mathbb{E}[G_k]$, thus
$$|x_1 - x_0| \geq (\mathbb{E}[G_k])_+.$$ Further, we define $\mathcal{E}_n = \{k \in \mathbb{N}, C_k \cap Z(y_n, \gamma) \neq \emptyset\}$ to be the cubes that intersect our cylinder $Z(y_n, \gamma)$. Therefore, we find with the Cauchy-Schwarz inequality
$$\mathbb{V} \mathbb{a} \mathbb{r}(\delta \log e_n) \geq \frac{pq}{p + q} \sum_{k \geq 1} (\mathbb{E}[G_k])^2$$
$$\geq \frac{pq}{p + q} \sum_{k \in \mathcal{E}_n} (\mathbb{E}[G_k])^2$$
$$\geq \frac{pq}{p + q} \left( \sum_{k \in \mathcal{E}_n} \mathbb{E}[G_k] \right)^2$$
$$\geq c_5 \left| y_n \right|^{1-(d-1)\gamma} \left( \frac{1}{|y_n|} \sum_{k \in \mathcal{E}_n} \mathbb{E}[G_k] \right)^2.$$ (4.5)

To prove (4.3) it remains to show that
$$\lim \inf_{n \to \infty} \frac{1}{|y_n|} \sum_{k \in \mathcal{E}_n} \mathbb{E}[G_k] > 0.$$ (4.6)

We denote the only element in $D_{0,k}$ by $\sigma_k^0$. For $\omega \in \Omega$ is
$$G_k(\omega_k) \geq \inf_{\sigma_k^1 \in D_{1,k}} \log \frac{e_\lambda(0, y_n, \sigma_k^0, \omega_k)}{e_\lambda(0, y_n, \sigma_k^1, \omega_k)} + \inf_{\sigma_k^2 \in D_{1,k}} \log \frac{e_\lambda(m_n, y_n + m_n, \sigma_k^0, \omega_k)}{e_\lambda(m_n, y_n + m_n, \sigma_k^1, \omega_k)}.$$ Thus, it suffices to prove
$$\lim \inf_{n \to \infty} \frac{1}{|y_n|} \sum_{k \in \mathcal{E}_n} \mathbb{E} \left[ \inf_{\sigma_k^1 \in D_{1,k}} \log \frac{e_\lambda(0, y_n, \sigma_k^0, \omega_k)}{e_\lambda(0, y_n, \sigma_k^1, \omega_k)} \right] > 0$$ (4.7)
and
$$\lim \sup_{n \to \infty} \frac{1}{|y_n|} \sum_{k \in \mathcal{E}_n} \mathbb{E} \left[ \sup_{\sigma_k^1 \in D_{1,k}} \log \frac{e_\lambda(m_n, y_n + m_n, \sigma_k^0, \omega_k)}{e_\lambda(m_n, y_n + m_n, \sigma_k^1, \omega_k)} \right] = 0.$$ (4.8)

The proof of claim (4.7) follows exactly in the same way as the proof of (3.6). So we want to show (4.8): By Lemma 3.1 and Lemma 3.2 follows, for $\sigma_k^1 \in D_{1,k}$,
$$\log \left( \frac{e_\lambda(m_n, y_n + m_n, \sigma_k^0, \omega_k)}{e_\lambda(m_n, y_n + m_n, \sigma_k^1, \omega_k)} \right)$$
$$- \log \left( 1 + \mathbb{E}_{m_n}^{k,1} \left[ \hat{H}_k \leq \mathcal{H}(y_n + m_n), \int_{\partial \hat{H}_k} g_{y_n + m_n, 0}(Z_{\tilde{H}_k}, z)(V_k^1 - V_k^0)(z) e_\lambda(z, y_n + m_n, \sigma_k^1, \omega_k) d\mathcal{H} \right] \right);$$
for all \( k \in \mathcal{E}_n \), \( \tilde{C}_k \) is not a neighboring box of our goal \( y_n + m_n \), so we can use Harnack’s inequality. Therefore the last member of the above equality is smaller than
\[
\leq \log \left(1 + c_\alpha M \hat{E}_{\alpha,1}^{k,1} \left[ \tilde{H}_k \leq H(y_n + m_n), \int_{\tilde{C}_k} g_{\alpha,1}^{y_n + m_n,0}(z)dz \right] \right).
\]
In the case \( d \geq 3 \) or \( \lambda > 0 \), \( g_{\alpha,1}^{y_n + m_n,0}(\cdot, \cdot) \) is smaller than \( g_{\alpha,1}^{y_n + m_n}(\cdot, \cdot, \omega = 0) \) the \( \lambda \)-Green function for Brownian motion, so the last expression is smaller than
\[
\leq \log \left(1 + c_\gamma \hat{P}_{\alpha,1}^{k,1} \left[ \tilde{H}_k \leq H(y_n + m_n) \right] \right) \leq c_\gamma \hat{P}_{\alpha,1}^{k,1} \left[ \tilde{H}_k \leq H(y_n + m_n) \right].
\]
For \( k \in \mathcal{E}_n \), we get with the above considerations, using the independence of the Poissonian process and Lemma 3.3
\[
\mathbb{E} \left[ \sup_{\sigma_k \in D_{1,k}} \log \frac{e_\alpha(m_n, y_n + m_n, \sigma_1^k, \sigma_2^k, \omega_k)}{e_\alpha(m_n, y_n + m_n, \sigma_1^k, \sigma_2^k, \omega_k)} \right] \leq c_\gamma \mathbb{E} \left[ \sup_{\sigma_k \in D_{1,k}} \hat{P}_{\alpha,1}^{k,1} \left[ \tilde{H}_k \leq H(y_n + m_n) \right] \right]
\leq c_\gamma \mathbb{E} \left[ \hat{P}_{\alpha,1}^{k,0} \left[ \tilde{H}_k \leq H(y_n + m_n) \right] 1_{D_k^0} \right]
\leq c_\gamma \mathbb{E} \left[ \hat{P}_{\alpha,1}^{k,0} \left[ \tilde{H}_k \leq H(y_n + m_n) \right] \right]
\equiv c_\gamma \mathbb{E} \left[ \hat{P}_{\alpha,1}^{k,0} \left[ \tilde{H}_k \leq H(y_n + m_n) \right] \right],
\]
Therefore to prove claim (4.8) it suffices to show that
\[
\lim_{n \to \infty} \frac{1}{|y_n|} \sum_{k \in \mathcal{E}_n} \mathbb{E} \left[ \hat{P}_{\alpha,1}^{k,0} \left[ \tilde{H}_k \leq H(y_n + m_n) \right] \right] = 0. \tag{4.9}
\]
We consider now the random lattice animal \( \mathcal{A} = \{ k \in \mathbb{N} ; \tilde{H}_k \leq H(y_n + m_n) \} \). By the Cauchy-Schwarz inequality and the fact that the distance between the two disjoint cylinders \( Z(y_n, \gamma) \) and \( Z(y_n, \gamma) + m_n \) is \( 2d \ell \), we find
\[
\frac{1}{|y_n|} \sum_{k \in \mathcal{E}_n} \mathbb{E} \left[ \hat{P}_{\alpha,1}^{k,0} \left[ \tilde{H}_k \leq H(y_n + m_n) \right] \right]
\leq \mathbb{E} \left[ \frac{1}{|y_n|} \hat{E}_{\alpha,1}^{y_n, m_n} \left[ \# \{ k \in \mathcal{E}_n ; \tilde{H}_k < H(y_n + m_n) \} \right] \right]
\leq 3^d \mathbb{E} \left[ \frac{1}{|y_n|} \hat{E}_{\alpha,1}^{y_n, m_n} \left[ |\mathcal{A}| \cdot 1_{A_m(y_n, \gamma)^c} \right] \right]
\leq 3^d \mathbb{E} \left[ \hat{E}_{\alpha,1}^{y_n, m_n} \left[ |\mathcal{A}|^2 / |y_n|^2 \right] \right]^{1/2} \mathbb{E} \left[ \hat{P}_{\alpha,1}^{y_n, m_n} \left[ A_m(y_n, \gamma)^c \right] \right]^{1/2}, \tag{4.10}
\]
where \( A_m(y_n, \gamma) \) denotes the event that the paths of Brownian motion starting in \( m_n \) with goal \( B(y_n + m_n, 1) \) do not leave the cylinder \( Z(y_n, \gamma) + m_n \). By translation invariance
is $\mathbb{E} \left[ \hat{P}_{m_n}^{y_n + m_n} \left| A_{m_n} (y_n, \gamma) \right. \right] = \mathbb{E} \left[ \hat{P}_{0_n}^{y_n} \left| A(y_n, \gamma) \right. \right]$, which tends to 0 as $n$ goes to infinity. Therefore it remains to show that the first term on the right-hand side of inequality (4.10) stays bounded for all large $n$. By Jensen’s inequality, Cauchy-Schwarz inequality and with the estimates (1.12) and (1.16) (using translation invariance), we see that for a suitable constant $c_8$:

\[
\mathbb{E} \left[ \hat{E}_{m_n}^{y_n + m_n} \left| \frac{|A|^2}{|y_n|^2} \right. \right] \leq c_8 \mathbb{E} \left[ \hat{E}_{m_n}^{y_n + m_n} \left| \exp \left\{ \frac{\gamma_3}{2} \left| A \right| \right\} \right. \right] \leq c_8 \left( \mathbb{E} \left[ \frac{1}{e^{\lambda(m_n, y_n + m_n)^2}} \right] \right)^{1/2}.
\]

The above expression is bounded, this completes the proof.

\[\square\]

# A MEASURABILITY

In this appendix we prove Lemma 2.1:

**Lemma 2.1.** For $\lambda \geq 0$ and $\gamma > 0$, the functions $(y, \omega) \rightarrow e_\lambda(0, y, \omega)$ and $(y, \omega) \rightarrow \hat{P}_{0}^{y} [A(y, \gamma)]$ are measurable in $\omega$ and for all $|y| > 1$ continuous in $y$.

**Proof.** For the measurability see Lemma 1.1 of [46]. Let us prove the continuity of $e_\lambda(0, y, \omega)$ in $y$. Define for $y, y' \in \mathbb{R}^d$, $w \in C(\mathbb{R}, \mathbb{R}^d)$

\[f(y, y', w) = \exp \left\{ - \int_0^{H(y)} (\lambda + V)(Z_s, \omega) ds \right\} \cdot 1_{\{H(y) < \infty\}}(w)\]

\[\quad - \exp \left\{ - \int_0^{H(y')} (\lambda + V)(Z_s, \omega) ds \right\} \cdot 1_{\{H(y') < \infty\}}(w).\]
First we choose \( w \in \{ H(y') \leq H(y) \} \):

\[
|f(y, y', w) \cdot 1_{\{H(y') \leq H(y)\}} |
\leq (1 - \exp \{- (\lambda + M)(H(y) - H(y'))\}) \cdot 1_{\{H(y) < \infty, H(y') \leq H(y)\}} + 1_{\{H(y) = \infty, H(y') < \infty\}}. \tag{A.2}
\]

Therefore, if we take the expectation with respect to the Brownian path measure \( P_0 \), we get the following upper estimate

\[
|E_0 [f(y, y', w) \cdot 1_{\{H(y') \leq H(y)\}}] |
\leq E_0 [(1 - \exp \{- (\lambda + M)(H(y) - H(y'))\}) \cdot 1_{\{H(y) < \infty, H(y') \leq H(y)\}}] + P_0 [H(y) = \infty, H(y') < \infty]. \tag{A.3}
\]

The second term on the right-hand side of (A.3) is zero for \( d = 2 \) and it tends to zero as \(|y' - y| \to 0\) in dimensions \( d \geq 3\). So we want to focus on the first term on the right-hand side of (A.3).

For all \( \varepsilon > 0 \) there exists a small \( b = b(\varepsilon) > 0 \) such that \( \exp \{- (\lambda + M)b\} \geq 1 - \varepsilon/8 \).

Thus, by the strong Markov property, there exists a \( \delta_1 > 0 \) such that

\[
E_0 [(1 - \exp \{- (\lambda + M)(H(y) - H(y'))\}) \cdot 1_{\{H(y) < \infty, H(y') \leq H(y)\}}] \\
\leq 1 - \exp \{- (\lambda + M)b\} + P_0 [H(y) < \infty, H(y') < H(y), H(y) - H(y') > b] \tag{A.4}
\]

\[
\leq \varepsilon/8 + \sup_{x \in \partial B(y',1)} P_x [b < H(y) < \infty] < \varepsilon/4 \quad \text{for all } y' \text{ with } |y' - y| \leq \delta_1.
\]

Of course, we have symmetry in \( y \) and \( y' \), so the same estimates hold in the case \( w \in \{ H(y) > H(y') \} \). The continuity of \( e_\lambda(0, \cdot, \omega) \) now easily follows.

To see the continuity of \( \hat{P}_0^w [A(y, \gamma)] \) in \( y \), define for \( y \in \mathbb{R}^d, w \in C(\mathbb{R}_+, \mathbb{R}^d) \),

\[
g(w, y) = \exp \left\{- \int_0^{H(y)} (\lambda + V)(Z_s, \omega) ds \right\} \cdot 1_{\{H(y) < \infty\}}(w) \tag{A.5}
\]

and

\[
h(w, y) = 1_{A(y, \gamma)} \cdot 1_{\{H(y) < \infty\}}(w). \tag{A.6}
\]

Using the continuity of \( e_\lambda(0, \cdot, \omega) \) and the tubular estimate (1.12), we see that it suffices to show that \( E_0 [g(\omega, \cdot)h(\omega, \cdot)] \) is continuous. For \( y, y' \in \mathbb{R}^d, \omega \in \Omega \), we have, using the triangle inequality,

\[
|E_0 [g(\omega, y)h(\omega, y)] - E_0 [g(\omega, y')h(\omega, y')]| \leq |E_0 [(g(w, y) - g(w, y')) h(w, y)]| + |E_0 [g(w, y') (h(w, y) - h(w, y'))]|. \tag{A.7}
\]
The first term on the right-hand side of (A.7) tends to zero as \(|y - y'| \to 0\), using the same observations as done in the proof of the continuity of \(e_1(0, y, \omega)\). So we want to focus on the second term on the right-hand side of (A.7). First we choose \(w \in \{H(y') < H(y)\}\), then we have

\[
|E_0 [g(w, y') (h(w, y) - h(w, y')) \cdot 1_{\{H(y') < H(y)\}}]| \\
\leq E_0 [|h(w, y) - h(w, y')| \cdot 1_{\{H(y') < H(y) < \infty\}}] + P_0 [H(y') < \infty, H(y) = \infty].
\]

For the second term on the right-hand side of (A.8) we use the same remark as after (A.3), whereas for the first term we see, using the strong Markov property, that it tends to zero as \(|y - y'| \to 0\). This finishes the proof of the lemma.

\(\square\)

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Scaling identity for crossing Brownian motion in a Poissonian potential

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Abstract

We consider $d$-dimensional Brownian motion in a truncated Poissonian potential $(d \geq 2)$. If Brownian motion starts at the origin and ends in the closed ball with center $y$ and radius 1, then the transverse fluctuation of the path is expected to be of order $|y|^\zeta$, whereas the distance fluctuation is of order $|y|^\chi$. Physics literature tells us that $\zeta$ and $\chi$ should satisfy a scaling identity $2\zeta - 1 = \chi$. We give here rigorous results for this conjecture.

0 INTRODUCTION AND RESULTS

In the present work we consider crossing Brownian motion in dimensions $d \geq 2$, evolving in a truncated Poissonian potential. Crossing Brownian motion is Brownian motion conditioned to reach a remote location. This model has been described in different articles of Sznitman, see e.g. [46]. Here, we want to study the relation between critical exponents for transverse fluctuations $\zeta$ and distance fluctuations $\chi$. Physical literature tells us that in many related models of growing interfaces $\zeta$ and $\chi$ should satisfy the scaling identity $2\zeta - 1 = \chi$ (see Krug-Spohn [29]). In mathematical literature one does not have many rigorously proven statements in that direction. In first-passage percolation Newman-Piza [35] have proved the statement $2\zeta_x \leq 1 \leq \chi$ (see Theorem 6 in [35]), which is only valid for certain well-behaved directions $\hat{x}$ (the definition of transverse fluctuation is also direction-dependent). In our model we prove that $2\zeta - 1 \leq \chi$ independent of directions (see Theorem 0.2 below). The new result in this paper is that we are able to provide a lower bound on $\zeta$ in terms of distance fluctuations (see Theorem 0.3 below). To my knowledge there is no similar result proven in any other related mathematical model.
Let us precisely describe the setting. For \( x \in \mathbb{R}^d \) we denote by \( P_x \) the Wiener measure on \( C(\mathbb{R}_+, \mathbb{R}^d) \) starting at \( x \), \( Z \) will denote the canonical process on \( C(\mathbb{R}_1, \mathbb{R}^d) \). We denote by \( \mathbb{P} \) the Poissonian law with fixed intensity \( \nu > 0 \) on the space \( \Omega \) of locally finite, simple, pure point measures on \( \mathbb{R}^d \). For \( M > 0 \), \( \omega = \sum_i \delta_{x_i} \in \Omega \) and \( x \in \mathbb{R}^d \) the truncated potential is defined as

\[
V(x, \omega) = \left( \sum_i W(x - x_i) \right) \wedge M = \left( \int_{\mathbb{R}^d} W(x - y) \omega(dy) \right) \wedge M,
\]

where the shape function \( W(\cdot) \geq 0 \) is bounded, measurable, compactly supported and not a.e. equal to zero. Furthermore we assume that

\[
W(\cdot) \text{ is rotationally invariant.}
\]

For \( \omega \in \Omega, \lambda \geq 0, x, y \in \mathbb{R}^d \), crossing Brownian motion in a Poissonian potential from \( x \) to \( y \) on \( C(\mathbb{R}_+, \mathbb{R}^d) \) is then defined by

\[
\hat{P}^y_x = \frac{1}{e_{\lambda}(x, y, \omega)} \exp \left\{ - \int_0^{H(y)} (\lambda + V)(Z_s, \omega) \, ds \right\} 1_{\{H(y) < \infty\}} P_x,
\]

where \( H(y) \) is the entrance time of \( Z \) into the closed Euclidean ball \( B(y, 1) \) and where \( e_{\lambda}(x, y, \omega) \) is the normalizing constant.

We define the critical exponent \( \xi_1 \) for transverse fluctuation as follows: We take \( y \) a non zero vector in \( \mathbb{R}^d \). Let \( l_y \) be the axis \( \{\alpha y; \alpha \in \mathbb{R}\} \). \( C(y, \gamma) \) is then the truncated cylinder with axis \( l_y \) and radius \( |y|\gamma \):

\[
C(y, \gamma) = \{ z \in \mathbb{R}^d; \text{dist}(l_y, z) \leq |y|\gamma \text{ and } -|y|\gamma \leq \langle z, y/|y| \rangle \leq |y| + |y|\gamma \},
\]

where \( \text{dist}(\cdot, \cdot) \) is the Euclidean distance and \( \langle \cdot, \cdot \rangle \) denotes the usual scalar product on \( \mathbb{R}^d \). The truncation of the ends of the cylinder will have a technical purpose during the calculations. \( A(y, \gamma) \) will be the event that the crossing Brownian motion does not leave the cylinder \( C(y, \gamma) \), i.e.,

\[
A(y, \gamma) = \{ w \in C(\mathbb{R}_+, \mathbb{R}^d); Z_s(w) \in C(y, \gamma) \text{ for all } s \leq H(y) \}.
\]

We then define the critical exponent for transverse fluctuation as follows

\[
\xi_1 = \inf \left\{ \gamma \geq 0; \lim_{y \to \infty} \mathbb{E} \left[ \hat{P}^y_x [A(y, \gamma)] \right] = 1 \right\}.
\]
From [53] Theorems 1.1 and 1.3 we know that

\begin{align}
\text{for all } d \geq 2, \quad \xi_1 &\leq 3/4, \\
\text{if } d \geq 3 \text{ or } \lambda > 0, \quad \xi_1 &\geq 1/(d + 1).
\end{align}

(0.7) \quad (0.8)

Our main goal is to improve these bounds, i.e., to get upper and lower bounds in terms of distance fluctuations. To define distance fluctuations we consider the following object, for \( \lambda \geq 0, \omega \in \Omega \) and \( x, y \in \mathbb{R}^d \) we define

\[ d_\lambda(x, y, \omega) = \max \left\{ - \inf_{B(x, 1)} \log e_\lambda(\cdot, y, \omega), - \inf_{B(y, 1)} \log e_\lambda(\cdot, x, \omega) \right\}. \]

(0.9)

Under mild conditions (see (1.7) of Sznitman [49]) \( d_\lambda(\cdot, \cdot, \omega) \) is a distance function on \( \mathbb{R}^d \). This distance measures the cost of linking two points with respect to the Poissonian cloud \( \omega \). The critical exponent for distance fluctuations is then defined via

\[ \chi^{(1)} = \inf \left\{ \kappa \geq 0; \lim_{y \to \infty} \mathbb{P} \left[ |d_\lambda(0, y) - M_\lambda(0, y)| \leq |y|^\kappa \right] = 1 \right\}, \]

(0.10)

where \( M_\lambda(0, y) \) is a median of \( d_\lambda(0, y) \) (see also (2.6) below). In two parts of the proofs we require more uniformity for different directions and different distances, therefore we have to define a second critical exponent measuring distance fluctuations:

\[ \chi^{(2)} = \inf \left\{ \kappa \geq 0; \lim_{r \to \infty} \mathbb{P} \left[ \sup_{x \in B(0, r)} |d_\lambda(0, x) - M_\lambda(0, x)| \leq r^\kappa \right] = 1 \right\}. \]

(0.11)

It is an open question whether \( \chi^{(1)} = \chi^{(2)} \), but we have at least the following relations:

**Proposition 0.1** For all \( d \geq 2 \),

\[ \chi^{(1)} < \chi^{(2)} < 1/2. \]

(0.12)

It is more involved to obtain a lower bound for distance fluctuations. Nevertheless we provide a simple approach to a non-trivial lower bound combining already known results and Theorem 0.2 (see Corollary 3.1 below). Coming back to the scaling identity we first state "the upper bound":

**Theorem 0.2** For all \( d \geq 2 \),

\[ \xi_1 \leq \frac{\chi^{(2)}}{2} + 1. \]

(0.13)
We remark that in [53] Theorem 1.1 we have proved a numerical upper bound on transverse fluctuation (see (0.7)) which is based on combining (0.12) and (0.13), but the result we obtain here is stronger and more general.

