Doctoral Thesis

The n-dimensional Laver and Miller ideals

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The \( n \)-dimensional Laver and Miller ideals

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Abstract

We investigate the ideals associated with finite powers of Laver forcing and Miller forcing and show that among them only the two-dimensional ideal $J(M^2)$ is a $\sigma$-ideal.

By a forcing iteration of $M^2$ we prove that it is consistent with ZFC to assume that the additivity number of $J(M^2)$ is less than its covering number. As a byproduct we get a rather simple model for the strict inequality of the cardinal invariants $\mathfrak{h}$ and $\mathfrak{s}$.

Zusammenfassung

Wir untersuchen die zu endlichen Potenzen von Laver-Forcing und Miller-Forcing gehörigen Ideale und zeigen, dass nur das zweidimensionale Miller-Ideal $J(M^2)$ ein $\sigma$-Ideal ist.

Mit Hilfe einer Forcing-Iteration von $M^2$ beweisen wir die Konsistenz mit ZFC der Aussage, dass die Additivitätszahl von $J(M^2)$ strikt kleiner als dessen Überdeckungszahl ist. Als Nebenprodukt erhalten wir ein relativ einfaches Modell für die Ungleichheit der kardinalen Invarianten $\mathfrak{h}$ und $\mathfrak{s}$. 
# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>vii</td>
</tr>
<tr>
<td>1 Preliminaries</td>
<td>1</td>
</tr>
<tr>
<td>2 The $n$-dimensional Laver and Miller ideals</td>
<td>9</td>
</tr>
<tr>
<td>3 Bounds for $\text{add}(J(\mathcal{M}^2))$</td>
<td>15</td>
</tr>
<tr>
<td>4 The consistency of $\text{add}(J(\mathcal{M}^2)) &lt; \text{cov}(J(\mathcal{M}^2))$</td>
<td>19</td>
</tr>
<tr>
<td>Bibliography</td>
<td>36</td>
</tr>
</tbody>
</table>
Introduction

Each of the classical tree forcings, such as Sacks forcing $\mathbb{S}$, Laver Forcing $\mathbb{L}$ or Miller forcing $\mathbb{M}$, is associated with a $\sigma$-ideal on $2^\omega$ (in the case of $\mathbb{S}$) or on $\omega^\omega$ (in the case of $\mathbb{L}$ or $\mathbb{M}$). These can be considered forcing ideals in the sense that a real (in $2^\omega$ or $\omega^\omega$) is generic for the respective forcing iff it avoids all the sets from the associated ideal. Such ideals were first studied by Marcone [Mar35], later by Veličković [Vel91], and Judah, Miller, Shelah [JMS92]. They all studied the Sacks ideal. The ideals associated with Laver or Miller forcing were investigated by Brendle, Goldstern, Johnson, Repický, Shelah and Spinas (see [GJS94], [GRSS95], [Spi95] and [Bre95], chronological order). In all these papers, among other things the additivity and covering coefficients were studied. Let $J(Q)$ denote the ideal associated with the forcing $Q$. The typical problem was whether for $Q \in \{\mathbb{S}, \mathbb{L}, \mathbb{M}\}$, in the trivial chain of inequalities

$$\omega_1 \leq \text{add}(J(Q)) \leq \text{cov}(J(Q)) \leq 2^\omega$$

any of the inequalities could consistently be strict. Here $\text{add}(J)$, $\text{cov}(J)$ denotes the additivity, covering coefficient of the ideal $J$, respectively. One of the main results of [JMS92] is that, letting $V \models \text{ZFC+GCH}$ and $S_{\omega_2}$ a countable support iteration of $\mathbb{S}$ of length $\omega_2$, then $V^{S_{\omega_2}}$ is a model for $\text{add}(J(\mathbb{S})) < \text{cov}(J(\mathbb{S}))$. The hard part of the argument is to show that $V^{S_{\omega_2}}$ is a model for $\text{add}(J(\mathbb{S})) = \omega_1$. Building on these arguments, analogies of this result were proved in [GRSS95] for $J(\mathbb{M})$ and $J(\mathbb{L})$: For $Q \in \{\mathbb{L}, \mathbb{M}\}$, letting $Q_{\omega_2}$ denote a countable support iteration of length $\omega_2$ of $Q$ and $V \models \text{ZFC+GCH}$, $V^{Q_{\omega_2}}$ is a model for $\text{add}(J(Q)) < \text{cov}(J(Q))$. Actually, for $Q = \mathbb{M}$ this result needs a lemma from [Spi95] which was not yet available in [GRSS95]. Again, the hard part was to show that $\text{add}(J(Q)) = \omega_1$ holds in the extension. For this, it was proved that the cardinal invariant $\mathfrak{h}$ which is defined as the distributivity
number of $\mathcal{P}(\omega)/\text{fin}$ is an upper bound of $\text{add}(\mathcal{J}(Q))$ in $V^{\mathcal{Q}_{\omega_2}}$. Then this was combined with, first, the difficult result from [JS90] that for any $Q \in \{L, M\}$, in the model $V^{\mathcal{Q}_{\omega_2}}$ there exists a nonmeasurable set of reals of size $\aleph_1$ and, secondly, with the easy result that such a set must have size at least $\text{h}$.

In the present work we study the ideals associated with finite powers $Q^n$, for any $Q \in \{L, M\}$. In Chapter 2 we show that $J(\mathcal{Q}^n)$ is not a $\sigma$-ideal for any $n \geq 2$ and that $J(\mathcal{M}^m)$ is not a $\sigma$-ideal for any $m \geq 3$. A deeper result says that $J(\mathcal{M}^2)$ is a $\sigma$-ideal. In the line of work outlined above we proceed to show that if $(\mathcal{M}^2)_{\omega_2}$ is a countable support iteration of $\mathcal{M}^2$ and $V \models \text{ZFC+CH}$, then $V^{(\mathcal{M}^2)_{\omega_2}}$ is a model for $\text{add}(J(\mathcal{M}^2)) < \text{cov}(J(\mathcal{M}^2))$. Part of the proof is completely analogous to [GRSS95], in particular that $\text{add}(J(\mathcal{M}^2)) \leq \text{h}$ and $\text{cov}(J(\mathcal{M}^2)) = \omega_2$ hold in $V^{(\mathcal{M}^2)_{\omega_2}}$. The first is shown in Chapter 3, the latter is the beginning of Chapter 4.

But there is a new point. Forcing with $\mathcal{M}^2$ makes the old reals a Lebesgue null set because it adds a nonsplit real, i.e. a set $a \in [\omega]^{\omega}$ for which there is no $b \in [\omega]^{\omega} \cap V$ with $b \cap a$ and $a \setminus b$ both infinite ($b$ does not split $a$). This implies that in $V^{(\mathcal{M}^2)_{\omega_2}}$ the splitting number $\mathfrak{s}$ is $\aleph_2$ and hence every set of reals of size $\aleph_1$ is a null set.

Nevertheless we are able to show that $\text{h} = \omega_1$ holds in $V^{(\mathcal{M}^2)_{\omega_2}}$ and therefore $\text{add}(J(\mathcal{M}^2)) = \omega_1$ as desired. This is the core of Chapter 4 and this thesis. The proof is a continuation of the difficult analysis of the forcing $\mathcal{M}^2$ which was carried out in [Spi1], [Spia]. As a byproduct we obtain a rather simple model for $\text{h} < \mathfrak{s}$. So far only very few models for this had been known, all of them due to Shelah and all of them involving creature forcings (see [She84], [BS87]).
Chapter 1

Preliminaries

Let us first fix our notation. Given a tree $p \subseteq \omega^{<\omega}$, the set of its infinite branches is denoted with $[p]$. We let $|\sigma|$ denote the length of $\sigma \in \omega^{<\omega}$. For $\sigma \in \omega^{<\omega}$ and $n \in \omega$, we let $\sigma \upharpoonright n$ be the sequence of length $|\sigma| + 1$ with initial segment $\sigma$ and last coordinate $n$. By $\text{succ}_p(\sigma)$ we denote the set of all $n < \omega$ with $\sigma \upharpoonright n \in p$. Then $\sigma$ is a splitnode of $p$ iff $\text{succ}_p(\sigma)$ has at least two elements. A tree $p \subseteq \omega^{<\omega}$ is called superperfect (or Miller), if $p \neq \emptyset$ and for every $\sigma \in p$ there exists $\tau \in p$ such that $\sigma \subseteq \tau$ and $\text{succ}_p(\tau)$ is infinite. Such $\tau$ are called infinite splitnodes. The set of all infinite splitnodes of $p$ is denoted by $\text{split}(p)$. For $\sigma \in \text{split}(p)$, by $\text{Succ}_p(\sigma)$ we denote the set of infinite successor splitnodes of $\sigma$, i.e. those $\tau \in \text{split}(p)$ such that $\sigma \subseteq \tau$ and for no $\varnothing$ with $\sigma \not\subseteq \varnothing \subseteq \tau$ do we have $\varnothing \in \text{split}(p)$. By $\text{Lev}_n(p)$ we denote the set of those $\sigma \in \text{split}(p)$ which have precisely $n$ proper initial segments which belong to $\text{split}(p)$. For convenience we shall always assume that a superperfect tree has only infinite splitnodes. Then $\text{st}(p)$ denotes the shortest splitnode of $p$, and $p^-$ denotes the set of nodes of $p$ which extend $\text{st}(p)$. If $\sigma \in p$ then $p(\sigma)$ consists of all $v \in p$ which are comparable with $\sigma$. A superperfect tree is called a Laver tree if for every $\sigma \in p^-$ the set $\text{succ}_p(\sigma)$ is infinite. We let $[\sigma]$ denote the set of all reals with initial segment $\sigma$.

The set of all superperfect trees will be denoted by $\mathcal{M}$, the set of all Laver trees by $\mathbb{L}$. Then $(\mathbb{L}, \subseteq)$ is Laver forcing (see [Lav76]), and $(\mathcal{M}, \subseteq)$ is Miller forcing (see [Mil84]). Let $Q$ be $\mathbb{L}$ or $\mathcal{M}$. For $p, q \in Q$ we shall write $p \leq q$ instead of $p \subseteq q$, and by $p \leq^\forall q$ we mean that $p \leq q$ and $p$ and $q$ have the
same $n + 1$ first splitnodes.

Then $Q^m$ carries the coordinate-wise ordering. For any $\langle p_0, \ldots, p_{m-1} \rangle, \langle q_0, \ldots, q_{m-1} \rangle \in Q^m$, by writing $\langle p_0, \ldots, p_{m-1} \rangle \leq^n \langle q_0, \ldots, q_{m-1} \rangle$ we mean $p_i \leq^n q_i$ for all $i < m$. A set $S \subseteq \omega^\omega$ is called superperfect iff it equals $[p]$ for some superperfect tree $p$. By $[p] \times^+ [q]$ we denote the upper half of the superperfect rectangle $[p] \times [q]$, i.e. the set of all $(x, y) \in [p] \times [q]$ with $x(|stp|) < y(|stq|)$. Similarly, $[p] \times^- [q]$ denotes the lower half of $[p] \times [q]$.

Let $\prec$ be the following wellordering of $\omega^{<\omega}$ in type $\omega$:

\[
\sigma \prec \tau \iff \max(|\sigma|, \max \text{ran}(\sigma)) < \max(|\tau|, \max \text{ran}(\tau)) \\
\vee \max(|\sigma|, \max \text{ran}(\sigma)) = \max(|\tau|, \max \text{ran}(\tau)) \land |\sigma| < |\tau| \\
\vee \max(|\sigma|, \max \text{ran}(\sigma)) = \max(|\tau|, \max \text{ran}(\tau)) \land |\sigma| = |\tau| \land \sigma \text{ precedes } \tau \text{ lexicographically}.
\]

Letting $\langle q_n : n < \omega \rangle$ be the $\prec$-increasing enumeration of $\omega^{<\omega}$ we write $\sharp_\omega, n = n$. Note that $\sharp_\omega \geq \max(|\sigma|, \max \text{ran}(\sigma))$. We shall often construct a superperfect tree $p$ by constructing inductively $\langle \sigma_n : n < \omega \rangle$, the set of splitnodes of $p$. This will be done so that for every $n$, the set of splitnodes and leaves of the tree generated by $\langle \sigma_i : i < n \rangle$ is isomorphic to the tree of the $n$ first (with respect to $\prec$) members of $\omega^{<\omega}$. Then we shall say that $\langle \sigma_i : i < n \rangle$ is an initial segment of the splitnodes of some superperfect tree or that $\langle \sigma_n : n < \omega \rangle$ is the increasing enumeration of the splitnodes of some superperfect tree. Note this does not imply that $\langle \sigma_i : i < n \rangle$ is $\prec$-increasing; however it can of course be arranged that it is.

The following is a basic concept for working with $\mathbb{M}_2$. In [Spib, Corollary 1.4] it has been shown that for every $\langle p, q \rangle \in \mathbb{M}_2$ there exists $\langle p', q' \rangle \leq^0 \langle p, q \rangle$ such that for every $(x, y) \in [p'] \times^+ [q']$ there exists an increasing sequence $\langle k_i : i < \omega \rangle$ such that $k_0 = |stq|, k_1 > |stp|$ and for all $n < \omega$ we have that the following hold:

1. $k_{2n} = \min\{i < \omega : y(i) > \sharp x \upharpoonright k_{2n+1} \}$,
2. $k_{2n+1} = \min\{i < \omega : x(i) > \sharp y \upharpoonright k_{2n+2} \}$,
3. $k_{2n+1} < y(k_{2n}) < k_{2n+2} < x(k_{2n+1}) < k_{2n+3}$,
4. $x \upharpoonright k_{2n+1} \in \text{split}(p'), y \upharpoonright k_{2n} \in \text{split}(q')$. 
In this case we say that \((x, y)\) oscillates infinitely often above \((stp, stq)\). Hence every \((x, y) \in [p'] \times [q']\) has a unique associated sequence \(\langle k_i : i < \omega \rangle\) which is determined solely by \((x, y)\) and \((stp, stq)\). Letting \(\sigma = stp\) and \(\tau = stq\), the sequence

\[
\langle \sigma, \tau, x \upharpoonright k_1, y \upharpoonright k_2, x \upharpoonright k_3, y \upharpoonright k_4, \ldots \rangle
\]

is called the type\(_{\sigma, \tau}\)-sequence of \((x, y)\). Let \(t_{p, q}^{\sigma, \tau}\)-pair\((x, y) = (\sigma, \tau)\), and for all \(n > 0\), let \(t_{p, q}^{\sigma, \tau}\)-(2\(n + 1\))-pair\((x, y) = \langle x \upharpoonright k_{2n+1}, y \upharpoonright k_{2n+1} \rangle\) and \(t_{p, q}^{\sigma, \tau}\)-(2\(n + 2\))-pair\((x, y) = \langle x \upharpoonright k_{2n+2}, y \upharpoonright k_{2n+2} \rangle\). Using this we can define a partial function

\[
t_{p, q}^{\sigma, \tau} : (\sigma, \tau) \to (\omega^\omega)^2
\]

by letting \(t_{p, q}^{\sigma, \tau}(\mu, \nu) = n\) iff there exists \((x, y) \in [p'] \times [q']\) such that \((\mu, \nu) = t_{p, q}^{\sigma, \tau}\)-pair\((x, y)\).

