Hubbard model: Crossover from one to two Dimensions

A dissertation submitted to the
ETH Zürich
for the degree of the
Doctor of Natural Sciences

presented by
URS LEDERMANN
Dipl. Phys. ETH
born April 25th, 1973
Swiss citizen

Accepted on the recommendation of
Prof. T.M. Rice, Examiner
Prof. M. Sigrist, Co-Examiner

2001
# Contents

Abstract v

Zusammenfassung vii

Preface ix

1 Introduction 1
   1.1 Electrons in one dimension .................................. 1
   1.2 Hubbard model ............................................... 3
      1.2.1 HTSC and the Hubbard model ............................. 3
      1.2.2 Large $U$ limit: $t$-$J$ model .......................... 5
      1.2.3 Single chain ............................................. 5
      1.2.4 2D case: Renormalization group ......................... 6
   1.3 Ladders ..................................................... 8
   1.4 Dimensional crossover ...................................... 10

2 Fermions in one dimension 13
   2.1 Luttinger liquid ............................................. 13
   2.2 Renormalization group ...................................... 15
   2.3 Model with spin: Spin-charge separation, gaps etc. ... 18
      2.3.1 Bosonization ............................................ 19
      2.3.2 Repulsive interactions: LL and spin-charge separation . 20
      2.3.3 Attractive interactions: Superconductivity .......... 21

3 Two-leg ladder 23
   3.1 Spinless two-leg ladder .................................... 24
      3.1.1 Hamiltonian ............................................. 24
      3.1.2 RG equations ........................................... 26
## CONTENTS

3.1.3 Bosonization .................................................. 28  
3.1.4 Interchain-pairing superconducting phase ................. 29  
3.1.5 Mixed SC+CDW phase ....................................... 33  
3.2 Spin-1/2 two-leg ladder ....................................... 34  
3.2.1 Hamiltonian and current algebra ......................... 35  
3.2.2 Hole doped ladder .......................................... 36  
3.2.3 Half-filled ladder .......................................... 38  
3.3 Conclusions and discussion ................................... 41  

4 Half-filled N-leg ladders ........................................ 43  
4.1 Hamiltonian .................................................... 44  
4.2 Groundstate .................................................... 47  
4.2.1 Three-leg ladder ............................................ 47  
4.2.2 Four-leg ladder ............................................. 50  
4.2.3 The case N > 4 ............................................. 51  
4.3 Dimensional crossover ........................................ 52  
4.3.1 ISL vs. AFM ................................................. 53  
4.3.2 Determination of the crossover energy .................. 54  
4.4 2D-like AFM phase ............................................ 56  
4.4.1 Asymptotic ratios of the couplings ...................... 56  
4.4.2 Effective low-energy Hamiltonian ......................... 59  
4.4.3 Physical properties: Bosonization ....................... 60  
4.5 Conclusions and discussion ................................... 64  

5 Hole doped N-leg ladders ........................................ 67  
5.1 Lightly doped case: Holes and hole pairs ................... 67  
5.1.1 Hole doping the groundstate .............................. 68  
5.1.2 Hole doping 2D-like AFMs ................................. 75  
5.2 Intermediate regime: Superconductivity .................... 77  
5.3 Large dopings: Fermi liquid ................................. 80  
5.4 Conclusions and discussion ................................... 82  

6 Conclusions and outlook ......................................... 83  

A Renormalization group .......................................... 85  
A.1 Current algebra ............................................... 85  
A.2 RG equations ................................................... 86
CONTENTS

B Correlation functions ......................................... 93
  B.1 Massless boson ............................................. 93
  B.2 Spinless two-leg ladder .................................... 94

Curriculum Vitae .................................................. 105
Seite Leer / Blank leaf
Abstract

In this thesis, we investigate coupled one-dimensional electron systems (in the following called N-leg ladders) and the crossover to 2D as the number of coupled chains N goes to infinity (dimensional crossover). The focus will be on the half-filled (i.e., one electron per lattice site) and lightly doped, weakly interacting Hubbard ladders. In this regime, the Hubbard model exhibits a strong competition between antiferromagnetic, umklapp, and Cooper processes and therefore allows for a variety of different phases. For the determination of the physical properties, we use a combination of (one-loop) renormalization group and bosonization techniques.

After an introduction to the physics of the single chain and Luttinger liquids, we discuss in detail the two-leg ladder. For the spinless two-leg ladder, we obtain that a finite interchain hopping $t_\perp > U$ ($U$ is the interaction strength) has the effect of rendering the ladder superconducting. Furthermore, we find a phase where superconductivity coexists with charge density wave correlations. For the spin-1/2 case, we revisit previous results.

We then treat ladders with $N > 2$. First, we study in detail the half-filled case. For a small on-site repulsion $U$, the N-leg ladders are equivalent to a (weakly interacting) N-band model, characterized by Fermi velocities $v_j$. At half-filling, due to nesting, $v_j = v_{N+1-j}$. Carefully examining the renormalization group equations, we find in the groundstate a decoupling into band pairs $(j, N+1-j)$ (plus a single band for N odd), i.e., the band pair $(j, N+1-j)$ flows at the energy $t e^{-\alpha v_j/U}$ ($\alpha \sim 1$) to a two-leg ladder fixed point and becomes frozen out. As a result, we recover the odd-even effect that is present in Heisenberg spin-ladders: even-leg ladders have a spin-gap, while odd-leg ladders exhibit one gapless spinon-mode [for N odd, the band $(N+1)/2$ behaves as a single chain].

The 2D Hubbard model is expected to be an antiferromagnetic Mott insulator with two gapless magnon-modes. Therefore, the spin-gap present
in even-leg ladders (and the odd-even effect) has to vanish for increasing $N$. In the small-$U$-case, we obtain a double-exponential decrease of the spin-gap as a function of $N$, $E_c \sim t \exp[-a \exp(bN)]$ ($a \ll 1$ and $b \sim 1$). Furthermore, we find an analytical expression for the effective Hamiltonian of the 2D-like antiferromagnetic Mott insulator that appears at energies above $E_c$. Interestingly, the charge-sector is the same as in the two-leg ladder, i.e., there is phase coherence between the bands $j$ and $N + 1 - j$.

Finally, we investigate the doping away from half-filling. The lightly doped case, we treat as a perturbation of the half-filled low-energy Hamiltonian. Similar as at half-filling, we obtain an odd-even effect: even-leg ladders become a 1D superconductor (Luther-Emery liquid), while odd-leg ladders become a Luttinger liquid. The same phases have been found in numerical treatments of the strongly interacting $t$-$J$ ladders. For increasing doping, phase coherence between all band pairs sets in and the ladders become a 2D-like superconductor, described by an (effective) $d$-wave BCS Hamiltonian. We find, that the origin of the renormalization group instability is a Kohn-Luttinger-type attraction mediated by short-range antiferromagnetic fluctuations.
Zusammenfassung

In der vorliegenden Doktorarbeit untersuchen wir gekoppelte eindimensionale Elektronensysteme (im folgenden N-Bein Leitern genannt) und den Übergang zu 2D wenn die Anzahl gekoppelter Ketten $N$ nach Unendlich strebt (dimensionaler Übergang). Das Schweremittel werden wir auf die halb gefüllten (d.h., ein Elektron pro Gitterplatz) und leicht gedopten, schwach wechselwirkenden Hubbard Leitern legen. In diesem Parameterbereich herrscht im Hubbard-Modell eine starke Konkurrenz zwischen antiferromagnetischen, umklapp und Cooper Prozessen und somit sind verschiedene Phasen möglich. Für die Bestimmung der physikalischen Eigenschaften benützen wir eine Kombination von Renormierungsgruppen und Bosonizierungs-Techniken.

Nach einer Einführung in die Physik einer einzelnen Hubbard-Kette und Luttinger liquids, diskutieren wir im Detail die 2-Bein Leiter. Für die spinlose 2-Bein Leiter erhalten wir, dass ein endliches Matrixelement für Interkettenhüpfprozesse $t_\perp > U$ ($U$ ist die Wechselwirkungsstärke) die Leiter supraleitend macht. Zudem finden wir eine Phase wo Supraleitung mit Ladungsduchwellen-Korrelationen koexistiert. Für den Spin-1/2 Fall diskutieren wir frühere Resultate.

Leitern mit einer ungeraden Anzahl Beinen ein Spin-Mode masselos ist [für $N$ ungerade verhält sich das Band $(N + 1)/2$ wie eine einzelne Kette].


Für kleines $U$ erhalten wir eine doppelt exponentielle Abnahme der Masse als Funktion von $N$, $E_c \sim t \exp[-a \exp(bN)]$ ($a \ll 1$ und $b \sim 1$). Wir finden zudem einen analytischen Ausdruck für den effektiven Hamilton des 2D-artigen antiferromagnetischen Mott-Isolators, der bei Energien oberhalb $E_c$ erscheint. Interessanterweise ist der Ladungssektor der gleiche, wie in der 2-Bein Leiter, d.h., es herrscht Phasenkohärenz zwischen den Bändern $j$ und $N + 1 - j$.

Preface

This thesis was motivated by a discrepancy between previous numerical and analytical results for the groundstate of the lightly doped 3 and 4-leg Hubbard ladders. In the first part of my Ph.D., I have carefully reexamined this problem and obtained general results for the half-filled and lightly doped Hubbard ladders, which are in agreement with the numerical treatments. This work was initiated by T.M. Rice and carried out in collaboration with K. Le Hur. Additionally, I found some new phases in the two-band model of spinless fermions (together with K. Le Hur). The next project which I got from T.M. Rice was about Li-doped 2D antiferromagnets. Unfortunately, the previous work which I should extend, turned out to be wrong (the error was found by my collaborator, A. Läuchli). My research about this topic is not included in the thesis. After a research-free time in the Boulder Summer school, my interest turned back to Hubbard ladders. I investigated what happens when the number of legs increases, $N \to \infty$ (dimensional crossover), and found as — on my opinion — key results, effective low-energy Hamiltonians for the half-filled and hole doped (large) $N$-leg Hubbard ladders which can be generalized to the (true) 2D case.

In these days, the Hubbard model is usually studied in order to explain the HTSCs. In particular, the underdoped phase is still a matter of debate in the community. Here, I would like to give a systematic investigation of the Hubbard model — which should remain true whatever the theory for the HTSCs will be — and do not have the intention to explain certain HTSC-experiments.

The structure of the thesis is as follows.

Chapter 1 contains a general introduction about Luttinger liquids, the Hubbard model, ladders, etc.

In chapter 2, we discuss the Luttinger liquid both for the spinless and the spin-1/2 case and study the single Hubbard chain by renormalization
group techniques. We restrict ourselves to topics which are relevant for the following.

Chapter 3 is devoted to the two-leg ladder. In the first part, we treat the spinless two-leg ladder (these results have been published in Phys. Rev. B). In the second part, we revisit the spin-1/2 two-leg ladder.

Chapter 4 and 5 contain the main part of this thesis. There, we give a systematic investigation of the half-filled (chapter 4) and hole doped (chapter 5), weakly interacting $N$-leg Hubbard ladders and their crossover to 2D as $N \to \infty$. The results for the groundstate of the 3 and 4-leg ladder have been published in Phys. Rev. B and the work about the dimensional crossover from 1D to 2D will appear in Phys. Rev. B.

In Chapter 6, we finally draw some conclusions and present ideas about possible future work.

In Appendix A, we give the renormalization group equations for the half-filled $N$-leg Hubbard ladders.

In Appendix B, we calculate various correlation functions.

Acknowledgments

I would like to thank my supervisor, T.M. Rice, for guidance during my Ph.D. time, K. Le Hur for many fruitful collaborations, and my diploma advisor, G. Blatter, for teaching me the "basics" of research.

In addition, I profited from discussions with M. Sigrist (co-examiner), C. Honerkamp, and M. Troyer.