Our main result in this paper is the following lower bound:

**Theorem 0.3** \(\text{For all } d \geq 2,\)

\[
\xi_1 \geq \frac{\chi^{(1)} + \chi^{(2)}}{2}. \tag{0.14}
\]

If we managed to prove the identity \(\chi^{(1)} = \chi^{(2)}\) it would follow that the conjectured scaling identity \(2\xi_1 - 1 = \chi^{(2)}\) holds true.

Let us briefly describe how this article is organised and how the main ideas of the proofs work.

In Section 1 we prove Theorem 0.3. The proof uses the same geometric construction as Theorem 1 of [30] (see also (4.2) of [53]), but the analysis of the proof is quite different: For \(y - (|y|, 0, \ldots, 0), \gamma > \xi_1,\) we consider two parallel disjoint cylinders \(C(y, \gamma)\) and \(C(y, \gamma) \parallel m\) with \(m = (0, 3|y|^{\gamma} + c, 0, \ldots, 0).\) In (4.2) of [53] we have proved that \(|-\log e_\gamma(0, y) + \log e_\gamma(m, m + y)| \leq c|y|^{\gamma}\) via a non-probabilistic argument using the fact that \(\text{dist}(0, m) \leq c|y|^{\gamma}.\) The main step here is to improve this upper bound by considering "better" paths for the crossings. We will compare \(d_\gamma(0, y)\) and \(d_\gamma(m, m + y)\) to their medians \(M_\gamma(0, y) = M_\gamma(m, m + y).\) By our choice of \(m,\) \(d_\gamma(0, y)\) and \(d_\gamma(m, m + y)\) are nearly independent. So for \(\kappa < \chi^{(1)}\) we manage to find a set of cloud configurations \(\omega\) with positive \(\mathbb{P}\)-probability such that \(d_\gamma(0, y, \omega) \geq M_\gamma(0, y)\) and \(d_\gamma(m, m + y, \omega) < M_\gamma(0, y) - |y|^\kappa,\) i.e., the costs (distances) are much smaller to pass the upper cylinder than the lower cylinder for these \(\omega\)'s (see Lemma 1.3). For such cloud configurations we analyze the tendency for crossings in \(d_\gamma(0, y)\) to pass through the upper cylinder, i.e., on the one hand we have to do a detour to walk to the upper cylinder but on the other hand we gain passing the upper cylinder because the cloud configuration is more favorable there (see Figure 1 and Lemma 1.4). But walking through the upper cylinder stands in contrast to the fact that for \(\gamma > \xi_1\) the path stays in the lower cylinder with large \(\mathbb{P}\)-probability. This will lead to the claimed relation between the critical exponents. Let us point out that considering two (nearly) disjoint cylinders is also a key step in slightly different models to obtain quantitative lower bounds on transverse fluctuation (see Licea-Newman-Piza [30] and Withrich [53, 56]).

In Section 2 we prove Theorem 0.2. The proof uses essentially the same geometric construction which has been used to prove Theorem 1.1 of [53]. We consider the event that the path leaves the cylinder \(C(y, \gamma)\) at \(z \in \partial C(y, \gamma).\) Due to the fact that we can compare \(M_\gamma(0, y)\) to the Euclidean norm we get an upper bound on the probability of
this event (via triangle inequalities). Choosing $\kappa > \chi^{(2)}$ and $\gamma > \frac{(\kappa + 1)}{2}$ the probability to leave $C(y, \gamma)$ at $z \in \partial C(y, \gamma)$ will be less than $c \exp\left\{c'|y|^\kappa - c''|y|^{2\gamma - 1}\right\}$, which tends to zero as $|y| \to \infty$ by our choice of $\kappa$ and $\gamma$. Thus we see that this choice of $\gamma$ enforces $\gamma \geq \xi_1$, which proves the claim of Theorem 0.2.

In the first part of Section 3 we give the proof of Proposition 0.1. It will use Corollary 3.5 of Sznitman [49]. In the second part of Section 3 we state and prove a non-trivial lower bound on distance fluctuations (see Corollary 3.1 below).

1 PROOF OF THE LOWER BOUND

In this section we want to prove "the lower bound" of the scaling identity. We start with some generalities. For the normalizing constant $e_\lambda(0, y, \omega)$ we have the following lower bound, using a tubular estimate on Brownian motion (Girsanov’s formula, see e.g. (1.35) of [49]), there exist constants $c_1 \in (0, 1)$ and $c_2 > 0$ such that for $x, y \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$e_\lambda(x, y, \omega) \geq c_1 \exp\{-c_2|x - y|\}. \quad (1.1)$$

Furthermore we have Harnack’s inequality (see for instance [46] after (1.28)), telling how to compare $e_\lambda(\cdot, y, \omega)$ for two different starting points. There exists $c_3 \in (0, \infty)$ such that for $|y| \geq 3$ and $\omega \in \Omega$ we have

$$\frac{\sup_{x \in B(0, 1)} e_\lambda(x, y, \omega)}{\inf_{x \in \mathbb{R}^d} e_\lambda(x, y, \omega)} \leq c_3. \quad (1.2)$$

In Theorem 1.4 of [46] and Proposition 1.3 of [49], Sznitman has proved that the distance functions $d_\lambda(\cdot, \cdot, \omega)$ satisfy a shape theorem: There exist norms $\alpha_\lambda(\cdot)$ on $\mathbb{R}^d$ (Lyapounov
coefficients) such that: on a set of full \( \mathbb{P} \) -measure we have
\[
\lim_{y \to -\infty} \frac{1}{|y|} |d_{\lambda}(0, y, \omega) - \alpha_{\lambda}(y)| = 0,
\]
the convergence also holds in \( L^1(\mathbb{P}) \), and one can replace \( d_{\lambda}(0, y, \omega) \) by \( - \log e_{\lambda}(0, y, \omega) \).

We remark that under \((0.2)\), \( \alpha_{\lambda}(\cdot) \) is proportional to the Euclidean norm on \( \mathbb{R}^d \), hence we can write \( \alpha_{\lambda}(x) = \alpha_{\lambda}|x| \), where \( \alpha_{\lambda} \) is a fixed constant. This fact will give the possibility to compare our random distance function \( d_{\lambda}(\cdot, \cdot, \omega) \) to the usual Euclidean distance.

Finally we quote the (for us) important part of Lemma 1.2 of Sznitman [49]. For
\[
|x - y| > 4 \quad \text{and} \quad \omega \in \Omega
\]
\[
|\log e_{\lambda}(x, y, \omega) - d_{\lambda}(x, y, \omega)| \leq c_4(1 + F_\lambda(x) + F_\lambda(y)),
\]
where for \( x \in \mathbb{R}^d \) and \( \omega \in \Omega \)
\[
F_\lambda(x) \begin{cases} c(d, \lambda), & \text{if } d \geq 3 \text{ or } \lambda > 0, \\ \leq c(W, M) \left(1 + \log^+(\log \text{dist}(x, \text{supp}\omega))\right), & \text{if } d = 2 \text{ and } \lambda = 0, \end{cases}
\]
provided \( \text{supp}\omega \) denotes the support of \( \omega \).

**Lemma 1.1** For \( d \geq 2 \) and \( \lambda \geq 0 \) we have
\[
\lim_{r \to \infty} \mathbb{P}\left[ \sup_{z \in \mathcal{B}(0, r)} F_\lambda(z) \leq o(\log r) \right] = 1.
\]

**Proof.** We only consider the case \( d = 2 \) and \( \lambda = 0 \) because otherwise \( F_\lambda(\cdot) \) is a constant. Choose \( M_1 \in (0, \infty) \), then for \( r > 0 \) we have
\[
\mathbb{P}\left[ \text{dist}(0, \text{supp}\omega) \geq M_1 (\log r)^{1/2} \right] = \exp\{-\nu v_2 M_1^2 \log r\} = r^{-\nu v_2 M_1^2},
\]
where \( \nu \) is the intensity of the Poisson distribution, and where \( v_2 \) stands for the volume of the two dimensional unit ball. Define \( \mathcal{B}(0, r) = \mathcal{B}(0, r) \cap \mathbb{Z}^d \), hence
\[
\mathbb{P}\left[ \sup_{z \in \mathcal{B}(0, r)} \text{dist}(z, \text{supp}\omega) \geq M_1 (\log r)^{1/2} \right] \leq \sum_{z \in \mathcal{B}(0, r)} \mathbb{P}\left[ \text{dist}(z, \text{supp}\omega) \geq M_1 (\log r)^{1/2} \right] = |\mathcal{B}(0, r)| r^{-\nu v_2 M_1^2} = c_5 r^{2-\nu v_2 M_1^2}.
\]
If we choose \( M_1 \) fixed but sufficiently large the claim of the lemma follows.

\( \Box \)
Proof of Theorem 0.3. Suppose $\xi_1 < 1$, otherwise there is nothing to prove, because (using Proposition 0.1) we know that $\chi^{(2)} < 1/2$ in all dimensions $d$. We choose $b$ and $\gamma$ with

$$\xi_1 < \gamma < b < 1.$$  \hfill (1.8)

Therefore we know that for all sequences $(y_n)_n$ with $|y_n| \to \infty$,

$$\lim_{n \to \infty} E \left[ \hat{p}^{y_n}_0 [A(y_n, \gamma)] \right] = 1.$$ \hfill (1.9)

Thanks to the rotational invariance, we can restrict ourselves (w.l.o.g.) to goals $y$ of the form $(|y|, 0, \cdots, 0)$. For such $y$ we define

$$m = m(y) = (0, 3|y|^\gamma + 2a, 0, \cdots, 0),$$ \hfill (1.10)

where $a$ is minimal, such that $W(x) = 0$ for all $x \in B(0, a)^c$. The main geometric construction is to look at two parallel disjoint cylinders $C_1$ and $C_2$ (see Figure 1 and (3.2) of [30]):

$$C_1 - C(y, \gamma) \quad \text{and} \quad C_2 - C(y, \gamma) + m.$$ \hfill (1.11)

For $D$ a Borel set of $\mathbb{R}^d$, $x, z \in \mathbb{R}^d$ and $\omega \in \Omega$ we define

$$e_\lambda^D(x, z) = E_x \left[ \exp \left\{ - \int_0^{H(z)} (\lambda + V)(Z_s, \omega) \right\}, Z_s \in D \text{ for all } s \leq H(z) < \infty \right],$$ \hfill (1.12)

i.e., the crossing Brownian motion does not leave the set $D$. Using this notation, we write for $y = (|y|, 0, \cdots, 0) \in \mathbb{R}^d$

$$\hat{p}^y_0 [A(y, \gamma)] = \frac{e_\lambda^{C_1}(0, y)}{e_\lambda(0, y)} = \frac{e_\lambda^{C_1}(0, y)}{e_\lambda^{C_2}(m, m + y)} \frac{e_\lambda^{C_2}(m, m + y)}{e_\lambda(0, y)}.$$ \hfill (1.13)

We remark that by our choice of $C_1$ and $C_2$, $e_\lambda^{C_1}(0, y)$ and $e_\lambda^{C_2}(m, m + y)$ are independent random variables for all large $|y|$. This will be a key element in the arguments (see Lemma 1.3 below).

So let us first transform the last term in (1.13): For $x \in \mathbb{R}^d$, $r > 0$ let $\partial B(x, r)$ be a finite subset of $\partial B(x, r)$ such that $|\partial B(x, r)| \leq c_0 r^{d-1}$ and $\bigcup_{x' \in \partial B(x, r)} B(x', 1) \supset \partial B(x, r)$. We will use the terminology that $\partial B(x, r)$ is a finite covering of $\partial B(x, r)$. Because $1 > b > \gamma$, we know that for $|y| > 1$, $\partial B(m, |y|^\gamma) \cap \partial C_2 \not= \emptyset$. We define the following non-empty set $\Lambda$ (see Figure 2),

...
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\[ \Lambda = \left\{ (z_1, z_2) \in \partial B(m, |y|) \times \partial B(y + m, |y|); \ B(z_i, 1) \cap C_2 \neq \emptyset \text{ for } i = 1, 2 \right\}. \]  

(1.14)

This \( \Lambda \) divides \( C_2 \) into three parts for large \( |y| \). The middle part will have length of order \(|y|\), whereas the outer parts will have lengths of order \(|y|^b\). Using strong Markov property, Harnack’s inequality (1.2) and the fact that the crossing Brownian motion stays in the cylinder \( C_2 \), we have the following estimate

\[ e_{\lambda}^C(m, m + y) \leq c_3^2 \sum_{(z_1, z_2) \in \Lambda} e_{\lambda}(m, z_1)e_{\lambda}(z_1, z_2)e_{\lambda}(z_2, y + m). \]  

(1.15)

On the other hand we have for any \((z_1, z_2) \in \Lambda\) that (detour through the upper cylinder),

\[ e_{\lambda}(0, y) \geq c_3^{-2}e_{\lambda}(0, z_1)e_{\lambda}(z_1, z_2)e_{\lambda}(z_2, y). \]  

(1.16)

Using (1.15) and (1.16) we conclude that (observe that the middle part \( e_{\lambda}(z_1, z_2) \) with length of order \(|y|\) cancels out),

\[ \frac{e_{\lambda}^C(m, m + y)}{e_{\lambda}(0, y)} \leq c_3^4 \sum_{(z_1, z_2) \in \Lambda} \frac{e_{\lambda}(m, z_1)e_{\lambda}(z_2, y + m)}{e_{\lambda}(0, z_1)e_{\lambda}(z_2, y)}. \]  

(1.17)

Next we explain our choice of \( 1 > b > \gamma \): For any \((z_1, z_2) \in \Lambda\) let \( \tilde{z}_1 = \tilde{z}_1(z_1) \) be the point on the line from 0 to \( z_1 \) such that dist\((0, \tilde{z}_1) = \text{dist}(m, z_1) = |y|^b \) and let \( \tilde{z}_2 = \tilde{z}_2(z_2) \) be the point on the line from \( y \) to \( z_2 \) such that dist\((y, \tilde{z}_2) = \text{dist}(m + y, z_2) = |y|^b \) (see Figure 3).

**Lemma 1.2** For all large \( |y| \), \((z_1, z_2) \in \Lambda\) we have the following upper bound:

\[ \text{dist}(z_1, \tilde{z}_1) \leq 33|y|^{2\gamma - b} \quad \text{and} \quad \text{dist}(z_2, \tilde{z}_2) \leq 33|y|^{2\gamma - b}. \]  

(1.18)

**Proof.** Our goal is to find an upper bound on the Euclidean distance \( \text{dist}(z_1, \tilde{z}_1) = |z_1 - \tilde{z}_1| \) for all possible choices of \( z_1 \). We define \( z_1^p \) to be the orthogonal projection of
Figure 3: The two parallel cylinders $C_1$ and $C_2$.

$z_1$ onto the first hyperplane. For all large $|y|$ we define $\alpha = \alpha(z_1)$ to be the angle in $[0, \pi)$ between the vector $z_1$ and the first hyperplane (see Figure 3). Hence, $\cos \alpha = \frac{|z_1|^\gamma}{|z_1|}$.

Therefore we know that

$$|z_1| = \frac{|z_1|^\gamma}{\cos \alpha} \leq \frac{|y|^b}{\cos \alpha}. \quad (1.19)$$

On the other hand we have for all large $|y|$, that

$$|z_1|^2 \geq |y|^{2b} - (d-1)|y|^{2\gamma} \geq 1/2|y|^{2b}. \quad (1.20)$$

If $z_1^2$ denotes the second coordinate of $z_1$, we have for all large $|y|$ that

$$\frac{1}{\cos \alpha} - 1 \leq \frac{1}{\cos^2 \alpha} - 1 = \tan^2 \alpha = \frac{|z_1|^2}{|z_1|^2} \leq \frac{(4|y|^\gamma + 3\delta)^2}{1/2|y|^{2b}}. \quad (1.21)$$

Therefore, using (1.19) and (1.21), we have for all large $|y|$,

$$|z_1 - \bar{z}_1| = |z_1| \quad |y|^b \leq |y|^b \left( \frac{1}{\cos \alpha} - 1 \right) \leq 33|y|^{2\gamma-b}. \quad (1.22)$$
Of course, the situation for \( z_2 \) is analogue. This finishes the proof of Lemma 1.2.

Thus, we fix for every \((z_1, z_2) \in \Lambda\) such a pair \((\tilde{z}_1, \tilde{z}_2) \in \partial B(0, |y|^{b}) \times \partial B(y, |y|^{b})\). Using the strong Markov property, Harnack’s inequality (1.2), (1.1) and the above lemma, we have for \((z_1, z_2) \in \Lambda\)

\[
e_{\lambda}(0, z_1) \geq c_{3}^{-1}e_{\lambda}(0, \tilde{z}_1)e_{\lambda}(\tilde{z}_1, z_1) \geq c_{7}e_{\lambda}(0, \tilde{z}_1) \exp\{-c_{8}|y|^{2\gamma-b}\} \quad \text{for all large } |y|,
\]

and analogously we get

\[
e_{\lambda}(z_2, y) \geq c_{7}e_{\lambda}(\tilde{z}_2, y) \exp\{-c_{8}|y|^{2\gamma-b}\} \quad \text{for all large } |y|.
\]

Therefore in view of (1.13), (1.17), (1.23) and (1.24) we have that for all large \(|y|\)

\[
\hat{P}_{0}^{|y|}[A(y, \gamma)] \leq c_{9} \exp\{c_{10}|y|^{2\gamma-b}\} \frac{e_{\lambda}^{C_{1}}(0, y)}{e_{\lambda}^{C_{2}}(m, m + y)} \sum_{(z_1, z_2) \in \Lambda} \frac{e_{\lambda}(m, z_1)e_{\lambda}(z_2, m + y)}{e_{\lambda}(0, \tilde{z}_1)e_{\lambda}(\tilde{z}_2, y)}. \tag{1.25}
\]

This upper bound is the key expression we want to further analyze. We aim to find a sequence \((y_n)_n\) with \(|y_n| \to \infty\) such that we get good controls on both fractions on the right-hand side of (1.25) (see Lemmas 1.3 and 1.4 below). We also remark that \(\text{dist}(0, z_1) = \text{dist}(m, z_1) = \text{dist}(\tilde{z}_2, y) = \text{dist}(z_2, m + y) = \gamma y_b\), thus to derive an upper bound on the last part of (1.25) we use an uniform control for distance fluctuations via \(\chi^{(2)}\), whereas for the first part in the right member of (1.25) we will use independence of \(e_{\lambda}^{C_{1}}(0, y)\) and \(e_{\lambda}^{C_{2}}(m, m + y)\):

The next lemma shows for which sequence \((y_n)_n\) we want to consider (1.25).

**Lemma 1.3** For any \(\kappa < \chi^{(1)}\) there exists an \(\varepsilon > 0\) and a sequence \((y_n)_n \geq 1\) with \(y_n = (|y_n|, 0, \ldots, 0)\) and \(|y_n| \to \infty\) such that for all large \(n\)

\[
\mathbb{P}\left[ \frac{c_{1}(0, y_n)}{e_{\lambda}^{C_{2}}(m_n, m_n + y_n)} \leq e^{\exp\{-|y_n|^\kappa\}} \right] \geq \varepsilon,
\]

where \(m_n = m(y_n)\) is defined in (1.10).