Suppose that \(t_{p, q}^{\sigma, \tau}(\mu, \nu) = 2n\) for some \((\mu, \nu) \in p \times q\). Then there exists a unique sequence \(\langle \mu_0, \nu_0, \ldots, \mu_n, \nu_n \rangle\), which is the initial sequence of length \(2n\) of the type\(_{\sigma, \tau}\)-sequence of \((x, y)\) for some \((x, y) \in [p'] \times [q']\) that is infinitely oscillating over \((\sigma, \tau)\). We will call this sequence the type\(_{\sigma, \tau}\)-sequence of \((\mu, \nu)\) and let \(t_{p, q}^{\sigma, \tau}\)-pair\((\mu, \nu) = t_{p, q}^{\sigma, \tau}\)-pair\((x, y)\) for all \(i \leq 2n\). A similar remark applies in case \(t_{p, q}^{\sigma, \tau}(\mu, \nu)\) is odd.

If \((\sigma, \tau)\) or \((p', q')\) are clear from context, we omit them in the above notation.

For any \((p, q) \in M^2\), \((\sigma, \tau) \in p \times q\), let

\[TP_{\sigma, \tau}(p, q) = \text{dom}(t_{p, q}^{\sigma, \tau}) \cap \text{split}(p(\sigma)) \times \text{split}(q(\tau))\]

the set of all the type pairs of \((p, q)\) above \((\sigma, \tau)\), and for any \(n < \omega\) let \(TP_{\sigma, \tau}^n(p, q)\) denote the set of all pairs in \(TP_{\sigma, \tau}(p, q)\) with type \(n\). Then for some \((\mu, \nu) \in TP_{\sigma, \tau}(p, q)\) with \(t_{p, q}^{\sigma, \tau}(\mu, \nu) = n\), say, \(n\) even, we denote the set of possible successive oscillation points as

\[
\text{Sop}_{\sigma, \tau}^n(\mu, \nu) = \{ \mu' \in \text{split}(p(\mu)^-) : t_{p, q}^{\sigma, \tau}(\mu', \nu) = n + 1 \land t_{p, q}^{\sigma, \tau}\)-pair\((\mu', \nu) = (\mu, \nu) \},
\]

and for odd \(n\) we define \(\text{Sop}_{\sigma, \tau}^n(\mu, \nu)\) symmetrically. If it is clear from the context, we will omit some or all of the indices.

Note that if \((p', q')\), \((\sigma, \tau)\) are as above and \((u, v) \leq (p', q')\), then in general we have that \(TP_{\sigma, \tau}(p', q') \cap (u \times v) \neq TP_{\sigma, \tau}(u, v)\). However, there
exists \langle u', v' \rangle \leq^0 \langle u, v \rangle \text{ such that for almost all } n < \omega \text{ and all } \langle x, y \rangle \in [u'] \times [v'], \text{ we have } \text{tp}_{\sigma, \tau, n}\text{-pair}(x, y) \in \text{TP}_{\sigma, \tau}(u', v').

For the rest of this work we tacitly assume that we always work with elements \langle p, q \rangle which have the property of the above \langle p', q' \rangle. \text{ Since the set of all such } \langle p, q \rangle \text{ is dense in } \mathbb{M}^2, \text{ forcing with this partial order is isomorphic to forcing with } \mathbb{M}^2.

Let \langle p, q \rangle \in \mathbb{M}^2, \phi \text{ a statement in the forcing language of } \mathbb{M}^2 \text{ and } * \in \{+, -\}. \text{ Define}

\begin{align*}
\langle p, q \rangle \models^* \phi
\end{align*}

iff \forall \langle (g_0, g_1) \rangle \neq \emptyset, \text{ for all pairs of } \mathbb{M}^2\text{-generic reals } (g_0, g_1) \in [p] \times^* [q]. \text{ By \cite[Main Lemma 4.2]{Spib}, } \mathbb{M}^2 \text{ has the weak decision property, i.e. if } p \in \mathbb{M}^2 \text{ and } \phi_0 \text{ and } \phi_1 \text{ are statements in the forcing language of } \mathbb{M}^2 \text{ with } p \models \phi_0 \lor \phi_1, \text{ then there exist } q \leq^0 p \text{ and } i_+, i_- \in 2 \text{ (which are not necessarily equal) with } q \models^+ \phi_{i_+} \text{ and } q \models^− \phi_{i_-}.

We will often use the following result from \cite{Mil84}:

**Lemma 1 (Miller)** For every coloring of the splitnodes of a superperfect tree, there exists a superperfect subtree such that all splitnodes have the same color.

\text{From \cite{Spia}, we have the following}

**Fusion Property of } \mathbb{M}^2. \text{ Suppose that } P \text{ is a property of elements of } \mathbb{M}^2 \text{ such that for every } \langle u, v \rangle \in \mathbb{M}^2 \text{ there exists } \langle u', v' \rangle \leq^0 \langle u, v \rangle \text{ which has } P. \text{ Then for every } \langle u, v \rangle \in \mathbb{M}^2 \text{ there exists } \langle u', v' \rangle \leq^0 \langle u, v \rangle \text{ such that for every } \langle \mu, v \rangle \in \text{TP}(u', v'),

- \text{ if } \text{tp}_{\text{stv}}(\mu, v) \text{ is even and if } \langle u'', v'' \rangle \leq^0 \langle u'(\mu), v'(\nu) \rangle \text{ is defined implicitly by letting } u'' \times v'' \text{ be the downward closure of }

\begin{align*}
\{ \langle \rho, \xi \rangle \in u'(\mu)^− \times v'(\nu) : \rho(|\mu|) > \sharp \nu \},
\end{align*}

\text{then } \langle u'', v'' \rangle \text{ has } P, \text{ and}

- \text{ if } \text{tp}_{\text{stv}}(\mu, v) \text{ is odd and if } \langle u'', v'' \rangle \leq^0 \langle u'(\mu), v'(\nu) \rangle \text{ is defined implicitly by letting } u'' \times v'' \text{ be the downward closure of }

\begin{align*}
\{ \langle \rho, \xi \rangle \in u'(\mu) \times v'(\nu)^− : \xi(|\nu|) > \sharp \mu \},
\end{align*}

\text{then } \langle u'', v'' \rangle \text{ has } P.
Let \( \langle P_\alpha, Q_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle \) be a countable support iteration of \( M_2 \) of length \( \omega_2 \), hence for all \( \beta < \omega_2 \), \( \| P_\beta \| \equiv Q_\beta \) is \( M_2 \). For \( \bar{p} \in P_{\omega_2} \) we write
\[
\bar{p} = \langle (\bar{p}(\alpha))^0, \bar{p}(\alpha)^1 : \alpha \in \omega_2 \rangle
\]
with
\[
\bar{p} \upharpoonright \alpha \models_{P_\alpha} \langle (\bar{p}(\alpha))^0, \bar{p}(\alpha)^1 \rangle \in M_2.
\]
If \( G \) is a \( P_{\omega_2} \)-generic filter over \( V \), for \( \alpha \in [1, \omega_2] \), let \( G_\alpha = \{ p \upharpoonright \alpha : p \in G \} \) (which by [Bau83, Theorem 1.2.] is a \( P_\alpha \)-generic filter), and let
\[
\langle (g_0^\beta, g_1^\beta) : \beta < \alpha \rangle
\]
be the pairs of generic reals determined by \( G_\alpha \).

**Definition 2** Let \( F \in [\omega_2]^{<\omega} \), \( \bar{p}, \bar{q} \in P_{\omega_2} \), \( *_\alpha \in \{+,-\} \) for \( \alpha \in F, n \in \omega \) and \( \phi \) a statement in the forcing language of \( P_{\omega_2} \).

1. \( \bar{p} \leq^n_F \bar{q} \) iff \( \bar{p} \leq \bar{q} \) and for every \( \alpha \in F, \bar{p} \upharpoonright \alpha \models_{P_\alpha} \bar{p}(\alpha) \leq^n \bar{q}(\alpha) \).
2. \( \bar{p} \models_{\langle *_\alpha, \alpha \in F \rangle} \phi \) iff for all \( P_{\omega_2} \)-generic \( G \) with \( \bar{p} \in G \) and such that
\[
g_0^\alpha (\text{st} \bar{p}(\alpha)^0[G_\alpha]) < g_1^\alpha (\text{st} \bar{p}(\alpha)^1[G_\alpha]) \iff *_\alpha = +
\]
holds for every \( \alpha \in F \), we have \( V[G] \models \phi \).

**Lemma 3** Suppose \( \bar{p} \in P_{\omega_2} \), \( F \in [\omega_2]^{<\omega} \), \( \phi_i \) statements in the forcing language of \( P_{\omega_2} \) for \( i < 2 \) such that
\[
\bar{p} \models_{P_{\omega_2}} \phi_0 \lor \phi_1.
\]
Also let \( *_\alpha \in \{+,-\} \) for all \( \alpha \in F \). There exists \( \bar{q} \leq^0_F \bar{p} \) and \( i < 2 \) such that
\[
\bar{q} \models_{\langle *_\alpha, \alpha \in F \rangle} \phi_i.
\]

**Proof.** By induction on \( \text{max}(F) \). Let \( \alpha = \text{max}(F) \). Let \( G_{\alpha + 1} \) be \( P_{\alpha + 1} \)-generic over \( V \) such that \( \bar{p} \upharpoonright \alpha + 1 \in G_{\alpha + 1} \). In \( V[G_{\alpha + 1}] \) find \( q^\alpha \) and \( l < 2 \) such that
\[
q^\alpha \leq \bar{p} \upharpoonright [\alpha + 1, \omega_2)[G_{\alpha + 1}] \quad \text{and} \quad q^\alpha \models_{P_{\omega_2}/G_{\alpha + 1}} \phi_l.
As $G_{\alpha+1}$ was arbitrary, in $V[G_\alpha]$ there exist $P_{\alpha+1}/G_\alpha$-names $\dot{\imath}, \dot{q}^\alpha$ such that

$$\dot{p}(\alpha)[G_\alpha] \models_{P_{\alpha+1}/G_\alpha} "\dot{q}^\alpha \models_{P_{\alpha}/G_\alpha} \Phi_\alpha".$$  

(We use that $P_{\alpha+1}/G_\alpha \cong Q_\alpha[G_\alpha] = (M^2)^V[G_\alpha]$). Since $M^2$ has the weak decision property, in $V[G_\alpha]$ there is $k < 2$ and $q(\alpha) \leq^0 p(\alpha)$ with $q(\alpha) \in (M^2)^V[G_\alpha]$ such that

$$q(\alpha) \models \langle \langle \dot{q} \rangle \rangle \dot{\imath} = k.$$  

In $V$ there are $P_\alpha$-names $\dot{q}(\alpha)$ and $\dot{k}$ such that

$$\dot{p} \upharpoonright \alpha \models_{P_\alpha} "\dot{k} \in 2 \land \dot{q}(\alpha) \models \langle \langle \dot{q} \rangle \rangle \dot{\imath} = \dot{k}".$$  

If $F \cap \alpha \neq 0$ then by induction hypothesis (and trivially otherwise) there exist $i < 2$ and $q_\alpha \leq^0 p \upharpoonright \alpha$ $\dot{p} \upharpoonright \alpha$ such that

$$q_\alpha \models \langle \langle q_\alpha \rangle \rangle \dot{\imath} = i.$$  

Finally, let

$$\tilde{q} = q_\alpha \sim (\dot{q}(\alpha)) \dot{q}^\alpha,$$

then $\tilde{q}$ is as desired.  \[\square\]

For $a, b \in [\omega]^{<\omega}$ we will write $a \lhd b$ iff $a$ is an initial segment of $b$, i.e. $a \subseteq b$ and $\min(a \setminus b) >\max(a)$.

**Definition 4** Let $J \subseteq \omega$ and $\langle s_j : j \in J \rangle$ a family of finite sets in $\omega$. Then

$$\lim_{j \in J} s_j = s$$

for some $s \subseteq \omega$, iff the following holds:

1. $s = \bigcup_{j_0 \in J} \bigcap_{j \in J \setminus j_0} s_j,$

2. if $s$ is finite, then $s = \bigcap_{j \in J} s_j.$

Thus $s$ is the limit point of the characteristic functions of the $s_j$ in $2^\omega$, where $2^\omega$ is equipped with the product topology of the discrete topology on 2.
Remark 5

1. By König’s Lemma, for every sequence \( \{s_j : j \in J\} \) as above and \( k \in \omega \), there exists \( J' \subseteq J \) such that \( \lim_{j \in J'} s_j \) exists.

2. If \( s \) is finite, then we have

\[
\forall l > \max(s) \exists j_0 \forall j \in J \setminus j_0 (l \not\in s_j),
\]

so by cutting off an initial part of the sequence we can get an arbitrarily long gap between \( \max(s) \) and \( \min(s_j \setminus s) \), for all \( j \).
Chapter 2

The \( n \)-dimensional Laver and Miller ideals

For \( Q \in \{L, M\} \), let \( J(Q^n) \) be the set of all \( X \subseteq (\omega^\omega)^n \) with the property that for every \( \langle p_i : i < n \rangle \in Q^n \) there exists \( \langle q_i : i < n \rangle \in Q^n \) extending \( \langle p_i : i < n \rangle \) such that \( X \cap \prod_{i < n} [q_i] = \emptyset \). So \( J(Q^n) \) is an ideal in \((\omega^\omega)^n\), which we call the \( n \)-dimensional \( Q \)-ideal. It is easy to see that \( J(\langle L \rangle) \) and \( J(\langle M \rangle) \) are \( \sigma \)-ideals. In this chapter we shall show that also \( J(\langle M^2 \rangle) \) is a \( \sigma \)-ideal, but \( J(\langle L^2 \rangle) \) and \( J(\langle M^3 \rangle) \) are not.

The additivity (\textit{add}) of any ideal is defined as the minimal cardinality of a family of sets belonging to the ideal whose union does not. The covering number (\textit{cov}) is defined as the least cardinality of a family of sets from the ideal whose union is the whole set on which the ideal is defined.