Finally, I thank the people of the ITP for many challenging soccer games.
Chapter 1

Introduction

1.1 Electrons in one dimension

For a wide class of many-electron systems in 2 and 3 dimensions, the (repulsive) electron-electron interactions have no dramatic consequences on the low-energy physics. Such systems can be described by the Landau Fermi liquid theory. One key property of this theory is the one to one correspondence between the noninteracting particles and the interacting (quasi) particles. It was a major surprise when it turned out that the low-energy physics of one-dimensional electron systems is never similar to the noninteracting case. These systems belong to a new universality class, the Luttinger liquid. In this work, we concentrate on the natural extension of 1D electron systems, i.e., $N$ coupled chains of electrons — such systems show a rich phase diagram as a function of $N$ and the number of electrons per lattice-site respectively the band filling.

In the 1940s and 1950s, Landau introduced a phenomenological concept to describe the low-energy excitations of many fermion systems such as $^3$He [1]. Over the years, Landau's Fermi liquid theory has then become the standard theory for a wide class of metals and the phenomenological approach has been justified for different microscopic models of interacting electrons in 2D and 3D [2]. Apart from the one to one correspondence of noninteracting and interacting particles, another characteristic property of the Fermi liquid theory is the discontinuity of the momentum distribution at the Fermi surface and therefore a quasiparticle pole in the Green's function. The effect of the interactions is a renormalization of masses, energies etc. (with respect to the
noninteracting case). Furthermore (in contrast to the free Fermi gas), there exist collective excitations, the so-called zero sound modes.

Models resulting in what is called today a Luttinger liquid have been introduced and developed in 1950 by Tomonaga [3] and in 1963 by Luttinger [4] (the correct solution of his model is due to Mattis and Lieb [5]). In contrast to the Fermi liquid, Luttinger liquids do not have (even at $T = 0$) a step-like discontinuity in the momentum distribution at the Fermi surface and consequently also no quasiparticle pole in the Green's function. The low-energy excitations are bosonic charge and spin density waves. Furthermore, the velocities of spin and charge are usually different, what is called spin-charge separation. In the absence of gaps, correlation functions decay with a power-law, where the exponent is nonuniversal and interaction dependent. Subsequently, a wide class of one-dimensional systems could be solved by what is called today bosonization techniques and were shown to belong to the universality class of the Luttinger liquid [6].

In the early 1970s, the first materials of predominantly 1D character were discovered and 1D electron systems and Luttinger liquids were no more only of purely theoretical interest. The most important examples are the organic conductor TTF-TCNQ [7, 8] and the so-called Bechgaard salts, where the series $(\text{TMTSF})_2X$ is superconducting and the series $(\text{TMTTF})_2X$ is not [9, 10]. Parts of the (complicated) phase diagram of these compounds can be described by 1D models [11]. More recently, interest turned towards carbon nanotubes. There is strong evidence that a certain class of these quantum wires behaves Luttinger liquid like [12]. As a result, there are nowadays quite a few materials which can be described by 1D models. Next, we briefly discuss some mechanisms which lead to the opening of gaps in the spin or charge sector. Such gaps will in the following chapters play a crucial role.

Similar as for the 2D and 3D case, attractive interactions lead in 1D to the opening of a spin-gap and dominant superconducting correlations appear. Since there is no long-range order in 1D, it is the correlation function with the slowest power-law decay, i.e., the smallest exponent, which determines the phase. Such a system has been investigated by Luther and Emery [13] for an exactly solvable case. We call therefore a 1D superconductor a Luther-Emery liquid.

A charge-gap can occur for a partially filled band, if the interactions are strong enough or the band filling is commensurate with the lattice: the system becomes Mott insulating. There are two classes of Mott insulators.
1.2. HUBBARD MODEL

In 1D, it is common that Mott insulators have a spin-gap and short-range magnetic correlations; we call such a system an insulating spin-liquid. In 2D (and 3D), quantum fluctuations are weaker, such that Mott insulators can exhibit long-range magnetic order which is then accompanied by gapless magnon (i.e., spin one) modes [14].

Summarizing, 1D electron systems exhibit not only unusual physical properties and are therefore interesting objects to study (from a theoretical point of view), but there are nowadays also many experimental applications.

1.2 Hubbard model

Introduced by Hubbard and others in the 1960s [15], the Hubbard model and its variations have become one of the most studied models in condensed matter physics. Today's interest in the Hubbard model is mainly due to the fact that the physics of the high temperature superconductors (HTSCs) can to some extent be described by the 2D Hubbard model. In this work, we will concentrate on the physical properties of the (quasi-1D) Hubbard model rather than on the fitting or explanation of experimental data.

1.2.1 HTSC and the Hubbard model

In 1986, Bednorz and Müller found in compounds based on La$_2$CuO$_4$ a superconducting transition temperature of $\sim$35K [16]. Soon after even higher transition temperatures were found in YBCO, BSCCO, and TBCCO (93, 110, and 130 K) [17]. In all these compounds, the physical properties depend strongly on the concentration of carriers $x$, the (hole) doping. For La$_2$CuO$_4$, this concentration can be varied by replacing La by Sr, resulting in La$_{2-x}$Sr$_x$CuO$_4$. It is now well established, that the undoped compounds ($x = 0$) are Mott insulating spin-1/2 Heisenberg antiferromagnets and that upon doping, the antiferromagnetic correlations become rapidly suppressed [18]. Furthermore, the antiferromagnetic correlations become incommensurate [19, 20], i.e., the peak in the magnetic structure factor (which is measured in neutron scattering experiments) shifts away from $(\pi, \pi)$; for LSCO, the incommensurability is in good approximation proportional to the doping [19]. Superconductivity with $d$-wave symmetry of the order parameter then sets in at sufficient doping, $x \sim 0.1$, i.e., the $k$-dependence of the order parameter is of the form $\Delta(k_x, k_y) \propto k_x^2 - k_y^2$. 
Between the antiferromagnetic and the superconducting phase, there are experimental signs of the opening of a spin-gap below a crossover temperature $T^*$ [21]. The nature of this pseudo-gap phase is still controversial. Fig. 1.1 gives an overview.

From the $d$-wave symmetry of the order parameter, the importance of the antiferromagnetic fluctuations, and other experimental findings, one comes to the conclusion, that HTSCs do not belong to the universality class of the BCS superconductors — in the sense that the pairing is not caused by phonons and the normal state is not a Fermi liquid [17]. In 1987, Anderson proposed, that the appropriate model for these HTSC compounds might be the 2D Hubbard model [22]. Assuming that the relevant physics takes place in the CuO$_2$ planes, a microscopic derivation showed that the 2D large-$U$ Hubbard model is indeed a promising candidate [23].

The Hamiltonian of the Hubbard model takes the form

$$H = -t \sum_{\langle i,j \rangle, s} (d_{i,s}^* d_{j,s} + H.c.) + U \sum_i d_{i,s}^* d_{i,s} d_{i,s}^* d_{i,s}.$$  \hspace{1cm} (1.1)

Here, $i$ and $j$ run over all lattice sites, $\langle \ldots \rangle$ denotes nearest neighbors and $s$ the spin. The first part describes the hopping of electrons between adjacent sites on the lattice (with hopping matrix element $t$) and the second part the on-site repulsion ($U > 0$) of electrons on the same site. The model can be studied for different fillings: One electron per lattice site corresponds to
half-filling; less (more) than one electron per site to hole (electron) doping. In the following, we briefly review some of the known results for the 1D and 2D Hubbard model.

1.2.2 Large $U$ limit: $t$-$J$ model

In the large $U$ case, the Hubbard model scales on the $t$-$J$ model with Hamiltonian (for a derivation, see Ref. [24])

$$H = -t \sum_{\langle i,j \rangle,s} P \left( d_i^\dagger d_j + \text{H.c.} \right) + J \sum_{\langle i,j \rangle} (S_i \cdot S_j - n_i n_j / 4),$$

where $J = 4t^2 / U$; $S_i$ is the spin operator and $n_i$ the particle density operator. Furthermore, due to the large $U$, doubly occupied sites are excluded by the Gutzwiller projector $P$. At half-filling, hopping is thus completely suppressed and the system is an insulating (spin-1/2) quantum antiferromagnet with Hamiltonian

$$H_{\text{AF}} = J \sum_{\langle i,j \rangle} S_i \cdot S_j.$$  

After the discovery of HTSC, considerable interest was devoted to the 2D Heisenberg antiferromagnet [25, 26]. It is now well-accepted that the ground-state ($T = 0$) exhibits long-range order and two gapless magnon modes. At finite temperature, $T > 0$, the spin-spin correlations decay exponentially with coherence length $\xi_{\text{AF}} \propto e^{\alpha J / T}$ ($\alpha \sim 1$) — in agreement with measurements in HTSC compounds [18].

1.2.3 Single chain

One-dimensional lattice models can often exactly be solved by what is called Bethe Ansatz — a method introduced in 1931 by Bethe in order to solve the spin-1/2 Heisenberg chain [27].

In 1968, Lieb and Wu solved the single Hubbard chain by Bethe Ansatz [28] and showed, that it is at half-filling Mott insulating for any non zero value of $U > 0$. However, the nature of the low-energy excitations remained undetermined. It was only later, when it turned out, that the Hubbard chain belongs to the Luttinger liquid universality class [29]. Away from half-filling, there is one gapless charge and one gapless spinon mode (i.e., spin-1/2 mode). At half-filling charge excitations are gapped, but the spin excitations stay gapless.
Figure 1.2: The (noninteracting) Fermi surface (solid line at a distance $k_F$) of the half-filled Hubbard model (left) and of the free electron gas (right). At low energies, the physics is determined by states close to the Fermi surface, i.e., within the shaded region.

Note that the $t$-$J$ chain can only be solved exactly at the supersymmetric point, $t = J$ [30].

1.2.4 2D case: Renormalization group

For the 2D Hubbard model, there exist few rigorous analytical results and numerical treatments are either restricted to small clusters or to higher temperatures [31]. Promising results come from renormalization group studies (for an introduction and overview about renormalization group, see Refs. [32, 33]).

For a weakly interacting system, the low-energy physics is determined by $k$-states close (i.e., within a distance $\Lambda$) to the Fermi surface (see Fig. 1.2). The interacting part consists of various, physically distinct, scattering processes: forward and backward (Cooper) scattering, and sufficiently close to half-filling (momentum non-conserving) umklapp scattering and antiferromagnetic processes, see Fig. 1.3. When lowering the energy-scale (i.e., $\Lambda$), some of these processes are enhanced and others suppressed. The processes which dominate in the low-energy limit determine the phase. The renormalization group method is a controlled way of calculating differential equations (with respect to $\Lambda$), which give the change of the couplings when lowering the energy. The numerical/analytical solution of these equations then allows one to derive the dominating phase. The best understood examples are the Fermi liquid and the BCS superconductor in 2D and 3D. For the Fermi liq-
In the Hubbard model, 4 physically distinct scattering processes take place (the square is the Fermi surface at half-filling): forward, backward (Cooper), umklapp, and antiferromagnetic processes (the dashed arrows denote momentum non-conserving processes). In the half-filled and lightly doped case, there is a strong competition between Cooper, umklapp, and antiferromagnetic interactions.

Renormalization group studies of the 2D weakly interacting Hubbard model found $d$-wave superconductivity for sufficient hole doping and dominant antiferromagnetism at and close to half-filling, similar to what has been measured in HTSCs [34, 35, 36]. There is strong evidence, that the $d$-wave instability is caused by short-range antiferromagnetic spin-fluctuations, i.e., a Kohn-Luttinger-type attraction [37] is generated by antiferromagnetic processes [38]. Note that the difference to the Fermi liquid comes from the shape of the noninteracting Fermi surface. In the half-filled and lightly doped Hubbard model, the Fermi surface is nested (close to the umklapp surface), such that additional scattering processes appear. Although the (one-loop) renormalization group method is restricted to weak interactions, it is reasonable to assume that a renormalization group instability persists when increasing the interaction strength, i.e., that the resulting phase diagram will qualitatively be the same.