**Proof.** Choose \(\kappa' \in (\kappa, \chi^{(1)})\). Then by the definition of \(\chi^{(1)}\) and rotational invariance there exists an \(\varepsilon_1\) and a sequence \((y_n)_n \geq 1\) with \(y_n = (|y_n|, 0, \ldots, 0)\) and \(|y_n| \to \infty\) such that for all large \(n\)

\[
\mathbb{P}\left[ |d_{\lambda}(0, y_n) - M_{\lambda}(0, y_n)| \geq |y_n|^\kappa' \right] \geq 2\varepsilon_1.
\]

\[
\hat{P}_{0}^{|y|}[A(y, \gamma)] \leq c_{9} \exp\{c_{10}|y|^{2\gamma-b}\} \frac{c_{1}(0, y)}{e_{\lambda}^{C_{2}}(m, m + y)} \sum_{(z_1, z_2) \in \Lambda} \frac{e_{\lambda}(m, z_1)e_{\lambda}(z_2, m + y)}{e_{\lambda}(0, \tilde{z}_1)e_{\lambda}(\tilde{z}_2, y)}. \tag{1.25}
\]
Now there are two cases: there exists a subsequence \((y_{n_k})_k\) of \((y_n)_n\) such that either
\[ P \left[ d_{\lambda}(0, y_{n_k}) - M_{\lambda}(0, y_{n_k}) \geq |y_{n_k}|^{\kappa'} \right] \geq \varepsilon_1 \quad \text{for all } k, \quad (1.28) \]
or
\[ P \left[ d_{\lambda}(0, y_{n_k}) - M_{\lambda}(0, y_{n_k}) \leq -|y_{n_k}|^{\kappa'} \right] \geq \varepsilon_1 \quad \text{for all } k. \quad (1.29) \]

First we consider the case \((1.28)\) and we assume that \((1.28)\) holds for all \(y_n, n \geq 1\). Let \(\tilde{\omega}\) be an independent copy of \(\omega\). Then we have, using translation invariance, independence and the median property, that
\[
P \times \tilde{P} \left[ d_{\lambda}(0, y, \omega) - d_{\lambda}(m, m + y, \tilde{\omega}) \geq |y|^{\kappa'} \right] \\
\geq P \times \tilde{P} \left[ \left\{ d_{\lambda}(0, y, \omega) \geq M_{\lambda}(0, y) + |y|^{\kappa'} \right\} \cap \left\{ d_{\lambda}(0, y, \tilde{\omega}) \leq M_{\lambda}(0, y) \right\} \right] \\
\geq P \left[ d_{\lambda}(0, y, \omega) \geq M_{\lambda}(0, y) + |y|^{\kappa'} \right] \times 1/2. \quad (1.30)\]

In the second case we assume that \((1.29)\) holds for all starting points \(m_n\) and goals \(m_n + y_n\) (using translation invariance). As above
\[
P \times \tilde{P} \left[ d_{\lambda}(0, y, \omega) - d_{\lambda}(m, m + y, \tilde{\omega}) \geq |y|^{\kappa'} \right] \\
\geq P \times \tilde{P} \left[ \left\{ d_{\lambda}(m, m + y, \tilde{\omega}) \leq M_{\lambda}(0, y) - |y|^{\kappa'} \right\} \cap \left\{ d_{\lambda}(0, y, \omega) \geq M_{\lambda}(0, y) \right\} \right] \\
\geq P \left[ d_{\lambda}(m, m + y, \omega) \leq M_{\lambda}(0, y) - |y|^{\kappa'} \right] \times 1/2. \quad (1.31)\]

Choosing the sequence \((y_n)_n\) and \(\varepsilon_1\) such that either \((1.28)\) or \((1.29)\) holds for all \(n \geq 1\), we get the following lower bound for all \(n\),
\[ P \times \tilde{P} \left[ d_{\lambda}(0, y_n, \omega) - d_{\lambda}(m_n, m_n + y_n, \tilde{\omega}) \geq |y_n|^{\kappa'} \right] \geq \varepsilon_1/2, \quad (1.32) \]
i.e., there exists a subset of cloud configuration-pairs \((\omega, \tilde{\omega})\) with \(P \times \tilde{P}\)-probability \(\geq \varepsilon_1/2\), such that for all these cloud-pairs the distance to pass the lower cylinder is much bigger than the distance to pass the upper cylinder. To keep the calculations transparent we only consider the case \(d \geq 3\) or \(\lambda > 0\) (the case \(d = 2\) and \(\lambda = 0\) can be handled analogously using Lemma 1.1, see also [53] at the end of Section 2). Using (1.4) we have
\[
P \times \tilde{P} \left[ -\log e_{\lambda}(0, y_n, \omega) + \log e_{\lambda}(m_n, m_n + y_n, \tilde{\omega}) \geq 1/2|y_n|^{\kappa'} \right] \\
\geq P \times \tilde{P} \left[ d_{\lambda}(0, y_n, \omega) - d_{\lambda}(m_n, m_n + y_n, \tilde{\omega}) - 2c_4(1 + 2c(d, \lambda)) \geq 1/2|y_n|^{\kappa'} \right] \quad (1.33)\]
\[ \geq P \times \tilde{P} \left[ d_{\lambda}(0, y_n, \omega) - d_{\lambda}(m_n, m_n + y_n, \tilde{\omega}) \geq |y_n|^{\kappa'} \right] \geq \frac{\varepsilon_1}{2} \quad \text{for all large } n. \]
Next we use the properties of the choice of $\gamma > \xi_1$ (see remark after (1.13)). For $\delta \in (0, 1)$ we have that
\[
\mathbb{P} \left[ \frac{e_{\lambda}^{C_1}(0, y_n, \omega)}{e_{\lambda}^{C_2}(m_n, m_n + y_n, \omega)} \leq \exp\{-|y_n|^\kappa\} \right] \geq \mathbb{P} \left[ \frac{e_{\lambda}(0, y_n, \omega)}{e_{\lambda}(m_n, m_n + y_n, \omega)} \cdot \exp\{-|y_n|^\kappa\} \right].
\]
Using $\kappa' > \kappa$, translation invariance and (1.33), we have for large $n$ that the above expression is greater or equal to
\[
\mathbb{P} \left[ \frac{e_{\lambda}(0, y_n, \omega)}{e_{\lambda}(m_n, m_n + y_n, \omega)} \leq \exp\{-1/2|y_n|^\kappa'\} \right] + \mathbb{P} \left[ \frac{e_{\lambda}^{C_1}(0, y_n)}{e_{\lambda}^{C_1}(0, y_n)} \leq 1 + \delta \right] - 1
\geq \varepsilon_1/2 + \mathbb{P} \left[ \frac{e_{\lambda}(0, y_n)}{e_{\lambda}(0, y_n)} \geq 1/(1 + \delta) \right] - 1.
\]
Our final claim is that $\mathbb{P} \left[ \frac{e_{\lambda}^{C_1}(0, y_n)}{e_{\lambda}(0, y_n)} \geq 1/(1 + \delta) \right]$ tends to 1 as $n$ goes to infinity. But this can be seen using Chebychev’s inequality and the fact that $\gamma > \xi_1$. This finishes the proof of Lemma 1.3.

\[\square\]

**Lemma 1.4** For $\kappa > \chi^{(2)}(\kappa)$ we have that for any sequence $(y_n)_{n\geq 1}$ with $y_n - (|y_n|, 0, \ldots, 0)$ and $|y_n| \to \infty$ that
\[
\lim_{n \to \infty} \mathbb{P} \left[ \sum_{(z_1, z_2) \in \Lambda} e_{\lambda}(m, z_1) e_{\lambda}(z_2, m + y) \leq \exp\{6|y|^\kappa\} \right] = 1,
\]
where $m = m(y_n)$ is defined in (1.10).

**Proof.** For simplicity we skip the $n$ in the notation. We choose $\kappa' \in (\chi^{(2)}, \kappa)$, and we introduce the following notation, let $y^b = (|y|^b, 0, \ldots, 0)$. We have that
\[
\mathbb{P} \left[ \sum_{(z_1, z_2) \in \Lambda} e_{\lambda}(m, z_1) e_{\lambda}(z_2, m + y) \leq \exp\{6|y|^\kappa\} \right] \geq \mathbb{P} \left[ e_{\lambda}|y|^{2\kappa(b-1)} \sup_{(z_1, z_2) \in \Lambda} e_{\lambda}(0, z_1) e_{\lambda}(z_2, y) \leq \exp\{6|y|^\kappa\} \right].
\]
Thus, if $|y|$ is sufficiently large the above expression is bigger than (using translation invariance in the second step)

\[
\geq \mathbb{P}\left[ \sup_{(z_1,z_2) \in A} \frac{e_\lambda(m,z_1) e_\lambda(z_2,m+y)}{e_\lambda(0,z_1) e_\lambda(z_2,y)} \exp\{5|y|^{b\kappa}\} \right]
\geq 2\mathbb{P}\left[ \sup_{z_1 \in \partial B(0,|y|^b)} -\log e_\lambda(0,\bar{z}_1) - M_\lambda(0,|y|^b) \right] \leq 5/4|y|^{b\kappa} - 3 \tag{1.36}
\]

Using Lemma 1.1 and because we have chosen $\kappa > \chi^{(2)}$ the above tends to 1 as $n \to \infty$. This finishes the proof of Lemma 1.4.

\[\square\]

Let us continue with the proof of Theorem 0.3. We choose $\kappa_1$ and $\kappa_2$ as follows:

\[\kappa_1 < \chi^{(1)} \leq \chi^{(2)} < \kappa_2. \tag{1.37}\]

In view of (1.25) we have that

\[
\mathbb{P}\left[ P_{\tilde{y}} [A(y,\gamma)] \leq c_9 \exp \{c_{10}|y|^{2\gamma-b} - |y|^\kappa_1 + 6|y|^{b\kappa_2}\} \right] \geq \mathbb{P}\left[ \frac{e_{\lambda}(0,y)}{e_{\lambda}(m,m+y)} \sum_{(z_1,z_2) \in A} \frac{e_\lambda(m,z_1) e_\lambda(z_2,m+y)}{e_\lambda(0,z_1) e_\lambda(z_2,y)} \leq \exp \{-|y|^\kappa_1 + 6|y|^{b\kappa_2}\} \right].
\]

Now we choose a sequence $(y_n)_n$ and $\varepsilon > 0$ such that Lemma 1.3 holds for our choice of $\kappa_1 < \chi^{(1)}$, hence using Lemmas 1.3 and 1.4 we see that for all large $n$

\[
\mathbb{P}\left[ P_{\tilde{y}_n} [A(y_n,\gamma)] \leq c_9 \exp \{c_{10}|y_n|^{2\gamma-b} - |y_n|^\kappa_1 + 6|y_n|^{b\kappa_2}\} \right] \geq \varepsilon/2. \tag{1.39}
\]

In view of (1.9) we get that $\kappa_1 \leq \max\{2\gamma - b, b\kappa_2\}$ for all $\kappa_1 < \chi^{(1)}$, for all $\kappa_2 > \chi^{(2)}$, for all $\gamma \in (\xi_1,1)$ and for all $b \in (\gamma,1)$. Therefore

\[\chi^{(1)} \leq \max\{2\xi_1 - b, b\chi^{(2)}\}, \quad \text{for all } b \in (\xi_1,1). \tag{1.40}\]

Consider the function $f(b) = \max\{2\xi_1 - b, b\chi^{(2)}\}$ (see Figure 4). $f(b)$ is minimal for $b = b_0 = \frac{2\xi_1 - \chi^{(2)}}{1 + \chi^{(2)}}$. Observe that $b_0 < \xi_1$ is impossible because otherwise $1 < \chi^{(2)}$ which contradicts Proposition 0.1. Thus $b_0 > \xi_1$. Now there are two cases:
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Figure 4: The function $f(b)$.

1. Either $b_0 = \frac{2\xi_1}{1 + \chi^{(2)}} \geq 1$: In this case letting $b$ tend to 1 we get

$$\xi_1 \geq \frac{\chi^{(2)} + 1}{2} \geq \frac{\chi^{(1)}}{\chi^{(2)}} \frac{\chi^{(2)} + 1}{2}, \quad (1.41)$$

which is the claim of the theorem in the first case.

2. Or $b_0 = \frac{2\xi_1}{1 + \chi^{(2)}} \in (\xi_1, 1)$, and hence $\xi_1 < \frac{\chi^{(2)} + 1}{2}$. Now choosing $b = b_0$ in (1.40), we find

$$\chi^{(1)} \leq \max\{2\xi_1 - b_0, b_0\chi^{(2)}\} = 2\xi_1 \left( \frac{\chi^{(2)}}{1 + \chi^{(2)}} \right), \quad (1.42)$$

from which the claim of the theorem follows in the second alternative.

This finishes the proof of the Theorem 0.3.

Remark. Of course one could relax the assumptions for $\chi^{(2)}$ to prove Lemma 1.4 (and hence Theorem 0.3). All we require is an uniform control on the fluctuation of $d_\lambda(0, \cdot)$ on the boundary of $B(0, r)$. We stated the Theorem as above, because we need the stronger assumptions to prove Theorem 0.2 and because we think that the different $\chi$'s are in fact equal.

2 PROOF OF THE UPPER BOUND

In this section we prove "the upper bound" of the scaling identity. The proof of Theorem 0.2 is essentially the same as the proof of Theorem 1.1 in [53]. We will give it for the
reader's convenience and because there appear some additional difficulties handling with medians. The key step is Lemma 2.2 of [53]. It states the following: If $\gamma \in (0,1]$, then there exists a constant $c \in (0,\infty)$ such that for all $y \in \mathbb{R}^d$ with $|y| > 1$ and for all $z \in \partial C(y, \gamma)$ the following is true

$$|0 - z| + |z - y| \geq |0 - y| + c|y|^{2\gamma - 1},$$

(2.1)
i.e., it describes the Euclidean cost of a detour through some point $z$ on the boundary of the cylinder $C(y, \gamma)$ when going from the origin to $y$.

First we define two subsets of $\Omega$, which will be considered throughout the whole section. For $\kappa > 0$, $x, y \in \mathbb{R}^d$ and $r > 0$ define

$$F_{\kappa}(y) = \{\omega \in \Omega; \ |d_\lambda(0,y,\omega) - M_\lambda(0,y)| \leq |y|^\kappa \}$$

(2.2)and

$$G_{\kappa}(x, r) = \left\{ \sup_{y_1 \in B(x,r)} |d_\lambda(x,y_1,\omega) - M_\lambda(x,y_1)| \leq r^\kappa \right\}.$$  

(2.3)

Of course, if we choose $\kappa_1 > \chi^{(1)}$ and $\kappa_2 > \chi^{(2)}$ then we have that

$$\lim_{y \to \infty} \mathbb{P}[F_{\kappa_1}(y)] = 1$$

(2.4)and

$$\lim_{r \to \infty} \mathbb{P}[G_{\kappa_2}(x, r)] = \lim_{r \to \infty} \mathbb{P}[G_{\kappa_2}(0, r)] = 1.$$  

(2.5)

Next we state two lemmas. They will provide us with controls on corrections to "triangle inequalities" for the median $M_\lambda(\cdot, \cdot)$. At this stage we must tell, how we want to choose the median $M_\lambda(x, y)$ in cases where it is not unique. Because the law of $d_\lambda(\cdot, \cdot, \omega)$ is shift and rotationally invariant and because it is symmetric, i.e., $d_\lambda(x,y,\omega) = d_\lambda(y,x,\omega)$ for all $\omega \in \Omega$, we want the medians to have the same features. So we choose $M_\lambda(\cdot, \cdot)$ such that it is shift and rotationally invariant and such that

$$M_\lambda(x, y) = M_\lambda(y, x) \quad \text{for all} \ x, y \in \mathbb{R}^d.$$  

(2.6)

A possible choice for $M_\lambda(x, y)$ is the minimal $m$ such that $m$ is a median for $d_\lambda(x, y, \omega)$.

**Lemma 2.1** There exists $c_{12}$ such that for all $\kappa > \chi^{(2)}$ there exists $C_1 \in (0, \infty)$ such that for all $|y| \geq 2C_1$ and $z$ on the straight line between $0$ and $y$ with $\text{dist}(0, z) \geq C_1$ and $\text{dist}(z, y) \geq C_1$ the following is true,

$$M_\lambda(0, z) + M_\lambda(z, y) \leq M_\lambda(0, y) + c_{12}|y|^\kappa.$$  

(2.7)
Proof. Choose \( R \in (1, |y| - 1) \) and consider the surfaces \( \partial B_1 = \partial B(0, R) \) and \( \partial B_2 = \partial D(y, |y| - R) \) (see Figure 5). Hence,

\[
\mathbb{P}_0^y [ H(\partial B_1) \leq H(\partial B_2) \leq H(y) ] = 1, \tag{2.8}
\]

where \( H(\partial B_i) = \inf\{ s \geq 0; \ Z_s \in \partial B_i \} \) for \( i = 1, 2 \). Using the strong Markov property and (2.8) we have for all \( \omega \in \Omega \),

\[
e_\lambda(0, y, \omega) = e_\lambda(0, y, \omega) \mathbb{P}_0^y [ H(\partial B_1) \leq H(\partial B_2) \leq H(y) ]
\leq E_0 \left[ \exp \left\{ - \int_0^{H(\partial B_1)} (\lambda + V)(Z_s, \omega) ds \right\} \right] \sup_{z \in \partial B_2} e_\lambda(z, y, \omega) \tag{2.9}
\leq c_{13} |y|^{d-1} \sup_{z \in \partial B_1} e_\lambda(0, z, \omega) \sup_{z \in \partial B_2} e_\lambda(z, y, \omega),
\]

where in the last step we have used that the volume of the surface of \( \partial B_1 \) is at most of order \( |y|^{d-1} \). As in the proof of Lemma 1.3 we only treat the case where \( d \geq 3 \) or \( \lambda > 0 \) (the case \( d = 2 \) and \( \lambda = 0 \) goes analogously using Lemma 1.1). Taking the logarithm on both sides of (2.9) and using (1.4) we get for all \( \omega \in \Omega \),

\[
d_\lambda(0, y) \geq -\log \left( c_{13} |y|^{d-1} \right) + \inf_{z \in \partial B_1} d_\lambda(0, z) + \inf_{z \in \partial B_2} d_\lambda(z, y) + c_4 (2 + 4c(d, \lambda)). \tag{2.10}
\]

Using (2.5) and the fact that the medians are deterministic and rotationally invariant, we see that the claim follows for \( R \) and \( |y| - R \) sufficiently large.

\[ \square \]

From Proposition 0.1 we know that \( \chi^{(1)} \leq 1/2 \).
Lemma 2.2 There exists $c_{14}$ such that for all $\kappa \in (\chi^{(1)}, 1)$ there exists $C_2 \in (0, \infty)$ such that for all $|z| > C_2$ and all $|y| > C_2$ with $|z| < 2|y|$ the following is true,

$$M_\lambda(0, z) \mid M_\lambda(z, y) \geq M_\lambda(0, y) \quad c_{14}|y|^\kappa. \quad (2.11)$$

Proof. $\mathbb{P}$-a.s. $d_\lambda(\cdot, \cdot, \omega)$ is a distance function, hence on a set of full $\mathbb{P}$-measure we have that for all $z, y \in \mathbb{R}^d$

$$d_\lambda(0, y, \omega) \leq d_\lambda(0, z, \omega) \mid d_\lambda(z, y, \omega). \quad (2.12)$$

Choose $\kappa \in (\chi^{(1)}, 1)$ and $\varepsilon \in (0, 1/4)$. So we have that for $\min\{|z|, |y|\}$ sufficiently large,

$$\mathbb{P} \left[ F_\kappa(y) \cap F_\kappa(z) \cap \{d_\lambda(z, y) \leq M_\lambda(z, y)\} \right] \geq 2 - \varepsilon + 1/2 - 2 > 1/4. \quad (2.13)$$

The claim follows as in Lemma 2.1 for all $|z|$ sufficiently large and $|z| \leq 2|y|$. 

Finally we have all the tools to proof Theorem 0.2.

Proof of Theorem 0.2. We choose

$$1 > \kappa > \chi^{(2)} \quad \text{and} \quad 2\gamma - 1 > \kappa. \quad (2.14)$$

We will show that in this case $\gamma \geq \xi_1$. Hence $2\xi_1 - 1 \leq \chi^{(2)}$, which will finish the proof of the theorem.

To keep the calculations transparent we only consider the case $d > 3$ or $\lambda > 0$ (otherwise use Lemma 1.1). Defining the event $N_\kappa(y) = F_\kappa(y) \cap G_\kappa(0, 2|y|) \cap G_\kappa(y, 2|y|)$ we find, using (2.4) and (2.5), that

$$\lim_{y \to \infty} \mathbb{P} \left[ N_\kappa(y) \right] = 1. \quad (2.15)$$

We define $C_y$ a finite subset of $\partial C(y, \gamma)$ such that $|C_y| \leq c_{15}|y|^{d-1}$ and $\bigcup_{z \in C_y} B(z, 1) \supset \partial C(y, \gamma)$. Therefore using (2.12) of [53], Harnack's inequality and (1.4) we find that

$$\mathbb{E} \left[ \hat{P}_0^y \left[ A(y, \gamma)^c \right] \right] \leq \mathbb{E} \left[ \hat{P}_0^y \left[ A(y, \gamma)^c \right], N_\kappa(y) \right] + \mathbb{P} \left[ N_\kappa(y)^c \right]
\leq c_{16}|y|^{d-1} \sup_{z \in C_y} \exp \left\{ M_\lambda(0, y) - M_\lambda(0, z) - M_\lambda(z, y) + 5|y|^\kappa \right\} + \mathbb{P} \left[ N_\kappa(y)^c \right],$$

where $c_{16} = c_9 c_{15} \exp \{3(1 + 2c(d, \lambda))\}$. In view of (2.15) it suffices to prove that

$$f(y) \overset{def}{=} |y|^{d-1} \sup_{z \in C_y} \exp \left\{ M_\lambda(0, y) - M_\lambda(0, z) - M_\lambda(z, y) + 5|y|^\kappa \right\} \to 0 \text{ as } y \to \infty. \quad (2.17)$$
Because our model is rotationally invariant we can restrict ourselves to $y$ of the form $(|y|,0,\ldots,0)$. For $z \in C_y$ we define $z_1 = (|z|,0,\ldots,0)$ and $z_2 = (|y - z|,0,\ldots,0)$. Using Lemmas 2.1 and 2.2 we have for $|y|$ sufficiently large, $z \in C_y$ that

$$
M_\lambda(0,z) + M_\lambda(z,y) = M_\lambda(0,z_1) + M_\lambda(z_1,z_1 + z_2) \geq M_\lambda(0,z_1 + z_2) - 2c_{14}|y|^\kappa
$$

$$
\geq M_\lambda(0,y) + M_\lambda(y,z_1 + z_2) - 2(c_{12} + c_{14})|y|^\kappa
$$

(2.18)

In view of (2.17) we have for all large $|y|$ (set $c_{17} = 5 + 2c_{12} + 2c_{14}$)

$$
f(y) \leq |y|^{d-1} \sup_{z \in C_y} \exp\{-M_\lambda(0,z_1 + z_2 - y) + c_{17}|y|^\kappa\}.
$$

(2.19)

From Sznitman’s shape theorem (see (1.3)) we know that: There exists $c_{18}$ such that for all large $|y|$, we have

$$
M_\lambda(0,y) \geq c_{18}|y|.
$$

(2.20)

Hence we can write for (2.17) that for $|z_1 + z_2 - y|$ large

$$
f(y) \leq |y|^{d-1} \sup_{z \in C_y} \exp\{-c_{18}(|z| + |z - y| - |y|) + c_{17}|y|^\kappa\},
$$

(2.21)

from which the claim follows, using Lemma 2.2 of [53] (see also (2.1)). This finishes the proof of Theorem 0.2.

3 RELATIONS FOR $\chi$

In this section we will give the proof of Proposition 0.1. In the second part of this section we will give some further numerical aspects for distance fluctuations.