\textbf{Theorem 1} \( J(\langle L^2 \rangle) \) and \( J(\langle M^3 \rangle) \) are not \( \sigma \)-ideals.

\textbf{Proof.} We have to show the following:

1. \( \text{add}(J(\langle L^2 \rangle)) = \omega_0 \),
2. \( \text{add}(J(\langle M^3 \rangle)) = \omega_0 \).
1. For any $\sigma, \tau \in \omega^{<\omega}$ and $(x, y) \in [\sigma] \times [\tau]$, we define $\text{Osc}_{\sigma, \tau}(x, y) \in 3^\omega$ such that

$$\text{Osc}_{\sigma, \tau}(x, y)(n) = \begin{cases} 
0 & \text{if } x(n + |\sigma|) < y(n + |\tau|), \\
1 & \text{if } y(n + |\tau|) < x(n + |\sigma|), \\
2 & \text{if } x(n + |\sigma|) = y(n + |\tau|).
\end{cases}$$

Fix $\alpha \in 2^\omega$ and let

$$X_{\sigma, \tau}^\alpha = \{ (x, y) \in (\omega^\omega)^2 : \text{Osc}_{\sigma, \tau}(x, y) = \alpha \}.$$

For $\sigma, \tau \in \omega^{<\omega}$ we show $X_{\sigma, \tau}^\alpha \in J(\mathbb{L}^2)$. For $(p, q) \in \mathbb{L}^2$, if $(\sigma, \tau) \in p \times q$, then choose $m \in \text{succ}_p(\sigma)$ and $n \in \text{succ}_q(\tau)$ such that if $\alpha(0) = 0$ then $m > n$ else $n > m$, hence $[p(\sigma \cap m)] \times [q(\tau \cap n)] \cap X_{\sigma, \tau}^\alpha = \emptyset$.

On the other hand $X = \bigcup_{\sigma, \tau} X_{\sigma, \tau}^\alpha$ is not in $J(\mathbb{L}^2)$. Indeed, for $(p, q) \in \mathbb{L}^2$ one can easily construct some reals $(x, y) \in [p] \times [q]$ with $\text{Osc}_{\sigma, \tau}(x, y) = \alpha$.

2. We define the partial function $f : (\omega^\omega)^3 \to 2^\omega$ essentially as in [VW98, proof of Theorem 1]: For $x, y, z \in \mathbb{M}$, wherever it is defined let

$$l_0 = \min \{ i : z(i) > \max\{x(i), y(i)\} \},$$
$$m_0 = \min \{ i : x(i) > z(l_0) \},$$
$$n_0 = \min \{ i : y(i) > z(l_0) \},$$
$$l_{k+1} = \min \{ i : z(i) > \max\{x(m_k), y(n_k)\} \},$$
$$m_{k+1} = \min \{ i : x(i) > z(l_k) \},$$
$$n_{k+1} = \min \{ i : y(i) > z(l_k) \},$$

and

$$f(x, y, z)(k) = \begin{cases} 
0, & x(m_k) \leq y(n_k), \\
1, & \text{otherwise}.
\end{cases}$$

Now for $\varrho, \sigma, \tau \in \omega^{<\omega}$ and $(x, y, z) \in [\varrho] \times [\sigma] \times [\tau]$ we define

$$f_{\varrho, \sigma, \tau}(x, y, z) = f(x_\varrho, y_\sigma, z_\tau)$$

where $x_\varrho(n) = x(n + |\varrho|)$, $y_\sigma(n) = y(n + |\sigma|)$ and $z_\tau(n) = z(n + |\tau|)$.

We fix $\alpha \in 2^\omega$ and let

$$X_{\varrho, \sigma, \tau}^\alpha = \{ (x, y, z) \in [\varrho] \times [\sigma] \times [\tau] : f_{\varrho, \sigma, \tau}(x, y, z) = \alpha \}.$$
Then on one hand, for $\varrho, \sigma, \tau \in \omega^{<\omega}$, we show $X^\omega_{\varrho, \sigma, \tau} \in J(M^3)$. Let $(r, s, t) \in M^3$ with $(\varrho, \sigma, \tau) \in \text{split}(r) \times \text{split}(s) \times \text{split}(t)$. Choose $\varrho_0 \in \text{Succ}_r(\varrho)$ and $\sigma_0 \in \text{Succ}_s(\sigma)$ arbitrarily and $\tau_0 \in \text{Succ}_t(\tau)$ such that

$$\tau_0(\vert\tau\vert) > \max\{\sharp \varrho_0, \sharp \sigma_0\}.$$ 

Choose $\varrho_1 \in \text{Succ}_r(\varrho_0)$ and $\sigma_1 \in \text{Succ}_s(\sigma_0)$ such that if $\alpha(0) = 0$ then

$$\varrho_1(\vert\varrho_0\vert) > \sigma_1(\vert\sigma_0\vert) > \tau_0.$$ 

So for any $(x, y, z) \in [\varrho_1] \times [\sigma_1] \times [\tau_0]$ we have $f_{\varrho, \sigma, \tau}(x, y, z)(0) = 1 \neq \alpha(0)$ and therefore $[r(\varrho_1)] \times [s(\sigma_1)] \times [t(\tau_0)] \cap X = \emptyset$.

On the other hand, with the coding argument in [VW98, proof of Theorem 1], we see that $\bigcup_{\varrho, \sigma, \tau} X^\omega_{\varrho, \sigma, \tau}$ is not in $J(M^3)$. □

**Remark 6** For any $Q \in \{\mathbb{L}, M\}$ and $n < \omega$, we have $\text{cov}(J(Q^n)) \geq \omega_1$. Indeed, for a sequence $(X_n : n < \omega)$ in $J(Q^n)$, let $(p_i^k : i < n) = (\omega^{<\omega})^n$, and for $k < \omega$ let $(p_i^{k+1} : i < n) \in Q^n$ be an extension of $(p_i^k : i < n)$ with $\text{st}(p_i^{k+1}) \supseteq \text{st}(p_i^k)$, all $i$, and $\prod_{i<n}[p_i^{k+1}] \cap X_k = \emptyset$. Let $x_i$ be the unique elements of $\bigcap_i [p_i^k]$ for $i < n$, hence $(x_i : i < n) \not\in \bigcup_k X_k$.

**Theorem 2** $J(M^2)$ is a $\sigma$-ideal.

**Lemma 7** For any $(p, q) \in M^2$ and $X \in J(M^2)$,

(a) there exists $(p', q') \leq_0 (p, q)$ such that $[p'] \times^+ [q'] \cap X = \emptyset$,

(b) there exists $(p', q') \leq_0 (p, q)$ such that $[p'] \times^- [q'] \cap X = \emptyset$.

**Corollary 8** For any $(p, q) \in M^2$ and $X \in J(M^2)$ there exists $(p', q') \leq_0 (p, q)$ such that $[p'] \times [q']$ is disjoint to $X$.

**Proof of Theorem 2 from Corollary 8.** Let $(X_n : n < \omega) \subseteq J(M^2)$. For any $(p, q) \in M^2$, let $\mu_0 = \text{stp}$ and $\nu_0 = \text{stq}$, let $\mu_1$ and $\nu_1$ be the $\prec$-minimal members of $\text{Levi}(p)$ and $\text{Levi}(q)$, respectively. By applying Corollary 8 four times, we get $(p^0, q^0) \leq_0 (p, q)$ with $\mu_1 \in \text{split}(p^0)$, $\nu_1 \in \text{split}(q^0)$ such that the following products are disjoint to $X_0$:

$$[p^0(\mu_1)] \times [q^0(\nu_1)], \quad [p^0 \setminus p^0(\mu_1)] \times [q^0(\nu_1)],$$

$$[p^0(\mu_1)] \times [q^0 \setminus q^0(\nu_1)], \quad [p^0 \setminus p^0(\mu_1)] \times [q^0 \setminus q^0(\nu_1)].$$
and therefore \([p^0] \times [q^0] \cap X_0 = \emptyset\). In this way one can easily construct a fusion sequence \(\langle p^n, q^n \rangle : n < \omega \rangle \) in \(M^2\) together with \(\langle \mu_n : n < \omega \rangle\) and \(\langle v_n : n < \omega \rangle\) with the following properties:

1. \(\langle \mu_n : n < \omega \rangle\) and \(\langle v_n : n < \omega \rangle\) are increasing enumerations of the splitnodes of two superperfect trees,
2. \(\langle p^{n+1}, q^{n+1} \rangle \leq_0 \langle p^n, q^n \rangle \leq_0 \langle p, q \rangle\),
3. \(\langle \mu_i, v_i \rangle \in p^n \times q^n\), for all \(i \leq n\),
4. \([p^n] \times [q^n] \cap X_n = \emptyset\).

Note that at each step we have to extend our trees only finitely many times.

Finally let \(p'\) and \(q'\) be the trees determined by its sets of splitnodes \(\{\mu_n : n < \omega\}\) and \(\{v_n : n < \omega\}\), respectively. Hence \(\langle p', q' \rangle \leq_0 \langle p, q \rangle\) and \([p'] \times [q'] \cap \bigcup_{n < \omega} X_n = \emptyset\) and therefore \(\bigcup_{n < \omega} X_n \in J(M^2)\).

**Proof of Lemma 7.** Let \(\sigma = \text{stp}\) and \(\tau = \text{stq}\). Recall that \(\langle p, q \rangle\) is such that all \((x, y) \in [p] \times [q]\) are infinitely oscillating above \([\sigma, \tau]\).

Suppose that there does not exist any \(\langle p', q' \rangle \leq_0 \langle p, q \rangle\) such that \([p'] \times [q'] \cap X = \emptyset\). We shall construct some \(\langle p', q' \rangle \leq_0 \langle p, q \rangle\) such that for every \(\mu \in \text{split}(p_1)\) there does not exist \(\langle p', q' \rangle \leq_0 \langle p_1(\mu), q_1 \rangle\) with \([p'] \times [q'] \cap X = \emptyset\). Note that this is equivalent to saying that for every \(\langle \mu, v \rangle \in \text{TP}_{\sigma, \tau}(p_1, q_1)\) for no \(\langle p', q' \rangle \leq_0 \langle p_1(\mu), q_1(v) \rangle\) we have \([p'] \times [q'] \cap X = \emptyset\).

We recursively construct sequences \(\langle \sigma_n : n < \omega \rangle\), \(\langle \xi_n : n < \omega \rangle\) in \(\omega^{<\omega}\) and \(\langle u_n : n < \omega \rangle\), \(\langle v_n : n < \omega \rangle\) in \(M\), such that the following holds for all \(n\):

1. \(\langle \sigma_n : n < \omega \rangle\) is the increasing enumeration of the splitnodes of some superperfect tree,
2. \(\sigma_n > \xi_{n-1}(|\tau|)\) and if \(k < n\) is maximal such that \(\sigma_k \subseteq \sigma_n\), then \(\sigma_n \in \text{split}(u_k)\) and \(u_n \leq_0 u_k(\sigma_n)\) (hence \(\text{st}(u_n) = \sigma_n\)),
3. \(v_{n+1} \leq_0 v_n \leq_0 q\),
4. \(\xi_n\) is the leftmost element of \(\text{Succ}_{v_n}(\tau)\) and \(\xi_n(|\tau|) < \xi_{n+1}(|\tau|)\),
5. if $k < n$ is maximal such that $\sigma_k \subseteq \sigma_n$ and there exists $\langle p', q' \rangle \leq^0 \langle u_k(\sigma_n), v_{n-1} \rangle$ such that $[p']^+ [q'] \cap X = \emptyset$, then $\langle u_n, v_n \rangle$ equals such $\langle p', q' \rangle$.

Let $\sigma_0 = \sigma, u_0 = p, v_0 = q$ and let $\xi_0$ be the $\prec$-minimal element of $\text{Succ}_\tau(q)$. At stage $n \geq 1$ suppose $\sigma_k$ needs an extension. Choose $\sigma_n \in \text{split}(u_k)$ such that $\not\exists \sigma_n > \xi_{n-1}(|\tau|)$. Let $v'$ be the downward closure of all $\xi \in \text{split}(v_{n-1})$ with $\xi(|\tau|) > \xi_{n-1}(|\tau|)$ and find $\langle u_n, v_n \rangle \leq^0 \langle u_k(\sigma_n), v' \rangle$ with possibly $[u_n]^{-}[v_n] \cap X = \emptyset$. Let $\xi_n$ be the leftmost member of $\text{Succ}_{\tau_0}(\tau)$.

Finally let $u \in \mathbb{M}$ be determined by $\text{split}(u) = \{\sigma_n : n < \omega\}$ and let $v = \bigcup_{n<\omega} v_n(\xi_n) \in \mathbb{M}$.

Now for all $n < \omega$ we color $\sigma_n$ by “yes” if at step $n$ we could find $\langle u_n, v_n \rangle$ such that $[u_n]^{-}[v_n] \cap X = \emptyset$, otherwise color $\sigma_n$ by “no”. By Lemma 1 there exists $u' \leq^0 u$ such that every splitnode of $(u')^{-}$ has the same color. Choose $\langle p_1, q_1 \rangle \leq^0 \langle u', v \rangle$ with all branches in the upper half infinitely oscillating.

In order to get a contradiction suppose that every splitnode of $p'_1$ has color “yes”. Fix $\langle x, y \rangle \in [p_1]^{-}[q_1]$. Then we get some $n < \omega$ such that $\text{tp}_{\sigma, \tau-1}\text{pair}(x, y) = \langle \sigma_n, \tau \rangle$. Thus $y(|\tau|) > \not\exists \sigma_n$, which, by 2, is $> \xi_{n-1}(|\tau|)$. Hence $\langle x, y \rangle \in [u_n]^{-}[v_n]$, and therefore we have $\langle x, y \rangle \not\in X$, since $\sigma_n$ is colored “yes”. As $(x, y)$ was arbitrary, we have shown that $[p_1]^{-}[q_1] \cap X = \emptyset$. So since $\langle p_1, q_1 \rangle \leq^0 \langle p, q \rangle$ we get a contradiction to our assumption.

So we have proved part (a) of the following:

**Claim 8.1** Suppose $X \in J(\mathbb{M}^2)$ and $\langle p, q \rangle \in \mathbb{M}^2$ and let $\sigma = \text{stp}, \tau = \text{stq}$.