The pseudo-gap phase in the lightly doped HTSCs is still a matter of debate. From the theoretical point of view, the strong competition between umklapp, antiferromagnetic, and Cooper processes in the lightly doped Hubbard model makes it difficult to obtain conclusive results. A recent renor-
malization group study of the Hubbard model including next nearest neighbor hopping suggests the formation of an insulating spin-liquid around the saddle-points \((\pi, 0)\) and \((0, \pi)\) \[36\].

A major problem in 2D is the infinite number of different interactions. This renders it difficult (or impossible) to derive an effective low-energy Hamiltonian — which can then be analyzed by other methods. In 2D, the only way to determine the phase is by examining the flow of the various susceptibilities. In (quasi) 1D the situation is different. There is only a limited number of interactions and bosonization techniques allow for a rather rigorous and analytical determination of the correlation functions and therefore the low-energy physics.

To conclude this section: Although a lot is known about the (2D) Hubbard model, it is far from being “solved”. In particular, there is no low-energy theory as it is the case for BCS superconductors or Fermi liquids.

### 1.3 Ladders

Growing interest in coupled 1D electron systems emerged in the early 1990s. “Coupled” means, e.g., spin-spin or density-density interactions between sites on different chains or hopping of electrons between different chains. We call in the following a system consisting of \(N\) coupled 1D systems a \(N\)-leg ladder.

There were basically two reasons for this increasing interest. First, the problem of HTSC motivated the search for a non-Fermi liquid (i.e., possible Luttinger liquid) phase in 2D. In particular Anderson argued, that spin-charge separation would be a key element in order to understand the HTSCs, such that a 2D Luttinger liquid might be a good starting point \[39\]. However, it soon turned out, that already (infinitesimal) small coupling between two Luttinger liquids leads to the opening of gaps and the system is (for example) driven towards a superconducting phase \[40, 41\].

The second reason based on ideas of Dagotto et al. \[42\] and Rice et al. \[43\], that two-leg ladder materials, such as \(\text{SrCu}_2\text{O}_3\), become superconducting when doped with holes. An intuitive understanding is possible as follows. The undoped compounds can be modeled by a two-leg antiferromagnetic ladder, with exchange constants \(J\) along and \(J_\perp\) between the legs, see Fig. 1.4. For large \(J_\perp \gg J\) the electrons form singlets, resulting in a spin-gap \(\sim J_\perp\). The groundstate thus exhibits only short-range correlations (insulating spin-liquid). For \(J_\perp \gg J, t, t_\perp\), doped holes tend to break as few singlets...
1.3. LADDERS

Figure 1.4: On the two-leg $t$-$J$ ladder, the electrons can hop between neighboring sites (hopping matrix elements $t$ and $t_\perp$) and are subject to an antiferromagnetic interaction with strength $J$ and $J_\perp$. At half-filling, there is one electron per site, the ladder is insulating and has a spin-gap. Upon hole doping, the holes form pairs (solid balls) and dominant $d$-wave like superconducting correlations arise.

as possible and therefore form pairs. Numerical calculations have then confirmed that $d$-wave like superconducting correlations indeed dominate upon doping and that the physics in the isotropic case $J \sim J_\perp$ is qualitatively the same as in the large $J_\perp$ case \cite{44, 45, 46}. For the weakly interacting Hubbard model (respectively for two coupled Luttinger liquids), a series of analytical works found the same phases \cite{40, 47, 48, 49}. As a result, for a wide parameter range, the Hubbard respectively $t$-$J$ two-leg ladder is an insulating spin-liquid at half-filling and a $d$-wave like superconductor upon hole doping. While for the (undoped) spin-ladders, theory and experiment are in agreement \cite{50}, superconductivity has up to now only been observed in a two-leg ladder material under high pressure \cite{51}.

The behavior of the two-leg spin ladder is thus entirely different from the single spin-chain and the same is true for the (doped) Hubbard model: The single chain belongs to the Luttinger liquid universality class and the two-leg ladder to the Luther-Emery liquid class of systems. Investigating spin-ladders with more than two legs, it soon turned out, that spin-ladders indeed exhibit an odd-even effect. Even-leg ladders have a spin-gap and odd-leg ladders one gapless spinon mode \cite{52}. This behavior is reminiscent of the "Haldane conjecture" for spin-$S$ spin-chains \cite{53}; in fact, a spin-$S$ chain can be rewritten as a system of $2S$ coupled spin-$1/2$ systems, see Ref. \cite{54}. Experimental applications are 3-leg ladder materials such as Sr$_2$Cu$_3$O$_5$ \cite{50}.
A similar odd-even effect has been found upon doping: Numerical works for the 3-leg $t$-$J$ ladder obtained a Luttinger liquid phase at small doping — similar to the single chain [55, 56]. In contrast, the 4-leg ladder becomes a 1D superconductor (Luther-Emery liquid) upon hole doping [57, 58]. Without giving a detailed derivation, this odd-even effect was also noted in an analytical work about $t$-$J$ ladders [59]. Arrigoni [60] and Lin et al. [61] studied the weakly interacting 3 and 4-leg Hubbard ladders by renormalization group techniques. However, these phases do not appear in their phase diagrams. This leads to the question whether the phases for small and large $U$ are different or not.

In chapter 4 and 5 of this work, we carefully examine the half-filled and lightly doped $N$-leg Hubbard ladders and show, that this discrepancy comes from the fact that previous works have neglected the umklapp interactions present close to half-filling. In particular, we obtain for small $U$ at and close to half-filling the same groundstate as has been found for large $U$: At half-filling, the Hubbard ladders are Mott insulating and exhibit a spin-gap for $N$ even and one gapless spinon mode for $N$ odd. Upon doping away from half-filling even-leg ladders become superconducting (Luther-Emery liquid) and odd-leg ladders are a Luttinger liquid.

1.4 Dimensional crossover

As we have discussed, the physical properties of (quasi) 1D electron systems are often different from their 2D counterpart. Spin-ladders (and the half-filled Hubbard model) exhibit an odd-even effect, while the corresponding 2D system has two gapless magnon modes. Similarly, conducting 1D electron systems are Luttinger liquids and 2D systems Fermi liquids.

It is therefore an interesting question for itself how the 1D system evolves towards the 2D counterpart. Furthermore, in contrast to 2D, for 1D systems, analytical methods such as bosonization usually allow for a rather rigorous determination of the low-energy physics. One can therefore hope that the study of the dimensional crossover from 1D to 2D gives insight in the 2D case.

Coupled Luttinger liquids and their crossover to 2D were studied by many different authors. The first investigations were done in the context of quasi-1D conductors [62, 63]; more recent works were motivated by the HTSCs [64, 65, 66]. The authors came always to the same conclusion: the Luttinger
liquid does not survive in 2D (an exception are maybe strongly anisotropic systems [67]).

In antiferromagnetic spin-ladders, the spin-gap present in even-leg ladders vanishes exponentially, $e^{-N}$, as a function of the number $N$ of coupled spin-chains. This can be shown using the non-linear sigma model [68]. For the (weakly interacting) Hubbard model, the dimensional crossover away from half-filling (i.e., without umklapp and antiferromagnetic processes) was studied in Ref. [61]; the authors concluded that the large $N$ limit is a Fermi liquid.

In chapter 4 and 5, we investigate the dimensional crossover of the half-filled and lightly doped $N$-leg Hubbard ladders as $N \to \infty$, properly taking into account the umklapp and antiferromagnetic interactions. We will see, that in this regime, the Hubbard model exhibits a variety of interesting (non-Fermi liquid) physics. At half-filling, we obtain that the spin-gap vanishes double-exponentially as a function of $N$, reflecting the fact that the 2D energy-scale is already an exponential function of $t/U$. Furthermore, we find analytical expressions for the antiferromagnetic (effective) Hamiltonian at half-filling and the $d$-wave superconductor that appears upon hole doping.
Chapter 2

Fermions in one dimension

In this chapter, we give an introduction and overview about the physics of 1D fermi-systems (single chain), where we limit the discussion to topics which are relevant for the treatment of N-leg ladders. An extensive review is Ref. [69].

First, following the historical development, we discuss the Luttinger liquid (LL) and bosonization techniques for the spinless case. We then give an introduction to the renormalization group (RG) method, where we take as example the single Hubbard chain. In the last section, we extend the concept of bosonization to spin-1/2 systems.

2.1 Luttinger liquid

Originally, (field theoretical) models of 1D (spinless) fermions were introduced in 1950 by Tomonaga [3] and later in 1963 by Luttinger [4] mainly for theoretical interests. These particular models were exactly solvable, but physical applications were in these days not in sight (the correct solution of Luttinger’s model is due to Mattis and Lieb [5]).

The Luttinger model has the form of the relativistic Dirac Hamiltonian [the two component spinor is $(\Psi_R, \Psi_L)$], where

\[ H_0 = -iv \int_0^{L_0} dx \Psi_R(x)^\dagger \partial_x \Psi_R(x) - \Psi_L(x)^\dagger \partial_x \Psi_L(x) \]  

(2.1)

and the interacting part is

\[ H_{\text{int}} = \int dx dy \Psi_R(x)^\dagger \Psi_R(x) \Psi_L(y)^\dagger \Psi_L(x) \]  

(2.2)
Going over to $k$-space, we obtain

$$H = v \sum_k \left[ \Psi^\dagger_R(k) \Psi_R(k) - \Psi^\dagger_L(k) \Psi_L(k) \right] + \frac{1}{L_0} \sum_{k,k',q} V(q) \Psi^\dagger_R(k) \Psi_R(k + q) \Psi^\dagger_L(k' + q) \Psi_L(k'). \quad (2.3)$$

The model is exactly solvable when we let go the sum over the momenta to $\pm \infty$ (field theoretical limit). The next step is to postulate a groundstate: the filled Dirac (Fermi) sea, i.e., states below a momentum $k_F$ are filled and states above $k_F$ are empty.

The key finding by Mattis and Lieb was that the operators ($h = R/L$)

$$\rho_h(q) = \sum_k \Psi^\dagger_h(k + q) \Psi_h(k) \quad (2.4)$$

fulfill bosonic commutation relations ($\pm$ for $h = R/L$)

$$[\rho_h(q), \rho_{h'}(-q')] = \pm \delta_{h,h'} \delta_{qq'} \frac{qL_0}{2\pi}, \quad (2.5)$$

and obey with $H_0$ the commutation relations

$$[H_0, \rho_h(q)] = \pm vq \rho_h(q). \quad (2.6)$$

As a result, $H_0$ can be rewritten in terms of bosonic operators $\rho_h$,

$$H_0 = \frac{2\pi v}{L_0} \sum_{q > 0, h} \rho_h(q) \rho_h(-q). \quad (2.7)$$

Since the interacting part takes the form

$$H_{\text{int}} = \frac{1}{L_0} \sum_q V(q) \rho_R(q) \rho_L(-q), \quad (2.8)$$

it is straightforward to diagonalize $H$ and we find the dispersion relation

$$\omega(q) = |q| \sqrt{v^2 - \left[ \frac{V(q)}{2\pi} \right]^2}. \quad (2.9)$$

The low-lying excitations are then (bosonic) charge density waves (CDW).
It is convenient to rewrite the Hamiltonians (2.7) and (2.8) using the Fourier transformations of the operators $\rho_R \pm \rho_L$, i.e., field operators $\Phi$ and $\Pi$ which satisfy the canonical commutation relations, $[\Phi(x), \Pi(y)] = i\delta(x-y)$,

$$\partial_x \Phi(x) = \frac{1}{L_0} \sum_q e^{-iqx}[\rho_R(q) + \rho_L(q)]$$

$$\Pi(x) = \frac{1}{L_0} \sum_q e^{-iqx}[\rho_R(q) - \rho_L(q)].$$ (2.10)

The Hamiltonian takes the form (for $V = \text{const}$)

$$H = \frac{u}{2} \int dx \left[ \frac{1}{K} (\partial_x \Phi)^2 + K \Pi^2 \right],$$ (2.11)

where $K$ is the Luttinger liquid parameter (LLP) and $u$ the renormalized velocity,

$$K = \sqrt{\frac{2\pi v - V}{2\pi v + V}} \quad \text{and} \quad u = \sqrt{v^2 - \left(\frac{V}{2\pi}\right)^2}.$$ (2.12)

The field $\Phi$ is related to the particle density, $\partial_x \Phi = \sqrt{\pi/4}[\rho(x) - \rho_0]$, and the field $\Pi$ to the current density, $j = uK\Pi$. We will discuss the physical properties of the Luttinger model in more detail for the spin-1/2 case, see below (for the correlation functions of the spinless case, see Appendix B). We note that the spinless case can also be treated by (conventional) many-body techniques, see Ref. [70].