Proof of Proposition 0.1. The first inequality in (0.12) is clear, we shall only prove $\chi^{(2)} \leq 1/2$, this fact will follow from Corollary 3.5 of [49]. We choose $\kappa > \kappa' > 1/2$. From Sznitman [49] Corollary 3.5 we know that $P$-a.s. for large $|y|$, 

$$
|d_\lambda(0,y) - \alpha_\lambda(y)| \leq |y|^{\kappa'}.
$$

(3.1)

The first step we do is to divide the event $\{\sup_{y \in B(0,R)} |d_\lambda(0,y) - \alpha_\lambda(y)| \leq R^{\kappa'}\}$ into two events. Choose $r$ such that $r \log r = R^{\kappa'}$ then

$$
P \left[ \sup_{y \in B(0,R)} |d_\lambda(0,y) - \alpha_\lambda(y)| \leq R^{\kappa'} \right] \geq P \left[ \sup_{y \in B(0,r)} |d_\lambda(0,y) - \alpha_\lambda(y)| \leq R^{\kappa'} \right] + P \left[ \sup_{y \in \partial B(0,r_0)} |d_\lambda(0,y) - \alpha_\lambda(y)| \leq r_0^{\kappa'}, \text{ for all } r_0 \geq r \right] - 1.
$$

(3.2)
Since $\kappa' > 1/2$, it follows from (3.1) that the last term in (3.2) tends to 1 as $R$ (resp., $r$) tends to infinity. For the first term in (3.2) we have that
\[
P\left[ \sup_{y \in B(0, r)} |d_\lambda(0, y) - \alpha_\lambda(y)| \leq R^{\kappa'} \right] \geq \mathbb{P} \left[ \sup_{y \in B(0, r)} | - \log e_\lambda(0, y) - \alpha_\lambda(y)| + \sup_{y \in B(0, r)} c_4 (1 + F_\lambda(0) + F_\lambda(y)) \leq R^{\kappa'} \right].
\]

Both members on the right-hand side of (3.3) are of order $o(R^{\kappa'})$ with $\mathbb{P}$-probability tending to 1 as $r \to \infty$. This is so because for the first term in the right-hand side of (3.3) we know that for $y \in B(0, r)$, $\alpha_\lambda(y) = o(R^{\kappa'})$ and $- \log e_\lambda(0, y, \omega) = o(R^{\kappa'})$ for all $\omega \in \Omega$ (see (1.1)). Whereas for the second term in the right-hand side we use Lemma 1.1.

Hence for $\kappa' > 1/2$ we have that
\[
\lim_{R \to \infty} \mathbb{P} \left[ \sup_{y \in B(0, R)} |d_\lambda(0, y) - \alpha_\lambda(y)| \leq R^{\kappa'} \right] = 1. \tag{3.4}
\]

Next we compare the Lyapunov coefficient to the median, we have
\[
M_\lambda(0, y) - \alpha_\lambda(y) = M_\lambda(0, y) - d_\lambda(0, y) + d_\lambda(0, y) - \alpha_\lambda(y), \tag{3.5}
\]
and in view of (3.1) and the definition of the median the right member is smaller than $|y|^{\kappa'}$ with positive $\mathbb{P}$-probability. For this we easily deduce that for large $R$
\[
\sup_{y \in B(0, R)} M_\lambda(0, y) - \alpha_\lambda(y) \leq R^{\kappa'}. \tag{3.6}
\]

We get the bound with the opposite sign in analogous fashion. Hence using triangle inequality we see that
\[
\lim_{R \to \infty} \mathbb{P} \left[ \sup_{y \in B(0, R)} |M_\lambda(0, y) - d_\lambda(0, y)| \leq R^{\kappa'} \right] \geq \lim_{R \to \infty} \mathbb{P} \left[ \sup_{y \in B(0, R)} |M_\lambda(0, y) - \alpha_\lambda(y)| + \sup_{y \in B(0, R)} |\alpha_\lambda(y) - d_\lambda(0, y)| \leq R^{\kappa'} \right] = 1. \tag{3.7}
\]

Hence $\chi^{(2)} \leq 1/2$. This finishes the proof of Proposition 0.1.

Remarks. Of course one is also interested in positive lower bounds for $\chi^{(1)}$, resp., $\chi^{(2)}$. Without much calculation we get an approach to a nontrivial lower bound (for
an analogous result in first-passage percolation see Newman-Piza [35], Theorem 7). We define

\[ \bar{\lambda} = \sup \left\{ \kappa \geq 0, \limsup_{y \to \infty} |y|^{-2\kappa} \text{Var}(-\log e^{x}(0, y)) > 0 \right\}. \]  \hspace{1cm} (3.8)

So \( \bar{\lambda} \geq \lambda_{(1)} \) (this can easily be seen using Chebychev's inequality). In Theorem 1.2 of [53] we have proved that in dimension \( d = 2 \) we have for \( \gamma > \xi_{1} \) that

\[ \liminf_{y \to \infty} |y|^{-1+\gamma} \text{Var}(-\log e^{x}(0, y)) > 0, \]  \hspace{1cm} (3.9)

hence

\[ \bar{\lambda} \geq (1 - \xi_{1})/2. \]  \hspace{1cm} (3.10)

Therefore combining (3.10) and Theorem 0.2 we have the following result:

**Corollary 3.1** For \( d = 2 \) we have the following lower bound,

\[ \max \{ \bar{\lambda}, \lambda^{(2)} \} \geq 1/5. \]  \hspace{1cm} (3.11)
Superdiffusive behavior of two-dimensional Brownian motion in a Poissonian potential

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Abstract

We consider d-dimensional Brownian motion in a truncated Poissonian potential conditioned to reach a remote location. If Brownian motion starts at the origin and ends in an hyperplane at distance $L$ from the origin, the transverse fluctuation of the path is expected to be of order $L^{\xi}$. We are interested in a lower bound for $\xi$. We first show that $\xi \geq 1/2$ in dimensions $d \geq 2$ and then we prove superdiffusive behaviour for $d = 2$, resulting in $\xi \geq 3/5$.

0 INTRODUCTION AND RESULTS

In the present work we want to focus on a special model in the theory of random motions in a random potential. Throughout this paper we consider Brownian motion in a truncated Poissonian potential. The Brownian motion will start at the origin and will be stopped when reaching a fixed hyperplane at distance $L$ from 0 (see for instance Sznitman [46]). Our purpose is to study certain fluctuation properties of the paths when they feel the presence of a truncated Poissonian potential, as $L \to \infty$.

Let $P$ stand for the Poisson law with fixed intensity $\nu > 0$ on the space $\Omega$ of simple pure point measures $\omega$ on $\mathbb{R}^d$, $d \geq 2$. The soft obstacles are generated by a fixed shape function $W(\cdot) \geq 0$, which is measurable, bounded, compactly supported and not a.e. equal to zero. Furthermore we assume that:

\[
W(\cdot) \text{ is rotationally invariant.} \tag{0.1}
\]

For $\omega - \sum_i \delta_{x_i} \in \Omega$ and $x \in \mathbb{R}^d$, we define the truncated Poissonian potential with fixed truncation level $M > 0$ as:

\[
V(x, \omega) = \left( \sum_i W(x - x_i) \right) \wedge M = \left( \int_{\mathbb{R}^d} W(x - y)\omega(dy) \right) \wedge M. \tag{0.2}
\]

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Our aim is to have a penalty on the Brownian path when it experiences the potential. For \( x \in \mathbb{R}^d \) we denote by \( P_x \) the Wiener measure on \( C(\mathbb{R}_+, \mathbb{R}^d) \) starting at \( x \), and by \( Z_r \) the canonical process on \( C(\mathbb{R}_+, \mathbb{R}^d) \). For \( \theta \in [0, 2\pi) \), \( L > 0 \), let \( \Lambda(\theta, L) \) be the half-space,

\[
\Lambda(\theta, L) = \{ x \in \mathbb{R}^d; \langle x, \hat{x}(\theta) \rangle \geq L \},
\]

where \( \hat{x}(\theta) = (\cos \theta, \sin \theta, 0, \ldots, 0) \), and where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product on \( \mathbb{R}^d \). If \( \theta = 0 \), we write \( \Lambda_L \) for \( \Lambda(0, L) \). For \( \lambda \geq 0, L > 0, \theta \in [0, 2\pi) \) and \( \omega \in \Omega \) the new path measure on \( C(\mathbb{R}_+, \mathbb{R}^d) \) is then defined by,

\[
d\tilde{P}_0^{\partial \Lambda(\theta, L)} = \frac{1}{\epsilon_0(0, \partial \Lambda(\theta, L), \omega)} \exp \left\{ - \int_0^{H(\partial \Lambda(\theta, L))} (\lambda + V)(Z_s, \omega) ds \right\} dP_0,
\]

where \( \epsilon_0(0, \partial \Lambda(\theta, L), \omega) \) is the normalizing constant and where \( H(\partial \Lambda(\theta, L)) = \inf\{s \geq 0, Z_s \in \Lambda(\theta, L)\} \) is the entrance time of \( Z(\omega) \) in the half-space \( \Lambda(\theta, L) \).

From Sznitman [46], Corollary 1.9, we know that we have a shape theorem: on a set of full \( \mathbb{P} \)-measure for \( \lambda \geq 0 \),

\[
\lim_{L \to \infty} \frac{1}{L} \log \epsilon_0(0, \partial \Lambda(\theta, L)) = -\alpha_0(\hat{x}(\theta)),
\]

and the Lyapounov coefficients \( \alpha_\lambda(\cdot) \) are norms on \( \mathbb{R}^d \). In the present article we restrict ourselves to rotationally invariant \( W(\cdot) \), hence the Lyapounov coefficients are proportional to the Euclidean norm. This gives us the possibility to compare \( -\log \epsilon_0(0, \partial \Lambda(\theta, L)) \) to the Euclidean norm.

Our main interest in this article is to study transverse fluctuations. Two natural questions arise: (1) “Where do the paths end in \( \partial \Lambda(\theta, L) \)?” (2) If we examine the line through the origin perpendicular to \( \partial \Lambda(\theta, L) \), “how far do the paths fluctuate from this line before they hit the goal”? These two questions have a similar flavour to questions studied by Licea-Newman-Piza [30] in the context of first-passage percolation.

For the first question we define a critical exponent \( \xi^{(1)} \): it describes the behaviour of the hitting distribution of the hyperplane \( \partial \Lambda(\theta, L) \) for large \( L \). It is a scale for the concentration of the hitting distribution of \( \partial \Lambda(\theta, L) \) for the perturbed Brownian motion. For \( \theta \in [0, 2\pi) \), \( L > 0 \) and \( \gamma > 0 \), we denote by \( B_\theta(L, \gamma) \) the event that the hitting point \( Z_{\mathcal{H}(\partial \Lambda(\theta, L))} \) is within distance \( L^\gamma \) of \( L\hat{x}(\theta) \in \partial \Lambda(\theta, L) \), i.e.,

\[
B_\theta(L, \gamma) = \{ w \in C(\mathbb{R}_+, \mathbb{R}^d); \text{dist}(Z_{\mathcal{H}(\partial \Lambda(\theta, L))}, L\hat{x}(\theta)) \leq L^\gamma \}.
\]

The first critical exponent \( \xi^{(1)} \) is then defined as follows (thanks to the rotational invari-
Figure 1: Construction of $A_\theta(L, \gamma)$ and $B_\theta(L, \gamma)$

 ance of our model we can restrict ourselves to the angle \( \theta = 0 \):

\[
\xi^{(1)} = \inf \left\{ \gamma \geq 0; \limsup_{L \to \infty} \mathbb{E} \left[ \hat{P}_0^{\partial \Lambda_L} \left[ B_0(L, \gamma) \right] \right] = 1 \right\}.
\] (0.7)

Observe that $\mathbb{E} \left[ \hat{P}_0^{\partial \Lambda_L} \left[ Z_{H(\Lambda_L)} \subset \right] \right]$ is symmetric around $L \hat{x}(0) \in \partial \Lambda_L$; therefore it is meaningful to use symmetric sets around $L \hat{x}(0) \in \partial \Lambda_L$ in the definition (0.6).

For the second question we consider the following construction: For $\theta \in [0, 2\pi)$, $L > 0$ and $\gamma > 0$, we consider $\ell_\theta$ the line $\{ \alpha \hat{x}(\theta) \in \mathbb{R}^d; \alpha \in \mathbb{R} \}$ and the truncated cylinder of radius $L \gamma$ and symmetry axis $\ell_\theta$, $Z(\hat{x}(\theta), L \gamma) = \{ z \in \Lambda(\theta, -L); \text{dist}(z, \ell_\theta) \leq L \gamma \}$. The truncation is there for technical purposes. The boundary of $Z(\hat{x}(\theta), L \gamma)$ is denoted by $\partial Z(\hat{x}(\theta), L \gamma)$.

$A_\theta(L, \gamma)$ is the event that the perturbed Brownian path starting at the origin with goal $\partial \Lambda(\theta, L)$ does not leave the cylinder $Z(\hat{x}(\theta), L \gamma)$, i.e.,

\[
A_\theta(L, \gamma) = \{ w \in C(\mathbb{R}_+, \mathbb{R}^d); Z_s(w) \in Z(\hat{x}(\theta), L \gamma) \text{ for all } s \leq H(\partial \Lambda(\theta, L)) \}.
\] (0.8)

The second critical exponent is then defined via

\[
\xi^{(2)} = \inf \left\{ \gamma \geq 0; \limsup_{L \to \infty} \mathbb{E} \left[ \hat{P}_0^{\partial \Lambda_L} \left[ A_0(L, \gamma) \right] \right] = 1 \right\}.
\] (0.9)

Since $A_\theta(L, \gamma)$ is a subset of $B_\theta(L, \gamma)$, we have

\[
\xi^{(2)} \geq \xi^{(1)}.
\] (0.10)
Our main results are

**Theorem 0.1** In dimensions $d \geq 2$,\n
$$\xi^{(1)} \geq 1/2. \quad (0.11)$$

Using (0.10) we already see that $\xi^{(2)} \geq 1/2$ for all dimensions $d \geq 2$. In fact

**Theorem 0.2** If $d = 2$ and $\lambda > 0$,\n
$$\xi^{(2)} \geq 3/5. \quad (0.12)$$

The arguments of the proof of this superdiffusive lower bound (Theorem 0.2) also work in general dimensions $d \geq 3$ (see the proofs of Lemmas 2.1 and 2.2 below). But in this case we do not get a superdiffusive lower bound, for $d \geq 3$, $\xi^{(2)} \geq 3/(d + 3)$, which is not new in view of (0.10) and (0.11).

![Figure 2: The two hyperplanes $\partial \Lambda(0, L)$ and $\partial \Lambda(\theta_L, L)$](image)

Let us briefly describe the main ideas used to prove the two theorems. At the heart of both proofs lies the following geometric construction. We look at the two hyperplanes $\partial \Lambda(0, L)$ and $\partial \Lambda(\theta_L, L)$ where the choice of the order $\gamma$ of the angle $\theta_L = L^{-\gamma}$ will be the crucial step in the argument (see Figure 2). We prove Theorem 0.1 by contradiction along the following lines. If $\xi^{(1)}$ were less than $1/2$, then the hitting distribution of the hyperplane would be too concentrated. To see this we look at two hyperplanes $\partial \Lambda(0, L)$ and $\partial \Lambda(\theta_L, L)$, with $\theta_L = 8L^{-1/2-\bar{\varepsilon}}$ where $\varepsilon > 0$ is small (see (1.4) and (1.6) below).
In each of the two "goals" we consider sets $G_0(L) \subset \partial \Lambda(0, L)$ and $G_{\theta(L)}(L) \subset \partial \Lambda(\theta(L), L)$, where the hitting probabilities of $Z_H(\partial \Lambda(0, L))$ are concentrated. On the one hand the distance between $G_0(L)$ and $G_{\theta(L)}(L)$ is very large (it is of order $L^{1/2-\bar{\epsilon}}$), but on the other hand $G_0(L)$ (resp., $G_{\theta(L)}(L)$) is very close to $\partial \Lambda(\theta(L), L)$ (resp., $\partial \Lambda(0, L)$). Indeed, with our choice of $\theta(L)$, $G_0(L)$ is within a distance $cL^{-2\bar{\epsilon}}$ of $\partial \Lambda(\theta(L), L)$ for large $L$. This leads to a contradiction because it doesn't cost enough to go from $G_0(L)$ to $\partial \Lambda(\theta(L), L)$, i.e., the probability of hitting $\partial \Lambda(\theta(L), L)$ outside of $G_{\theta(L)}(L)$ is too large.

The proof of Theorem 0.2 will use the same geometric construction as above but $\theta(L)$ is chosen to have a different order. Assuming $\xi^{(2)}$ is less than 3/5, we choose $\theta(L) = L^{-(1-\gamma)}$ for $\gamma \in (\xi^{(2)}, 3/5)$ (see (2.2) below). Thus the convergence of the angle $\theta(L)$ to zero is slower than in the first construction. This choice of $\theta(L)$ enables us to derive an upper and a lower bound on $\text{Var} \left( -\log e_1(0, \partial \Lambda(0, L)) + \log e_1(0, \partial \Lambda(\theta(L), L)) \right)$. In Lemma 2.1 below we get an upper bound due to the fact that the two cylinders (containing "most" of the paths) are close enough, i.e., the distance between the concentration sets $G_0(L)$ and $\partial \Lambda(\theta(L), L)$ is of order $L^{3\gamma-1}$. On the other hand Lemma 2.2 below provides a lower bound relying on the fact that the two cylinders are disjoint enough. This bound is proven by a martingale argument. Comparing the exponents of the lower and the upper bound will provide the superdiffusive claim.

It is worth pointing out that the martingale argument gives a lower bound on $\text{Var} \left( -\log e_1(0, \partial \Lambda(0, L)) + \log e_1(0, \partial \Lambda(\theta(L), L)) \right)$ of the same order $L^{1-\gamma}$ (see Lemma 2.2 below) as the lower bound on $\text{Var} \left( -\log e_1(0, \partial \Lambda(0, L)) \right)$ obtained by the same martingale technique (see Theorem 1.2 of [53]). It is a natural question whether the above variance of the difference itself has the same order as $\text{Var} \left( -\log e_1(0, \partial \Lambda(0, L)) \right)$?

As mentioned above, the results we present have a similar flavour to questions which have been studied in the context of first-passage percolation. However in first-passage percolation on the square lattice the rotational invariance is lost. This turns out to be a serious problem and one has to choose "well behaved" directions to prove these statements. In our model the directions do not play any role and the geometric ideas are more transparent.

Let us give a general remark on the conjectured behaviour of $\xi$. In the physics literature questions related to fluctuations of growing surfaces and questions about transverse fluctuations have been extensively analyzed (see e.g., [29]). For a broad variety of models, whose exponent $\xi$ should have the same value as in our model, physics literature predicts $\xi = 2/3$ in dimension $d = 2$ (see [21], [24], [22], [25]), whereas in higher dimensions there are conflicting predictions, but nevertheless $\xi$ should be $\geq 1/2$ (for a discussion see [30]). In [53] we have studied a slightly different defined critical exponent $\xi$ for a point-to-point
model. For $\xi$ we have found an upper bound of $3/4$ for all dimensions $d \geq 2$ (Theorem 1.1 in [53]). These bounds on $\xi, \xi^{(1)}$ and $\xi^{(2)}$ are a first approach to the expected behaviour of $\xi$. In the point-to-plane model the geometry is somewhat easier to control. The arguments we use here to prove the diffusive and the superdiffusive lower bound do not work in the point-to-point model. In fact in Theorem 1.3 of [53] we have proved a weaker (subdiffusive) lower bound for the point-to-point model. The point-to-plane model here has a somewhat easier geometric behaviour. Using curvature properties we manage to improve the upper bound on the variance (compare (4.2) of [53] to Lemma 2.1), whereas the technique to prove the lower bound on the variance is the same (see (4.3) of [53] and Lemma 2.2).

We close this section with some remarks on how this article is organized and on the notation we use. In Section 1 we prove the diffusive statement and in Section 2 the superdiffusive lower bound. We usually denote positive constants by $c_1, c_2, \ldots$. These constants will depend only on the invariant parameters of our model, namely $d, \nu, W(\cdot), M$ and $\lambda$.

1 THE DIFFUSIVE LOWER BOUND

In this section we want to prove Theorem 0.1. As mentioned in the previous section we will see that the hitting distribution of the hyperplane can not be too concentrated. During the proof we will also see the real advantage of using the point-to-plane model. It is an open problem to translate this proof to the point-to-point model.

Before we start with the proof, we introduce some notations for the concentration set: For $\theta \in [0, 2\pi)$, $L > 0$ and $\gamma > 0$, we consider a subset $G_\theta(L, \gamma)$ of $\partial A(\theta, L)$, which is of diameter $2L^\gamma$ and which is symmetric around $L\hat{x}(\theta) \in \partial A(\theta, L)$:

$$G_\theta(L, \gamma) = \partial A(\theta, L) \cap Z(\hat{x}(\theta), L^\gamma).$$

When $\theta = 0$, we drop the subscript $\theta$, of course

$$B_\theta(L, \gamma) = \{Z_{H(\partial A, \theta, L)} \in G_\theta(L, \gamma)\}.$$  

Proof of Theorem 0.1. We will prove this theorem by contradiction. Suppose

$$\xi^{(1)} < 1/2.$$  

We choose

$$\gamma = 1/2 - \tilde{\varepsilon} \in (\xi^{(1)}, 1/2).$$
Therefore by the definition of $\xi^{(1)}$ we have: there exists a sequence $(L_n)_n$ with $L_n$ tending to infinity as $n \to \infty$, such that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\mathbb{E}\left[\hat{P}_0^{\partial \Lambda L_n} [Z_{H(\partial \Lambda L_n)} \in G(L_n, \gamma)]\right] \geq 1 - \varepsilon,$$

where $G(L_n, \gamma)$ is the deterministic set in the hyperplane $\partial \Lambda(0, L_n)$ defined in (1.1). We define

$$\theta_{L_n} = 8L_n^{-1/2-\varepsilon},$$

where $\Lambda_n = \Lambda(0, L_n)$, $\Lambda'_n = \Lambda(\theta_{L_n}, L_n)$, $G_n = G_0(L_n, \gamma)$ and $G'_n = G_{\theta_{L_n}}(L_n, \gamma)$. $\mathcal{P}_n$ is the intersection of $\Lambda'_n$, $(\Lambda'_n)^c$, $\Lambda^c(-\pi/2, 0)$ and $\Lambda^c(\theta_{L_n} + \pi/2, 0)$. $\mathcal{P}_n$ intersected with the 1-2-hyperplane (called $P_n$) is a polygon with the following properties (see Figure 3): We want to enumerate its vertices as $0, z_1, z_2, z_3$ in a counterclockwise way so that $0$ is the origin. $P_n$ is a convex polygon with

$$\text{dist}(0, z_1) = \text{dist}(0, z_3) = L_n,$$

and

$$\text{dist}(z_1, z_2) = \text{dist}(z_2, z_3) = L_n \tan(\theta_{L_n}/2) \geq 2L_n^{1/2-\varepsilon} \quad \text{for all large } n.$$
Therefore, because \( \text{diam}(G_n) \) and \( \text{diam}(G'_n) \) are equal to \( L_n^{1-\varepsilon} \) for all \( n \) (where \( \text{diam}(\cdot) \) is the diameter of the set \( \cdot \) restricted to the 1-2-hyperplane) and because \( G_n \) (resp., \( G'_n \)) is symmetric with respect to \( z_1 \) (resp., \( z_3 \)), we know that

\[
\text{dist}(G_n, G'_n) \geq \sqrt{2} L_n^{1-\varepsilon} \quad \text{for all large } n,
\]

and

\[
\forall x \in G_n \quad \forall x' \in G'_n \quad \text{dist}(x, \partial \Lambda_n') \leq 9 L_n^{1-\varepsilon} \sin(\theta_{L_n}) \leq 72 L_n^{-2\varepsilon} \quad \text{for all large } n.
\]

So the distance between \( G_n \) and \( G'_n \) tends to infinity, but \( G_n \) is very close to \( \partial \Lambda_n' \) and \( G'_n \) is very close to \( \partial \Lambda_n \).

