Doctoral Thesis

Abelian varieties and identities for hypergeometric series

Author(s):
Archinard, Natália

Publication Date:
2000

Permanent Link:
https://doi.org/10.3929/ethz-a-004070858

Rights / License:
In Copyright - Non-Commercial Use Permitted
Abelian Varieties and Identities for Hypergeometric Series

A dissertation submitted to the

Swiss Federal Institute of Technology Zurich

for the degree of

Doctor of Mathematics

presented by

Natália Archinard

Dipl. Math. Univ. Genève

born on the 11th of June 1973

citizen of Geneva (GE)

accepted on the recommendation of

Prof. Dr. Gisbert Wüstholz, examiner

Prof. Dr. Paula B. Cohen, co-examiner

2000
Abstract

This work is devoted to hypergeometric series.

In the first chapter, we construct explicitly abelian varieties associated to Appell-Lauricella hypergeometric series. From our general construction, we deduce the constructions of abelian varieties for Gauss' hypergeometric series and for Beta-functions.

In the second chapter, we produce identities involving one Gauss' hypergeometric series on one hand and, on the other hand, the modular $J$-invariant and the Dedekind $\eta$-function. With such identities, we are able to show the algebraicity of some Gauss' hypergeometric series on some infinite subset of $\mathbb{Q}$ and to calculate explicitly the algebraic value of $F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; z\right)$ at some points of the corresponding subset.

Zusammenfassung

Diese Arbeit ist hypergeometrischen Reihen gewidmet.

Im ersten Kapitel bauen wir abelsche Varietäten, die in engem Zusammenhang mit Appell-Lauricella hypergeometrischen Reihen stehen. Als spezielle Fälle unserer allgemeinen Konstruktion erhalten wir abelsche Varietäten einerseits für die Gauss'sche hypergeometrische Reihe und anderseits für die Beta-Funktion.

Im zweiten Kapitel beweisen wir verschiedene Identitäten. Jede Identität bringt eine Gauss'sche hypergeometrische Reihe in Verbindung mit der $J$-Modulfunktion und der Dedekind'schen $\eta$-Funktion. Mit Hilfe solcher Identitäten zeigen wir in gewissen Fällen, dass die entsprechende hypergeometrische Reihe einen algebraischen Wert in jedem Punkt einer gewissen unendlichen Untermenge von $\mathbb{Q}$ annimmt. Ferner berechnen wir explizit ein paar algebraische Auswertungen von $F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; z\right)$ für $z$ aus der entsprechenden Untermenge.
Contents

Introduction 1

1 Abelian Varieties Associated to Hypergeometric Series 5

1.1 Definitions and Basic Facts on Hypergeometric Series .......... 5
1.2 Motivation and Structure of the Chapter 8
1.3 Preparatory Material ............................................. 9

1.3.1 Affine Algebraic Varieties ................................. 9
1.3.2 Projective Algebraic Varieties .............................. 10
1.3.3 Irreducibility and Dimension .............................. 11
1.3.4 Algebraic Curves ............................................. 12
1.3.5 Morphisms and Birational Equivalence .................... 13
1.3.6 Blow-up .................................................... 14
1.3.7 Desingularization of a Plane Curve ....................... 16
1.3.8 Riemann Surfaces and Nonsingular Algebraic Curves over \( \mathbb{C} \) ................................................ 18
1.3.9 Topological Invariants of a Riemann Surface .......... 22
1.3.10 Tangent Space to an Algebraic Variety and Differential Maps 28
1.3.11 Differential Forms on Algebraic Varieties 33
1.3.12 Differential Forms on Riemann Surfaces 36
1.3.13 Integration of Holomorphic Differential 1-Forms on Riemann Surfaces 39
1.3.14 Sheaves 42
1.3.15 Complex Tori and Abelian Varieties over $\mathbb{C}$ 43
1.3.16 Jacobian Varieties over $\mathbb{C}$ 46
1.3.17 Algebraic Appendix 49

1.4 Abelian Varieties for Appell-Lauricella Hypergeometric Series 53
1.4.1 Motivation and Structure of the Section 53
1.4.2 Reduction 54
1.4.3 Definition of the Family of Curves 56
1.4.4 Singular Points on $C_N$ 57
1.4.5 Structure of the Desingularization 57
1.4.6 Genus of $X_N$ 67
1.4.7 Actions of $\mu_N$ 71
1.4.8 Basis of Regular Differential Forms on $X_N$ 73
1.4.9 New Forms 83
1.4.10 New Jacobian 85

1.5 Abelian Varieties for Gauss' Hypergeometric Series 86
1.5.1 Motivation 86
2 Identities for some Gauss' Hypergeometric Series

2.1 Introduction .................................................. 101

2.2 Theoretical Preliminaries ................................... 102
    2.2.1 Elliptic Curves over C ............................... 102
    2.2.2 Isomorphisms between Elliptic Curves ............... 104
    2.2.3 Change of Basis of a Lattice ......................... 104
    2.2.4 From a Lattice Function to a Modular one, 
        Definition of Modular Functions and Forms .......... 105
    2.2.5 Eisenstein Series ................................... 106
    2.2.6 The Discriminant Function Δ ........................ 108
    2.2.7 The J-Function ..................................... 108
    2.2.8 The Dedekind η-Function ............................ 109
2.3 The Method of Beukers and Wolfart

2.4 Sixteen Solutions

2.4.1 List of Sixteen Solutions

2.4.2 Linear Differential Equation of Order Two, Riemann Scheme and Kummer's 24 Solutions

2.4.3 A New Eye on our Solutions

2.5 Twenty-two Identities

2.5.1 About $J = 0$

2.5.2 About $J = 1$

2.5.3 About $J = \infty$

2.6 Some Applications of our Identities

2.6.1 Algebraicity of some Hypergeometric Values

2.6.2 Some Algebraic Evaluations

Acknowledgments

Curriculum Vitae
Introduction

This work is divided in two parts having the common goal of contributing to the knowledge on hypergeometric functions. The first chapter constructs tools for the study of Appell-Lauricella hypergeometric series (these are hypergeometric series in several variables) and their monodromy groups, while the second chapter is devoted exclusively to Gauss’ hypergeometric series (which is the classical hypergeometric series in one variable).

From the time Schwarz [S873] proved that the algebraic Gauss’ hypergeometric series are exactly those having a finite monodromy group, it has mostly been believed that no transcendental hypergeometric series could take an algebraic value at an algebraic point. Surprising enough, Wolfart [Wo85], [Wo88] showed that the subset of \( \mathbb{Q} \) on which the Gauss’ hypergeometric series \( F(a, b, c; z) \) with \( a, b, c \in \mathbb{Q} \) takes algebraic values and the complement of this subset are dense in \( \mathbb{Q} \) exactly when the monodromy group of \( F \) is arithmetic and some conditions on the parameters \( a, b, c \) hold. By Takeuchi [Ta77] and because the monodromy group is triangular, this happens for 85 isomorphy classes of monodromy groups.

The idea in the proof of Wolfart is to interpret Euler’s integral representation of \( F(a, b, c; z) \) with \( a, b, c \in \mathbb{Q}, \ z \in \mathbb{Q} \), as a quotient of periods on abelian varieties defined over \( \mathbb{Q} \) and to use a consequence ([WW85] Satz 2) of Wüstholz’s Analytic Subgroup Theorem ([Wu89] Hauptsatz) to decide when this quotient can be algebraic. To hypergeometric series having the same smallest common denominator \( N \) of \( a, b, c \) (hence having \( SL_2(\mathbb{R}) \)-conjugated monodromy group) is associated a family of abelian varieties of dimension depending on \( N \) only. The abelian variety associated to an \( F \) is an essential factor of the Jacobian variety of the curve on which the numerator of the integral representation of \( F \) is a period (up to an algebraic factor). The denominator appears to be (up to an algebraic factor) a period on a component of the specialization at \( z = 0 \) of the above curve. For this case,
the essential factor of the Jacobian had already been constructed and its dimension calculated by Gross and Rohrlich [GR78].

Later, Cohen and Wolfart [CW90] have used this family of abelian varieties associated to a triangular group $\Delta$, interpreted as the monodromy group of a hypergeometric differential equation, in order to construct explicitly a modular embedding of $\Delta$, that is an embedding into a modular (hence arithmetic) group. The same authors [CW93] have generalized their work to monodromy groups of hypergeometric functions in two variables, defining at this occasion the families of abelian varieties for hypergeometric series in two variables and giving their dimension.

The families of curves arising from hypergeometric series in several variables have also been considered by Deligne and Mostow [DM86] in their study of the monodromy groups of these functions.

In this work, we construct abelian varieties for hypergeometric series in $r - 1$ variables. In order to do this, we first construct explicitly the desingularization of the singular curve associated to the hypergeometric series, calculate the genus of the desingularized curve and a basis of holomorphic differential 1-forms on it. We then define the action of the group of $N$-th roots of unity on the vector space of holomorphic differential 1-forms and calculate the decomposition of this linear representation in irreducible linear subrepresentations as well as the dimension of the isotypical components. Further, we select some isotypical components. Their direct sum defines an abelian subvariety of the Jacobian variety of the nonsingular curve. We calculate the dimension of this subvariety as a function of $N$ (and $r$) only.

Our results show the coherence of the definitions given in [Wo85], [Wo88], [CW93] and confirm the assertions given in there about the dimension. On the other hand, they correct the assertions of [Wo88] about the order of differential 1-forms and the dimension of the isotypical components. The result on the dimension given in [GR78] appears as a special case of our dimension’s result.

In two cases of Gauss’ hypergeometric series with monodromy group isomorphic to $SL_2(Z)$, Beukers and Wolfart [BW86] were able to determinate explicitely an infinite subset of $\mathbb{Q}$ on which the hypergeometric series takes algebraic values. In order to do this, they first determinate in each case an infinite subset on which the hypergeometric series may take algebraic values (the key here is Wüstholz’s theorem). Secondly and thanks to an idea of Beukers, they prove an identity relating the hypergeometric series with some modular functions. This identity for the
Introduction

hypergeometric series enables Beukers and Wolfart to show that the series indeed takes an algebraic value at each point of the selected subset. In each case, they calculate one algebraic value of the hypergeometric series. Using the identity for $F\left(\frac{1}{12}, \frac{5}{12}, 1; \frac{J-1}{J}\right)$, Flach [Fl89] calculated two more algebraic values of this series. Using only classical transformations for hypergeometric series and some known values of the complete elliptic integral $K(k) = \frac{1}{2} \pi F\left(\frac{1}{2}, \frac{1}{2}, 1; k^2\right)$, Joyce and Zucker [JZ91] calculated two new algebraic values, one of them correcting one of Flach’s ones. They also expressed four more values of this series in terms of values of the Gamma-function.

In our turn, we have calculated four new algebraic evaluations of $F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; z\right)$. For this, we have calculated the value of $J(mi)$ and $\frac{\eta(mi)}{\eta(i)}$ for some $m \in \mathbb{N}$ in using long known results on modular equations (see Weber [We]) and inserted them into the identity for $F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{J-1}{J}\right)$. Moreover, using the same idea as Beukers and Wolfart, we have produced twenty new identities for fifteen hypergeometric series. Some of them have allowed us to show the algebraicity of the corresponding hypergeometric series at the points of some infinite subset of $\mathbb{Q}$. 
Seite Leer / Blank leaf
Chapter 1

Abelian Varieties Associated to Hypergeometric Series

1.1 Definitions and Basic Facts on Hypergeometric Series

The Gauss' hypergeometric series was first introduced by Euler (1778) as the following power series in the complex variable $z$

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a; n)(b; n)}{(c; n)(1; n)} z^n,$$

where $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$ and

$$(x; n) := \begin{cases} x(x + 1) \cdots (x + n - 1) & \text{if } n > 0 \\ 1 & \text{if } n = 0. \end{cases}$$

If $a$ or $b$ is a nonpositive integer, then the series reduces to a polynom. In any case, this series defines a holomorphic function of $z$ in the open unit disc. If $Re(c - a - b) > 0$, the series also converges on the unit circle ([IKSY] 1.2). When $Re(c) > Re(b) > 0$ and $|z| < 1$, the hypergeometric series has an integral representation

$$F(a, b, c; z) = \frac{1}{B(b, c - b)} \int_{0}^{1} x^{b-1}(1 - x)^{c-b-1}(1 - zx)^{-a} dx, \quad (1.1)$$

where $B(a, b)$ is the Beta function.
where \( B(b, c - b) = \int_0^1 x^{b-1} (1 - x)^{c-b-1} dx \) is the classical Beta-function. It can be shown that the hypergeometric series satisfies a linear differential equation of order 2 with three regular singularities for which \( F(a, b, c; z) \) is the holomorphic solution at 0 taking value 1 (more on this aspect in §2.4.2).

The path of integration of the integral in the numerator of (1.1) is the open real segment \((0, 1)\) in \(\mathbb{C}\). The integrant \( x^{b-1} (1 - x)^{c-b-1} (1 - zx)^{-a} dx \) can be viewed as a differential 1-form on the (singular) algebraic projective curve \(C(N, z)\) defined by the equation

\[
y_N = x^A (1 - x)^B (1 - zx)^C,
\]

where \(N\) is the smallest common denominator of \(a, b, c\) and

\[
A := N(1 - b), \quad B := N(1 + b - c), \quad C := Na.
\]

If \(1 - b, 1 + b - c, a\) are not integers (i.e. \(N \nmid A, B, C\)), we can replace the integral \( \int_0^1 x^{b-1} (1 - x)^{c-b-1} (1 - zx)^{-a} dx \), up to an algebraic factor, by a period \(\int_{\gamma} \frac{dx}{y}\) on \(C(N, z)\), where \(\gamma\) is a cycle on \(C(N, z)\) whose image under the projection \((x, y) \mapsto x\) is a double contour loop (or “Pochhammer” loop) around 0 and 1. More precisely, the relation is

\[
\int_{\gamma} \frac{dx}{y} = (1 - \zeta_N^{-A})(1 - \zeta_N^{-B}) \int_0^1 x^{b-1} (1 - x)^{c-b-1} (1 - zx)^{-a} dx,
\]

where \(\zeta_N = e^\frac{2\pi i}{N}\). In particular, \(\int_{\gamma} \frac{dx}{y}\) is defined for \(z \in \mathbb{C} - \{0, 1\}\) without conditions on \(Re(b), Re(c)\).

In an analogous way, the integral \(B(b, c - b)\) appearing in the denominator of (1.1) can be considered, up to an algebraic factor, as a period on a projective algebraic curve \(C(M, 0)\) defined by the equation

\[
y_M = x^P (1 - x)^Q,
\]

where \(M\) is the smallest common denominator of \(b\) and \(c\) and

\[
P := M(1 - b), \quad Q := M(1 + b - c).
\]

Because \(M|N, C(M, 0)\) is isomorphic to a component of \(C(N, 0)\). (Remark that even if \(C(N, z)\) is irreducible for \(z \neq 0, 1\), \(C(N, 0)\) may not be so!)
1.1. Definitions and Basic Facts on Hypergeometric Series

The Gauss’ hypergeometric series has been generalized in many ways to series in several variables. We will however be concerned only by the following Appell-Lauricella function $F_1$ (see [CW91] §1) of the $d$ complex variables $x_2, \ldots, x_{d+1}$

$$F_1(a, b_2, \ldots, b_{d+1}; c; x_2, \ldots, x_{d+1}) = \sum_{n_2, \ldots, n_{d+1}=0}^{\infty} \frac{(a; \sum_j n_j) \prod_j (b_j; n_j)}{(c; \sum_j n_j) \prod_j (1; n_j)} \prod_{j=2}^{d+1} x_j^{n_j},$$

where the parameters $a, b_2, \ldots, b_{d+1}, c$ lie in $\mathbb{C}$, $c \neq 0, -1, -2, \ldots$ and $j$ runs from 2 to $d + 1$. This series converges when, $\forall j = 2, \ldots, d + 1$, $|x_j| < 1$. It is also denoted by $F_d$ if $d > 2$. If $\text{Re}(c) > \text{Re}(a) > 0$, it has the following integral representation

$$\frac{1}{B(a, c - a)} \int_1^{\infty} u^{-c + \sum_j b_j} (u - 1)^{c - a - 1} \prod_{j=2}^{d+1} (u - x_j)^{-b_j} du. \quad (1.2)$$

If the exponents are not integers, the integral can be replaced up to an algebraic factor by a period

$$\int \frac{u^{-\mu_0} (u - 1)^{-\mu_1}}{B(a - \mu, c - a)} \prod_{j=2}^{d+1} (u - x_j)^{-\mu_j} du$$

of a differential 1-form on a curve along a cycle on this curve whose projection in $\mathbb{C}$ is a double contour loop. The relations between the parameters are

$$\mu_0 = c - \sum_{j=2}^{d+1} b_j, \quad \mu_1 = 1 + a - c \quad \text{and, for } j \in \{2, \ldots, d + 1\}, \quad \mu_j = b_j.$$  

You will find more on the definition of this curve in §1.4.1.

Remark 1. Because $F(a, b, c; z)$ is symmetric in $a$ and $b$, it has also the integral representation

$$\frac{1}{B(a, c - a)} \int_0^1 x^{a-1} (1 - x)^{c-a-1} (1 - zx)^{-b} dx.$$  

If $\frac{1}{a}$ replaces $x$ in the integral, this becomes

$$\frac{1}{B(a, c - a)} \int_1^{\infty} u^{-c+b} (u - 1)^{c-a-1} (u - z)^{-b} du,$$

showing agreement with the generalization (1.2).
1.2 Motivation and Structure of the Chapter

As quoted in the Introduction, the goal of this chapter is to construct abelian varieties for hypergeometric series in several variables. We have seen in Section 1.1 that the hypergeometric series in several variables has an integral representation which can be interpreted, under some conditions on the parameters, as a quotient of algebraic multiples of periods on algebraic curves, hence on their Jacobian variety. If the parameters and the variables lie in \( \mathbb{Q} \), the curves and their Jacobian variety are defined over \( \mathbb{Q} \). Then, according to Wüstholz’s result (Satz 2 in [WW85]), such a quotient

\[
\frac{\int \omega_1}{\int \omega_2}
\]

of periods \( \int \omega_i \) on abelian varieties \( A_i \) (with \( A_2 \) simple) can only be algebraic if \( A_2 \) is isogenous to a factor of \( A_1 \). This criterion gives a powerful tool for testing the algebraicity of hypergeometric series.

In the case of hypergeometric series in one variable, this has strongly been used by Wolfart (in [Wo85], [W088], [BW86]), after he has defined the abelian variety associated to the numerator of the integral representation of the Gauss’ hypergeometric series. The abelian variety associated to the denominator was already known, since it was constructed by Rohrlich and Gross in [GR78].

In this chapter, we construct precisely the abelian variety on which an algebraic multiple of the numerator of the integral representation of the hypergeometric series is a period. This is done in Section 1.4 for Appell-Lauricella hypergeometric series (i.e. for hypergeometric series in any number of variables). Section 1.3 introduces the theoretical material necessary for the constructions and the results of Section 1.4. Section 1.5 shows how the abelian varieties for the case of one variable (i.e. for Gauss’ hypergeometric series) can be deduced from our general construction. The results obtained in this section will respectively prove and correct some assertions of Wolfart, [W088]. The abelian varieties associated to the denominator (which is always a Beta-function!) will also appear as special cases of our results. This is shown in Section 1.6. Finally, in Section 1.7, the definition of the family of curves treated in Section 1.4 is enlarged by admitting some “degenerated” curves. These curves can be considered as fibers above \( (\mathbb{P}^1)^{r+1} \). The genus of the irreducible fibers and the dimension of the corresponding abelian varieties are calculated.
1.3 Preparatory Material

Throughout this section $k$ will be an algebraically closed field.

1.3.1 Affine Algebraic Varieties

Let $k$ be an algebraically closed field and $T \subseteq k[x_1, \ldots, x_n]$ a set of polynomials in $n$ variables over $k$. The common roots in $k^n$ of all polynomials in $T$ build a subset $Z(T)$ of $k^n$:

$$Z(T) := \{ P \in k^n; f(P) = 0, \forall f \in T \}.$$

**Definition 1.1.** A subset $Y \subseteq k^n$ is called an algebraic set, if there exists a subset $T \subseteq k[x_1, \ldots, x_n]$ such that $Y = Z(T)$.

The intersection of any family of algebraic sets is an algebraic set and the union of a finite number of algebraic sets is again algebraic, hence we can define a topology on $k^n$ by defining the closed subsets to be the algebraic sets. (Note that $\emptyset = Z(1)$ and $k^n = Z(0)$). This topology on $k^n$ is called the Zariski topology.

**Definition 1.2.** The set $k^n$ equipped with the Zariski topology will be called the affine n-space over $k$ and denoted by $A^n_k$ (or simply $A^n$).

**Definition 1.3.** An affine algebraic variety (or affine variety) over $k$ is a nonempty closed subset of $A^n_k$ (together with the induced topology).

The set of all polynomials in $k[x_1, \ldots, x_n]$ which vanish at all points of an affine algebraic variety $X \subseteq A^n_k$ constitute an ideal of $k[x_1, \ldots, x_n]$:

$$J(X) := \{ f \in k[x_1, \ldots, x_n]; f(P) = 0, \forall P \in X \}.$$

According to Hilbert's so-called Nullstellensatz, which implies that any polynomial $g$ in $k[x_1, \ldots, x_n]$ which vanishes at all points of $Z(J(X))$ has a power which belongs to $J(X)$, one can associate a unique ideal $I(X)$ to $X$ by demanding it to be radical (that means $I(X) = \sqrt{I(X)}$, where $\sqrt{I(X)} = \{ f \in k[x_1, \ldots, x_n]; \exists r > 0$ s.t. $f^r \in I(X) \}$).

**Remark 2.** $I(X)$ is finitely generated, because the ring $k[x_1, \ldots, x_n]$ is noetherian. In particular, each affine variety is the locus of zeros of a finite number of polynomials.

**Definition 1.4.** If $X \subseteq A^n_k$ is an affine variety, the affine coordinate ring of $X$ is defined to be the quotient ring $k[x_1, \ldots, x_n]/I(X)$. It is denoted by $k[X]$. 


1.3.2 Projective Algebraic Varieties

The procedure to define a projective algebraic variety is very similar, the main difference is that we work in the projective space $\mathbb{P}^n(k)$ (or $k\mathbb{P}^n$ or simply $\mathbb{P}^n$), what forces us to use homogenous polynomials.

First recall that $\mathbb{P}^n(k)$ is the set of equivalence classes of nonzero $(n + 1)$-tuples $(a_0, \ldots, a_n)$, $a_i \in k$, under the equivalence relation

\[(a_0, \ldots, a_n) \sim (a'_0, \ldots, a'_n) \iff \exists \lambda \in k^* \text{ s.t. } a_i = \lambda a'_i, \forall i \in \{0, \ldots, n\}.
\]

The class of $(a_0, \ldots, a_n)$ is denoted by $(a_0 : \ldots : a_n)$ and is called a point in $\mathbb{P}^n(k)$, while by a set of homogenous coordinates for this point, we mean any representant of its class.

Let $S := k[x_0, \ldots, x_n]$ and $S_d$ be the set of all linear combinations of monomials in $x_0, \ldots, x_n$ of total degree $d$, then $S_d \subseteq S$ with the property that for any $e, f \geq 0$, $S_e \cdot S_f \subseteq S_{e+f}$. This gives the ring $S$ the structure of a graded ring. An element of $S_d$ is called a homogenous polynomial of degree $d$.

Anyone polynomial in $S$ would not define a function on $\mathbb{P}^n(k)$, because it has to take the same value on all representants of the same equivalence class in $(k^{n+1} - \{0\})/\sim = \mathbb{P}^n(k)$.

However, the vanishing of a homogenous polynomial is well-defined and, for any set $T$ of homogenous polynomials in $S$, we denote the set of their common zeros by

\[Z(T) := \{P \in \mathbb{P}^n(k); f(P) = 0, \forall f \in T\},\]

and define

**Definition 1.5.** A subset $Y$ of $\mathbb{P}^n(k)$ is an algebraic set, if there exists a set $T$ of polynomials in $k[x_0, \ldots, x_n]$ such that $Y = Z(T)$.

The algebraic sets in $\mathbb{P}^n(k)$ satisfy the properties required for them to be the closed sets of a topology on $\mathbb{P}^n(k)$. This topology on $\mathbb{P}^n(k)$ is called the Zariski topology.

**Definition 1.6.** $\mathbb{P}^n(k)$ together with the Zariski topology is called the projective $n$-space.
Definition 1.7. A **projective algebraic variety** (or projective variety) over $k$ is a nonempty closed subset of $\mathbb{P}^n(k)$ together with the induced topology.

Let $Y$ be a projective algebraic variety in $\mathbb{P}^n(k)$ and define the **homogenous ideal** $I(Y)$ of $Y$ to be the ideal in $k[x_0, \ldots, x_n]$ generated by

$$\{ f \in k[x_0, \ldots, x_n]; f \text{ homogenous and } f(P) = 0, \forall P \in Y \}.$$

As in the affine case, $Y$ is actually the locus of a finite number of homogenous polynomials.

**Definition 1.8.** The ring $k[x_0, \ldots, x_n]/I(Y)$ is called the **homogenous coordinate ring** of $Y$. It is denoted by $k[Y]$.

**Remark 3.** $\mathbb{P}^n(k)$ is covered by a finite number of open subsets, each of them homeomorphic to $\mathbb{A}^n_k$ and, consequently, every projective variety admits a finite covering by open sets homeomorphic to affine varieties.

**Terminology 1.** The qualifier **algebraic** will be used for a variety to mean that it is either affine algebraic or projective algebraic.

### 1.3.3 Irreducibility and Dimension

**Definition 1.9.** A nonempty subset in a topological space $X$ is **irreducible** if $X = X_1 \cup X_2$ with $X_1$ and $X_2$ closed subsets of $X$ implies $X_1 = X$ or $X_2 = X$. A nonempty subset of $X$ will be called **reducible** if it is not irreducible.

**Definition 1.10.** Let $X$ be a topological space. The dimension $\dim(X)$ of $X$ is defined to be the supremum of all integers $n$ such that there exists a chain $Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_n$ of distinct irreducible closed subsets of $X$.

In a noetherian topological space $X$ (that is $X$ satisfies the descending chain condition i.e. for any sequence $Y_1 \supseteq Y_2 \supseteq \ldots$ of closed subsets, there exists an integer $r$ such that $Y_r = Y_{r+1} = \ldots$), every nonempty closed subset $Y$ can be expressed as a finite union $Y_1 \cup \ldots \cup Y_n$ of irreducible closed subsets $Y_i$. If we require that $Y_i \nsubseteq Y_j$ for $i \neq j$, then the $Y_i$'s are uniquely determined and are called the **irreducible components** of $Y$. 

1. Abelian Varieties Associated to Hypergeometric Series

**Definition 1.11.** The reducibility nature of an algebraic variety is that as a topological space.

It can be shown that $\mathbb{A}^n_k$ and $\mathbb{P}^n(k)$ are noetherian topological spaces. Thus, each affine (resp. projective) algebraic variety can be expressed uniquely as a finite union of irreducible affine (resp. projective) varieties, no one containing another.

**Definition 1.12.** The dimension of an irreducible affine (resp. projective) algebraic variety is its dimension as a topological space. The dimension of a reducible affine (resp. projective) algebraic variety will be the maximum of the dimensions of its irreducible components. ([Sh1] I.6.1)

### 1.3.4 Algebraic Curves

**Definition 1.13.** An affine (resp. projective) algebraic curve over $k$ is an affine (resp. projective) algebraic variety over $k$ of dimension 1. An affine (resp. projective) algebraic plane curve is an algebraic curve in $\mathbb{A}^2_k$, (resp. $\mathbb{P}^2(k)$).

An affine (resp. projective) algebraic plane curve is then the locus of zeros of one nonconstant polynomial in $k[x, y]$ (resp. one nonconstant homogenous polynomial in $k[x_0, x_1, x_2]$). Moreover, it is irreducible if and only if the polynomial is irreducible and the irreducible components of the curve correspond bijectively to the irreducible factors of the polynomial ([Sh1] I.1.1).

Given now an affine plane curve $C$ defined by the equation $f(x, y) = 0$, where $f \in k[x, y]$, we would like to show how we can associate to $C$ a projective plane curve and what this means geometrically. First observe that $\mathbb{A}^2_k$ embeds to $\mathbb{P}^2(k)$ by $(x, y) \mapsto (1 : x : y)$. Write $x := \frac{x_1}{x_0}, \ y := \frac{x_2}{x_0}$ and define $f_{\text{hom}} \in k[x_0, x_1, x_2]$ by

$$f_{\text{hom}}(x_0, x_1, x_2) := x_0^d f\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right),$$

where $d$ is the degree of $f$. One can show that the projective plane curve $Z(f_{\text{hom}}) \subseteq \mathbb{P}^2(k)$ is the disjoint union of $C = Z(f) \subseteq \mathbb{A}^2_k$ (points with coordinate $x_0 \neq 0$) with a finite number of points lying at infinity (points with coordinate $x_0 = 0$) and is actually its closure with respect to the Zariski topology.
1.3.5 Morphisms and Birational Equivalence

In this paragraph, we introduce different kinds of maps that exist between algebraic varieties.

**Definition 1.14.** Let $Y$ be an affine (resp. projective) algebraic variety over $k$ and $P \in Y$. A function $f : Y \to \mathbb{A}_k^1$ is said to be **regular at $P$** if there is an open neighbourhood $U$ of $P$ and polynomials $g, h \in k[x_1, \ldots, x_n]$ (resp. homogenous polynomials $g, h \in k[x_0, \ldots, x_n]$ of the same degree) such that $h$ does not take the value zero on $U$ and $f = \frac{g}{h}$ on $U$.

Let $W$ be an open subset of $Y$, a function $Y \to \mathbb{A}_k^1$ is called **regular** on $W$ if it is regular at every point of $W$. The set of all regular functions on $W$ is a ring, which will be denoted by $\mathcal{O}(W)$ (or $\mathcal{O}_Y(W)$).

**Remark 4.** A regular function is continuous (for the Zariski topology). It follows that two regular functions on an algebraic variety $Y$ which agree on a dense open subset of $Y$ are equal.

**Remark 5.** ([Sh1] 1.2.2) If $X$ is an affine variety over $k$, then the coordinate ring of $X$ and the ring of regular functions on $X$ are isomorphic via the map

$$k[X] \longrightarrow \mathcal{O}(X)$$

$$[p] \longmapsto p|_X.$$

It is also interesting to consider functions which are not regular on the whole of $X$, but almost everywhere, that is on a dense subset of $X$. In view of Remark 4, we will have to identify functions which agree on a dense subset of $X$.

**Definition 1.15.** Let $X$ be an algebraic variety over $k$. We define an equivalence relation on the set of pairs $(U, f)$, where $U$ is an open dense subset of $X$ and $f$ a regular function on $U$, in the following way:

$$(U, f) \sim (V, g) \iff f|_{U \cap V} = g|_{U \cap V}.$$

The equivalence classes for this relation are called **rational functions** on $X$.

Let's now define maps between algebraic varieties which preserve the structure, or, weaker, which preserve it almost everywhere.
Definition 1.16. Let $X$ and $Y$ be two algebraic varieties. A morphism $\varphi : X \rightarrow Y$ is a continuous map such that for every open subset $V \subseteq Y$ and for every regular function $f : V \rightarrow \mathbb{A}^1_k$, the function $f \circ \varphi : \varphi^{-1}(V) \rightarrow \mathbb{A}^1_k$ is regular.

Definition 1.17. A morphism is an isomorphism if it admits an inverse which is again a morphism.

Definition 1.18. Let $X$ and $Y$ be two algebraic varieties. A rational map is an equivalence class of pairs $(U, \varphi)$, where $U$ is an open dense subset of $X$ and $\varphi$ a morphism $U \rightarrow Y$, under the equivalence relation:

$$(U, \varphi) \sim (V, \psi) \iff \varphi|_{U \cap V} = \psi|_{U \cap V}.$$ 

A rational map will often be denoted by a representative of its equivalence class.

Rational maps allow us to define an equivalence relation between algebraic varieties, which provides a tool for the classification of algebraic varieties.

Definition 1.19. A birational map $\varphi : X \rightarrow Y$ is a rational map which admits an inverse. By an inverse, we mean a rational map $\psi : Y \rightarrow X$ such that $\psi \circ \varphi = id_X$ and $\varphi \circ \psi = id_Y$ as rational maps (i.e. the compositions are regarded on the intersections of the domains of definition). Two algebraic varieties $X$ and $Y$ are said to be birationally equivalent (or birational), if there exists a birational map between them.

1.3.6 Blow-up

As we have seen in Remark 3, §1.3.2, a projective variety is locally homeomorphic to an affine variety. Because the blow-up is a local construction, we do not lose generality if we restrict ourselves to the blow-up of a point in $A^n_k$. Up to a linear change of coordinate, we can even suppose this point to lie at 0. For our purpose, it is sufficient to explain the construction in the case $n = 2$ (for general $n$, see [H] I.4).

This construction furnishes an example of a rational map which may not be birational. It also provides a better understanding of rational maps in general, but this is not the route that we will explore (for more information, consult [Ha] Thm 7.21). Instead, we will use it in resolving singularities on a plane curve (see §1.3.7). More generally, the blow-up is the key tool in resolving singularities on varieties.
1.3. Preparatory Material

First consider the product \( A^2 \times \mathbb{P}^1 \), which is an open dense subset of the projective variety \( \mathbb{P}^2 \times \mathbb{P}^1 \), and the closed subset
\[
X = \{(x_1, x_2; y_1 : y_2) \in A^2 \times \mathbb{P}^1; x_1y_2 = x_2y_1\}
\]
of \( A^2 \times \mathbb{P}^1 \). The projection \( A^2 \times \mathbb{P}^1 \to A^2 \) onto the first factor induces a morphism \( \varphi : X \to A^2 \).

**Definition 1.20.** The set \( X \) together with the morphism \( \varphi : X \to A^2 \) is called the blow-up of \( A^2 \) at the point \((0, 0)\).

A nonzero point in \( A^2 \) has exactly one preimage under \( \varphi \), while the point \((0, 0) \in A^2 \) (which will be denoted by 0) has infinitely many preimages which are in bijection with the lines in \( A^2 \) going through 0. One can show that \( \varphi \) induces an isomorphism of varieties from \( X \setminus \{0\} \) to \( A^2 \setminus \{0\} \) and that \( X \setminus \varphi^{-1}\{0\} \) is dense in \( X \).

Also useful in the practice is the description of \( \varphi \) in the local coordinates of \( X \). For \( i = 1, 2 \), let \((U_i, \psi_i)\) be the chart of \( X \) given by \( U_i = \{(x_1, x_2; y_1 : y_2) \in X; y_i \neq 0\} \) (note that \( X = U_1 \cup U_2 \)) and \( \psi_i((x_1, x_2; y_1 : y_2)) = \left(\frac{y_i}{y_j}, x_i\right) \), where \( j \) is the element of \( \{1, 2\} \) different from \( i \). Write \((u_i, v_i)\) for the coordinates of the chart \( U_i, i = 1, 2 \). Then \( \varphi \) is locally given by
\[
\varphi \circ \psi_1^{-1}(u_1, v_1) = (v_1, u_1v_1) \quad \text{and} \quad \varphi \circ \psi_2^{-1}(u_2, v_2) = (u_2v_2, v_2).
\]

**Definition 1.21.** Let \( Y \) be a closed subvariety of \( A^2 \) containing the point 0. Let's define the blow-up \( Y' \) of \( Y \) at the point 0 to be the closure of \( \varphi^{-1}(Y \setminus \{0\}) \), where \( \varphi : X \to A^2 \) is the blow-up of \( A^2 \) at 0, as described above. The restriction of \( \varphi \) to \( Y' \) is sometimes again denoted by \( \varphi \).

\( Y' \) is a closed subvariety of \( X \) and the morphism \( \varphi_{|Y'} \) induces an isomorphism from \( Y' \setminus \varphi^{-1}\{0\} \) to \( Y \setminus \{0\} \), so that \( Y \) and \( Y' \) are birationally equivalent.

**Definition 1.22.** The blow-up \( Y' \) of \( Y \) at 0 is also called the proper (or strict) transform of \( Y \) in the blow-up of \( A^2 \) at 0, while the inverse image \( \varphi^{-1}\{0\} \subset X \) is called the exceptional divisor of the blow-up.

An example of blowing-up will be given at the end of the next paragraph.
1.3.7 Desingularization of a Plane Curve

It is now the last possible stage to define singular points. For commodity reasons and because only such singularities will here be met, we will only define singular points on plane curves.

**Definition 1.23.** Let $C$ be an affine (resp. projective) plane curve given by the polynomial equation $f(x, y) = 0$ (resp. $F(x_0, x_1, x_2) = 0$). A **singular point** (or **singularity**) on $C$ is a point on $C$ whose coordinates annihilate the two (resp. three) partial derivatives of $f$ (resp. $F$).

A point on $C$ which is not singular is called **nonsingular** or **regular** as well as a curve which has no singular point.

The following notion gives a measure of "how singular" is a point or rather how serious is the singularity.

**Definition 1.24.** The multiplicity $v_P(C)$ of a point $P$ on $C$ is the order of the first nonvanishing term in the Taylor developpement of $f$ (resp. $F$) at $P$.

Note that one has

$$v_P(C) = 1 \iff P \text{ is regular.}$$

Let's now explain how one can resolve singularities on an irreducible affine plane curve $C$ by blowing-up ([Sh1] IV.4.1). Let $P$ be any singular point on $C$ and blow it up. For the preimages of $P$ under the blow-up $\varphi : C' \to C$, one has

$$v_P(C) \geq \sum_{Q \in \varphi^{-1}(P)} v_Q(C').$$

If $P$ is regular, it has exactly one preimage. Suppose then that $P$ is singular, then, if it has more than one preimage, the multiplicities of these are smaller than that of $P$. Blow then up the $Q$'s that are again singular on $C'$. Continue to blow up each time you get a singular preimage. It can be shown that, for each singular preimage, you will once get more than one preimage and so on inductively. Thus, the process must stop after a finite number of blowing-ups, because the multiplicity decreases but remains positive by definition. Applying the same to the eventual other singular points of $C$ (which are in finite number), we get the following theorem.
Theorem 1.1. ([H] V.3.8) Let $C$ be an irreducible curve in a surface $X$ [i.e. an algebraic variety of dimension 2, e.g. $\mathbb{A}^2_k$ or $\mathbb{P}^2(k)$]. Then there exists a finite sequence of blow-ups (of suitable points)

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$$

such that the strict transform $\tilde{C}$ of $C$ on $X_n$ is nonsingular.

The same is also true if $C$ is reducible ([H] V.3.8.1 and 3.9 or [BK] §8.4) and more generally, every curve is birationally equivalent to a nonsingular projective curve ([H] I.6.11).

Definition 1.25. In the context of the theorem, but with $C$ not necessarily irreducible, write $\pi$ for the restriction to $C$ of the composition map and call $\tilde{C}$ the desingularization of $C$ and $\pi : \tilde{C} \to C$ the desingularization map. ($\pi$ is actually a morphism, because it is the composition of morphisms.)

The desingularization $\tilde{C}$ of a plane curve is again a plane curve (because it is birationally equivalent to it) and is unique up to isomorphism ([Ful] Ch7 §5). Though it is in general not easy to find its equation, it may be in the practice sufficient to understand its local structure.

Consider the resolution of a singular point $P$ on a plane curve $C$. According to the theorem, this is obtained after blowing-up this point and its preimages a finite number of times. Since the intersection of two irreducible components of $C$ not contained in each other is a singular point, the proper transforms on $\tilde{C}$ of the irreducible components of $C$ do not meet. They are actually severed one from another by blowing-up as soon as their slopes at the center of the blow-up are different ([H] I.4.9.1). As the blow-up of $C$ at $P$ is defined to be the closure of the inverse image of $C - \{P\}$, it implies that each of the proper transforms of the branches of $C$ carries exactly one preimage of $P$ (which is the point added by closure). This can be transposed locally in the following way.

- The inverse images under $\pi$ of the local branches of $C$ at $P$ have no intersection point on $\tilde{C}$.
- They correspond bijectively to the irreducible factors of the polynomial defining $C$ locally at $P$.
- Each of them carries exactly one preimage of $P$ under the desingularization map $\pi$. 
Example 1. Consider the affine plane curve \( C \) in \( \mathbb{A}^2_{\mathbb{R}} \) which is the locus of zeros of the polynomial \( f(x, y) = y^6 - x^4 \). The sole singular point of \( C \) is \((0, 0)\). As the polynomial \( f \) reduces into two irreducible factors \( y^3 - x^2 \) and \( y^3 + x^2 \), the desingularization \( \tilde{C} \) of \( C \) will have two branches, no one of them meeting the other one, and each of them carrying exactly one preimage of \((0, 0)\) (written \( 0 \) from now on). According to Theorem 1.1, a finite number of blow-ups will be sufficient to desingularize \( C \). For the successive blow-ups, the local coordinates given in §1.3.6 will be used and for the drawings, \( \mathbb{A}^2_{\mathbb{R}} \) will be identified with \( \mathbb{R}^2 \).

Figure 1.1 shows how the curve \( C : y^6 - x^4 = 0 \) looks like. Let's blow up \( C \) at the point \( 0 \). The strict transform of \( C \) is \( C' : t^2 - s^4 = 0 \) (cf Figure 1.2) and the preimage on \( C' \) of \( 0 \) is \( 0 \). It is still singular on \( C' \), though it is not on any of the irreducible components of \( C' \). Blowing up \( C' \) at \( 0 \), we get a similar situation (cf Figure 1.3): the preimage of \( 0 \) on the strict transform \( C'' : q^2 - p^2 = 0 \) of \( C' \) is \( 0 \) and is singular only because it lies at the intersection of two irreducible components of \( C'' \) (each of them being nonsingular). We then have to blow up again at \( 0 \). At least (see Figure 1.4), the two irreducible components of the strict transform \( C''' \) have no intersection point (and stay nonsingular), hence \( C''' \) is nonsingular. Each of these components carries exactly one preimage of \( 0 \), these being \((0, 1)\) and \((0, -1)\).

1.3.8 Riemann Surfaces and Nonsingular Algebraic Curves over \( \mathbb{C} \)

We first recall the definition of a Riemann surface and then explain how an irreducible nonsingular algebraic curve defined over \( \mathbb{C} \) can be given the structure of a Riemann surface.

Definition 1.26. ([FK] 1.1) A Riemann surface \( M \) is a connected Hausdorff topological space on which there exists a maximal set of charts \( \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A} \) (i.e. \( \{U_\alpha\}_{\alpha \in A} \) is an open cover of \( M \) and \( \forall \alpha \in A, \exists V_\alpha \) an open subset of \( \mathbb{C} \) such that \( \varphi_\alpha : U_\alpha \rightarrow V_\alpha \) is a homeomorphism) with the property that, if \( U_\alpha \cap U_\beta \neq \emptyset \), then the so-called transition function

\[ \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \rightarrow \varphi_\alpha(U_\alpha \cap U_\beta) \]

is holomorphic. For each \( \alpha \), the elements of \( V_\alpha \) are called local coordinates or local parameters of \( M \) on \( U_\alpha \) and \( U_\alpha \) is called the domain of the chart \( (U_\alpha, \varphi_\alpha) \).
1.3. Preparatory Material

Figure 1.1: $y^6 = x^4$

Figure 1.2: $t^2 = s^4$
1. Abelian Varieties Associated to Hypergeometric Series

Figure 1.3: $q^2 = p^2$

Figure 1.4: $v = \pm 1$
Hence a Riemann surface is a connected one-dimensional complex analytic manifold, or also a connected two-real-dimensional differential manifold on which the transition functions are not only $C^1$ but even holomorphic.

Classical examples are the complex plane and the Riemann sphere $\mathbb{C} \cup \{\infty\}$. A less classical one is the desingularisation of an algebraic curve defined over $\mathbb{C}$ as we will see in the second part of this paragraph.

Originally, Riemann had defined such surfaces in order to "uniformize" multivalued complex functions of one variable, such as taking the complex $n$-th root or the logarithm of a complex number. Indeed, these functions can be defined as uniform functions on a finite covering of $\mathbb{C}$.

**Definition 1.27.** Let $M$ and $N$ be Riemann surfaces. A continuous map $f : M \to N$ is called **holomorphic** (or analytic) if for every chart $(U, \varphi)$ on $M$ and any chart $(V, \psi)$ on $N$ such that $U \cap f^{-1}(V) \neq \emptyset$, the map

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(V)$$

is holomorphic. A holomorphic map $M \to \mathbb{C}$ is called a **holomorphic function** on $M$.

A similar definition is applied to a function $W \to \mathbb{C}$ defined on an open subset $W$ of a Riemann surface by considering the intersections with $W$ of the domains of the charts of $M$.