### 2.2 Renormalization group

The concept of the LL was first introduced for spinless fermions, where only forward scattering takes place. In the case of spin-1/2 fermions, a realistic model has to include backscattering. Except for an exactly solvable case [13], this model cannot be solved by bosonization techniques alone, but in addition, renormalization group techniques have to be applied. Here and in the following chapters, we will see that the combination of RG and bosonization methods is a powerful tool for the analysis of quasi-1D electron systems. Originally, RG techniques were developed in particle physics to investigate the high-energy behavior [71]. However, they can equally well be applied for the study of the low-energy condensed matter physics [32].
In this and the next section, we show, using RG and an extension of the bosonization techniques, that backscattering leads for attractive interactions to the opening of a spin-gap. In contrast, for repulsive interactions, backscattering has no effect on the low-energy physics and we recover the LL.

In the early 1970s, Menyhard and Solyom [72] applied RG techniques to a realistic problem of spin-1/2 fermions in 1D — equivalent to the Hubbard chain. Here, we revisit this problem using a more modern derivation. Fourier transforming the Hamiltonian of the single Hubbard chain, see Eq. (1.1), we obtain for $H_0$ (we set the lattice parameter equal to 1)

$$H_0 = -2t \sum_{k,s} \cos(k) \Psi_s^\dagger(k) \Psi_s(k).$$  

(2.13)

For a small on-site repulsion $U \ll t$, the low-energy properties only depend on the states close to the Fermi points $\pm k_F$, where $k_F$ is determined by the band filling $n$, $k_F = \pi n/2$. This allows us to linearize the dispersion relation at $\pm k_F$ and to introduce operators for right movers and left movers at the Fermi level, $\Psi_{R/Ls}$. The Hamiltonian is then rewritten as

$$H_0 = v \sum_s \int_{-\Lambda}^{\Lambda} dk \, k \left[ \Psi_{Rs}^\dagger(k) \Psi_{Rs}(k) - \Psi_{Ls}^\dagger(k) \Psi_{Ls}(k) \right],$$  

(2.14)

where $v = 2t \sin(k_F)$ and $\Lambda$ is a momentum cutoff, $\Lambda \ll 1$. Rewriting the interacting part with the operators $\Psi_{R/Ls}$, we find

$$H_{\text{Int}} = \left( \prod_i \int_{-\Lambda}^{\Lambda} \frac{dk_i}{2\pi} \right) \delta(k_1 + k_2 - k_3 - k_4) \mathcal{H}_{\text{Int}},$$  

(2.15)

where

$$\mathcal{H}_{\text{Int}} = g_1 \Psi_{Rs}^\dagger(k_1) \Psi_{Rs'}^\dagger(k_2) \Psi_{Rs'}(k_3) \Psi_{Rs}(k_4)$$  

$$+ g_2 \Psi_{Rs'}^\dagger(k_1) \Psi_{Ls'}^\dagger(k_2) \Psi_{Ls'}(k_3) \Psi_{Rs}(k_4)$$  

$$+ \delta_{k_F,\pi/2} g_3 \left[ \Psi_{Rs}^\dagger(k_1) \Psi_{Rs}^\dagger(k_2) \Psi_{Ls}(k_3) \Psi_{Ls}(k_4) + R \leftrightarrow L \right]$$  

$$+ g_4 \left[ \Psi_{Rs}^\dagger(k_1) \Psi_{Rs}^\dagger(k_2) \Psi_{Rs}(k_3) \Psi_{Rs}(k_4) + R \leftrightarrow L \right].$$  

(2.16)

where again $\bar{s} = \downarrow$ for $s = \uparrow$ and vice versa. Away from half-filling, i.e., $k_F \neq \pi/2$, the umklapp term $g_3$ vanishes. The $g_4$ term includes completely chiral interactions. It is nowadays well accepted that such terms have no relevant
2.2. RENORMALIZATION GROUP

effect on the low-energy physics, but only renormalize (overall) velocities. Here and in the following chapters, we therefore drop such terms. The \( g_2 \) interaction corresponds to forward scattering and can (similar to the spinless case) be rewritten as a product of bosonic density operators. The \( g_1 \) term describes backscattering (or Cooper scattering). Next, we study the model including only \( g_1 \) and \( g_2 \).

We perform a Kadanoff-Wilson type high-energy mode elimination, see, e.g., the review by Shankar [32]. We denote by \( S' \) the action corresponding to \( H \) and to the cutoff \( \Lambda \). Formally, the action \( S' \) of the energy-reduced system is given by

\[
e^{-S'} = \int \mathcal{D} \Psi e^{-S},
\]

where the integration \( \mathcal{D} \Psi \) is carried out over fields \( \Psi_{R/L}(k) \) with \( \Lambda/z < |k| < \Lambda (z > 1) \). Decomposing \( \Psi = \tilde{\Psi} + \Psi' \), we obtain

\[
\int \mathcal{D} \Psi e^{-S} = e^{-S_0 - S_{10}} \int \mathcal{D} \bar{\Psi} e^{-\bar{S}_0 - \sum_{j=1}^{4} S_{ij}},
\]

where \( S_{ij} \) denotes the interacting part containing \( j \) times the field \( \Psi \) (respectively \( \Psi' \)). For the interacting part, the integration has to be done perturbatively. In leading (one-loop) order, the only contributions come from connected diagrams containing 4 \( \Psi \) fields,

\[
\int \mathcal{D} \bar{\Psi} e^{-\bar{S}_0 - \sum_{j=1}^{4} S_{ij}} = \langle e^{-\sum_{j=1}^{4} S_{ij}} \rangle_0 \approx e^{\langle S_{12}^2 \rangle_0/2},
\]

where the average is taken over \( \bar{S}_0 \). As a result,

\[
S' \approx S'_0 + S_{10} - 1/2 \langle S_{12}^2 \rangle_0.
\]

In our case,

\[
S_0 = \sum_{\hbar=R/L,\delta} \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \tilde{\Psi}_{h\delta}^\dagger(-i\omega \pm vk)\Psi_{h\delta},
\]

where \( \pm \) corresponds to right and left movers respectively; the propagator is

\[
P = \frac{1}{-i\omega \pm vk}.
\]

Calculating \( 1/2 \langle \ldots \rangle_0 \), we set external energies and momenta to 0. Properly taking into account particle-particle and particle-hole diagrams (they
differ in an overall minus sign, we obtain the following contributions to $S'$, respectively to the $g_1$ and $g_2$ interactions

$$ \delta g_1 = -\frac{\ln z}{\pi v} g_1^2, \quad \delta g_2 = -\frac{\ln z}{2\pi v} g_1^2. $$

(2.23)

Iterating this treatment, i.e., $\Lambda \to \Lambda/z$, $\Lambda/z \to \Lambda/z^2$ etc., results in two differential equations (RG equations, RGEs) for the couplings,

$$ \frac{dg_1(l)}{dl} = -\frac{1}{\pi v} g_1^2(l), \quad \frac{dg_2(l)}{dl} = -\frac{1}{2\pi v} g_2^2(l), $$

(2.24)

where $l = \ln z$. Apparently, $g_1 - 2g_2 = \text{const}$ and

$$ g_1(l) = \frac{g_1(0)}{1 + g_1(0) l / (\pi v)}. $$

(2.25)

For repulsive interactions, $g_1(0) = g_2(0) = U > 0$, $g_1$ vanishes in the low-energy limit and $g_2 = U/2$, while for attractive interactions, $U < 0$, the couplings diverge at a finite energy-scale, $E \propto e^{-\pi v / |U|}$. In the next section, extending our bosonization scheme, we will investigate these RG "fixed points" in more detail. One-loop RGEs are clearly not sufficient to proof the stability of a particular fixed point, i.e., when the couplings remain finite. However, in the case when some couplings diverge at a finite energy, it is reasonable to assume that the divergence persists when including higher orders. In the analysis of the Hubbard model, we always find such RG instabilities (i.e., divergent couplings). Therefore, we do not go beyond one-loop order.

### 2.3 Model with spin: Spin-charge separation, gaps etc.

In the last section, we have studied the RG flow of the Hubbard chain for attractive and repulsive interactions. In the attractive case, the divergence of backscattering at finite energy suggests the crossover to a new phase. The works of Mattis [73], Luther and Emery [13], Coleman [74], and Mandelstam [75] lead to a generalization of the bosonization scheme which allows in particular to bosonize backscattering. The resulting Hamiltonian is found to be the quantum sine-Gordon model.
2.3. **MODEL WITH SPIN: SPIN-CHARGE SEPARATION, GAPS ETC.**

2.3.1 **Bosonization**

Next, we bosonize the (weakly interacting) Hubbard chain including forward and backscattering. The noninteracting part and forward scattering can similarly be bosonized as for the spinless case. More generally, the fermionic operators $\Psi_{R/Ls}$ can be expressed in terms of field operators $\Phi_\nu$ and $\Pi_\nu$, where $\nu = \rho, \sigma$ for charge and spin degrees of freedom, as follows,

$$\Psi_{R/Ls} = \eta_s \sqrt{2\pi\alpha} \exp \left\{ i\sqrt{\pi/2} [\pm(\Phi_\rho + s\Phi_\sigma) - (\theta_\rho + s\theta_\sigma)] \right\}, \quad (2.26)$$

where $s = \pm$ for spin up/down and $\theta_\nu$ is the dual field of $\Phi_\nu$, $\partial_x^2\theta_\nu = \Pi_\nu$. The $\eta_s$ are Majorana (real) fermionic operators ("Klein factors"), which are necessary to fulfill the anticommutation relations ($\eta_\nu^2 = 1$) and $\alpha$ is a short-distance cutoff. Note that bosonization applies for lattice models in the continuum limit. The above expression allows not only to bosonize backscattering (and umklapp processes), but also simplifies the calculation of correlation functions, see below.

The noninteracting Hamiltonian including density-density interactions takes a similar form as in the spinless case,

$$\hat{H}_0 = \sum_\nu \frac{u_\nu}{2} \int dx \left[ \frac{1}{K_\nu} (\partial_x^2 \Phi_\nu)^2 + K_\nu \Pi_\nu^2 \right], \quad (2.27)$$

where the LLPs and the velocities are

$$K_\nu = \sqrt{\frac{2\pi\nu + g_\nu}{2\pi\nu - g_\nu}} \text{ and } u_\nu = \sqrt{v^2 - \left(\frac{g_\nu}{2\pi}\right)^2}, \quad (2.28)$$

where again $g_\rho = g_1 - 2g_2$ and $g_\sigma = g_1$. There is an additional $g_1$-term, which cannot be rewritten in terms of density operators,

$$\hat{H}_\text{int} = \frac{2g_1}{(2\pi\alpha)^2} \int dx \cos \left(\sqrt{8\pi}\Phi_\sigma\right). \quad (2.29)$$

Apparently, the total Hamiltonian is the sum of two commuting Hamiltonians, one including only charge degrees of freedom and one including only spin degrees of freedom.