(a) Suppose that for no $\langle p', q' \rangle \leq^0 \langle p, q \rangle$ do we have $[p']^+ [q'] \cap X = \emptyset$. Then there exists $\langle p_1, q_1 \rangle \leq^0 \langle p, q \rangle$ such that for every $\langle \mu, \tau \rangle \in \text{TP}^1_{\sigma, \tau}(p_1, q_1)$ there is no $\langle p', q' \rangle \leq^0 \langle p_1(\mu), q_1 \rangle$ with $[p']^{-}[q'] \cap X = \emptyset$.

(b) Suppose that for no $\langle p', q' \rangle \leq^0 \langle p, q \rangle$ do we have $[p']^- [q'] \cap X = \emptyset$. Then there exists $\langle p_1, q_1 \rangle \leq^0 \langle p, q \rangle$ such that for every $\langle \mu, \tau \rangle \in \text{TP}^1_{\sigma, \tau}(p_1, q_1)$ there is no $\langle p', q' \rangle \leq^0 \langle p_1(\mu), q_1 \rangle$ with $[p']^+ [q'] \cap X = \emptyset$.

The proof for part (b) is the same as for part (a) with “+” and “−” exchanged.
Now by recursion we build a fusion sequence \((p_n, q_n) : n < \omega\) such that for every odd \(n < \omega\) we get

\((-)_n\) for every \(\langle \mu, \nu \rangle \in TP^\mu_{\sigma, \tau}(p_n, q_n)\), for no \(\langle p', q' \rangle \leq^0 \langle p_n(\mu), q_n(\nu) \rangle\) we have \([p'] \times^- [q'] \cap X = \emptyset\),

and for every even \(n\) we get

\((+_n)\) for every \(\langle \mu, \nu \rangle \in TP^\mu_{\sigma, \tau}(p_n, q_n)\), for no \(\langle p', q' \rangle \leq^0 \langle p_n(\mu), q_n(\nu) \rangle\) we have \([p'] \times^+ [q'] \cap X = \emptyset\).

Let \(\langle p', q' \rangle\) be the fusion, i.e. \(p' = \bigcap_{n < \omega} p_n\) and \(q' = \bigcap_{n < \omega} q_n\), so \(\langle p', q' \rangle \leq^0 \langle p, q \rangle\). Now choose \(i \in \text{succ}_p(\sigma), j \in \text{succ}_q(\tau)\) with \(i < j\). Find \(\langle u, v \rangle \leq \langle p'(\sigma^- i), q'(\tau^- j) \rangle\) with \([u] \times [v] \cap X = \emptyset\). Then \(\langle stu, stv \rangle\) has some type \(\sigma, \tau\), say \(n\). As \(\langle u, v \rangle \leq^0 \langle p_n(stu), q_n(stv) \rangle\) we get a contradiction to \((-)_n\), where \(* \in \{+, -\}\) depending on the parity of \(n\).

The construction of the fusion sequence is a recursion of length \(\omega^n\), in the way it was explicitly done in [Spib]. The reason why we get length \(\omega^n\) for the recursion is that we have to go through all type-pairs and for each \(n\), the type-\(n\)-pairs are canonically wellordered in type \(\omega^n\).

We show how to get \(\langle p_2, q_2 \rangle\) with \((+_2)\). Suppose we have \(\langle \mu, \tau \rangle \in TP^1_{\sigma, \tau}(p_1, q_1)\). So by \((-)\), for no \(\langle p', q' \rangle \leq^0 \langle p_1(\mu), q_1 \rangle\) do we have \([p'] \times^- [q'] \cap X = \emptyset\) and equivalently \([q'] \times^+ [p'] \cap X^{-1} = \emptyset\). By Claim 8.1 we obtain \(\langle q^\mu, p^\mu \rangle \leq^0 \langle q_1, p_1(\mu) \rangle\) such that for all \(\langle v, \mu \rangle \in TP^1_{\tau, \mu}(q^\mu, p^\mu)\) we do not have \(\langle q', p' \rangle \leq^0 \langle q^\mu(v), p^\mu \rangle\) with \([q'] \times^- [p'] \cap X^{-1} = \emptyset\). So in particular we have that for all pairs \(\langle \mu', v' \rangle \in TP^2_{\sigma, \tau}(p^\mu, q^\mu)\) with \(\mu' = \mu\), for no \(\langle p', q' \rangle \leq^0 \langle p^\mu(\mu), q^\mu(v') \rangle\), \([p'] \times^+ [q'] \cap X = \emptyset\). So with a fusion over all \(\mu \in \text{split}(p_1)\) we get \(\langle p_2, q_2 \rangle\) such that \((+_2)\) holds. \(\square\)
Chapter 3

Bounds for add\((J(\mathbb{M}^2))\)

We show that under the assumption \(b = c\), \(\text{add}(J(\mathbb{M}^2))\) is less or equal to the collapsing number \(\kappa(\mathbb{M}^2)\) of the two-dimensional Miller forcing, which is defined as the least cardinal to which forcing with \(\mathbb{M}^2\) collapses the continuum. Then we will show that \(h\), the distributivity number of \(\mathcal{P}(\omega)/\text{fin}\), is an upper bound for \(\kappa(\mathbb{M}^2)\).

**Lemma 9** Suppose \(b = c\). For every dense open \(D \subseteq \mathbb{M}^2\) there exists a maximal antichain \(A \subseteq D\) such that

\[
\forall (u, v), (p, q) \in A ([u] \times [v] \cap [p] \times [q] = \emptyset),
\]

\[
\forall (p, q) \in \mathbb{M}^2 [ [p] \times [q] \subseteq \bigcup \{[u] \times [v] : (u, v) \in A \}]
\]

\[
\Rightarrow \exists A' \in [A]^{\leq c} ([p] \times [q] \subseteq \bigcup \{[u] \times [v] : (u, v) \in A' \})
\]

**Proof.** Let \(\mathbb{M}^2 = \{\langle p_\alpha, q_\alpha \rangle : \alpha \in c\}\). Inductively we will define a set \(S \subseteq c\) and sequences \(\langle u_\alpha, v_\alpha \rangle : \alpha \in S\) and \(\langle x_\alpha, y_\alpha \rangle : \alpha \in c\). Finally we will let \(A = \langle u_\alpha, v_\alpha \rangle : \alpha \in S\).

Let \(0 \in S\) and choose \((x_0, y_0) \in [p_0] \times [q_0]\) arbitrarily.

**Fact 1.** Every Miller tree contains \(c\) extensions such that every two of them do not contain a common branch. The analogous thing holds for \(\mathbb{M}^2\).
So clearly we may find \((u_0, v_0) \in D\) such that \((x_0, y_0) \notin [u_0] \times [v_0]\).

Now suppose we got \((x_\gamma, y_\gamma) : \gamma < \alpha) and \((u_\gamma, v_\gamma) : \gamma < \alpha) for \alpha < c and look at \(p_\alpha, q_\alpha\).

First we choose \((x_\alpha, y_\alpha) \in [p_\alpha] \times [q_\alpha]\) such that if possible \((x_\alpha, y_\alpha) \notin \bigcup [u_\gamma] \times [v_\gamma] : \gamma < \alpha) and distinguish the following two cases:

**Case 1.** The condition \((p_\alpha, q_\alpha)\) is compatible with some \((u_\gamma, v_\gamma), \gamma < \alpha\). Then declare \(\alpha \notin S\).

**Case 2.** We have that \((p_\alpha, q_\alpha)\) is incompatible with all \((u_\gamma, v_\gamma), \gamma < \alpha\). Then declare \(\alpha \in S\). Note that for each \(\gamma < \alpha\), we must have incompatibility in at least one of the two coordinates. So we let \(B_0 = \{\gamma < \alpha : p_\alpha \perp_M u_\gamma\}\) and \(B_1 = \alpha \setminus B_0\). For each \(\gamma < \alpha\), if \(p_\alpha\) and \(v_\gamma\) are incompatible in \(M\), then by [Kec95, Corollary 21.19, p. 161] we know that \([p_\alpha] \cap [u_\gamma]\) is \(\sigma\)-bounded by some \(f_\gamma \in \omega^{\omega_0}\), and analogously if \(q_\alpha\) and \(v_\gamma\) are incompatible in \(M\), then \([q_\alpha] \cap [v_\gamma]\) is \(\sigma\)-bounded by some \(g_\gamma \in \omega^{\omega_0}\). By \(b = c\), \(\{f_\gamma : \gamma \in B_0\}\) is \(\sigma\)-bounded by some \(h_0 \in \omega^{\omega_0}\) and \(\{g_\gamma : \gamma \in B_1\}\) is \(\sigma\)-bounded by some \(h_1 \in \omega^{\omega_0}\). We can find a pair of superperfect trees \((p', q') \leq (p_\alpha, q_\alpha)\) such that for all \((x, y) \in [p'] \times [q']\), \(x \not\leq^* h_0\) and \(y \not\leq^* h_1\), hence \([p'] \times [q'] \cap [u_\gamma] \times [v_\gamma] = \emptyset\) for all \(\gamma < \alpha\). By Fact 1, we can choose \((u_\alpha, v_\alpha) \in D\) such that \((u_\alpha, v_\alpha) \leq (p', q')\) and \([u_\alpha] \times [v_\alpha] \cap \{(x, y) : \gamma \leq \alpha\} = \emptyset\). This finishes the construction.

Now let \(A = \{(u_\alpha, v_\alpha) : \alpha \in S\} \subseteq D\). Since every \((p_\alpha, q_\alpha)\) is compatible with some \((u_\gamma, v_\gamma), \gamma < \alpha\) (Case 1) or contains the condition \((u_\alpha, v_\alpha)\) (Case 2), and for all \(\alpha, \gamma \in S\) with \(\gamma < \alpha\) we have \([u_\alpha] \times [v_\alpha] \cap [u_\gamma] \times [v_\gamma] = \emptyset\), we conclude \(A\) is a maximal antichain.

Condition (3.1) clearly holds by construction. For (3.2), if \([p_\alpha] \times [q_\alpha] \not\subseteq \bigcup [u_\gamma] \times [v_\gamma] : \gamma \in S \cap \alpha), \alpha < c\), then it was possible to choose \((x_\alpha, y_\alpha) \in [p_\alpha] \times [q_\alpha] \setminus \bigcup [u_\gamma] \times [v_\gamma] : \gamma \in S \cap \alpha). By construction no \((u_\beta, v_\beta), \beta \geq \alpha\), contains \((x_\alpha, y_\alpha),\ hence \([p_\alpha] \times [q_\alpha] \not\subseteq \bigcup [u_\gamma] \times [v_\gamma] : \gamma \in S\). □

Note that in Lemma 9 it actually is sufficient to assume \(d = c\). For this we apply [Spi95, Lemma 2.1] in Case 2 with \(u_\gamma : \gamma \in B_0\) and \(p_\alpha\) to obtain \(p'\), and also with \(u_\gamma : \gamma \in B_1\) and \(q_\alpha\) to obtain \(q'\).
Theorem 3  Suppose $b = c$. Then $\text{add}(J(M^2)) \leq \kappa(M^2)$.

Proof. We may assume $\kappa(M^2) < c$. Let $\dot{f}$ be a $M^2$-name such that $\Vdash_{M^2}\dot{f} : \kappa(M^2) \rightarrow c$ is onto. For $\alpha < \kappa(M^2)$ let

$$D_\alpha = \{ (p, q) \in M^2 : \exists \beta(p, q) \Vdash_{M^2} \dot{f}(\alpha) = \beta \}.$$ 

For $(p, q) \in D_\alpha$ write $\beta_{p, q} = \beta_{p, q}(\alpha)$ for the unique $\beta$ satisfying $\langle p, q \rangle \Vdash_{M^2} \dot{f}(\alpha) = \beta$.

Clearly $D_\alpha$ is dense and open. So we may choose a maximal antichain $A_\alpha \subseteq D_\alpha$ as in Lemma 9. Let

$$X_\alpha = \omega^\alpha \setminus \bigcup \{ [p] \times [q] : (p, q) \in A_\alpha \}.$$ 

Then $X_\alpha \in J(M^2)$. We claim that $X = \bigcup \{ X_\alpha : \alpha < \kappa(M^2) \} \not\in J(M^2)$. Suppose on the contrary $X \in J(M^2)$. So we may find $(u, v) \in M^2$ such that $[u] \times [v] \cap X = \emptyset$ and hence $[u] \times [v] \subseteq \bigcup \{ [p] \times [q] : (p, q) \in A_\alpha \}$ for each $\alpha$. By the choice of $A_\alpha$, letting

$$B_\alpha = \{ \beta_{p, q}(\alpha) : (p, q) \in A_\alpha, (p, q) \text{ compatible with } (u, v) \},$$

each of these sets is bounded in $c$. Since $c$ is regular by our assumption $b = c$, we can find $v < c$ such that for all $\alpha < \kappa(M^2)$, $B_\alpha \subseteq v$. So easily conclude that

$$(u, v) \Vdash_{M^2} \text{ "ran}(\dot{f}) \subseteq v < c".$$ 

This is a contradiction.  

Theorem 4  $\kappa(M^2) \leq \mathfrak{h}$.

Proof. We work in $V$. Let $(A_\alpha : \alpha < \mathfrak{h})$ be a family of mad families such that:

1. if $\alpha < \beta < c$, then $A_\beta$ refines $A_\alpha$,
2. there exists no mad family refining all the $A_\alpha$,
3. $\bigcup \{ A_\alpha : \alpha < \mathfrak{h} \}$ is dense in $(\omega^\omega, \subseteq^*)$. 


That such a sequence exists was shown in [BPS80].

Since \( h \) is regular, for every \( p \in \mathbb{M} \) there exists \( \alpha < h \) such that for each \( \sigma \in \text{split}(p) \) there is \( a \in \mathbb{A}_\alpha \) with \( a \subseteq^* \text{succ}_p(\sigma) \). Hence, writing \( \mathbb{M}_\alpha \) for the set of those \( p \in \mathbb{M} \) for which \( \alpha \) has the property just stated, we conclude \( \mathbb{M} = \bigcup \{ \mathbb{M}_\alpha : \alpha < h \} \).

For each \( a \in \mathbb{A}_\alpha \) choose \( \mathcal{B}_a = \{ B^a_p : p \in \mathbb{M} \} \), a mad family on \( a \).