Define \( \Omega_n = \{\omega \in \Omega; e_{\lambda}(0, \partial \Lambda_n, \omega) \leq e_{\lambda}(0, \partial \Lambda'_n, \omega)\} \). By symmetry we have \( \mathbb{P}[\Omega_n] \geq 1/2 \) for all \( n \). We define for \( \omega \in \Omega \) and \( n \in \mathbb{N} \),

\[
f_n(\omega) = 1_{\Omega_n}(\omega) \cdot \frac{e_{\lambda}(0, \partial \Lambda_n, \omega)}{e_{\lambda}(0, \partial \Lambda'_n, \omega)} \leq 1.
\]

Further we define

\[
\tilde{\delta} = \mathbb{P}_0[H(\partial \Lambda(0, 1/2)) \leq 1] > 0.
\]

Now we try to find a contradiction for arbitrarily large \( n \). In Lemma 1.1 we prove that for all \( \varepsilon \in (0, 1/2) \) we have that \( \mathbb{E} \left[ \hat{P}_{0}^{\partial \Lambda_n} \left[ Z_H(\partial \Lambda_n) \in G'_n \right] \cdot f_n(\omega) \right] > \exp\left\{ -(\lambda + M) \right\} (1/2 - \varepsilon)^{\tilde{\delta}} \) for all large \( n \). This is due to the fact that the angle \( \theta_{L_n} \) tends fastly to zero. In Lemma 1.2 we prove that for all \( \varepsilon > 0 \) we have that \( \mathbb{E} \left[ \hat{P}_{0}^{\partial \Lambda_n} \left[ Z_H(\partial \Lambda_n) \in G'_n \right] \cdot f_n(\omega) \right] < \varepsilon \) for all large \( n \). This is so because the distance between the goals \( G_n \) and \( G'_n \) is big for large \( n \). Combining these two lemmas leads to a contradiction. This finishes the proof of Theorem 0.1.

\[\square\]

**Lemma 1.1** Assume \( (1.3) \), then for all \( \varepsilon \in (0, 1/2) \) there exists \( N_1 \in \mathbb{N} \) such that for all \( n \geq N_1 \) we have that

\[
\mathbb{E} \left[ \hat{P}_{0}^{\partial \Lambda_n} \left[ Z_H(\partial \Lambda_n) \in G'_n \right] \cdot f_n(\omega) \right] > \exp\left\{ -(\lambda + M) \right\} (1/2 - \varepsilon)^{\tilde{\delta}}.
\]

**Proof of Lemma 1.1.** We have

\[
\hat{P}_{0}^{\partial \Lambda_n} \left[ Z_H(\partial \Lambda_n) \in G'_n \right] \geq \hat{P}_{0}^{\partial \Lambda_n} \left[ Z_H(\partial \Lambda_n) \in G_n, H(\partial \Lambda_n) \leq H(\partial \Lambda'_n) < \infty \right] + \hat{P}_{0}^{\partial \Lambda_n} \left[ Z_H(\partial \Lambda_n) \in G'_n, H(\partial \Lambda_n) < H(\partial \Lambda'_n) \leq H(\partial \Lambda'_n) + 1 \right].
\]

\[\square\]
Considering the first term on the right-hand side of (1.12), we find
\[\hat{P}_0^{\partial_n} [Z_{H(\partial\Lambda_n)} \in G'_n, \ H(\partial\Lambda_n) \leq H(\partial\Lambda'_n) < \infty] \geq \frac{1}{c_{\lambda}(0, \partial\Lambda_n)} E_0 \left[ \exp \left\{ - \int_0^{H(\partial\Lambda_n)} (\lambda + V)(Z_s) \, ds \right\} \right] E_n \left[ \exp \left\{ - \int_0^{H(\partial\Lambda'_n)} (\lambda + V)(Z_s) \, ds \right\} \right], \]
whereas for the second term on the right-hand side of (1.12), we have the following lower bound (using the strong Markov property),
\[\hat{P}_0^{\partial_n} [Z_{H(\partial\Lambda_n)} \in G'_n, \ H(\partial\Lambda_n) < H(\partial\Lambda'_n) < \infty + 1] \geq \exp\left\{ - (\lambda + M) \right\} \inf_{x \in G'_n} P_x [H(\partial\Lambda_n) \leq 1] \cdot E_n \left[ \exp \left\{ - \int_0^{H(\partial\Lambda_n)} (\lambda + V)(Z_s) \, ds \right\} \right] E_n \left[ \exp \left\{ - \int_0^{H(\partial\Lambda'_n)} (\lambda + V)(Z_s) \, ds \right\} \right].\]

Adding these two estimates, and using the fact that for all large \( n \) the distance between \( z \in G'_n \) and \( \partial\Lambda_n \) is less than 1/2, we find (using rotational invariance and (1.5))
\[\mathbb{E} \left[ \hat{P}_0^{\partial_n} [Z_{H(\partial\Lambda_n)} \in G'_n] \cdot f_n(\omega) \right] \geq \exp\left\{ - (\lambda + M) \right\} \tilde{\delta} \cdot \mathbb{E} \left[ \hat{P}_0^{\partial_n} [Z_{H(\partial\Lambda_n)} \in G'_n] \cdot f_n \right] \geq \exp\left\{ - (\lambda + M) \right\} \tilde{\delta} \mathbb{E} \left[ \hat{P}_0^{\partial_n} [Z_{H(\partial\Lambda_n)} \in G'_n] \cdot \mathbb{E} \left[ \hat{P}_0^{\partial_n} [Z_{H(\partial\Lambda_n)} \in G'_n] \cdot \mathbb{E} [\xi_n] \right] \right] \geq \exp\left\{ - (\lambda + M) \right\} \tilde{\delta} (1 - \varepsilon - 1/2) \quad \text{for all large} \ n. \]
This finishes the proof of the lemma.

\[\square\]

**Lemma 1.2** Assume (1.3). For any \( \varepsilon > 0 \) there exists \( N_2 \in \mathbb{N} \) such that for all \( n \geq N_2 \) we have that
\[\mathbb{E} \left[ \hat{P}_0^{\partial_n} [Z_{H(\partial\Lambda_n)} \in G'_n] \cdot f_n(\omega) \right] < \varepsilon. \quad (1.14)\]

**Proof of Lemma 1.2.** Using (1.5), we have for all large \( n \):
\[\mathbb{E} \left[ \hat{P}_0^{\partial_n} [Z_{H(\partial\Lambda_n)} \in G'_n] \cdot f_n(\omega) \right] \leq \mathbb{E} \left[ \hat{P}_0^{\partial_n} [Z_{H(\partial\Lambda_n)} \in G_n, \ Z_{H(\partial\Lambda'_n)} \in G'_n] \cdot f_n(\omega) \right] + \hat{P}_0^{\partial_n} [Z_{H(\partial\Lambda_n)} \notin G_n] \quad (1.15)\]
\[\leq \mathbb{E} \left[ \hat{P}_0^{\partial_n} [Z_{H(\partial\Lambda_n)} \in G_n, \ Z_{H(\partial\Lambda'_n)} \in G'_n] \cdot f_n(\omega) + \hat{P}_0^{\partial_n} [Z_{H(\partial\Lambda_n)} \notin G_n] \right] + \varepsilon/3.\]
We split the first term on the right-hand side of (1.15) into two terms with respect to the event \( \{H(\partial \Lambda_n) \leq H(\partial \Lambda_n')\} \) (resp., \( \{H(\partial \Lambda_n) > H(\partial \Lambda_n')\}\)). Using the strong Markov property, and the fact that \( Z \) behaves as the unperturbed Brownian motion after time \( H(\partial \Lambda_n) \), we have

\[
E \left[ \hat{\xi}_{0}^{\partial \Lambda_n}[Z_{H(\partial \Lambda_n)} \in G_n, \ Z_{H(\partial \Lambda_n')} \in G_n'] \right] \\
\leq \sup_{z \in G_n} E_z \left[ Z_{H(\partial \Lambda_n)} \in G_n' \right]. \tag{1.16}
\]

For the second term we use the strong Markov property and the definition of \( f_n(\omega) \) to conclude,

\[
E \left[ \hat{\xi}_{0}^{\partial \Lambda_n}[Z_{H(\partial \Lambda_n')} \in G_n, \ Z_{H(\partial \Lambda_n')} > H(\partial \Lambda_n') \cdot f_n(\omega)] \right] \\
\leq \frac{1}{e_{\lambda}(0, \partial \Lambda_n)} E_0 \left[ \exp \left\{- \int_0^{H(\partial \Lambda_n')} (\lambda + V)(Z_s) ds \right\}, \ Z_{H(\partial \Lambda_n')} \in G_n' \right] \cdot 1_{\Omega_n} \\
\sup_{z \in G_n} E_z \left[ \exp \left\{- \int_0^{H(\partial \Lambda_n')} (\lambda + V)(Z_s) ds \right\}, \ Z_{H(\partial \Lambda_n')} \in G_n \right] \\
\leq \sup_{z \in G_n} E_z \left[ Z_{H(\partial \Lambda_n)} \in G_n \right]. \tag{1.17}
\]

Using standard estimates on the Brownian motion we conclude that the expressions on the right-hand side of (1.16) and (1.17) are smaller than \( \epsilon/3 \) for all large \( n \). Sticking all these results together we find that for all large \( n \) the claim of the lemma is true.

\[\Box\]

2 THE SUPERDIFFUSIVE LOWER BOUND

In this section we prove the superdiffusive lower bound for \( d = 2 \) and \( \lambda > 0 \) given in Theorem 0.2. The proof will be a combination of the ideas we have used to prove the subdiffusive lower bound for \( \xi_0 \) in Theorem 1.3 of [53] (there we have used two parallel disjoint cylinders) and the ideas to prove the diffusive lower bound \( \xi^{(1)} \geq 1/2 \). Actually we refine the technique used in Theorem 1.3 of [53] using also curvature properties. As mentioned in the introduction we will look again at the polygon construction used in the proof of Theorem 0.1, but we will choose \( \theta_L \) to be of a different order (\( \theta_L = L^{-1-n} \)), \( \gamma_1 > \xi^{(2)} \). This way we find two almost disjoint cylinders such that in Lemma 2.1 we are able to show that for large \( L \), \( \text{Var}(-\text{log} e_{\lambda}(0, \partial \Lambda(0, L)) + \text{log} e_{\lambda}(0, \partial \Lambda(\theta_L, L)) \leq c_1 L^{4n-2} \). This will use the fact that the two cylinders are close together. In Lemma 2.2 we will get
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a lower bound on the same variance of order \( L^{1-\gamma_1} \). Combining these two result will lead us to the claim of Theorem 0.2.

Proof of Theorem 0.2. In view of (0.10) we know that in all dimensions \( d \geq 2, \xi^{(2)} \geq 1/2 \). Suppose \( \xi^{(2)} < 3/5 \). We will show that this leads to a contradiction. We take

\[ \gamma_1 \in (\xi^{(2)}, 3/5). \]  

(2.1)

Then we use the polygon construction presented in the proof of Theorem 0.1. We define

\[ \theta_L = 8L^{-(1-\gamma_1)}, \]  

(2.2)

\( \Lambda_L - \Lambda(0, L) \) and \( \Lambda'_L = \Lambda(\theta_L, L) \). \( P_L \) is then the intersection of \( \Lambda'_L \), \( (\Lambda'_L)^c \), \( \Lambda^c(-\pi/2, 0) \) and \( \Lambda^c(\theta_L + \pi/2, 0) \) (\( P_L \) is the same polygon as in the previous section). We denote its corner as before in a counterclockwise way by 0, \( z_1, z_2 \) and \( z_3 \), where 0 is the origin. The following properties are true

\[ \text{dist}(0, z_1) = \text{dist}(0, z_3) = L, \]  

(2.3)

and

\[ \text{dist}(z_1, z_2) = \text{dist}(z_2, z_3) = L \tan(\theta_L/2) \geq 2L^{\gamma_1} \quad \text{for all large } L. \]  

(2.4)

Therefore, we see that for all large \( L \),

\[ \text{dist}(z_1, z_3) \geq 3L^{\gamma_1}. \]  

(2.5)

Next we choose

\[ \gamma_2 \in (\xi^{(2)}, \gamma_1). \]  

(2.6)

By the definition of \( \xi^{(2)} \), there exists a sequence \( (L_n)_{n \geq 1} \) with \( L_n \to \infty \) such that

\[ \mathbb{E} \left[ \int_0^{\partial \Lambda_{L_n}} [A(L_n, \gamma_2)] \right] \to 1 \quad \text{for } n \to \infty. \]  

(2.7)

Using (2.4) and (2.5) we see that \( z_2 \notin \hat{Z}(0, L_n \gamma_2) \cup \hat{Z}(\theta_{L_n}, L_n \gamma_2) \). We will see that these two cylinders are "well separated", i.e., that we can apply the same methods as in the proof of Theorem 1.3 of [53] for parallel cylinders: We find a lower bound for the variance of the difference between the two random variables \( -\log e_\lambda(0, \partial \Lambda_{L_n}) \) and \( -\log e_\lambda(0, \partial \Lambda'_{L_n}) \) (see Lemma 2.2) as well as an upper bound (see Lemma 2.1). We slightly modify our goals \( \partial \Lambda_{L_n} \) and \( \partial \Lambda'_{L_n} \). Look at the line-segment with endpoints \( z_1 + (z_1 - z_2) \) and \( z_2 + 2(z_2 - z_1) \).
(resp., \(z_3 + (z_3 - z_2)\) and \(z_2 + 2(z_2 - z_3)\)), let \(\partial_n\) (resp., \(\partial'_n\)) be the closure of the 1/2-neighborhood of first (resp., second) line-segment intersected with \(\Lambda L_n\) (resp., \(\Lambda'_L_n\)) (see Figure 4). For \(\lambda > 0\), \(\omega \in \Omega\) and \(n \geq 1\), we define our new random variables as follows,

\[-\log \hat{e}_{\lambda,n}(\omega) = -\log e_{\lambda}(0, \partial_n, \omega), \tag{2.8}\]

and

\[-\log \hat{e}'_{\lambda,n}(\omega) = -\log e_{\lambda}(0, \partial'_n, \omega), \tag{2.9}\]

where

\[e_{\lambda}(0, \partial_n, \omega) = E_0 \left\{ \exp \left\{ - \int_{0}^{H(\partial_n)} (\lambda + V)(Z_s, \omega)ds \right\}, H(\partial_n) < \infty \right\}, \tag{2.10}\]

with \(H(\partial_n) = \inf\{s \geq 0; Z_s \in \partial_n\}\), \(e_{\lambda}(0, \partial'_n, \omega)\) is defined analogously with respect to \(\partial'_n\). The path measure starting in 0 conditioned to reach \(\partial_n\) (resp., \(\partial'_n\)) in finite time with respect to the Poissonian potential \((\omega \in \Omega)\) will be denoted by \(\hat{P}_{\partial_n}^{\theta_n} = \hat{P}_{\partial'_n}^{\theta_n}(\omega)\) (resp., by \(\hat{P}_{0}^{\partial_n}\)). Define

\[A_{\partial_n}(L_n, \gamma_2) = \{w \in C(\mathbb{R}^+, \mathbb{R}^d); w(0) = 0\ \text{and}\ Z_s(w) \in Z(\hat{x}(\theta), L^*_n) \} \tag{2.11}\]

for all \(s \leq H(\partial_n)\).

For all \(w \in C(\mathbb{R}^+, \mathbb{R}^d)\) we have \(H(\partial_n) \geq H(\partial L_n)\), whereas if \(w \in A_{\partial_n}(L_n, \gamma_2)\) we have a strict equality, \(H(\partial_n) = H(\partial L_n)\), therefore using (2.7) we see that

\[\mathbb{E} \left[ \hat{P}_{0}^{\partial_n}[A_{\partial_n}(L_n, \gamma_2)] \right] \to 1 \quad \text{for} \ n \to \infty. \tag{2.12}\]
Next we state the two main lemmas of this proof.

**Lemma 2.1** There exists a constant $c_1 \in (0, \infty)$ such that for all large $n$,

$$\text{Var} (-\log \hat{e}_{\lambda,n} + \log \hat{\varepsilon}_{\lambda,n}') \leq c_1 (L_{n}^{2\gamma -1})^2. \quad (2.13)$$

**Proof of Lemma 2.1.** Of course, $\partial_n$ is within distance $32L_{n}^{2\gamma -1}$ from $\partial'_n$ for large $n$ and also $\partial'_{n}$ is within distance $32L_{n}^{2\gamma -1}$ from $\partial_n$ for large $n$. Therefore using the strong Markov property, we see that for any $\omega \in \Omega$:

$$| -\log \hat{e}_{\lambda,n} + \log \hat{\varepsilon}_{\lambda,n}' | \leq \sup_{z \in \partial_n \cap \partial'_n} -\log e_{\lambda}(z, \partial'_n, \omega) + \sup_{z \in \partial'_n} -\log e_{\lambda}(z, \partial_n, \omega). \quad (2.14)$$

Now using a tubular estimate for Brownian motion (see (1.35) in [49]) and the fact that $\partial_n$ is in distance $32L_{n}^{2\gamma -1}$ of $\partial'_n$ and vice versa, we see that there exists $c_2 \in (0, \infty)$ such that

$$| -\log \hat{e}_{\lambda,n} + \log \hat{\varepsilon}_{\lambda,n}' | \leq c_2 L_{n}^{2\gamma -1} \text{ for all large } n. \quad (2.15)$$

Therefore (2.13) holds. \hfill \Box

**Lemma 2.2** There exists a constant $c_3 \in (0, \infty)$ such that for all large $n$,

$$\text{Var} (-\log \hat{e}_{\lambda,n} + \log \hat{\varepsilon}_{\lambda,n}') \geq c_3 L_{n}^{1-\gamma}. \quad (2.16)$$

Lemmas 2.1 and 2.2 lead to the conclusion that $4\gamma_1 - 2 \geq 1 - \gamma_2$, which contradicts $\gamma_1 < 3/5$. This finishes the proof of Theorem 0.2. \hfill \Box

So it remains to proof Lemma 2.2. We remark, as mentioned before, that this proof is analogous to the proof of formula (4.3) of [53].

**Proof of Lemma 2.2.** By $a = a(W) > 0$ we denote the smallest possible $a \in \mathbb{R}_+$ such that $W(\cdot) = 0$ on $\bar{B}(0, a)^c$, where $\bar{B}(0, a)$ is the closed Euclidean ball with center $0 \in \mathbb{R}^d$ and radius $a > 0$. We then start by introducing a paving of the plane $\mathbb{R}^2$. Let $(q_k)_{k \in \mathbb{N}}$ be a deterministic ordering of the points in $\mathbb{Z}^2$. For $k \in \mathbb{N}$, we consider the cubes of size $l$ and center $q_k$,

$$C_k = \{z \in \mathbb{R}^2; -l/2 \leq z - lq_k < l/2, \text{ for } i = 1, 2\},$$

where $l = l(d, \nu, a) \in (d(4 + 8a), \infty)$ is fixed but sufficiently large, such that (1.26) of [49] holds. The closed $a$-neighborhood of $C_k$ is denoted by $\tilde{C}_k$. The entrance time of the
motion into the closed cube $C_k$ will be denoted by $H_k = H(C_k)$ and analogously $\tilde{H}_k$ will be the entrance time of the motion into the close cube $\tilde{C}_k$. If $U \subset \mathbb{R}^2$ is a subset, we define $\mathcal{F}(U)$ to be the $\sigma$-algebra generated by all $\omega(A)$, where $A \in \mathcal{B}(\mathbb{R}^2)$ and $A \subset U$. Then we introduce a filtration $(\mathcal{F}_k)_{k}$ on our probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

$$\mathcal{F}_k = \mathcal{F}(\cup_{i=1}^{k} C_i) \quad \text{for } k \geq 1 \quad \text{and} \quad \mathcal{F}_0 = \{\emptyset, \mathcal{F}\},$$

$$\mathcal{G}_k = \mathcal{F}(C_k) \quad \text{for } k \geq 1.$$

Of course we have that for $k \geq 1$, $\mathcal{F}_k = \mathcal{G}_1 \vee \mathcal{G}_2 \vee \cdots \vee \mathcal{G}_k$, where $\mathcal{G}_j \vee \mathcal{G}_i$ is the smallest $\sigma$-algebra containing $\mathcal{G}_i$ and $\mathcal{G}_j$.