Let's consider now an affine algebraic plane curve $X$ defined over $\mathbb{C}$ (i.e. $X \subset \mathbb{A}^2_\mathbb{C}$) and write $X(\mathbb{C})$ for the set of complex points of $X$. If $x$ is a nonsingular point of $X$ and $t_1, t_2$ coordinates on $\mathbb{A}^2_\mathbb{C}$, then by the Implicit Functions Theorem, one of the two coordinates (say $t_2$) can be written in the neighbourhood of $x$ as a holomorphic function of the other. Hence the projection on the first factor defines a homeomorphism of $\{(t_1, t_2) \in X(\mathbb{C}); |t_i| < \eta, i = 1, 2\}$ (where $\eta > 0$) onto an open domain of $\mathbb{C}$. This defines a new topology on $X(\mathbb{C})$, which is called the **complex topology**. The same can be done for a projective plane curve and, more generally, for any algebraic variety over $\mathbb{C}$ ([Sh2] VII.1.1). In particular, if the curve $X$ has only nonsingular points, $X(\mathbb{C})$ is locally homeomorphic to a domain of $\mathbb{C}$.

Let then $X$ be a nonsingular projective algebraic curve defined over $\mathbb{C}$ and $X(\mathbb{C})$ the set of complex points of $X$ with the complex topology defined above. The following properties hold.
1. Abelian Varieties Associated to Hypergeometric Series

- $X(\mathbb{C})$ is compact ([Sh1] II.2.3).
- $X(\mathbb{C})$ is triangulable (for a definition see §1.3.9) ([Sh2] VII.3.2).

If $X$ is moreover irreducible, then

- $X(\mathbb{C})$ is connected ([Sh2] VII.2.2).

Suppose now that $X$ is irreducible. In order for $X(\mathbb{C})$ to be a Riemann surface, the transition functions must be holomorphic. To show that they actually are, we use the fact that if $f$ is a regular function at $x \in X$ and $\varphi : U \to V \subseteq \mathbb{C}$ a homeomorphism from an open neighbourhood $U$ of $x$ and an open subset $V$ of $\mathbb{C}$, then the function $f \circ \varphi^{-1}$ is analytic in a neighbourhood of $\varphi(x)$. Conclude by remembering that a projective curve is locally isomorphic to an affine one and that, in this case, the maps are regular functions, since they are projections on the first (or second) factor. (See also [Mir] I.2 §5 for the affine case and I.3 §2 for the projective one.)

This also holds in bigger dimension: for any nonsingular algebraic variety $X$ over $\mathbb{C}$, $X(\mathbb{C})$ has the structure of a complex analytic variety ([Sh2] VIII.1.1).

Hence, if $X$ is a nonsingular irreducible projective algebraic curve, then the set $X(\mathbb{C})$ of complex points of $X$ admits a structure of compact Riemann surface.

Remark 6. Let $X$ be a nonsingular projective algebraic curve over $\mathbb{C}$ and $X(\mathbb{C})$ the associated complex analytic surface. If $f$ is a regular function on $X$, then it is a holomorphic function on $X(\mathbb{C})$, because it is locally the quotient of a polynomial by a nonvanishing polynomial. By Serre’s GAGA theorem, the converse also holds (see [Sh2] VIII.3.1).

1.3.9 Topological Invariants of a Riemann Surface

In this paragraph, we shall discuss some topological invariants of Riemann surfaces, such as the Euler characteristic and the first homology group with integer coefficients. These objects can be defined for any topological space, but are easier to calculate if the space is triangulable. Since any Riemann surface is triangulable, we will define these objects through the use of a triangulation. We naturally begin by explaining what a triangulation is ([AS] ChI §22).
1.3. Preparatory Material

Definition 1.28. An (abstract) simplicial complex $K$ is a set $V(K)$ together with a family of finite subsets of $V(K)$, called the simplices, such that the following properties hold:

1. Each $\alpha \in V(K)$ belongs at least to one and at most to a finite number of simplices.

2. Every subset of a simplex is a simplex.

The dimension of a simplex is one less than the number of its elements. A simplex of dimension $q$ is also called a $q$-simplex. The dimension of the simplicial complex $K$ is the maximum of the dimensions of its simplices (or infinite, if there is no maximum).

To each $q$-simplex $\sigma$ of a simplicial complex $K$ can be associated a closed subset $|[\sigma]|$ of $\mathbb{R}^{q+1}$ in such a way that, $\forall \sigma_1, \sigma_2 \in K$, $|\sigma_1 \cap \sigma_2| = |\sigma_1 \cap \sigma_2|$. The set $|K| := \bigcup_{\sigma \in K} |\sigma|$ can be given a topology. Together with this topology, $|K|$ is called the geometric realization of $K$.

Definition 1.29. Let $F$ be a differentiable manifold of dimension 2. A triangulation of $F$ is the data of a simplicial complex $K$ of dimension 2 together with a homeomorphism of $|K|$ onto $F$.

It can be shown that every Riemann surface $M$ admits a triangulation ([Sp] Ch 9-3) and that $M$ is compact if and only if it admits a finite triangulation (that is a triangulation whose underlying simplicial complex contains only a finite number of points ([Sp] Ch5-1)).

We would like now to define the homology of a simplicial complex $K$ with coefficients in a ring $R$. For this, we have to define ordered $q$-simplices and identify some of them.

An ordered $q$-simplex is a $(q+1)$-uplet $(v_0, ..., v_q) \in V(K)^{q+1}$ such that $\{v_0, ..., v_q\}$ lies in $K$. Two ordered $q$-simplices $(v_0, ..., v_q)$ and $(v_0', ..., v_q')$ will be equivalent if and only if there exists an even permutation $s \in A_{q+1}$ such that $\forall i = 0, ..., q$, $v_i' = v_{s(i)}$. A class for this equivalence relation is called an oriented $q$-simplex of $K$. Now, define $C_q(K, R)$ to be the free $R$-module on the set of oriented $q$-simplices of $K$. Let $\sigma$ be an oriented $q$-simplex, we want to identify the $q$-simplex $\sigma^#$ having
the inverse orientation with the the opposite $-\sigma$ of $\sigma$ in $\mathcal{C}_q(K, R)$. Let then $I$ be the $R$-submodule of $\mathcal{C}_q(K, R)$ generated by $\{\sigma + \sigma^a; \sigma$ oriented $q$-simplex of $K\}$, we set

$$C_q(K, R) := \mathcal{C}_q(K, R)/I.$$ 

A class $[v_0, ..., v_q] \in C_q(K, R)$ is called a \textbf{q-chain}.

We have a \textbf{boundary operator} $\partial_q : C_q(K, R) \to C_{q-1}(K, R)$ given by

$$\partial_q([v_0, ..., v_q]) = \sum_{i=0}^{q} (-1)^i [v_0, ..., \hat{v}_i, ..., v_q],$$

where the hat on $v_i$ means that the element $v_i$ is omitted. This is a homomorphism of $R$-modules. The image of a q-chain is a (q-1)-chain called its \textbf{boundary}. One verifies that $\partial_{q-1} \circ \partial_q = 0$ (i.e. $\text{Im}\partial_q \subset \text{Ker}\partial_{q-1}$). Hence we have a \textbf{chain complex}

$$\ldots \to C_{q+1}(K, R) \xrightarrow{\partial_{q+1}} C_q(K, R) \xrightarrow{\partial_q} C_{q-1}(K, R) \to \ldots \to C_0(K, R) \xrightarrow{\partial_0} 0,$$

where $\partial_0$ denotes the zero homomorphism (cf [AS] ChI 23C).

This sequence is not necessarily exact and its nonexactness is measured by the $R$-modules

$$H_q(K, R) := \text{Ker}(\partial_q)/\text{Im}(\partial_{q+1}) \text{ where } q \in \mathbb{N}.$$ 

\textbf{Definition 1.30.} For $q \in \mathbb{N}$, the $R$-module $H_q(K, R)$ is called the \textbf{q-th (or q-dimensional) homology R-module} of the simplicial complex $K$. The elements of $\text{Ker}(\partial_q)$ are called \textbf{q-cycles} and those of $\text{Im}(\partial_{q+1})$ \textbf{q-boundaries}.

For a triangulable surface (or manifold) $F$, we define its homology to be that of its triangulation. It can be shown that these $R$-modules are actually isomorphic to given $R$-modules depending directly on the topology of $F$ (these are the so-called \textbf{singular homology R-modules}) and hence independent of the choice of the triangulation. This justifies the following definition.

\textbf{Definition 1.31.} Let $M$ be a Riemann surface and $R$ a ring. Let the simplicial complex $K$ furnish a triangulation of $M$, then we define the \textbf{q-th (or q-dimensional) simplicial homology R-module} $H_q(M, R)$ to be $H_q(K, R)$. 

Remark 7. Since a Riemann surface $M$ can be triangulable by a simplicial complex of dimension 2, we have,

$$\forall q \geq 3, \quad H_q(M, R) = 0.$$  

From now on suppose that $R = \mathbb{Z}$ ([AS] ChI 25). What can be said about $H_i(M, R)$ when $i \in \{0, 1, 2\}$?

- Note that any nonzero multiple of a 0-simplex (vertex) $\alpha_0$ is nonzero in $H_0(M, R)$. On the other hand, take any other vertex $\alpha_1$. In view of the connectedness of $M$, there exists a finite sequence of 1-simplices which begins at $\alpha_0$ and ends at $\alpha_1$. Hence, $\alpha_1 \sim \alpha_0$ and $H_0(M, R) = R[\alpha_0] \cong R$.

- It can be shown that $H_2(M, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 1 if and only if $M$ is compact. In all other cases, $H_2(M, \mathbb{Z}) = 0$.

- For a compact Riemann surface $M$, $H_1(M, \mathbb{Z})$ is a free $\mathbb{Z}$-module of finite rank.

**Definition 1.32.** Let $M$ be a compact Riemann surface. For $i \in \{0, 1, 2\}$, the (finite) rank of $H_i(M, \mathbb{Z})$ over $\mathbb{Z}$ is called the $i$-th Betti number and denoted by $b_i$.

The Betti numbers of $M$ are topological invariants of $M$ (that means that they depend only on the topology of $M$ and are invariants under any homeomorphism).

**Definition 1.33.** ([Sp] 5-9) Let $K$ be a simplicial complex giving a triangulation of the compact Riemann surface $M$. Write $n_q$ for the number of $q$-simplices in $K$. Then, we define

$$\chi(M) := n_0 - n_1 + n_2.$$  

This number is called the **Euler characteristic** of $M$.

This definition is actually independent of the choice of the triangulation of $M$. Indeed, the Euler-Poincaré formula

$$\chi(M) = b_0 - b_1 + b_2$$  

shows that $\chi(M)$ depends only on the topology of $M$ and is therefore a topological invariant of $M$ (see [Sp] 5-9).
For a compact Riemann surface $M$, we have seen that $b_0 = b_2 = 1$. In this case, it follows from the formula that

$$b_1 = 2 - \chi(M).$$

Next, we will explain how the Euler characteristic of a Riemann surface can be deduced from that of another compact Riemann surface if there exists a nonconstant holomorphic map from one to the other. This is the object of Hurwitz formula.

We first begin with a somehow more general definition.

**Definition 1.34.** ([FK] 1.2.4) Let $M$ and $M^*$ be two manifolds. Then $M^*$ is said to be a (ramified) covering manifold of $M$ if there exists a map $f : M^* \rightarrow M$ (called covering map) such that for each point $P^* \in M^*$, there exists a local coordinate $z^*$ around $P^*$, which vanishes at $P^*$, and a local coordinate $z$ around $f(P^*)$, which vanishes at $f(P^*)$, and an integer $n > 0$ such that $f$ is given in terms of these coordinates by

$$z = (z^*)^n.$$

The integer $n$ depends only on the point $P^*$. If $n > 1$, then $P^*$ is said to be a ramification point of order $n$. $n$ is called the ramification index (or ramification order) of $P^*$ and is denoted by $r_f(P^*)$. If $n = 1$, $P^*$ is said to be regular and we set $r_f(P^*) = 1$. If $f$ has no ramification point, then the covering is called unramified (or smooth).

**Remark 8.** ([Sp] 4-1) The expression $z^* \mapsto (z^*)^n$ of $f$ in local coordinates shows that the ramification points are isolated. Indeed, the chart at $P^*$ can be chosen such that the restriction of the map $z^* \mapsto (z^*)^n$ to any open subset in the domain $U$ of the chart which does not contain 0 is in bijection with its image (restriction to a branch of the n-th root) and consequently each point in $U$ distinct from 0 parametrizes a regular point on $M^*$.

If $M^*$ is compact, the above implies that the covering map $f$ has only finitely many ramification points.

Returning to the case of Riemann surfaces, suppose that $M$ and $N$ are compact Riemann surfaces and $f : M \rightarrow N$ a holomorphic map, then $f$ is either constant or surjective. If $f$ is nonconstant, it can be shown that at each point of $M$, there exist local coordinate as in the definition. Moreover, such a map takes each values
the same number of times counting multiplicities ([FK] I.1.6), i.e \( \exists m \in \mathbb{N} \) such that \( \forall Q \in N \), we have

\[
m = \sum_{P \in f^{-1}(\{Q\})} r_f(P).
\]

The nonnegative integer \( m \) is called the **degree** of the map \( f \) and \( f \) a **m-sheeted cover** of \( N \) by \( M \).

**Remark 9.** A point \( Q \in N \) has exactly \( m \) distinct preimages if and only if \( \forall P \in f^{-1}(\{Q\}) \), \( r_f(P) = 1 \), that is all the preimages of \( Q \) are regular points of \( f \).

Let’s summarize all of this in a proposition.

**Proposition 1.2.** Let \( M, N \) be two compact Riemann surfaces, then any nonconstant holomorphic map \( f : M \to N \) is a covering map and its degree equals the number of preimages of each point of \( N \), except for a finite number of isolated ones.

If there exists a map as in the proposition, then the Euler characteristics of \( M \) and \( N \) respectively are related in the following way.

**Proposition 1.3. **Hurwitz formula

Let \( M \) and \( N \) be two compact Riemann surfaces and suppose that there is a nonconstant holomorphic map \( f : M \to N \). Write \( m \) for the degree of \( f \) and \( P_1, \ldots, P_s \in M \) for its ramification points. Then one has

\[
\chi(M) = m \cdot \chi(N) - \sum_{i=1}^{s} (r_f(P_i) - 1).
\]

**Proof.** (FK I.2.7) Let \( K \) be a simplicial complex furnishing a triangulation of \( N \). Since the ramification points of \( f \) are isolated, so are their images and we can suppose these to coincide with 0-simplices of \( K \) if only to refine the triangulation. (Recall that the Euler characteristic is independent of the choice of the triangulation.) For \( q \in \{0, 1, 2\} \), write \( n_q \) for the number of \( q \)-simplices in this triangulation and lift it to a triangulation of \( M \) via the map \( f \). This triangulation of \( M \) has

\[
\begin{align*}
m \cdot n_2 & \text{ 2-simplices,} \\
m \cdot n_1 & \text{ 1-simplices,} \\
m \cdot n_0 - \sum_{i=1}^{s} (r_f(P_i) - 1) & \text{ 0-simplices.}
\end{align*}
\]
Conclude by summing
\[ \chi(M) = m(n_0 - n_1 + n_2) - \sum_{i=1}^s (r_f(P_i) - 1). \]

1.3.10 Tangent Space to an Algebraic Variety and Differential Maps

We first give a geometric definition of the tangent space to an affine algebraic variety (cf [Sh1] II.1.2) and then relate it to an algebraic object, what will justify the general definition.

Let \( X \subset \mathbb{A}^n_k \) be an affine algebraic variety over \( k \). According to §1.3.1, it is the locus of a finite number of polynomials in \( k[x_1, ..., x_n] \). Let \( F_1, ..., F_m \) be these polynomials and fix a point \( P \) on \( X \). Up to a linear change of coordinates, we can suppose this point to lie at 0. A line \( L \) in \( \mathbb{A}^n_k \) going through 0 is a set \( \{ ta; t \in k \} \), where \( a \in \mathbb{A}^n_k \) is a fixed nonzero point. The intersection of \( X \) with \( L \) is given by the equations

\[ F_1(ta) = ... = F_m(ta) = 0. \]

\( a \) being fixed, the equations are polynomial in the variable \( t \) and have \( t = 0 \) as solution. Write \( f_i(t) := F_i(ta) \) for \( i \in \{1, ..., m\} \).

**Definition 1.35.** The **intersection multiplicity** of the line \( L \) with \( X \) at 0 is the multiplicity of \( t = 0 \) as a common root of all \( f_i \); \( i = 1, ..., m \). It is independent of the choice of generators of \( I(X) \) and nonnegative, since \( 0 \in L \cap X \). If \( \forall i \in \{1, ..., m\} \), \( f_i = 0 \), the multiplicity is set to be infinite.

We can now define the tangency to \( X \) of a line going through 0 and the tangent space to \( X \) at 0.

**Definition 1.36.** A line \( L \) is said to be **tangent** to \( X \) at 0 if its intersection multiplicity with \( X \) is \( > 1 \).

**Definition 1.37.** The geometric locus of all points lying on lines tangent to \( X \) at 0 is called the **tangent space** to \( X \) at 0 and is denoted by \( \Theta_{X,0} \) or simply \( \Theta_0 \).
Since $0$ is a root of each $F_i$, $i = 1, ..., m$ (because $0 \in X$), the $F_i$'s have constant term equal to 0. Thus, for every $i$, we can write

$$F_i = L_i + H_i$$

where $L_i$ is the linear term of $F_i$ and $\deg(H_i) \geq 2$.

Now, because $\forall i = 1, ..., m$, $L_i(ta) = tL_i(a)$, we have

$$f_i(t) = tL_i(a) + H_i(ta), \quad \forall i = 1, ..., m,$$

and the order of $0$ as a common root of the polynomials $f_i$, $i = 1, ..., m$, is $\geq 2$ if and only if $L_i(a) = 0$, $\forall i = 1, ..., m$. Hence, the tangent space to $X$ at $0$ is given by a system of linear equations

$$L_1(a) = ... = L_m(a) = 0 \quad (1.4)$$

and therefore is a linear subspace of $\mathbb{A}^n_\mathbb{k}$.

The tangent space to $X$ at any point $P$ of $X$ can be defined in an analogous way. It is also a vector space over $k$ and will be denoted $\Theta_{X,P}$ or simply $\Theta_P$.

Let $X$ still be an affine algebraic variety in $\mathbb{A}^n_\mathbb{k}$ and $g$ be a regular map on $X$, i.e. $g \in \mathcal{O}(X)$. Using the isomorphism in Remark 5, §1.3.5, we can identify $g$ with the restriction to $X$ of a polynomial $G$ in $k[x_1, ..., x_n]$. For such a polynomial and a point $P$ on $X$ with coordinates $(p_1, ..., p_n)$, we define $d_P G$ to be the linear form on the $k$-vector space $\Theta_{X,P}$ defined by the term of first order in the Taylor development of $G$ at $P$. That is, for $x = (x_1, ..., x_n) \in \Theta_{X,P}$, we have

$$d_P G(x) := \sum_{i=1}^n \frac{\partial G}{\partial x_i}(P)(x_i - p_i).$$

Note that for $F, H \in k[x_1, ..., x_n]$, we have

$$d_P(F + H) = d_P F + d_P H \quad \text{and} \quad d_P(F \cdot H) = H(P)d_P F + F(P)d_P H.$$

Write again $F_1, ..., F_m$ for the generators of the radical ideal $I(X)$ of $X$, then, in view of (1.4), the tangent space to $X$ at $P$ is given by the equations

$$d_P F_i(x) = 0 \quad \text{for} \quad i = 1, ..., m, \quad (1.5)$$
because $L_i$ is precisely the linear term of $F_i$. Now, by Remark 5, we know that $G$ is unique up to addition by an element of $I(X) = \langle F_1, \ldots, F_m \rangle$. Let $G' := G + \sum_{i=1}^{m} A_i F_i$ with $A_i \in k[x_1, \ldots, x_n]$, for $i = 1, \ldots, m$, then
\[
\frac{d_p G'}{d_p G} = \frac{d_p G' + \sum_{i=1}^{m} (F_i(P)) d_p A_i + A_i(P) d_p F_i}{d_p G}.
\]
But $P \in X$ implies $F_i(P) = 0, \forall i = 1, \ldots, m$ and besides the linear forms $d_p F_i$ are zero on $\Theta_{x,P}$ by definition of $\Theta_{x,P}$. Hence
\[
\frac{d_p G'}{d_p G} = \frac{d_p G}{d_p G}
\]
and we can define

**Definition 1.38.** Let $X$ be an affine variety in $\mathbb{A}^n$ and $g \in \mathcal{O}(X)$ represented by $G \in k[x_1, \ldots, x_n]$, then we define the **differential** $d_p g$ of $g$ at $P$ to be the linear form $d_p G$ on $\Theta_P$.

As seen above, this definition is independant of the representant of the class of $G$ in $k[X] = k[x_1, \ldots, x_n]/I(X) \simeq \mathcal{O}(X)$. This definition furnishes a homomorphism of $\mathcal{O}(X)$ to the $k$-vector space $\Theta_P^*$ of linear forms on $\Theta_P$. Remark that if $\alpha \in k$, then $d_p \alpha = 0$, thus for $f \in \mathcal{O}(X)$, $f(P) \in k$ and we have
\[
d_p (f - f(P)) = d_p f.
\]
It is therefore sufficient to restrict the study to the image of the set of regular functions vanishing at $P$. This set $\tilde{\mathcal{M}}_P = \{ f \in \mathcal{O}(X); f(P) = 0 \}$ is an ideal in $\mathcal{O}(X)$ and the quotient $\tilde{\mathcal{M}}_P/\tilde{\mathcal{M}}_P^2$ is a $k$-vector space. The homomorphism $d_p : \tilde{\mathcal{M}}_P \to \Theta_P^*$, $f \mapsto d_p f$ is surjective and its kernel equals $\tilde{\mathcal{M}}_P^2$. Hence it induces an isomorphism of $k$-vector spaces ([Sh1] II.1.3.1)
\[
d_p : \tilde{\mathcal{M}}_P/\tilde{\mathcal{M}}_P^2 \sim \Theta_P^*.
\]
The dual spaces are then also isomorphic
\[
\Theta_P \simeq (\tilde{\mathcal{M}}_P/\tilde{\mathcal{M}}_P^2)^*.
\]
We would like to use this isomorphism in order to define the tangent space to a projective variety $Y$. We know (cf Remark 3) that such a variety is covered by a finite number of affine ones, but the above cannot be applied directly to any affine neighbourhood of $P \in Y$, because it could depend on the choice of this affine neighbourhood. This leads us to the definition of a local algebraic invariant of a point on an algebraic variety.
**Definition 1.39.** ([Sh1] II.1.1) Given an algebraic variety $Y$ and a point $P$ on $Y$, consider the set of pairs $(U, f)$, where $U$ is an open neighbourhood of $P$ and $f \in \mathcal{O}(U)$, and the following equivalence relation on this set

$$(U, f) \sim (V, g) \iff \exists W \text{ open } \subset U \cap V \text{ s.t. } P \in W \text{ and } f|_W = g|_W.$$ 

Write $\mathcal{O}_P$ (or $\mathcal{O}_{Y, P}$) for the set of equivalence classes for this relation. It is a ring for the operation induced from the usual multiplication and addition on representants and has a unique maximal ideal

$$\mathcal{M}_P = \{((U, f)) \in \mathcal{O}_P; f(P) = 0\}.$$

Hence $\mathcal{O}_P$ is a local ring which is called the **local ring** of $P$.

When $Y$ is an affine variety, we have

$$\mathcal{M}_P / \mathcal{M}_P^2 \simeq \mathcal{O}_P / \mathcal{M}_P^2 \tag{1.7}$$

and remembering the isomorphism (1.6), we define

**Definition 1.40.** ([Sh1] II.1.3) Let $Y$ be an affine or projective variety and $P \in Y$, the **tangent space** to $Y$ at $P$ will be $\Theta_P / \Theta_P^*\mathcal{O}_P$, where $\mathcal{M}_P$ is the maximal ideal of the local ring $\mathcal{O}_P$. It will be denoted by $\Theta_P$ (or $\Theta_{Y, P}$).

By (1.6) and (1.7), $\Theta_{Y, P}$ is then also the tangent space in the previous sense to any affine neighbourhood of $P$.

Now, if $Y$ is any algebraic (i.e. affine or projective) variety and $f \in \mathcal{O}(Y)$, how can we associate to $f$ an element of the dual space to $\Theta_{Y, P}$? In the affine case, for a polynomial $G \in k[x_1, \ldots, x_n]$ and a point $P = (p_1, \ldots, p_n)$, we had define $d_P G$ to be the term of first order in the Taylor development

$$G = G(P) + \sum_{i=1}^{n} \frac{\partial G}{\partial x_i}(P)(x_i - p_i) + H,$$

of $G$ at $P$. Here $H$ is of degree $\geq 2$ and $H(P) = 0$, hence $H \in \mathcal{M}_P^2$. This implies

$$G - G(P) \equiv \sum_{i=1}^{n} \frac{\partial G}{\partial x_i}(P)(x_i - p_i) \pmod{\mathcal{M}_P^2}$$
and, by definition,

\[ d_PG \equiv G - G(P) \pmod{\mathcal{M}^2_P}. \]

This allows us to extend Definition 1.38 as follows.

**Definition 1.41.** Let \( Y \) be an affine or projective algebraic variety, \( P \in Y \) and \( f \in \mathcal{O}(Y) \) (or even in \( \mathcal{O}_P(Y) \)), we define the **differential** \( d_P f \) of \( f \) at \( P \) to be the following element of \((\Theta_{Y,P})^*\)

\[ f - f(P) \pmod{\mathcal{M}^2_P}. \]

Note that for \( f, g \in \mathcal{O}(Y) \) and \( P \in Y \), we have

\[ \begin{align*}
  d_P(f + g) &= d_Pf + d_Pg \\
  d_P(f \cdot g) &= f(P)d_Pg + g(P)d_Pf.
\end{align*} \]

Let's now define the differential of a morphism between algebraic varieties ([Sh1] II.1.3). Let \( X, Y \) be two algebraic varieties and \( \Phi : X \to Y \) a morphism. If \( f \) is a regular function on \( Y \), then the composition \( f \circ \Phi \) is regular on \( X \), hence \( \Phi \) induces a linear map \( \Phi^* : \mathcal{O}(Y) \to \mathcal{O}(X) \) by sending \( f \) to \( f \circ \Phi \). Obviously, for each \( P \in X \), \( \Phi^*(\mathcal{M}_{\Phi(P)}) \subset \mathcal{M}_P \) and \( \Phi^*(\mathcal{M}^2_{\Phi(P)}) \subset \mathcal{M}^2_P \). Hence \( \Phi^* \) factorizes and, for each \( P \in X \), induces

\[ \Phi^* : \mathcal{M}_{\Phi(P)}/\mathcal{M}^2_{\Phi(P)} \to \mathcal{M}_P/\mathcal{M}^2_P. \]

To a linear function \( g : \mathcal{M}_P/\mathcal{M}^2_P \to k \), we associate \( g \circ \Phi^* \in (\mathcal{M}_{\Phi(P)}/\mathcal{M}^2_{\Phi(P)})^* \cong \Theta_{Y,\Phi(P)} \) and define

**Definition 1.42.** Let \( X, Y \) be algebraic varieties, \( P \in X \) and \( \Phi : X \to Y \) a morphism of algebraic varieties, then the induced linear map on the tangent spaces

\[ \begin{align*}
  \Theta_{X,P} &\to \Theta_{Y,\Phi(P)} \\
  g &\mapsto g \circ \Phi^*
\end{align*} \]

is called the **differential** of \( \Phi \) at \( P \) and denoted by \( d_P\Phi \).

Note that if \( \Phi \) is the identity on \( X \), then for every \( P \in X \), \( d_P\Phi \) is the identity on \( \Theta_{X,P} \). If \( Z \) is another algebraic variety and \( \Psi : Y \to Z \) a morphism, then the differential \( d_P(\Psi \circ \Phi) : \Theta_{X,P} \to \Theta_{Z,\Psi(\Phi(P))} \) is given by

\[ d_P(\Psi \circ \Phi) = d_{\Phi(P)}\Psi \circ d_P\Phi. \]
In particular, if $\Phi : X \to Y$ is an isomorphism of varieties, then its differential $d_P \Phi : \Theta_{X,P} \to \Theta_{Y,\Phi(P)}$ is an isomorphism of $k$-vector spaces.

**Remark 10.** ([Mil] §4 p.63) A morphism $\varphi : X \to Y$, where $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ are affine algebraic varieties, is given in each coordinate by a regular function. Write $\varphi_i$ for the regular function in the $i$-th coordinate of $\varphi$ and $Y_i$ for the projection $(y_1, \ldots, y_n) \mapsto y_i$. Then the differential $d_P \varphi$ of $\varphi$ at $P \in X$ is the map $\Theta_{X,P} \to \Theta_{Y,\varphi(P)}$ such that

$$d_{\varphi(P)} Y_i \circ d_P \varphi = \sum_{j=1}^m \frac{\partial \varphi_i}{\partial x_j}(P) d_P X_j,$$

where $X_i$ is the projection $(x_1, \ldots, x_m) \mapsto x_i$ and $d_P X_i : T_P X \to k$ the linear map sending $\left( \begin{array}{c} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_m \end{array} \right)$ to $x_i - p_i$.

### 1.3.11 Differential Forms on Algebraic Varieties

Given an algebraic variety $X$ over $k$, we have shown in Paragraph 1.3.10 how can be associated to each point $P$ of $X$ an element of $\Theta^1_{X,P}$. This was done through the use of any regular function $f$ on $X$; indeed, for each $P \in X$, $d_P f \in \Theta^1_{X,P}$. For an open set $V$ of $X$, the set $\Phi[V]$ of functions sending each point $P$ of $V$ to an element of $\Theta^1_{X,P}$ is an abelian group for the usual addition of functions and a $\mathcal{O}(V)$-module for the action $(f \cdot \varphi)(P) := f(P)\varphi(P)$, where $f \in \mathcal{O}(V)$.

**Definition 1.43.** An element of $\Phi[V]$ will be called a **differential form** (or differential 1-form) on $V$.

The set $\Phi[X]$ would actually be too big to be interesting, therefore, we would like to distinguish special elements of $\Phi[X]$, whose variation on $P$ is better contolled, just as the regular functions on $X$ are better understood that any one function from $X$ to $\mathbb{A}^1$. Among the elements of $\Phi[X]$, we distinguish those that are, in substance, locally given by differentials of regular functions on $X$. Indeed, let's set

**Definition 1.44.** ([Sh1] III.5.1) Let $X$ be an algebraic variety and $V$ an open subset of $X$. Then a function $\varphi$ sending each $P \in V$ to an element of $\Theta^1_{X,P}$ is said to be a **regular differential form** (or regular differential 1-form) on $V$, if each $p \in V$ has a neighbourhood $U \subseteq V$ such that the restriction of $\varphi$ to $U$ belongs to the
$\mathcal{O}(U)$-submodule of $\Phi[U]$ generated by the elements $df$ with $f \in \mathcal{O}(U)$, where $df$ is the function on $U$ such that, for $Q \in U$, $df(Q) = d_Q f$.

The set of all regular differential forms on $V$ is a module over $\mathcal{O}(V)$, and, in particular, a $k$-vector space, which will be denoted by $\Omega^1[V]$ (or $\Omega^1_X[V]$).

If $\omega \in \Omega^1[X]$, then for each $P$ in $X$ there exist a neighbourhood $U$ of $P$ and $f_1, \ldots, f_m, g_1, \ldots, g_m \in \mathcal{O}(U)$ such that

$$\omega|_U = \sum_{i=1}^{m} (f_i \cdot dg_i) \quad \text{i.e.} \quad \forall Q \in U, \quad \omega(Q) = \sum_{i=1}^{m} f_i(Q) d_Q g_i.$$

If $X$ is a nonsingular projective algebraic curve, one can show that the $k$-vector space $\Omega^1[X]$ has finite dimension ([Sh1] III.6.3).

**Definition 1.45.** Let $X$ be a nonsingular projective algebraic curve. Then the number $\dim_k(\Omega^1[X])$ is called the *genus* of $X$ and denoted by $g[X]$ or simply $g$ if there is no ambiguity.

**Remark 11.** If $X$ is irreducible and defined over $\mathbb{C}$, one can show ([Sh2] VII.3.3) that

$$\chi(X(\mathbb{C})) = 2 - 2g[X], \quad (1.8)$$

where $\chi(X(\mathbb{C}))$ is the Euler characteristic of the Riemann surface $X(\mathbb{C})$ (see §§1.3.8 and 1.3.9 for the definitions).

**Remark 12.** ([Sh1] III.5.1) If $C$ is an algebraic plane curve over $k$ given locally by the affine equation $f(x, y) = 0$ and $(x_0, y_0)$ a nonsingular point on $X$, then at least one of the two derivatives of $f$ does not vanish at $(x_0, y_0)$. Without restricting the generality, suppose that $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$. Then by the Implicit Functions Theorem, $y$ can be written in a neighbourhood $U$ of $(x_0, y_0)$ as a holomorphic (hence regular) function of $x$ and, because the differentials of the local parameters at $(x_0, y_0)$ form a basis of $\Theta^*_p$ for every $P$ at which all parameters are regular, any regular differential form on $C$ can be written in $U$ as

$$f dx,$$

where $f$ is a regular function on $U$. 
In §1.3.10, we have seen that a morphism $\Phi : X \to Y$ between algebraic varieties induces for each $P \in X$ a linear map $d_P \Phi : \Theta_{X,P} \to \Theta_{Y,\Phi(P)}$. Hence, if $\omega$ is a differential form on $Y$, then for each point $Q \in Y$, $\omega(Q) \in \Theta^*_{Y,Q}$. This implies $\omega(\Phi(P)) \circ d_P \Phi \in \Theta^*_{X,P}$. This defines a differential form on $X$, which is called the pull-back of $\omega$ (with respect to $\Phi$) and denoted by $\Phi^* \omega$. According to the definition, we have

$$\forall P \in X, \quad (\Phi^* \omega)(P) = \omega(\Phi(P)) \circ d_P \Phi \quad \text{i.e.} \quad \Phi^* \omega = (\omega \circ \Phi) \circ d \Phi,$$

where $d \Phi$ is the function on $X$ such that for $P \in X$, $d \Phi(P) = d_P \Phi$. If $\Psi : Y \to Z$ is another morphism of algebraic varieties and $\eta$ a differential form on $Z$, then the following holds

$$\forall P \in X, \quad (\Phi^* (\Psi^* \eta))(P) = ((\Psi \circ \Phi)^* \eta)(P).$$

Indeed, both sides are equal to

$$\eta((\Psi \circ \Phi)(P)) \circ d_P (\Psi \circ \Phi).$$

Suppose now that $\omega$ is a regular differential form on $Y$. An interesting question to ask is whether the pull-back of $\omega$ under a morphism $\Phi : X \to Y$ is again regular. To answer this question, we will work locally. Indeed, choose a point $S \in X$ and an open neighbourhood $V$ of $\Phi(S)$ in $Y$. Then there exist $f_1, \ldots, f_m, g_1, \ldots, g_m \in \Theta_Y(V)$ such that

$$\forall Q \in V, \quad \omega(Q) = \sum_{i=1}^m f_i(Q) \cdot d_Q g_i,$$

so that, $\forall P \in \Phi^{-1}(V)$,

$$(\Phi^* \omega)(P) = \omega(\Phi(P)) \circ d_P \Phi$$

$$= \sum_{i=1}^m f_i(\Phi(P)) \cdot d_{\Phi(P)} g_i \circ d_P \Phi$$

$$= \sum_{i=1}^m (f_i \circ \Phi)(P) \cdot d_P (g_i \circ \Phi).$$

Now, because $\Phi$ is a morphism and of the regularity on $V$ of the $f_i$'s and $g_i$'s, the $f_i \circ \Phi$'s and $g_i \circ \Phi$'s are regular on $\Phi^{-1}(V)$ and $\Phi^* \omega$ is a regular differential form on $\Phi^{-1}(V) \ni S$.

In this way, we have shown that the pull-back of a regular differential form under a morphism is again a regular differential form.
1.3.12 Differential Forms on Riemann Surfaces

As in the case of algebraic varieties, we first have to define the tangent space to a Riemann surface.

**Definition 1.46.** ([Sp] 5-4) Let $M$ be a Riemann surface. A (differentiable) path on $M$ is a pair $(c, I_c)$, where $I_c$ is an open interval in $\mathbb{R}$ and $c$ a continuous map from $I_c$ to $M$ which satisfies

$$\forall P \in M \text{ such that } \exists t_0 \in I_c \text{ with } c(t_0) = P \text{ and for each chart } (U, \varphi) \text{ of } M \text{ with } P \in U, \text{ the composition } \varphi \circ c \text{ is a differentiable function on } \{t \in I_c; \ c(t) \in U\} \text{ and } (\varphi \circ c)'(t_0) \neq 0.$$ 

A pair $(c, I_c)$, where $I_c = (a, b)$ is an open interval on $M$ and $c$ a continuous map from $I_c$ to $M$, is called a piecewise differentiable path on $M$ if there exists a partition $a = t_0 < t_1 < ... < t_r = b$ of $I_c$ such that, for each $i$ in $\{1, ..., r\}$, the restriction of $c$ to the open subinterval $I_{c,i} := (t_{i-1}, t_i)$ furnishes a differentiable path $(c|_{I_{c,i}}, I_{c,i})$ on $M$.

Fix a point $P$ on $M$. On the set of paths on $M$ going through $P$ (it can be supposed that $0 \in I_c$ and $c(0) = P$), we would like to identify the differentiable paths having the same tangent direction at $P$. Hence, we set

$$(c_1, I_{c_1}) \sim (c_2, I_{c_2}) \iff \exists \text{ a chart } (U, \varphi) \text{ of } M \text{ such that } P \in U \text{ and } \frac{d}{dt}(\varphi \circ c_1)|_{t=0} = \frac{d}{dt}(\varphi \circ c_2)|_{t=0}.$$ 

In view of this equivalence relation, we define

**Definition 1.47.** Let $M$ be a Riemann surface and $P \in M$. The tangent space $T_PM$ to $M$ at $P$ will be the set of equivalence classes of differentiable paths on $M$ going through $P$ under the above equivalence relation.

**Remark 13.** If $V$ is the topological space $\mathbb{C}^n$, then $\forall P \in V$, $T_P V$ is in bijection with $\mathbb{C}^n$ via the well-defined map $\alpha_P : T_P V \to \mathbb{C}^n$ given by $\langle (c, I_c) \rangle \mapsto \lim_{t \to 0} \frac{c(t)-c(0)}{t}$.

**Definition 1.48.** Let $M, N$ be two Riemann surfaces, $P \in M$ and $\Phi : M \to N$ a holomorphic map. Then we define the tangent map of $\Phi$ at $P$ to be the map $T_P \Phi : T_PM \to T_{\Phi(P)}N$

$$(\langle (c, I_c) \rangle) \mapsto (\Phi \circ c, I_c).$$
Remark 14. If $\Phi : M \rightarrow N$ and $\Psi : N \rightarrow L$ are holomorphic maps between Riemann surfaces, then for $P \in M$, $T_P (\Psi \circ \Phi) = T_{\Phi(P)} \Psi \circ T_P \Phi$ and the tangential map at $P \in M$ of the identity map on $M$ is the identity map on $T_PM$. Hence, if $\Phi : M \rightarrow N$ is a biholomorphic map, then $\forall P \in M$, $T_P \Phi : T_PM \rightarrow T_{\Phi(P)} N$ is a bijection.

Using the above definition, we define the differential of a holomorphic function

**Definition 1.49.** Let $f : M \rightarrow \mathbb{C}$ be a holomorphic function on a Riemann surface $M$ and $\forall Q \in \mathbb{C}$, $\alpha_Q$ the bijection $T_Q \mathbb{C} \rightarrow \mathbb{C}$, then we define the differential $d_pf$ of $f$ at $P \in M$ to be the composition

$$\alpha_{f(P)} \circ T_pf : T_PM \rightarrow \mathbb{C}.$$  

Remark 15. Let $M$ be a Riemann surface, $P \in M$ and $(U, \varphi)$ a chart on $M$ such that $P \in U$. Then $\varphi : U \rightarrow \varphi(U)$ is biholomorphic and

$$d_p\varphi = \alpha_{\varphi(P)} \circ T_p\varphi : T_PM \rightarrow T_{\varphi(P)}\mathbb{C} \rightarrow \mathbb{C}$$

is a composition of bijections. Hence a $\mathbb{C}$-vector space structure can be defined on $T_PM$ by setting the bijection $d_p\varphi$ to be linear. It can be shown that this structure is independent of the choice of the chart.

The special case $\varphi = \text{id}$ implies that, for each $P \in \mathbb{C}$, $\alpha_P$ is itself linear, because in this case we have $d_{\varphi(P)}\varphi = \alpha_P \circ \text{id}_{T_{\varphi^{-1}(P)}M}$.

Remark 16. Let $\Phi : M \rightarrow N$ be a holomorphic map between Riemann surfaces and $P \in M$. Let further $(U, \varphi)$ be a chart of $M$ at $P$ and $(V, \psi)$ a chart of $N$ at $\Phi(P)$. Then the composition

$$d_{\Phi(P)}\psi \circ T_P\Phi \circ (d_p\varphi)^{-1} : \mathbb{C} \rightarrow \mathbb{C}$$

equals $\alpha_{(\psi \circ \Phi)(P)} \circ T_{\psi(P)}(\psi \circ \Phi \circ (\varphi^{-1})) \circ (\alpha_{\varphi(P)})^{-1}$ and is given by

$$z \mapsto \frac{\partial}{\partial z} (\psi \circ \Phi \circ (\varphi^{-1}))_{\varphi(P)} \cdot (z - \varphi(P)).$$

This being a linear transformation of $z$, $T_P\Phi$ will be linear for the vector space structures on $T_PM$ resp. $T_{\Phi(P)}N$ for which $d_p\varphi$ resp. $d_{\Phi(P)}\psi$ is linear.
The differential $dpf$ at $P$ of a holomorphic function $f$ on a Riemann surface $M \ni P$ being the composition of two linear functions it is linear and defines an element of $(TpM)^*$. Therefore, as for algebraic varieties, we have here associated to a "good" function and to a point on the variety a linear form on the tangent space to the variety at this point. Going further with this analogy, we define

**Definition 1.50.** Let $M$ be a Riemann surface and $V$ an open subset of $M$. Then a function $\omega$ sending each point $P$ of $V$ to an element of $(TpM)^*$ is called a holomorphic differential form (or holomorphic differential 1-form) on $V$ if, for each chart $(U, \varphi)$ of $M$ with $V \cap U \neq \emptyset$, there exists a holomorphic function $f$ on $V \cap U$ such that

$$\omega|_{V \cap U} = f \cdot d\varphi.$$ 

Then, for each $P \in V \cap U$, we have

$$\omega(P) = f(P)dp\varphi.$$ 

**Remark 17.** The set of holomorphic differential forms on an open subset $V$ of a Riemann surface $M$ is a $\mathbb{C}$-vector space, which will be denoted by $\Omega^1_{an}[V]$. It can be shown ([AS] 24A) that, if $M$ is compact, then $\Omega^1_{an}[M]$ has finite dimension.

**Definition 1.51.** Let $M$ be a compact Riemann surface, then the nonnegative integer $\dim \Omega^1_{an}[M]$ is called the genus of $M$ and denoted by $g(M)$.

One can shows that the first Betti number of a compact Riemann surface $M$ actually equals $2g(M)$ ([Sp] 5-9) and according to formula (1.3) in §1.3.9, we have

$$\chi(M) = 2 - 2g(M). \quad (1.9)$$

Comparing with (1.8) in §1.3.11, we deduce that the genus of a nonsingular irreducible projective algebraic curve defined over $\mathbb{C}$ equals the genus of the associated compact Riemann surface $X(\mathbb{C})$, i.e.

$$g[X] = g(X(\mathbb{C})). \quad (1.10)$$

This shows that the vector space of regular differential forms on $X$ and the vector space of holomorphic differential forms on $X(\mathbb{C})$ have the same dimension. The following remark makes the comparision even stronger.
Remark 18. Remember Remark 6 at the end of §1.3.8 which asserts that the regular functions on a nonsingular projective algebraic curve \( X \) over \( \mathbb{C} \) are holomorphic functions on \( X(\mathbb{C}) \) and vice versa. This can actually be transposed to differential forms. Indeed, having regard to Remark 12, §1.3.11, a regular differential form \( \omega \) on \( X \) is locally given on an open subset \( U \) of \( X \) by \( f \, dt \), where \( f \in \mathcal{O}(U) \) and \( t \) a local parameter regular on \( U \). Thus \( f \) and \( t \) are holomorphic on \( U \) equipped with the complex structure induced from that of \( X(\mathbb{C}) \) and so are their restrictions to the intersections with the domains of the charts. Hence \( \omega \) is holomorphic on \( X(\mathbb{C}) \). The converse also holds (by the same arguments and Remark 6) and we have

\[
\Omega^1[X] = \Omega^1_{\text{reg}}[X(\mathbb{C})].
\]

1.3.13 Integration of Holomorphic Differential 1-Forms on Riemann Surfaces

In this section, we would like to define the integral of a holomorphic 1-form on a Riemann surface \( M \) along an element of \( H_1(M, \mathbb{Z}) \). To this aim, there will first be associated a piecewise differentiable path to a 1-cycle on \( M \).