Now we will define \( \mathbb{M}'_\alpha = \{ u^\alpha_p : p \in \mathbb{M}_\alpha \} \) such that \( u^\alpha_p \) extends \( p \) for every \( p \in \mathbb{M}_\alpha \) and \( p_1 \neq p_2 \) implies \( u^\alpha_{p_1} \perp u^\alpha_{p_2} \). For \( p \in \mathbb{M}_\alpha \), \( u^\alpha_p \) will be defined as follows:

For each \( \sigma \in \text{split}(p) \) let \( C^\alpha_\sigma(p) = \text{succ}_\sigma(\sigma) \cap B^a_p \) where \( a \in \mathbb{A}_\alpha \) is such that \( a \subseteq^* \text{succ}_p(\sigma) \). So clearly \( C^\alpha_\sigma(p) \) is infinite. Now \( u^\alpha_p \) is the unique superperfect tree extending \( p \) with \( \text{st}(u^\alpha_p) = \text{st}p \) and for each \( \sigma \in \text{split}(u^\alpha_p) \) we have \( \text{succ}_{u^\alpha_p}(\sigma) = C^\alpha_\sigma(p) \).

It is not difficult to see that \( \mathbb{M}'_\alpha \) has the stated properties.

Now we are ready to define a \( \mathbb{M}^2 \)-name \( \dot{f} \) such that \( \models_{\mathbb{M}^2} \dot{f} : h^V \rightarrow c^V \) is onto": For each \( (p, q) \in \mathbb{M}_\alpha \times \mathbb{M} \), let \( \{ r^\xi_{\alpha}(p, q) : \xi < c \} \subseteq \mathbb{M}^2 \) be a maximal antichain below \( (u^\alpha_p, q) \) and define \( \dot{f} \) in such a way that \( r^\xi_{\alpha}(p, q) \models_{\mathbb{M}^2} \dot{f}(\alpha) = \xi \). As \( \bigcup \{ \mathbb{M}'_\alpha : \alpha < h \} \) is dense in \( \mathbb{M} \) and therefore \( \bigcup \{ \mathbb{M}'_\alpha \times \mathbb{M} : \alpha < h \} \) is dense in \( \mathbb{M}^2 \), it is easy to check that \( \dot{f} \) is as desired. \( \square \)
Chapter 4

The consistency of $\text{add}(J(\mathcal{M}^2)) < \text{cov}(J(\mathcal{M}^2))$

**Theorem 5** Suppose that $P_{\omega_2}$ is a countable support iteration of length $\omega_2$ of $\mathcal{M}^2$ and $V \models \text{ZFC} + \text{GCH}$. Then $V^{P_{\omega_2}} \models \text{cov}(J(\mathcal{M}^2)) = \omega_2$.

**Proof.** In $V^{P_{\omega_2}}$ let $\langle X_\alpha : \alpha < \omega_1 \rangle \in V^{P_{\omega_2}}$ be a sequence of sets in $J(\mathcal{M}^2)$. For every $\alpha < \omega_2$, there exists a function $f_\alpha : \mathcal{M}^2 \rightarrow \mathcal{M}^2$ such that $f_\alpha(p) \leq p$ and $[f_\alpha(p)]^0 \times [f_\alpha(p)]^1 \cap X_\alpha = \emptyset$, for all $p \in \mathcal{M}^2$. In $V$ we have a $P_{\omega_2}$-name $\check{f}_\alpha$ for $f_\alpha$, so $\check{f}_\alpha$ is a set of pairs $\langle p, q \rangle$ such that $\Vdash_{P_{\omega_2}} \check{p}, \check{q} \in \mathcal{M}^2$. Because $P_{\omega_2}$ is proper we can assume without loss of generality that $\check{p}$ and $\check{q}$ are hereditarily countable, for all $\langle \check{p}, \check{q} \rangle \in \check{f}_\alpha$. Let

$$C_\alpha = \{ \beta < \omega_2 : \text{if } \langle \check{p}, \check{q} \rangle \in \check{f}_\alpha \text{ and } \check{p} \in V^{P_\beta}, \text{ then } \check{q} \in V^{P_\beta} \}.$$

Then $C_\alpha$ is $\omega_1$-club, i.e. unbounded in $\omega_2$ and closed under increasing $\omega_1$-sequences. Hence, letting $C = \bigcap_{\alpha < \omega_1} C_\alpha$, $C$ is $\omega_1$-club as well. Fix $\beta \in C$ and let $\check{f}_\alpha^\beta = \{ \langle \check{p}, \check{q} \rangle \in \check{f}_\alpha : \check{p} \in V^{P_\beta} \}$. Hence $f_\alpha \upharpoonright \mathcal{M}^2(V^{P_\beta}) = f_\alpha^\beta \in V^{P_\beta}$, for all $\alpha < \omega_1$. Let $\langle \check{g}_0^\beta, \check{g}_1^\beta \rangle$ be the $P_\beta$-name for the pair of generic reals determined by $\check{P}_\beta$. We claim

$$\Vdash_{P_{\omega_2}} \langle \check{g}_0^\beta, \check{g}_1^\beta \rangle \not\in \bigcup_{\alpha < \omega_1} X_\alpha.$$
Suppose we have $p \in P_{\omega_2}$ and $\alpha < \omega_1$ such that $p \Vdash_{P_{\omega_2}} \langle \hat{g}_0^\beta, \hat{g}_1^\beta \rangle \in X_\alpha$. Let $G$ be a $P_{\omega_2}$-generic filter containing $p$. We work in $V[G]$, where $p \Vdash [\beta, \omega_2) \Vdash_{P_{\omega_2}/G} \langle \hat{g}_0^\beta, \hat{g}_1^\beta \rangle \in X_\alpha$. Let $q = f_\alpha^\beta(p(\beta))$ and let $p' = q \upharpoonright [\beta, \omega_2)$, so $p' \leq p \Vdash [\beta, \omega_2)$ and since $p' \Vdash \langle \hat{g}_0^\beta, \hat{g}_1^\beta \rangle \in [q^0] \times [q^1]$, we have $p' \Vdash \langle \hat{g}_0^\beta, \hat{g}_1^\beta \rangle \notin X_\alpha$. Hence we get a contradiction. □

**Theorem 6** Suppose that $P_{\omega_2}$ and $V$ are as in Theorem 5. Then $V^{P_{\omega_2}} \models \text{add}(J(M_2)) = \omega_1$.

The proof consists of a series of lemmas after the following result.

**Theorem 7** Forcing with $M_2$ adds a nonsplit real, i.e. in $V^{M_2}$ there exists a set $a \in [\omega]^{\omega}$ such that for all $b \in [\omega]^{\omega} \cap V$ either $a \subseteq^* b$ or $|a \cap b| < \omega$. Hence $V^{P_{\omega_2}} \models s = \omega_2$.

**Proof.** Let $G$ be an $M_2$-generic filter over $V$ and $\langle g_0, g_1 \rangle$ the generic pair of reals determined by $G$. By genericity there exists $(p, q) \in G$ and $p, \tau \in q$ such that every $(x, y) \in \langle p, q \rangle$ oscillates infinitely often above $\langle \sigma, \tau \rangle$, hence the same holds for $\langle g_0, g_1 \rangle$. Let

$$a = \langle g_0 \upharpoonright k_{2n+1} : n < \omega \rangle,$$

the sequence of the left oscillation points of $\langle g_0, g_1 \rangle$ above $\langle \sigma, \tau \rangle$. Fix $b \in [\omega^{<\omega}]^\omega$. By Lemma 1 the set of $\langle p', q' \rangle$ such that either $\text{split}(p') \subseteq b$ or $\text{split}(p') \cap b = \emptyset$ is dense below $\langle p, q \rangle$. Hence such $\langle p', q' \rangle$ belongs to $G$. Since $\{g_0 \upharpoonright k_{2n+1} : n < \omega\} \subseteq^* \text{split}(p')$, we get that $a$ is as desired.

Now assume $X \subseteq [\omega]^{\omega}$ is a splitting family in $V^{P_{\omega_2}}$ with $|X| = \omega_1$. For some $\alpha < \omega_2$ we have that $X \subseteq V^{P_\alpha}$. So $Q_\alpha$ adds some real that is not split by any $b \in X$. □

**Lemma 10** Forcing with $M_2$ adds a dominating real, i.e. some function $f$ such that for any $g \in \omega^{\omega} \cap V$ we have $f \geq^* g$. Hence $V^{P_{\omega_2}} \models b = c$.

**Proof.** Let $\langle g_0, g_1 \rangle$ be an $M_2$-generic pair of reals. As in the proof of Theorem 7, find $(p, q)$ and $\sigma \in p, \tau \in q$ such that $\langle g_0, g_1 \rangle \in [p(\sigma)] \times [q(\tau)]$ and every $(x, y) \in [p(\sigma)] \times [q(\tau)]$ oscillates infinitely often above $\sigma, \tau$. Let
\langle k_n : n < \omega \rangle$ be the type-sequence of $\langle g_0, g_1 \rangle$. Define $f = \langle k_{2n+1} : n < \omega \rangle$.

Note that for every $g \in \omega^\omega \cap V$, the set of all $\langle p', q' \rangle \leq \langle p, q \rangle$ such that

$$\forall n \forall \varnothing \in \text{Lev}_n(p') \left( |\varnothing| > g(n) \right)$$

is dense below $\langle p, q \rangle$. Now it is easy to see that by genericity this implies $g \leq^* f$.

Combining this result with the results in Chapter 3, we get

$$V^{\mathbb{M}_2} \models \text{add}(J(\mathbb{M}^2)) \leq h.$$ 

In order to show that the additivity number stays small in the iterated forcing extension by $\mathbb{M}_2$, we want to show that $h = \omega_1$ holds in that model. In the following lemma we establish the main construction for the proof.

**Lemma 11** Suppose $\dot{a}$ is a $\mathbb{P}_{\omega_2}$-name and $p \in \mathbb{P}_{\omega_2}$ with $\dot{p} \forces \dot{a} \in [\omega]^\omega$. Then for every finite $F \subseteq \omega_2 \setminus \{0\}$, there exist $r \in \mathbb{P}_{\omega_2}$ with

$$r \leq_0 \left( \bigcup_{i \in F} \check{p} \right)$$

and refining finite partitions

$$\dot{\Gamma}_n = \{ \dot{r}_i^n : i < n + 1 \}, \quad n < \omega$$

of the Boolean completion of $\mathbb{P}_{\omega_2}$ below $r \upharpoonright [1, \omega_2)$, and, letting $\langle u, v \rangle = r(0)$, $\sigma = \text{stu}$, $\tau = \text{stv}$, there exists a set

$$\{ w_i^{\mu, v}, \dot{y}_i^{\mu, v} \in \mathcal{P}(\omega) : \exists n < \omega \langle \langle u, v \rangle \in \text{TP}_{\sigma, \tau}^n(u, v) \wedge i < n+2 \wedge i' < n+1 \rangle \},$$

and a countable set $\mathcal{D}$ of disjoint families in $[\omega]^\omega$ such that for any $\langle \mu, v \rangle \in \text{TP}^n(u, v)$, $n < \omega$, and any $i < n+2$ and $i' < n+1$, letting $J = \text{succ}_u(\mu) \setminus \sharp v$ and $J' = \text{succ}_v(\mu) \setminus \sharp u$, the following properties hold:

1. Suppose $n$ is even. For any sequence $\langle G^j : j \in J \rangle$ of $\mathbb{P}_{\omega_2}$-generic filters over $V$ such that $\langle u(\mu \land j), v \rangle \cap \check{r}_i^{n+1} \in G^j$ and $\text{tp}_{\sigma, \tau}^n$-pair$(g_0^j, g_1^j) = \langle \mu, v \rangle$ where $G_1^j = \langle g_0^j, g_1^j \rangle$ and $\langle \mu^j, v \rangle$ is the type-$(n + 1)$-pair of $\langle g_0^j, g_1^j \rangle$, we have

$$w_i^{\mu, v} = \lim_{j \in J} \dot{a}([G^j] \cap |\mu^j|).$$
Suppose $n$ is odd, for any sequence $\langle G^j : j \in J' \rangle$ of $P_{\omega_2}$-generic filters over $V$ such that $(u, \nu \nu) \in \mathcal{P}^n_{\omega_1} \subseteq G^j$ and $\text{tp}_{\sigma, \tau} \eta$-pair$(g^j_0, g^j_1) = \langle \mu, \nu \rangle$ where $G^j_1 = \langle g^j_0, g^j_1 \rangle$ and $\langle \mu, \nu \rangle = \text{tp}_{\sigma, \tau} (n + 1)$-pair$(g^j_0, g^j_1)$, we have

$$w^\mu_{i, \nu} = \lim_{j \in J'} \hat{a}[G^j] \cap |\nu^j|.$$ 

2. Suppose $n > 0$ and even. For any sequence $\langle G^j : j \in J' \rangle$ of $P_{\omega_2}$-generic filters over $V$ such that $(u, \nu \nu) \in \mathcal{P}^n_{\omega_1} \subseteq G^j$ and $\text{tp}_{\sigma, \tau} \eta$-pair$(g^j_0, g^j_1) = \langle \mu, \nu \rangle$ with $G^j_1 = \langle g^j_0, g^j_1 \rangle$ and some $\nu^j \supseteq \nu$ (hence $\nu^j \subseteq \nu^j$), we have

$$y^\mu_{i, \nu} = \lim_{j \in J'} \hat{a}[G^j] \cap j.$$ 

If $n$ is odd, for any sequence $\langle G^j : j \in J' \rangle$ of $P_{\omega_2}$-generic filters over $V$ such that $(u, \nu \nu) \in \mathcal{P}^n_{\omega_1} \subseteq G^j$ and $\text{tp}_{\sigma, \tau} \eta$-pair$(g^j_0, g^j_1) = \langle \mu, \nu \rangle$ where $G^j_1 = \langle g^j_0, g^j_1 \rangle$ and some $\nu^j \supseteq \nu$ (hence $\nu^j \subseteq \nu^j$), we have

$$y^\mu_{i, \nu} = \lim_{j \in J'} \hat{a}[G^j] \cap j.$$ 

3. Suppose $w^\mu_{i, \nu}$ is finite. If $n$ is even, then either all $y^\mu_{i, \nu}$ with $\mu' \in \text{Sop}^{\omega}_0(\mu, \nu)$ are finite or they are all infinite. In the finite case we have either one of the following:

(a) (empty case) For all $\mu'$ as above, we have $y^\mu_{i, \nu} = w^\mu_{i, \nu}$. Let the set $\{r^n_{i_0}, r^n_{i_1}\}$ consist of those elements of $\mathcal{P}^n_{\omega_1}$ which are below $r^n_{i_1}$, so possibly $i_0 = i_1$. Then for $j < 2$, we have either

i. for all $\mu' \in \text{Sop}(\mu, \nu)$, $w^\mu_{i, \nu} \cap w^\mu_{i, \nu} = \emptyset$, or

ii. $w^\mu_{i, \nu} \cap w^\mu_{i, \nu} = \emptyset$.