Note that we use here abelian bosonization, i.e., non-abelian symmetries which are present in the (original) fermionic Hamiltonian are (usually) broken. In particular, the SU(2) spin-symmetry is broken down to U(1). This has the effect that the $x, y$-part of spin-spin correlation functions behave differently as the $z$-part, see Ref. [69]. Non-abelian bosonization schemes have the disadvantage to be too complicated for practical applications [76].
2.3.2 Repulsive interactions: LL and spin-charge separation

For repulsive interactions, the RG flow of the couplings is such that $g_1 = 0$ and $g_2 = U/2$ at low energies and the effective low-energy Hamiltonian is given by $\tilde{H}_0$ alone, see Eq. (2.27). As a result, the repulsive Hubbard model belongs to the LL universality class. The low-energy excitations are (noninteracting) charge and spin density waves (CDWs and SDWs). Note that the velocities for charge-modes and spin-modes are different: $u_\sigma = v$ and $u_\rho < v$. The system exhibits spin-charge separation. The “particles” carrying charge and spin are called holons and spinons.

In 1D, there is no long-range order (even at $T = 0$); in the absence of gaps, the correlation functions decay algebraically, $\propto x^{-\kappa}$, where the exponent $\kappa$ is interaction-dependent. The correlations which exhibit the slowest decay (i.e., the smallest exponent) are on long distances the dominant one’s and therefore determine the phase.

Since the Hamiltonian $\tilde{H}_0$ is Gaussian in the fields $\Phi_\nu$ and $\Pi_\nu$, the calculation of correlation functions for the LL is straightforward. The action corresponding to $\tilde{H}_0$ is

$$S = -\sum_\nu \frac{1}{2K_\nu} \int dx d\tau \Phi_\nu \left( \frac{1}{u_\nu} \partial_\tau^2 + u_\nu \partial_x^2 \right) \Phi_\nu,$$

where $\tau = it$. The Green’s function is given by

$$G_\nu(x, \tau) = \frac{K_\nu}{4\pi} \ln \left( \frac{R^2}{x^2 + u_\nu^2 \tau^2 + \alpha^2} \right),$$

where $\alpha$ is a short-distance cutoff and $R$ the radius of the integration boundary in the complex plane (we finally take $R \to \infty$, corresponding to the usual thermodynamic limit).

There are various operators, where the correlations are of interest: The CDW operator at $2k_F$ is

$$O_{\text{CDW}} = \sum_s \Psi_{R_s}^\dagger \Psi_{L_s} \propto e^{-i\sqrt{2\pi}\Phi_\rho} \cos(\sqrt{2\pi}\Phi_\sigma),$$

the SDW operator reads

$$O_{\text{SDW}} = \sum_s \Psi_{R_s}^\dagger \Psi_{L_s} \propto e^{-i\sqrt{2\pi}\Phi_\rho} \cos(\sqrt{2\pi}\theta_\sigma),$$
and the singlet-pairing operator is

\[ O_{ss} = \sum_s \Psi_{Rs} \Psi_{Ls} \propto e^{-i \sqrt{2 \pi} \theta} \cos(\sqrt{2 \pi} \Phi_\sigma). \quad (2.34) \]

To leading order, the decay of the correlation functions is then (for more details, see Appendix B)

\[ \langle O_{cdw}(z) O_{cdw}(0) \rangle \propto x^{-1-K_\sigma}, \quad (2.35) \]
\[ \langle O_{sdw}(z) O_{sdw}(0) \rangle \propto x^{-1-K_\rho}, \quad (2.36) \]
\[ \langle O_{ss}(x) O_{ss}(0) \rangle \propto x^{-1-1/K_\rho}, \quad (2.37) \]

where we used that for repulsive interactions \( K_\sigma = 1 \). Since \( K_\rho < 1 \), the dominant correlations are charge and spin density waves.

The Green's function and the momentum distribution function can be calculated similarly. In particular, the momentum distribution function in the vicinity of \( k_F \) shows that the above model is not a FL

\[ n(k) = n(k_F) - \text{const} \times \text{sign}(k - k_F)(k - k_F)^\eta, \quad (2.38) \]

where \( \eta \approx U^2/(8\pi^2v^2) \). For an interacting system, \( U \neq 0 \), the momentum distribution is continuous at \( k_F \) (for \( T = 0 \)).

### 2.3.3 Attractive interactions: Superconductivity

For attractive interactions, the physical properties are entirely different. The RG flow is such that \( g_1 \) diverges at a finite energy-scale. One should not overinterpret this divergence. The one-loop RGEs cease to be valid long before the couplings become infinite. The “divergence” signals a RG instability and the crossover to a new phase. In 1D, the properties of this “strong coupling” phase can be derived from the bosonized form of the Hamiltonian. Developing the cosine-term to leading order in \( \Phi_\sigma \) gives a first insight:

\[ g_1 \cos(\sqrt{8 \pi} \Phi_\sigma) \approx g_1 \left( 1 - 4 \pi \Phi_\sigma^2 \right), \quad (2.39) \]

i.e., when \( g_1 \) is not vanishing but growing, the spin-part describes a massive field theory and we obtain a gap in the excitation spectrum.
More precisely, the spin-part of the Hamiltonian is equal to the Hamiltonian of the quantum sine-Gordon model,

$$H_{SG} = \int dx \frac{\gamma}{2} \left( (\partial_x \Phi)^2 + \Pi^2 \right) - g \cos(\beta \Phi),$$

(2.40)

which is one of the best studied exactly solvable models — as well classically and quantum mechanically, see Refs. [77, 78]. The excitation spectrum is gapless for $\beta > \sqrt{8\pi}$ and gapped for $\beta < \sqrt{8\pi}$. There are two types of gapped (solitonic) excitations. For $\sqrt{4\pi} < \beta < \sqrt{8\pi}$, the excitations are massive fermionic particles (kinks) with relativistic dispersion relation. For $\beta < \sqrt{4\pi}$, there appear additional particle-antiparticle bound states (breathers).

In our case, the canonical transformation $\Phi_\sigma = \sqrt{K_\sigma} \Phi_\sigma, \Pi_\sigma = \tilde{\Pi}_\sigma / \sqrt{K_\sigma}$ brings the spin-part of the Hamiltonian in a similar form as $H_{SG}$ with $\beta = \sqrt{8\pi} K_\sigma$. Since for attractive interactions $K_\sigma > 1$, see Eq. (2.28), i.e., $\beta < \sqrt{4\pi}$, the spin-part has a gap.

The size of the gap follows from the RGEs. The couplings become large, when the energy approaches the scale of divergence, $E \sim \epsilon e^{-\pi/|U|}$. In other words, the energy $E$ is the scale where the gap opens, i.e., the gap $\Delta$ is of the order of $E$, $\Delta \sim E \sim \epsilon e^{-\pi/|U|}$. This exponential dependence on the interaction strength is generic for small $U$, e.g., the BCS gap has the same form.

Next, we discuss the correlation functions. In order to minimize the energy, the $\Phi_\sigma$ field is pinned, i.e.,

$$\langle \cos(\beta \Phi_\sigma) \rangle \approx 1; \quad (2.41)$$

quasiclassically, we may say that $\Phi_\sigma \approx 0$ (up to $2\pi / \beta$). The commutation relation for $\Phi_\sigma$ and $\Pi_\sigma$ implies, that the dual field, $\theta_\sigma$, is disordered, i.e., correlation functions including this field decay exponentially on a length-scale $\propto \Delta^{-1}$. The SDW correlations, see Eq. (2.33), are therefore strongly suppressed. For attractive interactions, the charge-LLP is $K_\rho > 1$ implying that the singlet pairing correlation function, Eq. (2.34), dominates over the CDW correlations, Eq. (2.32). Since we can set $\Phi_\rho = 0$, the integration has only to be done over the charge fields and we find

$$\langle O_{SS}^\dagger(x) O_{SS}(0) \rangle \propto x^{-1/K_\rho}. \quad (2.42)$$

For attractive interactions, the single Hubbard chain exhibits dominant superconducting correlations. A 1D superconductor with this property belongs to the universality class of the Luther-Emery liquid.
Chapter 3

Two-leg ladder

The half-filled and hole doped two-leg (Hubbard, $t$-$J$) ladder is one of the widely studied and well understood models in condensed matter physics. Coupled chain problems were on the one hand motivated by the question of the stability of the LL fixed point of the single chain [40, 41], and on the other hand by the possibility of superconductivity in the two-leg ladder [42, 43].

In our case, the two-leg ladder is an important starting point for the study of general $N$-leg ladders. The reason is twofold: First, the physics of the two-leg ladder is already nontrivial and allows for a discussion of many interesting phenomenons. Second, as we show in the next chapters, the Hamiltonian of the half-filled $N$-leg ladder decouples for low energies into a sum of (effective) two-leg ladders Hamiltonians.

The weakly interacting two-chain (and $N$ chain) problem can be treated for two different limits: First, transverse hopping is added to two (uncoupled) LLs, i.e., $t \perp < U < t$ and second, interactions are included in a noninteracting system (which is after diagonalization a two-band model), $U < t \perp, t$. Here, we concentrate on the second case. Using RG and bosonization techniques, we discuss two systems in more detail: In Sec. 1, the spinless two-leg ladder away from half-filling (Ledermann and Le Hur, Ref. [79]) and in Sec. 2, the (half-filled) spin-1/2 two-leg ladder, where we follow the works of Lin, Balents, and Fisher [49, 80].

We will introduce the notation of current algebra, which is more appropriate for such problems than the “g-ology” used for the single chain.
3.1 Spinless two-leg ladder

The first analytical works about coupled chains were done for spinless fermions; the reason was mainly a technical one: the RGEs largely simplify. Physically, spinless fermions can, e.g., be considered as completely polarized spin-1/2 fermions in a high magnetic field. In the next chapters, we will give another useful application: bound hole-pairs in N-leg ladders behave at low dopings as spinless fermions. Possible binding of these pairs can be studied in a N/2-band model of spinless fermions.

Similar to spin-1/2 fermions, the LL fixed point of the single chain is already destroyed by a (infinitesimal) small transverse hopping $t_{\perp}$. In contrast to spin-1/2 fermions, the ratio of the interaction strength $U$ and the hopping $t_{\perp}$ has relevant consequences on the phase diagram. While the hole doped spin-1/2 two-leg ladder is superconducting both for $t_{\perp} < U$ [48] and $t_{\perp} > U$ [49], the spinless two-leg ladder is a CDW metal for $t_{\perp} < U$ [41, 78, 81]. Here, we show that superconductivity with unconventional pairing arises for $t_{\perp} > U$. Furthermore, between the metallic phase at high doping and the superconductor at low doping, we find an intermediate phase, where one part of the Fermi surface (FS) is superconducting and another part a CDW metal.