According to [Sp] 9-3, each Riemann surface can be triangulated in such a way that the geometric realization of each 1-simplex furnishes, via the homeomorphism to \( M \), a differentiable path on \( M \). From now on, we will suppose the triangulation to satisfy this. If the 1-simplex is oriented, the paramatrization of the path has to agree with the orientation. To the sum of two oriented 1-simplices, the endpoint of the former being the initial point of the latter, is associated the concatenation of the corresponding pathes ([Sp] 5-8). Hence to each 1-cycle is associated a closed piecewise differentiable path on \( M \). (In view of this, the 1-cycle and the associated path will sometimes be denoted by the same symbol.)

Let \((U, \varphi)\) be a chart on the Riemann surface \( M \) and \( \omega \) a holomorphic differential 1-form on \( M \). Then there exists a holomorphic function \( f \) on \( U \) such that

\[
\omega|_U = f \, d\varphi.
\]

Let further \( \gamma \) be a 1-cycle on \( M \) contained in \( U \) and suppose that the associated closed piecewise differentiable path \( c : I \to U \) is differentiable. By the way, \( I \) can be supposed to be \((0, 1)\) if only to be reparametrized. In this situation, we define
the integral of $\omega$ along $\gamma$ to be the complex number

$$\int_\gamma \omega := \int_0^1 f(c(t)) \, d_\gamma (\varphi \circ c).$$

This generalizes to the case where $c$ is piecewise differentiable and $c(I)$ not contained in the domain of a unique chart. Indeed, there exists a partition $0 = t_0 < t_1 < \ldots < t_r = 1$ of $(0, 1)$ such that the restriction $c_i$ of $c$ to the subinterval $I_i := (t_{i-1}, t_i)$ is a differentiable path whose image is contained in the domain $U_i$ of a unique chart $(U_i, \varphi_i)$ (cf [Mir] IV 3, Lemma 3.7). For $i = 1, \ldots, r$, write $f_i$ for the holomorphic function on $U_i$ satisfying

$$\omega|_{U_i} = f_i d\varphi_i.$$

Then we set

$$\int_\gamma \omega := \sum_{i=1}^r \int_{t_{i-1}}^{t_i} f_i(c_i(t)) \, d_\gamma (\varphi_i \circ c_i).$$

Consider now a class $\mathcal{C}$ in $H_1(M, \mathbb{Z})$, then two représentants $\gamma_1, \gamma_2 \in Ker(\partial_1)$ of $\mathcal{C}$ differ from a boundary, i.e. $\exists \delta \in C_2(M, \mathbb{Z})$ s.t. $\gamma_1 = \gamma_2 + \partial_2 \delta$. Because $\omega$ is holomorphic, we have $\int_{\partial_2 \delta} \omega = 0$ ([Sp] 6-5, Thm 6-9) and

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega + \int_{\partial_2 \delta} \omega = \int_{\gamma_2} \omega.$$

This shows that the integrals of a holomorphic 1-form along two homologous 1-cycles are equal ([Sp] 6-3, Thm 6-2).

**Definition 1.52.** Let $M$ be a Riemann surface, $\omega$ a holomorphic differential 1-form on $M$ and $\mathcal{C} \in H_1(M, \mathbb{Z})$. For any 1-cycle $\gamma$ representing the class $\mathcal{C}$, the above allows us to set

$$\int_{\mathcal{C}} \omega := \int_{\gamma} \omega.$$

Note that integration is $\mathcal{C}$-linear in the 1-forms and $\mathbb{Z}$-linear in the 1-cycles (cf [Mir] IV 3).

Taking the homology of a simplicial complex defines a covariant fonctor (between the category of simplicial complices with so-called simpicial applications and the
category of $R$-modules with $R$-homomorphisms). Since we are only interested in the first homology $R$-module of Riemann surfaces, we will use the association we have made above of a piecewise differentiable path to a 1-cycle. Namely, consider two Riemann surfaces $M$ and $N$ and a holomorphic map $\Phi : M \to N$. Take then two homologous 1-cycles $\gamma_i$, $i = 1, 2$, on $M$ and their associated piecewise differentiable paths $(c_i, I)_1, i = 1, 2$. Then the compositions $\Phi \circ c_i$ for $i = 1, 2$ are piecewise differentiable paths on $N$ which correspond to homologous 1-cycles on $N$. More precisely, we define

**Definition 1.53.** Let $M, N$ be two Riemann surfaces and $\Phi : M \to N$ a holomorphic map. Let $\mathcal{C} = [\gamma] \in H_1(M, \mathbb{Z})$ and $(c, I)$ the piecewise differentiable path associated to $\gamma$. Then $\Phi \circ c$ defines a class in $H_1(N, \mathbb{Z})$ which depends only on $\mathcal{C}$ and is called the **push-forward** of $\mathcal{C}$. It is denoted by $\Phi_* \mathcal{C}$ (or sometimes by $\Phi_* \gamma$ by abuse of notation).

**Remark 19.** The push-forward of 1-cycles is adjoint to the pull-back of differential 1-forms in the sense that

$$\int_{\Phi_* \gamma} \omega = \int_{\gamma} \Phi^* \omega.$$ 

This can be verified by an easy calculation.

**Remark 20.** If $\Phi : M \to N$ is an analytic isomorphism, we can define the **pull-back** of $\mathcal{D} \in H_1(N, \mathbb{Z})$ under $\Phi$ to be the push-forward of $\mathcal{D}$ under $\Phi^{-1}$. Indeed, let's set

$$\Phi^* \mathcal{D} := (\Phi^{-1})_* \mathcal{D}.$$ 

One observes that first pushing forward and then pulling back (or the converse) is the identity.

We end this section by introducing an often used terminology.

**Definition 1.54.** ([Mir] VIII 1) Let $M$ be a Riemann surface and $\Omega^1_{an}[M]$ the $\mathbb{C}$-vector space of holomorphic differential 1-forms on $M$. Then a linear form $\lambda : \Omega^1_{an}[M] \to \mathbb{C}$ is called a **period** on $M$ if there exists $\mathcal{C} \in H_1(M, \mathbb{Z})$ such that,

$$\forall \omega \in \Omega^1_{an}[M], \quad \lambda(\omega) = \int_{\mathcal{C}} \omega.$$
1. Abelian Varieties Associated to Hypergeometric Series

1.3.14 Sheaves

The notions and notations introduced in this paragraph will hardly be used as they are here presented. It is however useful to introduce them briefly, in order to be able to recognize them and translate them into more friendly objects.

**Definition 1.55.** ([H] III) Let's give us a topological space $X$ and for each open subset $U$ of $X$ an abelian group $\mathcal{F}(U)$. If an open subset $V$ is included in $U$, let's give us a morphism of abelian groups $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ ($\rho_{UV}$ can be viewed as a restriction and $\rho_{UV}(s)$ for $s \in \mathcal{F}(U)$ is therefore sometimes denoted by $s|_V$). Suppose that

- $\mathcal{F}(\emptyset) = \{0\}$,
- for every open subset $U \subseteq X$, $\rho_{UU} = id_{\mathcal{F}(U)}$,
- if $W \subseteq V \subseteq U$ are open subsets of $X$, then $\rho_{WV} = \rho_{VV} \circ \rho_{UV}$

and for every open subset $U \subseteq X$ and every open cover $\{V_i\}_{i \in I}$ of $U$, we have

- if $s \in \mathcal{F}(U)$ satisfies $s|_{V_i} = 0$, $\forall i \in I$, then $s = 0$ and
- for each family $\{s_i; s_i \in \mathcal{F}(V_i)\}_{i \in I}$ satisfying $\forall i, j \in I, s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, there exists an element $s \in \mathcal{F}(U)$ such that $\forall i \in I, s|_{V_i} = s_i$. (Note that the previous condition implies the unicity of $s$.)

Such a data is called a **sheaf of abelian groups** on $X$ and denoted by $\mathcal{F}$.

**Terminology 2.** If $\mathcal{F}$ is a sheaf of abelian groups on a topological space $X$ and $U$ an open subset of $X$, then the elements of $\mathcal{F}(U)$ are called **sections** of $\mathcal{F}$ over $U$ and $\mathcal{F}(U)$ is sometimes also denoted by $\Gamma(U, \mathcal{F})$. The elements of $\mathcal{F}(X)$ are usually called **global sections** of $\mathcal{F}$.

**Example 2.** Let $X$ be an algebraic variety over $k$. On the underlying topological space can be defined a sheaf (of rings) by associating to each open subset $U$ of $X$ the ring of regular functions $\mathcal{O}(U)$ on $U$ and, for $V \subseteq U$ open, defining $\rho_{UV} : \mathcal{O}(U) \to \mathcal{O}(V)$ by $\rho_{UV}(f) = f|_V$. Indeed, the first three conditions are easily verified, while for the two last ones, note that a function which is defined on a union of open sets and regular (resp. zero) on each of them is actually regular (resp. zero trivially) on this union, because regularity is a local property. This sheaf $\mathcal{O}$ is called the **sheaf of regular functions** on $X$. 

Example 3. Let $X$ be an algebraic variety over $k$. To each open subset $U$ of $X$, associate the $k$-vector space $\Omega^1[U]$ of regular differential 1-forms on $U$ and, if $V \subseteq V$ is open, define $\rho_{UV} : \Omega^1[U] \to \Omega^1[V]$ by $\rho_{UV}(\omega) = \omega|_V$ which is a regular differential 1-form on $V$. The first four properties are easily verified. For the last one, define $s$ by $s(P) := s_1(P)$ if $P \in V_i$. This is well-defined, because any two elements of the family agree on the intersection of their domains of definition. This makes $s$ to a regular differential 1-form on the whole of $U$, because regularity of a differential form is a local property. Hence, we get a sheaf on $X$, which will be called the sheaf of regular differential forms (or 1-forms) on $X$ and denoted by $\Omega$ (or $\Omega^1_X$).

The key in the two above examples is that regularity (of functions or differential forms) on an algebraic variety is a local property. Since the holomorphy of functions and differential forms on a Riemann surface $M$ is also a local property, one can define with them sheaves on $M$ in a similar way as what we have done here. The sheaf $\Omega^1_{\text{an}}$ (or $\Omega^1_{M,\text{an}}$) defined by associating to an open subset $U$ of $M$ the $\mathbb{C}$-vector space $\Omega^1_{\text{an}}[U]$ will be called the sheaf of holomorphic differential forms (or 1-forms) on $M$.

1.3.15 Complex Tori and Abelian Varieties over $\mathbb{C}$

Definition 1.56. ([Mu] 12.1) Let $k$ be a field (not necessarily algebraically closed). An abelian variety $A$ over $k$ is a projective algebraic variety over $k$, with a group law $\mu : A \times A \to A$ such that $\mu$ and the map $\nu : A \to A$ given $x \mapsto x^{-1}$ are morphisms (of algebraic varieties).

An abelian variety is then a group object in the category of projective algebraic varieties. The group law actually forces $A$ to be nonsingular.

Definition 1.57. ([BL] Ch1) A lattice $\Lambda$ in a complex vector space $V$ of dimension $d$ is a discrete free group of maximal rank.

Equivalently, $\Lambda$ is a free abelian group of rank $2d$. As such, its generators generate $V$ over $\mathbb{R}$. Moreover, $\Lambda$ acts on $V$ by addition, hence the following definition makes sense.
Definition 1.58. A quotient $V/\Lambda$, where $\Lambda$ is a lattice in a $\mathbb{C}$-vector space $V$, is called a complex torus.

A complex torus $T := V/\Lambda$ can be given the structure of a complex manifold of dimension $\dim(V)$. The complex topology on $V/\Lambda$ is obtained by requiring the quotient map $V \to V/\Lambda$ to be continuous. $T$ also inherits a group structure given by a holomorphic map $T \times T \to T$. The map $T \to T$, $x \mapsto x^{-1}$ is also holomorphic. Hence $T$ is a group variety in the category of complex manifolds (such a variety is called a Lie-group). It can be proved that $T$ is moreover connected, compact and abelian.

Let $A$ be an abelian variety over $\mathbb{C}$. Then the complex manifold $A(\mathbb{C})$ (see Definition §1.3.8) is analytically isomorphic to a complex torus $V/\Lambda$, where $\dim(V)$ is the dimension of $A$ ([Mu] 12.5). The elements of $\Lambda$ are called periods on $A$. Conversely, it can be asked whether any complex torus can be realized as an algebraic projective variety. The answer to this question is in general negative. In order to give the necessary and sufficient condition on $V/\Lambda$ for it to be embedded in a projective space, we first recall some notions of linear algebra.

Definition 1.59. ([Ke] 1.2) A Hermitian form $H$ on a complex vector space $V$ is a pairing $V \times V \to \mathbb{C}$ such that

1. $H$ is $\mathbb{C}$-linear in the first variable and
2. $\forall v, w \in V, \, H(v, w) = H(w, v)$.

This implies

1. $\forall \lambda \in \mathbb{C}, \forall v, w \in V, \, H(v, \lambda w) = \lambda H(v, w)$ and
2. $\forall v \in V, \, H(v, v) \in \mathbb{R}$.

Write $E$ for the imaginary part of $H$ (that is, $\forall v, w \in V, \, E(v, w) = \text{Im}(H(v, w))$), then $E : V \times V \to \mathbb{R}$ is a real bilinear form on $V$ with the properties

1. $\forall v \in V, \, E(v, v) = 0$ and
2. $\forall v, w \in V, \, E(iv, iw) = E(v, w)$. 


1.3. **Preparatory Material**

**Remark 21.** ([Ke] §1.3) $H$ can be recovered from $E$. More precisely, given a \( \mathbb{R} \)-bilinear form \( E : V \times V \rightarrow \mathbb{R} \) satisfying 1. and 2., there exists a unique Hermitian form \( H : V \times V \rightarrow \mathbb{C} \) whose imaginary part is $E$. (Define $H$ by $H(v, w) := E(i v, w) + iE(v, w)$.)

**Remark 22.** Let $H$ and $E$ be as above and set $\text{Ker} H = \{ v \in V ; \forall w \in V, H(v, w) = 0 \}$ and $\text{Ker} E = \{ v \in V ; E(v, w) = 0, \forall w \in V \}$. Then, we have

\[
\text{Ker} H = \text{Ker} E.
\]

Clearly, $\text{Ker} E \supset \text{Ker} H$. For the converse, take $z \in V$ such that, for each $w \in V$, $E(z, w) = 0$ and write $H(z, w) = E(i z, w) + iE(z, w)$. By 2. and the assumption, $H(z, w) = E(-z, iw) = -E(z, iw) = 0$. Hence, $\text{Ker} E \subset \text{Ker} H$.

Consequently, $E$ is nondegenerate if and only if $H$ is nondegenerate.

**Theorem 1.4.** ([Mu] 4.1) A complex torus $T = V/\Lambda$ can be given the structure of an algebraic variety if and only if there exists a positive definite Hermitian form $H$ on $V$ such that its imaginary part $\text{Im}(H)$ takes integral values on $\Lambda \times \Delta$. In this case, $T$ can be realized as a projective algebraic variety.

Note that for a complex torus, being a complex projective algebraic variety is equivalent to being an abelian variety (over $\mathbb{C}$) ([Ke] §2.4). Hence, this theorem answers our question. Given $H$ as in the theorem, the construction of the embedding of $T$ into a projective space is due to Lefschetz.

**Remark 23.** By the above two remarks, the data of a positive definite Hermitian form $H$ on $V \times V$ with imaginary part taking integral values on $\Lambda \times \Lambda$ is equivalent to the data of a positive definite alternating $\mathbb{R}$-bilinear form $E : V \times V \rightarrow \mathbb{R}$ such that $E(\Lambda \times \Lambda) \subset \mathbb{Z}$ and $E(i v, i w) = iE(v, w)$, $\forall v, w \in V$.

**Definition 1.60.** ([Mu] 5.9) Such a form $E$ on $T := V/\Lambda$ as in the above remark is called a **Riemann form** on $T$ or a **polarisation** of $T$.

Let's restate the theorem in using this definition:

A complex torus is an abelian variety if and only if it admits a polarisation.

We intend now to define an equivalence relation on complex tori ([BL] Ch1§2).
1. Abelian Varieties Associated to Hypergeometric Series

**Definition 1.61.** Let $T$, $T'$ be two complex tori. A **homomorphism** from $T$ to $T'$ is a holomorphic map $T \to T'$ which is compatible with the group structures.

**Definition 1.62.** An **isogeny** from a complex tori $T$ to a complex tori $T'$ is a surjective homomorphism $T \to T'$ with finite kernel.

**Definition 1.63.** Let $f : T \to T'$ be an isogeny between complex tori. We define the **exponent** of $f$ to be the smallest integer $n$ satisfying $nx = 0, \forall x \in Ker f$. (Such a $n$ exists and is positive, because $Ker f$ is finite.)

**Proposition 1.5.** Let $f : T \to T'$ be an isogeny of exponent $e$ between complex tori. Then there exists an isogeny $g : T' \to T$, unique up to isomorphism, such that $g \circ f$ is the multiplication by $e$ on $T$ and $f \circ g$ the multiplication by $e$ on $T'$.

Since, moreover, the composition of two isogenies is again an isogeny, saying that the complex tori $T$ is equivalent to the complex tori $T'$ if there exists an isogeny from $T$ to $T'$ defines an equivalence relation on the set of complex tori. Thus, the following definition makes sense.

**Definition 1.64.** Two complex tori $T$, $T'$ are called **isogenous**, if there exists an isogeny from $T$ to $T'$. We will say that two abelian varieties over $\mathbb{C}$ are **isogenous**, if their corresponding complex tori are so.

**Remark 24.** Let $A$ be an abelian variety over $k$, then any abelian subvariety $B$ of $A$ has a complement $C$, i.e. there exists an abelian subvariety $C$ of $A$ such that $A = B + C$ and $B \cap C$ is finite. Hence, $A$ is isogenous to $B \oplus C$ ([Mu] 12.2). In the case where $A$ is defined over $\mathbb{C}$, we can view this in terms of complex tori ([Ke] §9.2). For example, writing $X = V/L$ for the complex torus isomorphic to $A(\mathbb{C})$, then any subspace $W$ of $V$ defines a subtorus $Y = W/(W \cap L)$ of $X$. Applying the above to $Y$, there exists a subtorus $Z$ of $X$ such that $X = Y + Z$ and the intersection $Y \cap Z$ is finite. Hence, $X$ is isogenous to $Y \oplus Z$ and $\dim X = \dim Y + \dim Z$.

The next paragraph provides an example of abelian variety.

### 1.3.16 Jacobian Varieties over $\mathbb{C}$

We first give the definition as a complex torus of the Jacobian variety of a compact Riemann surface, explain how it can be given the structure of an abelian variety and
finally sketch the relation with the definition of the Jacobian variety of a nonsingular algebraic projective curve defined over \( \mathbb{C} \).

Consider a compact Riemann surface \( M \) of genus \( g := g(M) \). Then from Definition 1.51, §1.3.12, we have \( \dim_{\mathbb{C}}(\Omega^1_{an}[M]) = g \) and from the comparison of (1.3), §1.3.9, with (1.9), §1.3.12, we have \( \text{rank}_{\mathbb{Z}} H_1(M, \mathbb{Z}) = 2g \).

Suppose that \( g(M) > 0 \), write \( a_1, \ldots, a_{2g} \) for a system of représentants of \( \mathbb{Z} \)-generators of \( H_1(M, \mathbb{Z}) \) and \( \omega_1, \ldots, \omega_g \) for a basis of \( \Omega^1_{an}[M] \). Recall (cf §1.3.13) that the integral \( \int_{a_i} \omega_j \) does not depend on the choice of the représentant \( a_i \) of the class \([a_i]\). For \( i = 1, \ldots, 2g \), write

\[
A_i := \left( \int_{a_1} \omega_1, \ldots, \int_{a_i} \omega_g \right) \in \mathbb{C}^g
\]

and \( \Lambda := \mathbb{Z}A_1 \oplus \cdots \oplus \mathbb{Z}A_{2g} \).

It can be shown (cf [Mu] 3.8) that \( \Lambda \) is a lattice in \( V := \mathbb{C}^g \). Hence, the quotient \( V/\Lambda \) is a complex torus. The bases can actually be chosen (cf [Fu2] 20d) in order that, for \( i = 1, \ldots, g \) and \( j = 1, \ldots, g \), the following hold

\[
\int_{a_i} \omega_j = \delta_{ij}, \quad (1.11)
\]

where \( \delta_{ij} \) is the Kronecker's symbol. For the rest of the story, we will suppose that our systems of generators satisfy (1.11). Then, for \( i = 1, \ldots, g \), \( A_i \) is the \( i \)-th vector \( e_i \) of the standard basis of \( \mathbb{C}^g \) and \( A_{g+1}, \ldots, A_{2g} \) determine \( \Lambda \). As \( \Lambda \) is a lattice in \( V \), its generators over \( \mathbb{Z} \) generate \( V \) over \( \mathbb{R} \). Write \( A_1^*, \ldots, A_{2g}^* \) for the dual basis of \( V^* \) viewed as \( \mathbb{R} \)-vector space and, for \( u, v \in V \), set

\[
E(u, v) := \sum_{i=1}^{g} (A_{i+g}^*(u)A_i^*(v) - A_i^*(u)A_{i+g}^*(v)).
\]

One can show (cf [Mu] 6.10), that this \( \mathbb{R} \)-bilinear form on \( V \times V \) satisfies the conditions for being a Riemann form (see definition in §1.3.15) on the quotient \( V/\Lambda \). Consequently, this quotient can be realized as a projective variety and it is isomorphic to an abelian variety defined over \( \mathbb{C} \).

**Definition 1.65.** Let \( M \) be a compact Riemann surface of genus \( g \) and \( \Lambda = \mathbb{Z}A_1 \oplus \cdots \oplus \mathbb{Z}A_{2g} \subset \mathbb{C}^g \) be the lattice defined above. Then the abelian variety \( \mathbb{C}^g/\Lambda \) is called the **Jacobian variety** of \( M \) and is usually denoted by \( \text{Jac}(M) \) (or \( J(M) \)).
Now, we would like to give another interpretation of the Jacobian variety of a compact Riemann surface $M$ in terms of the holomorphic 1-forms on $M$ ([Mu] 6.12). For this, first remark that, if $\gamma_1, \ldots, \gamma_{2g}$ form a system of $\mathbb{Z}$-generators of $H_1(M, \mathbb{Z})$, then the map

$$H_1(M, \mathbb{Z}) \rightarrow \Lambda$$

$$\gamma_i = [a_i] \mapsto \lambda_i = (\int_{a_i} \omega_1, \ldots, \int_{a_i} \omega_g)$$

is clearly surjective and preserves the $\mathbb{Z}$-module structures. Since both objects are free $\mathbb{Z}$-modules of rank $2g$, it is actually an isomorphism.

On the other hand, $\Omega^1_{an}[M]$ is a complex vector space of dimension $g(M)$ and so is its dual $(\Omega^1_{an}[M])^*$. It remains to show how $H_1(M, \mathbb{Z})$ embeds in $(\Omega^1_{an}[M])^*$ as a lattice. The map $\gamma : H_1(M, \mathbb{Z}) \rightarrow (\Omega^1_{an}[M])^*$ is given by

$$\gamma_i = [a_i] \mapsto \langle \gamma_i, \cdot \rangle : \omega \mapsto \int_{a_i} \omega.$$ 

In particular, for $i = 1, \ldots, g$, we have $\gamma_i = ^*\omega_i$, because our systems of generators satisfy (1.11), and we can apply the so-called Riemann period relations to show that

$$\exists \lambda_1, \ldots, \lambda_{2g} \in \mathbb{R} \text{ s.t. } \sum_{i=1}^{2g} \lambda_i \gamma_i = 0 \implies \lambda_1 = \ldots = \lambda_{2g} = 0.$$ 

Hence, the vectors $\langle \gamma_i, \cdot \rangle$, $i = 1, \ldots, 2g$, are $\mathbb{R}$-linearly independant and generate $(\Omega^1_{an}[M])^*$ over $\mathbb{R}$. This shows that $\gamma(H_1(M, \mathbb{Z}))$ is a lattice in $(\Omega^1_{an}[M])^*$.

This implies

$$\text{Jac}(M) \simeq (\Omega^1_{an}[M])^* / \gamma(H_1(M, \mathbb{Z}))$$

and, because $\Omega^1_{an}[M] = \Gamma(M, \Omega^1_{M, an})$ (cf Terminology 2, §1.3.14), it justifies the following definition.

**Definition 1.66.** ([CS] VII by J.-S. Milne §2) Let $X$ be a nonsingular algebraic projective curve over $\mathbb{C}$. Let $X(\mathbb{C})$ be the associated one-dimensional complex analytic manifold (which is compact, cf §1.3.8), $\Omega^1_{X(\mathbb{C}), an}$ be the sheaf of holomorphic differential 1-forms on $X(\mathbb{C})$ and $\Gamma(X(\mathbb{C}), \Omega^1_{X(\mathbb{C}), an})$ the $g(X(\mathbb{C}))$-dimensional $\mathbb{C}$-vector space of global sections of $\Omega^1_{X(\mathbb{C}), an}$. Then the **Jacobian variety** of $X$ is defined to be the quotient

$$\Gamma(X(\mathbb{C}), \Omega^1_{X(\mathbb{C}), an})^* / \gamma(H_1(X(\mathbb{C}), \mathbb{Z})).$$
1.3. Preparatory Material

It is denoted by $Jac(X)$ (or $J(X)$) and is an abelian variety.

**Remark 25.** Remembering that $\Omega^1_{X(C),an}[X(C)] = \Omega^1[X]$ (cf Remark 18, §1.3.12), we can also write the Jacobian variety of $X$ as

$$\Omega^1[X]^*/i(H_1(X(C),\mathbb{Z})).$$

**Remark 26.** According to Definition 1.54, §1.3.13, the periods on a Riemann surface $M$ are the linear forms in $\Omega^1_{an}[M]^*$ of the form $\langle \gamma, \cdot \rangle$ for one $\gamma \in H_1(M, \mathbb{Z})$. Such elements are also the periods on the abelian variety $Jac(M)$ (cf §1.3.15).

**Remark 27.** Let $M$ be a compact Riemann surface of genus $g > 0$ and $P_0$ a fixed point on $M$. Let further $\omega$ be a holomorphic differential 1-form on $M$ and $P$ any point on $M$, then the integral $\int_{P_0}^P \omega$ depends on the choice of the path on $M$ going from $P_0$ to $P$. But since the difference of the integrals along two such paths is the integral along an element of $H_1(M, \mathbb{Z})$, we have a well-defined map

$$\varphi : M \to \mathbb{C}^g/\Lambda = Jac(M)$$

$$P \mapsto (\int_{P_0}^P \omega_1, \ldots, \int_{P_0}^P \omega_g),$$

where $\Lambda$ is the periods lattice and $\omega_1, \ldots, \omega_g$ form a basis of $\Omega^1_{an}[M]$. One can show that $\varphi$ is an embedding and that the induced map

$$\varphi^* : \Omega^1_{an}[Jac(M)] \to \Omega^1_{an}[M]$$

is an isomorphism.

In the case where $M$ has genus 1, it can be shown that $\varphi$ itself is an isomorphism. It follows that each compact Riemann surface of genus 1 is isomorphic to a complex torus $\mathbb{C}/\Lambda$ and thus to an elliptic curve over $\mathbb{C}$ (compare with §2.2.1).

1.3.17 Algebraic Appendix

This paragraph concerns exclusively linear representations of finite groups. We just intend to recall some definitions and properties and give an example that will be used in Section 1.4.
Definition 1.67. ([Se2] 1.1) Let $G$ be a finite group and $k$ a (commutative) field. Then a linear representation of $G$ is a finitely dimensional $k$-vector space $V$ together with a group-homomorphism

$$
\rho : G \longrightarrow GL(V)
$$

$$
\rho_s.
$$

This is actually equivalent to the data of a linear action

$$
G \times V \longrightarrow V
$$

$$(s,v) \longmapsto \rho_s(v).$$

In particular, $V$ is a $G$-module. The dimension of $V$ over $k$ is also called the degree (or dimension) of the representation.

Definition 1.68. ([Se2] 1.3) A linear subrepresentation $U$ of $V$ is a linear subspace of $V$, which is invariant under the action of $G$. Equivalently, $U$ is a sub-$kG$-module of $V$.

Definition 1.69. A linear representation $V$ of $G$ is said to be irreducible, if $V \neq \{0\}$ and the sole submodule of $V$ are $V$ and $\{0\}$.

From now on, we will suppose $k = \mathbb{C}$, because the following is not true for any field. Namely, $k$ may have to be algebraically closed or of characteristic not dividing the order of $G$.

It can be shown ([Se2] 1.4) that the complement as a linear subspace of any sub-$\mathbb{C}G$-module of $V$ is $G$-invariant. Hence, any linear representation admits a decomposition as a direct sum of irreducible subrepresentations.

Let $\mathbb{C}G$ be the $\mathbb{C}$-vector space on the set $\{e_s\}_{s \in G}$. Then, setting $s \cdot e_t = e_{st}$ for $s, t \in G$ and extending $\mathbb{C}$-linearly to $\mathbb{C}G$, makes $\mathbb{C}G$ into a linear representation of $G$. It is called the (left-) regular representation of $G$.

One can show ([Se2] 2.4) that each irreducible $\mathbb{C}G$-module $U$ is isomorphic to an irreducible factor in the decomposition of $\mathbb{C}G$. More precisely, if $d$ is the degree of $U$, then there are $d^2$ factors isomorphic to $U$. This shows, in particular, that the isomorphism classes of irreducible $\mathbb{C}G$-modules are in finite number and that

$$
|G| = \sum_{\text{Irred } \mathbb{C}G\text{-module}} \dim(U)^2. \tag{1.12}
$$
1.3. Preparatory Material

Now, given any linear representation $V$ of $G$, its decomposition in irreducible subrepresentations is not necessarily unique. But ([Se2] 2.6), if you take one such decomposition, say $V = U_1 \oplus \ldots \oplus U_n$, and write $W_1, \ldots, W_h$ for the irreducible $\mathbb{C}G$-modules and, moreover, for each $i \in \{1, \ldots, h\}$ such that there exists $k \in \{1, \ldots, h\}$ with $U_k \cong W_i$, you write $V_i$ for the direct sum of the $U_k$'s that are isomorphic to $W_i$, then

$$V = \bigoplus_i V_i.$$ 

This decomposition does not depend on the choice of the initial decomposition and is unique.

**Definition 1.70.** ([Se2] 2.6) A linear representation which is the direct sum of pairwise isomorphic subrepresentations (such as $V_i$) is called **isotypical** and the decomposition of $V$ in isotypical components is called **canonical**.

To a linear representation $V$ over $\mathbb{C}$ of a finite group $G$, one can associate a function $G \to \mathbb{C}$ which characterizes the representation.

**Definition 1.71.** ([Se2] 2.1) The **character** $\chi$ of a linear representation $\rho : G \to GL(V), s \mapsto \rho_s$, is the function

$$G \to \mathbb{C}$$

$$s \mapsto Tr(\rho_s).$$

A character of a linear representation of $G$ is also called **character of $G$** and the dimension of $V$ the **degree** of $\chi$.

One can show ([Se2] 2.3) that two linear representations of $G$ are isomorphic if and only if their characters are equal.

**Definition 1.72.** A character of $G$ is said to be **irreducible**, if it is the character of an irreducible representation of $G$.

**Remark 28.** Because the trace is invariant under conjugation, so are the characters. Such a function is called **central**. It can be shown ([Se2] 2.5) that the irreducible characters of $G$ form a basis of the $\mathbb{C}$-vector space of central functions on $G$. This shows that the number of irreducible characters of $G$ equals the number of conjugacy classes in $G$. 
Let $h$ be the number of conjugacy classes in $G$ and $\chi_1, \ldots, \chi_h$ the irreducible characters of $G$. Write $d_i$ for the degree of $\chi_i$. Then (1.12) can be written as

$$|G| = \sum_{i=1}^{h} d_i^2. \quad (1.13)$$

Suppose now that $G$ is abelian. This is equivalent to the fact that each conjugacy class of $G$ contains exactly one element. In this case, their number equals $|G|$. Together with (1.13), we get

**Proposition 1.6.** ([Se2] 3.1)

$G$ is abelian $\iff$ Each irreducible linear representation of $G$ has degree 1.

**Corollary 1.7.** The irreducible characters of an abelian group are homomorphisms.

**Example 4.** Consider the group $\mu_N$ of complex $N$-th roots of unity. Let $\mu_N$ be generated by the primitive $N$-th root $\zeta_N$, $\chi$ be an irreducible character of $\mu_N$ and set $\chi(\zeta_N) = \nu$. Since $\chi$ is a group-homomorphism, $\forall k \in \mathbb{Z}$, $\chi(\zeta_N^k) = \nu^k$ and, in particular, $\nu^N = 1$. Hence, $\nu$ is also a $N$-th root of unity and the $N$ irreducible characters $\chi_0, \ldots, \chi_{N-1}$ of $\mu_N$ are given by

$$\chi_n(\zeta_N^k) = \zeta_N^{kn},$$

where $n, k \in \{0, \ldots, N - 1\}$. 


1.4 Abelian Varieties for Appell-Lauricella Hypergeometric Series

1.4.1 Motivation and Structure of the Section

The Appell-Lauricella hypergeometric series \( F\left(a, b_1, ..., b_r; c; \lambda_2, ..., \lambda_r\right) \) in the \( r - 1 \) variables \( \lambda_2, ..., \lambda_r \) (cf Section 1.1) has the integral representation

\[
\frac{1}{B(c - a, a)} \int_{1}^{\infty} x^{c - a} \prod_{j=2}^{r} (x - \lambda_j)^{-b_j} dx.
\]

The above integrals converge for \( \text{Re}(c) > \text{Re}(a) > 0 \), but it is sufficient that

\[
a, -c + \sum_{j=2}^{r} b_j, c - a, b_2, ..., b_r \notin \mathbb{Z}
\]

in order to replace the above quotient up to an algebraic factor by a quotient of periods on algebraic curves. The algebraic multiple of the integral on the numerator is an integral

\[
\int_{\mathcal{Y}} x^{-\mu_0} (x - 1)^{-\mu_1} \prod_{j=2}^{r} (x - \lambda_j)^{-\mu_j} dx, \tag{1.14}
\]

where \( \mathcal{Y} \) is the lifting on the following algebraic curve of a Pochhammer cycle around 1 and \( \infty \)

\[
y^N = x^{N\mu_0} (x - 1)^{N\mu_1} \prod_{j=2}^{r} (x - \lambda_j)^{N\mu_j}. \tag{1.15}
\]

Here, \( N \) is the smallest common denominator of

\[
\mu_0 = c - \sum_{j=2}^{r} b_j, \\
\mu_1 = 1 + a - c, \\
\mu_j = b_j, \text{ for } j = 2, ..., r.
\]

The integral (1.14) can be written as \( \int_{\mathcal{Y}} \frac{dx}{y} \) and is the integral of a differential 1-form on the projective algebraic curve defined affinely by (1.15). If, for \( k = 0, ..., r \), we write \( A_k := N\mu_k \), then the condition that \( \mu_0, ..., \mu_r \notin \mathbb{Z} \) implies that

\[
N \nmid A_0, ..., A_r.
\]
Moreover, since $\sum_{k=0}^{r} A_k = N(a + 1)$, the condition $a \notin \mathbb{Z}$ implies

$$N \uparrow \sum_{k=0}^{r} A_k.$$ 

In the next paragraph, we first show that we can in fact suppose the exponents $A_0, ..., A_r$ to be positive. Then in §1.4.3 we define the family of curves for which we will construct abelian varieties of dimension depending only on $N$. Since we will start from a singular curve $C_N$, we have to desingularize it. This is done in §1.4.5 after we have determined the possible singular points of $C_N$ (cf §1.4.4).

In §1.4.6, we calculate the genus of the nonsingular curve $X_N$. In order to find a basis of the vector space $\Omega^1[X_N]$ of regular differential 1-forms on $X_N$, we use the fact that the group $\mu_N$ of $N$-th roots of unity acts on $X_N$ and therefore on $\Omega^1[X_N]$. These actions are defined in §1.4.7. In §1.4.8, we calculate a basis for $\Omega^1[X_N]$ and the dimension of the isotypical components $V_n$ for the action of $\mu_N$ on $\Omega^1[X_N]$, first in the general case $n \in \{0, ..., N-1\}$, then in the special case where $(n, N) = 1$. For this last case, we show that the sum $\dim V_n + \dim V_{N-n}$ is constant. In §1.4.9, we explain why we will only be interested in the $V_n$'s with $(n, N) = 1$ and in §1.4.10 define with them an abelian subvariety of the Jacobian variety of $X_N$, which will be called the “New Jacobian” of $X_N$. Its dimension is calculated as a function of $N$ and $r$.

### 1.4.2 Reduction

The curve we will be interested in is the desingularization $X'_N$ of the projective curve $C'_N$ with affine equation

$$y^N = \prod_{i=0}^{r}(x - \lambda_i)^{A_i},$$

where $N \in \mathbb{N}$, $A_0, ..., A_r \in \mathbb{Z}$. In this paragraph, we want to show that, to our aim, it is sufficient to suppose that $A_0, ..., A_r$ are positive. The case $A_i = 0$ is implicitly excluded, because we do not want to talk about a factor that does not appear.

Indeed, for each $i \in \{0, ..., r\}$, write $A_i = k_i N + r_i$, where $0 \leq r_i \leq N - 1$ and
Note that if $A_i < 0$ then $k_i < 0$. Hence, the equation reads
\[ y^N \prod_{A_i < 0} (x' - \lambda_i)^{-k_i} = \prod_{A_i < 0} (x' - \lambda_i)^{r_i} \prod_{A_i > 0} (x' - \lambda_i)^{A_i}. \]
If we denote by $C_N$ the projective curve with affine equation
\[ y^N = \prod_{A_i < 0} (x - \lambda_i)^{r_i} \prod_{A_i > 0} (x - \lambda_i)^{A_i}, \]
we have a map $\rho : C'_N \to C_N$ given by
\[ (x', y') \mapsto (x', y \prod_{A_i < 0} (x' - \lambda_i)^{-k_i}) =: (x, y) \]
\[ \infty \mapsto \infty. \]
It is well-defined, because if $(x', y') \in C'_N$, then
\[ y^N = (y' \prod_{A_i < 0} (x' - \lambda_i)^{-k_i})^N = \prod_{A_i < 0} (x' - \lambda_i)^{r_i} \prod_{A_i > 0} (x' - \lambda_i)^{A_i} \]
and $(x, y) \in C_N$.

$A_i < 0$ implies $-k_i > 0$ and $\rho$ is a morphism. It has a rational inverse given by
$(x, y) \mapsto (x, y \prod_{A_i < 0} (x - \lambda_i)^{k_i})$. Write $\pi' : X'_N \to C'_N$ and $\pi : X_N \to C_N$ for the desingularization maps and remember that they are birational morphisms. Then $\rho \circ \pi'$ is a birational morphism from $X'_N$ to $C_N$ and there exists a unique isomorphism $\tilde{\rho} : X'_N \to X_N$ such that the following diagram commutes
\[ \begin{array}{ccc}
X'_N & \xrightarrow{\tilde{\delta}} & X_N \\
\downarrow{\pi'} & & \downarrow{\pi} \\
C'_N & \xrightarrow{\rho} & C_N.
\end{array} \]
Consequently
\[ \tilde{\rho}^* (\Omega^1[X_N]) = \Omega^1[X'_N] \]
and the existence of an abelian subvariety of $Jac(X_N)$ on which an integral $\int \omega$, $\omega \in \Omega^1[X_N]$, lives as a period implies the existence of an abelian subvariety of $Jac(X'_N)$ of the same dimension on which $\int \tilde{\rho}^* \omega$ lives as a period.

From now on we will then suppose $A_0, ..., A_r > 0$. 
1.4.3 Definition of the Family of Curves

In this paragraph, we give the precise definition of the family of curves that will be considered in this section.

Let \( r \in \mathbb{N} \cup \{0\}, N, A_0, ..., A_r \in \mathbb{N} \) and

\[
N \nmid A_0, ..., A_r, \sum_{k=0}^{r} A_k.
\]

Note that, in particular, \( N \neq 1 \), \( \sum_{k=0}^{r} A_k \). Let give us also \( \lambda_0, ..., \lambda_r \in \mathbb{C} \) such that, \( \forall i, j \in \{0, ..., r\} \) with \( i \neq j \), \( \lambda_i \neq \lambda_j \). We define \( C_N \) to be the projective plane curve with affine equation

\[
y^N = \prod_{i=0}^{r} (x - \lambda_i)^{A_i},
\]

and \( X_N \) to be its desingularization (cf §1.3.7). \( C_N \) and \( X_N \) are projective algebraic plane curves defined over \( \mathbb{C} \).

In order that the set of complex points of \( X_N \) form a Riemann surface, we have to suppose \( X_N \) to be irreducible, that is

\[
(N, A_0, ..., A_r) = 1.
\]

This hypothesis will also be used in §1.4.8.

To get the projective equation of \( C_N \), we set \( x := \frac{x_1}{x_0}, y := \frac{x_2}{x_0} \) and substitute this in (1.16). We get:

- **Case 1:** \( N - \sum_{k=0}^{r} A_k > 0; \quad C_N : x_2^N = x_0^{N - \sum_{k=0}^{r} A_k} \prod_{i=0}^{r} (x_1 - \lambda_i x_0)^{A_i} \)
- **Case 2:** \( N - \sum_{k=0}^{r} A_k < 0; \quad C_N : x_2^N x_0^{-N + \sum_{k=0}^{r} A_k} = \prod_{i=0}^{r} (x_1 - \lambda_i x_0)^{A_i}. \)

By definition, the points at infinity are those with coordinate \( x_0 = 0 \). In each case, there is only one point at infinity. It is \((0 : 1 : 0)\) in the first case and \((0 : 0 : 1)\) in the second one.

**Remark 29.** The case \( N - \sum_{k=0}^{r} A_k = 0 \) is excluded by our hypotheses.
Recall that each time we need to know the local behaviour of $C_N$ at an affine point, it is sufficient to consider the affine equation of $C_N$ and the affine coordinates of this point.

**Remark 30.** When we do not wish to distinguish the cases $N - \sum_{k=0}^{r} A_k > 0$ and $N - \sum_{k=0}^{r} A_k < 0$, the point at infinity will be denoted by $\infty$.

### 1.4.4 Singular Points on $C_N$

Recall that the singular points are those whose coordinates annihilate all the partial derivatives of the polynomial defining the curve (cf § 1.3.7).

- **Affine Singularities:**
  \[ f(x, y) := y^N - \prod_{i=0}^{r} (x - \lambda_i)^{A_i} \]
  \[ \frac{\partial f}{\partial x}(x, y) = - \sum_{j=0}^{r} r A_j (x - \lambda_j)^{A_j-1} \prod_{i=0}^{r} (x - \lambda_i)^{A_i} = 0 \]
  \[ \Leftrightarrow \exists j \in \{0, \ldots, r\} \text{ s.t. } x = \lambda_j \text{ and } A_j > 1, \]
  \[ \frac{\partial f}{\partial y}(x, y) = N y^{N-1} = 0 \Leftrightarrow y = 0 \text{ (because } N > 1 \text{ per hypothesis).} \]

Since for $(x, y) \in C_N, x = \lambda_i$ implies $y = 0$, we have

\[ (\lambda_j, 0) \text{ is singular } \Leftrightarrow A_j > 1. \]

- **Singularity at Infinity:** Calculating as above, but on the projective equations, one finds in both cases

\[ \infty \text{ is singular } \Leftrightarrow |N - \sum_{k=0}^{r} A_k| > 1. \]

### 1.4.5 Structure of the Desingularization

In this paragraph, we construct the desingularization $X_N$ of $C_N$ by gluing local desingularizations. These are constructed explicitly in the first subparagraph. By
the way, we get the number of preimages on $X_N$ of each potentially singular point of $C_N$ under the desingularization map $\pi$. This will be used in the computation of the genus of $X_N$ in §1.4.6. In the second subparagraph, we explain how the nonsingular projective curve $X_N$ can be constructed by gluing these local desingularizations. In the last one, we calculate the compositions of $\pi$ with local parametrizations of $X_N$ at the $\pi$-preimages of the potentially singular points of $C_N$. These will be used in §1.4.8 to calculate the order of the pull-back on $X_N$ of some differential forms on $C_N$.

**Local Desingularizations**

Let $P$ be a potentially singular point of $C_N$ (cf §1.4.4). We will work locally at $P$ by restricting to an open neighbourhood of $P$ isomorphic to an affine curve.