Moreover, for all $k \in \text{succ}(\mu)$, letting

$$d^\mu_k(\mu, \nu) = \bigcup \left\{ w^\mu_{i, \nu} \cap w^\mu_{i, \nu} : j < 2 \right\},$$

we have $\{d^\mu_k(\mu, \nu) : k \in \text{succ}(\mu)\} \in \mathcal{D}$. 

(b) (constant case) For every $\mu', \mu'' \in \text{Sop}(\mu, v)$ with $\mu' \leq \mu''$, we have $y_i^{\mu', v} \land w_i^{\mu', v} = y_i^{\mu', v} \land w_i^{\mu', v} \neq \emptyset$. Let

$$d^\ell(\mu, v) = \bigcup \{y_i^{\mu', v} \land w_i^{\mu', v} : \mu' \in \text{Sop}(\mu, v)\}.$$ 

(c) (tree case) For every $\mu', \mu'' \in \text{Sop}(\mu, v)$ we have $y_i^{\mu', v} \land w_i^{\mu', v} \neq \emptyset$ and if $\mu' \subseteq \mu''$, we have $y_i^{\mu', v} \land y_i^{\mu', v} \neq \emptyset$. Then, for all $k \in \text{succ}_\mu(\mu)$, letting

$$d^g_k(\mu, v) = \bigcup \{y_i^{\mu', v} \land w_i^{\mu', v} : \mu' \in u(\mu \smallsetminus k) \cap \text{Sop}(\mu, v)\},$$

we have $\{d^g_k(\mu, v) : k \in \text{succ}_\mu(\mu)\} \in \mathcal{D}$.

For odd $n$, we have the symmetric properties and $d^\ell(\mu, v)$ and disjoint families $\{d^g_k(\mu, v) : k \in \text{succ}_\mu(\mu)\} \in \mathcal{D}$.

Moreover, for any $\mu \in \text{split}(u)$ and $v \in \text{split}(v)$, letting

$$D_\mu = \bigcup \{d^\ell(\mu, \hat{v}) : \hat{v} \in \text{split}(v) \land \text{tp}(\mu, \hat{v}) = n \text{ even} \land \langle \mu, \hat{v} \rangle \text{ of constant case for } i < n + 2\},$$

$$E_v = \bigcup \{d^\ell(\mu, v) : \mu \in \text{split}(u) \land \text{tp}(\mu, v) = n \text{ odd} \land \langle \mu, v \rangle \text{ of constant case for } i < n + 2\},$$

then we have

$$\{D_\mu, E_v : \mu \in \text{split}(u) \land v \in \text{split}(v)\} \in \mathcal{D}.$$

**Proof.** Fix $G_1 = (g_0, g_1)$, a $\mathcal{Q}_0$-generic filter over $V$ containing $\mathcal{P}(0)$. Let $(p, q) = \mathcal{P}(0)$, $\sigma = \text{stp}$ and $\tau = \text{stq}$; for the rest of the proof, the types will always refer to $\sigma, \tau$. Let $(k_n : n \in \omega)$ enumerate the lengths of the oscillation points of $(g_0, g_1)$, i.e. for all $n \in \omega$, if $(\mu, v)$ is the type-$n$-pair of $(g_0, g_1)$, then let $k_n = \max\{|\mu|, |v|\}$.

In $V[G_1]$ we consider $(\dot{a}[G_1] \cap k_i : i < \omega)$, which is a $P_{\omega_2}/G_1$-name for a member in $\Pi_{i<\omega}2^{|k_i|}$. (We identify $\mathcal{P}(k_i)$ with $2^{|k_i|}$.) By [Bau83, Theorem 5.2.], $P_{\omega_2}/G_1$ is isomorphic to $(P_{\omega_2})^{|G_1|}$. It is straightforward to apply Lemma 3 to build a fusion sequence $(p_j : j < \omega)$ with $p_j \leq^0_{\mathcal{P}} \mathcal{P} \upharpoonright [1, \omega_2)$ such that
1. \( p_0 = \overline{p} \upharpoonright [1, \omega_2) \),
2. for every \( n \) there exists a partition

\[
\Gamma_n = \{ r^n_i : i < n + 1 \}
\]

of the Boolean completion of \( P_{\omega_2}/G_1 \) below \( p_{n+1} \) and

\[
B_n = \{ b^n_i : i < n + 1 \} \subseteq 2^{2^n}
\]

such that

\[
\forall i < n + 1 \left( r^n_i \equiv_{P_{\omega_2}/G_1} \dot{a}[G_1] \upharpoonright k_n = b^n_i \right),
\]

and if \( n > 0 \), then \( \Gamma_n \) refines \( \Gamma_{n-1} \).

Now let \( \dot{q} \) be the infimum of \( \langle p_n : n < \omega \rangle \). As \( G_1 \) was arbitrary, in \( V \) there exist \( Q_0 \)-names \( \langle \dot{k}_n : n < \omega \rangle \), \( \dot{\Gamma}_n = \langle \dot{r}_n^i : i < n + 1 \rangle \), \( \dot{B}_n = \langle \dot{b}_n^i : i < n + 1 \rangle \), \( \langle \dot{p}_n : n < \omega \rangle \) and \( \dot{q} \) such that \( \overline{p}(0) \) \( Q_0 \)-forces that \( \langle \dot{k}_n : n < \omega \rangle \) are the lengths of the oscillation points of \( \langle \dot{s}_0^0, \dot{s}_1^0 \rangle \), \( \langle \dot{p}_j : j < \omega \rangle \) is a fusion sequence in \( P_{\omega_2}/\dot{G}_1 \) with \( \dot{p}_j \leq \dot{p} \upharpoonright [1, \omega_2) \) and infimum \( \dot{q} \) and \( \dot{\Gamma}_n \) is a partition of the Boolean completion of \( P_{\omega_2}/\dot{G}_1 \) below \( \dot{p}_{n+1} \), such that \( \dot{\Gamma}_{n+1} \) refines \( \dot{\Gamma}_n \), and

\[
\dot{B}_n = \{ \dot{b}_n^i : i < n + 1 \} \subseteq 2^{2^n} \text{ such that }
\]

\[
\forall i < n + 1 \left( r^n_i \equiv_{P_{\omega_2}/\dot{G}_1} \dot{a} \upharpoonright \dot{k}_n = \dot{b}_n^i \right),
\]

for all \( n < \omega \). So \( \overline{p}(0) \) forces that for all \( n \), \( \dot{\Gamma}_n \) is a partition below \( \dot{q} \) and actually \( \dot{\Gamma}_{n+1} \) refines \( \dot{\Gamma}_n \).

Since \( M^2 \) has the weak decision property, for every \( \langle \mu, v \rangle \in \text{TP}^\omega (p, q) \), \( n < \omega \), there exists an extension \( \langle u', v' \rangle \leq^0 \langle p(\mu), q(v) \rangle \) such that every generic \( \langle g_0, g_1 \rangle \) thru \( \langle u', v' \rangle \) with \( \langle \mu, v \rangle \) as type pair gives \( B_n \) the same value, say

\[
\langle b^n_i(\mu, v) : i < n + 1 \rangle.
\]

Also if \( n > 0 \) and even, then there exists \( \langle u'', v'' \rangle \leq \langle u', v' \rangle \) such that every generic \( \langle g_0, g_1 \rangle \) thru \( \langle u'', v'' \rangle \) which has \( \langle \mu, v' \rangle \) as type-\( n \)-pair for some \( v' \supseteq v \cap j \), gives \( B_n \upharpoonright j \) the same value, say

\[
\langle s^n_{i,j}(\mu, v) : i < n + 1 \rangle.
\]

To show the latter, again we use the weak decision property to decide \( B_n \upharpoonright j \) above every \( \langle \mu, v' \rangle \) with \( v' \supseteq v \cap j \) together with Lemma 1 applied on the
trees \( v'(v \cap j) \). Symmetrically we get \( \{ s_i^{n_j} (\mu, v) : i < n + 1 \} \) in case \( n \) is odd.

So by the Fusion Property of \( M^2 \), we can assume without loss of generality that for every \( \langle \mu, v \rangle \in TP^q (p, q) \), \( \langle p(\mu), q(v) \rangle \) decides \( \hat{B}_n \) and \( \hat{B}_n \upharpoonright j \) with \( j \in \text{sucq}_q(v) \) for even \( n \) and \( j \in \text{succ}_p(\mu) \) for odd \( n \), as explained above.

By Remark 5.1 and the Fusion Property, without loss of generality we may assume that for every \( \langle \mu, v \rangle \in TP^q (p, q) \), \( i < n + 2 \) and \( i' < n + 1 \), we have \( w_i^{\mu, v} \) and \( y_i^{\mu, v} \), such that if \( n \) is even, then

1. for every \( \langle \mu^j : j \in \text{succ}_p(\mu) \rangle \) with \( \text{tp}(\mu^j, v) = n + 1 \) and \( \mu^j \supseteq \mu \cap j \),
   \[
   w_i^{\mu, v} = \lim_{j \in \text{succ}_p(\mu)} b_i^{n+1}(\mu^j, v),
   \]

2. \( y_0^{\mu, v} = \emptyset \) and if \( n > 0 \) then \( y_i^{\mu, v} = \lim_{j \in \text{succ}_q(v)} s_i^{n_j}(\mu, v) \),

and symmetrically for odd \( n \). So 1 and 2 of the Lemma hold. Note that if \( \langle \mu, v \rangle \in TP^p (p, q) \), \( i < \text{tp}(\mu, v) + 2 \) and \( i' < \text{tp}(\mu, v) + 3 \) such that \( r_i^n \) is compatible with \( r_i^{n+1} \), and \( \mu', \mu'', \mu''' \in \text{Sop}(\mu, v) \) with \( \mu' \subseteq \mu'' \subseteq \mu''' \), in case the following are finite sets we have

\[
\begin{align*}
& w_i^{\mu, v} \preceq w_i^{\mu', v} \quad \text{and} \quad w_i^{\mu, v} \preceq y_i^{\mu', v} \preceq y_i^{\mu'', v} \preceq \ldots \preceq w_i^{\mu'''},
\end{align*}
\]

Now we fix \( \langle \mu, v \rangle \in TP^p (p, q) \) with, say, even type \( n \), also fix \( i < n + 2 \). If \( w_i^{\mu, v} \) is finite, by Lemma 1 we can find \( u^0 \leq p(\mu) \), such that either for all \( \mu' \in \text{Sop}^{\mu,v} (\mu, v) \) we have that \( y_i^{\mu', v} \) is infinite or for all such \( \mu' \), \( y_i^{\mu', v} \) is finite. In the second case, again by Lemma 1 we can prepare \( u^0 \) (i.e. extend \( u^0 \) to some \( u' \leq^0 u^0 \)) such that we are either in the empty, constant or tree case of Lemma 11.

In the empty case, we look at \( r_i^{n+1} \in \hat{\Gamma}_{n+1} \). Since \( \hat{\Gamma}_{n+2} \) is a refinement of \( \hat{\Gamma}_{n+1} \), at most one member of \( \hat{\Gamma}_{n+1} \) gets split into two members of \( \hat{\Gamma}_{n+2} \), and all other members of \( \hat{\Gamma}_{n+1} \) belong to \( \hat{\Gamma}_{n+2} \). So there are \( i_0, i_1 < n + 3 \) such that \( \{ r_i^{n+2}, r_i^{n+2} \} \) is a partition of \( r_i^{n+1} \). In particular, for \( j < 2 \), below \( r_i^{n+2} \) we have that \( b_i^{n+1} \) is an initial segment of \( \hat{B}_i^{n+2} \) and therefore \( w_i^{\mu, v} \preceq w_i^{\mu', v} \) for all \( \mu' \in \text{Sop}(\mu, v) \).
Claim 11.1 There exists $u^1 \leq^0 u^0$ such that the following hold:

1. for $j < 2$, we have either $w_{i_j}^{\mu',v} \setminus w_i^{\mu,v} \neq \emptyset$ for all $\mu' \in \text{Sop}^{u^1,q}(\mu,v)$, or $w_{i_j}^{\mu',v} \setminus w_i^{\mu,v} = \emptyset$ for all such $\mu'$,

2. for all $k \in \text{succ}_u(\mu)$, letting

$$d_k^1(\mu,v) = \bigcup \{ w_{i_j}^{\mu',v} \setminus w_i^{\mu,v} : j < 2 \land \mu' \in u^1(\mu \cup k) \cap \text{Sop}(\mu,v) \},$$

then for all $k, l \in \text{succ}_u(\mu)$, if $k \neq l$ then $d_k^1(\mu,v) \cap d_l^1(\mu,v) = \emptyset$.

Note that either all $d_k^1(\mu,v)$ are infinite (if the new pieces are not empty for at least one $j < 2$) or all $d_k^1(\mu,v)$ are finite (if all the new pieces are empty for both $j = 0$ and $j = 1$).

PROOF. By Lemma 1, we can find $u' \leq^0 u^0$ such that 1 holds. For the disjointness, by induction we construct the splitnodes $\langle \sigma_l : l < \omega \rangle$ of $u^1$ such that

1. $\sigma_l \in \text{Sop}^{u',q}(\mu,v),$

2. if $0 < k < l$, then for $j < 2$ with $w_{i_j}^{\sigma_l,v} \setminus w_i^{\mu,v} \neq \emptyset$ and for any $j' < 2$ we have

$$\min(w_{i_j}^{\sigma_l,v} \setminus w_i^{\mu,v}) > \max(w_{i_j}^{\sigma_{l'},v} \setminus w_i^{\mu,v}).$$

Fix $\sigma_0 \in \text{Sop}^{u',q}(\mu,v)$ arbitrarily. At stage $l$, assume $\sigma_l$ needs an extension. For $j < 2$, suppose $w_{i_j}^{\mu,v} \setminus w_i^{\mu,v} \neq \emptyset$ for all $\mu' \in \text{Sop}(\mu,v)$. Suppose $\sigma_l = \mu$. As $w_i^{\mu,v}$ is finite, we can apply Remark 5.2, hence by construction we can choose $k \in \text{succ}_u(\mu)$ large enough, such that for every $\mu' \in u'(\mu \cup k) \cap \text{Sop}(\mu,v)$, $\min(w_{i_j}^{\mu,v} \setminus w_i^{\mu,v})$ is as large as we like. For $\sigma_l \supseteq \mu$, note that for all $\mu' \in \text{split}(u'(\sigma_l))$, $w_{i_j}^{\mu',v} \setminus w_i^{\mu,v} = w_{i_j}^{\mu',v} \setminus w_i^{\mu'',v}$, where $\mu''$ is the predecessor splitnode of $\mu'$. Therefore, with the same argument as before we can make the new piece of some $w_{i_j}^{\mu',v}$ start as late as we like. Therefore we can choose $\sigma_l \in \text{Succ}_u(\sigma_l)$ as desired. □

In the tree case we have the following:
Claim 11.2 There exists $u^2 \leq^0 u^1$ such that for all $k \in \text{succ}_{u^2}(\mu)$, letting

$$d^e_k(\mu, v) = \bigcup \{y_i^{\mu', v} \setminus w_i^{\mu', v} : \mu' \in u^2(\mu^\sim k) \cap \text{Sop}(\mu, v)\},$$

we have $d^e_k(\mu, v) \cap d^e_{k'}(\mu, v) = \emptyset$ for $k \neq 1$ and all $d^e_k(\mu, v)$ are infinite.