3.1.1 Hamiltonian

The noninteracting two-leg ladder of spinless fermions is given by the Hamiltonian

$$H_0 = -t \sum_{x,i} d_i^\dagger(x+1) d_i(x) + H.c. - t_{\perp} \sum_x d_1^\dagger(x) d_2(x) + H.c.,$$

where $t$ and $t_{\perp}$ are the hopping matrix elements along and between the chains and $d_i^\dagger(x)$ creates a fermion in the chain $i$ at the rung $x$. We are going to consider small repulsive interactions $0 < \epsilon < t, t_{\perp}$. In this limit, it is a good approach to diagonalize first $H_0$ by a canonical transformation,

$$\Psi_j(x) = \frac{1}{\sqrt{2}} [d_1(x) \pm d_2(x)],$$

where $\pm$ corresponds to $j = 1, 2$. Going over to the momentum space, we find a decoupling into two bands (we set the lattice parameter equal to 1),

$$H_0 = \sum_{j=1,2} \int dk \epsilon_j(k) \Psi_j^\dagger(k) \Psi_j(k),$$
3.1. SPINLESS TWO-LEG LADDER

where the dispersion relations are

\[ \epsilon_j(k) = \mp t_\perp - 2t \cos(k). \]  \tag{3.4}

Band 1 is the bonding and band 2 the antibonding band. By analogy with the 2D case, the associated transverse momenta are denoted as \( k_\perp = 0, \pi \). Since we are discussing only the low-energy physics, we linearize the dispersion relation \( \epsilon_j \) around the Fermi momenta \( \pm k_F \), which are determined by the chemical potential \( \mu = \epsilon_j(k_F) \) and the band filling \( n, k_{F1} + k_{F2} = 2n\pi \). For the operator \( \Psi_j \) at \( \pm k_{Fj} \), we write \( \Psi_{R/Lj}(k) = \Psi_j(\pm k_{Fj} + k) \). The Fermi velocities \( v_j = 2t \sin(k_{Fj}) \) can be expressed by \( n, t, \) and \( t_\perp \),

\[ v_{1,2} = 2t \sin \left[ \pi n \pm \arcsin \left( \frac{t_\perp}{2t \sin(\pi n)} \right) \right]. \tag{3.5} \]

For spinless ladders, the band filling is \( 0 \leq n \leq 1 \) and the hole doping away from half filling is \( \delta = 0.5 - n \). The effect of the interchain hopping \( t_\perp \) is included in the velocities \( v_j \). We like to point out that the velocities are not equal for finite \( t_\perp \),

\[ v_1 - v_2 = \frac{2t_\perp}{\tan(\pi n)}. \tag{3.6} \]

For \( v_1 = v_2 \) and without making the link to ladders, this two-band model has first been studied in Ref. [82]. We will see that the difference in the velocities has the remarkable effect of driving the system to a superconducting state for repulsive interactions, \( U > 0 \). We note that both bands are partially filled, when \( t_\perp < 2t \sin(\pi n) \).

We do not consider the half-filled case, \( k_{F1} + k_{F2} = \pi \) \( (v_1 = v_2) \), where the ladder is insulating, allowing us to neglect umklapp processes (we also exclude the particular points \( k_{Fj} = \pi/2 \)). Including all interactions allowed by symmetry (leaving away completely chiral one's), in momentum space, the Hamiltonian is given by

\[ H = H_0 + H_{\text{Int}}, \ ]

and

\[ H_0 = \sum_{j=1,2} v_j \int dk_1 k \left[ \Psi_{Rj}^\dagger(k) \Psi_{Rj}(k) - \Psi_{Lj}^\dagger(k) \Psi_{Lj}(k) \right], \tag{3.7} \]

\[ H_{\text{Int}} = \int dk_1 dk_2 dk_3 dk_4 \delta(k_1 + k_3 - k_2 - k_4) \times \left[ c_1 \Psi_{R1}^\dagger(k_1) \Psi_{R1}(k_2) \Psi_{L1}^\dagger(k_3) \Psi_{L1}(k_4) + c_2 (1 \leftrightarrow 2) \right. \]

\[ + f_{12} \left( \Psi_{R1}^\dagger(k_1) \Psi_{R2}(k_2) \Psi_{L2}^\dagger(k_3) \Psi_{L1}(k_4) + 1 \leftrightarrow 2 \right) \]

\[ + c_{12} \left( \Psi_{R1}^\dagger(k_1) \Psi_{R2}(k_2) \Psi_{L1}^\dagger(k_3) \Psi_{L2}(k_4) + 1 \leftrightarrow 2 \right). \tag{3.8} \]
The bare values of the couplings are chosen as

$$c_1 = c_2 = f_{12} = c_{12} = U > 0. \quad (3.9)$$

We will see, that the $c_{12}$ interaction (pair hopping of left/right going quasi-particles from band 1 to band 2) is the most relevant in determining the low-energy physics.

### 3.1.2 RG equations

We give the RGEs and solve them analytically. We find, that for $v_1/v_2 < 7$, all couplings diverge, while for $v_1/v_2 > 7$, $c_{12} \to 0$ and the other couplings remain of the order of $U$.

To one-loop order, the RGEs are derived and given in Refs. [40, 82]. In addition, we can infer the general form to all orders: The model with the couplings $c_1$, $c_2$, and $f_{12}$ alone is exactly solvable (by bosonization, see below) and in particular at a RG fixed point. Products of couplings without at least one $c_{12}$ do therefore not appear in the RGEs for $c_1$, $c_2$, and $f_{12}$. Including the one-loop exact results, the particular form of the $c_{12}$ interaction then implies that the RGEs (to all orders) have the following form,

$$\begin{align*}
\frac{dc_1}{dl} &= -\frac{1}{2\pi v_2} c_{12}^2 \left[1 + O(g_\alpha/t)\right] \\
\frac{dc_2}{dl} &= -\frac{1}{2\pi v_1} c_{12}^2 \left[1 + O(g_\alpha/t)\right] \\
\frac{df_{12}}{dl} &= \frac{1}{\pi(v_1 + v_2)} c_{12}^2 \left[1 + O(g_\alpha/t)\right] \\
\frac{dc_{12}}{dl} &= \frac{c_{12}}{\pi} \left[\frac{2f_{12}}{v_1 + v_2} - \frac{c_1}{2v_1} - \frac{c_2}{2v_2} + O(g_\alpha^2/t^2)\right].
\end{align*} \quad (3.10)$$

The energy-scale (temperature-scale) is related to $l$ by $E \sim l^{-1}$ and $O(g_\alpha^n/t^n)$ denotes higher order terms in the couplings ($g_\alpha = c_1$, $c_2$, $f_{12}$, or $c_{12}$). The plus and minus signs result from particle-hole respectively particle-particle diagrams.

Keeping only the terms quadratic in the coupling constants, we transform the set of 4 differential equations into one differential equation for $f_{12}$. Note that the equations for $c_1$, $c_2$, and $f_{12}$ are almost the same; in a first step, we thus obtain a system of two differential equations, where the one for $f_{12}$ is
the same as before and the one for \( c_{12} \) reads

\[
\frac{dc_{12}}{dl} = c_{12} \left[ B f_{12} - \frac{(v_1 + v_2)U}{\pi v_1 v_2} \right],
\]

(3.11)

where

\[
B = \frac{4v_1 v_2 + (v_1 + v_2)^2}{2\pi v_1 v_2 (v_1 + v_2)}.
\]

(3.12)

From these two equations, we then find

\[
\left[ B f_{12} - \frac{(v_1 + v_2)U}{\pi v_1 v_2} \right] \frac{df_{12}}{dl} = \frac{c_{12}}{\pi (v_1 + v_2)} \frac{dc_{12}}{dl},
\]

(3.13)

and finally

\[
\frac{1}{B} \frac{df_{12}}{dl} = (f_{12} - CU)^2 + DU^2,
\]

(3.14)

where

\[
C = \frac{2(v_1 + v_2)^2}{4v_1 v_2 + (v_1 + v_2)^2},
\]

(3.15)

and

\[
D = \frac{-v_1^4 + 6v_1^3 v_2 + 6v_1^2 v_2^2 + 6v_1 v_2^3 - v_2^4}{[4v_1 v_2 + (v_1 + v_2)^2]^2}.
\]

(3.16)

The solution of Eq. (3.14) is qualitatively different for \( D < 0 \) and \( D > 0 \). For \( D > 0 \), all couplings diverge, while for \( D < 0 \), \( c_{12} \) scales to zero and the others remain of the order of \( U \). Solving the equation

\[
x^4 - 6x^3 - 6x^2 - 6x + 1 = 0,
\]

(3.17)

for \( x = v_1/v_2 \), we obtain the exact transition ratio \( v_1/v_2 \). For comparable velocities,

\[
1/7 \approx 0.14327 \ldots < v_1/v_2 < 6.9798 \ldots \approx 7,
\]

(3.18)

we find \( D > 0 \) resulting in

\[
f_{12}(l) = U \left\{ C + \sqrt{D} \tan \left[ B \sqrt{DU}l - \arctan \left( \frac{C - 1}{\sqrt{D}} \right) \right] \right\}
\]

(3.19)

and

\[
c_{12}(l) = U \sqrt{\pi (v_1 + v_2) BD} \left\{ 1 + \tan \left[ B \sqrt{DU}l - \arctan \left( \frac{C - 1}{\sqrt{D}} \right) \right] \right\}^{1/2}.
\]

(3.20)
Figure 3.1: Phase diagram of the spinless two-leg ladder for $t_\perp/t = 1$ and $U/t = 0.2$. The doping away from half filling is $\delta$ and $T$ is the energy (or temperature) scale (the $T_0 \sim t$ is a high temperature cutoff). The solid line shows the crossover to a superconducting (SC) phase with interchain pairing and the dashed line displays where the crossover from a metallic phase with charge density wave (CDW) excitations to a mixed phase with coexistence of SC and CDW excitations occurs. The dotted line separates the region where both bands are partially filled, from the region where one band is empty. At half-filling, the ladder is insulating.

For ratios $v_1/v_2 > 7(< 1/7)$, we find $c_{12} \to 0$ and

$$f_{12} \to (C - \sqrt{-D})U,$$  \hspace{1cm} (3.21)

for $l \to \infty$. Similar as $f_{12}$, the couplings $c_1$ and $c_2$ stay of the order of $U$ (e.g., $U < f_{12} < 1.4U$). Since $c_{12}$ flows to zero and the form of the RGEs to all orders is such that couplings whatever the order is, are always multiplied at least once with $c_{12}$, the fixed point is stable.

### 3.1.3 Bosonization

Using bosonization techniques and the above RG results, we derive the low-energy physics. When doping the half-filled ladder, a superconducting phase
3.1. SPINLESS TWO-LEG LADDER

with interchain pairing arises. Well away from half-filling, the system undergoes a transition to the $c_{12} = 0$ phase which exhibits two gapless modes. Fig. 3.1 gives an overview.

For spinless fermions, the operators $\Psi_{R/Lj}$ are rewritten according to

$$\Psi_{R/Lj}(x) = \frac{\eta_{R/Lj}}{\sqrt{2\pi\alpha}} \exp \left\{ i\sqrt{\pi} [\mp \Phi_j(x) + \theta_j(x)] \right\}$$

(3.22)

and currents $J_{hj} = \Psi_{hj}^\dagger \Psi_{hj}$, where $h = R/L$, become

$$J_{Lj} + J_{Rj} = \frac{1}{\sqrt{\pi}} \partial_x \Phi_j \quad \text{and} \quad J_{Lj} - J_{Rj} = \frac{1}{\sqrt{\pi}} \Pi_j.$$  

(3.23)

Fourier transforming the Hamiltonian $H = H_0 + H_{\text{int}}$, see Eqs. (3.7) and (3.8), back to $x$-space and using the above bosonization rules, we obtain (for the Klein factors, we choose the gauge $\eta_{R1}\eta_{L1}\eta_{R2}\eta_{L2} = 1$)

$$H = \sum_{j=1,2} \int dx \left( \frac{v_j}{2} + \frac{c_j}{4\pi} (\partial_x \Phi_j)^2 + \frac{v_j}{2} - \frac{c_j}{4\pi} \right) \Pi_j^2$$

$$+ \int dx \left( \frac{f_{12}}{2\pi} (\partial_x \Phi_1 \partial_x \Phi_2 - \Pi_1 \Pi_2) - \frac{c_{12}}{(2\pi\alpha)^2} \cos \left( \sqrt{4\pi}(\theta_1 - \theta_2) \right) \right).$$

(3.24)

A flow to strong coupling of $c_{12}$ results (quasiclassically) in a "pinning" of $\theta_1 - \theta_2 = 0$ in order to minimize the energy, and a single gapless mode, while for $c_{12} \rightarrow 0$, two gapless modes are present. Note that for (quasi-)1D systems with a non-commensurate band filling (i.e., without umklapp interactions), the Lieb, Schultz, Mattis theorem tells us that there is always at least one gapless mode (see Ref. [83]).

3.1.4 Interchain-pairing superconducting phase

We show that as a result of a finite interchain hopping $t_\perp$, doping away from half-filling, superconducting correlations dominate. The pairing takes place between left (right) going particles in chain one and right (left) going particles in chain two. For $t_\perp \ll t$, we recover previous results [41].