1. **Above $P_j := (1 : \lambda_j : 0)$**

   In the neighbourhood of an affine point, we have the classical isomorphism

   $$C_N - \{\infty\} \sim C_{\text{aff}}$$

   

   $$(x_0 : x_1 : x_2) \mapsto \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right).$$

   It induces a morphism $\kappa_0 : C_{\text{aff}} \to C_N$ given by $(x, y) \mapsto (1 : x : y)$. Setting $x := \frac{x_1}{x_0}, y := \frac{x_2}{x_0}$, we recover the affine equation of $C_N$ (cf §1.4.3):

   $$C_{\text{aff}} : y^N = \prod_{i=0}^{r}(x - \lambda_i)^{A_i}.$$  

   Let's fix $j \in \{0, ..., r\}$ and work locally in a neighbourhood of $(\lambda_j, 0)$ on which $g_j(x) := \prod_{i \neq j}(x - \lambda_i)^{A_i} \neq 0$. Set

   $$N' := \frac{N}{(N, A_j)} \quad \text{and} \quad A' := \frac{A_j}{(N, A_j)}.$$  

   Then there exist $n, m \in \mathbb{Z}$ such that

   $$nN' + mA' = 1.$$
and we have

\[(x, y) \in C_{aff} \implies \begin{cases} y^{N'} = (x - \lambda_j)^{N'}u & \text{and} \\ u^{(N, A_j)} = g_j(x). \end{cases} \]

Remark that

\[y^{mN'}(x - \lambda_j)^{nN'} = (x - \lambda_j)u^m\]

and that \(u = y^{N'}(x - \lambda_j)^{-A'}\). Hence, if we set \(z := y^m(x - \lambda_j)^n\) and define

\[X_j := \{(x, u, z) \in \mathbb{C}^3; \ z^{N'} = (x - \lambda_j)u^m, \ u^{(N, A_j)} = g_j(x), \ g_j(x) \neq 0\}\]

and \(C_{aff, j} := C_{aff} - \{(x, y) \in C_{aff}; \ g_j(x) = 0\}\). Then the rational map

\[v_j : C_{aff, j} \to X_j \quad (x, y) \mapsto (x, y^{N'}(x - \lambda_j)^{-A'}, y^m(x - \lambda_j)^n)\]

becomes a morphism on the open dense subset \(\hat{C}_{aff} := C_{aff, j} - \{\lambda_j, 0\}\) of \(C_{aff, j}\). This morphism has an inverse given by the morphism

\[\tau_j : X_j \to C_{aff, j} \quad (x, u, z) \mapsto (x, u^mz^{A'}).\]

In particular, \(\tau_j\) is a birational morphism, which restricts to an isomorphism from \(\hat{X}_j := X_j - (\tau_j^{-1}\{(\lambda_j, 0)\})\) to \(\hat{C}_{aff} = C_{aff} - \{(\lambda_k, 0); k = 0, \ldots, r\}\). Moreover, since \(C_{aff, j}\) is isomorphic to \(C_{N - \{\infty\}}\) under \(\kappa_0\), \(\hat{C}_{aff}\) is isomorphic to \(C_{N - \{P_0, \ldots, P_r, \infty\}}\) and

\[\hat{X}_j \sim C_{N - \{P_0, \ldots, P_r, \infty\}}.\]

In particular, \(X_j\) is birationally equivalent to \(C_{N}\) under the morphism \(\pi_j := \kappa_0 \circ \tau_j\).

**Remark 31.** The point \((\lambda_j, 0) \in C_{aff, j}\) has exactly \((N, A_j)\) preimages under \(\tau_j\), which are the points \((\lambda_j, u, 0)\), where \(u\) runs among the \((N, A_j)\)-th roots of \(g_j(\lambda_j)\). (They are distinct, because \(g_j(\lambda_j) \neq 0\).) \(P_j \in C_{N}\) has then also \((N, A_j)\) \(\pi_j\)-preimages on \(X_j\).
Remark 32. $X_j$ is nonsingular. Indeed, calculating the Jacobian matrix of $X_j$, we get
\[
\begin{pmatrix}
  u^m \\
  m(x - \lambda_j)u^{m-1} - (N, A_j)u^{(N, A_j)-1} \\
  -N'z^{N'-1} \\
  0
\end{pmatrix}
\]

Remember that $u \neq 0$ on $X_j$. If $x = \lambda_j$, then the above square looks like
\[
\left( \begin{array}{cc}
  \neq 0 & \neq 0 \\
  0 & \neq 0
\end{array} \right)
\]
and has rank 2. If $x \neq \lambda_j$, then $z \neq 0$ and the low square looks like
\[
\left( \begin{array}{cc}
  \neq 0 & \neq 0 \\
  0 & 0
\end{array} \right)
\]
and the matrix has rank 2.

\section*{Above Infinity}

- Case I: $N - \sum_{k=0}^{r} A_k > 0$

In this case, the projective equation of $C_N$ is
\[
x_2^N = x_0^{N-A_0} \prod_{i=0}^{r} (x_1 - \lambda_i x_0)^{A_i}
\]
and the point at infinity has coordinates $(0 : 1 : 0)$. We choose the neighbourhood of $(0 : 1 : 0)$ on $C_N$, on which $x_1 \neq 0$, and have the isomorphism
\[
C_N \cap \{x_1 \neq 0\} \sim C_{\infty 1}
\]

$(x_0 : x_1 : x_2) \mapsto (\frac{x_0}{x_1}, \frac{x_2}{x_1}) =: (x, y)$.

Its inverse is given by $\kappa_1 : (x, y) \mapsto (x : 1 : y)$. Remark that the affine potentially singular points with coordinate $x_1 \neq 0$ also lie on $C_N \cap \{x_1 \neq 0\}$. In the coordinates $(x, y)$, the equation of $C_{\infty 1}$ is
\[
y^N = x^{N-A_0} \prod_{i=0}^{r} (1 - \lambda_i x)^{A_i},
\]
the point at infinity is $(x, y) = (0, 0)$ and the affine potentially singular points are the points $(\frac{1}{\lambda_k}, 0)$, for each $k \in \{0, \ldots, r\}$ such that $\lambda_k \neq 0$. If we set $h(x) := \prod_{i=0}^{r} (1 - \lambda_i x)^{A_i}$,
\[
N' := \frac{N}{(N, N - \sum A_k)} \quad \text{and} \quad A' := \frac{N - \sum A_k}{(N, N - \sum A_k)},
\]
there exist \( n, m \in \mathbb{Z} \) such that

\[ nN' + mA' = 1 \]

and we have

\[ (x, y) \in C_{\infty 1} \implies \begin{cases} y^{N'} = x^{A'}u \\
u^{N,N-\sum A_k} = h(x) \end{cases} \]

Note that

\[ y^N x^{A'} = xu^m \]

and that \( u = y^{N'} x^{-A'} \). Set further \( z := y^m x^n \),

\[ X_{\infty 1} := \{ (x, u, z) \in \mathbb{C}^3; z^{N'} = xu^m, u^{N,N-\sum A_k} = h(x), h(x) \neq 0 \} \]

and \( C_{\infty 1}' := C_{\infty 1} - \{ (x, y); h(x) = 0 \} \). Then we have a rational map

\[ v_{\infty 1} : C_{\infty 1}' \to X_{\infty 1} \]

which restricts to a morphism on the open dense subset \( \tilde{C}_{\infty 1} := C_{\infty 1}' - \{(0, 0)\} \) of \( C_{\infty 1}' \). This morphism has an inverse given by the morphism

\[ \tau_{\infty 1} : X_{\infty 1} \to C_{\infty 1}' \]

\[ (x, u, z) \mapsto (x, z^{A'} u^n). \]

\( \tau_{\infty 1} \) is then a birational morphism and restricts to an isomorphism of \( \tilde{X}_{\infty 1} := X_{\infty 1} - \tau_{\infty 1}^{-1} \{(0, 0)\} \) to \( \tilde{C}_{\infty 1} \). Remembering that \( C_{\infty 1} \) is isomorphic to \( C_N \cap \{ x \neq 0 \} \) under \( \kappa_1 \), then \( \tilde{C}_{\infty 1} \) is isomorphic to \( C_N - \{ P_0, \ldots, P_r, \infty \} \) and so is \( \tilde{X}_{\infty 1} \), i.e.

\[ \tilde{X}_{\infty 1} \xrightarrow{\sim} C_N - \{ P_0, \ldots, P_r, \infty \}. \]

In particular, \( X_{\infty 1} \) is birationally equivalent to \( C_N \) under the morphism \( \pi_{\infty 1} := \kappa_1 \circ \tau_{\infty 1} \).
Remark 33. The point at infinity, which has coordinates \((x, y) = (0, 0)\) on \(C_{\infty 1}\), has \((N, N - \sum A_k)\) preimages under \(\tau_{\infty 1}\). They are \((0, u, 0)\), where \(u\) is a \((N, N - \sum A_k)\)-th root of \(h(0)\) and \(h(0) \neq 0\). This implies that the point at infinity on \(C_N\) has also \((N, N - \sum A_k)\) preimages on \(X_{\infty 1}\).

Remark 34. One verifies that \(X_{\infty 1}\) is nonsingular.

\(- \text{ Case 2: } N - \sum_{k=0}^{r} A_k < 0\)

In this case, the projective equation of \(C_N\) is

\[ x_2^N x_0^{N - \sum A_k} = \prod_{i=0}^{r} (x_1 - \lambda_i x_0)^{A_i} \]

and the point at infinity \((0 : 0 : 1)\). We choose the neighbourhood of \((0 : 0 : 1)\) on \(C_N\), on which \(x_2 \neq 0\). On this neighbourhood, there is no other potentially singular point than \((0 : 0 : 1)\), because they all have coordinate \(x_2 = 0\). We have the isomorphism

\[ C_N \cap \{ x_2 \neq 0 \} \longrightarrow C_{\infty 2} \]

\[ (x_0 : x_1 : x_2) \longmapsto (\frac{x_0}{x_2}, \frac{x_1}{x_2}) =: (x, y), \]

with inverse \(\kappa_2 : (x, y) \mapsto (x : y : 1)\). In these coordinates, the equation of \(C_{\infty 2}\) is

\[ C_{\infty 2} : x^{-N + \sum A_k} = \prod_{i=0}^{r} (y - \lambda_i x)^{A_i} \]

and the point at infinity \((x, y) = (0, 0)\). But the equation in this shape is not easy to handle. We will see that after having blowed up the point \((0, 0)\) on \(C_{\infty 2}\), everything becomes easier. In order to do this, we will use the expressions of the blow-up map in local coordinates as given in §1.3.6.

In the first coordinates' set, the point \((0, 0)\) has no preimage on the preimage of \(C_{\infty 2}\) bereft of the exceptional divisor. In the second coordinates’ set, the preimage of \(C_{\infty 2}\) under the map \(\varphi : (u, v) \mapsto (uv, v)\)
is given by
\[ u^{-N+\sum A_k} v^{-N+\sum A_k} = v^{\sum A_k} \prod_{i=0}^{r}(1 - \lambda_i u)^{A_i} \]

\[ \iff \begin{cases} v = 0 \text{ (exceptional divisor), or} \\ C'_{\infty 2} : u^{-N+\sum A_k} = v^{N} \prod_{i=0}^{r}(1 - \lambda_i u)^{A_i} \end{cases} \]

and the preimage of \((x, y) = (0, 0)\) is \((u, v) = (0, 0)\).

We can apply to \(C'_{\infty 2}\) the same procedure as in the other cases, though it will be slightly more technical. As usual, we begin by setting \(h(u) := \prod_{i=0}^{r}(1 - \lambda_i u)^{A_i}\),

\[ N' := \frac{N}{(N, -N + \sum A_k)} \quad \text{and} \quad A' := \frac{-N + \sum A_k}{(N, -N + \sum A_k)} \]

and letting \(n, m \in \mathbb{Z}\) be such that \(nN' + mA' = 1\). Then \((u, v) \in C'_{\infty 2}\) implies

\[ \begin{cases} u^{A'} = v^{N'} w \\ w^{(N, -N + \sum A_k)} = h(u) \\ h(u) \neq 0. \end{cases} \]

Note that \(u^{nA'} v^{mA'} = uw^n\) and that \(w = u^{A'} v^{-N'}\). Set \(z := u^n v^m\) and define

\[ X'_{\infty 2} := \{(u, v, w, z) \in \mathbb{C}^4; z^{A'} = uw^n, \ w^{(N, -N + \sum A_k)} = h(u), \ h(u) \neq 0\}. \]

Then we have a rational map

\[ \nu'_{\infty 2} : C'_{\infty 2} \rightarrow X'_{\infty 2} \]

\[ (u, v) \mapsto (u, v, u^{A'} v^{-N'}, u^n v^m), \]

which restricts to a morphism on the open dense subset \(C'_{\infty 2} - \{(0, 0)\}\) of \(C'_{\infty 2}\). This morphism has an inverse given by the morphism

\[ \tau'_{\infty 2} : X'_{\infty 2} \rightarrow C'_{\infty 2} \]

\[ (u, v, w, z) \mapsto (w^m z^{N'}, v). \]
Hence, \( r'_{\infty 2} \) is a birational morphism and restricts to an isomorphism of \( \hat{X}_{\infty 2} := X_{\infty 2} - \{(r'_{\infty 2})^{-1}(0, 0)\} \) to \( C'_{\infty 2} - \{(0, 0)\} \). Remembering that the blow-up map \( \varphi : C'_{\infty 2} \rightarrow C_{\infty 2} \) is a birational morphism and restricts to an isomorphism on \( C'_{\infty 2} - \{\varphi^{-1}(0, 0)\} \) and that \( \varphi^{-1}(0, 0) = \{(0, 0)\} \), we get a birational morphism

\[
\tau_{\infty 2} := \varphi \circ r'_{\infty 2} : X_{\infty 2} \rightarrow C_{\infty 2},
\]

which induces an isomorphism from \( \hat{X}_{\infty 2} \) to \( C_{\infty 2} := C_{\infty 2} - \{(0, 0)\} \).

Now, since \( C_{\infty 2} \) is isomorphic to \( C_N - \{P_0, ..., P_r\} \) under \( \kappa_2 \), \( C_{\infty 2} \) is isomorphic to \( C_N - \{P_0, ..., P_r, \infty\} \) and so is \( \hat{X}_{\infty 2} \), i.e.

\[
\hat{X}_{\infty 2} \sim C_N - \{P_0, ..., P_r, \infty\}.
\]

The birational morphism from \( X_{\infty 2} \) to \( C_N \) is given by \( \pi_{\infty 2} := \kappa_2 \circ \tau_{\infty 2} \).

Remark 35. \( X_{\infty 2} \) is nonsingular and the \( \tau_{\infty 2} \)-preimages of \( (0, 0) \in C_{\infty 2} \) are \( (0, 0, w, 0) \), where \( w \) runs through the \( (N, -N + \sum A_k) \)-th roots of \( h(0) \). Since \( h(0) \neq 0 \), their number is \( (N, -N + \sum A_k) = (N, N - \sum A_k) \).

Construction of \( X_N \) by Gluing

We refer here to the construction described in [Sh2] V.3.2. Let \( X_{\infty} \) resp. \( \pi_{\infty} \) denote \( X_{\infty 1} \) resp. \( \pi_{\infty 1} \) in the case \( N - \sum A_k > 0 \) and \( X_{\infty 2} \) resp. \( \pi_{\infty 2} \) in the case \( N - \sum A_k < 0 \).

Remember that, for each \( j \in \{0, ..., r, \infty\} \), the morphism \( \pi_j : X_j \rightarrow C_N \) restricts to an isomorphism of the open dense subset \( \hat{X}_j \) of \( X_j \) to \( C_N - \{P_0, ..., P_r, \infty\} \) and that \( X_j \) and \( C_N \) are birationally equivalent. Then one can define an equivalence relation on the disjoint union \( \bigsqcup_{j \in \{0, ..., r, \infty\}} X_j \) by setting, for \( Q_j \in \hat{X}_j, Q_k \in \hat{X}_k \) with \( j, k \in \{0, ..., r, \infty\} \) and \( j \neq k \),

\[
Q_j \sim Q_k \iff \pi_j(Q_j) = \pi_k(Q_k).
\]

Moreover, the functions \( \pi_j, j \in \{0, ..., r, \infty\} \), induce a well-defined function \( \pi \) on the quotient \( X := \bigsqcup_{j \in \{0, ..., r, \infty\}} X_j/\sim \) by setting, for \( \mathcal{C} \in X \) and \( Q_j \in X_j \) with \( [Q_j] = \mathcal{C} \),

\[
\pi(\mathcal{C}) := \pi_j(Q_j).
\]
By definition of the equivalence relation, this is independent of the choice of the representant of the class \( C \).

On the set \( X \), we have the quotient topology and can define a sheaf induced from the sheaf of regular functions on each \( X_j \), for which \( \pi \) is again a birational and finite morphism. This implies that \( X \) is again a projective curve. Since \( X \) is moreover nonsingular, because so is each \( X_j \), and, as we have seen, birationally equivalent to \( C_N \), it provides a model of the desingularization of \( C_N \), hence is isomorphic to \( X_N \). The desingularization map is given by the map \( \pi \) such that \( \pi |_{X_j} = \pi_j \) for each \( j \). Another isomorphic construction is given in [Sh1] II.5.3 Thm 6, Thm 7.

Since, to our purpose, we only need to know the desingularization up to isomorphism, \( X \) will be identified with \( X_N \) in the following.

**Compositions of \( \pi \) with Local Parametrizations of \( X_N \).**

Because the restriction of \( \pi \) to \( X_N - \{ \pi^{-1}((P_0, ..., P_r, \infty)) \} \) is an isomorphism to \( C_N - \{ P_0, ..., P_r, \infty \} \), we only need to know the compositions of \( \pi \) with local parametrizations at the points of \( \pi^{-1}((P_0, ..., P_r, \infty)) \), which are isolated on \( X_N \).

- **Above \( P_j = (1 : \lambda_j : 0) \)**

  Fix \( j \in \{0, ..., r\} \) and remember that the local desingularization above \( P_j \) is

  \[
  X_j = \{(x, u, z) \in \mathbb{C}^3; \quad z^{N'j} = (x - \lambda_j)u^m, \quad u^{(N', A_j)} = g_j(x), \quad g_j(x) \neq 0\}.
  \]

  We have then the composition

  \[
  \pi_j = \kappa_0 \circ \tau_j : \quad X_j \longrightarrow C_N \\
  (x, u, z) \longmapsto (1 : x : u^n z^{A_j}).
  \]

  Recall that \( X_j \) is nonsingular and open in an affine variety, hence each point has a neighbourhood for the complex topology which is isomorphic to an open neighbourhood on \( \mathbb{C} \) (cf §1.3.8). Choose a complex open neighbourhood \( U_j \) of \( s = 0 \) in \( \mathbb{C} \) on which \( g_j(s^{N'} + \lambda_j) \neq 0 \). Then the image of \( U_j \) under \( s \mapsto g_j(s^{N'} + \lambda_j) \) is included in \( \mathbb{C} \) bereft of a half-line through 0. Thus, branches of roots of \( g_j(s^{N'} + \lambda_j) \) can be well-defined as holomorphic
functions of $s$ on $U_j$. Then choosing fixed branches, we have a well-defined holomorphic function
\[
\varphi_j : s \mapsto (s^{N'} + \lambda_j, g_j(s^{N'} + \lambda_j)^{\frac{1}{(N,A_j)}}, sg_j(s^{N'} + \lambda_j)^{\frac{m}{N}})
\]
from $U_j$ to $X_j$. Indeed, $u^{(N,A_j)} = g_j(s^{N'} + \lambda_j) = g_j(x)$ and
\[
z^{N'} = s^{N'} g_j(s^{N'} + \lambda_j)^{\frac{m}{N}}
= (s^{N'} + \lambda_j - \lambda_j)g_j(s^{N'} + \lambda_j)^{\frac{m}{(N,A_j)}} = (x - \lambda_j)u^m.
\]
On the image of $\varphi_j$, we have a well-defined holomorphic inverse map
\[
(x, u, z) \mapsto zg_j(x)^{-\frac{m}{N}}.
\]
Hence, $\varphi_j$ is an analytic isomorphism and a local parametrization of $X_j$ at $\varphi(0) = (\lambda_j, g_j(\lambda_j)^{\frac{1}{(N,A_j)}}, 0)$, one of the $\pi$-preimages of $P_j \in C_N$. Remark again that the choices of branches for the $(N, A_j)$-th root of $g_j(s^{N'} + \lambda_j)$ are in bijective correspondence with the $\pi$-preimages on $X_j$ of $P_j$.

The expression of $\pi$ in this local parameter $s$ at each $\pi$-preimage of $P_j$ is given by the composition $\pi_j \circ \varphi_j : U_j \rightarrow C_N$
\[
s \mapsto (1 : s^{\frac{N}{(N,A_j)}} + \lambda_j : s^{\frac{A_j}{(N,A_j)}} g_j(s^{\frac{N}{(N,A_j)}} + \lambda_j)^{\frac{m}{N}}).
\]

\textit{Above Infinity}

- \textit{Case 1: $N - \sum_{k=0}^N A_k > 0$}

We are looking for an analytic parametrization of $X_{\infty 1}$ at the points $(0, u, 0)$, where $u^{(N,N-\sum A_k)} = h(0)$. Let's choose a complex neighbourhood $U_{\infty}$ of $s = 0$ in $\mathbb{C}$ on which $h(s^{N'}) \neq 0$, $N'$ being here \(\frac{N}{(N,N-\sum A_k)}\). On such a neighbourhood, we can define roots of $h(s^{N'})$ as analytic functions of $s$. Fix an $N$-th root $h(s^{N'})^{\frac{1}{N}}$. Then, for each branch of the $(N, N - \sum A_k)$-th root of $h(s^{N'})$, the map
\[
\varphi_{\infty 1} : U_{\infty} \rightarrow X_{\infty 1}
\]
\[
s \mapsto (s^{N'}, h(s^{N'})^{\frac{1}{(N,N-\sum A_k)}}, sh(s^{N'})^{\frac{m}{N}})
\]
1.4. Abelian Varieties for Appell-Lauricella Hypergeometric Series

is a well-defined analytic map such that \( \varphi_{\infty 1}(0) = (0, u, 0) \), where \( u \) is the corresponding \((N, N - \sum A_k)\)-th root of \( h(0) \). \( \varphi_{\infty 1} \) has an analytic inverse on its image, which is given by

\[
(x, u, z) \mapsto z h(x)^{-\frac{N}{N'}}.
\]

This shows that \( \varphi_{\infty 1} \) is a local parametrization of \( X_{\infty 1} \) at the preimage \((0, h(0), \frac{N - \sum A_k}{N'}) \) of \( \infty \). Since \( \pi_{\infty 1} = \kappa_1 \circ \tau_{\infty 1} : X_{\infty 1} \to C_N \) is given by \((x, u, z) \mapsto (x : 1 : z^{N'})\), \( \pi_{\infty 1} \circ \varphi_{\infty 1} : U_{\infty} \to C_N \) is given by

\[
s \mapsto \left( s^{\frac{N}{N'}} \right) \frac{1}{s^{\frac{N - \sum A_k}{N'}}} \frac{h(s^{N'})^{-\frac{1}{N'}}}{h(s^{\frac{N - \sum A_k}{N'}})} \]

(1.18)

and is a local expression of \( \pi \) at each point on \( X_N \) lying above \((0 : 1 : 0)\).

Case 2: \( N - \sum_{k=0}^{L} A_k < 0 \)

On \( U_{\infty} \) defined as in Case 1, we have a well-defined holomorphic map

\[
\varphi_{\infty 2} : U_{\infty} \to X_{\infty 2}
\]

\[
s \mapsto \left( s^{N'}, s^{\frac{-N + \sum A_k}{N'}} \frac{h(s^{N'})^{-\frac{1}{N'}}}{h(s^{\frac{-N + \sum A_k}{N'}})} \right),
\]

for a fixed choice of the branch of the roots of \( h(s^{N'}) \). On its image, this map has an analytic inverse given by

\[
(u, v, w, z) \mapsto z h(x)^{-\frac{1}{N'} N''}. \]

Each choice of the branch of the \((N', -N' + \sum A_k)\)-th root of \( h(s^{N'}) \) corresponds to a \( \pi \)-preimage of \((0 : 0 : 1) \) at which \( \varphi_{\infty 2} \) is a local parametrization. The composition of \( \varphi_{\infty 2} \) with \( \pi \) equals \( \kappa_2 \circ \varphi \circ \tau_{\infty 2} \circ \varphi_{\infty 2} : U_{\infty} \to C_N \) and is given by

\[
s \mapsto \left( s^{\frac{N}{N' - \sum A_k}} \right) \frac{-N' + \sum A_k}{s^{\frac{-N + \sum A_k}{N'} - 1}} \frac{1}{h(s^{\frac{N}{N' - \sum A_k}})} \]

(1.19)

1.4.6 Genus of \( X_N \)

The aim of this paragraph is to calculate the genus \( g[X_N] \) of \( X_N \). The curve \( X_N \) is irreducible, nonsingular and defined over \( \mathbb{C} \), hence the set \( X_N(\mathbb{C}) \) of complex
points of $X_N$ can be given the structure of a Riemann surface, which is compact (cf §1.3.8).

Since $g(X_N(\mathbb{C})) = g[X_N]$ (cf (1.10) in §1.3.12), we can use the Hurwitz formula (Proposition 1.3, §1.3.9) to calculate $\chi(X_N(\mathbb{C}))$ and then apply (1.9), §1.3.12.

Consider the projection $p : C_N \to \mathbb{P}^1_\mathbb{C}$ given by $(x_0 : x_1 : x_2) \mapsto (x_0 : x_1)$ and compose it with the desingularization map $\pi : X_N(\mathbb{C}) \to C_N$. The composition $\nu := p \circ \pi : X_N(\mathbb{C}) \to \mathbb{P}^1_\mathbb{C}$ is nonconstant and regular (hence holomorphic) and we can apply Hurwitz formula to it ($X_N(\mathbb{C})$ and $\mathbb{P}^1_\mathbb{C}$ being compact Riemann surfaces). The degree of $\nu$ is $N$, because each affine point $(x, y) \in C_N$ with $x \neq \lambda_0, \ldots, \lambda_r$ has $N$ distinct preimages on $X_N$ corresponding to the $N$-th roots of $\prod_{i=0}^{r}(x - \lambda_i)^{A_i}$.

It remains to calculate the ramification indices.

- Each point $Q \in X_N(\mathbb{C})$ such that $\pi(Q) = P$ is nonsingular is a regular point of the covering $\nu$. Indeed, if $P \in C_N$ is nonsingular and $P = (x, y)$ (resp. $\infty$), then $y$ (resp. $x_1$ and $x_2$) can be written as a function of $x$ (resp $x_0$) in a neighbourhood of $P$. $\pi$ being locally an isomorphism at $Q$, $x$ (resp. $x_0$) can also be taken as a local parameter of $X_N(\mathbb{C})$ at $Q$. In this parameter, $\pi$ is given by $x \mapsto (x, y(x))$ (resp. by $x_0 \mapsto (x_0 : x_1(x_0) : x_2(x_0))$) and $\nu$ by $x \mapsto x$ (resp. $x_0 \mapsto x_0$). Hence $r_\nu(Q) = 1$.

- For $j \in \{0, \ldots, r\}$, let $Q_j$ be one $\pi$-preimage of $(\lambda_j, 0) \in C_N$. Remembering the composition (1.17) of $\pi$ with a local parametrization of $X_N$ at $Q_j$ given in §1.4.5 and composing it with $p$, we get

$$s \mapsto (1 : s^{\lambda_j/N} + \lambda_j).$$

If we now choose the chart $(x_0 : x_1) \mapsto \frac{x_1 - \lambda_j}{x_0}$ on $\{(x_0 : x_1) \in \mathbb{P}^1_\mathbb{C} : x_0 \neq 0\}$ and compose it with the above map, we get the expression of $\nu$ in local coordinates as

$$s \mapsto s^{\lambda_j/N}.$$ 

This shows that each $\pi$-preimage of $(\lambda_j, 0)$ has ramification index equals to $\frac{N}{(N, A_j)}$. 

1. Abelian Varieties Associated to Hypergeometric Series
In the cases of the points above $\infty$, we will choose the chart $(U_1, \psi)$ on $\mathbb{P}^{1}_C$, where $U_1 := \{(x_0 : x_1) \in \mathbb{P}^{1}_C; x_1 \neq 0\}$ and $\psi : (x_0 : x_1) \mapsto \frac{x_0}{x_1}$.

- **Case 1:** $N - \sum_{k=0}^{r} A_k > 0$
  The composition of $p$ with the composition (1.18) of $\pi$ with a local parametrization of $X_N$ at each preimage of $\infty$ reads
  \[ s \mapsto \left( s \left( \frac{N}{(N, -N + \sum A_k)} \right) : 1 \right). \]
  Composing it with $\psi$, we get the expression of $\nu$ in local coordinates at each $\pi$-preimage of $\infty$ as
  \[ s \mapsto s \left( \frac{N}{(N, -N + \sum A_k)} \right). \]

- **Case 2:** $N - \sum_{k=0}^{r} A_k < 0$
  In this case, we have to be a little more careful, because the map $p$ is not defined at $\infty$. If only to consider momentarily the restriction of $p$ to the punctured Riemann surface $X_N(\mathbb{C}) - \{\pi^{-1}(0 : 0 : 1)\}$, we can suppose $s \neq 0$ and consider the composition of (1.19) with $p$, which is
  \[ \nu \circ \varphi_{\infty 2} : s \mapsto \left( s \left( \frac{\sum A_k}{(N, -N + \sum A_k)} : s \left( \frac{-N + \sum A_k}{(N, -N + \sum A_k)} \right) \right) = \left( s \left( \frac{\sum A_k}{(N, -N + \sum A_k)} : 1 \right) \right). \]
  If $s$ tends to 0, $\nu \circ \varphi_{\infty 2}(s)$ tends to $(0 : 1)$, the point at infinity in $\mathbb{P}^{1}_C$. Hence, we can extend the map continuously by setting $0 \mapsto (0 : 1)$. The composition with $\psi$ of the extended map is
  \[ s \mapsto s \left( \frac{\sum A_k}{(N, -N + \sum A_k)} \right) \]
  and the ramification index of each point lying above $\infty$ is $N/(N, -N + \sum A_k)$.

Since $(N, -N + \sum A_k) = (N, N - \sum A_k)$, we have found in both cases that each $\pi$-preimage of $\infty$ has ramification index equals to $(N, N - \sum A_k)$ under $\nu$.

**Summary 1.**

<table>
<thead>
<tr>
<th>point $P$ of $C_N$</th>
<th>nb of $\pi$-preimages $Q$</th>
<th>$r_\nu(Q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\lambda_j, 0), \ j \in {0, ..., r}$</td>
<td>$(N, A_j)$</td>
<td>$\frac{N}{(N, A_j)}$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>$(N, N - \sum_{k=0}^{r} A_k)$</td>
<td>$\frac{N}{(N, N - \sum_{k=0}^{r} A_k)}$</td>
</tr>
<tr>
<td>other points</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Theorem 1.8. Let $X_N$ be the desingularization of the irreducible projective algebraic plane curve $C_N$ defined over $\mathbb{C}$ by the affine equation

$$y^N = \prod_{i=0}^{r}(x - \lambda_i)^{A_i},$$

where $\lambda_0, ..., \lambda_r \in \mathbb{C}$ are such that, for all $i, j \in \{0, ..., r\}$ with $i \neq j$, $\lambda_i \neq \lambda_j$ and $N, A_0, ..., A_r \in \mathbb{N}$ such that

$$N \neq \sum_{k=0}^{r} A_k \text{ and } (N, A_0, ..., A_r) = 1.$$

Then the Euler characteristic of $X_N(\mathbb{C})$ is given by

$$\chi(X_N(\mathbb{C})) = -rN + (N, N - \sum_{k=0}^{r} A_k) + \sum_{j=0}^{r} (N, A_j),$$

and the genus of $X_N$ by

$$g[X_N] = g(X_N(\mathbb{C})) = 1 + \frac{1}{2}(rN - (N, N - \sum_{k=0}^{r} A_k) - \sum_{j=0}^{r} (N, A_j)).$$

Proof. We apply the Hurwitz formula (Proposition 1.3, §1.3.9) to the covering map

$$\nu = p \circ \pi : X_N(\mathbb{C}) \rightarrow \mathbb{P}^1_{\mathbb{C}},$$

where $\pi : X_N \rightarrow C_N$ is the desingularization map and $p : C_N \rightarrow \mathbb{P}^1_{\mathbb{C}}$ the projection given by $(x_0 : x_1 : x_2) \mapsto (x_0 : x_1)$. As seen above, it has degree $N$ and the only possible ramification points lie above $\infty \in \mathbb{P}^1_{\mathbb{C}}$ and $(1 : \lambda_j)$, $j \in \{0, ..., r\}$. Using the ramification indices calculated above, the number of preimages calculated in §1.4.5 (all recalled in Summary 1) and the fact that $\chi(\mathbb{P}^1_{\mathbb{C}}) = 2$, we get, by Hurwitz’s formula

$$\chi(X_N(\mathbb{C})) = 2N - (N, N - \sum A_k)(\frac{N}{(N, N - \sum A_k)} - 1) - \sum_{j=0}^{r} (N, A_j)(\frac{N}{(N, A_j)} - 1)$$

$$= -rN + (N, N - \sum A_k) + \sum_{j=0}^{r} (N, A_j).$$
To get the second formula, remember that \( g[X_N] = g(X_N(\mathbb{C})) \) (by (1.10), §1.3.12) and that \( X(X_N(\mathbb{C})) = 2 - 2g(X_N(\mathbb{C})) \) (cf (1.9), §1.3.12). Hence
\[
g(X_N(\mathbb{C})) = 1 - \frac{1}{2} \chi(X_N(\mathbb{C})) = 1 + \frac{1}{2}(rN - (N, N - \sum_{k=0}^{r} A_k) - \sum_{j=0}^{r}(N, A_j)).
\]

### 1.4.7 Actions of \( \mu_N \)

Let \( \mu_N \) be the group of complex \( N \)-th roots of unity. In this paragraph, we define an action of \( \mu_N \) on \( X_N \) and show how it induces a linear action on the \( \mathbb{C} \)-vector space \( \Omega^1[X_N] \) of regular differential 1-forms on \( X_N \).

For \( \zeta \in \mu_N \) and an affine point \((x, y) \in C_N\), define
\[
\zeta \cdot (x, y) := (x, \zeta^{-1}y).
\]
Further, set \( \zeta \cdot \infty = \infty \), \( \forall \zeta \in \mu_N \). Because \((\zeta^{-1})^N = 1\), \( \zeta \cdot (x, y) \in C_N \) and this is an action, because \( \mu_N \) is abelian and 1 acts as the identity. Moreover, since \( \mu_N \) is included in the definition field \( \mathbb{C} \) of \( C_N \), for each \( \zeta \in \mu_N \), the map
\[
\varphi_{\zeta} : C_N \longrightarrow C_N
\]
\[
(x, y) \longmapsto \zeta \cdot (x, y)
\]
\[
\infty \longmapsto \infty
\]
is a morphism of algebraic varieties.

Now, we want to extend this action to an action on \( X_N \). Remember that the desingularization map \( \pi : X_N \rightarrow C_N \) restricts to an isomorphism on the dense subset \( \pi^{-1}(C_N^{reg}) \) of \( X_N \), where \( C_N^{reg} \) is the set of regular points in \( C_N \). Let \( P \in X_N \), if \( \pi(P) \) is regular, set
\[
\zeta \cdot P := \pi^{-1}(\zeta \cdot \pi(P)).
\]
If \( P \in X_N \) is such that \( \pi(P) = \infty \) or \( \exists j \in \{0, ..., r\} \) with \( \pi(P) = (\lambda_j, 0) \), set
\[
\zeta \cdot P := P.
\]
Note that, if \((\lambda, 0)\) (respectively \(\infty\)) is regular on \(C_N\), this last definition is coherent with the above one. This defines an action on \(X_N\).

For \(\xi \in \mu_N\), set

\[
\Phi_\xi : X_N \longrightarrow X_N
\]

\[
P \longmapsto \xi \cdot P.
\]

Then \(\Phi_\xi\) makes the following diagram commute

\[
\begin{array}{ccc}
X_N & \xrightarrow{\Phi_\xi} & X_N \\
\downarrow{\pi} & & \downarrow{\pi} \\
C_N & \xrightarrow{\varphi_\xi} & C_N.
\end{array}
\] (1.20)

Because \(\varphi_\xi\) and \(\pi\) are morphisms, \(\Phi_\xi\) will also be a morphism.

Let \(\omega\) be a regular differential form on \(X_N\). Since \(\Phi_\xi : X_N \rightarrow X_N\) is a morphism, the pull-back \(\Phi_\xi^\ast\omega\) is again regular on \(X_N\) (see the end of §1.3.11). Hence, the following map is well-defined

\[
(\xi, \omega) \longmapsto \Phi_\xi^\ast\omega.
\]

It defines an action of \(\mu_N\) on \(\Omega^1[X_N]\), which is linear, because, for every \(\xi \in \mu_N\), the map \(\Omega^1[X_N] \rightarrow \Omega^1[X_N], \omega \mapsto \Phi_\xi^\ast\omega\) is linear. Further, since the \(\mathbb{C}\)-vector space \(\Omega^1[X_N]\) is finitely dimensional (cf §1.3.11), it furnishes, together with the above action, a linear representation of \(\mu_N\) (in the sense of §1.3.17).

As seen in §1.3.17, such a linear representation admits a decomposition in isotypical components (the so-called "canonical" decomposition). The isotypical component, which is the direct sum of irreducible representations of character \(\chi_n\) for one \(n \in \{1, \ldots, N\}\) (cf Example 4 in §1.3.17), will be denoted by \(V_n\). In these terms, we can write the canonical decomposition of \(\Omega^1[X_N]\) as

\[
\Omega^1[X_N] = \bigoplus V_n,
\] (1.21)

where the sum is taken over the indices \(n\) in \([0, \ldots, N - 1]\) for which \(\dim V_n > 0\).

In the next paragraph, the dimension of \(V_n\) will be calculated for each \(n \in \{0, \ldots, N - 1\}\). It corresponds to the number of irreducible subrepresentations of \(\Omega^1[X_N]\) having character \(\chi_n\).
1.4.8 Basis of Regular Differential Forms on $X_N$

In this section, we intend to calculate a basis of the $\mathbb{C}$-vector space $\Omega^1[X_N]$ of regular differential 1-forms on $X_N$. In view of the decomposition (1.21), §1.4.7, it is sufficient to find a basis of $V_n$ for each $n \in \{0, \ldots, N - 1\}$. Once this being done, we will calculate $\dim_{\mathbb{C}} V_n$ by counting the basis elements and also the sum $\dim V_n + \dim V_{N-n}$, when $(n, N) = 1$.

Let's first make use of a results that goes back to Abel and Riemann and that is stated in Satz 1, §9.3 of [BK] in the following way.

**Proposition 1.9.** The nonvanishing holomorphic differential 1-forms on the Riemann surface $C'$, which is the desingularization of an irreducible algebraic plane curve $C$ with affine equation $f(x, y) = 0$, where the coordinates are chosen in such a way that $\frac{\partial f}{\partial y}$ is not identically zero, are given by [the pull-backs under the desingularization map of]

$$\frac{\Phi(x, y)dx}{\frac{\partial f}{\partial y}(x, y)},$$

where $\Phi(x, y) = 0$ is the equation of an adjoint curve to $C$ of degree $(\deg f) - 3$.

We do not want to introduce what an adjoint curve is, but this proposition allows us to choose a basis of regular differential 1-forms on $X_N$ among the regular pull-backs under $\pi$ of the differential forms

$$\frac{\Phi(x, y)dx}{y^{N-1}}, \quad (1.22)$$

on $C_N$, where $\Phi(x, y) \in \mathbb{C}[x, y]$.

Since, for a point $(x, y) \in C_N$, each potence $y^{kN}, k \in \mathbb{N}$, can be replaced by a polynomial expression in $x$, we can suppose that

$$\Phi(x, y) = \Phi_0(x) + \Phi_1(x)y + \ldots + \Phi_{N-1}(x)y^{N-1}.$$  

That is

$$\frac{\Phi(x, y)dx}{y^{N-1}} = \frac{\Phi_0(x)dx}{y^{N-1}} + \frac{\Phi_1(x)dx}{y^{N-2}} + \ldots + \frac{\Phi_{N-2}(x)dx}{y} + \Phi_{N-1}(x)dx.$$
Hence, if the regular pull-backs of the differential forms (1.22) generates \( \Omega^1[X_N] \), so do the regular pull-backs of the differential forms

\[
\Psi(x) dx \over y^n,
\]

where \( n \in \{0, \ldots, N - 1\} \) and \( \Psi(x) \in \mathbb{C}[x] \). Further, the polynomials \( \Psi(x) \) will be replaced by polynomials that are fitter to reflect the topology of \( C_N \) (resp. \( X_N \)) and that also generate the ring \( \mathbb{C}[x] \). Namely, polynomials of the form

\[
\prod_{i=0}^{r}(x - \lambda_i)^{a_i} \in \mathbb{C}[x], \quad a_i \in \mathbb{Z}.
\]

This discussion may be summarized by saying that the regular pull-backs under \( \pi \) of the following differential forms on \( C_N \)

\[
\omega_n(x, y) := \prod_{i=0}^{r}(x - \lambda_i)^{a_i} \over y^n,
\]

with \( a_i \in \mathbb{Z} \) and \( n \in \{0, \ldots, N - 1\} \), generate \( \Omega^1[X_N] \).

### Regularity Conditions for \( \pi^*\omega_n \)

Let's now fix \( n \) in \( \{0, \ldots, N - 1\} \). We are looking for conditions on \( a_0, \ldots, a_r \) for \( \pi^*\omega_n \) to be regular on \( X_N \).

- **On the dense subset** \( U := C_N - \{\infty, (\lambda_i, 0); j = 0, \ldots, r\} \) of \( C_N \), the differential form \( \omega_n \) is obviously regular, because \( (x, y) \mapsto x \) and \( (x, y) \mapsto \frac{1}{y^n} \) are regular functions on \( U \). Since the desingularization map \( \pi : X_N \to C_N \) is a morphism, the pull-back \( \pi^*\omega_n \) is regular on \( \pi^{-1}(U) \).

- **Above(1 : \lambda_j : 0)**
  Let \( j \in \{0, \ldots, r\} \) and \( Q_j \in X_N \) be such that \( \pi(Q_j) = (\lambda_j, 0) \). As calculated in (1.17) (last subparagraph of §1.4.5), the composition of \( \pi \) with the local parametrization \( \varphi_j \) of \( X_N \) at \( Q_j \) reads (we use here the affine coordinates on \( C_N \))

\[
s \mapsto \left( s^{N(A_j)} + \lambda_j, s^{A_j} g_j(s^{N(A_j)} + \lambda_j)^{1/2} \right),
\]

where \( g_j \) is a polynomial in \( s \) that depends on \( \lambda_j \) and \( A_j \).
where \(g_j(x) := \prod_{i \neq j} (x - \lambda_i)^{A_i}\) and \(s\) takes values in a neighbourhood \(U_j\) of 0 in \(\mathbb{C}\) on which \(g_j(s_{(N,A_j)} + \lambda_j) \neq 0\). By definition, we have

\[
((\pi_j \circ \varphi_j)^*(\omega_n))(s) = \omega((\pi_j \circ \varphi_j)(s)) \circ ds(\pi_j \circ \varphi_j)
\]

and by Remark 10, §1.3.10, \(d(\pi_j \circ \varphi_j)(s) \circ ds(\pi_j \circ \varphi_j) = \frac{\partial((\pi_j \circ \varphi_j)^*)}{\partial s}(s) d_s s\). Hence

\[
((\pi_j \circ \varphi_j)^*(\omega_n))(s)
= \frac{N}{(N,A_j)} \prod_{i=0}^{n-1} (s_{(N,A_j)}^{N/(N,A_j)} + \lambda_j - \lambda_i)^{A_i} s_{(N,A_j)}^{N/(N,A_j)-1} g_j(s_{(N,A_j)}^{N/(N,A_j)} + \lambda_j)^{-\frac{n}{N}} ds
= C(s) s^{N/(N,A_j)} (N,A_j)^{-1} ds,
\]

where \(C(s) = \frac{N}{(N,A_j)} \prod_{i \neq j} (s_{(N,A_j)}^{N/(N,A_j)} + \lambda_j - \lambda_i)^{A_i} g_j(s_{(N,A_j)}^{N/(N,A_j)} + \lambda_j)^{-\frac{n}{N}}\) does not take the value zero on \(U_j\) and is regular (because \(g_j(s_{(N,A_j)}^{N/(N,A_j)} + \lambda_j) \neq 0\) on \(U_j\)).

Remark that this amounts to replacing \(x\) and \(y\) by their expressions in \(s\) and \(dx\) by \((\pi_j \circ \varphi_j)^*(\omega_n)\) in \(\omega_n(x, y)\). Since \(\varphi_j\) is an analytic isomorphism, it is an algebraic morphism and so is its inverse. This has the consequence that \(\pi^* \omega_n\) is regular at \(Q_j = \varphi_j(0)\) exactly when \((\pi_j \circ \varphi_j)^*(\omega_n)\) is regular at 0 (because \(\pi_j^* \omega_n = (\varphi_j^{-1})^* ((\pi_j \circ \varphi_j)^*(\omega_n))\) and \((\pi_j \circ \varphi_j)^*(\omega_n) = (\varphi_j)^* (\pi_j^* \omega_n)\). Hence, we have

\[
\pi^* \omega_n\text{ is regular at } Q_j \iff a_j \geq \frac{nA_j + (N, A_j)}{N} - 1.
\]

Note that this condition ensures the regularity of \(\pi^* \omega_n\) at each \(\pi\)-preimage of \((\lambda_j, 0)\).