Using the filtration, we see (as in [53] formulas (3.3) and (4.4)) that the following martingale estimate holds,

$$\text{Var} \left( -\log \tilde{\lambda}_{\Delta,n} + \log \tilde{\lambda}_{\Delta,n}^{\prime} \right) \geq \sum_{k \geq 1} \text{Var} \left( \mathbb{E}[ -\log \tilde{\lambda}_{\Delta,n} + \log \tilde{\lambda}_{\Delta,n}^{\prime} | \mathcal{G}_k ] \right). \quad (2.17)$$

Next, for $k \in \mathbb{N}$, we introduce (as in [53]) the events $D_{0,k}$, $D_{1,k}$, $D_0^k$, $D_1^k$: If $\omega \in \Omega$ is a cloud configuration, we write $\tilde{\omega}_k$ for the restriction of $\omega$ to $C_k^c$ and we write $\omega_k$ for the restriction of $\omega$ to $C_k$, so we have the following decomposition $\omega = (\omega_k, \tilde{\omega}_k) \in \Omega$. Now we define the events, $D_{0,k} = \{ \omega_k; \omega_k(\tilde{C}_k) = 0 \}$ and $D_{1,k} = \{ \omega \in \Omega; \omega_k \in D_{0,k} \}$. These are the events that no point of the cloud falls into the closed cube $\tilde{C}_k$. The disjoint events on $C_k$ will be $D_{1,k} - \{ \omega_k; \omega_k(B(lqk,1) \geq 1) \}$ and $D_{1,k}^k = \{ \omega \in \Omega; \omega_k \in D_{1,k} \}$. These are the events that we have at least one point of the cloud in the center of $C_k$ (i.e., in the closed ball with radius 1 around $lq_k$). Of course, $D_0^k$ and $D_1^k$ are disjoint, $\mathcal{G}_k$ measurable and $p = \mathbb{P}[D_0^k] > 0$, $q = \mathbb{P}[D_1^k] > 0$. Using Lemma 3 of [35], we find that

$$\text{Var} \left( \mathbb{E}[ -\log \tilde{\lambda}_{\Delta,n} + \log \tilde{\lambda}_{\Delta,n}^{\prime} | \mathcal{G}_k ] \right) \geq \frac{pq}{p+q} (x_1 - x_0)^2, \quad (2.18)$$

with $x_\delta$ the numbers

$$x_\delta = \mathbb{E} \left[ \mathbb{E}[ -\log \tilde{\lambda}_{\Delta,n} + \log \tilde{\lambda}_{\Delta,n}^{\prime} | \mathcal{G}_k ] \cdot 1_{D_0^k} \right] \quad \text{for } \delta = 0, 1. \quad (2.19)$$

We define the random variable $\Psi_{k,n}(\tilde{\omega}_k)$ on $C_k^c$ as follows, for $k \in \mathbb{N}$, $\omega \in \Omega$, $\sigma_k^0 \in D_0^k$ (notice that $D_0^k$ contains only one element),

$$\Psi_{k,n}(\tilde{\omega}_k) = \inf_{\omega_k \in D_{1,k}} -\log \frac{\tilde{\lambda}_{\Delta,n}(\omega_k, \tilde{\omega}_k)}{\tilde{\lambda}_{\Delta,n}(\omega_k, \tilde{\omega}_k)} + \log \frac{\tilde{\lambda}_{\Delta,n}(\sigma_k^0, \tilde{\omega}_k)}{\tilde{\lambda}_{\Delta,n}(\sigma_k^0, \tilde{\omega}_k)} \quad (2.20)$$
We remark that $\Psi_{k,n}$ is measurable. This can be seen by using an approximation of the cloud configurations by cloud configurations with rational coordinates. As in (3.4) of [53], we see that
\begin{equation}
|x_1 - x_0| \geq \mathbb{E}[\Psi_{k,n}]_+ = \max \{\mathbb{E}[\Psi_{k,n}], 0\}.
\end{equation}

Next we define the finite set of all the labels $k \in \mathbb{N}$, such that the boxes $C_k$ intersect the truncated cylinder $Z(\hat{x}(0), L_{t_n}^\epsilon) \cap \Lambda_{t_n}^\epsilon \cap \Lambda_{t_n/2}$:
\begin{equation}
\mathcal{E}_n = \{k \in \mathbb{N}; C_k \cap (Z(\hat{x}(0), L_{t_n}^\epsilon) \cap \Lambda_{t_n}^\epsilon \cap \Lambda_{t_n/2}) \neq \emptyset\}.
\end{equation}

Thus the number of points $k \in \mathbb{N}$, which lie in $\mathcal{E}_n$, is bounded from above by $c_4 L_{t_n}^{1+\gamma_2}$. Therefore we find, using (2.17), (2.18), (2.21) and the Cauchy-Schwarz inequality,
\begin{equation}
\forall n \in \mathbb{N} \left( - \log \hat{\epsilon}_{\lambda,n} + \log \hat{\epsilon}_{\lambda,n}' \right) \geq \frac{pq}{p+q} \sum_{k \in \mathcal{E}_n} \left( \mathbb{E}[\Psi_{k,n}]_+ \right)^2 \geq \frac{pq}{c_4(p+q)} L_{t_n}^{1-\gamma_2} \left( \frac{1}{L_n} \sum_{k \in \mathcal{E}_n} \mathbb{E}[\Psi_{k,n}]_+ \right)^2 \geq \frac{pq}{c_4(p+q)} L_{t_n}^{1-\gamma_1} \left( \frac{1}{L_n} \sum_{k \in \mathcal{E}_n} \mathbb{E}[\Psi_{k,n}]_+ \right)^2,
\end{equation}
where in the last step we have used that $\gamma_2 < \gamma_1$. To prove the lower bound (2.16) we have to verify
\begin{equation}
\liminf_{n \to \infty} \frac{1}{L_n} \sum_{k \in \mathcal{E}_n} \mathbb{E}[\Psi_{k,n}] > 0.
\end{equation}

Now
\begin{equation}
\Psi_{k,n}(\hat{\omega}_k) \geq \inf_{\omega_k \in D_{1,k}} \log \left( \frac{\hat{\epsilon}_{\lambda,n}(\omega_k, \hat{\omega}_k)}{\hat{\epsilon}_{\lambda,n}(\sigma^0_k, \hat{\omega}_k)} \right) - \sup_{\omega_k \in D_{1,k}} \log \left( \frac{\hat{\epsilon}_{\lambda,n}'(\sigma^0_k, \hat{\omega}_k)}{\hat{\epsilon}_{\lambda,n}'(\omega_k, \hat{\omega}_k)} \right).
\end{equation}

Therefore it suffices to show that
\begin{equation}
\liminf_{n \to \infty} \frac{1}{L_n} \sum_{k \in \mathcal{E}_n} \mathbb{E} \left[ \inf_{\omega_k \in D_{1,k}} \log \left( \frac{\hat{\epsilon}_{\lambda,n}(\sigma^0_k, \hat{\omega}_k)}{\hat{\epsilon}_{\lambda,n}(\omega_k, \hat{\omega}_k)} \right) \right] > 0,
\end{equation}
and
\begin{equation}
\lim_{n \to \infty} \frac{1}{L_n} \sum_{k \in \mathcal{E}_n} \mathbb{E} \left[ \sup_{\omega_k \in D_{1,k}} \log \left( \frac{\hat{\epsilon}_{\lambda,n}'(\sigma^0_k, \hat{\omega}_k)}{\hat{\epsilon}_{\lambda,n}'(\omega_k, \hat{\omega}_k)} \right) \right] = 0.
\end{equation}
The proofs of (2.26) and (2.27) are exactly the same as the proofs of (4.7) and (4.8) in [53]. Therefore we will only give the structure of the rest of the proof. Using Lemmas 3.1, 3.2 and 3.3 of [53], Harnack’s inequality and the fact that \( \lambda > 0 \), the claims (2.26) and (2.27) can be reduced to the following two statements,

\[
\liminf_{n \to \infty} \frac{1}{L_n} \sum_{k \in \mathcal{E}_n \setminus N(\partial_n)} \mathbb{E} \left[ \tilde{H}_k \leq H(\partial_n) \right] > 0,
\]

and

\[
\lim_{n \to \infty} \frac{1}{L_n} \sum_{k \in \mathcal{E}_n} \mathbb{E} \left[ \tilde{H}_k \leq H(\partial_n') \right] = 0.
\]

\( N(\partial_n) \) denotes the following set: Choose \( R \) minimal such that \( \tilde{C}_k \subset B(lq_k, R) \). \( \tilde{C}_k \) is a neighboring box of the goal \( \partial_n \), if \( B(lq_k, R + 2) \cap \partial_n \neq \emptyset \).

\[
N(\partial_n) = \{ k \in \mathbb{N}; \tilde{C}_k \text{ is a neighboring box of } \partial_n \}.
\]

**Proof of (2.28).** Using the properties of \( A_{\partial_n}(L_n, \gamma_2) \) we find

\[
\frac{1}{L_n} \sum_{k \in \mathcal{E}_n \setminus N(\partial_n)} \mathbb{E} \left[ \tilde{P}_0^{\partial_n} \left[ H_k \leq H(\partial_n) \right] \right] \geq \frac{1}{L_n} \mathbb{E} \left[ \tilde{E}_0^{\partial_n} \left[ \sum_{k \in \mathcal{E}_n \setminus N(\partial_n)} 1_{\{H_k \leq H(\partial_n)\}} 1_{A_{\partial_n}(L_n, \gamma_2)} \right] \right] \geq \frac{1}{L_n} c_0 \frac{L_n}{2} \mathbb{E} \left[ \tilde{P}_0^{\partial_n} \left[ A_{\partial_n}(L_n, \gamma_2) \right] \right],
\]

which stays strictly positive as \( n \) tends to infinity.

**Proof of (2.29).** For all large \( n \) the following is true

\[
\bigcup_{k \in \mathcal{E}_n} \tilde{C}_k \cap Z(\hat{\theta}(\theta_{L_n}), L_n^{\gamma_2}) = \emptyset.
\]

Thus, if \( |\mathcal{A}_n| \) is the number of visited boxes \( C_k \) before reaching the goal \( \partial_n' \), we have for all large \( n \) (using Cauchy-Schwarz inequality),

\[
\frac{1}{L_n} \sum_{k \in \mathcal{E}_n} \mathbb{E} \left[ \tilde{P}_0^{\partial_n} \left[ H_k \leq H(\partial_n') \right] \right] \leq \frac{3^2}{L_n} \mathbb{E} \left[ \tilde{E}_0^{\partial_n} \left[ \mathcal{A}_n 1_{A_{\partial_n}(L_n, \gamma_2)} \right] \right] \leq 3^2 \mathbb{E} \left[ \tilde{E}_0^{\partial_n} \left[ (|\mathcal{A}_n|/L_n)^2 \right] \right]^{1/2} \mathbb{E} \left[ \tilde{P}_0^{\partial_n} \left[ A_{\partial_n}(L_n, \gamma_2) \right] \right]^{1/2}.
\]

The last term on the right-hand side of (2.33) tends to zero as \( n \) goes to infinity, whereas the first term stays bounded (see (1.31) of [49]). This finishes the proof of Lemma 2.2.

\( \square \)

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Geodesics and crossing Brownian motion in a soft Poissonian potential

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Abstract

We compare the model of crossing Brownian motion in a soft Poissonian potential to the model of continuum first-passage percolation among soft Poissonian obstacles. In both models we construct (via a shape theorem) a deterministic norm on $\mathbb{R}^d$ called Lyapounov coefficient, resp., time-constant. The main theorem of this article claims that the properly rescaled Lyapounov coefficient converges to the time-constant as the strength of the potential tends to infinity.

Résumé. – Nous comparons le modèle d'un mouvement Brownien traversant un potentiel Poissonien, au modèle continu de percolation de premier passage associé à des obstacles Poissoniens. Dans chacun des modèles, la distance naturelle d'un point à l'origine se comporte asymptotiquement comme une norme déterministe sur $\mathbb{R}^d$ appelée respectivement coefficient de Liapounov, et constante-temps. Le théorème principal de cet article montre que le coefficient de Liapounov convenablement normalisé converge vers la constante-temps lorsque la force du potentiel tend vers l'infini.

0 INTRODUCTION AND RESULTS

In this article we consider crossing Brownian motion evolving in a soft Poissonian potential. Crossing Brownian motion describes the evolution of Brownian motion in a Poissonian potential conditioned to reach a remote location. Various properties of this model have been studied in the literature (see, e.g., Sznitman [46, 50]). The exponential decay of the normalizing constant of crossing Brownian motion, as its goal is tending to infinity, is described by a deterministic norm $\alpha_{\lambda,\beta}(\cdot)$, called the Lyapounov coefficient or norm. If we increase the strength of the Poissonian potential by a factor $\beta$, we observe that "non-optimal" paths lose weight in probability. This strengthening of the potential
leads to the idea of comparing the Lyapounov coefficient (after rescaling with $\beta^{-1/2}$) with the random Riemannian distance associated with the Poissonian potential, where only optimal paths (geodesics) survive. Considering this second model, we observe (as in first passage percolation on $\mathbb{Z}^d$) that the random distance $g_\lambda(0, x) \sim \mu_\lambda(x)$, as $|x| \to \infty$, where $\mu_\lambda$ is a deterministic norm on $\mathbb{R}^d$. Our main theorem (see Theorem 0.3 below) states that $\beta^{-1/2} \alpha_{\lambda, \beta} \to \mu_\lambda$, as $\beta \to \infty$.

Let us precisely describe the setting. For $x \in \mathbb{R}^d$ ($d \geq 1$), we denote by $P_x$ the Wiener measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ starting at site $x$. $Z$ denotes the canonical process on $C(\mathbb{R}_+, \mathbb{R}^d)$. We denote by $\mathbb{P}$ the Poissonian law with fixed intensity $\nu > 0$ on the space $\Omega$ of locally finite, simple, pure point measures on $\mathbb{R}^d$. For a cloud configuration $\omega = \sum_i \delta_{x_i} \in \Omega$, $\lambda > 0$ and $x \in \mathbb{R}^d$ the soft Poissonian potential is defined as

$$q(x, \omega) = \lambda + V(x, \omega) = \lambda + \sum_i W(x - x_i) = \lambda + \int_{\mathbb{R}^d} W(x - y)\omega(dy), \quad (0.1)$$

where the shape function $W(\cdot) \geq 0$ is bounded, continuous, compactly supported and not a.e. equal to zero. For $\lambda, \beta > 0$, $\omega \in \Omega$, $x, y \in \mathbb{R}^d$ our main object of interest will be the normalizing constant for crossing Brownian motion in the soft Poissonian potential:

$$e_{\lambda, \beta}(x, y, \omega) = E_x \left[ \exp \left\{ - \int_0^{H(y)} \beta q(Z_s, \omega)ds \right\}, H(y) < \infty \right], \quad (0.2)$$

where $H(y)$ denotes the entrance time of $Z$ into the closed ball $\overline{B}(y, 1)$. $\beta$ will be the parameter measuring the strength of the soft potential, which we let tend to infinity; $\lambda$ will be kept fixed.

The function $u(x) = e_{\lambda, \beta}(x, 0, \omega)$ appears as the $\beta q(\cdot, \omega)$-equilibrium potential of the set $\overline{B}(0, 1)$, which satisfies in a weak sense the following second order equation (see Proposition 2.3.8 of [50] or Proposition 2.3 below):

$$\begin{cases} -\frac{1}{2} \Delta u + \beta qu = 0 & \text{in } \overline{B}(0, 1)^c, \\ u = 1 & \text{on } \partial B(0, 1), \\ u = 0 & \text{at infinity } (\lambda > 0). \end{cases} \quad (0.3)$$

Furthermore, $-\log e_{\lambda, \beta}(x, y, \omega)$ has the nice property that it measures the distance between $x$ and $y$ with respect to the potential $\beta q(\cdot, \omega) = \beta (\lambda + V(\cdot, \omega))$ for our crossing Brownian motion: $-\log e_{\lambda, \beta}(\cdot, \cdot, \omega)$ is up to a small correction term a distance function on $\mathbb{R}^d$ which increases if we add additional points to the Poissonian cloud $\omega$ (for more details see Sznitman [50], formula (5.2.3), and Proposition 5.2.2).

From Sznitman [46] we have a shape theorem, describing for typical cloud configurations $\omega$ the principal behavior of $-\log e_{\lambda, \beta}(x, 0, \omega)$ as $x$ tends to infinity:
Theorem 0.1 For $\lambda \geq 0$ and $\beta > 0$, there exists a deterministic norm $\alpha_{\lambda,\beta}(\cdot)$ on $\mathbb{R}^d$ such that, on a set of full $\mathbb{P}$-measure, we have

$$\lim_{x \to \infty} \frac{1}{|x|} \left| -\log e_{\lambda,\beta}(x, 0, \omega) - \alpha_{\lambda,\beta}(x) \right| = 0. \quad (0.4)$$

The convergence takes place in $L^1(\mathbb{P})$ as well.

We call $\alpha_{\lambda,\beta}(\cdot)$ the Lyapunov coefficients. We now consider the following continuum first-passage percolation model on $\mathbb{R}^d$, obtained by considering the Riemannian distance between $x$ and $y$ with respect to the metrics $ds^2 = 2q(\cdot, \omega) dx^2$:

$$\mathcal{g}_\lambda(x, y, \omega) = \inf_{\gamma \in \mathcal{P}(x, y, 1)} \left\{ \int_0^1 \sqrt{2q(\gamma_s, \omega)|\gamma_s|ds} \right\}, \quad (0.5)$$

where $\mathcal{P}(x, y, 1)$ is the set of Lipschitz paths $\gamma$ leading in time 1 from $x$ to $y$, i.e., $\gamma_0 = x$ and $\gamma_1 = y$. Of course, $\mathcal{g}_\lambda$ also fulfills a shape theorem:

Theorem 0.2 For $\lambda > 0$, there exists a deterministic norm $\mu_\lambda(\cdot)$ on $\mathbb{R}^d$ such that, on a set of full $\mathbb{P}$-measure, we have

$$\lim_{x \to \infty} \frac{1}{|x|} \left| \mathcal{g}_\lambda(x, 0, \omega) - \mu_\lambda(x) \right| = 0. \quad (0.6)$$

The convergence also holds in $L^1(\mathbb{P})$.

Following the terminology of the first-passage percolation model on the lattice $\mathbb{Z}^d$ (see Hammersley-Welsh [18], Kesten [26]) we call $\mu_\lambda(\cdot)$ the time-constant. The connection between the two models comes in the following theorem, which is our main result:

Theorem 0.3 For $\lambda > 0$ fixed,

$$\lim_{\beta \to \infty} \frac{1}{\sqrt{\beta}} \alpha_{\lambda,\beta}(e) - \mu_\lambda(e), \quad e \in \mathbb{R}^d, \quad (0.7)$$

where the above convergence is uniform on the unit-sphere and hence uniform on every compact subset of $\mathbb{R}^d$.

Let us point out that we obtain a much sharper lower bound than upper bound on the difference between $-\beta^{-1/2} \log e_{\lambda,\beta}(x, 0, \omega)$ and $\mathcal{g}_\lambda(x, 0)$ (compare Theorem 1.1 to Corollary 2.7).

Zerner [57] has studied a discrete related model (see Proposition 9 of [57]). He considers a random walk on $\mathbb{Z}^d$ with random potentials $\omega(x)$ at the sites $x \in \mathbb{Z}^d$. He obtains a similar result to Theorem 0.3, for increasing strength of the potentials, however the normalizing
factor turns out to be $\beta$ instead of $\beta^{1/2}$ (In the continuous model Brownian motion can choose the velocity at which it passes the obstacles). The proof in the discrete model is much simpler, the quantity replacing the left member of (0.7) turns out to decrease to its limit as $\beta$ tends to infinity. We do not know whether this is the case in our model.

This article is organised as follows: In Section 1 we prove the upper bound of Theorem 0.3 (see Corollary 1.3 below). The main idea is to consider "nearly optimal tubes", along which the crossing paths should move. For Brownian motion restricted to these tubes, we use the Cameron-Martin-Girsanov transformation (see Freidlin-Wentzell [14] and Carmona-Simon [8]). This classical construction gives a bound, which is not sharp, but which is sufficient for our purpose.

In Section 2 we prove the lower bound of Theorem 0.3 (see Corollary 2.7 below). The main control comes from the use of estimates of Agmon [1] for $u(\cdot) = e_{\lambda,\beta}(\cdot; 0, \omega)$, a non-negative, bounded weak solution of the second order elliptic equation (0.3). It provides for all $\varepsilon \in (0, 1)$ an $L^2$-bound on the product $f = u_{\varepsilon}^{(1-\varepsilon)\sqrt{\beta}\xi}(\cdot, \rho)$. Having this $L^2$-bound we use Harnack type inequalities to get pointwise upper bounds on $u(\cdot) = e_{\lambda,\beta}(\cdot; 0, \omega)$ in terms of $\sqrt{\beta}\xi(\cdot, 0)$. From this we will easily deduce the proof of Corollary 2.7.

Finally in Section 3 we give a short proof of the shape theorem in continuum first-passage percolation (see Theorem 0.2). In Appendix A we provide the proof of the estimates of Agmon (see Lemma 2.2) for the reader’s convenience.

1 THE UPPER BOUND ON $\alpha_{\lambda,\beta}$

From Sznitman [50], Lemma 4.5.2, we know that there exists $\Omega_0$, a subset of $\Omega$ with full $\mathbb{P}$-measure, such that for all $\omega \in \Omega_0$

$$\sup_{x \in [-l,l]^d} V(x, \omega) = o(\log l), \quad \text{as } l \to \infty. \quad (1.1)$$

For $\delta > 0$, $\omega \in \Omega$ and $x \in \mathbb{R}^d$, we define

$$V_\delta(x, \omega) = \sup_{y \in B(x,\delta)} V(y, \omega). \quad (1.2)$$

The geodesic distance $\varrho_{\lambda,\delta}$ is then defined, for $\lambda > 0$, $\omega \in \Omega$ and $x, z \in \mathbb{R}^d$, as

$$\varrho_{\lambda,\delta}(x, z, \omega) = \inf_{\gamma \in \mathcal{P}(x,z,1)} \left\{ \int_0^1 \sqrt{2(\lambda + V_\delta(\gamma_s, \omega))}{|\gamma_s|}ds \right\}. \quad (1.3)$$

The distance function $\varrho_{\lambda,\delta}(\cdot, \cdot)$ will play an important role, because if we consider Brownian motion moving in a tube around the geodesic path $\gamma$ from $x$ to $z$ (with respect to $\varrho_\lambda(\cdot, \cdot)$),
this motion will typically experience the potential \( q(\cdot, \omega) = \lambda + V(\cdot, \omega) \) in a neighborhood of \( \gamma \).