Using the canonical transformation $\Phi_\pm = (\Phi_1 \pm \Phi_2)/\sqrt{2}$ and $\Pi_\pm = (\Pi_1 \pm \Pi_2)/\sqrt{2}$, the Hamiltonian (3.24) takes the form

$$H = H_B + H_{SG} + H_{\text{mix}},$$

(3.25)
where $H_B$ is the Hamiltonian of a massless boson,

$$ H_B = \int dx \frac{u_+}{2} \left[ \frac{1}{K_+} \left( \partial_x \Phi_+ \right)^2 + K_+ \Pi_+^2 \right], $$

and $H_{SG}$ is the sine-Gordon Hamiltonian,

$$ H_{SG} = \int dx \left\{ \frac{u_-}{2} \left[ \frac{1}{K_-} \left( \partial_x \Phi_- \right)^2 + K_- \Pi_-^2 \right] - \frac{c_{12}}{(2\pi\alpha)^2} \cos \left( \sqrt{8\pi \theta_-} \right) \right\}, $$

and finally, $H_{\text{mix}}$ is a mixing term,

$$ H_{\text{mix}} = \int dx \left( v^c \partial_x \Phi_+ \partial_x \Phi_- + v^p \Pi_+ \Pi_- \right). $$

The velocities $u_\pm$, $v^c_\pm$, and the LLPs $K_\pm$ are given by

$$ u_\pm = \sqrt{\left( \frac{v_1 + v_2}{2} \right)^2 - \left( \frac{c_1 + c_2 \pm 2f_{12}}{4\pi} \right)^2}, $$

and

$$ v^c_\pm = \frac{v_1 - v_2}{2} \pm \frac{c_1 - c_2}{4\pi}, $$

and

$$ K_\pm = \sqrt{\frac{2\pi(v_1 + v_2) - (c_1 + c_2 \pm 2f_{12})}{2\pi(v_1 + v_2) + (c_1 + c_2 \pm 2f_{12})}}. $$

The mixing term hinders for $v_1 - v_2 \neq 0$ a (simple) analytical solution of the (classical) equations of motion. Since the $\theta_-$ field is pinned, the current density takes the form $j = u_+K_+\Pi_+$. At half-filling the $\Phi_+$ field is also pinned resulting in $j = 0$ and an insulating phase [82].

Next, we discuss the correlation functions. For repulsive interactions, the charge density and superconducting pairing fluctuations with the most divergent susceptibilities are the following ones: The CDW operator (at a wavevector $k_{F1} + k_{F2}$) is given by

$$ O_{\text{CDW}} = d_{R1}^id_{L1} - d_{R2}^id_{L2} = \Psi_{R2}^\dagger \Psi_{L1} + \Psi_{R1}^\dagger \Psi_{L2} $$

$$ \propto \exp \left( i\sqrt{2\pi \Phi_+} \right) \cos \left( \sqrt{2\pi \theta_-} \right), $$
3.1. SPINLESS TWO-LEG LADDER

and the superconducting pairing operator by

\[ O_{SC} = d_{R1}d_{L2} + d_{R2}d_{L1} = \Psi_{R1}\Psi_{L1} - \Psi_{R2}\Psi_{L2} \]
\[ \propto \exp \left(i\sqrt{2\pi\theta_+}\right) \cos \left(\sqrt{2\pi\theta_-}\right), \]  
(3.33)

where the \(d_{R/Li}\) are the annihilation operators for the fermions in chain \(i\).

The operator \(O_{CDW}\) represents an antisymmetric CDW and \(O_{SC}\) superconductivity with interchain pairing — previously called \(d\)-wave like due to the antisymmetry with respect to the bonding and antibonding band [81]. However, the operator \(O_{SC}\) has odd parity, \(O_{SC}(-x) = -O_{SC}(x)\), which one associates rather with \(p\)-wave like superconductivity in each band.

We obtain for the CDW correlation function (for a derivation, see Appendix B),

\[ \langle O_{CDW}^+(x)O_{CDW}(0) \rangle \propto x^{-\gamma} \]  
(3.34)

and for the superconducting pairing correlation function

\[ \langle O_{SC}^+(x)O_{SC}(0) \rangle \propto x^{-1/\gamma}, \]  
(3.35)

where the exponent is

\[ \gamma = \frac{K_+}{1 - \frac{K_+K_-}{2v_+v_-}(v_c^2)^2}. \]  
(3.36)

As a result, the finite \(t_+ \propto v_c^2\) leads to \(\gamma > 1\), implying that superconducting pairing correlations dominate.

The increase of \(\gamma\) can be understood as follows. The pinning of \(\theta_-\) allows us to set \(\Pi_- = 0\). The only coupling is then between the \(\Phi_\pm\) fields. The \(\Phi_-\) field fluctuates strongly and affects the \(\Phi_+\) correlation function with additional fluctuations thus increasing \(\gamma\) and stabilizing the superconductivity.

We furthermore find that in the low-energy regime \((l \to \infty)\), the LLP \(K_+\) becomes bigger than 1. In detail, whether \(K_+\) is bigger or smaller than 1 depends on the following sum [see Eq. (3.31)]

\[ c_1 + c_2 + 2f_{12} = \left(3 + \frac{v_1^2 + v_2^2}{2v_1v_2}\right)U - \frac{(v_1 - v_2)^2}{2v_1v_2}f_{12}. \]  
(3.37)

Since \(f_{12}\) increases for decreasing energy (temperature), the above sum becomes negative and \(K_+ > 1\).

\[ ^{1}\text{The difference of } O_{SC} \text{ to the bosonized form of } O_{S2} \text{ in Ref. [41] is due to an error in Ref. [41] (already noted in Ref. [81]). The difference to the fermionic form of } O_{SC_{4d}} \text{ in Ref. [81] is due to a mistake in Ref. [81] (private communication).} \]
Figure 3.2: The exponent $\gamma$ of the CDW correlation function, $\propto x^{-\gamma}$, is shown for $U/t = 0.2$ and different ratios of $t_\perp/t$. While for small ratios $t_\perp \ll t$, CDWs dominate for almost all physical values of the doping $\delta$, comparable ratios $t_\perp \sim t$ favor superconductivity already at lower doping. The superconducting correlations are strongest, when approaching the transition to the mixed SC+CDW phase, where the chemical potential is close to the bottom of the antibonding band ($\gamma \to 1.5$ for $v_1/v_2 \to 7$).

It is instructive to expand Eq. (3.36) leading order in $f_{12}/t$ and $t_\perp/t$ (we rewrite $c_1$ and $c_2$ in terms of $f_{12}$ and neglect $U \ll t$),

$$\gamma = 1 + \frac{1}{2} \left( \frac{v_1 - v_2}{v_1 + v_2} \right)^2 \left( 1 + \frac{v_1 + v_2}{2\pi v_1 v_2} f_{12} \right).$$

Neglecting the $f_{12} \sim t$ due to the relatively small prefactor, we rewrite Eq. (3.38) in terms of physical quantities,

$$\gamma = 1 + \frac{1}{8} \left( \frac{t_\perp}{t} \right)^2 \frac{\cot(\pi n)^2}{\sin(\pi n)^2 - (t_\perp/2t)^2}.$$

Leaving order in the doping $\delta = 0.5 - n$, we then find the particular simple form,

$$\gamma = 1 + \frac{\pi^2}{8} \left( \frac{t_\perp}{t} \right)^2 \delta^2.$$

(3.39)
3.1. SPINLESS TWO-LEG LADDER

The $s^2$ and the $t_{\perp}^2$ reflect the $v_1 \leftrightarrow v_2$ symmetry. Comparing Eq. (3.40) with the usual exponent $K$ of a LL, we deduce that the $t_1$ term produces an effective attraction between particles of chain one and chain two.

The exponent $\gamma$ as a function of the doping $\delta$ is shown in Fig. 3.2 for different ratios $t_\perp/t = 0.1, \ldots, 1.5$ (we have set $f_{12} = t$). It becomes smaller than 1 when approaching half-filling, $\gamma \approx K_+ \leq 1$ (a precursor of the insulating state), and 1.5 near the transition to the mixed phase. The insulating state at half-filling thus evolves over a phase with weak CDW correlations (this depends on the choice of $f_{12} \sim t$) to a superconducting phase; the pairing correlations are strongest, when the chemical potential lies near the bottom of the antibonding band ($v_1/v_2 \approx 7$).\footnote{The same finding was made in numerical calculations for the spin-1/2 two-leg ladder \cite{46}.} A comparison of Fig. 3.2 with Fig. 3.1 shows that in this case also the crossover temperature is highest.

For small $t_\perp/t$, we recover previous results; the bosonized Hamiltonian goes over to the one studied in Refs. \cite{41, 81} and the exponent $\gamma$ stays smaller than 1, $\gamma \approx K_+ \leq 1$, such that CDW correlations dominate (for almost all physical values of the doping $\delta$).

3.1.5 Mixed SC+CDW phase

For $v_1/v_2 > 7$, the fixed points of the couplings are such that $c_{12} = 0$, and $c_1$, $c_2$, and $f_{12}$ are of the order of $U$. We diagonalize the Hamiltonian (3.24) for $c_{12} = 0$ using a current representation of the fields. Defining $\mathbf{J} = (J_{R1}, J_{L1}, J_{R2}, J_{L2})$ and

$$M = \begin{pmatrix}
v_1 & c_1/2\pi & 0 & f_{12}/2\pi \\
c_1/2\pi & v_1 & f_{12}/2\pi & 0 \\
f_{12}/2\pi & 0 & v_2 & c_2/2\pi \\
f_{12}/2\pi & v_2 & 0 & v_2
\end{pmatrix},$$

the Hamiltonian (3.24) can be written as $H = \pi \int dx \mathbf{J}^T M \mathbf{J}$ and the new LLPs and velocities are determined by the eigenvalues of $M$, which are

$$\frac{v_1 + v_2}{2} + \frac{c_1 + c_2}{4\pi} \pm \frac{D_+}{2} \text{ and } \frac{v_1 + v_2}{2} - \frac{c_1 + c_2}{4\pi} \pm \frac{D_-}{2},$$

where

$$D_{\pm} = \sqrt{\left(v_1 - v_2 \pm \frac{c_1 - c_2}{2\pi}\right)^2 + 4 \left(\frac{f_{12}}{2\pi}\right)^2}. \tag{3.43}$$
Since $v_1 - v_2 \gg f_{12}$, the coupling $f_{12}$ does not contribute to $D_{\pm}$ leading order in $U$. The velocities and the LLPs thus have the same form as in two decoupled LLs,

$$u_j = \sqrt{v_j^2 - \frac{c_j^2}{(2\pi)^2}} \quad \text{and} \quad K_j = \sqrt{\frac{2\pi v_j - c_j}{2\pi v_j + c_j}} \approx 1 - \frac{c_j}{2\pi v_j}, \quad (3.44)$$

where in our case the $c_j$ are scale dependent. The new basis is "almost" the old one, $\Pi_1 \approx \Pi_1 + \epsilon\Pi_2$, where $\epsilon \sim U/t$ (and similarly for $\Pi_2$ and the fields $\Phi_j$) and the current density takes the usual form,

$$j = u_1 K_1 \Pi_1 + u_2 K_2 \Pi_2, \quad (3.45)$$

where the $\Pi_j$ obey the equations of motion $\partial_t^2 \Pi_j = u_j^2 \partial_x^2 \Pi_j$. It should be noted that here the Drude coefficient $u_j K_j$ is not equal to that of the free Fermi gas.