- **Above Infinity**

  First of all, we have to write the differential form \(\omega_n\) in projective coordinates. Setting \(x := \frac{y_1}{x_0}\) and \(y := \frac{y_2}{x_0}\), we get

\[
dx = \frac{1}{x_0} dx_1 - \frac{x_1}{x_0^2} dx_0
\]

and

\[
\omega_n(x_0, x_1, x_2) = x_2^{-n} x_0^{n-2} \sum_{k=0}^{\infty} \prod_{i=0}^{r}(x_1 - \lambda_i x_0)^{a_i}(x_0 dx_1 - x_1 dx_0).
\]
1. Abelian Varieties Associated to Hypergeometric Series

- Case 1: \( N - \sum_{k=0}^{r} A_k > 0 \)

The composition of \( \pi \) with the local parametrization \( \varphi_{\infty_1} \) of \( X_N \) at each preimage \( Q \) of \( (0 : 1 : 0) \) is given by (cf (1.18) §1.4.5)

\[
\begin{align*}
\pi_\ast & \omega_n \quad (s) \longleftrightarrow \left( s^{\frac{N}{N - \sum A_k}} : 1 : s^{\frac{N - \sum A_k}{N - \sum A_k}} h(s^{\frac{N}{N - \sum A_k}})^{\frac{1}{N}} \right),
\end{align*}
\]

where \( h(x) = \prod_{i=0}^{r} (1 - \lambda_i x)^{A_i} \) and \( h(s^{\frac{N}{N - \sum A_k}}) \neq 0 \) for \( s \in U_\infty \).

Noting that \( x_1 = 1 \Rightarrow dx_1 = 0 \) and inserting the expressions for \( x_0, x_1, x_2, dx_0 \) into that of \( \omega_n \), we get

\[
((\pi_\ast \varphi_{\infty_1})_\ast \omega_n)(s) = C(s) \left( \frac{n \sum A_k - N - N \sum a_i}{(N - N + \sum A_k)} - 1 \right) ds,
\]

where \( C \) is regular on \( U_\infty \) and \( C(s) \neq 0 \) for \( s \in U_\infty \). Therefore

\( \pi_\ast \omega_n \) is regular at \( Q \)

\[
\sum_{i=0}^{r} a_i \leq \frac{n \sum_{k=0}^{r} A_k - (N, N - \sum_{k=0}^{r} A_k)}{N}.
\]

- Case 2: \( N - \sum_{k=0}^{r} A_k < 0 \)

The composition of \( \pi \) with the local parametrization \( \varphi_{\infty_2} \) of \( X_N \) at each preimage \( Q \) of \( (0 : 0 : 1) \) is given, for \( s \in U_\infty \), by (cf (1.19) §1.4.5)

\[
\begin{align*}
\pi_\ast & \omega_n \quad (s) \leftrightarrow \left( s^{\frac{\sum A_k}{(N - N + \sum A_k)}} : s^{\frac{N + \sum A_k}{(N - N + \sum A_k)}} : h(s^{\frac{N}{(N - N + \sum A_k)}})^{\frac{1}{N}} \right).
\end{align*}
\]

Replacing \( x_0, x_1, x_2, dx_0, dx_1 \) by their expressions in \( s \), we get

\[
((\pi_\ast \varphi_{\infty_2})_\ast \omega_n)(s) = C(s) \left( \frac{n \sum A_k - N - N \sum a_i}{(N - N + \sum A_k)} - 1 \right) ds,
\]

where \( C \) is regular on \( U_\infty \) and \( C(s) \neq 0 \) for \( s \in U_\infty \). Thus, we get

\( (\pi_\ast \omega_n)(s) \) is regular at \( Q \)

\[
\sum_{i=0}^{r} a_i \leq \frac{n \sum_{k=0}^{r} A_k - (N, -N + \sum_{k=0}^{r} A_k)}{N} - 1.
\]
1.4. Abelian Varieties for Appell-Lauricella Hypergeometric Series

Summary 2. Since \((N, -N + \sum_{k=0}^r A_k) = (N, N - \sum A_k)\), we can summarize these conditions by saying that the pull-back under \(\pi : X_N \to C_N\) of the differential form

\[\omega_n(x, y) = \frac{\prod_{i=0}^r (x - \lambda_i)^{a_i} dx}{y^n}\]
on \(C_N\) is regular on \(X_N\) if and only if

\[
\sum_{i=0}^r a_i \leq n \sum_{k=0}^r A_k - (N, N - \sum_{k=0}^r A_k) - 1
\]
\[a_j \geq \frac{nA_j + (N, A_j)}{N} - 1, \forall j \in \{0, \ldots, r\}.\] (1.23)

These conditions will be referred to as the regularity conditions for \(\pi^*\omega_n\).

Remark 36. We would like now to show that the pull-back \(\pi^*\omega_n\) of a differential form \(\omega_n(x, y) = y^{-n} \prod_{i=0}^r (x - \lambda_i)^{a_i} dx\) belongs to the isotypical component \(V_n\) of character \(\chi_n\), if it satisfies the above conditions. If it is the case, \(\pi^*\omega_n \in \Omega^1[X_N]\) and it remains to study the action of \(\mu_N\) on \(\pi^*\omega_n\), for a fixed \(n \in \{0, \ldots, N - 1\}\).

Let \(\xi \in \mu_N\), then

\[\xi \cdot \pi^*\omega_n = \Phi_\xi^*(\pi^*\omega_n) = (\pi \circ \Phi_\xi)^*\omega_n = (\varphi_\xi \circ \pi)^*\omega_n\text{ by (1.20), §1.4.7.}\]

Now, for \(P \in X_N\), we have

\[
((\varphi_\xi \circ \pi)^*\omega_n)(P) = \omega_n((\varphi_\xi \circ \pi)(P)) \circ d_P(\varphi_\xi \circ \pi) = \omega_n(\varphi_\xi(\pi(P))) \circ d_P(\varphi_\xi \circ \pi) = \xi^n \omega_n(\pi(P)) \circ d_P \pi = \xi^n(\pi^*\omega_n)(P).
\]

Hence, for every \(\xi \in \mu_N\), we have

\[\xi \cdot (\pi^*\omega_n) = \chi_n(\xi) \pi^*\omega_n.\]

This shows that \(\pi^*\omega_n \in V_n\).
Dimension of $V_n$

Let $n$ be fixed in $\{0, \ldots, N - 1\}$. In order to determine the dimension of $V_n$, we will count the number of elements in a maximal family of linearly independent differential forms of the form $y^{-n} \prod_{i=0}^{r}(x - \lambda_i)^{a_i} dx$, where $n, a_0, \ldots, a_r \in \mathbb{Z}$ and satisfy the regularity conditions (1.23). Since $a_0, \ldots, a_r$ are integers and according to the regularity conditions, the maximal possible value $(\sum_{i=0}^{r} a_i)_{\max}$ of $\sum_{i=0}^{r} a_i$ and the minimal possible value $(a_j)_{\min}$ of $a_j$, $j \in \{0, \ldots, r\}$, are given by

\[
(\sum_{i=0}^{r} a_i)_{\max} = \left[ \frac{n \sum A_k - (N, N - \sum A_k)}{N} - 1 \right] \quad \text{and} \quad (a_j)_{\min} = -\left[ 1 - \frac{nA_j + (N, A_j)}{N} \right], \quad j \in \{0, \ldots, r\},
\]

where $[x]$ denotes the integral part of $x$.

Terminology 3. Let $x \in \mathbb{R}$, then $x$ admits a unique decomposition as

\[ x = [x] + \langle x \rangle, \]

where $[x] \in \mathbb{Z}$ and $\langle x \rangle \in [0, 1)$ are respectively called the integral part and the fractional part of $x$.

Write further $(\sum_{i=0}^{r} a_i)_{\min} := \sum_{i=0}^{r} (a_i)_{\min}$ and $\ell := (\sum_{i=0}^{r} a_i)_{\max} - (\sum_{i=0}^{r} a_i)_{\min}$.

If $\ell \geq 0$, there is at least one solution. Write $\omega_{\min}$ for the solution where each $a_i$ is minimal. Then $V_n = \langle x^k \omega_{\min} \rangle_{k=0, \ldots, \ell}$ and $\dim V_n = \ell + 1$. Indeed, one verifies that each possible value for $\sum_{i=0}^{r} a_i$ brings exactly one element in the maximal family of linearly independent differential forms. For instance, if $\exists j \in \{0, \ldots, r\}$ such that $(a_j)_{\min} + 1$ and $(\sum_{i=0}^{r} a_i)_{\min} + 1$ satisfy the regularity conditions, then

\[ y^{-n}(x - \lambda_j)^{(a_j)_{\min} + 1} \prod_{i \neq j}(x - \lambda_i)^{(a_i)_{\min}} = x\omega_{\min} - \lambda_j \omega_{\min} \in \omega_{\min}, x\omega_{\min} > . \]

Note that this is independent of $j$ and conclude by induction on $\ell$.

Theorem 1.10. Let $X_N$ be the curve defined in Theorem 1.8, §1.4.6, and recall (cf §1.4.7) that the space $\Omega^1[X_N]$ of regular differential 1-forms on $X_N$ furnishes a linear representation of $\mu_N$. Then, for $n \in \{0, \ldots, N - 1\}$, the isotypical component $V_n$ of character $\chi_n$ has dimension

\[
\dim V_n = \begin{cases} 
  d_n & \text{if } d_n > 0 \\
  0 & \text{otherwise,}
\end{cases}
\]
where
\[ d_n := \left[ n \sum A_k - (N, N - \sum A_k) \right] + \sum_{\ell=0}^{r} \left[ 1 - \frac{n A_\ell + (N, A_\ell)}{N} \right]. \]

**Proof.** Use \([x - 1] = [x] - 1\) to show that \(d_n = \ell + 1\) and apply the above reasoning.

**Remark 37.** If \(\dim V_n = 0\), then \(V_n\) does not appear in the canonical decomposition (1.21), §1.4.7, of \(\Omega^1[X_N]\).

**Remark 38.** Since by Definition 1.45, we have \(g[X_N] = \dim_{\mathbb{C}}(\Omega^1[X_N])\), Theorem 1.8 and Theorem 1.10 together imply the following relation
\[ 1 + \frac{1}{2} (rN - (N, N - \sum_{k=0}^{r} A_k) - \sum_{j=0}^{r} (N, A_j)) = \sum_{n \in \{0, \ldots, N-1\}} d_n. \]

**Dimension of \(V_n\), \((n, N) = 1\)**

Here will be used the conditions \(N \nmid A_0, \ldots, A_r, \sum_{k=0}^{r} A_k\). The goal here is to transform the formula for \(\dim V_n\) (cf Theorem 1.10) in the case where \((n, N) = 1\) into a more treatable form. Some preparatory lemmata are given in order to prove Theorems 1.15 and 1.16.

**Lemma 1.11.** Let \(x \in \mathbb{R}\), \(\ell \in \mathbb{Z}\), \(N, A \in \mathbb{N}\) and \(n \in \{0, \ldots, N-1\}\). Then we have

1. \([x + \ell] = [x] + \ell\),
2. \([x] = x - \langle x \rangle\),
3. \(\langle x + \ell \rangle = \langle x \rangle\).
4. If \(x \notin \mathbb{Z}\), then \(\langle -x \rangle = 1 - \langle x \rangle\).
5. If \(N \nmid A\) and \((n, N) = 1\), then \(\langle \frac{nA - (N, A)}{N} \rangle = \frac{nA}{N} - \frac{(N, A)}{N}\).
6. If \(N \nmid A\) and \((n, N) = 1\), then \(\left\lfloor \frac{nA + (N, A)}{N} \right\rfloor - 1\).
Proof. The first four points follow directly from the definitions. For point 5., write
\[ N' := \frac{N}{(N, A)}, \quad A' := \frac{A}{(N, A)}, \quad \text{and} \quad nA' = kN' + r, \quad \text{with} \quad k \in \mathbb{Z} \quad \text{and} \quad r \in \{1, \ldots, N'-1\}. \]

Note that \( r \neq 0 \), because \( N' \nmid nA \). Then
\[
\frac{nA'}{N'} = k + \frac{r}{N'} \quad \text{and} \quad \frac{nA' - 1}{N'} = k + \frac{r - 1}{N'}. 
\]

Since \( r - 1 \in \{0, \ldots, N' - 2\} \) and \( k \in \mathbb{Z} \), we have
\[
\frac{nA'}{N'} = \frac{r}{N'} \quad \text{and} \quad \frac{nA' - 1}{N'} = \frac{r - 1}{N'}. 
\]

This implies \( \frac{nA' - 1}{N'} = \frac{(nA') - 1}{N} \) or equivalently
\[
\frac{nA - (N, A)}{N} = \frac{nA}{N} - \frac{(N, A)}{N}. 
\]

6. With the same notations and hypotheses as above, we have \( \lfloor \frac{nA' - 1}{N} \rfloor = k \). Now,
\[
\lfloor \frac{-nA' - 1}{N} \rfloor = \lfloor -k - \frac{r + 1}{N'} \rfloor = -k + \lfloor -\frac{r + 1}{N'} \rfloor = -k - 1, 
\]
because \( -\frac{r + 1}{N'} \in [-1, 0) \). Hence,
\[
\lfloor \frac{-nA + (N, A)}{N} \rfloor = \lfloor \frac{nA - (N, A)}{N} - 1 \rfloor. 
\]

Lemma 1.12. Let \( n \in \{0, \ldots, N - 1\}, \quad N, A_0, \ldots, A_r \in \mathbb{N} \) and suppose \((n, N) = 1\) and \( N \nmid A_0, \ldots, A_r, \sum_{k=0}^{r} A_k \). Then, \( \forall j \in \{0, \ldots, N\} \), we have

1. \( \lfloor \frac{-nA_j + (N, A_j)}{N} \rfloor = \lfloor \frac{nA_j}{N} \rfloor - \lfloor \frac{nA_j}{N} \rfloor - 1 \) and
2. \( \lfloor \frac{n\sum_{k=0}^{r} A_k - (N, N - \sum_{k=0}^{r} A_k)}{N} \rfloor = \lfloor \frac{n\sum_{k=0}^{r} A_k}{N} \rfloor - \lfloor \frac{n\sum_{k=0}^{r} A_k}{N} \rfloor - 1 \).

Proof. The reference number refers to Lemma 1.11.

1. Fix \( j \in \{0, \ldots, N\} \), then
\[
\lfloor \frac{-nA_j + (N, A_j)}{N} \rfloor = \lfloor \frac{nA_j - (N, A_j)}{N} \rfloor - 1 \quad \text{by 5.} 
\]
\[
= \lfloor \frac{nA_j}{N} \rfloor - \lfloor \frac{nA_j}{N} \rfloor - 1 \quad \text{by 2.} 
\]
\[
= \lfloor \frac{nA_j}{N} \rfloor + \lfloor \frac{-(N, A_j)}{N} \rfloor - 1 \quad \text{by 5.} 
\]
\[
= \lfloor \frac{nA_j}{N} \rfloor - \lfloor \frac{nA_j}{N} \rfloor - 1. 
\]
2. \[
\left[ \frac{n \sum A_k - (N, N - \sum A_k)}{N} \right] = \frac{n \sum A_k - (N, N - \sum A_k)}{N} - \frac{n \sum A_k - (N, N - \sum A_k)}{N} - \left( \frac{n \sum A_k - (N, N - \sum A_k)}{N} \right) - \frac{n \sum A_k - (N, N - \sum A_k)}{N} - \left( \frac{n \sum A_k - (N, N - \sum A_k)}{N} \right).
\]

by 2.

Proposition 1.13. If \((n, N) = 1\) and \(N \nmid A_0, ..., A_r, \sum_{k=0}^r A_k\), then the integer \(d_n\) defined in Theorem 1.10 is equal to

\[
d_n = -\left( \frac{n \sum_{k=0}^r A_k}{N} \right) + \sum_{i=0}^r \left( \frac{n A_i}{N} \right).
\]

Proof.

\[
d_n = \left[ \frac{n \sum A_k - (N, N - \sum A_k)}{N} \right] + \sum_{i=0}^r \left[ 1 - \frac{n A_i + (N, A_i)}{N} \right]
\]

\[
= \left[ n \sum A_k - (N, N - \sum A_k) + r + 1 + \sum \left( \frac{n A_i + (N, A_i)}{N} \right) \right]
\]

\[
= \left( \frac{n \sum A_k}{N} - \left( \frac{n \sum A_k}{N} \right) + r + 1 + \sum \left( \frac{n A_i}{N} \right) - \sum \frac{n A_i}{N} - (r + 1) \right)
\]

\[
= -\left( \frac{n \sum A_k}{N} \right) + \sum \left( \frac{n A_i}{N} \right).
\]

The second and third equalities are respectively obtained by applying 1. Lemma 1.11 and Lemma 1.12. □

Under the hypotheses \((n, N) = 1\) and \(N \nmid A_0, ..., A_r, \sum_{k=0}^r A_k\), we still can get a better result on \(\dim V_n\). For this, we will use the following lemma.

Lemma 1.14. Let \(x_0, ..., x_r\) be real numbers. Then we have

\[
-\left( \sum_{i=0}^r x_i \right) + \sum_{i=0}^r \{x_i\} \in \{0, ..., r\}.
\]

Proof. First remark that

\[
\langle \sum x_i \rangle = \langle \sum [x_i] + \sum \{x_i\} \rangle = \langle \sum \{x_i\} \rangle,
\]

because of \(\sum [x_i] \in \mathbb{Z}\) applied to 3. Lemma 1.11. Then we have

\[
-\langle \sum x_i \rangle + \sum \{x_i\} = -\langle \sum \{x_i\} \rangle + \sum \langle x_i \rangle = \lfloor \sum \langle x_i \rangle \rfloor =: c,
\]
by the above and by definition. Pay attention to the fact that \( c \) is an integer. Since 
\[ \sum(x_i) \geq 0 \text{ and } -\left(\sum x_i\right) \in (-1, 0], \]
the integer \( c \) cannot be negative, because 
\(-1\) cannot be reached. Moreover, \( c \leq r \), because \( \sum(x_i) < r + 1. \) Hence \( c \in \{0, \ldots, r\}.\) \(\square\)

**Theorem 1.15.** Let the notations be as in Theorem 1.10 and suppose \((n, N) = 1\) and \(N \nmid A_0, \ldots, A_r, \sum_{k=0}^r A_k.\) Then we have

\[
\dim V_n = -\left\langle \frac{n \sum_{k=0}^r A_k}{N} \right\rangle + \sum_{i=0}^r \left\langle \frac{n A_i}{N} \right\rangle.
\]

**Proof.** By Proposition 1.13, we have

\[
d_n = -\left\langle \frac{n \sum_{k=0}^r A_k}{N} \right\rangle + \sum_{i=0}^r \left\langle \frac{n A_i}{N} \right\rangle
\]

and by Lemma 1.14, we know that \( d_n \in \{0, \ldots, r\}. \) Hence, by Theorem 1.10, 
\( \dim V_n = d_n.\) \(\square\)

\(\dim V_n + \dim V_{N-n}, (n, N) = 1\)

**Theorem 1.16.** Let the notations be as in Theorem 1.10 and suppose \((n, N) = 1\) and \(N \nmid A_0, \ldots, A_r, \sum_{k=0}^r A_k.\) Then we have

\[
\dim V_n + \dim V_{N-n} = r.
\]

**Proof.**

\[
\dim V_{N-n} = -\left\langle \frac{(N-n) \sum_{k=0}^r A_k}{N} \right\rangle + \sum_{i=0}^r \left\langle \frac{(N-n) A_i}{N} \right\rangle \quad (\text{by Theorem } 1.15)
\]

\[= -\left\langle \sum A_k - \frac{n \sum_{k=0}^r A_k}{N} \right\rangle + \sum (\frac{n A_i}{N} - \frac{A_i}{N}) \]

\[= -\left\langle -\frac{n \sum A_k}{N} \right\rangle + \sum (\frac{n A_i}{N}) \quad (\text{by 3. Lemma } 1.11)
\]

\[= -1 + \left\langle \frac{n \sum_{k=0}^r A_k}{N} \right\rangle + r + 1 - \sum \left\langle \frac{n A_i}{N} \right\rangle \quad (\text{by 4. Lemma } 1.11)
\]

\[= r - (-\left\langle \frac{n \sum_{k=0}^r A_k}{N} \right\rangle + \sum \left\langle \frac{n A_i}{N} \right\rangle) \quad (\text{by Theorem } 1.15).
\]
1.4.9 New Forms

We are now approaching our goal of constructing an abelian variety on which \( \int \pi^* \omega \) lives as a period. We could have taken the Jacobian variety of \( X_N \), but its dimension (equals to \( \dim \Omega^1[X_N] = g[X_N] \)) would have depended not only on \( N \) but also on \( A_0, ..., A_r \) (cf Theorem 1.8). That is the reason why we will restrict ourselves to a abelian subvariety of \( Jac(X_N) \), whose dimension depends on \( N \) only.

In order to define this subvariety, we will select regular differential forms on \( X_N \), which “do not come from under” and are therefore called “new”. This will be made more precise.

First of all, let’s work at the level of the singular curve \( C_N \), because it is here possible to work with explicit expressions for the differential forms, in coordinates that we choose to be affine.

Let \( d \in \mathbb{N} \). If \( d \mid N \), then we have a well-defined morphism

\[
\psi_d : \quad C_N \rightarrow C_d
\]

\[
(x, y) \mapsto (x, y^d)
\]

\[
\infty \mapsto \infty.
\]

Let \( (u, v) \in C_d \) be an affine point, then

\[
\psi_d^{-1}\{(u, v)\} = \{(u, v_0), (u, \zeta_N^d v_0), ..., (u, \zeta_N^{d-1} v_0)\},
\]

where \( v_0 \) is any fixed \( \frac{N}{d} \)-th root of \( v \) and \( \zeta_N^d := e^{2\pi i/d} \). We see that there is an open dense subset of \( C_d \) of points having \( \frac{N}{d} \) preimages. The other points have exactly one preimage. (\( \psi_d \) is actually a ramified topological covering.) The set of preimages of a point \( P \in C_d \) is called the fiber over \( P \) with respect to \( \psi_d \).

Remember that the group \( \mu_N \) of \( N \)-th roots of unity acts on \( C_N \) (cf §1.4.7) and remark that the subgroup \( I_d := \{\zeta_N^d\}, \zeta_N := e^{2\pi i/N} \), of index \( d \) in \( \mu_N \) acts transitively on each fiber by permutation.

In the same way we had defined the induced action of \( \mu_N \) on \( \Omega^1[X_N] \) (cf §1.4.7), we can deduce from the action

\[
\mu_N \times C_N \rightarrow C_N
\]

\[
(\zeta, P) \mapsto \varphi_\zeta(P)
\]
of $\mu_N$ on $C_N$ an action of $\mu_N$ on the set $\Phi[C_N]$ of differential forms on $C_N$ by setting

$$
\mu_N \times \Phi[C_N] \longrightarrow \Phi[C_N]
$$

$$(\zeta, \omega) \mapsto \varphi_{\zeta}^* \omega.
$$

Now, suppose that you have a differential form $\eta$ on $C_d$. It is clear that its pull-back $\psi_d^* \eta$ on $C_N$ is invariant under the action of the subgroup $I_d$, because $I_d$ preserves the fibers.

The converse is more subtle. Let $\omega \in \Phi[C_N]$ be invariant under the action of $I_d$. Does $\omega$ define a differential form $(\psi_d)_* \omega$ on $C_d$? The answer to this question is positive, because $I_d$ acts transitively on each fiber. Hence, for $Q \in C_d$, we can define $((\psi_d)_* \omega)(Q)$ to be the unique linear form on $\theta_{C_d,Q}$ such that, for $P \in C_N$ with $\psi_d(P) = Q$, $((\psi_d)_* \omega)(Q) \circ d_P \psi_d = \omega(P)$. This is well-defined, because $\forall P' \in C_N$ with $\psi_d(P') = Q$, $\exists \xi \in I_d$ such that $\varphi_{\xi}(P') = P$ and then

$$
\omega(P) = \omega(\varphi_{\xi}(P')) \circ d_P \varphi_{\xi} = (\varphi_{\xi}^* \omega)(P') = \omega(P'),
$$

by invariance of $\omega$ under $I_d$. Remark further that $\psi_d^*((\psi_d)_* \omega) = \omega$. Indeed, let $P \in C_N$, then

$$
\psi_d^*((\psi_d)_* \omega)(P) = ((\psi_d)_* \omega)(\psi_d(P)) \circ d_P \psi_d = \omega(P).
$$

The so-defined differential form $(\psi_d)_* \omega \in \Phi[C_d]$ is called the push-forward of $\omega$ with respect to $\psi_d$.

For a differential form $\omega$ on $C_N$ and $d|N$, we have shown

$$
\omega \text{ is fixed under the action of } I_d \text{ on } \Phi[C_N]
$$

$$
\iff \exists \eta \in \Phi[C_d] \text{ such that } \psi_d^* \eta = \omega.
$$

Let's now consider the differential form $\omega_n(x, y) = y^{-n} \prod_{i=0}^n (x - \lambda_i)^{a_i} \, dx$, $a_i \in \mathbb{Z}$, on $C_N$. Under which condition on $n$ is $\omega_n$ fixed by the action of $I_d$? Well,

$$
\forall \xi \in I_d, \varphi_{\xi}^* \omega_n = \omega_n \iff \forall \xi \in I_d, \xi^n \omega_n = \omega_n
$$

$$
\iff \forall k \in \{0, \ldots, N/d - 1\}, (\zeta_N^d)^{kn} \omega_n = \omega_n
$$

$$
\iff \exists \ell \in \mathbb{Z} \text{ s.t. } n = \ell N/d.
$$
The differential forms which satisfy this for a \( d \) dividing \( N \) and different from \( N \) are the ones we want to get rid of, because “they come from under”. This is equivalent to the fact that \( (N, n) \neq 1 \). Indeed, if the above equivalent conditions hold, \( \frac{N}{d} \) is \( \neq 1 \) and divides \( N \) and \( n \). Conversely, suppose that \( (N, n) \neq 1 \), then \( \omega_n \) is fixed under the action of \( I_{\frac{N}{(N,n)}} \). Hence, we define

**Definition 1.73.** A differential form \( \omega_n \) (resp. \( \pi^* \omega_n \)) on \( C_N \) (resp. \( X_N \)) such that \( (n, N) = 1 \) and the linear combinations of such differential forms are said to be **new**. The vector subspace of \( \Omega^1[X_N] \) consisting of all new forms on \( X_N \) which are holomorphic is

\[
\Omega^1[X_N]_{new} := \bigoplus V_n,
\]

where the sum is taken over the \( n \in \{0, \ldots, N-1\} \) such that \( \dim V_n > 0 \) and \( (n, N) = 1 \).

### 1.4.10 New Jacobian

In the previous paragraph, we have defined a vector subspace \( \Omega^1[X_N]_{new} \) of \( \Omega^1[X_N] \), consisting of the regular differential forms on \( X_N \), which are not the pull-back of a differential form on \( X_d \) for any \( d \) dividing \( N \).

According to Remark 25, §1.3.16, recall that the Jacobian variety \( Jac(X_N) \) of \( X_N \) is the following abelian variety

\[
\Omega^1[X_N]^*/\iota(H_1(X_N(\mathbb{C}), \mathbb{Z})).
\]

By Remark 24, §1.3.15, the vector subspace \( \Omega^1[X_N]_{new} \) defines an abelian subvariety of \( Jac(X_N) \). It will be called the **New Jacobian** of \( X_N \) and denoted by \( Jac_{new}(X_N) \).

Furthermore, \( Jac_{new}(X_N) \) has a complement \( C \) (as abelian variety) such that the abelian variety \( Jac(X_N) \) is isogenous to

\[
Jac_{new}(X_N) \oplus C
\]

and \( \dim(Jac(X_N)) = \dim(Jac_{new}(X_N)) + \dim C \). The dimension of the abelian variety \( Jac_{new}(X_N) \) equals that of the \( \mathbb{C} \)-vector space \( \Omega^1[X_N]_{new} \). Because

\[
\dim(\Omega^1[X_N]_{new}) = \sum_{0<n<N} \dim(V_n) = \frac{1}{2} \sum_{0<n<N} (\dim V_n + \dim V_{N-n})
\]
and because of the general assumptions \( N \nmid A_0, ..., A_r, \sum_{k=0}^{r} A_k \), we have \( \dim V_n + \dim V_{N-n} = r \) by Theorem 1.16, §1.4.8. Hence,

\[
\dim \text{Jac}_{\text{new}}(X_N) = \frac{r \varphi(N)}{2},
\]

where \( \varphi(N) := \sum_{(n,N)=1}^{\infty} 1 \) is the Euler function.

### 1.5 Abelian Varieties for Gauss’ Hypergeometric Series

#### 1.5.1 Motivation

Consider a Gauss’ hypergeometric series \( F(a, b, c; z) \), where \( a, b, c \in \mathbb{Q}, c \neq 0, -1, -2, ... \) and \( z \in \mathbb{C} \). If \( |z| < 1 \), the series converges and if \( \Re(c) > \Re(b) > 0 \), we can write

\[
F(a, b, c; z) = \frac{\mathcal{P}(z)}{\mathcal{P}(0)},
\]

where \( \mathcal{P}(z) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx \). As explained in Section 1.2, our goal is to find an abelian variety on which a given algebraic multiple of \( \mathcal{P}(z) \) is a period. In this section, we treat the case \( z \neq 0, 1 \). Let \( N \) be the smallest common denominator of \( a, b, c \) and set

\[
A := N(1-b), \quad B := N(1+b-c), \quad C := Na.
\]

Then \( A, B, C \in \mathbb{Z}, \ N \in \mathbb{N} \) and the affine equation

\[
y^N = x^A(1-x)^B(1-zx)^C
\]

defines an algebraic projective curve over \( \mathbb{C} \) denoted by \( C(N, z) \). With this notation, we have

\[
\mathcal{P}(z) = \int_0^1 \frac{dx}{y}.
\]
If $a, b, c - b \not\in \mathbb{Z}$, this integral can be replaced, up to a complex algebraic factor, by an integral $\int_{\gamma} \frac{dx}{y}$, where $\gamma$ is a closed path on $C(N, z)$ whose projection in $\mathbb{C}$ is a Pochhammer loop around $0$ and $1$. We have gained that this integral converges uniformly for every $z \in \mathbb{C} - \mathbb{R}_+$ without condition on $\text{Re}(b), \text{Re}(c)$.

Let $X(N, z)$ be the desingularization of $C(N, z)$, $\pi_z : X(n, z) \to C(N, z)$ the desingularization map and $\omega := \frac{dx}{y}$. If $\pi_z^* \omega$ is regular on $X(N, z)$ and if $\delta$ represents a class $[\ell]$ in $H_1(X(N, z)(\mathbb{C}), \mathbb{Z})$ such that $(\pi_z)_* \delta = \gamma$, then the integral

$$\int_{\delta} \pi_z^* \omega = \int_{(\pi_z)_* \delta} \omega$$

is independent of the choice of the representant of $[\ell]$ (cf §1.3.13) and is a period on $\text{Jac}(X(N, z))$. We will actually define an abelian subvariety on which $\int_{\delta} \pi_z^* \omega$ is still a period.

Throughout this section, we will assume that

$$N \not\divides A, B, C, A + B + C,$$

what implies

$$a, b, c - b, c - a \not\in \mathbb{Z}.$$

In prevision of Section 1.6, we will also assume

$$c \not\in \mathbb{Z}.$$

The excluded cases correspond to hypergeometric series, whose transcendental nature is well understood (cf [Wo88] §3). Moreover, we will suppose the curve to be irreducible, i.e.

$$(N, A, B, C) = 1.$$

In the next paragraph, we show that $X(N, z)$ is isomorphic to a given algebraic curve of the family considered in Section 1.4. This allows us to use in §1.5.3 our results of Section 1.4 to calculate the genus of $X(N, z)$, a basis of regular differential 1-forms and to define the New Jacobian of $X(N, z)$ on which $\int_{\delta} \pi_z^* \omega$ lives as a period, if $\pi_z^* \omega$ is regular on $X(N, z)$. This will respectively correct and prove some assertions of Wolfart, [Wo88].
1.5.2 The Isomorphism

As announced in the Motivation, the aim is here to find an isomorphism (of algebraic curves) between the projective algebraic curve $X(N, z)$ and the desingularization of the projective curve with affine equation

$$y^N = \prod_{i=0}^{2}(x - \lambda_i)^{A_i},$$

where $A_0, A_1, A_2 \in \mathbb{Z}$ and $N \in \mathbb{N}$. Let's first work at the level of the singular curves. Recall that $C(N, z)$ is affinely defined by

$$y^N = x^A(1 - x)^B(1 - zx)^C,$$

where $z \in \mathbb{C} - \{0, 1\}, A, B, C \in \mathbb{Z}, N \in \mathbb{N}, N \neq 1, A + B + C$. Define $C_N$ to be the projective curve with affine equation

$$y^N = x^{N-A-B-C}(x - 1)^B(x - z)^C,$$

$X_N$ its desingularization and $\pi : X_N \rightarrow C_N$ the desingularization map. Then the following map is well-defined

$$\kappa : C(N, z) \rightarrow C_N$$

$$(x_0 : x_1 : x_2) \mapsto (x_1 : x_0 : x_2).$$

Indeed, using the notations of §1.4.3 and the coordinates $(u_0 : u_1 : u_2)$ on $C_N$, we have $N - \sum_{k=0}^{2} A_k = A$ and

$$C_N : u_2^N = u_0^A u_1^{N-A-B-C}(u_1 - u_0)^B(u_1 - zu_0)^C.$$

For the moment, let's accept negative exponents and simply write a factor with negative exponent on the other side of the equality. With the same conventions, the equation of $C(N, z)$ in projective coordinates reads

$$C(N, z) : x_2^N = x_0^{N-A-B-C} x_1^A(x_0 - x_1)^B(x_0 - zx_1)^C.$$

Suppose that $(x_0 : x_1 : x_2) \in C(N, z)$ and write $(u_0 : u_1 : u_2) := (x_1 : x_0 : x_2)$ then $u_2^N = u_1^{N-A-B-C} u_0^A(u_1 - u_0)^B(u_1 - zu_0)^C$, hence $(u_0 : u_1 : u_2) \in C_N$ and $\kappa$ is well-defined.
1.5. Abelian Varieties for Gauss' Hypergeometric Series

$k$ is clearly a morphism of algebraic varieties, and because $k^{-1} = k$, it is also an isomorphism. This implies that the composition $k \circ \pi : X(N, z) \to X_N$ is a birational morphism. Since $X(N, z)$ is moreover nonsingular, it provides a model of the desingularization of $C_N$ and there exists a unique isomorphism $\tilde{k}$ from $X(N, z)$ to $X_N$ such that $\pi \circ \tilde{k} = k \circ \pi$ (cf. [Ful], Ch7§5, Thm3). Hence, $\Omega^1[X_N]$ and $\Omega^1(X(N, z))$ are in bijection under $\tilde{k}^*$. Remembering that in §1.4.2, we had reduced to positive exponents, we can here also suppose $N - A - B - C, B, C > 0$.

1.5.3 Genus and New Jacobian of $X(N, z)$

Let $X(N, z)$ with $z \neq 0, 1$ be the curve defined in the introduction of this section. In Paragraph 1.5.2, we have seen that $X(N, z)$ is isomorphic to the desingularization $X_N$ of the projective curve $C_N$ with affine equation

$$y^N = x^{N-A-B-C}(x-1)^B(x-z)^C$$

and that we can suppose

$$N - A - B - C, B, C > 0.$$

Moreover, the irreducibility of $X(N, z)$ is equivalent to that of $X_N$, so that we are allowed to apply the results of Section 1.4 to $X_N$.

Genus of $X(N, z)$

Because $X(N, z)$ and $X_N$ are isomorphic, we have

$$\chi(X(N, z)) = \chi(X_N) \quad \text{and} \quad g[X(N, z)] = g[X_N].$$

Applying Theorem 1.8, §1.4.8, we get

$$\chi(X(N, z)(\mathbb{C})) = -2N + (N, A) + (N, B) + (N, C) + (N, N - A - B - C)$$

and

$$g[X(N, z)] = N + 1 - \frac{1}{2}[(N, A) + (N, B) + (N, C) + (N, N - A - B - C)].$$
Regular Differential Forms on $X_N$

Write $\pi : X_N \to C_N$ for the desingularization map and, for $n \in \{0, \ldots, N-1\}$, let

$$\frac{x^{a_0}(x-1)^{a_1}(x-z)^{a_2}dx}{y^n}$$

define a (rational) differential form $\omega_n$ on $C_N$. Then by the regularity conditions (1.23), §1.4.8, $\pi^*\omega_n$ is regular on $X_N$ if and only if the following four conditions hold

$$a_0 \geq \frac{n(N-A-B-C)+(N,N-A-B-C)}{N} - 1,$$

$$a_1 \geq \frac{nB+(N,B)}{N} - 1,$$

$$a_2 \geq \frac{nC+(N,C)}{N} - 1,$$

$$a_0 + a_1 + a_2 \leq \frac{n(N-A)+(N,A)}{N} - 1.$$  (1.24)

If, for $n \in \{0, \ldots, N-1\}$, $V_n$ denotes the isotypical component of $\Omega^1[X_N]$ of character $\chi_n$, then, by Theorem 1.10, §1.4.8, we have

$$\dim V_n = \begin{cases} d_n & \text{if } d_n \geq 0 \\ 0 & \text{otherwise}, \end{cases}$$

where $d_n$ is equal to

$$d_n = \left[ \frac{n(A+B+C)-(N,N-A-B-C)}{N} \right] + \left[ 1 - \frac{nA+(N,A)}{N} \right] + \left[ 1 - \frac{nB+(N,B)}{N} \right] + \left[ 1 - \frac{nC+(N,C)}{N} \right]$$

(1.25)

(in order to get $d_n$ in this form use that, if $k \in \mathbb{Z}$, then $[x+k] = [x]+k$).

**Remark 39.** If $\eta_n$ denotes the (rational) differential form $y^{-n}x^{b_0}(1-x)^{b_1}(1-zx)^{b_2}dx$ on $C(N, z)$, the conditions (1.24) can be used to determine when $\pi^*_z\eta_n$ is holomorphic on $X(N, z)$. Indeed, $\pi^*_z\eta_n$ is holomorphic on $X(N, z)$ exactly when $\pi^*(\kappa^{-1})^*\eta_n$ is holomorphic on $X_N$, that is, when the following four conditions hold

$$b_0 \geq \frac{nA+(N,A)}{N} - 1,$$

$$b_1 \geq \frac{nB+(N,B)}{N} - 1,$$

$$b_2 \geq \frac{nC+(N,C)}{N} - 1,$$

$$b_0 + b_1 + b_2 \leq \frac{n(A+B+C)-(N,N-A-B-C)}{N} - 1.$$
This corrects the assertion of Wolfart ([Wo88] §4) on the holomorphy conditions for differential 1-forms, while his assertion on the dimension of the isotypical components $V_n$ for $n \in \{0, \ldots, N - 1\}$ is corrected by (1.25).

Supposing now that $(n, N) = 1$ and using Theorem 1.15, §1.4.8, we get

$$\dim V_n = \left\lfloor \frac{n(N-A)}{N} \right\rfloor + \left\lfloor \frac{n(N-A-B-C)}{N} \right\rfloor + \left\lfloor \frac{nB}{N} \right\rfloor + \left\lfloor \frac{nC}{N} \right\rfloor$$

and, using that, $\forall x \in \mathbb{R} \setminus \mathbb{Z}, \; \langle -x \rangle = 1 - \langle x \rangle$,

$$\dim V_n = \left\lfloor \frac{nA}{N} \right\rfloor + \left\lfloor \frac{nB}{N} \right\rfloor + \left\lfloor \frac{nC}{N} \right\rfloor - \left\lfloor \frac{n(A + B + C)}{N} \right\rfloor.$$ 

And finally, applying Theorem 1.16, §1.4.8, we have

$$\dim V_n + \dim V_{N-n} = 2$$

in the case where $(n, N) = 1$.

**New Jacobian of $X(N, z)$**

In paragraph 1.4.9, we have selected so-called new forms on $X_N$. In §1.4.10, it has been shown that the vector space $\Omega^1[X_N]_{\text{new}}$ generated by these new forms has dimension $\frac{1}{2}\varphi(N)$ (here $r = 2$) and the abelian subvariety $J_{ac_{\text{new}}}(X_N)$ of $J_{ac}(X_N)$ generated by $\Omega^1[X_N]_{\text{new}}$ has been called the new Jacobian of $X_N$. If we set

$$\Omega^1[X(N, z)]_{\text{new}} := \tilde{\kappa}^*(\Omega^1[X_N]_{\text{new}}),$$

we have

$$\dim(\Omega^1[X(N, z)]_{\text{new}}) = \varphi(N),$$

because $\tilde{\kappa}$ is an isomorphism.

**Definition 1.74.** The abelian subvariety of $J_{ac}(X(N, z))$ generated by the vector subspace $\Omega^1[X(N, z)]_{\text{new}}$ of $\Omega^1[X(N, z)]$ will be called the **New Jacobian** of $X(N, z)$. It has dimension $\varphi(N)$ and will be denoted by $J_{ac_{\text{new}}}(X(N, z))$. 

Conclusion

Recall that the following diagram commutes (cf §1.5.2)

\[
\begin{array}{ccc}
X(N, z) & \xrightarrow{\kappa} & X_N \\
\downarrow \pi_z & & \downarrow \pi \\
C(N, z) & \xrightarrow{\kappa} & C_N,
\end{array}
\]

where \( \pi, \pi_z \) are morphisms and \( \kappa, \kappa \) isomorphisms. Write \( \omega_1 \) for the differential form \( (\kappa^{-1})^* \omega \) on \( C_N \), where \( \omega \) is the differential form \( \frac{dx}{y} \) on \( C(N, z) \) (hence \( \kappa^* \omega_1 = \omega \)). One verifies that \( \pi^* \omega_1 \) is a new form on \( X_N \).

Remember that our goal was to construct an abelian variety on which \( \int y \omega \) lives as a period, if \( y \) is a given cycle on \( C(N, z) \). Let \( \pi_z^* y \) be a (not unique) cycle on \( X(N, z) \) satisfying \( \pi_z^* y = y \).

If \( \pi^* \omega_1 \) is regular on \( X_N \), our task is achieved. Indeed, in this case, \( \kappa^*(\pi^* \omega_1) \) is regular on \( X(N, z) \) and is an element of \( \Omega^1[X(N, z)]_{new} \). Remark that

\[
\kappa^*(\pi^* \omega_1) = (\pi \circ \kappa)^* \omega_1 = (\kappa \circ \pi_z)^* \omega_1 = \pi_z^*(\kappa^* \omega_1) = \pi_z^* \omega.
\]

Hence, the integral

\[
\int_{\pi_z^* y} \kappa^*(\pi^* \omega_1) = \int_{\pi_z^* y} \pi_z^* \omega = \int y \omega
\]

is a period on the abelian variety \( Jac_{new}(X(N, z)) \) of dimension \( \varphi(N) \).

1.6 Abelian Varieties for the Beta Function

1.6.1 Motivation

We would like here to construct an abelian variety on which an algebraic multiple of the integral in the denominator of the integral representation of the hypergeometric series \( F(a, b, c; z) \) lives as a period. The procedure will be analogous as that in Section 1.5. That is, the integral \( P(0) \) will be written as \( \int \frac{dx}{y} \), where \( (x, y) \) lies on a
certain curve which is isomorphic to a curve of a subfamily of the family for which
we have had results in Section 1.4.

First recall that the integral representation of \( F(a, b, c; z) \) reads
\[
F(a, b, c; z) = \frac{\mathcal{P}(z)}{\mathcal{P}(0)},
\]
where \( \mathcal{P}(z) = \int_0^1 x^{b-1}(1 - x)^{c-b-1}(1 -zx)^{-a}dx \). In Section 1.5, we had treated
the case of \( \mathcal{P}(z) \) with \( z \neq 0, 1 \). Let’s now treat the case of \( \mathcal{P}(0) \). First remark that
\[
\mathcal{P}(0) = B(b, c-b),
\]
where \( B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1 - x)^{\beta-1}dx \) is the classical Beta function, which con¬
verges if \( \text{Re}(\alpha) > 0 \) and \( \text{Re}(\beta) > 0 \). Set \( M \in \mathbb{N} \) for the smallest common denomi¬
nator of \( b \) and \( c \). Set further
\[
P := M(1-b), \quad Q := M(1+b-c)
\]
and define \( X(M, 0) \) to be the desingularization of the projective curve \( C(M, 0) \)
defined over \( \mathbb{C} \) by the affine equation
\[
y^M = x^P(1-x)^Q,
\]
where \( P, Q \in \mathbb{Z} \). Write \( \omega \) for the differential form \( \frac{dx}{y} \) on \( C(M, 0) \), then we have
\[
\mathcal{P}(0) = \int_0^1 \omega.
\]
If \( b, c - b \notin \mathbb{Z} \) and if \( \gamma \) is the lifting on \( C(M, 0) \) of a Pochhammer loop in \( \mathbb{C} \)
around 0 and 1, then the integral \( \int_{\gamma} \omega \) is equal to an algebraic multiple of \( \int_0^1 \omega \) and
converges without restriction on \( \text{Re}(b), \text{Re}(c) \). If \( \delta \) is a cycle on \( X(M, 0) \) such that
\( (\pi_0)_* \delta = \gamma \) and \( \pi_0 : X(M, 0) \to C(M, 0) \) the desingularization map, then
\[
\int_{\delta} \pi_0^* \omega = \int_{(\gamma)_0} \omega.
\]
Consequently, it is for us sufficient to construct an abelian variety on which \( \int_{\delta} \pi_0^* \omega \)
lives as a period.