Proof. Note that for such $u^2$ and $d^e_k(\mu, v)$, if $\mu' \in \text{Sop}(\mu, v) \cap u(\mu^\sim k) \cap \text{Succ}_{u^2}(\mu)$, then

$$d^e_k(\mu, v) = \bigcup \{y_i^{\mu', v} \setminus w_i^{\mu', v} : \mu' \in \text{split}(u^2(\mu^\prime)) \land q' \in \text{Succ}_{u^2}(q)\}$$

where $\mu'$ is the single element of $\text{Succ}_{u^2}(\mu) \cap \text{Sop}^{u^2,v}(\mu, v)$ with $\mu'(\mu') = k$. \(\square\)

For odd-type pairs, we proceed symmetrically. So, by applying the Fusion Property, we obtain some $(\hat{u}, \hat{v}) \leq^0 (p, q)$ such that 3 holds.

Finally we need the following:

Claim 11.3 There exists $(u, v) \leq^0 (\hat{u}, \hat{v}) \leq^0 (p, q)$ such that $\{D_\mu, E_\nu : \mu \in \text{split}(u) \land v \in \text{split}(v)\}$ is a disjoint family.

Proof. We construct $(u, v)$ by a simultaneous induction on splitnodes: Suppose we got the initial segments $(\sigma_l : l < k)\land (\tau_l : l < k)$ of the splitnodes of $u, v$, respectively, and $\sigma_l$ needs an extension (and therefore also $\tau_l$). Fix $\tau_l$ such that $\text{tp}(\sigma_l, \tau_l) = n$ is even and also fix $i < n + 2$. Since $w_i^{\sigma_l, \tau_l}$ is finite, we can choose $\sigma_k \in \text{Sop}(\sigma_l, \tau_l) \cap \text{Succ}_{\sigma_l}(\sigma_l)$ so much right that $\min(y_i^{\sigma_k, \tau_l} \setminus w_i^{\sigma_k, \tau_l})$ is as large as we like, say $m_*$. So we make sure $m_* > \max(y_i^{\sigma_k, \tau_l} \setminus w_i^{\sigma_k, \tau_l})$ for all $k'' < k' \leq k$ such that $\sigma_k'' \subsetneq \sigma_{k'}$ and $\text{tp}(\sigma_k, \tau_k) = n'$ is even and all $i' < n' + 2$, and also $m_* > \max(y_i^{\sigma_k, \tau_k} \setminus w_i^{\sigma_k, \tau_k})$ for all $l'' < l' < k$ and $\tau_{l''} \subsetneq \tau_{l'}$ and $\text{tp}(\sigma_{l''}, \tau_{l''}) = n'$ is odd and $i' < n' + 2$. Get $\tau_k$ analogously. \(\square\)

Note that for all $(\mu, v) \in \text{TP}(u, v)$ and $i < \text{tp}(\mu, v) + 2$ as in the empty or constant case, all $k \in \text{succ}_{\mu}(\mu)$ for even type or $k \in \text{succ}_{\nu}(v)$ for odd type, $d^e_k(\mu, v) \supseteq d^e_k(\mu, v)$. Hence we get

$$r = (u, v)^{\tilde{q}} \leq^0 (0) \cup \tilde{p}$$
as desired, which finishes the proof of Lemma 11. \(\Box\)

**Lemma 12 (CH)** There exists a Matrix \(\mathcal{A} = \{\mathcal{A}_\gamma : \gamma < \omega_1\}\) of mad families in \(\omega\) such that

\[
\models P_{\omega_2} \forall \dot{a} \in [\omega]^\omega \exists \gamma < \omega_1 \forall b \in \mathcal{A}_\gamma (|\dot{a} \smallsetminus b| = \omega).
\]

**Proof.** By CH we can enumerate \([\omega]^\omega\) by \([x_v : v < \omega_1]\) and enumerate the set of all disjoint families in \([\omega]^\omega\) by \([D_v : v < \omega_1]\). We construct a sequence \(\mathcal{A} = \{\mathcal{A}_\gamma : \gamma < \omega_1\}\) of mad families \(\mathcal{A}_\gamma\) in \(\omega\) such that the following hold:

\begin{enumerate}
  \item A1. if \(v < \gamma < \omega_1\), then \(\mathcal{A}_\gamma\) refines \(\mathcal{A}_v\),
  \item A2. \(\forall \gamma < \omega_1 \forall v < \gamma \forall b \in \mathcal{A}_\gamma (|x_v \smallsetminus b| = \omega)\),
  \item A3. \(\forall \gamma < \omega_1 \forall v < \gamma \forall b \in \mathcal{A}_\gamma \left[\exists d \in D_v (b \subseteq^* d) \land \forall d \in D_v (|b \cap d| < \omega)\right]\).
\end{enumerate}

For any \(\gamma < \omega_1\), we define some mad family \(\mathcal{A}' \subseteq [\omega]^\omega\) by a recursion of length \(\omega_1\). Let \(\{y_v : v < \omega_1\} = \{x_v : v < \omega_1\}\) be another enumeration such that \(\{y_v : v < \omega_1\} = D_{\gamma}\). Now for any \(v < \omega_1\), if \(y_v\) is almost disjoint to all \(b\) that are already in \(\mathcal{A}'\), then we extend \(\mathcal{A}'\) as follows: If \(x_{\gamma} \subseteq^* y_v\), then put into \(\mathcal{A}'\) two disjoint parts of \(y_v\) which are splitting \(x_{\gamma}\), otherwise just add \(y_v\) itself. Finally let \(\mathcal{A}_{\gamma}\) be a refinement of the countably many mad families \(\mathcal{A}'\) and \(\mathcal{A}_v\), \(v < \gamma\).

Note that in A3, \(\mathcal{A}_{\gamma}\) refines an extension of \(D_v\).

We claim that \(\mathcal{A}\) works for Lemma 12. Let \(\dot{a}\) be a \(P_{\omega_2}\)-name and \(\dot{p} \in P_{\omega_2}\) such that

\[
\dot{p} \models P_{\omega_2} \forall \dot{a} \in [\omega]^\omega \setminus V.
\]

We have to show that we can find some \(\gamma < \omega_1\) and an extension \(r \leq \dot{p}\) such that

\[
r \models P_{\omega_2} \forall b \in \mathcal{A}_{\gamma} (|\dot{a} \smallsetminus b| = \omega).
\]

For this we apply Lemma 11 with \(F = \emptyset\). Choose \(\gamma < \omega_1\) such that

1. all infinite \(w_i^{\mu, v}, y_i^{\mu, v}\) are in \(\{x_\beta : \beta < \gamma\}\) and
2. \(D \subseteq \{D_v : v < \gamma\}\).
In order to get a contradiction, suppose that for some $P_{\omega_2}$-extension $r' \leq r$, some $b \in A_y$ and $n_0 < \omega$, we have

$$r' \models P_{\omega_2} \quad \alpha \smallsetminus n_0 \subseteq b.$$  

(4.1)

We say $w_i^{\mu, v}$ or $y_i^{\mu, v}$ is relevant (for $r'$), iff $\langle \mu, v \rangle \in TP^0(r'(0))$ and $i < n + 2$, $i' < n + 1$ and $(r'(0)^0(\mu), r'(0)^1(\nu)) \cap \mathcal{P}^{n+1}_i \models r'$, $(r'(0)^0(\mu), r'(0)^1(\nu)) \cap \mathcal{P}^{n+1}_i \models r'$, respectively. We investigate two cases:

**Case 1.** There exists an infinite $w_i^{\mu, v}$ or $y_i^{\mu, v}$ that is relevant for $r'$.

Suppose $w_i^{\mu, v}$ is relevant and infinite for some $\langle \mu, v \rangle \in TP^0(r'(0))$ with even type $n$ and some $i < n + 2$. Since $|w_i^{\mu, v} \setminus b| = \omega$, we can pick some $k \in (w_i^{\mu, v} \setminus b) \setminus n_0$. But we can choose a $P_{\omega_2}$-generic filter $G$ containing $(r'(0)^0(\mu), r'(0)^1(\nu)) \cap \mathcal{P}^{n+1}_i$ with $g_0(\{\mu\})$ large enough, such that $k \in \dot{a}(G)$. So we get a contradiction to (4.1). For an infinite $y_i^{\mu, v}$ and also for odd type we proceed accordingly.

**Case 2.** All relevant $w_i^{\mu, v}$ and $y_i^{\mu, v}$ are finite. Therefore we are always in one of the three finite cases of Lemma 11.

**Claim 12.1** There exists $\langle u', v' \rangle \leq r'(0)$ such that for all $\langle \mu, v \rangle \in TP^0(u', v')$ with even type $n$ and $i < n + 2$,

1. if we have the empty case and $i^{n+2}_j : j < 2$ are below $i^{n+1}_i$, then for both $j < 2$ we have

$$\forall \mu' \in \text{Sop}^{u', v'}(\mu, v) \left[ (w_i^{\mu, v} \setminus w_i^{\mu, v}) \cap b = \emptyset \right].$$

2. in the tree case,

$$\forall \mu' \in \text{Succ}_{u'}(\mu) \cap \text{Sop}^{u', v'}(\mu, v) \exists \mu'', \mu''' \in \text{split}(r(0)^0) \left[ \mu \subseteq \mu'' \subseteq \mu''' \subseteq \mu' \wedge (y_i^{\mu''}, v \setminus y_i^{\mu''}, v) \cap b = \emptyset \right].$$

3. if we are in the constant case and for no $\hat{\mu} \in \text{split}(r(0)^0)$ do we have $b \subseteq^* D_{\hat{\mu}}$, then

$$\forall \mu' \in \text{Succ}_{u'}(\mu) \cap \text{Sop}^{u', v'}(\mu, v) \exists \mu'' \in \text{split}(r(0)^0) \left[ \mu \subseteq \mu'' \subseteq \mu' \wedge (y_i^{\mu''}, v \setminus y_i^{\mu''}, v) \cap b = \emptyset \right].$$
and the symmetric properties hold for odd-type pairs.

**Proof.** For any \( \langle \mu, v \rangle \in \text{TP}(r'(0)) \) of empty case, the \( d'_k(\mu, v) \) we got from Lemma 11 form a disjoint family, hence by the construction of \( \mathcal{A} \) we get that \( d'_k(\mu, v) \cap b \) is finite, so with the same construction as in the proof of Claim 11.1, we get \( (w_j^{\mu', v} \setminus w_j^{\mu, v}) \cap b = \emptyset \) for \( j < 2 \) and any \( \mu' \in \text{Sop}^{r', v'}(\mu, v) \) that is right enough (i.e. \( \mu'(\lceil \mu \rceil) \) large enough). For 2, just make the stem of \( u'(\mu \setminus k) \) long enough, for all \( k \in \text{succ}_{u'}(\mu) \). For 3, before we start, for any \( \hat{\mu} \in \text{split}(r(0)^0) \) let \( n_{\hat{\mu}} = \max(b \cap D_{\hat{\mu}}) \). Now make sure the stem of all \( u'(\mu \setminus k) \) is right enough and long enough such that for some appropriate \( \mu'' \subseteq \text{Sop}'(\mu_0) \) we have \( \min(y_j^{\mu'', v} \setminus w_j^{\mu, v}) > n_{\hat{\mu}}. \)

Now fix a \( P_{\omega_2} \)-generic filter \( G \) containing \( \langle u', v' \rangle \cap r' \cap [1, \omega_2) \) with \( G_1 = \langle g_0, g_1 \rangle \). For any \( \langle \mu, v \rangle \in \text{TP}^n(u, v) \), let \( i < n + 2 \) be the unique number such that we have \( \langle u', v' \rangle \cap r_i^{n+1} \in G \) and let \( w_{\mu, v} = w_i^{\mu, v} \), and similarly, for the unique \( i' < n + 1 \) with \( \langle u', v' \rangle \cap r_i' \in G \) let \( y_{\mu, v} = y_i^{\mu, v} \).

Let

\[
I = \{ \langle \mu, v \rangle : \exists n \exists (\mu', v')[(\mu, v) = \text{tp}-n\text{-pair}(g_0, g_1) \land (\mu', v') = \text{tp}-(n+1)\text{-pair}(g_0, g_1) \land w_{\mu', v'} \setminus w_{\mu, v} \neq \emptyset] \},
\]

the type pairs where the new pieces of \( w \) give a contribution to \( \check{a}[G] \). Note that \( I \) is infinite.

**Case 2a.** We have the constant case for infinitely many \( \langle \mu, v \rangle \in I \). Without loss of generality we may assume that they all have even type. If \( b \subseteq^* D_{\mu} \) for some \( \mu \), then clearly \( \check{a}[G] \setminus b \) is infinite, which contradicts (4.1). Otherwise for \( \omega \) many \( \langle \mu, v \rangle \in I \), if \( (\mu', v') = \text{tp}-(n+1)\text{-pair}(g_0, g_1) \), there exists \( \mu'' \subseteq \mu \) such that

\[
b \cap (y_{\mu'', v} \setminus w_{\mu, v}) = \emptyset
\]

with \( y_{\mu'', v} \setminus w_{\mu, v} \subseteq \check{a}[G] \), and all these \( y_{\mu'', v} \setminus w_{\mu, v} \) nonempty and pairwise disjoint. So again we get \( |\check{a}[G] \setminus b| = \omega \) in contradiction to (4.1).

**Case 2b.** We have the empty case or the tree case for almost all \( \langle \mu, v \rangle \in I \). Apply Claim 12.1 to get infinitely many nonempty, pairwise disjoint pieces into \( \check{a}[G] \setminus b \), so also in this case we get \( |\check{a}[G] \setminus b| = \omega \). □
Theorem 8 (GCH) There exists a $P_{\omega_2}$-name $\dot{\mathcal{M}}$ for a family of $\aleph_1$ mad families on $\omega$ with no refinement in $V^{P_{\omega_2}}$.