**Theorem 1.1** There exists \( c_1 = c_1(d) \subset (0, \infty) \) such that for all \( \delta \in (0, 1) \), \( \beta \geq 1 \), \( x \in \mathbb{R}^d \) and \( \omega \in \Omega_0 \),

\[ e_{\lambda, \beta}(x, 0, \omega) \geq c_1 \exp \left\{ -\sqrt{\beta} \mathcal{D}_{\lambda, \beta}(x, 0, \omega) \left( 1 + \frac{\lambda D}{2\lambda \beta \delta^2} \right) \right\}, \tag{1.4} \]

where \( \lambda D \) is the first Dirichlet eigenvalue of \( -\frac{1}{2} \Delta \) in the ball \( B(0, 1) \).

**Proof.** Choose \( \delta \in (0, 1) \), \( \beta \geq 1 \) and \( x \in \mathbb{R}^d \) fixed. Then pick \( t > 0 \) and \( \phi \in \mathcal{P}(x, 0, t) \). We define the following tube around \( \phi \) with radius \( \delta \) up to time \( t \):

\[ T(\phi, \delta, t) = \left\{ w \in C(\mathbb{R}_+, \mathbb{R}^d); \, |w(s) - \phi_s| < \delta \text{ for all } s \in [0, t] \right\}. \tag{1.5} \]

Of course, having chosen \( \delta < 1 \), if \( Z \in T(\phi, \delta, t) \) then \( Z_t \in B(0, 1) \), hence we see that on the event \( T(\phi, \delta, t), \Pi(0) \leq t \). Therefore we have for \( \omega \in \Omega_0 \),

\[
e_{\lambda, \beta}(x, 0, \omega) = E_x \left[ \exp \left\{ -\int_0^{H(0)} \beta q(Z_s, \omega) ds \right\}, \, H(0) < \infty \right]
\geq E_x \left[ \exp \left\{ -\int_0^t \beta (\lambda + V)(Z_s, \omega) ds \right\}, \, T(\phi, \delta, t) \right] \tag{1.6}
\geq \exp \left\{ -\int_0^t \beta (\lambda + V)(\phi_s, \omega) ds \right\} \times P_x \left[ T(\phi, \delta, t) \right].
\]

Consider the last term of the above inequality. For Brownian motion remaining in the tube \( T(\phi, \delta, t) \) we use Cameron-Martin-Girsanov’s formula to describe the density of the law of \( Z_\cdot - \phi \) with respect to \( P_0 \) (see, e.g., [14]). We obtain the following lower bound (with the obvious notation that \( T(0, \delta, t) = \{ w \in C(\mathbb{R}_+, \mathbb{R}^d); \, Z_s \in B(0, \delta) \text{ for all } s \in [0, t]\} \),

\[
P_x \left[ T(\phi, \delta, t) \right] = E_0 \left[ T(0, \delta, t), \, \exp \left\{ -\int_0^t 1/2|\dot{\phi}_s|^2 ds - \int_0^t \dot{\phi}_s dZ_s \right\} \right]
\geq \exp \left\{ -\int_0^t 1/2|\dot{\phi}_s|^2 ds \right\}
\times P_0 \left[ T(0, \delta, t) \right] \exp \left\{ -E_0 \left[ \int_0^t \dot{\phi}_s dZ_s \right] T(0, \delta, t) \right\}
\leq \exp \left\{ -\int_0^t 1/2|\dot{\phi}_s|^2 ds \right\} P_0 \left[ T(0, \delta, t) \right]
\geq c_1 \exp \left\{ -\int_0^t 1/2|\dot{\phi}_s|^2 ds \right\} \times \exp \left\{ -\frac{t\lambda P}{\delta^2} \right\}, \tag{1.7}
\]
where we have used Jensen's inequality in the second step and Brownian symmetry in the third step. Hence, we have for all $t > 0$ and $\phi \in \mathcal{P}(x, 0, t)$:

$$e_{\lambda, \beta}(x, 0, \omega) \geq c_1 \exp \left\{ - \int_0^t \beta(\lambda + V_\delta)(\phi_s, \omega) ds - \int_0^t 1/2|\dot{\phi}_s|^2 ds - \frac{t \lambda \rho}{\beta^2} \right\}. \quad (1.8)$$

Next we consider the two functionals

$$L_2(\phi) = \int_0^t \beta(\lambda + V_\delta)(\phi_s, \omega) ds + \int_0^t 1/2|\dot{\phi}_s|^2 ds, \quad (1.9)$$

$$L_1(\phi) = \int_0^t \sqrt{2\beta(\lambda + V_\delta)(\phi_s, \omega)}|\dot{\phi}_s| ds. \quad (1.10)$$

**Lemma 1.2** For $\beta > 0$, $\lambda > 0$, $\delta > 0$, $x, y \in \mathbb{R}^d$ and $\omega \in \Omega_0$ we have: There exists at least one minimizing element $\psi \in \mathcal{P}(x, y, 1)$ such that

$$\sqrt{\beta} \theta_{\lambda, \delta}(x, y) = \inf_{\phi \in \mathcal{P}(x, y, 1)} L_1(\phi) = L_1(\psi). \quad (1.11)$$

Furthermore, we have

$$\inf_{t > 0} \inf_{\phi \in \mathcal{P}(x, y, t)} L_2(\phi) = L_1(\psi). \quad (1.12)$$

**Proof of Lemma 1.2.** Choose $t > 0$, $\beta > 0$, $\lambda > 0$, $\delta > 0$, $x, y \in \mathbb{R}^d$ and $\omega \in \Omega_0$. Using the inequality $2ab \leq a^2 + b^2$, we see that for all Lipschitz paths $\phi$, with $\phi_0 = x$ and $\phi_t = 0$,

$$L_2(\phi) \geq L_1(\phi). \quad (1.13)$$

Using Theorem 1, page 261 of [16], Vol. II, we know that there exists at least one minimizing element $\psi \in \mathcal{P}(x, 0, t)$ such that

$$\inf_{\phi \in \mathcal{P}(x, 0, t)} L_1(\phi) = L_1(\psi), \quad \text{and} \quad \sqrt{2\beta(\lambda + V_\delta)(\psi_s, \omega)}|\dot{\psi}_s| = L_1(\psi)/t = \text{const.} \quad \text{for a.e. } s \in [0, t]. \quad (1.14)$$

We can now find a continuous, strictly increasing Lipschitz reparametrization $s = \sigma(u)$, such that for $\tilde{\psi}(u) = \psi(\sigma(u)) \in \mathcal{P}(x, 0, \sigma^{-1}(t))$:

$$|\dot{\tilde{\psi}}_u| = \sqrt{2\beta(\lambda + V_\delta)(\tilde{\psi}_u, \omega)} \quad \text{for a.e. } u \in [0, \sigma^{-1}(t)]. \quad (1.15)$$

In view of (1.13), and because $L_1$ is invariant under reparametrization:

$$\inf_{t > 0} \inf_{\mathcal{P}(x, 0, t)} L_2(\phi) \geq \inf_{t > 0} \inf_{\mathcal{P}(x, 0, t)} L_1(\phi) = L_1(\psi) = L_1(\tilde{\psi}) = L_2(\tilde{\psi}), \quad (1.16)$$

where in the last step we have used that $2ab = a^2 + b^2$ for $a = b$. This finishes the proof of Lemma 1.2.
Let us now see, how the claim of the theorem follows. We choose a minimizing element \( \tilde{\psi} \) of \( L_1 \) satisfying (1.16), where \( \tilde{t} \) denotes the first time that \( \tilde{\psi} \) reaches 0. We have

\[
\sqrt{\beta} g_{\lambda, \delta}(x, 0) = L_1(\tilde{\psi}) = \int_0^\tilde{t} 2\beta (\lambda + V_\delta(\tilde{\psi})_s) ds \geq 2\beta \lambda \tilde{t}.
\] (1.18)

Therefore, coming back to (1.8) with \( \tilde{\psi} \) in place of \( \phi \),

\[
e_{\lambda, \delta}(x, 0, \omega) \geq c_1 \exp \left\{ -\sqrt{\beta} g_{\lambda, \delta}(x, 0, \omega) - \frac{\tilde{t} \lambda_D}{\delta^2} \right\} 
\geq c_1 \exp \left\{ -\sqrt{\beta} g_{\lambda, \delta}(x, 0, \omega) \left( 1 + \frac{\lambda_D}{2\lambda \beta \delta^2} \right) \right\}.
\] (1.19)

This finishes the proof of Theorem 1.1.

\[\square\]

**Corollary 1.3** For \( \lambda > 0 \),

\[
\limsup_{\beta \to \infty} \sup_{e \in \partial B(0,1)} \frac{1}{\beta^{1/2}} \alpha_{\lambda, \delta}(e) - \mu_\lambda(e) \leq 0.
\] (1.20)

**Proof.** Choose a unit vector \( e \in \mathbb{R}^d \), \( n \in \mathbb{N} \) and \( \delta > 0 \). Using Theorem 1.1, we see that for all \( \omega \in \Omega_0 \),

\[
-\frac{1}{\beta^{1/2} n} \log e_{\lambda, \delta}(ne, 0, \omega) \leq \frac{1}{n} g_{\lambda, \delta}(ne, 0, \omega) \left( 1 + \frac{\lambda_D}{2\lambda \beta \delta^2} \right) - \frac{\log c_1}{\beta^{1/2} n}.
\] (1.21)

Using the shape theorems (see Theorems 0.1 and 0.2) on both sides of the above inequality, we see that for \( n \to \infty \)

\[
\frac{1}{\beta^{1/2}} \alpha_{\lambda, \delta}(e) \leq \mu_{\lambda, \delta}(e) \left( 1 + \frac{\lambda_D}{2\lambda \beta \delta^2} \right),
\] (1.22)

where \( \mu_{\lambda, \delta}(\cdot) \) denotes the time-constant for the continuum first-passage percolation model with respect to the potential \( V_\delta \). On the right-hand side of (1.22) only the second factor depends on \( \beta \), and the left-hand side is independent of \( \delta \), passing to the limit we conclude

\[
\limsup_{\beta \to \infty} \frac{1}{\beta^{1/2}} \alpha_{\lambda, \delta}(e) \leq \inf_{\delta > 0} \mu_{\lambda, \delta}(e) = \lim_{\delta \to 0} \mu_{\lambda, \delta}(e).
\] (1.23)

For \( m \in \mathbb{N} \), choose \( \delta = 1/m \). Of course, \( \mu_{\lambda, 1/m}(e) \) is non-increasing in \( m \), hence the claim of our corollary follows by a Dini type argument, if we manage to prove that

\[
\lim_{m \to \infty} \mu_{\lambda, 1/m}(e) = \mu_\lambda(e).
\] (1.24)
From Kingman's subadditive ergodic theorem (see for instance Liggett [31], p.277), we know that
\[ \mu_{\lambda,1/m}(e) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\ell_{\lambda,1/m}(n,e,0)] - \inf_{n \geq 1} \frac{1}{n} \mathbb{E}[\ell_{\lambda,1/m}(n,e,0)]. \] (1.25)
Therefore, we can exchange the following infima
\[ \lim_{m \to \infty} \mu_{\lambda,1/m}(e) = \inf_{m \geq 1} \mu_{\lambda,1/m}(e) = \inf_{n \geq 1} \inf_{m \geq 1} \frac{1}{n} \mathbb{E}[\ell_{\lambda,1/m}(n,e,0)] \]
\[ = \inf_{n \geq 1} \frac{1}{n} \mathbb{E}[\ell_{\lambda}(n,e,0)] = \mu_{\lambda}(e), \] (1.26)
to apply Lebesgue's dominated convergence theorem in the last step of (1.26), we have used that \( \ell_{\lambda,1/m}(n,e,0) \leq \ell_{\lambda,1}(n,e,0) \in L^1(\mathbb{P}) \) and that \( \ell_{\lambda,1/m}(n,e,0) \to \ell_{\lambda}(n,e,0) \mathbb{P}\text{-a.s.}, \) as \( m \to \infty \) (\( W \) has been chosen to be continuous). This finishes the proof of Corollary 1.3.

\[ \square \]

2. THE LOWER BOUND ON \( \alpha_{\lambda,\beta} \)

In this section we always work with typical cloud configurations \( \omega \subset \Omega_0 \) (see (1.1)). As a result of our assumptions on \( W, V(\cdot,\omega) \) is continuous for all \( \omega \in \Omega_0 \). We pick a fixed \( \omega \in \Omega_0, \lambda > 0, \) then \( q(x,\omega) \) is a strictly positive, continuous potential on \( \mathbb{R}^d \). Therefore, using Lemma 1.3 and Theorem 1.4 of Agmon [1], we see that
\[ \ell_{\lambda}(\cdot,0) \] is locally Lipschitz,
\[ \frac{1}{2} |\nabla \ell_{\lambda}(x,0)|^2 \leq q(x), \quad \text{a.e.} \] (2.1)
(2.2)

For \( D = \bar{B}(0,1)^c \) an open subset of \( \mathbb{R}^d \), \( H^1(D) \) denotes the Sobolev space of functions \( f \in L^2(D) \) such that the distributional derivatives of \( f \) are in \( L^2(D) \). The next theorem gives us an \( L^2 \)-bound for a non-negative, bounded weak solution of the second order equation (2.3) with \( q \) as above. The key estimate to prove the theorem is a simplified version of Theorem 1.5 given in Agmon [1] (see Lemma 2.2 below).

**Theorem 2.1** Take \( \beta \geq 1 \) and suppose that \( u \in H^1_{\text{loc}}(D) \) is a weak solution of
\[ -\frac{1}{2} \Delta u + \beta qu = 0 \text{ in } D = \bar{B}(0,1)^c, \] (2.3)
in the sense that $qu \in L^1_{\text{loc}}(D)$ and

$$
\int_D \frac{1}{2} \nabla u \nabla \phi + \beta qu \phi \, dx = 0
$$

(2.4)

for every $\phi \in C_0^\infty(D)$. Furthermore, suppose that there exists $b < \infty$ such that

$$
0 \leq u(x) \leq b \quad \text{for all } x \in D.
$$

(2.5)

Then there exists a constant $c(q, \lambda) > 0$ such that for all $\varepsilon \in (0, 1)$, $\delta \in (0, 1)$,

$$
\int_D |u(x)|^2 e^{2\varepsilon(1-\varepsilon)\sqrt{\beta_\varepsilon}}(x,0) \, dx \leq \frac{b^2 c(q, \lambda)}{2 \varepsilon} \left( \frac{1 + \delta}{\delta} \right)^2 \sup_{B(0,1+\frac{1}{\sqrt{\varepsilon}})} e^{2\sqrt{\beta_\varepsilon}}(x,0).
$$

(2.6)

The above theorem is a consequence of Agmon's Theorem 1.5 in [1]. We state here a simplified version, which is sufficient in our case of strictly positive potential $q$ and positive and bounded weak solutions $u$. As we shall see, Theorem 2.1 follows from

**Lemma 2.2** Let $\beta$, $D$ and $q$ be as above. For $\delta > 0$ we define

$$
D_\delta = \{ x \in D; \, \varrho_\varepsilon(x, \partial D) > \delta \}.
$$

(2.7)

Choose $\varepsilon \in (0, 1)$ and define for $x \in D$:

$$
h(x) = (1-\varepsilon) \sqrt{\beta_\varepsilon}(x,0).
$$

(2.8)

For $u \in H^1_{\text{loc}}(D)$, a weak solution of (2.3) satisfying (2.5), we have

$$
\int_{D_{\delta/\sqrt{\varepsilon}}} |u|^2 \left( \beta q - \frac{1}{2} |\nabla h|^2 \right) e^{2h} \, dx \leq \frac{1 + 2\delta}{\delta^2} \int_{D_{\delta/\sqrt{\varepsilon}}} |u|^2 \beta q e^{2h} \, dx.
$$

(2.9)

We provide the proof of Lemma 2.2 in Appendix A.

**Proof of Theorem 2.1.** In view of Lemma 2.2, we consider $h(x) = (1-\varepsilon) \sqrt{\beta_\varepsilon}(x,0)$. Of course $h(\cdot)$ is locally Lipschitz (see (2.1) and (2.2)) with

$$
\frac{1}{2} |\nabla h(x)|^2 \leq (1-\varepsilon)^2 \beta q(x), \text{ a.e.}
$$

(2.10)

Therefore, we see that

$$
\beta q(x) - \frac{1}{2} |\nabla h(x)|^2 \geq (2\varepsilon - \varepsilon^2) \beta q(x) \geq (2\varepsilon - \varepsilon^2) \beta \lambda, \text{ a.e.}
$$

(2.11)

Using Lemma 2.2, we conclude that

$$
\int_{D_{\delta/\sqrt{\varepsilon}}} |u|^2 e^{2h} \, dx \leq \frac{1}{(2\varepsilon - \varepsilon^2) \beta \lambda} \frac{1 + 2\delta}{\delta^2} \int_{D_{\delta/\sqrt{\varepsilon}}} |u|^2 \beta q e^{2h} \, dx.
$$

(2.12)
On the other hand, because $2\varepsilon - \varepsilon^2 < 1$ for $\varepsilon \in (0, 1)$, and because $q \geq \lambda$, we have that

$$\int_{D \setminus D_{\delta/\sqrt{3}}} |u|^2 e^{2h} dx \leq \frac{1}{(2\varepsilon - \varepsilon^2)\beta \lambda} \int_{D \setminus D_{\delta/\sqrt{3}}} |u|^2 \beta q e^{2h} dx.$$  \hfill (2.13)

In view of (2.12) and (2.13) we obtain:

$$\int_D |u|^2 e^{2h} dx \leq \left( \frac{1 + \delta}{\delta} \right)^2 \frac{1}{(2\varepsilon - \varepsilon^2)\lambda} \int_{D \setminus D_{\delta/\sqrt{3}}} |u|^2 q e^{2h} dx$$  \hfill (2.14)

$$\leq \left( \frac{1 + \delta}{\delta} \right)^2 \frac{b^2}{(2\varepsilon - \varepsilon^2)\lambda} \int_{B_{(0,1+\frac{1}{\sqrt{2}\lambda})}} q(x) dx \sup_{z \in B_{(0,1+\frac{1}{\sqrt{2}\lambda})}} e^{2\sqrt{3} q(z, 0)}.$$

Now the claim follows choosing $c(q, \lambda) = \lambda^{-1} \int_{B_{(0,1+\frac{1}{\sqrt{2}\lambda})}} q(x) dx$. \hfill \Box

Our next step is to prove that for $x \in D = \bar{B}(0, 1)^c$, $u(x) = e_{\lambda, \delta}(x, 0, w)$ is a non-negative, bounded weak solution of (2.3).

**Proposition 2.3** $u(\cdot) = e_{\lambda, \delta}(\cdot, 0, w)$ is a non-negative, bounded weak solution of (2.3) on $D = \bar{B}(0, 1)^c$ with bound $b = 1$ (see (2.5)).

**Proof.** Set $K = \bar{B}(0, 1)$. For $\lambda > 0$, $e_{\lambda, \delta}(\cdot, 0, w)$ appears as the $\beta(\lambda + V(\cdot, \omega)) = \beta q(\cdot)$-equilibrium potential of $K$ (see (2.3.26) of [50]). Using Proposition 2.3.8 of [50], we see that $e_{\lambda, \delta}(\cdot, 0, w)$ is continuous on $\mathbb{R}^d$, equals 1 on $K$ and is $\beta q(\cdot)$-harmonic on $D$. Using Proposition 2.5.1 and Theorem 1.4.9 of [50], we see that $e_{\lambda, \delta}(\cdot, 0, w) \in H^1(\mathbb{R}^d)$, which finishes the proof of Proposition 2.3. \hfill \Box

Applying Theorem 2.1 to $e_{\lambda, \delta}(\cdot, 0, w)$, we conclude the following corollary:

**Corollary 2.4** ($L^2$-bound) For all $\omega \in \Omega_0$, there exists $c(\omega, W, \lambda) < \infty$ such that for all $\beta \geq 1$, $\varepsilon \in (0, 1)$ and $\delta \in (0, 1)$ we have

$$\int_D |e_{\lambda, \delta}(x, 0, \omega)|^2 e^{2(1-\varepsilon)\sqrt{3} q(x, 0, \omega)} dx \leq \frac{c(\omega, W, \lambda)}{2\varepsilon - \varepsilon^2} \left( \frac{1 + \delta}{\delta} \right)^2 \sup_{z \in B_{(0,1+\frac{1}{\sqrt{2}\lambda})}} e^{2\sqrt{3} q(x, 0, \omega)}.$$  \hfill (2.15)

To derive a pointwise upper bound on $e_{\lambda, \delta}(\cdot, 0, w)$ we use Harnack type inequalities:
Lemma 2.5 There exist constants $c_2, c_3 \in (0, \infty)$ (depending only on the dimension $d$) such that for all $\omega \in \Omega$ and all $z \in \mathbb{R}^d \setminus \tilde{B}(0,3)$ the following estimate holds:

$$
\frac{\sup_{z_1 \in B(z,1)} e_{\lambda, \beta}(z_1, 0, \omega)}{\inf_{z_2 \in B(z,1)} e_{\lambda, \beta}(z_2, 0, \omega)} \leq c_2 \exp \left\{ c_3 \beta \left( \lambda + \sup_{x \in B(z,2)} V(x, \omega) \right) \right\}.
$$

(2.16)

For the proof of Lemma 2.5 we refer the reader to Sznitman [50], formula (5.2.22). This Harnack type inequality leads directly to the desired pointwise upper bound:

Theorem 2.6 (Pointwise bound) There exist constants $c_2, c_3 \in (0, \infty)$ such that for all $\omega \in \Omega$ we find a constant $c(\omega, W, \lambda) < \infty$ such that for all $\beta \geq 1$, $\varepsilon \in (0, 1)$, $\delta \in (0, 1)$ and $x \in B(0,3)^c$ the following is true

$$
- \log e_{\lambda, \beta}(x, 0, \omega) \geq (1 - \varepsilon) \sqrt{\beta} g_\lambda(x, 0, \omega) - \log \left( \frac{a_1 c_2}{\sqrt{d}} \right) - c_3 \beta \left( \lambda + \sup_{x \in B(z,2)} V(z, \omega) \right)
$$

$$
- \sup_{x \in B(0,1 + \frac{1-\delta}{2\sqrt{d}})} \sqrt{\beta} g_\lambda(z, 0, \omega) - \sup_{y, z \in B(x,1)} \sqrt{\beta} g_\lambda(y, z, \omega),
$$

(2.17)

where $a_1^2 = a_1 (\omega, \varepsilon, \delta)^2 = \frac{c(\omega, W, \lambda)}{2 \varepsilon - \delta^2} \left( \frac{1-\delta}{\delta} \right)^2$, and $v_d$ is the volume of the unit ball in $\mathbb{R}^d$.