Before going further, let’s recall the hypotheses (cf §1.5.1). They read
\[
a, b, c \in \mathbb{Q} - \mathbb{Z} \quad \text{and} \quad a - c, b - c \notin \mathbb{Z}
\]
and imply
\[
M \nmid P, \quad Q, \quad P + Q.
\]
In particular, \( M \neq 1, \quad P + Q \).
1.6.2 The Isomorphism

Consider the algebraic projective curve $C_M$ affinely defined over $\mathbb{C}$ by

$$y^M = x^{M-P-Q} (x - 1)^Q,$$

where $M \in \mathbb{N}$, $P, Q \in \mathbb{Z}$. Then, the projective equation of $C_M$ reads

$$x_2^M = x_0^P x_1^{M-P-Q} (x_1 - x_0)^Q,$$

with the convention that a factor with negative exponent should be written on the other side of the equality. With the same convention, the projective equation of $C(M, 0)$ is

$$x_2^M = x_0^{M-P-Q} x_1^P (x_0 - x_1)^Q,$$

and we can define a map

$$\kappa : C(M, 0) \rightarrow C_M$$

$$(x_0 : x_1 : x_2) \mapsto (x_1 : x_0 : x_2).$$

This map is an isomorphism, because it is a morphism equal to its inverse.

Write now $\pi : X_M \rightarrow C_M$ for the desingularization of $C_M$. Since $\kappa \circ \pi_0$ and $\pi$ are birational maps from a nonsingular projective curve to $C_M$, there exists a unique isomorphism $\tilde{\kappa} : X(M, 0) \rightarrow \tilde{X}_M$ such that the following diagram commutes

$$\begin{array}{ccc}
X(M, 0) & \longrightarrow & X_M \\
\pi_0 \downarrow & & \downarrow \pi \\
C(M, 0) & \longrightarrow & C_M.
\end{array}$$

In particular, this shows that $X(M, 0)$ and $X_M$ have same Euler characteristic and genus, and that $\tilde{\kappa}^* \Omega^1[X_M] = \Omega^1[X(M, 0)]$. We will here also suppose $X(M, 0)$ and $X_M$ to be irreducible, i.e.

$$(M, P, Q) = 1.$$

Thanks to §1.4.2, we can also suppose that $M-P-Q$, $Q > 0$ and apply our results of Section 1.4.
1.6.3 Genus and New Jacobian of $X(M, 0)$

Genus of $X(M, 0)$

Applying Theorem 1.8, §1.4.8, we get

$$\chi(X(M, 0)(\mathbb{C})) = -M + (M, P) + (M, Q) + (M, M - P - Q)$$

and

$$g[X(M, 0)] = 1 + \frac{1}{2} [M - (M, P) - (M, Q) - (M, M - P - Q)].$$

Regular Differential Forms on $X_M$

For $n \in \{0, \ldots, M - 1\}$, let $\omega_n$ be the (rational) differential form on $C_M$ defined by

$$\frac{x^{a_0}(x - 1)^{a_1} dx}{y^n}.$$

By the regularity conditions (1.23), §1.4.8, $\pi^* \omega_n$ is regular on $X_M$ if and only if

$$a_0 \geq \frac{n(M - P - Q) + (M, M - P - Q)}{M} - 1,$$

$$a_1 \geq \frac{nQ + (M, Q)}{M} - 1,$$

$$a_0 + a_1 \leq \frac{n(M - P) - (M, P)}{M} - 1. \quad (1.26)$$

Remark 40. For $n \in \{0, \ldots, M - 1\}$, let $\eta_n$ be the (rational) differential 1-form $y^{-n} x^{b_0}(1 - x)^{b_1} dx$ on $C(M, 0)$. Then $\pi_0^* \eta_n$ is regular on $X(M, 0)$ if and only if $\pi^*(\kappa^{-1})^* \eta_n$ is regular on $X_M$. By the conditions (1.26), this is the case exactly when the following three conditions hold.

$$b_0 \geq \frac{nP + (M, P)}{M} - 1,$$

$$b_1 \geq \frac{nQ + (M, Q)}{M} - 1,$$

$$b_0 + b_1 \leq \frac{n(P + Q) - (M, M - P - Q)}{M} - 1.$$
For \( n \in \{0, \ldots, M-1\} \), let \( V_n \) be the isotypical component of \( \Omega^1[X_M] \) with character \( \chi_n \). Then Theorem 1.10, §1.4.8, implies

\[
\dim V_n = \begin{cases} 
  d_n & \text{if } d_n \geq 0 \\
  0 & \text{otherwise}, 
\end{cases}
\]

where

\[
d_n = \frac{n(M-P)+(M,P)}{M} + \frac{1-n(M-P+Q)+(M,M-P-Q)}{M} + \frac{1-nQ+(M,Q)}{M}.
\]

Suppose now that \( (n, N) = 1 \), then it follows from Theorem 1.15, §1.4.8, that

\[
\dim V_n = \frac{n(M-P)}{M} + \frac{n(M-P+Q)}{M} + \frac{nQ}{M}.
\]

and from Theorem 1.16, §1.4.8, that

\[
\dim V_n + \dim V_{N-n} = 1.
\]

New Jacobian of \( X(M, 0) \)

According to §1.4.10, the vector space \( \Omega^1[X_M]_{\text{new}} \) of new differential forms on \( X_M \) has dimension \( \frac{\varphi(M)}{2} \). Set \( \Omega^1[X(M, 0)]_{\text{new}} := \tilde{k}^*(\Omega^1[X_M]_{\text{new}}) \). Because \( \tilde{k} \) is an isomorphism, \( \dim(\tilde{k}^*(\Omega^1[X_M]_{\text{new}})) = \dim(\Omega^1[X_M]_{\text{new}}) \) and

\[
\dim(\Omega^1[X(M, 0)]_{\text{new}}) = \frac{\varphi(M)}{2}.
\]

Definition 1.75. The abelian subvariety of \( Jac(X(M, 0)) \) which is generated by \( \Omega^1[X(M, 0)]_{\text{new}} \) will be called the New Jacobian of \( X(M, 0) \) and denoted by \( Jac_{\text{new}}(X(M, 0)) \). It has dimension \( \frac{\varphi(M)}{2} \).

Conclusion

Write \( \omega_1 \) for the differential 1-form \((\kappa^{-1})^*\omega \) on \( C_M \), where \( \omega \) is the differential form \( \frac{dx}{y} \) on \( C(M, 0) \). Then \( \omega_1 \) is a new form on \( C_M \) and, if \( \pi^*\omega_1 \) is regular on \( X_M \), then
1.7. Degenerated Fibers

In this section, we would like to enlarge the family of curves treated in Section 1.4 by considering the parameters $X_q, \ldots, X_r$ to lie in $\mathbb{P}_C^1$ and allowing them to be equal. The nondegenerated curves of this family will however be those of the family of Section 1.4, their genus and the dimension of their New Jacobian being then known. We here intend to calculate the genus and the dimension of the New Jacobian of the degenerated curves of the enlarged family. This is actually again an application of Section 1.4 and will be done in §1.7.2 after we have defined the enlarged family and introduced new notations in §1.7.1.

1.7.1 New Notations and Definitions

For $r \in \mathbb{N} \cup \{0\}$, $N \in \mathbb{N}$ and $u = (\lambda_0, A_0; \ldots; \lambda_r, A_r) \in (\mathbb{P}_C^1 \times \mathbb{N})^{r+1}$ such that $N = (A_0, \ldots, A_r)$ and $\sum_{k=0}^{r} A_k = (N, A_0, \ldots, A_r) = 1$, let us define $C_N(u)$ to be the projective plane curve with affine equation

$$y^N = \prod_{i=0}^{r} (x - \lambda_i)^{A_i}$$

and $X_N(u)$ its desingularization.

If, $\forall i = 0, \ldots, r, \lambda_i \neq \infty$ and, for all $i, j \in \{0, \ldots, r\}$ with $i \neq j$, $\lambda_i \neq \lambda_j$, then $C_N(u) = C_N$ and $X_N(u) = X_N$ are the curves defined in §1.4.3 and the results of Section 1.4 apply.
Remark 4.1. The projective equation of $C_n(u)$ defines an algebraic projective variety $\mathcal{F}$ in $\mathbb{P}_C^r \times (\mathbb{P}_C^1)^{r+1}$. For fixed $r \in \mathbb{N} \cup \{0\}$ and $N, A_0, \ldots, A_r \in \mathbb{N}$, the curves $C_N(u)$ are isomorphic to the fibers of the restriction to $\mathcal{F}$ of the projection $\mathbb{P}_C^r \times (\mathbb{P}_C^1)^{r+1} \to (\mathbb{P}_C^1)^{r+1}$. A fiber over $(\lambda_0, \ldots, \lambda_r)$ such that, $\forall i \in \{0, \ldots, r\}, \lambda_i \neq \infty$, and, $\forall i, j \in \{0, \ldots, r\}$ with $i \neq j, \lambda_i \neq \lambda_j$ will be said to be nondegenerated. It is then isomorphic to a curve of the family considered in Section 1.4. The other fibers are said to be degenerated. The same qualifiers will also be used for the curves themselves. In order to describe the degenerated curves, we introduce the following definition.

Definition 1.76. Let $m \geq 1$ be the number of pairwise distinct entries in the $(r+1)$-tuple $(\lambda_0, \ldots, \lambda_r) \in (\mathbb{P}_C^1)^{r+1}$ and call it the multiplicity of $u = (\lambda_0, A_0; \ldots; \lambda_r, A_r)$. Define the reduced form of $u$ to be the element $u_{\text{red}} := (\mu_0, B_0; \ldots; \mu_{m-1}, B_{m-1})$ of $(\mathbb{C} \times \mathbb{N})^{m-1} \times (\mathbb{P}_C^1 \times \mathbb{N})$, where $\forall i \in \{0, \ldots, m-1\} : \exists j \in \{0, \ldots, r\}$ such that $\mu_i = \lambda_j$ and $B_i = \sum A_j$, where the sum is taken over all the $j$'s in $\{0, \ldots, r\}$ such that $\lambda_j = \mu_i$. Moreover, $\forall i, j \in \{0, \ldots, m-1\}$ with $j \neq i : \mu_j \neq \mu_i$.

This definition is essentially only a new notation. Indeed, we have

$$C_N(u) = C_N(u_{\text{red}}) : y^N = \prod_{i=0}^{m-1} (x - \mu_i)^{B_i}$$

and $X_N(u) = X_N(u_{\text{red}})$.

1.7.2 Dimension of the New Jacobian and Genus of some Fibers

Let $C_N(u)$ be the curve defined in §1.7.1 and the notations and definitions be the same. We will first treat the case $\mu_{m-1} \neq \infty$. If $C_N(u)$ is irreducible (i.e. $(N, B_0, \ldots, B_{m-1}) = 1$) and if $N \neq \sum_{k=0}^{m-1} B_k$ (note that $\sum_{k=0}^{m-1} B_k = \sum_{j=0}^{m-1} A_j$), then the genus of $X_N(u)$ is given by

$$g(X_N(u)) = 1 + \frac{1}{2} [(m - 1)N - (N, N - \sum_{k=0}^{m-1} B_k) - \sum_{j=0}^{m-1} (N, B_j)]$$

(cf Theorem 1.8, §1.4.6). If moreover $N \nmid B_0, \ldots, B_{m-1}, \sum_{k=0}^{m-1} B_k$, then the results of Section 1.4 apply to the case $\mu_{m-1} \neq \infty$. In this case, we have

$$\dim(Jac_{\text{new}}(X_N(u))) = \frac{(m - 1)\varphi(N)}{2}$$
1.7. Degenerated Fibers

(cf §1.4.10). Note that this dimension depends only on the number of pairwise distinct factors in the equation of $C_N(u)$.

For the case $\mu_{m-1} = \infty$, replace $\mu_{m-1}$ by $\frac{z_1}{z_0}$ in the projective equation of $C_N(\mu_{red})$ and get

- **Case 1:** $N - \sum_{k=0}^{m-1} B_k > 0$:
  
  \[ x_2^{N-1} - x_0^{N-1} \sum_{k=0}^{m-1} B_k (x_1 z_0 - x_0 z_1)^{m-2} \prod_{i=0}^{m-2} (x_1 - \mu_i x_0) B_i \]

- **Case 2:** $N - \sum_{k=0}^{m-1} B_k < 0$:
  
  \[ x_2^{N-1} - x_0^{N-1} \sum_{k=0}^{m-1} B_k (x_1 z_0 - x_0 z_1)^{m-2} \prod_{i=0}^{m-2} (x_1 - \mu_i x_0) B_i. \]

Set $z_0 = 0$ and get

- **Case 1:** $0 = x_0^{N-1} \sum_{k=0}^{m-1} B_k z_0^{B_{m-1}} \prod_{i=0}^{m-2} (x_1 - \mu_i x_0) B_i$

- **Case 2:** $0 = x_0^{B_{m-1}} z_1^{B_{m-1}} \prod_{i=0}^{m-2} (x_1 - \mu_i x_0) B_i$

Since $z_1 \neq 0$, the solutions in both cases are $x_0 = 0$ or $\exists i \in \{0, \ldots, m-2\}$ such that $x_1 = \mu_i x_0$, i.e.

\[ C_N(u) = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2_C : x_0 = 0 \text{ or } \exists i \in \{0, \ldots, m-2\} \text{ s.t. } x_1 = \mu_i x_0\} \]

\[ = \{(0 : x_1 : x_2) \in \mathbb{P}^2_C \} \cup \bigcup_{i=0}^{m-2} \{(x_0 : \mu_i x_0 : x_2) \in \mathbb{P}^2_C \}. \]

This shows that for $\mu_{m-1} = \infty$, $C_N(u)$ is isomorphic to $m$ copies of $\mathbb{P}^1_C$ lying in $\mathbb{P}^2_C$. These copies are the irreducible components of $C_N(u)$ and the only singular points of $C_N(u)$ appear to be the intersection points of at least two distinct copies. These irreducible components are separated during the sequence of blow-ups and are isomorphic to the connected components of the desingularization $X_N(u)$ of $C_N(u)$, because each of them is nonsingular (cf §1.3.7). Each connected component of $X_N(u)$ has genus 0 and 0 is also the dimension of the Jacobian variety of $X_N(u)$. 
Chapter 2

Identities for some Gauss’ Hypergeometric Series

2.1 Introduction

In their article [BW86], Beukers and Wolfart proved two identities involving a hypergeometric series, the $J$-function and the Dedekind $\eta$-function. One of these reads: there exists $A \in \mathbb{C}$ such that, for all $\tau$ in the neighbourhood of $i$ given by $|1 - J(z)^{-1}| < 1$, we have

$$F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; 1 - J(\tau)^{-1}\right) = A(\tau + i)\eta(\tau)^2 J(\tau)^{\frac{1}{12}}.$$

To get such an identity, Beukers and Wolfart made use of a result of Klein and Fricke, which states that the periods on the “universal” elliptic curve satisfy a given linear differential equation of order two with regular singularities at 0, 1, $\infty$ only and of the theory of elliptic curves to get a local fundamental system to this differential equation. On the other hand, they found two solutions involving hypergeometric series and expressed them locally in terms of the fundamental system. This way, they obtained an identity for each solution.

In the Theoretical Preliminaries, we recall some theoretical facts used throughout the chapter. In Section 2.3, we explain and extend the method of Beukers and Wolfart. In Section 2.4, we give the list of the solutions involving hypergeometric
series that we have calculated to the above linear differential equation. We also explain how one can find them using Riemann's point of view and Kummer's idea. In Section 2.5, we produce identities for our solutions, most of them being new. Finally, Section 2.6 provides some applications of our identities. On one hand, some identities allow us to show the algebraicity of the corresponding hypergeometric series at the points of some infinite subset of \( \bar{Q} \). On the other hand, we are able to calculate explicitly some new algebraic evaluations of \( F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; z\right) \).

2.2 Theoretical Preliminaries

We recall here some definitions and facts on elliptic curves over \( \mathbb{C} \) and some associated modular functions and forms.

2.2.1 Elliptic Curves over \( \mathbb{C} \)

Let \( \Lambda \) be a lattice in \( \mathbb{C} \) and \( \wp_\Lambda \) the associated Weierstrass function on \( \mathbb{C} \)

\[
\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \left[ \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right].
\]

The series converges uniformly on each compact set not containing any lattice point and the only poles of \( \wp_\Lambda \) are double poles in each lattice point. Moreover \( \wp_\Lambda \) is periodic with respect to \( \Lambda \), what makes it an elliptic function. Differentiating term by term, one shows that \( \wp_\Lambda \) and \( \wp_\Lambda' \) satisfy the relation

\[
(\wp_\Lambda')^2 = 4\wp_\Lambda^3 - g_2(\Lambda)\wp_\Lambda - g_3(\Lambda),
\]

where \( g_2(\Lambda) = 60 \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^4} \) and \( g_3(\Lambda) = 140 \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^6} \). This means that the map \( z \mapsto (\wp_\Lambda(z), \wp_\Lambda'(z)) \) parametrizes points on the curve defined by the equation

\[
y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda). \quad (2.1)
\]

More precisely, writing \( E_\Lambda(\mathbb{C}) \) for the algebraic projective curve defined affinely by (2.1) over \( \mathbb{C} \), we have the analytic (group) isomorphism

\[
\Phi_\Lambda : \mathbb{C}/\Lambda \rightarrow E_\Lambda(\mathbb{C}) \quad \text{(for } z \notin \Lambda),
\]

\[
z \mod \Lambda \rightarrow (1 : \wp_\Lambda(z) : \wp_\Lambda'(z)) \text{ for } z \notin \Lambda.
\]

\[
0 \rightarrow (0 : 0 : 1).
\]
2.2. Theoretical Preliminaries

Each equation of the so-called Weierstrass' form $y^2 = 4x^3 - ax - b$ with $a, b \in \mathbb{C}$ and $a^3 - 27b^2 \neq 0$ defines an algebraic projective curve $E_{a,b}$ over $\mathbb{C}$, which is non-singular and called elliptic. For each such curve, there exists a unique lattice $\Lambda_{a,b}$ in $\mathbb{C}$, such that $\Phi_{\Lambda_{a,b}} : \mathbb{C}/\Lambda \rightarrow E_{a,b}$ is an analytic isomorphism. This lattice could be referred to as the "corresponding" lattice to the curve $E_{a,b}$. Its elements are called periods of $E_{a,b}$. In particular, $a = g_2(\Lambda_{a,b})$ and $b = g_3(\Lambda_{a,b})$ hold.

Let $E$ be an elliptic curve over $\mathbb{C}$ given in Weierstrass' form by $x, y$ coordinates. We are now wishing to determine its corresponding lattice $\Lambda$. The analytic isomorphism $\Phi_\Lambda : \mathbb{C}/\Lambda \rightarrow E(\mathbb{C})$ induces an isomorphism of $\mathbb{Z}$-modules

$$\Phi_\Lambda^* : H_1(E(\mathbb{C}), \mathbb{Z}) \rightarrow H_1(\mathbb{C}/\Lambda, \mathbb{Z})$$

$$[\gamma] \mapsto [\Phi_\Lambda^{-1} \circ \gamma]$$

(cf §1.3.13). Moreover, $H_1(\mathbb{C}/\Lambda, \mathbb{Z})$ is isomorphic to $\Lambda$ via the map

$$[\delta] \mapsto \int_\delta dz.$$

Hence, if $e_1, e_2$ form a $\mathbb{Z}$-basis of $H_1(E(\mathbb{C}), \mathbb{Z})$, then $\int_{\Phi_\Lambda^* e_1} dz, \int_{\Phi_\Lambda^* e_2} dz$ will form a $\mathbb{Z}$-basis of $\Lambda$. Now, for $i = 1, 2$, we have

$$\int_{e_i} \frac{dx}{y} = \int_{\Phi_\Lambda^* e_i} \Phi_\Lambda^* \left( \frac{dx}{y} \right) = \int_{\Phi_\Lambda e_i} dz.$$

This implies

$$\Lambda = \mathbb{Z} \int_{e_1} \frac{dx}{y} + \mathbb{Z} \int_{e_2} \frac{dx}{y}.$$

In fact, the inverse map of $\Phi_\Lambda$ is given by

$$E(\mathbb{C}) \rightarrow \mathbb{C}/\Lambda$$

$$P \mapsto \left( \int_0^P dz \right) (\text{mod } \Lambda).$$

Remark 42. The above map is a special case of the map given in Remark 27, §1.3.16, which embeds a Riemann surface into its Jacobian variety. In particular, we see that each elliptic curve defined over $\mathbb{C}$ is isomorphic to its Jacobian variety.
2.2.2 Isomorphisms between Elliptic Curves

Let $E : y^2 = 4x^3 - ax - b$ and $E' : y'^2 = 4x'^3 - a'x' - b'$ be two elliptic curves in Weierstrass’ form over a field $k$ of characteristic $\neq 2, 3$. They are isomorphic if and only if there exists $\mu \in k$ such that $a' = \mu^{-4}a$ and $b' = \mu^{-6}b$. If $k = \mathbb{C}$, we get in terms of the corresponding lattices $\Lambda$ for $E$ and $\Lambda'$ for $E'$

$$E \rightarrow E' \iff \exists \mu \in \mathbb{C}^\ast \text{ such that } \Lambda' = \mu \Lambda.$$ 

More precisely, we have the following commutative diagramm

$$
\begin{array}{ccc}
\mathbb{C}/\Lambda & \xrightarrow{\phi_\Lambda} & \mathbb{C}/\mu \Lambda \\
\Phi_\Lambda \downarrow & & \downarrow \Phi_{\mu \Lambda} \\
E(\mathbb{C}) & \longrightarrow & E'(\mathbb{C}) \\
(x, y) & \longrightarrow & (\mu^{-2}x, \mu^{-3}y).
\end{array}
$$

Corresponding to these variable transformations, the following homogeneity properties hold for any $\mu \in \mathbb{C}^\ast$ and $z \in \mathbb{C}$

$$
\begin{align*}
\varphi_{\mu \Lambda}(\mu z) &= \mu^{-2}\varphi_\Lambda(z) \\
g_2(\mu \Lambda) &= \mu^{-4}g_2(\Lambda) \\
g_3(\mu \Lambda) &= \mu^{-6}g_3(\Lambda).
\end{align*}
$$

2.2.3 Change of Basis of a Lattice

([Sei] VII §1) Let $\Lambda$ be a lattice in $\mathbb{C}$ with $\mathbb{Z}$-basis $\omega_1$ and $\omega_2$. Since $\omega_1$ and $\omega_2$ generate $\mathbb{C}$ over $\mathbb{R}$, they are $\mathbb{R}$-linearly independent. Hence $\text{Im}(\frac{\omega_1}{\omega_2}) \neq 0$ and reordering $\omega_1$ and $\omega_2$ if necessary, we can suppose $\text{Im}(\frac{\omega_1}{\omega_2}) > 0$.

Any new basis $\omega_1', \omega_2'$ of $\Lambda$ has the form $\omega_i' = a\omega_1 + b\omega_2, \omega_2' = c\omega_1 + d\omega_2$ with $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$. $\omega_1', \omega_2'$ is also positively oriented if and only if $\gamma$ has determinant 1, that is $\gamma \in SL_2(\mathbb{Z})$. The action by linear transformations of $SL_2(\mathbb{Z})$ on $\mathbb{H} := \{z \in \mathbb{C}; \text{Im}(z) > 0\}$ is defined by setting, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$,

$$
\gamma \cdot \tau := \frac{a \tau + b}{c \tau + d}.
$$
2.2. Theoretical Preliminaries

The quotient $\mathbb{H}/SL_2(\mathbb{Z})$ for this action is in bijection with the set of lattices in $\mathbb{C}$ defined up to homothety, hence also with the set of isomorphism classes of elliptic curves over $\mathbb{C}$.

Since $I$ and $-I$ are the only elements of $SL_2(\mathbb{Z})$ that act trivially on the whole of $\mathbb{H}$, the quotient group $PSL_2(\mathbb{Z}) := SL_2(\mathbb{Z})/\{\pm I\}$ acts freely on $\mathbb{H}$. It will be called the modular group (this denomination is sometimes used for $SL_2(\mathbb{Z})$ itself). A transformation of $\mathbb{H}$ induced by the action of an element of the modular group will be said to be modular.

Definition 2.1. Let $\Gamma$ be a discrete subgroup of $SL_2(\mathbb{Z})$. A fundamental domain $D$ for $\Gamma$ in $\mathbb{H}$ is a subset of $\mathbb{H}$ such that every orbit of $\Gamma$ has one representant in $D$ and two elements of $D$ are in the same orbit if and only if they lie on the boundary of $D$.

A fundamental domain for $PSL_2(\mathbb{Z})$ is the set of $z \in \mathbb{H}$ such that
\[-\frac{1}{2} \leq \Re(z) \leq \frac{1}{2} \quad \text{and} \quad |z| \geq 1.

Definition 2.2. The parabolic points of $\Gamma$ in a fundamental domain $D$ for $\Gamma$ are the real points together with $\infty$ which lie in the closure of $D$ for the topology of the Riemann sphere.

$\infty$ is the sole parabolic point of $SL_2(\mathbb{Z})/\{\pm 1\}$.

2.2.4 From a Lattice Function to a Modular one, Definition of Modular Functions and Forms

([Se1] VII §§2.1-2) In §2.2.1, we have met two functions $g_2, g_3$ defined on the set of lattices and satisfying some transformations properties under homothety of lattice. More generally, a function $F$ on the set $\mathcal{R}$ of lattices in $\mathbb{C}$ with complex values is said to be of weight $2k, k \in \mathbb{Z}$, if for every $\Lambda \in \mathcal{R}$ and $\mu \in \mathbb{C}^*$
\[ F(\mu \Lambda) = \mu^{-2k} F(\Lambda). \]

Write $\Lambda = Z\omega_1 + Z\omega_2$ with $\Im(\frac{\omega_1}{\omega_2}) > 0$ and let $F(\omega_1, \omega_2)$ stand for $F(\Lambda)$, then the above property shows that $\omega_2^{2k} F(\omega_1, \omega_2)$ depends only on $\frac{\omega_1}{\omega_2}$ and is invariant
under any change of basis of \( \Lambda \). Let \( f : \mathbb{H} \to \mathbb{C} \) be defined by

\[
f(\frac{\omega_1}{\omega_2}) = \omega_2^{2k} F(\omega_1, \omega_2),
\]

it will then satisfy, for any \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \) and \( z \in \mathbb{H} \)

\[
f(\frac{a \tau + b}{c \tau + d}) = (c \tau + d)^{2k} f(\tau).
\]

Noting that the transformation \( \tau \mapsto e^{2\pi i \tau} = q \) takes \( \mathbb{H} \) to the open disk \( D^{\ast} := \{ q \in \mathbb{C}; |q| = 1, q \neq 0 \} \) with the origin removed and that, \( \forall \tau \in \mathbb{H}, f(\tau + 1) = f(\tau) \),

we can consider \( f \) as a function \( f \) of \( q \) on \( D^{\ast} \).

**Definition 2.3.** Let \( k \) be an integer. A meromorphic function \( f : \mathbb{H} \to \mathbb{C} \) is said to be **modular of weight** \( 2k \) if \( f \) is meromorphic at zero and if, for every \( \gamma \in C \) and every \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \),

\[
f(\gamma \cdot \tau) = (c \tau + d)^{2k} f(\tau).
\]

In this case, \( f \) has a Laurent development \( \sum_{-\infty}^{\infty} a_n q^n \) at 0 and the nature of \( f \) at \( \infty \) will per definition be that of \( \tilde{f} \) at 0.

**Definition 2.4.** A modular function is called a **modular form**, if it is holomorphic in \( \mathbb{H} \) and at infinity. In this case, we set \( f(\infty) := \tilde{f}(0) \). If moreover it takes the value zero at infinity, it is called a **cusp form**.

### 2.2.5 Eisenstein Series

Remembering the functions \( g_2, g_3 \) we had met in \( \S 2.2.1 \), we consider here more generally the so-called **Eisenstein series**

\[
G_k(\Lambda) = \sum_{\omega \in \Lambda} \frac{1}{\omega^{2k}},
\]

for given integer \( k > 1 \) and lattice \( \Lambda \) in \( \mathbb{C} \), \( \sum' \) meaning summation over the non zero elements. Observing that

\[
G_k(\mu \Lambda) = \mu^{-2k} G_k(\Lambda),
\]
we can apply the procedure described in §2.2.4 to make it a function of \( \tau \in \mathbb{H} \)

\[
G_k(\omega_1, \omega_2) = \sum_{m,n \in \mathbb{Z}}' \frac{1}{(m\omega_1 + n\omega_2)^{2k}} = \frac{1}{\omega_2^{2k}} \sum_{m,n \in \mathbb{Z}}' \frac{1}{(m\frac{\omega_1}{\omega_2} + n)^{2k}}.
\]

Let's write again \( G_k \) for the corresponding function of \( \tau := \frac{\omega_1}{\omega_2} \in \mathbb{H} \)

\[
G_k(\tau) = \sum_{m,n \in \mathbb{Z}}' \frac{1}{(m\tau + n)^{2k}}.
\]

It can be shown that the Eisenstein series \( G_k \) is a modular form of weight \( 2k \). This authorizes us to write \( g_2 \) and \( g_3 \) as modular forms of weight 4 and 6 respectively as follows

\[
g_2(\tau) = 60G_2(\tau) \quad \text{and} \quad g_3(\tau) = 140G_3(\tau).
\]

\( g_2 \) and \( g_3 \) are holomorphic in \( \mathbb{H} \) and also at infinity. They have the following Laurent’s development in \( q := e^{2\pi i \tau} \)

\[
g_2(\tau) = \pi^4 \left[ \frac{4}{3} + 320q + \ldots \right]
\]

\[
g_3(\tau) = \pi^6 \left[ \frac{8}{3} - \frac{448}{3}q + \ldots \right].
\]

Setting \( q = 0 \), we get the values at infinity

\[
g_2(\infty) = \frac{4}{3} \pi^4 \quad \text{and} \quad g_3(\infty) = \frac{8}{27} \pi^6.
\]

For \( \rho \) denoting \( e^{\frac{2\pi i}{3}} \), \( g_2(\rho) \) is equal to

\[
60 \sum_{m,n \in \mathbb{Z}}' \frac{1}{(m\rho + n)^4} = 60 \sum_{m,n \in \mathbb{Z}}' \frac{\rho^8}{(m\rho^3 + n\rho^2)^4} = \rho^2 60 \sum_{m,n \in \mathbb{Z}}' \frac{1}{(m + n')^4}
\]

\[= \rho^2 g_2(\rho). \]

\( g_2(\rho) = 0 \), because \( \rho^2 \neq 0 \). Similarly, \( g_3(i) \) equals

\[
140 \sum_{m,n \in \mathbb{Z}}' \frac{1}{(mi + n)^6} = 140 \sum_{m,n \in \mathbb{Z}}' \frac{i^6}{(mi^2 + ni)^6} = -140 \sum_{m',n \in \mathbb{Z}}' \frac{1}{(ni + m')^6},
\]

what equals \( -g_3(i) \). This implies \( g_3(i) = 0 \) (cf [Fo] 67).

There is a well-known result establishing a relation between the weight of a modular function and its order at the points of \( \mathbb{H}/SL_1(\mathbb{Z}) \) (see for example [Se1] VII §3.1 or [La] 3§2). From this result, it can be deduced that the only zeros of \( g_2 \) (resp. \( g_3 \)) are congruent modulo \( SL_2(\mathbb{Z}) \) to \( \rho \) (resp. \( i \)) and are of order 1.
2.2.6 The Discriminant Function $\Delta$

Let's define $\Delta(\tau)$ to be the discriminant of the polynomial $4x^3 - g_2(\tau)x - g_3(\tau)$. That is

$$\Delta(\tau) := g_2(\tau)^3 - 27g_3(\tau)^2.$$ 

It is a modular form of weight 12, which vanishes only at infinity

$$\Delta(\tau) = \pi^{12}[4096e^{2\pi i \tau} + ...].$$

Hence, $\Delta$ is a cusp form and $\Delta(\infty) = 0$ with order 1. $q$ denoting $e^{2\pi i \tau}$, $\Delta$ has the following beautiful product expansion due to Jacobi

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$ 

2.2.7 The $J$-Function

The $J$-invariant of an elliptic curve $E$ in Weierstrass’ form

$$y^2 = 4x^3 - ax - b$$

with $a^3 - 27b^2 \neq 0, a, b \in \mathbb{C}$ is defined to be

$$J := \frac{a^3}{a^3 - 27b^2}.$$

As seen in §2.2.1, there exists $\tau \in \mathbb{H}$ such that, for $\Lambda := \mathbb{Z}\tau + \mathbb{Z}$, the map $\Phi_\Lambda : \mathbb{C}/\Lambda \rightarrow E(\mathbb{C})$ is an analytic isomorphism. Moreover $a = g_2(\Lambda) = g_2(\tau)$ and $b = g_3(\Lambda) = g_3(\tau)$. We thus define the $J$-function on $\mathbb{H}$ by

$$J(\tau) := \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}.$$ 

It is modular of weight 0 and has only one pole which lies at infinity

$$J(\tau) = \frac{1}{1728q} + c_0 + c_1q + ...,$$
2.2. Theoretical Preliminaries

where \( q = e^{2\pi i \tau} \) and \( c_i \in \mathbb{Q} \) for every \( i \). This implies that \( J \) takes on each value once and exactly once in a fundamental domain of \( SL_2(\mathbb{Z}) \) and that \( J : \mathbb{H}/SL_2(\mathbb{Z}) \to \mathbb{C} \) is an isomorphism. As seen in §2.2.3, \( \mathbb{H}/SL_2(\mathbb{Z}) \) is in bijection with the set of isomorphy classes of elliptic curves over \( \mathbb{C} \). Hence two elliptic curves in Weierstrass’ form over \( \mathbb{C} \) are isomorphic if and only if their \( J \)-invariants are equal.

Remembering that \( g_2(\rho) = 0 \) and \( g_3(i) = 0 \) with order 1 (cf §2.2.5), we get

\[
J(\rho) = 0 \quad \text{and} \quad J(i) = 1
\]

with order 3 and 2 respectively. As seen above, \( J(\infty) = \infty \) with order 1.

Remark 43. Let \( \mathbb{H}^* := \mathbb{H} \cup \{\infty\} \). \( SL_2(\mathbb{Z}) \) still acts on \( \mathbb{H}^* \). The quotient \( \mathbb{H}^*/SL_2(\mathbb{Z}) \) can be equipped with a structure of Riemann surface, for which it is compact. \( J \) extends then to an isomorphism (renoted \( J \))

\[
J : \mathbb{H}^*/SL_2(\mathbb{Z}) \rightarrow \mathbb{P}^1(\mathbb{C}) \simeq \mathbb{C} \cup \{\infty\}.
\]

2.2.8 The Dedekind \( \eta \)-Function

The \( \eta \)-function was defined by Dedekind (1877) to be

\[
\eta(\tau) := q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m),
\]

where \( q = e^{2\pi i \tau} \) and for \( \tau \in \mathbb{H} \). This infinite product is absolutely convergent since \( \text{Im}(\tau) > 0 \) and converges uniformely on every compact subset of \( \mathbb{H} \). It therefore defines an analytic function of \( \tau \in \mathbb{H} \). All terms of the product being distinct from zero, \( \eta \) does not take the value zero on \( \mathbb{H} \). The product expansion of the discriminant function (cf §2.2.6) induces the relation

\[
\Delta(\tau) = (2\pi)^{12} \eta(\tau)^{24}. \tag{2.2}
\]

Since \( \Delta \) is modular of weight 12, so is \( \eta(\tau)^{24} \). Therefore, for every \( \tau \in \mathbb{H} \) and \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \)

\[
\eta(\gamma \cdot \tau)^{24} = (c\tau + d)^{12} \eta(\tau)^{24}
\]
and there exists a 24-th root of unity $\varepsilon$ depending only on $\gamma$ such that, for all $\tau \in \mathbb{H}$,

$$\eta(\gamma \cdot \tau) = \varepsilon \sqrt{\tau + d} \eta(\tau).$$

Once a branch of the squared-root being fixed, the constant $\varepsilon$ may be determined explicitly in dependance of $a$, $b$, $c$, $d$ (see [Ra], Ch 9).

Particular cases are the transformations under the generators of $SL_2(\mathbb{Z})$ for which the following holds for every $\tau \in \mathbb{H}$

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}} \eta(\tau),$$

$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau} \eta(\tau)$$

(the branch of the squared-root being here positive for real positive arguments).

Remark 44. The first property could also have been derived directly from the definition of $\eta(\tau)$. More generally, one gets this way, for $c \in \mathbb{Z}$ and $\tau \in \mathbb{H}$,

$$\eta(\tau + c) = e^{\frac{2\pi i c}{12}} \eta(\tau).$$

Remark 45. Given a subgroup $\Gamma$ of finite index in $SL_2(\mathbb{Z})$, extending the definition given in §2.2.4, one can define modular forms with respect to $\Gamma$ and of real degree $-r$. Roughly speaking, such functions are meromorphic functions $f$ on $\mathbb{H}$ which satisfy, for each $\tau \in \mathbb{H}$ and $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma$,

$$f(\gamma \cdot \tau) = \nu(\gamma)(c\tau + d)^r f(\tau),$$

for a fixed branch of $(c\tau + d)^r$ and a complex number $\nu(\gamma)$ of module 1 independent of $\tau$. Moreover, $f$ shall have at most finitely many poles in $\mathcal{R} \cap \mathbb{H}$, where $\mathcal{R}$ is a fundamental domain for $\Gamma$, and be meromorphic at each parabolic point of $\mathcal{R}$. (Note that this is independent of the choice of $\mathcal{R}$.) Such a modular form is called entire if it is holomorphic in $\mathbb{H}$ and at each parabolic point of some fundamental domain. Moreover, if it has a zero at each parabolic point, it is called a cusp form. (For a precise definition, see [Kn] Ch2.)

The above shows that $\eta(\tau)$ is an entire modular form of degree $-\frac{1}{2}$ with respect to the full modular group $SL_2(\mathbb{Z})$. The development of $\eta$ at the only parabolic point $\infty$ of $SL_2(\mathbb{Z})$ is

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \sum_{m=-\infty}^{\infty} (-1)^m e^{\pi i m(3m+1)\tau}.$$
Setting \( n := \frac{3m^2 + m}{2} \), we see that the exponents are all of the form \( 2\pi i (n + \frac{1}{24}) \) with \( n + \frac{1}{24} \) strictly positive. Hence \( \eta(\infty) = 0 \) with order \( \frac{1}{24} \) (cf [Kn] Ch 3.3).

2.3 The Method of Beukers and Wolfart

In this section, we explain and generalize the proof of Proposition 2.1 of [BW86] to a more general solution of the differential equation (2.3) satisfied by the periods on the universal elliptic curve \( E_J \).

For \( \tau \in \mathbb{H} \), let's consider the elliptic curve \( E_\tau \) over \( \mathbb{C} \) given by the following equation

\[
y^2 = 4x^3 - g_2(\tau)x - g_3(\tau).
\]

Since the modular \( J \)-function is an invariant of the isomorphism class of \( E_\tau \), let's look for a representative \( E_J \) of this class which depends only on \( J = J(\tau) \). As seen in §2.2.7, the \( J \)-invariant of the curve \( y^2 = 4x^3 - ax - b \) is \( J = \frac{a^3}{a^3 - 27b^2} \). Requiring \( a = b = c \), the equation will only depend on \( J \). For \( J \neq 0, 1 \), we have

\[
J = \frac{c^3}{c^3 - 27c^2} \quad \iff \quad c = \frac{27J}{J - 1}
\]

and \( E_J \) is given by

\[
\tilde{y}^2 = 4\tilde{x}^3 - \frac{27J}{J - 1}(\tilde{x} + 1).
\]

The periods on \( E_J \) are functions of \( J \), which satisfy the differential equation

\[
\frac{d^2\Omega}{dJ^2} + \frac{1}{J} \frac{d\Omega}{dJ} + \frac{31J - 4}{144J^2(J - 1)^2} \Omega = 0,
\]

as was shown by Klein and Fricke in [KF] II §3. Now, since for each \( \tau \in \mathbb{H} \) such that \( J(\tau) = J \), the curve \( E_\tau \) is isomorphic to \( E_J \), the lattices \( \Lambda_\tau \) and \( \Lambda_J \) corresponding to \( E_\tau \) and \( E_J \) respectively (in the sense of §2.2.1) will be homothetic. We intend to calculate the factor of homothety.

As seen in §2.2.1, a basis of \( \Lambda_J \) is given by

\[
\int_{\mathcal{D}_i} \frac{d\tilde{x}}{\tilde{y}}, \quad i = 1, 2,
\]
where $\mathcal{D}_1, \mathcal{D}_2$ form a basis of $H_1(E_J(\mathbb{C}), \mathbb{Z})$. Let $\varphi_\tau : E_\tau \rightarrow E_J$ be the isomorphism and $\mathcal{C}_1, \mathcal{C}_2$ a basis of $H_1(E_\tau(\mathbb{C}), \mathbb{Z})$, then the push-forwards $(\varphi_\tau)_* \mathcal{C}_i$, $i = 1, 2$, will form a basis of $H_1(E_J(\mathbb{C}), \mathbb{Z})$. Noting that

$$\frac{27J}{J - 1} = \frac{g_2^3}{g_3^2},$$

we check that the isomorphism $\varphi_\tau$ is given by

$$(\tilde{x}, \tilde{y}) = \left(\frac{g_2(\tau)}{g_3(\tau)}x, \left(\frac{g_2(\tau)}{g_3(\tau)}\right)\frac{3}{2}y\right).$$

This implies, for $i = 1, 2$,

$$\int_{(\varphi_\tau)_* \mathcal{C}_i} \frac{d\tilde{x}}{\tilde{y}} = \int_{\mathcal{C}_i} \varphi_\tau^*(\frac{d\tilde{x}}{\tilde{y}}) = \sqrt{\frac{g_3(\tau)}{g_2(\tau)}} \int_{\mathcal{C}_i} \frac{dx}{y}.$$  

Hence

$$\Lambda_J = \mu(\tau) \Lambda_\tau,$$

with

$$\mu(\tau) := \sqrt{\frac{g_3(\tau)}{g_2(\tau)}}.$$ 

The determination of the squared-root does not play any role, because $\tau$ is fixed and for any lattice $\Lambda$, we have $-\Lambda = \Lambda$. Since 1, $\tau$ also generate $\Lambda_\tau$,

$$\mu(\tau) \text{ and } \tau \mu(\tau)$$

are periods on $E_J$. But because of the squared-root and of the multivaluedness of the association $J \mapsto \tau$, $\mu(\tau)$ and $\tau \mu(\tau)$ are multivalued functions of $J$. As such, they satisfy equation (2.3) and, since they are $\mathbb{C}$-linearly independent, their branches furnish local systems of solutions.

As we do better understand the $J$-function than $g_2$ and $g_3$, let's try to get rid of these ones. To this aim, we work out the following identity for $\mu(\tau)$.
2.3. The Method of Beukers and Wolfart

Identity 1.

\[ \mu(\tau) = \sqrt{\frac{g_3(\tau)}{g_2(\tau)}} \]

\[ = \left( g_2(\tau) \right)^{1/2} \left( \frac{J(\tau) - 1}{J(\tau)} \right)^{1/4} \quad \text{(by Identity 1)} \]

\[ = \left( J(\tau)^{1/2} \left( \frac{J(\tau) - 1}{2J(\tau)} \right)^{1/4} \Delta(\tau)^{1/12} \quad \text{(by definition of } J) \right. \]

These are multivalued functions of \( \tau \), whose domains of definition and determination will be discussed later.