**Proof.** For any $\alpha < \omega_2$, suppose

$$\models_{P_\alpha} ^{\omega} \{ \dot{x}_\nu^{\alpha} : \nu < \omega_1 \} \text{ enumerates } [\omega]^{\omega_0},$$

and

$$\models_{P_\alpha} ^{\omega} \{ \dot{D}_\nu^{\alpha} : \nu < \omega_1 \} \text{ enumerates the set of all disjoint families in } [\omega]^{\omega_0}.$$

We construct a sequence $(\dot{\mathcal{A}}^{\alpha} : \alpha < \omega_2)$ such that for every $\alpha < \omega_2$,

$$\dot{\mathcal{A}}^{\alpha} = (\dot{\mathcal{A}}_\gamma^{\alpha} : \gamma < \omega_1)$$

is a family of $P_\alpha$-names with the following properties in $V^{P_\alpha}$:

1. $\dot{\mathcal{A}}_\gamma^{\alpha}$ is a mad family on $\omega$ for all $\gamma < \omega_1$,
2. $\dot{\mathcal{A}}^{\alpha}$ is refining,
3. $\forall \beta < \alpha \forall \gamma < \omega_1 (\dot{\mathcal{A}}_\gamma^\beta \subseteq \dot{\mathcal{A}}_\gamma^{\alpha})$,
4. $\forall \gamma < \omega_1 \forall \nu < \gamma \forall b \in \dot{\mathcal{A}}_\gamma^{\alpha} \setminus \bigcup_{\beta < \alpha} \dot{A}_\gamma^\beta \ (|\dot{x}_\nu^{\alpha} \setminus b| = \omega)$,
5. $\forall \gamma < \omega_1 \forall \nu < \gamma \forall b \in \dot{\mathcal{A}}_\gamma^{\alpha} \setminus \bigcup_{\beta < \alpha} \dot{A}_\gamma^\beta$

$$[\exists d \in \dot{D}_\nu^{\alpha}(b \subseteq d) \lor \forall d \in \dot{D}_\nu^{\alpha}(|b \cap d| < \omega)].$$

Note that then for every $\alpha$, if $b' \in [\omega]^{\omega_0}$ is almost contained in some $b \in \dot{\mathcal{A}}_\gamma^{\alpha} \setminus \dot{\mathcal{A}}_\gamma^\beta$ (where $\gamma < \omega_1$ and $\beta < \alpha$) then it is not in $V[G_\beta]$.

We already have $\dot{\mathcal{A}}^0$ from Lemma 12. Suppose we got $\langle \dot{\mathcal{A}}^\beta : \beta < \alpha \rangle$ for some $\alpha < \omega_2$. We define $\dot{\mathcal{A}}^{<\alpha} = (\dot{\mathcal{A}}_\gamma^{<\alpha} : \gamma < \omega_1)$ by letting

$$\dot{A}_\gamma^{<\alpha} = \bigcup_{\beta < \alpha} \dot{A}_\gamma^\beta$$

for all $\gamma < \omega_1$. If $\alpha$ has uncountable cofinality, let

$$\dot{\mathcal{A}}^{\alpha} = \dot{\mathcal{A}}^{<\alpha}.$$
As then $[\omega]^{\omega} \cap V^{P_{\omega}} = \bigcup_{\beta < \alpha} [\omega]^{\omega} \cap V^{P_{\beta}}$, $\mathcal{A}^{\alpha}$ is a family of mad families.

If $\text{cf}(\alpha) \leq \omega$, extend each $\mathcal{A}^{\gamma}_{\gamma}$, $\gamma < \omega_1$, to get the mad families $\mathcal{A}^{\alpha}_{\gamma}$ as desired. Finally let $\mathcal{M} = \langle \mathcal{M}_{\gamma} : \gamma < \omega_1 \rangle$ with all $\mathcal{M}_{\gamma} = \mathcal{A}^{\omega_2}_{\gamma}$.

Suppose that for some $P_{\omega_2}$-name $\dot{a}$ and $p \in P_{\omega_2}$ we have

$$p \vDash P_{\omega_2} \dot{a} \in [\omega]^{\omega}.$$  

We shall find $q \leq p$ and $\gamma < \omega_1$ such that

$$q \vDash P_{\omega_2} \forall b \in \mathcal{M}_{\gamma} (\dot{a} \not\subseteq^* b).$$

We may assume that there exists $\alpha \in \omega_2$ with countable cofinality such that

$$p \vDash \dot{a} \in V[\dot{G}_{\alpha}] \setminus \bigcup_{\beta < \alpha} V[\dot{G}_{\beta}].$$

Hence we may assume that $\dot{a}$ is a $P_{\alpha}$-name, but not a $P_{\beta}$-name for any $\beta < \alpha$. By construction of $\mathcal{M}$, we know

$$\vDash P_{\omega_2} \forall \gamma < \omega_1 \forall b \in \mathcal{M}_{\gamma} \setminus \mathcal{A}^{\alpha}_{\gamma} (\dot{a} \not\subseteq^* b).$$

Hence if for some $q \leq p$, $\gamma < \omega_1$ and $\dot{b} \in \mathcal{M}_{\gamma}$, we have

$$q \vDash \dot{a} \subseteq^* \dot{b},$$

then in fact

$$q \models \alpha \vDash P_{\alpha} \dot{a} \subseteq^* \dot{b} \land \dot{b} \in \mathcal{A}^{\alpha}_{\gamma}.$$  

We may therefore assume that $p \in P_{\alpha}$.

Using the Lemma 11, we construct a fusion sequence $\langle (p_k, F_k) : k < \omega \rangle$ in $P_\alpha$ such that the following holds for all $\beta \leq \alpha$:

1. We have $p_0 \leq p$ and there exists $\gamma^* < \omega_1$ such that $p_0 \vDash P_{\alpha} \dot{a} \in \{ \dot{a}^{\gamma^*} : v < \gamma^* \}$.
2. If $\beta \in \text{supp}(p_k)$, then there exists $m > k$ such that

   a. for every $n < \omega$, we have hereditarily countable $P_{\beta}$-names $\dot{\Gamma}_{\alpha}(\beta)$ for a finite partition of $p \upharpoonright [\beta + 1, \alpha)$ in $V^{P_{\beta}}$, $P_{\beta}$-names $\dot{w}_{i}^{\mu,v}(\beta)$, $\dot{y}_{i}^{\mu,v}(\beta)$ for every $P_{\beta}$-name $\langle \mu, v \rangle$ for an element in $\text{TP}^{\alpha}(p_{m}(\beta))$ and $i < n + 2$, $i' < n + 1$, and also a $P_{\beta}$-name $\dot{D}(\beta)$ for a countable set of disjoint families in $[\omega]^{\omega}$, which are all obtained together with $p_{m}$ by applying Lemma 11 in $V^{P_{\beta}}$ to $\dot{a}[\dot{G}_{\beta}]$ and $p_{m-1} \upharpoonright [\beta, \alpha)$ and $F_k \leq \beta$. 

(b) there exists $\gamma_\beta < \omega_1$ such that
\[ p_m \models \beta \models_{p_\beta} \{ \dot{w}^{H,\nu}_i(\beta), \dot{y}^{H,\nu}_i(\beta) : (\mu, \nu) \in TP^\nu(p(\beta)) \}
\land i < n + 2, i' < n + 1 \} \cap [\omega]^{\omega} \subseteq \{ \dot{x}_{\nu}^{\beta} : \nu < \gamma_\beta \}^\omega, \]
and also
\[ p_m \models \beta \models_{p_\beta} \{ \dot{D}_\nu^\beta : \nu < \gamma_\beta \}^\omega, \]
3. For $\beta \in \text{supp}(p_k)$, all the (countably many) coordinates needed to evaluate $p_k(\beta)$, the families $\{ \dot{w}^{H,\nu}_i(\beta), \dot{y}^{H,\nu}_i(\beta) : (\mu, \nu) \in TP^\nu(p(\beta)) \land i < n + 2, i' < n + 1 \}$, $\{ \dot{\Gamma}_n(\beta) : n < \omega \}$ and $\dot{D}(\beta)$ and also their supremum belong to $\text{supp}(p_m)$, for some $m > k$. Hence if $\beta$ is this supremum and $\beta < \beta'$, then all these objects are $P_\beta$-names. In this case, we make sure that $m$ as above can be chosen such that for some $\gamma'_{\beta'}$ we have
\[ p_m \models \beta \models_{p_\beta} \{ \dot{D}_\nu^\beta : \nu < \gamma'_{\beta'} \}^\omega, \]
so $\dot{D}(\beta)$ has been taken care of before row $\gamma'_{\beta'}$ of the old $\dot{\mathcal{A}}_\gamma \setminus \dot{\mathcal{A}}^{<\beta}$.

Now suppose $q$ is the infimum of $\{ p_k : k < \omega \}$ and let
\[ \gamma = \sup \{ \gamma^*, \gamma_\beta, \gamma'_{\beta'} : \beta \in \text{supp}(q) \}. \]
In order to get a contradiction, suppose that for some $P_{\omega_2}$-extension $r \leq q$, $\beta < \omega_2, \dot{b} \in \dot{\mathcal{A}}^{\beta}_{\gamma} \setminus \dot{\mathcal{A}}^{<\beta}_{\gamma}$ and $n_0 < \omega$, we have
\[ r \models_{P_{\omega_2}} \dot{a} \upharpoonright n_0 \subseteq \dot{b}. \] (4.2)

**Case 1.** Suppose $\beta \in \text{supp}(q)$. We step into $V^{P_\beta}$ and proceed as in Lemma 12 to get a contradiction to (4.2).

**Case 2.** Suppose $\beta > \alpha$. Note that we may assume $\sup(\text{supp}(q)) = \alpha$. By (4.2), the new $\dot{b}$ almost contains the old $\dot{a}$, which by the construction of $\dot{\mathcal{M}}$ is a contradiction.

**Case 3.** Suppose $\alpha = \beta$, then we get a contradiction by 1 and the construction of $\dot{\mathcal{M}}$.

**Case 4.** Suppose $\beta \notin \text{supp}(q)$ and $\beta < \alpha$. Choose $\beta' \in \text{supp}(q)$ minimal such that $\beta' > \beta$. 
By 3 above, letting \( \beta \) the supremum of all coordinates needed to evaluate \( p_k(\beta') \) and all the families \( \{u_{i,v}^{\beta'}(\beta'), y_{i,v}^{\beta'}(\beta') : (\mu, \nu) \in \text{TP}(p(\beta')) \land i < \text{tp}(\mu, \nu) + 2, i' < \text{tp}(\mu, \nu) + 1\} \) and also \( \mathcal{D}(\beta') \), we have \( \beta < \beta \). In this case we step into \( V^{P_{\beta}} \). There we can evaluate these families. We use the term relevant as above.

Case 4a. For at least one \( (\mu, \nu) \in r(\beta') \) and \( i < \omega \) we have an infinite \( u_{i,v}^{\beta'}(\beta') \) or \( y_{i,v}^{\beta'}(\beta') \) that is relevant for \( r(\beta') \). As \( \hat{b} \) is added only at stage \( \beta \), by construction of \( \mathcal{M} \) we know that \( u_{i,v}^{\beta'}(\beta') \setminus \hat{b} \) or \( y_{i,v}^{\beta'}(\beta') \setminus \hat{b} \) is infinite. So since \( \hat{a} \) contains more and more of \( u_{i,v}^{\beta'}(\beta') \) or \( y_{i,v}^{\beta'}(\beta') \) as we go right with the succeeding splitnode of \( (\mu, \nu) \), we can force some point \( k \) above \( m_0 \) into \( \hat{a} \) which is outside \( \hat{b} \).

Case 4b. All relevant \( u_{i,v}^{\beta'}(\beta') \) and \( y_{i,v}^{\beta'}(\beta') \) are finite. By 3 we have that the disjoint family \( \mathcal{D}(\beta') \) of \( \omega \) belongs to \( V^{P_{\beta}} \) with

\[
\mathcal{D}(\beta') \subseteq \{ D^{\beta}_v : v < \gamma \}.
\]

At the later step \( \beta \), \( \hat{b} \) was put into \( \hat{A}_\beta \). Hence for any \( D \in \hat{D}(\beta') \), either \( \hat{b} \subseteq d \) for some \( d \in D \) or \( \hat{b} \cap d \) is finite for all \( d \in D \). So analogously to the proof of Claim 12.1 we can find some \( (\mu, \nu) \in \text{TP}(u, v) \) with even type \( n \) and all \( i < n + 1 \),

1. if we have the empty case and \( \mathcal{H}_n(\beta'), \mathcal{H}_{n+2}(\beta') \) are below \( \mathcal{H}_{n+1}(\beta') \), then for both \( j < 2 \) we have

\[
\forall \mu' \in \text{Sop}^{\mu,v}(\mu, \nu) \left[ (u_{i,j}^{\mu,v}(\beta') \setminus w_{i,v}^{\mu,v}(\beta')) \cap \hat{b} = \emptyset \right].
\]

2. in the tree case,

\[
\forall \mu' \in \text{Succ}_u(\mu) \cap \text{Sop}^{\mu,v}(\mu, \nu) \exists \mu'', \mu''' \in \text{split}(r(\beta')^0) \quad \left[ \mu \lneq \mu'' \lneq \mu''' \lneq \mu' \land (y_{i,v}^{\mu'',v}(\beta') \setminus y_{i,v}^{\mu'''(\beta')} \cap \hat{b} = \emptyset) \right].
\]

3. if we are in the constant case and for no \( \hat{\mu} \in \text{split}(r(0)^0) \) do we have \( \hat{b} \subseteq \hat{A}_{\hat{\mu}} \), then

\[
\forall \mu' \in \text{Succ}_u(\mu) \cap \text{Sop}^{\mu,v}(\mu, \nu) \exists \mu'' \in \text{split}(r(\beta')^0) \quad \left[ \mu \lneq \mu'' \lneq \mu' \land (y_{i,v}^{\mu''}(\beta') \setminus w_{i,v}^{\mu,v}(\beta')) \cap \hat{b} = \emptyset \right].
\]
and the symmetric properties hold for odd-type pairs. So again for some $P_{\omega_2}$-generic filter containing $r \upharpoonright \beta \prec (u, v) \upharpoonright r \upharpoonright [\beta + 1, \omega_2)$, we get that infinitely many nonempty disjoint sets of the form $u_{j, i}^{\mu, w} \setminus w_{i}^{\mu, w}$ are contained in $\dot{d}[G] \setminus \dot{b}$, which contradicts (4.2).

This completes the proof of Theorem 8 and hence of Theorem 6. \qed
Bibliography


Curriculum Vitae

May 1, 1969  Born in Visp, Switzerland
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