Proof. Choose $x \in \tilde{B}(0,3)^c$. Using the Cauchy-Schwarz inequality, we see that

$$
\inf_{z_1 \in B(z,1)} e_{\lambda, \beta}(z_1, 0, \omega) \leq \frac{1}{v_d} \int_{B(z,1)} e_{\lambda, \beta}(z, 0, \omega) dz
$$

$$
\leq \frac{1}{v_d^{1/2}} \left( \int_{B(z,1)} e_{\lambda, \beta}(z, 0, \omega)^2 dz \right)^{1/2}
$$

$$
\leq \frac{a_1}{v_d^{1/2}} \sup_{z \in B(0,1 + \frac{1-\delta}{2\sqrt{d}})} e^{\sqrt{\beta} g_\lambda(z, 0, \omega)} \sup_{y \in B(x,1)} e^{-(1-\varepsilon) \sqrt{\beta} g_\lambda(y, 0, \omega)},
$$

(2.18)

in the last step we have used Corollary 2.4. We easily obtain (2.17) with the help of Lemma 2.5 on the left-hand side of (2.18) and the triangle inequality for $g_\lambda$.

Corollary 2.7 For $\lambda > 0$ and $\beta \geq 1$ we have:

$$
P\text{-a.s. for } |x| \text{ large, } \quad - \log e_{\lambda, \beta}(x, 0, \omega) \geq \sqrt{\beta} g_\lambda(x, 0, \omega) - o(\log |x|),
$$

(2.19)

and

$$
\inf_{c \in \partial B(0,1)} \frac{1}{\beta^{1/2}} \alpha_{\lambda, \beta}(c) \quad \mu_\lambda(c) \geq 0.
$$

(2.20)
Proof. Choose $\delta = 1/2$ fixed, and $\varepsilon = |z|^{-1}$. For $\omega \in \Omega_0$ and $|x|$ large we have that (using Theorem 2.6)

$$\log e_{\lambda,\beta}(x, 0, \omega) \geq \sqrt{2} g_{\lambda}(x, 0, \omega) - o(\log |x|),$$

(2.21)

here we have used the control on the growth of $V$ (see (1.1)). Now we apply the shape theorems on both sides of (2.21) to conclude that our corollary holds true (the correction term $o(\log |x|)$ in (2.21) does not depend on the direction of $x$).

\[\square\]

Remark. In view of (2.21) we see that the lower bound on the distance $-\log e_{\lambda,\beta}(x, 0, \omega)$ in terms of the geodesic distance $g_{\lambda}(x, 0, \omega)$ differs only by a logarithmic term in $|x|$. On the other hand the upper bound (1.4) is much less sharp. Hence, we have to improve (1.4) if we want to define "quasi geodesics" for the crossing Brownian motion model, where a "quasi geodesic" is a tube of small radius (compared with $|x|$), such that with sufficiently large probability crossing Brownian motion moves along that tube to its goal $B(0, 1)$ (as $|x| \to \infty$).

3 SHAPE THEOREMS

In this section we prove Theorem 0.2. For the proof of Theorem 0.1 we refer the reader to Theorem 1.4 of Sznitman [46].

To prove Theorem 0.2 we use the technique given in Proposition 1.2 of [46] (see also Kesten [26], p. 158). The first step is to see that $g_{\lambda}(x, y, \omega)$ is a positive, subadditive, translation invariant, ergodic random variable which is in $L^1(\mathbb{P})$. Hence, we can apply Kingman’s subadditive ergodic theorem. The second step is to patch up the limits for different directions.

Proof of Theorem 0.2. Choose $\lambda > 0$ and $v \in \mathbb{R}^d \setminus \{0\}$ fixed, the doubly indexed sequence

$$X_{m,n}^\lambda = g_{\lambda}(mv, nv, \omega), \quad 0 \leq m \leq n,$$

(3.1)

satisfies the triangle inequality $X_{0,n}^\lambda \leq X_{0,m}^\lambda + X_{m,n}^\lambda$. It is easy to verify that this family satisfies the hypothesis of Kingman’s subadditive ergodic theorem (see Liggett [31], p. 277) giving for $v \in \mathbb{R}^d$ (the case $v = 0$ is trivial):

$$\mu_{\lambda}(v) = \lim_{n \to \infty} \frac{X_{0,n}^\lambda}{n} = \lim_{n \to \infty} \frac{\mathbb{E}[X_{0,n}^\lambda]}{n} = \inf_{n \geq 1} \frac{\mathbb{E}[X_{0,n}^\lambda]}{n} \quad \text{exists } \mathbb{P} \text{ – a.s. and in } L^1(\mathbb{P}).$$

(3.2)
For \( x, y \in \mathbb{R}^d \), consider the straight line \( r : x + s(y - x), s \in [0, 1] \). Then
\[
\varrho_\lambda(x, y) \leq \int_0^1 \sqrt{2q(r_s)} |r_s| ds \\
\leq |y - x| \int_0^1 \sqrt{2(\lambda + |W|_\infty \omega(B(r_s, a)))} ds. \tag{3.3}
\]

Hence, from (3.2) and (3.3) we have for an appropriate choice of \( k_1 = k_1(d, \nu, W, \lambda) \):
\[
\sqrt{2\lambda |v|} \leq \mu_\lambda(v) \leq k_1 |v|. \tag{3.4}
\]

Using the triangle inequality and translation invariance, it is easy to conclude that for \( v, v' \in \mathbb{R}^d \), \( p \geq 0 \) and \( q \geq 1 \) integers (see Sznitman [50], (5.2.36)-(5.2.39)):
\[
\begin{align*}
\mu_\lambda(v + v') &\leq \mu_\lambda(v) + \mu_\lambda(v'), \tag{3.5} \\
\mu_\lambda(-v) &\leq \mu_\lambda(v), \tag{3.6} \\
\frac{p}{q} \mu_\lambda(v) &\leq \mu_\lambda \left( \frac{p}{q} v \right). \tag{3.7}
\end{align*}
\]

As a result we can extend \( \mu_\lambda(\cdot) \) to a norm on \( \mathbb{R}^d \):
\[
\mu_\lambda(\gamma v) = |\gamma| \mu_\lambda(v), \quad \gamma \in \mathbb{R}, \ v \in \mathbb{R}^d, \tag{3.8}
\]
here we use \( \lambda > 0 \) to guarantee that we obtain a norm (see (3.4)).

The next step is to patch up the limits. Therefore we need the following maximal lemma (see also [50], Lemma 5.2.6):

**Lemma 3.1** There exists a constant \( A(d, \nu, W, \lambda) > 0 \), such that for large \( r \) and for \( \eta > 0 \),
\[
\mathbb{P} \left[ \sup_{||x|| < r} \varrho_\lambda(x, 0) > \eta \right] \leq (4r + 1)^d \exp \left\{ -\eta + Ar \right\}. \tag{3.9}
\]

We will prove this lemma below. Now we choose \( \eta = 2A\epsilon|p| \), \( r = \epsilon|p| \), \( p \in \mathbb{Z}^d \), \( \epsilon \in (0,1) \) fixed, so we have for large \( p \):
\[
\mathbb{P} \left[ \sup_{||y - p|| < \epsilon|p|} \varrho_\lambda(p, y) > 2A\epsilon|p| \right] \leq (4\epsilon|p| + 1)^d \exp \left\{ -A\epsilon|p| \right\}, \tag{3.10}
\]
which is summable in \( p \). It thus follows: on a set of full \( \mathbb{P} \)-measure,
\[
\text{for any } \epsilon \in \mathbb{Q} \cap (0,1), \text{ for large } p, \sup_{||y - p|| < \epsilon|p|} \varrho_\lambda(p, y, \omega) \leq 2A\epsilon|p|. \tag{3.11}
\]
Now we consider a fixed \( \omega \) in a set of full \( \mathbb{P} \)-measure such that (3.11) and (3.2) for all \( v \in \mathbb{Q}^d \) hold. It suffices to show that for any sequence \( x_k \to \infty \), with
\[
\frac{x_k}{|x_k|} \to c \in \partial B(0, 1),
\]
we have
\[
\lim_{k \to \infty} \frac{1}{|x_k|} \left| \varrho_\lambda(x_k, 0, \omega) - \mu_\lambda(x_k) \right| = 0.
\]
Choose \( \epsilon \in \mathbb{Q} \cap (0, 1) \), \( v \in \mathbb{Q}^d \), and an integer \( M \), with
\[
|v - \epsilon| < \epsilon/2 \quad \text{and} \quad Mv \in \mathbb{Z}^d.
\]
Define
\[
y_k = \frac{|x_k|}{M} \cdot Mv \in \mathbb{Z}^d.
\]
Then, as in [50], formulas (5.2.47)-(5.2.48), we have for large \( k \)
\[
|y_k - x_k| \leq \epsilon |x_k|/2 \quad \text{and} \quad |y_k - x_k| \leq \epsilon |y_k|.
\]
Hence, for large \( k \) (using the triangle inequality)
\[
\left| \frac{1}{|x_k|} \varrho_\lambda(x_k, 0, \omega) - \mu_\lambda \left( \frac{x_k}{|x_k|} \right) \right| \leq \frac{1}{|x_k|} \varrho_\lambda(y_k, x_k, \omega) + \frac{1}{|y_k|} \varrho_\lambda(y_k, 0, \omega) \left| \frac{|y_k|}{|x_k|} - 1 \right| + \left| \frac{1}{|y_k|} \varrho_\lambda(y_k, 0, \omega) - \mu_\lambda(v) \right| + \left| \mu_\lambda(v) - \mu_\lambda \left( \frac{x_k}{|x_k|} \right) \right|.
\]
In view of (3.11), (3.16), (3.2) and (3.14), we obtain
\[
\limsup_{k \to \infty} \left| \frac{1}{|x_k|} \varrho_\lambda(x_k, 0, \omega) - \mu_\lambda \left( \frac{x_k}{|x_k|} \right) \right| \leq 2A \epsilon + \left( \frac{\epsilon}{2} + \frac{\epsilon}{2} \right) \sup_{|\epsilon| = 1} \mu_\lambda(\epsilon).
\]
Letting \( \epsilon \) tend to 0 finishes the proof of Theorem 0.2.

Proof of Lemma 3.1. Once we have an inequality of the form (5.2.49)-(5.2.50) in [50], we prove Lemma 3.1 exactly by the same method as Lemma 5.2.6 of [50]. So our goal is to find such an inequality. Choose \( x, y \in \mathbb{R}^d \) such that \( |x - y| \leq \sqrt{d} \). In view of (3.3) we have
\[
\varrho_\lambda(x, y) \leq B(x, \omega),
\]
with \( B(x, \omega) = \sqrt{2d \left( \lambda + ||W||_\infty \omega(B(x, \sqrt{d} + a)) \right)} \). Thus, \( B(x, \omega) \) and \( B(y, \omega) \) are independent for \(|x - y| \geq L(d, a) = 2(\sqrt{d} + a + 1)\). Therefore the claim of our lemma follows as in Lemma 5.2.6 of [50], if we choose \( c \) (defined in (5.2.53), [50]) to be

\[
c = \log \mathbb{E} \left[ \exp \{ dB(0, \omega) \} \right] < \infty. \tag{3.20}
\]

\( \square \)

Remark. We have only treated the case \( \lambda > 0 \). The case \( \lambda = 0 \) is more delicate. If the intensity \( \nu \) of the Poissonian point process is small, then the probability that the origin belongs to an infinite cluster of the complement of the support of \( V(\cdot, \omega) \) is strictly positive (see Sarkar [38], Theorem 1 for \( d \geq 3 \), and Meester-Roy [32], Theorems 4.4 and 3.10 for \( d = 2 \)). In that case (\( \lambda = 0, \nu \) small) we expect, as in discrete first-passage percolation, that \( \mu_\lambda \) is not positive (see Theorem 6.1, p. 218, of Kesten [26]).

\section*{A PROOF OF LEMMA 2.2}

In this appendix we provide for the reader's convenience a proof of Lemma 2.2, which follows Theorem 1.5 of Agmon [1]. In our special setting (we assume that we have a bounded solution) we manage to simplify Agmon's proof substantially.

Proof of Lemma 2.2. We choose \( u \in H^1_{\text{loc}}(D) \) to be a non-negative, bounded weak solution of (2.3). \( L^\infty_\nu(D) \) denotes the space of essentially bounded functions with compact support in \( D \). By a density argument, we see that the integral equation (2.4) still holds true, if we choose \( \phi \in H^1(D) \cap L^\infty_\nu(D) \) (We smoothen \( \phi \) with a non-negative function \( \rho_\varepsilon \in C^\infty_c(D) \) with \( \text{supp} \rho_\varepsilon \subset B(0, \varepsilon) \) and \( |\rho_\varepsilon|_{L^1} = 1 \). Letting \( \varepsilon \) tend to 0 yields the claim). For \( \psi \) is a Lipschitz function with compact support in \( D \) we see that \( u\psi^2 \in H^1(D) \cap L^\infty_\nu(D) \), hence we will choose \( u\psi^2 \) as \( \phi \) in the integral equation (2.4):

\[
0 = \int_D \frac{1}{2} \nabla u \nabla (u\psi^2) + \beta qu^2 \psi^2 \, dx \\
\geq \int_D -\frac{1}{2} u^2 |\nabla \psi|^2 + \beta qu^2 \psi^2 \, dx, \tag{A.1}
\]

where the last expression follows from \( \nabla u \nabla (u\psi^2) = |\nabla (u\psi)|^2 \). \( u^2 |\nabla \psi|^2 \geq u^2 |\nabla \psi|^2 \).

Our aim is to choose a function \( \psi \), which will prove the claim. Choose

\[
\psi(x) = e^{g(x)} \chi(x), \tag{A.2}
\]

where we assume that \( g \) and \( \chi \) are Lipschitz functions, \( \chi \) is compactly supported with values in \([0, 1]\), and \( g \) satisfies \( 1/2 |\nabla g(x)|^2 < \beta q(x) \) for a.e. \( x \in D \). \( g \) will play the role of
the function $h$ in (2.8), and $\chi$ will be a smoothened characteristic function, such that $\psi$ lives only on a compact subset of $\mathbb{R}^d$. Using (A.1) we have

$$0 \geq \int_D \beta q u^2 e^{2q} \chi^2 - \frac{1}{2} u^2 |\nabla (e^2 \chi)|^2 \, dx$$

(A.3)

$$\geq \int_D (u \chi)^2 \left( \beta q - \frac{1}{2} |\nabla g|^2 \right) e^{2q} \, dx - \int_D \frac{1}{2} u^2 e^{2q} (|\nabla \chi|^2 + 2 \chi |\nabla g \nabla \chi|) \, dx.$$  

Now let $e^q \chi$ approximate $e^h$. For $i \in \mathbb{N}$ we choose $K_i = \tilde{B}(0, 2 + i) \setminus B^c(0, 1 + 1/i)$, then $\{K_i\}_i$ is a sequence of compact subsets in $D = \tilde{B}(0, 1)^c$, such that $K_i \subset K_j$, for all $i < j$, and $\cup_{i \geq 1} K_i = D$. For $\delta > 0$ we define

$$\chi_i(x) = \begin{cases} \sqrt{\beta} \phi_x(x, D \setminus K_i) & \text{if } \sqrt{\beta} \phi_x(x, D \setminus K_i) \in [0, \delta], \\ 1 & \text{otherwise.} \end{cases}$$

(A.4)

$\chi_i$ is a compactly supported Lipschitz function, which is 0 on $K_i^c$. Using Theorem 1.4 of [1] and the triangle inequality for the distance function $\varrho$, we see that

$$\frac{1}{2} |\nabla \chi_i|^2 \leq \frac{\beta}{\delta^2 q}, \text{ a.e.}$$

(A.5)

Next we choose $g$. For $j \geq 1$, we define

$$g_j(x) = h(x) \wedge \left( -\frac{1}{2} \sqrt{\beta} \varrho_x(x, 0) + j \right).$$

(A.6)

The truncation of $h$ will be important to obtain an integrable function (in order to apply Lebesgue’s dominated convergence theorem). We remark that $g_j$ is Lipschitz with

$$\frac{1}{2} |\nabla g_j|^2 < \beta q, \text{ a.e.}$$

(A.7)

This is true, of course, for $x \in D$ with $h(x) \neq -\frac{1}{2} \sqrt{\beta} \varrho_x(x, 0) + j$, but on the other hand, if $h(x) = -\frac{1}{2} \sqrt{\beta} \varrho_x(x, 0) + j$, then for $z \in \mathbb{R}^d$, and $t > 0$ small, $g_j(x + tz) - g_j(x) \leq h(x + tz) - h(x)$, from which follows that for all $z \in \mathbb{R}^d$

$$\langle z, (\nabla g_j(x) - \nabla h(x)) \rangle < 0, \text{ a.e.}$$

(A.8)

This implies $\nabla g_j(x) = \nabla h(x)$ a.e. in the second case. Therefore (A.7) also holds true in this second case. So $\psi = e^{q_i} \chi_i$ fulfills all the assumptions after (A.2). Using (A.3)

$$\int_D (u \chi_i)^2 \left( \beta q - \frac{1}{2} |\nabla g_j|^2 \right) e^{2q_i} \, dx \leq \int_D f_{i,j} \, dx,$$  

(A.9)
with
\[ f_{i,j}(x) = \frac{1}{2} u(x)^2 e^{2g_j(x) \left( |\nabla \chi_i(x)|^2 + 2\chi_i(x) |\nabla g_j(x) \nabla \chi_i(x)| \right)} . \] (A.10)

Next, we define the "inner" of \( K_i \) (with respect to \( \delta > 0 \)):
\[ K_i^I = \left\{ x \in K_i; \sqrt{\beta} g_\lambda(x,D \setminus K_i) > \delta \right\} . \] (A.11)

We remark that \( \nabla \chi_i = 0 \) a.e. on \( K^c_i \cup K^I_i \). Therefore, if we define the functions \( \kappa_i \):
\[ \kappa_i(x) = \begin{cases} 1 & \text{if } x \in K_i \setminus K^I_i, \\ 0 & \text{otherwise}, \end{cases} \] (A.12)

it follows that for all \( x \) with \( K_i(x) = 0 \): \( f_{i,j}(x) = 0 \). Hence, we can split the right member of (A.9) into two parts:
\[ \int_D f_{i,j} dx = \int_{D \setminus D_{\delta_i/\sqrt{\beta}}} f_{i,j} dx + \int_{D_{\delta_i/\sqrt{\beta}}} \kappa_i f_{i,j} dx. \] (A.13)

We claim that the rightmost term in the above equation tends to zero for \( i \to \infty \): on \( D_{\delta_i/\sqrt{\beta}} \) we know that \( \kappa_i \) tends to zero (as \( i \to \infty \)), and
\[ \kappa_i f_{i,j} \leq u^2 e^{2g_j/\delta} \left( \frac{1 + 2\delta}{\delta^2} \right) \leq e^{2j} u^2 e^{-\sqrt{\beta} g_\lambda(x,0)} q \beta \left( \frac{1 + 2\delta}{\delta^2} \right) . \] (A.14)

Because \( u \) is bounded, this last term is in \( L^1 \) for every \( j \) and every \( \omega \in \Omega_0 \). Applying Lebesgue's dominated convergence theorem to (A.9) and (A.13), we see that
\[ \limsup_{j \to \infty} \int_D (u \chi_i)^2 \left( \beta q - \frac{1}{2} |\nabla g_j|^2 \right) e^{2g_j} dx \leq \limsup_{i \to \infty} \int_{D \setminus D_{\delta_i/\sqrt{\beta}}} f_{i,j} dx \leq \frac{1 + 2\delta}{\delta^2} \int_{D \setminus D_{\delta_i/\sqrt{\beta}}} u^2 e^{2h} \beta q dx. \] (A.15)

To conclude the claim of the lemma, we remark that on \( D_{\delta_i/\sqrt{\beta}} \) we know that \( \chi_i \to 1 \) as \( i \to \infty \), \( g_j \to h \) as \( j \to \infty \) and we know that \( \beta q - \frac{1}{2} |\nabla g_j|^2 \geq 0 \) on \( D \), hence, using Fatou's lemma twice, we see that
\[ \int_{D_{\delta_i/\sqrt{\beta}}} u^2 \left( \beta q - \frac{1}{2} |\nabla h|^2 \right) e^{2h} dx \leq \liminf_{j \to \infty} \liminf_{i \to \infty} \int_D (u \chi_i)^2 \left( \beta q - \frac{1}{2} |\nabla g_j|^2 \right) e^{2g_j} dx \leq \frac{1 + 2\delta}{\delta^2} \int_{D \setminus D_{\delta_i/\sqrt{\beta}}} u^2 e^{2h} \beta q dx. \] (A.16)

This finishes the proof of Lemma 2.2.
References


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Curriculum vitae

I, Mario Valentin Wüthrich from Winterthur (ZH) and Trub (BE), was born on May 19, 1969. After primary school I spent two years at Kantonsschule Rychenberg in Winterthur. In 1984 I had the opportunity to spend one year abroad at Murrumbeena High School in Melbourne (Australia). From 1985 to 1989 I studied at Kantonsschule im Lee in Winterthur, where I graduated with a type C Matura.

I began my studies in mathematics at ETH Zürich in Autumn 1989. After the first two semesters I deferred my studies for one year to complete my military service, and to gain six months practical experience with Dr. A. Gisler at Winterthur Insurance Company. I continued to work with Dr. A. Gisler during my semester breaks.

My studies in mathematics at ETH Zürich recommenced in Autumn 1991, and were completed in March 1995. My diploma on the topic "First-passage percolation" was supervised by Prof. A.-S. Sznitman.

Since May 1995 I have worked as a teaching assistant in the Department of Mathematics at ETH Zürich. At the same time I have been working on this doctoral thesis.