On the other hand, suppose that we have a solution to \( (2.3) \) of the form

\[ J^\alpha (1 - J)^\beta F(a, b, c; z(J)), \]

where \( \alpha, \beta \in \mathbb{Q} \) and \( z(J) \) is one of the following transformation of \( \mathbb{C} \) preserving the set \( \{0, 1, \infty\} \)

\[ J, \quad \frac{1}{J}, \quad 1/J, \quad \frac{1}{1-J}, \quad \frac{J}{J-1}, \quad \frac{J-1}{J}. \]

For the moment, we do not wish to bother about domains of definition, indeterminacy and singularity problems and write such a solution in terms of our local solutions \( (2.5) \) considered as multivalued functions of \( J \). There exist \( A, B \in \mathbb{C} \) such that

\[ J^\alpha (1 - J)^\beta F(a, b, c; z(J)) = A \tau \mu(\tau) + B \mu(\tau). \]

Using Identity 1 and writing the left-hand member as a function of \( \tau \), we indeed lose the multivaluedness due to that of \( J \mapsto \tau \), but keep that due to the fractional exponents. Hence, there exist \( A, B \in \mathbb{C} \) such that

\[ J(\tau)^\alpha (1 - J(\tau))^\beta F(a, b, c; z(J(\tau))) = (A \tau + B) J(\tau)^{-\beta} (J(\tau) - 1)^{\frac{1}{12}} \Delta(\tau)^{\frac{1}{12}}, \]

for \( \tau \) to be precised.
2. Identities for some Gauss’ Hypergeometric Series

2.4 Sixteen Solutions

In this section, we calculate all solutions to Differential Equation (2.3) of the form
\[ G(J) := J^a(1 - J)^b F(a, b, c; z(J)), \]
with \( a, b \in \mathbb{Q} \) and
\[ z(J) \in \{ J, \frac{1}{J}, 1 - J, \frac{1}{1 - J}, \frac{J}{J - 1}, \frac{J - 1}{J} \}. \]

These are called Kummer’s solutions. We then recall some theoretical facts on linear differential equations of the kind of (2.3) together with the ideas of Riemann to characterize their solutions and of Kummer to find all solutions of this form. Finally, we study again Equation (2.3) with Riemann’s and Kummer’s tools.

2.4.1 List of Sixteen Solutions

In order to determine for which \( a, b, a, b, c \in \mathbb{Q} \), \( G(J) \) is a solution of Differential Equation (2.3), we use the fact that \( F(a, b, c; z) \) satisfies the hypergeometric differential equation
\[
\frac{u''(z)}{z(1-z)} + \frac{ab}{z(1-z)} u(z) = 0.
\]

This implies that \( G(J) \) satisfies the following differential equation
\[
\Omega''(J) + \Omega'(J) \left[ -2\alpha J^{-1} + 2\beta (1 - J)^{-1} - z''(z')^{-1} + \frac{c}{z(1-z)} z' - \frac{a+b+1}{1-z} \right] u'(z) = 0.
\]

For each value of \( z(J) \), we require the coefficients of this equation to be respectively equal to those of Differential Equation (2.3) and resolve with respect to \( a, b, c, a, \beta \). We get the following sixteen solutions.
2.4. Sixteen Solutions

\[ J^{-\frac{1}{6}}(1 - J)^{\frac{1}{4}} F\left(\frac{1}{12}, \frac{1}{12} \cdot \frac{2}{3} \mid J\right) \]  
(1)

\[ J^{\frac{1}{6}}(1 - J)^{\frac{1}{4}} F\left(\frac{5}{12}, \frac{5}{12} \cdot \frac{4}{3} \mid J\right) \]  
(2)

\[ J^{-\frac{1}{6}}(1 - J)^{\frac{3}{4}} F\left(\frac{7}{12}, \frac{7}{12} \cdot \frac{2}{3} \mid J\right) \]  
(3)

\[ J^{\frac{1}{6}}(1 - J)^{\frac{3}{4}} F\left(\frac{11}{12}, \frac{11}{12} \cdot \frac{4}{3} \mid J\right) \]  
(4)

\[ J^{-\frac{1}{6}}(1 - J)^{\frac{1}{4}} F\left(\frac{1}{12}, \frac{1}{12} \cdot \frac{1}{2} \mid 1 - J\right) \]  
(5)

\[ J^{\frac{1}{6}}(1 - J)^{\frac{1}{4}} F\left(\frac{5}{12}, \frac{5}{12} \cdot \frac{1}{2} \mid 1 - J\right) \]  
(6)

\[ J^{-\frac{1}{6}}(1 - J)^{\frac{3}{4}} F\left(\frac{7}{12}, \frac{7}{12} \cdot \frac{3}{2} \mid 1 - J\right) \]  
(7)

\[ J^{\frac{1}{6}}(1 - J)^{\frac{3}{4}} F\left(\frac{11}{12}, \frac{11}{12} \cdot \frac{3}{2} \mid 1 - J\right) \]  
(8)

\[ J^{-\frac{1}{4}}(1 - J)^{\frac{1}{4}} F\left(\frac{1}{12}, \frac{5}{12} \cdot \frac{1}{J} \mid J\right) \]  
(9)

\[ J^{-\frac{1}{2}}(1 - J)^{\frac{3}{4}} F\left(\frac{7}{12}, \frac{11}{12} \cdot \frac{1}{J} \mid J\right) \]  
(10)

\[ J^{\frac{1}{6}}(1 - J)^{\frac{1}{6}} F\left(\frac{1}{12}, \frac{7}{12} \cdot \frac{1}{1 - J} \mid 1 - J\right) \]  
(11)

\[ J^{\frac{1}{2}}(1 - J)^{-\frac{1}{6}} F\left(\frac{5}{12}, \frac{11}{12} \cdot \frac{1}{1 - J} \mid J\right) \]  
(12)

\[ J^{-\frac{1}{4}}(1 - J)^{\frac{1}{4}} F\left(\frac{5}{12}, \frac{1}{12} \cdot \frac{1}{2} \mid \frac{J - 1}{J}\right) \]  
(13)

\[ J^{-\frac{1}{2}}(1 - J)^{\frac{3}{4}} F\left(\frac{7}{12}, \frac{11}{12} \cdot \frac{3}{2} \mid \frac{J - 1}{J}\right) \]  
(14)

\[ J^{\frac{1}{6}}(1 - J)^{\frac{1}{6}} F\left(\frac{1}{12}, \frac{7}{12} \cdot \frac{2}{3} \mid \frac{J}{J - 1}\right) \]  
(15)

\[ J^{\frac{1}{2}}(1 - J)^{-\frac{1}{6}} F\left(\frac{5}{12}, \frac{11}{12} \cdot \frac{4}{3} \mid \frac{J}{J - 1}\right) \]  
(16)
Remark 46. The two solutions treated by Beukers and Wolfart [BW86] are (13) and (15).

2.4.2 Linear Differential Equation of Order Two, Riemann Scheme and Kummer’s 24 Solutions

Linear Differential Equations

Let’s first recall some facts on homogenous linear differential equations of order two ([Ah] Ch8 4)

\[ a_0(z)u'' + a_1(z)u' + a_2(z)u = 0, \]  \hspace{1cm} (2.7)

with single-valued analytic functions \( a_i \) for coefficients. Writing (2.7) in the form

\[ u'' + p(z)u' + q(z)u = 0, \]  \hspace{1cm} (2.8)

the nature of local solutions will be determined by that of \( p \) and \( q \).

Definition 2.5. A point \( z_0 \) is called an ordinary point of (2.8) if \( p \) and \( q \) are analytic in a neighbourhood of \( z_0 \).

It is known that in the neighbourhood of an ordinary point, there exist two linearly independent solutions, which are analytic in a neighbourhood of this point ([Ah] Ch8 §4.1).

Definition 2.6. If \( p \) or \( q \) has a pole at the point \( z_0 \), \( z_0 \) is said to be a singular point of equation (2.8).

There are different kind of singular points. The ones that interest us are the regular singular ones.

Definition 2.7. The point \( z_0 \) is called regular singular, if \( p \) has at most a simple pole at \( z_0 \) and \( q \) at most a double pole at \( z_0 \).

If \( z_0 \) is a regular singular point, the differential equation has solutions about \( z_0 \) of the form

\[ u(z) = (z - z_0)^a g(z), \]  \hspace{1cm} (2.9)
where \( \alpha \in \mathbb{Q} \) and \( g \) is analytic at \( z_0 \) and satisfy \( g(z_0) \neq 0 \). Write
\[
p(z) = \frac{p(z)}{z-z_0} + \ldots \quad \text{and} \quad q(z) = \frac{q(z)}{(z-z_0)^2} + \ldots
\]
for the developments of \( p \) and \( q \) at \( z_0 \). Then, if \((z - z_0)^\alpha g(z)\) is a solution at \( z_0 \) of Differential Equation (2.8), the fact that \( z_0 \) is regular singular implies the following condition on \( \alpha \)
\[
\alpha(\alpha - 1) - p_{-1}\alpha - q_{-2} = 0.
\]
This quadratic equation is called the **indicial equation** and its solutions \( \alpha_1, \alpha_2 \) the **characteristic exponents** to (2.8) at \( z_0 \). The two solutions \((z - z_0)^\alpha g_1(z)\) and \((z - z_0)^\beta g_2(z)\) are linearly independent if and only if \( \alpha_2 - \alpha_1 \notin \mathbb{Z} \). In the case of an integral difference \( \alpha_2 - \alpha_1 \geq 0 \), the existence of a solution corresponding to \( \alpha_2 \) (involving a logarithm) can still be stated ([Ah] Ch8 §4.2).

If the coefficients \( a_0, a_1, a_2 \) are polynomials, we are interested to know their behaviour “at infinity” ([Ah] Ch8 §4.3). This can be done either by going to homogeneous coordinates \((z_0 : z_1)\) with \( z = \frac{z_0}{z_1} \) or by setting \( z = \frac{1}{w} \). The nature of the original equation at infinity will be that of the new one at respectively \( z_0 = 0 \) or \( w = 0 \). Actually, this amounts to compactify the domain of definition of the equation, in going from the complex plane to the (complex) projective line.

The first non-trivial linear differential equation of order two is that with three regular singularities ([Ah] Ch8 §4.4). Since any linear transformation of the variable preserves the type of the equation and the character of its singularities, we can suppose the singularities to lie at \( 0, 1 \) and \( \infty \). Let \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \) be the pairs of characteristic exponents at \( 0, 1 \) and \( \infty \) respectively, the fact that the singularities are regular implies the relation
\[
\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 = 1 \tag{2.10}
\]
and that Differential Equation (2.8) can be written in the form
\[
u'' + \left( \frac{1 - \alpha_1 - \alpha_2}{z} + \frac{1 - \beta_1 - \beta_2}{z - 1} \right) u' + \left( \frac{\alpha_1 \alpha_2 + \beta_1 \beta_2 - \gamma_1 \gamma_2}{z^2} + \frac{\beta_1 \beta_2}{z(z - 1)^2} \right) u = 0. \tag{2.11}
\]
Let’s now assume that
\[
\alpha_1 - \alpha_2, \beta_1 - \beta_2 \text{ and } \gamma_1 - \gamma_2 \notin \mathbb{Z}. \tag{2.12}
\]
Then, in the neighbourhood of each regular singular point, there exist two linearly independent solutions of the form (2.9) and Differential Equation (2.11) can be reduced to

\[ u'' + \left( \frac{c}{z} + \frac{1-c+a+b}{z-1} \right) u' + \frac{ab}{z(z-1)} u = 0 \]

or equivalently

\[ z(1-z)u'' + [c - (a + b + 1)z]u' - abu = 0, \]

where

\[ a = \alpha_1 + \beta_1 + \gamma_1, \quad b = \alpha_1 + \beta_1 + \gamma_2 \text{ and } c = 1 + \alpha_1 - \alpha_2. \]

Since (2.10) implies \( \beta_2 - \beta_1 = c - a - b \), condition (2.12) translates then to

\[ c - 1, \quad a - b, \quad a + b - c \notin \mathbb{Z}. \quad (2.13) \]

This is the well-known hypergeometric differential equation whose pairs of characteristic exponents at 0, 1 and \( \infty \) are respectively

\[ 0, \quad \alpha_2 - \alpha_1, \quad 0, \quad \beta_2 - \beta_1, \quad \gamma_1 + \alpha_1 + \beta_1, \quad \gamma_2 + \alpha_1 + \beta_1, \]

i.e.

\[ 0, \quad 1 - c, \quad 0, \quad c - a - b, \quad a, b. \]

Its solutions are equal to those of (2.11) multiplied by \( z^{-\alpha_1}(z-1)^{-\beta_1} \). The hypergeometric series \( F(a, b, c; z) \) is the holomorphic solution at 0, where it takes the value 1.

### Riemann Scheme

Riemann was very keen on the idea that a complex function could be characterized by its behaviour at its singularities ([Ah] Ch8 §4.5). To a linear differential equation of order two and with regular singularities at 0, 1 and \( \infty \), such as Differential Equation (2.11), he associated the later so-called Riemann scheme

\[
\begin{pmatrix}
0 & 1 & \infty \\
\alpha_1 & \beta_1 & \gamma_1 \\
\alpha_2 & \beta_2 & \gamma_2
\end{pmatrix}
\]  

(2.14)
2.4. **Sixteen Solutions**

It is unique up to permutation of the columns and of the characteristic exponents at a same point. As seen above, such data characterize the local solutions to the equation. Riemann has denoted the set of all local solutions to Differential Equation (2.11) by the symbol

\[
P = \begin{pmatrix}
0 & 1 & \infty \\
\alpha_1 & \beta_1 & \gamma_1 \\
\alpha_2 & \beta_2 & \gamma_2 
\end{pmatrix}.
\]

With this notation, we can nicely summarize what we have just done in reducing Differential Equation (2.11) to the hypergeometric differential equation. Indeed, we have

\[
P \begin{pmatrix}
0 & 1 & \infty \\
\alpha_1 & \beta_1 & \gamma_1 \\
\alpha_2 & \beta_2 & \gamma_2 
\end{pmatrix} = z^{\alpha_1}(1-z)^{\beta_1} P \begin{pmatrix}
0 & 1 & \infty \\
\alpha_2 - \alpha_1 & \beta_2 - \beta_1 & \gamma_2 + \alpha_1 + \beta_1 
\end{pmatrix}.
\]

It can be shown that, to each Riemann scheme (2.14) satisfying condition (2.10), there exist a unique linear differential equation of order two with the given characteristic exponents at 0, 1 and \(\infty\) (see also [IKSY] 2.1.1).

**Kummer's 24 Solutions**

Let's consider the following hypergeometric differential equation

\[
u'' + \left(\frac{c}{z} + \frac{1-c+a+b}{z-1}\right)u' + \frac{ab}{z(z-1)}u = 0,
\]

and denote it by \(E(a, b, c)\). We have already mentioned that \(F(a, b, c; z)\) is the solution at 0 with characteristic exponent 0. Since \(1-c\) is the second characteristic exponent at 0, there exists a solution to (2.15) of the form \(z^{1-c}g(z)\), with \(g\) analytic at 0 and \(g(0) \neq 0\). Inserting it in (2.15), one finds that \(g\) satisfies a hypergeometric equation with characteristic exponents

\[
c - 1, 0, 0, c - a - b, a + 1 - c, b + 1 - c
\]

at 0, 1, \(\infty\) respectively. Hence

\[
g \in P \begin{pmatrix}
0 & 1 & \infty \\
c - 1 & 0 & a + 1 - c \\
0 & c - a - b & b + 1 - c 
\end{pmatrix}.
\]
i.e. $g$ is a solution of $E(a + 1 - c, b + 1 - c, 2 - c)$ and since it is holomorphic at 0 and $g(0) \neq 0$, we can set

$$g(z) = F(a + 1 - c, b + 1 - c, 2 - c; z).$$

In the same way, one can find other solutions to (2.15), only by exchanging the points 0, 1, $\infty$ and requiring that the only two characteristic exponents equal to zero correspond to the points 0 and 1 respectively. This means that the hypergeometric differential equation (2.15) is transformed into another hypergeometric differential equation. We present here Kummer's idea to this aim and follow the exposition of [IKSY] 2.1.3.

First remark that the full group of permutations of the set $\{0, 1, \infty\}$ is

$$H := \text{Aut}(\mathbb{P}_C^1 - \{0, 1, \infty\}) = \{z \mapsto \frac{1}{z}, \frac{1}{1-z}, \frac{z-1}{z}, \frac{z}{z-1}\}.$$

To each $h \in H$, we can associate a not unique vector $\mu = (\mu_0, \mu_1, \mu_\infty)$ satisfying $\mu_0 + \mu_1 + \mu_\infty = 0$ (in order for the sum of all characteristic exponents to still be equal to 1, cf Condition (2.10)) such that

$$\begin{bmatrix}
  h(0) \\
  h(1) \\
  h(\infty)
\end{bmatrix} = \begin{bmatrix}
  0 + \mu_{h(0)} \\
  0 + \mu_{h(1)} \\
  a + \mu_{h(\infty)}
\end{bmatrix} = \begin{bmatrix}
  1-c + \mu_{h(0)} \\
  c-a-b + \mu_{h(1)} \\
  b + \mu_{h(\infty)}
\end{bmatrix},$$

is in the form $E(a', b', c')$ for some $a', b', c' \in \mathbb{C}$. In our above example, we had $h = \text{identity}$ and $\mu = (c-1, 0, 1-c)$.

To each $h$, there correspond actually exactly four vectors $\mu$, since there are four ways to annihilate two among the four characteristic exponents corresponding together to 0 and 1 ($\mu_\infty$ being then determined by the condition $\mu_0 + \mu_1 + \mu_\infty = 0$). This produces 24 solutions to (2.15), each of which involving a hypergeometric series. These 24 solutions are called Kummer's 24 solutions.

Writing $u_{z_0}(z, \alpha)$ for the solution at $z_0$ with characteristic exponent $\alpha$, we tabulate the 24 solutions. The first four equalities hold because these functions are holomorphic solutions at $z_0 = 0$ having constant term equal to 1 and we know that there is only one solution to (2.15) with this property. These first four equalities imply the subsequent ones.
\[ u_0(z, 0) = F(a, b, c; z) \]
\[ = (1 - z)^{-a-b} F(c - b, c - a, c; z) \]
\[ = (1 - z)^{-a} F(c - b, a, c; \frac{z}{1-z}) \]
\[ = (1 - z)^{-b} F(c - a, b, c; \frac{z}{1-z}) \]

\[ u_0(z, 1 - c) = z^{1-c} F(a - c + 1, b - c + 1, 2 - c; z) \]
\[ = z^{1-c} (1 - z)^{-a-b} F(1 - a, 1 - b, 2 - c; z) \]
\[ = z^{1-c} (1 - z)^{-c-a} F(1 - b, a + 1 - c, 2 - c; \frac{z}{1-z}) \]
\[ = z^{1-c} (1 - z)^{-c-b} F(1 - a, b + 1 - c, 2 - c; \frac{z}{1-z}) \]

\[ u_1(z, 0) = F(a, b, a + b - c + 1; 1 - z) \]
\[ = z^{1-c} F(b + 1 - c, a + 1 - c, a + b + 1 - c; 1 - z) \]
\[ = z^{-a} F(a + 1 - c, a, a + b + 1 - c; \frac{z}{1-z}) \]
\[ = z^{-b} F(b + 1 - c, b, a + b + 1 - c; \frac{z}{1-z}) \]

\[ u_1(z; c - a - b) = (1 - z)^{-c-a-b} F(c - a, c - b, c + 1 - a - b; 1 - z) \]
\[ = z^{1-c} (1 - z)^{-c-a} F(1 - a, 1 - b, c + 1 - a - b; 1 - z) \]
\[ = z^{b-c} (1 - z)^{-c-a} F(1 - b, c - b, c + 1 - a - b; \frac{z}{1-z}) \]
\[ = z^{a-c} (1 - z)^{-c-a} F(1 - a, c - a, c + 1 - a - b; \frac{z}{1-z}) \]

\[ u_\infty(z; a) = z^{-a} F(a, a - c + 1, a + 1 - b; \frac{1}{z}) \]
\[ = (-z)^{b-c} (1 - z)^{-c-a} F(1 - b, c - b, a + 1 - b; \frac{1}{z}) \]
\[ = (1 - z)^{-a} F(a, c - b, a + 1 - b; \frac{1}{1-z}) \]
\[ = (-z)^{1-c} (1 - z)^{-c-a} F(a + 1 - c, 1 - b, a + 1 - b; \frac{1}{1-z}) \]

\[ u_\infty(z; b) = z^{-b} F(b, b - c + 1, b + 1 - a; \frac{1}{z}) \]
\[ = (-z)^{a-c} (1 - z)^{-c-a} F(1 - a, c - a, b + 1 - a; \frac{1}{z}) \]
\[ = (1 - z)^{-b} F(b, c - a, b + 1 - a; \frac{1}{1-z}) \]
\[ = (-z)^{1-c} (1 - z)^{-c-b} F(b + 1 - c, 1 - a, b + 1 - a; \frac{1}{1-z}) \]
2.4.3 A New Eye on our Solutions

Since Differential Equation (2.3) has only three singularities, which are regular and lie at 0, 1, and ∞, we can find some solutions by Kummer’s method and with Riemann’s notation as described in the previous section.

In order to determine the characteristic exponents of Differential Equation (2.3), we compare its coefficients with those of Differential Equation (2.11) and resolve with respect to \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \). We get

\[
\begin{align*}
\frac{1}{6}, & \quad -\frac{1}{6} \quad \text{at } 0, \\
\frac{3}{4}, & \quad \frac{1}{4} \quad \text{at } 1, \\
0, & \quad 0 \quad \text{at } \infty.
\end{align*}
\]

Hence the Riemann scheme associated to (2.3) is

\[
\begin{pmatrix}
0 & 1 & \infty \\
\frac{1}{6} & \frac{3}{4} & 0 \\
-\frac{1}{6} & \frac{1}{4} & 0
\end{pmatrix}.
\]

Wanting to reduce to a hypergeometric equation, we use

\[
P \begin{pmatrix}
0 & 1 & \infty \\
\frac{1}{6} & \frac{3}{4} & 0 \\
-\frac{1}{6} & \frac{1}{4} & 0
\end{pmatrix} \cdot J = J^{\frac{1}{6}}(1 - J)^{\frac{3}{2}} P \begin{pmatrix}
0 & 1 & \infty \\
0 & 0 & \frac{11}{12} \\
-\frac{1}{3} & -\frac{1}{2} & \frac{11}{12}
\end{pmatrix} \cdot J,
\]

where

\[
\begin{pmatrix}
0 & 1 & \infty \\
0 & 0 & \frac{11}{12} \\
-\frac{1}{3} & -\frac{1}{2} & \frac{11}{12}
\end{pmatrix} = E(\frac{11}{12}, \frac{11}{12}, \frac{4}{3}).
\]

Remark 47.

- One immediately notes that, since \( a = b \), our equation does not belong to the general case, where we could ensure the existence at each regular singular
2.4. Sixteen Solutions

point of two linearly independent solutions of the form $z^a g(z)$, with $g$ analytic and not zero at the point. However, we can still use Riemann's notation and Kummer's method to get new solutions to the equation. We just have to pay attention to the fact that this method will not produce two linearly independent solutions at infinity, because the characteristic exponents at infinity differ from an integer. This is also the reason why we only have 16 solutions, since the solutions which are symmetric in $a$ and $b$ fall together.

- The orders of the generators of the monodromy group of the hypergeometric equation being the absolute values of the inverses of the characteristic exponents differences at each singular point respectively, we can read them easily in the Riemann scheme, they read

$$3, \quad 2, \quad \infty.$$ 

Let's now tabulate Kummer's solutions to Differential Equation (2.3) (on the right) by multiplying those to $E(\frac{11}{12}, \frac{11}{12}, \frac{4}{3})$ (on the left) by $J^{\frac{1}{6}} (1 - J)^{\frac{1}{3}}$:

$$u_0(J, 0) =$$

$$\begin{align*}
F(\frac{11}{12}, \frac{11}{12}, \frac{4}{3}; J) & \quad J^{\frac{1}{6}} (1 - J)^{\frac{1}{3}} F(\frac{11}{12}, \frac{11}{12}, \frac{4}{3}; J) \\
(1 - J)^{-\frac{1}{6}} F(\frac{5}{12}, \frac{5}{12}, \frac{4}{3}; J) & \quad J^{\frac{1}{6}} (1 - J)^{\frac{1}{3}} F(\frac{5}{12}, \frac{5}{12}, \frac{4}{3}; J) \\
(1 - J)^{-\frac{1}{6}} F(\frac{5}{12}, \frac{11}{12}, \frac{4}{3}; J; J^{-1}) & \quad J^{\frac{1}{6}} (1 - J)^{\frac{1}{3}} F(\frac{5}{12}, \frac{11}{12}, \frac{4}{3}; J; J^{-1})
\end{align*}$$

$$u_0(J, 1 - c) =$$

$$\begin{align*}
J^{-\frac{1}{3}} F(\frac{7}{12}, \frac{7}{12}, \frac{2}{3}; J) & \quad J^{-\frac{1}{3}} (1 - J)^{\frac{1}{6}} F(\frac{7}{12}, \frac{7}{12}, \frac{2}{3}; J) \\
J^{-\frac{1}{3}} (1 - J)^{-\frac{1}{3}} F(\frac{1}{12}, \frac{1}{12}, \frac{2}{3}; J) & \quad J^{-\frac{1}{3}} (1 - J)^{\frac{1}{6}} F(\frac{1}{12}, \frac{1}{12}, \frac{2}{3}; J) \\
J^{-\frac{1}{3}} (1 - J)^{-\frac{1}{3}} F(\frac{7}{12}, \frac{11}{12}, \frac{2}{3}; J; J^{-1}) & \quad J^{-\frac{1}{3}} (1 - J)^{\frac{1}{6}} F(\frac{7}{12}, \frac{11}{12}, \frac{2}{3}; J; J^{-1})
\end{align*}$$

$$u_1(J, 0) =$$

$$\begin{align*}
F(\frac{11}{12}, \frac{11}{12}, \frac{3}{2}; 1 - J) & \quad J^{\frac{1}{3}} (1 - J)^{\frac{1}{6}} F(\frac{11}{12}, \frac{11}{12}, \frac{3}{2}; 1 - J) \\
J^{-\frac{1}{3}} F(\frac{7}{12}, \frac{7}{12}, \frac{3}{2}; 1 - J) & \quad J^{-\frac{1}{3}} (1 - J)^{\frac{1}{6}} F(\frac{7}{12}, \frac{7}{12}, \frac{3}{2}; 1 - J) \\
J^{-\frac{1}{3}} F(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}; J; J^{-1}) & \quad J^{-\frac{1}{3}} (1 - J)^{\frac{1}{6}} F(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}; J; J^{-1})
\end{align*}$$
\[ u_1(J, c - a - b) = \begin{cases} 
(1 - J)^{-\frac{1}{3}} F\left(\frac{5}{12}, \frac{5}{12}, \frac{1}{2}; 1 - J\right) & J^{\frac{1}{3}} (1 - J)^{\frac{1}{3}} F\left(\frac{5}{12}, \frac{5}{12}, \frac{1}{2}; 1 - J\right) \\
(1 - J)^{-\frac{1}{3}} F\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{2}; 1 - J\right) & J^{\frac{1}{3}} (1 - J)^{\frac{1}{3}} F\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{2}; 1 - J\right) \\
(1 - J)^{-\frac{5}{12}} F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{2}{1 - J}\right) & J^{\frac{5}{12}} (1 - J)^{\frac{5}{12}} F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{2}{1 - J}\right) 
\end{cases} \]

\[ u_\infty(J, a) = u_\infty(J, b) = \begin{cases} 
J^{-\frac{1}{12}} F\left(\frac{11}{12}, \frac{7}{12}, 1; \frac{1}{J}\right) & J^{\frac{1}{12}} (1 - J)^{\frac{1}{12}} F\left(\frac{11}{12}, \frac{7}{12}, 1; \frac{1}{J}\right) \\
J^{-\frac{5}{12}} (1 - J)^{-\frac{1}{12}} F\left(\frac{1}{12}, \frac{5}{12}, 1; \frac{1}{J}\right) & J^{\frac{5}{12}} (1 - J)^{-\frac{5}{12}} F\left(\frac{1}{12}, \frac{5}{12}, 1; \frac{1}{J}\right) \\
(1 - J)^{-\frac{11}{12}} F\left(\frac{5}{12}, \frac{11}{12}, 1; \frac{1}{1 - J}\right) & J^{\frac{11}{12}} (1 - J)^{-\frac{11}{12}} F\left(\frac{5}{12}, \frac{11}{12}, 1; \frac{1}{1 - J}\right) \\
J^{-\frac{1}{6}} (1 - J)^{-\frac{5}{6}} F\left(\frac{1}{12}, \frac{7}{12}, 1; \frac{1}{1 - J}\right) & J^{\frac{1}{6}} (1 - J)^{-\frac{1}{6}} F\left(\frac{1}{12}, \frac{7}{12}, 1; \frac{1}{1 - J}\right). 
\end{cases} \]

2.5 Twenty-two Identities

As announced in the introduction of this chapter (Section 2.1), we shall now make use of our extension (cf Section 2.3) of Beukers' and Wolfart's method in order to produce at least one identity for each of our solutions to Differential Equation (2.3).

According to Section 2.3, for each solution \( J^\alpha (1 - J)^\beta F(a, b, c; z(J)) \), there exist \( A, B \in \mathbb{C} \) such that

\[ J(\tau)^\alpha (1 - J(\tau))^\beta F(a, b, c; z(J(\tau))) = (A \tau + B) J(\tau)^{-\frac{1}{2}} (J(\tau) - 1)^{\frac{1}{4}} \Delta(\tau)^{\frac{1}{12}}. \]

(2.16)

We have now to choose a domain of definition on which both members of the equality may be well-defined as holomorphic functions of \( \tau \). Since the hypergeometric series converges in the unit disk, the first step is to restrict to \( \tau \in \mathbb{H} \) such that \( |z(J(\tau))| < 1 \), that is to work locally at a point \( \tau_0 \in \mathbb{H} \) such that \( z(J(\tau_0)) = 0 \).

**Notation 1.** For \( \tau_0 \in \{i, \rho, -\rho, \infty\} \) and \( z(J) \) a transformation of \( J \) among those of (2.6) such that \( z(J(\tau_0)) = 0 \), \( C_{z(J), \tau_0} \) will denote the neighbourhood of \( \tau_0 \) in \( \{ \tau \in \mathbb{H}; |z(J(\tau))| < 1 \} \).

Note that \( C_{z(J), \tau_0} \) is connected, because \( J \) resp. \( \frac{1}{J} \), \( \frac{1}{1 - J} \) behave in such neighbourhoods of \( i, \rho, -\rho \) resp. \( \infty \) like \( n \)-fold coverings (\( n = 2 \) at \( i \), \( 3 \) at \( \rho \) and \( -\rho \) resp. \( 1 \).
2.5. Twenty-two Identities

at \( \infty \). In particular, \( C_{z(J),\tau_0} \) is invariant under each modular transformation fixing \( \tau_0 \), because it is connected and because \( J \) is invariant under such a transformation. If such a modular transformation is applied to Identity (2.16), the invariance of the left-hand side will imply a relation between the constants \( A \) and \( B \). This is the method we will use to have stronger identities. The solutions in a neighbourhood of \( J = 0, 1, \infty \) respectively will be treated separately.

### 2.5.1 About \( J = 0 \)

The solutions about \( J = 0 \) are of the form \( J^\alpha (1 - J)^\beta F(\alpha, b, c; z(J)) \), with

\[
z(J) \in \{ J, \frac{J}{J-1} \} \quad \text{and} \quad \alpha \in \left\{ \frac{1}{6}, -\frac{1}{6} \right\}.
\]

If \( \tau \) is such that \( |J(x)| < 1 \) or \( |J(x)| < 1 \), then \( J(x) \neq 1 \). This implies that, if \( z(J) \in \{ J, \frac{J}{J-1} \} \) and \( \tau_0 \in \{ \rho, -\bar{\rho} \} \), then each branch of \( (J(\tau) - 1)^{\frac{1}{6} - \alpha} \) is well-defined on \( C_{z(J),\tau_0} \) as holomorphic function of \( \tau \). Because \( \Delta \) does not vanish on \( \mathbb{H} \), the same holds for \( \Delta(\tau)^{\frac{1}{12}} \). There is only one term to worry about, which is \( J(\tau)^{-\frac{1}{6} - \alpha} \), but it will appear to be easy to handle, because the only values taken \( \alpha \) are \( -\frac{1}{6} \) and \( \frac{1}{6} \).

### In the Neighbourhood of \( \rho \)

The modular transformation \( \tau \mapsto -\frac{1}{\tau + 1} \) fixes \( \rho \) and therefore preserves \( C_{z(J),\rho} \). Since \( J \) is invariant under such a transformation, so is the left-hand side of (2.16) and so must be the right-hand side. Hence, the following condition must hold for all \( \tau \in C_{z(J),\rho} \) and for one 12-th root \( \zeta \) of unity.

\[
A \tau + B = \zeta \left( -\frac{A}{\tau + 1} + B \right) (\tau + 1)
\]

\[
\Leftrightarrow \quad A \tau + B = B \zeta \tau + \zeta (B - A)
\]

\[
\Rightarrow \quad A = \zeta B \quad \text{and} \quad B = \zeta (A - B)
\]

\[
\Rightarrow \quad \zeta^{-1} = 1 - \zeta
\]

\[
\Leftrightarrow \quad \zeta \in \{ \rho, -\bar{\rho} \}
\]

and \( B = -\bar{\rho} A \) or \( B = -\rho A \).
2. Identities for some Gauss’ Hypergeometric Series

Considering our solutions about \( J = 0 \), we distinguish two cases.

- **Case 1:** \( \alpha = \frac{1}{6} \), \( \mathcal{F}(a, b, c; z(J(\tau))) = (A\tau + B) J(\tau)^{-\frac{1}{3}} (J(\tau) - 1)^{\frac{1}{12}} \Delta (\tau)^{-\frac{1}{12}} \).
  
  The left-hand side and \( (J - 1)^{\frac{1}{12}} \Delta^{\frac{1}{12}} \) are holomorphic at \( \rho \), but \( J^{-\frac{1}{3}} \) has a pole of order 1 at this point. Hence, \( A\tau + B \) must have a zero of order 1 at \( \rho \), that is
  
  \[ B = -\rho A \]

  must hold. Then \( (A\tau + B) J(\tau)^{-\frac{1}{3}} \) is holomorphic and well-defined on \( C_{z(J(\tau), \rho)} \).

- **Case 2:** \( \alpha = -\frac{1}{6} \), \( \mathcal{F}(a, b, c; z(J(\tau))) = (A\tau + B)(J(\tau) - 1)^{\frac{1}{12}} \Delta^{\frac{1}{12}} \).

  Since the left-hand side and \( (J - 1)^{\frac{1}{12}} \Delta^{\frac{1}{12}} \) are holomorphic and nonzero at \( \rho \), so must \( A\tau + B \) be. This implies the relation
  
  \[ B = -\bar{\rho} A \]

  This shows that in both cases, the identity (2.16) is well-defined on \( C_{z(J(\tau), \rho)} \). Since this domain is connected, \( \Delta^{\frac{1}{12}} \) can be replaced on it by \( \eta^2 \) up to a unique complex constant. Hence, we have shown that

  \[ \exists A \in \mathbb{C} \text{ such that } \forall \tau \in C_{J, \rho} \text{ we have} \]

  \[ F\left(\frac{1}{12}, \frac{1}{12}, \frac{2}{3}; J(\tau)\right) = A(\tau - \bar{\rho}) \eta(\tau)^2 \]  \hspace{1cm} (1)

  \[ F\left(\frac{5}{12}, \frac{5}{12}, \frac{4}{3}; J(\tau)\right) = A(\tau - \rho) J(\tau)^{-\frac{1}{3}} \eta(\tau)^2 \]  \hspace{1cm} (2)

  \[ F\left(\frac{7}{12}, \frac{7}{12}, \frac{2}{3}; J(\tau)\right) = A(\tau - \bar{\rho})(J(\tau) - 1)^{-\frac{1}{12}} \eta(\tau)^2 \]  \hspace{1cm} (3)

  \[ F\left(\frac{11}{12}, \frac{11}{12}, \frac{4}{3}; J(\tau)\right) = A(\tau - \rho) J(\tau)^{-\frac{1}{3}} (J(\tau) - 1)^{-\frac{1}{12}} \eta(\tau)^2 \]  \hspace{1cm} (4)

  and \( \exists A \in \mathbb{C} \) such that \( \forall \tau \in C_{J, \rho} \):

  \[ F\left(\frac{1}{12}, \frac{7}{12}, \frac{2}{3}; J(\tau) - 1\right) = A(\tau - \bar{\rho})(J(\tau) - 1)^{\frac{1}{12}} \eta(\tau)^2 \]  \hspace{1cm} (15)

  \[ F\left(\frac{5}{12}, \frac{11}{12}, \frac{4}{3}; J(\tau) - 1\right) = A(\tau - \rho) J(\tau)^{-\frac{1}{3}} (J(\tau) - 1)^{\frac{5}{12}} \eta(\tau)^2. \]  \hspace{1cm} (16)
Remark 48. Note that these identities correspond to Kummer’s identities listed at the end of §2.4.2. For example

\begin{align*}
(2) \quad & (J(\tau) - 1)^{-\frac{1}{2}} F\left(\frac{5}{12}, \frac{5}{12}, \frac{4}{3}; J(\tau)\right) & = & A(\tau - \rho) J(\tau)^{-\frac{1}{3}} \\
(4) \quad & F\left(\frac{11}{12}, \frac{11}{12}, \frac{4}{3}; J(\tau)\right) & = & (J(\tau) - 1)^{-\frac{1}{2}} \eta(\tau)^2.
\end{align*}

In the case \(\alpha = -\frac{1}{6}\), the constant \(A\) is very easy to calculate. Indeed, setting \(\tau = \rho\) in each identity, we get

\[ F\left(\frac{1}{12}, \frac{11}{12}, \frac{2}{3}; J(\tau)\right) = \frac{\tau - \rho}{i\sqrt{3}} \left(\frac{\eta(\tau)}{\eta(\rho)}\right)^2 \]

and

\[ F\left(\frac{1}{12}, \frac{7}{12}, \frac{2}{3}; J(\tau) - 1\right) = \frac{\tau - \rho}{i\sqrt{3}} (1 - J(\tau))^{\frac{1}{3}} \left(\frac{\eta(\tau)}{\eta(\rho)}\right)^2. \]

The branches of the roots are here determined by \((1)_{\frac{1}{2}} = 1\) resp. \((1)_{\frac{1}{3}} = 1\).

For the identities where \((\tau - \rho) J(\tau)^{-\frac{1}{3}}\) appears, the constant can be calculated in dependance of the residue \(c(-1)(J^{-\frac{1}{3}}, \rho)\) of \(J^{-\frac{1}{3}}\) at \(\rho\). Since \(J^{-\frac{1}{3}}\) has a pole of order \(1\) at \(\rho\), it has a Laurent-expansion at \(\rho\) of the form

\[ \frac{c(-1)(J^{-\frac{1}{3}}, \rho)}{\tau - \rho} + h(\tau) \]

where \(h(\tau) = \sum_{k=0}^{\infty} c_k(J^{-\frac{1}{3}}, \rho)(\tau - \rho)^k\) is the entire part and

\[ c(-1)(J^{-\frac{1}{3}}, \rho) = \frac{1}{2\pi i} \int_{|z - \rho| = r} J(z)^{-\frac{1}{3}} dz. \]
the residue. The radius $r$ satisfies $0 < r < R_\rho$, where $R_\rho$ is the radius of convergence of $h(\tau)$. Hence, for $\tau \in D(\rho, R_\rho)$, the open disc centered at $\rho$ with radius $R_\rho$, the product $(\tau - \rho)J(\tau)^{-\frac{1}{2}}$ equals
\[ c_{-1}(J^{-\frac{1}{2}}, \rho) + (\tau - \rho)h(\tau) \]
and takes the value $c_{-1}(J^{-\frac{1}{2}}, \rho)$ at $\rho$. This allows us to calculate the constant $A$ in each of the identities (2), (4) and (16). We get
\[ \forall \tau \in \mathbb{C}_{J, \rho}: \]
\[ F \left( \frac{5}{12}, \frac{5}{12}, -\frac{4}{3}; J(\tau) \right) = c_{-1}(J^{-\frac{1}{2}}, \rho)^{-1}(\tau - \rho)J(\tau)^{-\frac{1}{2}} \left( \frac{\eta(\tau)}{\eta(\rho)} \right)^2 \]  
(2)
\[ F \left( \frac{11}{12}, \frac{11}{12}, -\frac{4}{3}; J(\tau) \right) = c_{-1}(J^{-\frac{1}{2}}, \rho)^{-1}(\tau - \rho)J(\tau)^{-\frac{1}{2}}(1 - J(\tau))^{-\frac{1}{2}} \left( \frac{\eta(\tau)}{\eta(\rho)} \right)^2 \]  
(4)
and $\forall \tau \in \mathbb{C}_{J, -\rho}$:
\[ F \left( \frac{5}{12}, \frac{11}{12}, -\frac{4}{3}; J(\tau) - 1 \right) = c_{-1}(J^{-\frac{1}{2}}, \rho)^{-1}(\tau - \rho)J(\tau)^{-\frac{1}{2}}(1 - J(\tau))^{\frac{5}{12}} \left( \frac{\eta(\tau)}{\eta(\rho)} \right)^2. \]
(16)

The roots are determined respectively by $(1)\frac{5}{6} = (1)\frac{5}{12} = 1$.

**In the Neighbourhood of $-\tilde{\rho}$**

Let's now work in the neighbourhood $C_{z(J), -\tilde{\rho}}$ of $-\tilde{\rho}$. Since $F(a, b, c; z(J(\tau)))$ and $(J(\tau) - 1)^{\frac{1}{\beta} - \Delta(\tau)^{\frac{1}{\beta}}}$ are well-defined and holomorphic functions of $\tau$ in $C_{z(J), -\tilde{\rho}}$, so must be $(A\tau + B)J(\tau)^{-\frac{1}{\beta} - \alpha}$.

- The case $\alpha = \frac{1}{6}$ is the easiest one. Indeed, in this case, $J^{-\frac{1}{6} - \alpha} = J^{-\frac{1}{3}}$ has a pole of order 1 at $-\tilde{\rho}$. Hence, $A\tau + B$ must have a zero of order 1 at $\tau = -\tilde{\rho}$, i.e.
\[ B = -\tilde{\rho}A \]
must hold. Under this condition, \((A\tau + B)J(\tau)^{-\frac{1}{2}}\) is well-defined and holomorphic on \(C_{\zeta(J),-\bar{\rho}}\).

- In the case \(\alpha = -\frac{1}{6}\), \(J^{-\frac{1}{6}-\alpha}\) does not appear in the identity and no holomorphy condition implies any relation on \(A\) and \(B\). Remarking that the modular transformation \(\tau \mapsto \frac{\tau - 1}{\tau}\) fixes \(-\bar{\rho}\) and preserves \(C_{\zeta(J),-\bar{\rho}}\) and applying it to both members of the identity

\[
F(a, b, c; z(J(\tau))) = (A\tau + B)(J(\tau) - 1)^{\frac{1}{4} - \beta} \Delta(\tau)^{\frac{1}{12}},
\]

we get the following condition for the right-hand side to be invariant. It must be valid for all \(\tau\) in \(C_{\zeta(J),-\bar{\rho}}\) and one 12-th root \(\zeta\) of unity.

\[
A\tau + B = (A \frac{\tau - 1}{\tau} + B)\zeta \tau
\]

\(\Rightarrow\) \(A\tau + B = \zeta(A + B)\tau - \zeta A\)

\(\Rightarrow\) \(A = \zeta A + \zeta B\) and \(B = -\zeta A\)

\(\Rightarrow\) \(1 + \zeta^2 = \zeta\) and \(B = -\zeta A\)

\(\Rightarrow\) \(\zeta \in \{-\bar{\rho}, -\rho\}\) and \(B = -\zeta A\)

\(\Rightarrow\) \(B = \bar{\rho}\) or \(B = \rho A\).

Since the left-hand side of (2.20) is nonzero at \(-\bar{\rho}\), so must be the right-hand side, hence

\(B = \rho A\).

Our identities are valid on \(C_{\zeta(J),-\bar{\rho}}\). On such a connected domain, \(\Delta^{\frac{1}{12}}\) can be replaced by \(\eta^2\) up to a unique complex constant. Then

\(\exists A \in \mathbb{C}\) such that \(\forall \tau \in C_{\zeta(J),-\bar{\rho}}:\)

\[
F\left(\frac{1}{12}, \frac{1}{12}, \frac{2}{3}; J(\tau)\right) = A(\tau + \rho)\eta(\tau)^2
\]

(1’)

\[
F\left(\frac{5}{12}, \frac{5}{12}, \frac{4}{3}; J(\tau)\right) = A(\tau + \bar{\rho})J(\tau)^{-\frac{1}{2}}\eta(\tau)^2
\]

(2’)

\[
F\left(\frac{7}{12}, \frac{7}{12}, \frac{2}{3}; J(\tau)\right) = A(\tau + \rho)(J(\tau) - 1)^{-\frac{1}{2}}\eta(\tau)^2
\]

(3’)

\[
F\left(\frac{11}{12}, \frac{11}{12}, \frac{4}{3}; J(\tau)\right) = A(\tau + \bar{\rho})J(\tau)^{-\frac{1}{2}}(J(\tau) - 1)^{-\frac{1}{2}}\eta(\tau)^2
\]

(4’)

2.5. Twenty-two Identities
and $\exists A \in \mathbb{C}$ such that $\forall \tau \in C_{J^{\frac{1}{3}}, -\rho}$:

$$
F\left(\frac{1}{12}, 1; 2; \frac{J(\tau)}{J(\tau) - 1}\right) = A(\tau + \rho)(J(\tau) - 1)^{\frac{1}{13}} \eta(\tau)^2
$$  \hspace{1cm} (15')

$$
F\left(\frac{5}{12}, 11; 4; \frac{J(\tau)}{J(\tau) - 1}\right) = A(\tau + \rho)J(\tau)^{-\frac{1}{3}}(J(\tau) - 1)^{\frac{5}{13}} \eta(\tau)^2.
$$  \hspace{1cm} (16')

In the cases where $\alpha = -\frac{1}{6}$, the constant $A$ is easy to calculate. One gets

$\forall \tau \in C_{J, -\rho}$:

$$
F\left(\frac{1}{12}, 1, 2; \frac{J(\tau)}{J(\tau) - 1}\right) = \frac{\tau + \rho}{i \sqrt{3}} \left(\frac{\eta(\tau)}{\eta(-\rho)}\right)^2
$$  \hspace{1cm} (2.21)

$$
F\left(\frac{7}{12}, 7, 3; \frac{J(\tau)}{J(\tau) - 1}\right) = \frac{\tau + \rho}{i \sqrt{3}} (1 - J(\tau))^{-\frac{1}{3}} \left(\frac{\eta(\tau)}{\eta(-\rho)}\right)^2
$$  \hspace{1cm} (2.22)

and $\forall \tau \in C_{J^{\frac{1}{3}}, -\rho}$:

$$
F\left(\frac{1}{12}, 7, 2; \frac{J(\tau)}{J(\tau) - 1}\right) = \frac{\tau + \rho}{i \sqrt{3}} (1 - J(\tau))^{\frac{1}{3}} \left(\frac{\eta(\tau)}{\eta(-\rho)}\right)^2.
$$  \hspace{1cm} (2.23)

Again, the the branches of the roots are here determined by $(1)^{\frac{1}{2}} = 1$ resp. $(1)^{\frac{1}{13}} = 1$. Note that $\eta(-\rho)^3$ equals $-i \rho \eta(\rho)^2$ (cf §2.2.8).

For the identities where $(\tau + \rho)J(\tau)^{-\frac{1}{3}}$ appears, the constant can be calculated in dependance of the residue of $J^{-\frac{1}{3}}$ at $-\rho$. Let $R_{-\rho} > 0$ be such that, for $\tau$ in $D(-\rho, R_{-\rho})$, we have

$$
J(\tau)^{-\frac{1}{3}} = \frac{c_{-1}(J^{-\frac{1}{3}}, -\rho)}{\tau + \rho} + g(\tau)
$$

and the entire part $g(\tau) = \sum_{k \geq 0} c_k(J^{-\frac{1}{3}}, -\rho)(\tau + \rho)^k$ converges in $D(-\rho, R_{-\rho})$. The residue at $-\rho$ is

$$
c_{-1}(J^{-\frac{1}{3}}, -\rho) = \frac{1}{2\pi i} \int_{|z + \rho| = r} \frac{dz}{J(z)^{\frac{1}{3}}},
$$
2.5. Twenty-two Identities

where $0 < r < R_\tau$. The product $(\tau + \rho)J(\tau)^{-\frac{1}{2}}$ takes then the value $c_{-1}(J^{-\frac{1}{2}}, -\rho)$ at $-\rho$. Setting $\tau = -\rho$ in Identities (2'), (4') and (16'), we get

\[
\forall \tau \in C_{J, -\rho}:
F\left(\frac{5}{12}, \frac{5}{12}, \frac{11}{3}; J(\tau)\right) = c_{-1}(J^{-\frac{1}{2}}, -\rho)^{-1}(\tau + \rho)J(\tau)^{-\frac{1}{2}}\left(\eta(\tau)\right)^2
\]

\[
F\left(\frac{11}{12}, \frac{11}{12}, \frac{11}{3}; J(\tau)\right) = c_{-1}(J^{-\frac{1}{2}}, -\rho)^{-1}(\tau + \rho)J(\tau)^{-\frac{1}{2}}(1 - J(\tau))^{-\frac{1}{2}}\left(\eta(\tau)\right)^2.
\]

and $\forall \tau \in C_{J, \rho}$:

\[
F\left(\frac{5}{12}, \frac{11}{12}, \frac{4}{3}; J(\tau) - 1\right) = c_{-1}(J^{-\frac{1}{2}}, -\rho)^{-1}(\tau + \rho)J(\tau)^{-\frac{1}{2}}(1 - J(\tau))^{-\frac{1}{2}}\left(\eta(\tau)\right)^2.
\]

The branches of the roots are determined respectively by $1^{\frac{1}{2}} = 1^{\frac{1}{2}} = 1$.

2.5.2 About $J = 1$

We here deal with solutions $J^\alpha(1 - J)^\beta F(a, b, c; z(J))$ where,

\[
z(J) \in \{1 - J, \frac{J - 1}{J}\} \text{ and } \beta \in \left\{\frac{1}{4}, \frac{3}{4}\right\}.
\]

We here choose to work in the neighbourhood $C_{z(J), i}$ of $\tau = i$. Since $J$ and $\Delta$ do not take the value 0 on $C_{z(J), i}$, a unique determination of $J(\tau)^{-\frac{1}{2}}$ resp. of $\Delta(\tau)^{\frac{1}{2}}$ can be well-defined as holomorphic function of $\tau$ on $C_{z(J), i}$. Since $F(a, b, c; z(J(\tau)))$ is holomorphic on $C_{z(J), i}$, $(At + B)(J(\tau) - 1)^{-\frac{1}{2}}$ must also be so.

- In the case $\beta = \frac{3}{4}$, $(J(\tau) - 1)^{-\frac{1}{2}}$ has a pole of order 1 at $i$. Hence, $At + B$ must have a zero of order 1 at $i$, i.e.

\[
B = -iA
\]

must hold. With this relation, $(At + B)(J(\tau) - 1)^{-\frac{1}{2}}$ is well-defined and holomorphic on $C_{z(J), i}$. 

• In the case $\beta = \frac{1}{4}$, (2.16) reads

$$F(a, b, c; z(J(\tau))) = (A\tau + B)J(\tau)^{-\frac{1}{6}}\Delta(\tau)^{\frac{1}{12}}$$

and both members are holomorphic on $C_{z(J),i}$ without condition on $A$ and $B$. But a condition can still be obtained by applying the modular transformation $\tau \mapsto -\frac{1}{\tau}$ on $C_{z(J),i}$. This transformation indeed preserves $C_{z(J),i}$, because it is continuous and fixes $i$, while $C_{z(J),i}$ is connected. The invariance of $J$ under this transformation implies that of $(A\tau + B)\Delta(\tau)^{\frac{1}{12}}$. Hence, the following condition must hold for all $\tau \in C_{z(J),i}$ and one 12-th root $\zeta$ of unity

$$A\tau + B = \zeta(-\frac{A}{\zeta} + B)\tau$$

$$\Rightarrow A\tau + B = -\zeta A + \zeta B\tau$$

$$\Rightarrow A = \zeta B$$

$$\Rightarrow B = iA \text{ or } B = -iA.$$

Because $J$ and $F(a, b, c; z(J))$ are nonzero at $i$, so must be $(A\tau + B)\Delta(\tau)^{\frac{1}{12}}$ and the following relation must hold

$$B = iA.$$

In each case, we have shown that the identity is valid on $C_{z(J),i}$. Since this domain is connected, $\Delta^{\frac{1}{12}}$ can be replaced on it by $\eta^2$ up to a unique complex constant. Hence, we have

$$\exists \lambda \in \mathbb{C} \text{ such that } \forall \tau \in C_{z(J),i}:$$

$$F(\frac{1}{12}, \frac{1}{12}, \frac{1}{2}; 1 - J(\tau)) = A(\tau + i)\eta(\tau)^2$$

$$F(\frac{5}{12}, \frac{5}{12}, \frac{1}{2}; 1 - J(\tau)) = A(\tau + i)J(\tau)^{-\frac{1}{2}}\eta(\tau)^2$$

$$F(\frac{7}{12}, \frac{7}{12}, \frac{3}{2}; 1 - J(\tau)) = A(\tau - i)(J(\tau) - 1)^{-\frac{1}{2}}\eta(\tau)^2$$

$$F(\frac{11}{12}, \frac{11}{12}, \frac{3}{2}; 1 - J(\tau)) = A(\tau - i)(J(\tau) - 1)^{-\frac{1}{2}}J(\tau)^{-\frac{1}{2}}\eta(\tau)^2$$
and $\exists A \in \mathbb{C}$ such that $\forall \tau \in C_{\frac{1}{12}, i}$:

$$F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{J(\tau) - 1}{J(\tau)}\right) = A(\tau + i)J(\tau)\frac{1}{2i} \eta(\tau)^2$$

(13)

$$F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}, \frac{J(\tau) - 1}{J(\tau)}\right) = A(\tau - i)(J(\tau) - 1)^{-\frac{1}{2}}J(\tau)^{\frac{7}{12}} \eta(\tau)^2.$$

(14)

When $\beta = \frac{1}{4}$, it is easy to determinate the constant $A$. Indeed, setting $\tau = i$ in Identities (5),(6),(13), we get in each case $A = \frac{1}{2i^2}$ and, eventually,

$\forall \tau \in C_{1-i}:

F\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{2}, 1 - J(\tau)\right) = \frac{\tau + i}{2i} \left(\frac{\eta(\tau)}{\eta(i)}\right)^2$

(2.24)

and $\forall \tau \in C_{\frac{1}{12}, i}:

F\left(\frac{5}{12}, \frac{5}{12}, \frac{1}{2}, 1 - J(\tau)\right) = \frac{\tau + i}{2i} J(\tau)^{-\frac{1}{2}} \left(\frac{\eta(\tau)}{\eta(i)}\right)^2$

(2.25)

The roots are here determined respectively by $(1)^{\frac{1}{2}} = 1$ and $(1)^{\frac{1}{12}} = 1$.

**Remark 49.** Identities (13) and (15) are stated respectively in Proposition 2.1 and in the last remark of [BW86].

In the other cases, the constant can be determined in dependance of the residue of $(J(\tau) - 1)^{-\frac{1}{2}}$ at $\tau = i$. Since $J(i) = 1$ with order $2$, $(J(\tau) - 1)^{-\frac{1}{2}}$ has a pole of first order at $i$. Then there exists $R_i > 0$ such that, for $\tau \in D(i, R_i)$,

$$(J(\tau) - 1)^{-\frac{1}{2}} = \frac{c_{-1}(J - 1)^{-\frac{1}{2}}, i}{\tau - i} + f(\tau),$$

where the entire part $f(\tau)$ is convergent in $D(i, R_i)$ and

$$c_{-1}((J - 1)^{-\frac{1}{2}}, i) = \frac{1}{2\pi i} \int_{|z - i| = r} \frac{dz}{(J(z) - 1)^{\frac{1}{2}}}.$$
for any $r$ satisfying $0 < r < R_1$. The product $(\tau - i)(J(\tau) - 1)^{-\frac{1}{2}}$ takes then the value $c_{-1}((J - 1)^{-\frac{1}{2}}, i)$ at $i$. This implies, in each case among (7), (8), (14), that $A = c_{-1}((J - 1)^{-\frac{1}{2}}, i)^{-1}\eta(i)^{-2}$. Hence

$$\forall \tau \in C_{1-J,i}:$$

$$F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}; 1 - J(\tau)\right) = c_{-1}((J - 1)^{-\frac{1}{2}}, i)^{-1}(\tau - i)(J(\tau) - 1)^{-\frac{1}{2}}\left(\frac{\eta(\tau)}{\eta(i)}\right)^2$$

(7)

$$F\left(\frac{11}{12}, \frac{11}{12}, \frac{3}{2}; 1 - J(\tau)\right) = c_{-1}((J - 1)^{-\frac{1}{2}}, i)^{-1}(\tau - i)(J(\tau) - 1)^{-\frac{1}{2}}J(\tau)^{-\frac{1}{2}}\left(\frac{\eta(\tau)}{\eta(i)}\right)^2$$

(8)

and $\forall \tau \in C_{1-J,i}$:

$$F\left(\frac{7}{12}, \frac{11}{12}, \frac{3}{2}; \frac{J(\tau) - 1}{J(\tau)}\right) = c_{-1}((J - 1)^{-\frac{1}{2}}, i)^{-1}(\tau - i)(J(\tau) - 1)^{-\frac{1}{2}}J(\tau)^{-\frac{1}{2}}\left(\frac{\eta(\tau)}{\eta(i)}\right)^2.$$  

(14)

2.5.3 About $J = \infty$

We consider here the solutions $J^\alpha(1 - J)^\beta F(a, b, c; z(\tau))$ with

$$z(\tau) \in \{\frac{1}{J}, \frac{1}{1 - J}\} \text{ and } \alpha + \beta = 0.$$  

The functions $J$ and $\Delta$ are invariant under the modular transformation $\tau \mapsto \tau + 1$ and admit a Fourier expansion in $\eta := e^{2\pi i \tau}$ (cf §2.2.4) which is called expansion at infinity. These are given in §2.2.7 and §2.2.6 respectively and show that $\Delta$ (resp. $J$ and $1 - J$) has a zero (resp. pole) of order 1 at infinity. The application of the modular transformation $\tau \mapsto \tau + 1$ to Identity (2.16) and the invariance of $J$ and $\Delta$ under it show that $A\tau + B$ must also be invariant, hence that

$$A = 0.$$  

One verifies that $BJ(\tau)^{-\frac{1}{2} - \alpha}(J(\tau) - 1)^{\frac{1}{2} - \beta}\Delta(\tau)^{\frac{1}{2}}$ has then a pole of order $-\frac{1}{6} - \alpha + \frac{1}{2} - \beta - \frac{1}{12} = 0$ at infinity, because $\alpha + \beta = 0$. This corresponds to the holomorphy of $F(a, b, c; z(J(\tau)))$ at $\infty$. Hence, both members are well-defined.
and holomorphic in $C_{\tau(j),\infty}$. Replacing $\Delta_{12}^\frac{1}{2}$ by $\eta^2$ up to a complex constant, which is unique in $C_{\tau(j),\infty}$, we have

$\exists B \in \mathbb{C}$ such that $\forall \tau \in C_{\frac{1}{12},\infty}$:

$$F\left(\frac{5}{12}, \frac{11}{12}, 1; \frac{1}{J(\tau)}\right) = BJ(\tau)^{\frac{7}{12}}(J(\tau) - 1)^{-\frac{1}{2}}\eta(\tau)^2$$  \hspace{1cm} (10)$$

and $\exists B \in \mathbb{C}$ such that $\forall \tau \in C_{\frac{1}{12},\infty}$:

$$F\left(\frac{1}{12}, \frac{7}{12}, 1; \frac{1}{1 - J(\tau)}\right) = BJ(\tau)^{-\frac{1}{2}}(J(\tau) - 1)^{\frac{5}{12}}\eta(\tau)^2.$$  \hspace{1cm} (12)$$

Since $\alpha + \beta = 0$ in each case, the constant term of the development at infinity of $J^{-\frac{1}{6} - a}(J - 1)^{\frac{1}{4} - \beta}^2$ is $(1728^{-\frac{1}{6} - a + \frac{1}{4} - \beta})^{-1} = 1728^{-\frac{1}{12}}$. This implies that, in each case, $B = 1728^{\frac{1}{12}} = \sqrt[12]{12}$. Inserting this value, we get

$\forall \tau \in C_{\frac{1}{12},\infty}$:

$$F\left(\frac{5}{12}, \frac{11}{12}, 1; \frac{1}{J(\tau)}\right) = \sqrt[12]{12}J(\tau)^{\frac{5}{12}}(J(\tau) - 1)^{-\frac{1}{2}}\eta(\tau)^2.$$  \hspace{1cm} (2.28)$$

and $\forall \tau \in C_{\frac{1}{12},\infty}$:

$$F\left(\frac{1}{12}, \frac{7}{12}, 1; \frac{1}{1 - J(\tau)}\right) = \sqrt[12]{12}(J(\tau) - 1)^{\frac{5}{12}}\eta(\tau)^2.$$  \hspace{1cm} (2.29)$$

and $\forall \tau \in C_{\frac{1}{12},\infty}$:

$$F\left(\frac{5}{12}, \frac{11}{12}, 1; \frac{1}{1 - J(\tau)}\right) = \sqrt[12]{12}(J(\tau) - 1)^{\frac{5}{12}}\eta(\tau)^2.$$  \hspace{1cm} (2.30)$$
2. Identities for some Gauss' Hypergeometric Series

2.6 Some Applications of our Identities

2.6.1 Algebraicity of some Hypergeometric Values

Applying some classical facts about elliptic functions to some of our identities, we will deduce the algebraicity of the values of the corresponding hypergeometric series at certain points.

Let us first recall two facts:

1. Let $A, B, D \in \mathbb{Z}$ with $AD > 0$ and for $x \in \mathbb{H}$, set $\varphi(x) = \left( \frac{B + c}{D} \right)^2$, then the function $\varphi$ is integral over $\mathbb{Z}[1728]$ (see [La] 12§2 Thm 2).

2. If $\tau \in \mathbb{H}$ is imaginary quadratic, then $J(\tau)$ is an algebraic integer (see [La] 5§2 Thm 4).

**Proposition 2.1.** Let $\tau \in \mathbb{Q}(i) \cap C_{1-J,i}$, then $1 - J(\tau) \in \bar{\mathbb{Q}}$.

$$F\left( \frac{1}{12}, \frac{1}{12}, \frac{1}{2}; 1 - J(\tau) \right) \in \bar{\mathbb{Q}} \quad \text{and} \quad F\left( \frac{5}{12}, \frac{5}{12}, \frac{1}{2}; 1 - J(\tau) \right) \in \bar{\mathbb{Q}}.$$  

**Proof.** Let $\tau \in C_{1-J,i} \cap \mathbb{Q}(i)$. The first assertion follows from the second fact, since $Im(\tau) > 0$ and $\mathbb{Q}(i)$ is a quadratic extention of $\mathbb{Q}$. Further, there exist $A, B, D \in \mathbb{Z}$ such that $\tau = \frac{A+iB}{D}$. Note that $Im(\tau) > 0$ implies $AD > 0$. By the first fact, $(\frac{B+c}{D})^2$ is then integral over $\mathbb{Z}[1728] = \mathbb{Z}[1728]$ and, in particular, algebraic. Now, it remains to apply this to Identities (2.24) and (2.25) respectively. \hfill \Box

**Proposition 2.2.** Let $\tau \in \mathbb{Q}(i) \cap C_{\eta^{-1},i}$, then $\frac{J(\tau) - 1}{J(\tau)} \in \bar{\mathbb{Q}}$ and

$$F\left( \frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{J(\tau) - 1}{J(\tau)} \right) \in \bar{\mathbb{Q}}.$$  

**Proof.** As in the proof of Proposition 2.1, we get, for $\tau \in \mathbb{Q}(i)$ with $Im(\tau) > 0$, the algebraicity of $\frac{J(\tau) - 1}{J(\tau)}$ and $(\frac{B+c}{D})^2$ by the two facts. If $\tau \in \mathbb{Q}(i) \cap C_{\eta^{-1},i}$, we conclude by using Identity (2.26). \hfill \Box
2.6. Some Applications of our Identities

Proposition 2.3. Let \( \tau \in \mathbb{Q}(i\sqrt{3}) \cap (C_{J,\rho} \cup C_{J,-\rho}) \). Then \( J(\tau) \in \overline{\mathbb{Q}} \).

\[
F\left(\frac{1}{12}, \frac{1}{12}, 1, \frac{2}{3}; J(\tau)\right) \in \overline{\mathbb{Q}} \quad \text{and} \quad F\left(\frac{7}{12}, \frac{7}{12}, 1, \frac{2}{3}; J(\tau)\right) \in \overline{\mathbb{Q}}.
\]

Proof. By the second fact, \( J(\tau) \in \overline{\mathbb{Q}} \) follows from \( [\mathbb{Q}(i\sqrt{3}) : \mathbb{Q}] = 2 \) and \( \text{Im}(\tau) > 0 \). If \( \tau \in \mathbb{Q}(i\sqrt{3}) \cap C_{J,\rho} \), then \( \exists \alpha, \beta, \gamma \in \mathbb{Z} \) such that \( \tau = \frac{\alpha + \beta i}{\gamma} \) (because \( \mathbb{Q}(i\sqrt{3}) = \mathbb{Q}(\rho) \)) and \( \text{Im} (\tau) = \frac{\sqrt{3}}{2\gamma} > 0 \). Thus \( AD > 0 \), so that we can apply the first fact to \( \left(\frac{\eta(\tau)}{\eta(-\rho)}\right)^2 \) with \( \mathbb{Z}[1728J(\rho)] = \mathbb{Z} \). In this case, we conclude by using Identities (2.17) and (2.18) respectively. In the case where \( \tau \in \mathbb{Q}(i\sqrt{3}) \cap C_{J,-\rho} \), using that \( \mathbb{Q}(i\sqrt{3}) = \mathbb{Q}(-\rho) \), we can write \( \tau = \frac{-A + Bi}{\gamma} \) with \( \alpha, \beta, \gamma \in \mathbb{Z} \) and \( AD > 0 \), hence \( \left(\frac{\eta(\tau)}{\eta(-\rho)}\right)^2 \in \overline{\mathbb{Q}} \). We conclude by using Identity (2.21) resp. (2.22) together with the fact that \( J(\tau) \in \overline{\mathbb{Q}} \).

Proposition 2.4. Let \( \tau \in \mathbb{Q}(i\sqrt{3}) \cap (C_{J\tau,\rho} \cup C_{J\tau,-\rho}) \), then \( \frac{J(\tau)}{J(\tau) - 1} \in \overline{\mathbb{Q}} \) and

\[
F\left(\frac{1}{12}, \frac{7}{12}, 2, \frac{2}{3}; \frac{J(\tau)}{J(\tau) - 1}\right) \in \overline{\mathbb{Q}}.
\]

Proof. By the second fact, for \( \tau \in \overline{\mathbb{Q}}(i\sqrt{3}) \) with \( \text{Im}(\tau) > 0 \), we have \( \frac{J(\tau)}{J(\tau) - 1} \in \overline{\mathbb{Q}} \). Moreover, we have shown in the proof of Proposition 2.3, that, for such a \( \tau \), \( \left(\frac{\eta(\tau)}{\eta(-\rho)}\right)^2 \) and \( \left(\frac{\eta(\tau)}{\eta(-\rho)}\right)^2 \) are algebraic. If \( \tau \in \mathbb{Q}(i\sqrt{3}) \cap C_{J\tau,\rho} \), the result follows from Identity (2.19), while if \( \tau \in \mathbb{Q}(i\sqrt{3}) \cap C_{J\tau,-\rho} \), it follows from Identity (2.23).

Remark 50. In [BW86], the result of Proposition 2.2 is asserted for each \( \tau \in \mathbb{Q}(i) \cap \mathbb{H} \) such that \( |J(\tau) - 1| < 1 \) (cf Thm1) and that of Proposition 2.4 for each \( \tau \) in \( \mathbb{Q}(i\sqrt{3}) \cap \mathbb{H} \) (cf Thm2) such that \( |J(\tau) - 1| < 1 \).

2.6.2 Some Algebraic Evaluations

Using some long known results on modular equations (see [We]), we will calculate the values of \( J(mi) \) and \( \frac{\eta(mi)}{\eta(i)} \) for some \( m \in \mathbb{N} \). These values will be found to be applicable only in the identities where \( z(J) = \frac{J-1}{J} \), because this will be the
only transformation \( z(J) \) among \( \{ J, 1 - J, \frac{J}{J - 1}, \frac{J}{J - 1} \} \) for which \( |z(J(mi))| < 1 \). Inserting these values into Identity (2.26), we will get some values of \( F(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; z) \) which will be algebraic, showing agreement with Proposition 2.2.

The nontrivial calculations of this paragraph have been made using Mathematica.

Gathering of Datas

The functions of \( \tau \in \mathbb{H} \)

\[
f(\tau) = e^{-\frac{\pi i}{3}} \frac{\eta(\frac{1+\tau}{2})}{\eta(\tau)} \quad \text{and} \quad f_1(\tau) = \frac{\eta(\frac{\tau}{2})}{\eta(\tau)}
\]

satisfy the following relations with the \( J \)-function

\[
J(\tau) = \frac{(f(\tau)^{24} - 16)^3}{2633 f(\tau)^{24}}.
\]

\[
J(\tau) = \frac{(f_1(\tau)^{24} + 16)^3}{2633 f_1(\tau)^{24}}.
\]

\[
J\left(\frac{\tau}{2}\right) = \frac{(256 + f_1(\tau)^{24})^3}{2633 f_1(\tau)^{48}}
\]

(cf [We] §34 (9) for the definitions and for the relations §54 (5) and §72 (3), (4) respectively).

The following datas of Tabelle VI, pp721-726, [We] will also be used.

\[
f_1(2i) = \sqrt{8}
\]

\[
f(3i)^3 = \sqrt{2}(1 + \sqrt{3})
\]

\[
f_1(4i)^4 = 2\sqrt{8}(1 + \sqrt{2})
\]

\[
f(5i) = 8^{-\frac{1}{4}}(1 + \sqrt{5})
\]

\[
f_1(6i)^3 = \sqrt{8}x, \text{ where } x^2 - 4x - 2 = 2\sqrt{3}(x + 1)
\]

\[
f(7i)^2 = \sqrt{2}x, \text{ where } x + \frac{1}{x} = 2 + \sqrt{7}
\]

\[
f_1(10i) = 2^{-\frac{1}{8}}x, \text{ where } x^2 - x - 1 = \sqrt{5}(x + 1).
\]
2.6. Some Applications of our Identities

It can be shown (cf [We] §72) that the function $27\left(\frac{\eta(5\tau)}{\eta(\tau)}\right)^6$ satisfies the equation

$$x^4 + 18x^2 + 12\frac{3}{2}(J(\tau) - 1)\frac{3}{2}x - 27 = 0. \quad (2.41)$$

Moreover, $5\left(\frac{\eta(5\tau)}{\eta(\tau)}\right)^2$ satisfies

$$x^6 + 10x^3 - 12J(\tau)\frac{3}{2}x + 5 = 0, \quad (2.42)$$

while $7\left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^2$ satisfies

$$x^8 + 14x^6 + 63x^4 + 70x^2 + 12\frac{3}{2}(J(\tau) - 1)\frac{3}{2}x - 7 = 0. \quad (2.43)$$

**Calculation of $J(mi)$ and $(\frac{\eta(mi)}{\eta(i)})^2$ for some $m \in \mathbb{N}$**

The values of $J(mi)$ for $m \in \{2, 3, 4, 5\}$ is obtained easily by inserting the values (2.34) to (2.37) into formulas (2.31) or (2.32) respectively. This yields

$$\begin{align*}
J(2i) &= \frac{(8^3 + 16)^3}{2^63^38^3} = \frac{(11)^3}{2^3} \\
J(3i) &= \frac{(1 + \sqrt{3})^8 - 4)^3}{2^23^3(1 + \sqrt{3})^8} = \frac{133283}{3} + \frac{2^37^219 \cdot 31\sqrt{3}}{3^2} \\
J(4i) &= \frac{(2^3(1 + \sqrt{2})^6 + 1)^3}{2^33^3(1 + \sqrt{2})^6} = \frac{2^5 \cdot 181 \cdot 210319 + 3^47^2(11)^219 \cdot 59\sqrt{2}}{2^5} \\
J(5i) &= \frac{(1 + \sqrt{5})^{24} - 2^{22})^3}{2^{42}3^3(1 + \sqrt{5})^{24}} = 1637 \cdot 2659 \cdot 2927 + 2^43^47^2 \cdot 23 \cdot 47 \cdot 83\sqrt{5}. 
\end{align*}$$

The value of $\frac{\eta(2i)}{\eta(i)}$ is given by $f_1(2i) = \frac{\eta(i)}{\eta(2i)}$, hence

$$\left(\frac{\eta(2i)}{\eta(i)}\right)^2 = 2^{-\frac{3}{4}} \quad (2.44)$$

**Remark 51.** Using the definition of the $\eta$-function (cf §2.2.8), one can verify that, for $m, n \in \mathbb{N}$, the quotient $\frac{\eta(mi)}{\eta(ni)}$ is real and positive.
In view of the above remark, (2.37) implies \((\frac{\eta(4i)}{\eta(i)})^2 = (2 + \sqrt{2})^{\frac{1}{2}}\) and using \((\frac{\eta(4i)}{\eta(i)})^2 = (\frac{\eta(4i)}{\eta(2i)})^2 (\frac{\eta(2i)}{\eta(i)})^2\) together with (2.44), we have

\[
\left(\frac{\eta(4i)}{\eta(i)}\right)^2 = \frac{1}{2^{\frac{1}{2}} (1 + \sqrt{2})^{\frac{1}{2}}}.
\]

(2.45)

According to (2.41), the function \(3(\frac{\eta(3i)}{\eta(i)})^2\) satisfies the equation

\[
z^{12} + 18z^6 - 27 = 0,
\]

because \(J(i) = 1\). Solving this equation with respect to \(z\), one gets only one real and positive solution, which is \(3(-3 + 2\sqrt{3})^\frac{1}{6}\). This implies

\[
\left(\frac{\eta(3i)}{\eta(i)}\right)^2 = 3^{-\frac{5}{6}} (-3 + 2\sqrt{3})^{\frac{1}{6}}.
\]

(2.46)

Similarly, Equation (2.42) has only \(\frac{1}{2}(-1 + \sqrt{5})\) as real positive solution. Hence,

\[
\left(\frac{\eta(5i)}{\eta(i)}\right)^2 = \frac{-1 + \sqrt{5}}{2 \cdot 5}.
\]

(2.47)

The next step is to find the value of \(f_1(6i)^3\). According to Remark 51, we know that it is real and positive. Since the equation \(x^2 - 2x - 2 = 2\sqrt{3}(x + 1)\) of (2.38) has a positive and a negative real solution, we only need to choose the positive one and the positive real 8-th root of 8 to get

\[
f_1(6i)^3 = 2^\frac{1}{8}(2 + \sqrt{3} + \sqrt{9 + 6\sqrt{3}}).
\]

From this, the value of \(J(6i)\) is obtained by using Formula (2.32). Indeed,

\[
J(6i) = \frac{((2 + \sqrt{3} + \sqrt{9 + 6\sqrt{3}})^8 + 2)^3}{3^3(2 + \sqrt{3} + \sqrt{9 + 6\sqrt{3}})^8}.
\]

Since \(f_1(6i)^3 = (\frac{\eta(3i)}{\eta(6i)})^3\), we get \((\frac{\eta(6i)}{\eta(3i)})^2 = 2^{-\frac{1}{3}}(2 + \sqrt{3} + \sqrt{9 + 6\sqrt{3}})^{-\frac{2}{3}}\) and together with (2.46)

\[
\left(\frac{\eta(6i)}{\eta(i)}\right)^2 = \frac{(-3 + 2\sqrt{3})^{\frac{1}{6}}}{2^{\frac{1}{4}} 3^{\frac{5}{6}} (2 + \sqrt{3} + \sqrt{9 + 6\sqrt{3}})^{\frac{1}{3}}}.
\]
2.6. Some Applications of our Identities

In view of (2.43) and of Remark 51, \(7\left(\frac{\eta(7i)}{\eta(i)}\right)^2\) is a real positive solution of

\[x^8 + 14x^6 + 63x^4 + 70x^2 - 7 = 0.\]

Again, there is only one real and positive solution. It reads

\[\sqrt{-\frac{7}{2} + \sqrt{7} + \frac{1}{2}\sqrt{-7 + 4\sqrt{7}}} \text{ and implies}
\]

\[\left(\frac{\eta(7i)}{\eta(i)}\right)^2 = \frac{1}{7}\sqrt{-\frac{7}{2} + \sqrt{7} + \frac{1}{2}\sqrt{-7 + 4\sqrt{7}}}.
\]

Now, we would like to know the value of \(J(7i)\). In order to do this, let’s use (2.39) and solve \(x + \frac{1}{x} = 2 + \sqrt{7}\) with respect to \(x\). The solutions are

\[\frac{1}{2}(2 + \sqrt{7} - \sqrt{7 + 4\sqrt{7}}) \quad \text{and} \quad \frac{1}{2}(2 + \sqrt{7} + \sqrt{7 + 4\sqrt{7}})
\]

and are both real and positive. To determinate which of them equals \(f(7i)^2 \frac{1}{\sqrt{2}}\), first consider that

\[f(7i) = e^{\frac{7\pi}{12}} \prod_{k=0}^{\infty} (1 + e^{-7\pi(2k+1)}).
\]

It is sufficient to evaluate \(e^{\frac{7\pi}{12}} (1 + e^{-7\pi})^2 (1 + e^{-21\pi})^2 (1 + e^{-35\pi})^2 \frac{1}{\sqrt{2}}\) to find a very good approximation of

\[\frac{1}{2}(2 + \sqrt{7} + \sqrt{7 + 4\sqrt{7}}).
\]

This implies that

\[f(7i)^2 = 2^{-\frac{1}{2}}(2 + \sqrt{7} + \sqrt{7 + 4\sqrt{7}})
\]

and together with Formula (2.31)

\[J(7i) = \frac{(2 + \sqrt{7} + \sqrt{7 + 4\sqrt{7}})^{12} - 2^{10}}{2^{18}3^3(2 + \sqrt{7} + \sqrt{7 + 4\sqrt{7}})^{12}}.
\]

According to Formula (2.33), \(f_1(8i)\) is solution of

\[(256 + z^{24})^3 - 2^63^3J(4i)z^{48} = 0.
\]
This equation has only two real positive solutions, which are

\[
(2(-140 + 99\sqrt{2}))^{\frac{1}{24}} \quad \text{and} \quad 2^{\frac{1}{24}} \left( 2^4 \cdot 257 \cdot 2441 + 2^2 3^2 \cdot 11 \cdot 17923\sqrt{2} \right.
\]
\[
\left. + 3^2 \sqrt{2^2 \cdot 3 \cdot 221047 \cdot 937823 + 11 \cdot 17 \cdot 139 \cdot 67672987\sqrt{2}} \right)^{\frac{1}{24}}.
\]

Well, let's try to get an approximation of the value of

\[
f_1(8i) = e^{\frac{\pi}{3}} \prod_{k=0}^{\infty} (1 - e^{-8\pi(2k+1)}).
\]

The evaluation of

\[
e^{\frac{\pi}{3}} (1 - e^{-8\pi})(1 - e^{-24\pi})(1 - e^{-40\pi})
\]
gives a good approximation of the second solution found above. Hence,

\[
\left( \frac{\eta(8i)}{\eta(4i)} \right)^2 = 2^{-\frac{11}{24}} \left( 2^4 \cdot 257 \cdot 2441 + 2^2 3^2 \cdot 11 \cdot 17923\sqrt{2} \right.
\]
\[
\left. + 3^2 \sqrt{2^2 \cdot 3 \cdot 221047 \cdot 937823 + 11 \cdot 17 \cdot 139 \cdot 67672987\sqrt{2}} \right)^{-\frac{1}{12}}
\]

and together with (2.45)

\[
\left( \frac{\eta(8i)}{\eta(i)} \right)^2 = 2^{-\frac{61}{32}} (1 + \sqrt{2})^{-\frac{1}{3}} \left( 2^4 \cdot 257 \cdot 2441 + 2^2 3^2 \cdot 11 \cdot 17923\sqrt{2} \right.
\]
\[
\left. + 3^2 \sqrt{2^2 \cdot 3 \cdot 221047 \cdot 937823 + 11 \cdot 17 \cdot 139 \cdot 67672987\sqrt{2}} \right)^{-\frac{1}{12}}.
\]

Inserting the value of \( f_1(8i) \) in Formula (2.32), we get

\[
J(8i) = \frac{(2^7 \gamma + 1)^3}{2^{53} 3^\gamma}, \quad \text{where}
\]
2.6. Some Applications of our Identities

\( Y := 2^4 \cdot 257 \cdot 2441 + 2^2 \cdot 3^2 \cdot 11 \cdot 17923\sqrt{2} \\
+ 3^2 \sqrt{2^2 \cdot 3 \cdot 221047 \cdot 937823 + 11 \cdot 17 \cdot 139 \cdot 67672987\sqrt{2}} \)

In order to calculate \( \tau(10i) \), we refer to (2.40) and solve equation \( x^2 - x - 1 = \sqrt{5}(x + 1) \) with respect to \( x \). Only one solution is real and positive. It reads \( \frac{1}{2}(1 + \sqrt{5} + \sqrt{10 + 6\sqrt{5}}) \). Hence, \( f_1(10i) = 2^{-\frac{9}{8}}(1 + \sqrt{5} + \sqrt{10 + 6\sqrt{5}}) \) and

\[
\left( \frac{\eta(10i)}{\eta(5i)} \right)^2 = \frac{2^{\frac{9}{8}}}{(1 + \sqrt{5} + \sqrt{10 + 6\sqrt{5}})^2}.
\]

Together with (2.47), we deduce

\[
\left( \frac{\eta(10i)}{\eta(i)} \right)^2 = \frac{2^{\frac{1}{4}}(-1 + \sqrt{5})}{5(1 + \sqrt{5} + \sqrt{10 + 6\sqrt{5}})^2}.
\]

and with Formula (2.32)

\[
J(10i) = \frac{(1 + \sqrt{5} + \sqrt{10 + 6\sqrt{5}})^{24} + 2^{31})^3}{2^{60}3^3(1 + \sqrt{5} + \sqrt{10 + 6\sqrt{5}})^{24}}.
\]

Some Algebraic Evaluations of \( F(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; z) \)

Considering the results of the previous subparagraph, one sees that \( z(J) = \frac{J-I}{J} \) is the only transformation among \( \{J, 1 - J, \frac{J-I}{J}, \frac{J}{J-1}\} \) for which \( |z(J(mi))| < 1 \) when \( m = 2, 3, 4, 5, 6, 7, 8 \) or 10.
The following simplifications can be calculated:

\[
\frac{J(2i) - 1}{J(2i)} = \frac{1323}{1331} = \frac{3^3 \cdot 7^2}{11^3}
\]

\[
\frac{J(3i) - 1}{J(3i)} = \frac{2^3 \cdot 7^2 (2^2 \cdot 11093 - 3 \cdot 19 \cdot 31 \sqrt{3})}{(11 \cdot 23)^3}
\]

\[
\frac{J(4i) - 1}{J(4i)} = \frac{3^3 \cdot 7^2 \cdot 11^2 (83967 - 2^4 \cdot 3 \cdot 19 \cdot 59 \sqrt{2})}{(23 \cdot 47)^3}
\]

\[
\frac{J(5i) - 1}{J(5i)} = \frac{2^4 \cdot 3^3 \cdot 7^2 (2^3 \cdot 5 \cdot 5237 \cdot 22067 - 3 \cdot 23 \cdot 47 \cdot 83 \sqrt{5})}{(11 \cdot 59 \cdot 71)^3}
\]

Applying the above results to Identity (2.26), we find the following evaluations of
\[F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{J(\tau) - 1}{J(\tau)}\right)\] at \(\tau = mi\) for \(m = 2, 3, 4, 5, 6, 7, 8, 10\) respectively.

For \(\tau = 2i\), we get

\[
F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{3 \cdot 7^2}{11^3}\right) = \frac{3}{2} \cdot \frac{11^{1/4}}{2^{2}}
\]

then, successively, for \(\tau = 3i\):

\[
F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{2^3 \cdot 7^2 (2^2 \cdot 11093 - 3 \cdot 19 \cdot 31 \sqrt{3})}{(11 \cdot 23)^3}\right) = \frac{2}{3} \cdot \frac{(3 \cdot 7 + 2^2 \cdot 5 \sqrt{3})^{1/4}}{2}
\]

\(\tau = 4i\):

\[
F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{3^3 \cdot 7^2 \cdot 11^2 (83967 - 2^4 \cdot 3 \cdot 19 \cdot 59 \sqrt{2})}{(23 \cdot 47)^3}\right) = \frac{5}{2^3} \cdot \frac{(7 \cdot 13 + 2^2 \cdot 3 \cdot 5 \sqrt{2})^{1/4}}{2}
\]

\(\tau = 5i\):

\[
F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{2^4 \cdot 3^3 \cdot 7^2 (2^3 \cdot 5 \cdot 5237 \cdot 22067 - 3 \cdot 23 \cdot 47 \cdot 83 \sqrt{5})}{(11 \cdot 59 \cdot 71)^3}\right) = \frac{3}{5} \cdot \frac{(7 \cdot 23 + 2^3 \cdot 3 \cdot 5 \sqrt{5})^{1/4}}{2}
\]
2.6. Some Applications of our Identities

\( \tau = 6i: \)

\[
F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{(2 + (2 + \sqrt{3} + \sqrt{3} + 2\sqrt{3})^3 - 27(2 + \sqrt{3} + \sqrt{9} + 6\sqrt{3})^3)}{(2 + (2 + \sqrt{3} + \sqrt{9} + 6\sqrt{3})^3)} \right) = \frac{7}{2^2 \cdot 3} \left( 3 \cdot 7 \cdot 11 + 2^2 \cdot 5 \cdot 7\sqrt{3} + 2^3 \cdot 3 \cdot 5 \sqrt{2} \cdot 3 + \frac{7\sqrt{3}}{2} \right)^{\frac{1}{3}}.
\]

\( \tau = 7i: \)

\[
F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{(2 + \sqrt{7} + \sqrt{7} + 4\sqrt{7})^{12} - 2^{10})^3 - 2^{18}3^3(2 + \sqrt{7} + \sqrt{7} + 4\sqrt{7})^{12}}{(2 + \sqrt{7} + \sqrt{7} + 4\sqrt{7})^{12} - 2^{10})^3 \right) = \frac{2^2}{7} \left( 7 \cdot 43 + 2^3 \cdot 3 \cdot 5\sqrt{7} + 2^2 \cdot 3 \cdot 5 \sqrt{7} + \frac{8}{\sqrt{7}} \right)^{\frac{1}{3}}.
\]

\( \tau = 8i: \)

\[
F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{(2^2\gamma + 1)^3 - 2^53^3\gamma}{(2^7\gamma + 1)^3} \right) = \frac{3^\frac{3}{2}(2^7\gamma + 1)^{\frac{1}{3}}}{2^{\frac{10}{3}}(1 + \sqrt{2})^{\frac{3}{2}}\gamma^{\frac{3}{2}}}, \text{ where }
\]

\[
\gamma = 2^4 \cdot 257 \cdot 2441 + 2^23^2 \cdot 11 \cdot 17923\sqrt{2}
\]

\[
+ 3^2\sqrt{2^2 \cdot 3 \cdot 221047 \cdot 937823 + 11 \cdot 17 \cdot 139 \cdot 67672987\sqrt{2},
\]

and, finally, for \( \tau = 10i: \)

\[
F\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}, \frac{\Sigma^{24} + 2^{31})^3 - 2^{60}3^3\Sigma^{24}}{(\Sigma^{24} + 2^{31})^3} \right) = \frac{11(-1 + \sqrt{5})(\Sigma^{24} + 2^{31})^{\frac{1}{2}}}{2^{\frac{11}{2}} \cdot 3^{\frac{1}{2}} \cdot 5\Sigma^4}, \text{ where }
\]

\[
\Sigma := 1 + \sqrt{5} + \sqrt{10 + 6\sqrt{5}}.
\]

**Remark 52.** The evaluation at \( \tau = 2i \) is that given by Beukers and Wolfart ([BW86] Theorem 3). The evaluation at \( \tau = 3i \) was found by Flach [Fl89]. Flach also asserted a value for \( \tau = 4i \), but this was found to be false by Joyce and Zucker [JZ91]. For this, they used another method which also produced the evaluation at \( \tau = 5i \). Our evaluations at \( \tau = 6i, 7i, 8i, 10i \) are new.
Seite Leer / Blank leaf
Bibliography


Seite Leer / Blank leaf
Acknowledgments

My first thought goes to Prof. Eva Bayer-Flückiger whose intervention at the beginning made all of this possible. Her enthusiasm and dynamism were very encouraging.

I wish to express my sincere gratitude to Prof. Gisbert Wüstholz for having taken me under his directorship and proposed to me an interesting subject which lies at the confluence of algebra, number theory and geometry.

The sporadic meetings I had with Prof. Paula Cohen were always a great source of motivation. I am also obliged to her for some pertinent remarks on previous versions of this work.

Many thanks to Prof. Horst Knörrer for some fruitful discussions and to Prof. Richard Pink for his time and help in solving a problem.

Special thanks to my teaching assistant colleagues for many discussions, in particular to Walter Gubler, Mauro Triulzi and Thomas Loher. Thanks to Roberto Ferretti for a useful reference.

The last thoughts are dedicated to my family for their patience and support and for the encouragement given to me by the numerous mathematicians among them.
Seite Leer / Blank leaf
Curriculum Vitae

I was born in Geneva on the 11th of June 1973. In this town, I attended primary and secondary school and obtained the Maturité Scientifique in June 1992 at the Collège Calvin. In October 1992, I began my studies in mathematics at the University of Geneva. My diploma thesis “Puissance d’une Matrice carrée et Polynômes de Dickson de deuxième Espèce” was written under the direction of Prof. Michel Kervaire and completed in March 1996. Having obtained the Diplôme de Mathematicien in July 1996, I moved to Zurich in September to begin a PhD under the direction of Prof. Gisbert Wüstholz. From that time, I have been employed as a teaching assistant at the Swiss Federal Institute of Technology of Zurich (ETHZ).