In a short range of ratios, $7 < v_1/v_2 < 8$, the low temperature fixed point of the couplings is such that $c_1 < 0$ and $c_2 > 0$ (for $v_1/v_2 = 7$, we find $c_1 = -0.57U$ and $c_2 = 0.78U$), implying $K_1 > 1$ and $K_2 < 1$ — a coexistence of pairing correlations in the bonding with charge density correlations in the antibonding band. Rewriting the superconducting pairing operator $O_{\text{SC}} = \Psi_{R1} \Psi_{L1}$ in terms of the chain operators $d_{R/L} \bar{d}$ shows, that there are as well intra and inter-chain superconducting pairs. We interpret this coexistence as a precursor of the superconducting phase at $v_1/v_2 < 7$, i.e., pairing correlations first appear in the bonding band (at a momentum $k_\perp = 0$), and then also in the antibonding (at a momentum $k_\perp = \pi$) and phase coherence between the bands sets in for increasing $c_{12}$. In Fig. 3.1, the dashed line shows, where $K_1 > 1$.

For ratios $v_1/v_2 > 8$, we find $c_j > 0$ and in both bands CDW correlations dominate, $K_j < 1$ (for $v_1/v_2 \gg 8$, the fixed points are $c_1 \approx c_2 \approx U$). This corresponds to the usual 1D metallic (LL) phase.

### 3.2 Spin-1/2 two-leg ladder

The weakly interacting spin-1/2 two-leg Hubbard ladder and two coupled LLs have been studied by many different authors [40, 47, 48, 49, 84, 85]. Here, following the works of Lin, Balents, and Fisher [49, 80], we show, that the half-filled ladder is an insulating spin-liquid (ISL) and that upon hole doping...
the ladder becomes \(d\)-wave like superconducting. We give the bosonized form of the various interactions.

### 3.2. SPIN-1/2 TWO-LEG LADDER

#### 3.2.1 Hamiltonian and current algebra

The Hamiltonian of the two-leg Hubbard ladder is given by

\[
H = -t \sum_{x,j,s} d_{j,s}^{\dagger}(x + 1)d_{j,s}(x) + H.c. - t_\perp \sum_{x,s} d_{1,s}^{\dagger}(x)d_{2,s}(x) + H.c. \\
+ U \sum_{x,i} d_{i,s}^{\dagger}(x) d_{i,s}(x) d_{i+1,s}(x) d_{i+1,s}(x).
\]  

(3.46)

Similar as for the spinless case, we proceed by diagonalizing \(H_0\), resulting in a weakly interacting two-band model, see above.

For an increasing number of couplings, it is advantageous to organize the interacting part in a form which manifestly takes into account the symmetries, in particular the U(1) charge and the SU(2) spin symmetry (the “g-ology” notation used in the previous chapter does not so). We therefore employ the notation of current algebra, i.e., we use the U(1) charge currents

\[
J_{hij} = \sum_s \Psi_{h,s}^\dagger \Psi_{hjs},
\]  

(3.47)

and the SU(2) spin currents

\[
J_{hij}^p = \frac{1}{2} \sum_{s,s'} \Psi_{h,s}^\dagger \tau_p^{s,s'} \Psi_{hjs}.
\]  

(3.48)

where \(h = R/L\) and the \(\tau^p\) are the Pauli matrices \((p = x, y, z)\). Away from half-filling, the interacting part consists of U(1) and SU(2) Cooper and forward scattering processes (with couplings \(c_{ij}^{\sigma}\) and \(f_{ij}^{\sigma}\) respectively) within a band and between two different bands

\[
H_{i,c} = \sum_{ij \neq j} \int dx \left( f_{ij}^{\sigma} J_{Rij} J_{Lij} - f_{ij}^{\sigma} J_{Rij} \cdot J_{Lij} \\
+ c_{ij}^{\sigma} J_{Rij} J_{Lij} - c_{ij}^{\sigma} J_{Rij} \cdot J_{Lij} \right) \\
+ \sum_{j=1,2} \int dx \left( c_{jj}^{\sigma} J_{Rjj} J_{Ljj} - c_{jj}^{\sigma} J_{Rjj} \cdot J_{Ljj} \right).
\]  

(3.49)

Due to symmetry, \(c_{ij}^{\rho \sigma} = c_{ij}^{\rho \sigma} \) and \(f_{ij}^{\rho \sigma} = f_{ij}^{\rho \sigma} \). The Hubbard initial-values are \(4f_{ij}^{\sigma} = f_{ij}^{\sigma} = 4c_{ij}^{\sigma} = c_{ij}^{\sigma} = U\).
At half-filling, \( k_{F1} + k_{F2} = \pi \), and additional interband umklapp interactions have to be included. Using the umklapp currents

\[
I_{hij} = \sum_{s,s'} \Psi_{hs} \epsilon_{ss'} \Psi_{hs'}
\]  

(3.50)

and

\[
I_{hij}^p = \frac{1}{2} \sum_{s,s'} \Psi_{hs} (\epsilon \tau^p)_{ss'} \Psi_{hs'}
\]

(3.51)

where \( \epsilon = -i \tau^y \) (note that \( I_{hij} = I_{hji} \) and \( I_{hij}^p = -I_{hji}^p \)), the additional terms are (for \( u_{121}^o \), we differ in notation from Ref. [80] by a factor 2)

\[
H_{1,u} = \int dx \left[ u_{1221}^o \left( I_{12}^1 \cdot I_{21}^1 + H.c. \right) - u_{1221}^\sigma \left( I_{12}^1 \cdot I_{21}^1 + H.c. \right) + u_{122}^1 \left( J_{11}^1 I_{11}^1 + H.c. \right) \right].
\]

(3.52)

The initial-values are \( 2u_{121}^o = 2u_{121}^\sigma = U \) and \( u_{121}^o = 0 \).

The \( g_{1,2} \) couplings of the single chain are related to the \( c_{ij}^{\sigma} \) couplings by

\[
c_{11}^o = g_1^o / 2 - g_2^o \quad \text{and} \quad c_{11}^\sigma = 2g_1^o \quad (\text{see chapter} \ 2).
\]

### 3.2.2 Hole doped ladder

Using RG and bosonization techniques, we determine the low-energy physics of the hole doped two-leg ladder (i.e., without umklapp processes). In contrast to the spinless case, the (one-loop) RGEs (see Appendix A and Ref. [49]) cannot be solved analytically, such that one has to perform a numerical integration.

A characteristic property of the RG flow in the N-leg ladders is, that the couplings flow towards universal ratios as the initial-value \( U/t \) respectively the energy \( E \sim te^{-t/U} \) is reduced. The expression for the low-energy Hamiltonian is then (almost) independent of \( t, t_\perp \), and \( U \). In particular, for the hole doped two-leg ladder,

\[
4c_{12}^o = c_{12}^o \sim t, \quad c_{11}^o / v_1 = c_{22}^o / v_2 \sim -1, \quad \text{and} \quad f_{12}^o \approx 0 \quad (3.53)
\]

at energies below \( E \sim te^{-t/U} \), see Ref. [49].

Next, we bosonize the two-leg ladder Hamiltonian; we find that the flow to strong coupling of the Cooper processes \( c_{12}^{\sigma} \) leads to the opening of a spin-gap
and to phase coherence between the two bands. Introducing bosonic fields for each band, the fermionic operators transform according

\[ \Psi_{R/L,j} = \frac{\eta_{js}}{\sqrt{2\pi\alpha}} \exp \left\{ i\sqrt{\pi/2} \left[ \pm(\Phi_{\rho j} + s\Phi_{\sigma j}) - (\theta_{\rho j} + s\theta_{\sigma j}) \right] \right\}. \]  

(3.54)

The single-band charge currents become

\[ J_{L,jj} + J_{R,jj} = \sqrt{\frac{2}{\pi}} \partial_x \Phi_{\rho j} \quad \text{and} \quad J_{L,jj} - J_{R,jj} = \sqrt{\frac{2}{\pi}} \Pi_{\rho j}. \]  

(3.55)

and the spin currents are

\[ J_{L,jj}^x + J_{R,jj}^x = \frac{1}{\sqrt{2\pi}} \partial_x \Phi_{\sigma j} \quad \text{and} \quad J_{L,jj}^z - J_{R,jj}^z = \frac{1}{\sqrt{2\pi}} \Pi_{\sigma j}. \]  

(3.56)

The noninteracting part, \( H_0 \), takes the usual form

\[ H_0 = \sum_{\nu=\rho,\sigma} \sum_{j=1,2} \frac{\alpha}{2} \int dx \left[ (\partial_x \Phi_{\nu j})^2 + \Pi_{\nu j}^2 \right]. \]  

(3.57)

The single-band U(1) and the z-part of the SU(2) processes become

\[ c_{\rho j}^\nu J_{R,jj}^\nu J_{L,jj}^\nu = \frac{c_{\rho j}^\nu}{2\pi} \left[ (\partial_x \Phi_{\rho j}) - \Pi_{\rho j}^2 \right] \]

\[ c_{\sigma j}^\nu J_{R,jj}^\nu J_{L,jj}^\nu = \frac{c_{\sigma j}^\nu}{8\pi} \left[ (\partial_x \Phi_{\sigma j}) - \Pi_{\sigma j}^2 \right]. \]  

(3.58)

Similarly, the two-band U(1) and the z-part of the SU(2) processes are written as

\[ f_{\rho j}^{ij} J_{R,i}^\rho J_{L,j}^\rho = \frac{f_{\rho j}^{ij}}{2\pi} \left( \partial_x \Phi_{\rho i} \partial_x \Phi_{\rho j} - \Pi_{\rho i} \Pi_{\rho j} \right) \]

\[ f_{\sigma j}^{ij} J_{R,i}^\sigma J_{L,j}^\sigma = \frac{f_{\sigma j}^{ij}}{8\pi} \left( \partial_x \Phi_{\sigma i} \partial_x \Phi_{\sigma j} - \Pi_{\sigma i} \Pi_{\sigma j} \right). \]  

(3.59)

These expressions allow us to calculate the LLPs, see below.

Using the transformation \( \Phi_{\nu \pm} = (\Phi_{\nu 1} \pm \Phi_{\nu 2})/\sqrt{2} \) and \( \Pi_{\nu \pm} = (\Pi_{\nu 1} \pm \Pi_{\nu 2})/\sqrt{2} \), the processes which contain cosine terms, and therefore lead to the opening of gaps, take the form (we drop overall prefactors and use the "gauge" \( \eta_1 \eta_4 \eta_2 \eta_4 = 1 \))

\[ \mathcal{H}_{I,cf} = -c_{12}^\sigma \cos(\beta_{\rho -}) \left[ 2 \cos(\beta_{\Phi_{\sigma +}}) + \cos(\beta_{\Phi_{\sigma -}}) + \cos(\beta_{\theta_{\sigma +}}) \right] + 2f_{12}^\sigma \cos(\beta_{\theta_{\sigma -}}) \cos(\beta_{\Phi_{\rho +}}) + c_{11}^\sigma \cos(\sqrt{2} \beta_{\Phi_{\sigma 1}}) + c_{22}^\sigma \cos(\sqrt{2} \beta_{\Phi_{\sigma 2}}) - 4c_{12}^\sigma \cos(\beta_{\theta_{\sigma -}}) \left[ \cos(\beta_{\Phi_{\sigma -}}) - \cos(\beta_{\theta_{\sigma -}}) \right], \]  

(3.60)
where $\beta = \sqrt{4\pi}$. At low energies, the couplings flow towards the ratios (3.53), such that the cosine-part simplifies

$$\tilde{\mathcal{H}}_{t,\phi} = -|c_{11}'| \cos(\sqrt{2}\beta \Phi_{\sigma 1}) - |c_{22}'| \cos(\sqrt{2}\beta \Phi_{\sigma 2}) - 2c_{12}' \cos(\beta \theta_{\rho -}) [\cos(\beta \Phi_{\sigma +}) + \cos(\beta \Phi_{\sigma -})].$$

(3.61)

Minimizing the energy leads to a pinning of the spin-fields, $\Phi_{\sigma 1} = \Phi_{\sigma 2} = 0$ and of the difference between the charge-fields, $\theta_{\rho -} = 0$. Since both spin-fields are pinned, the ladder has a spin-gap. The pinning of the difference between the two charge fields results in phase coherence between the two bands and only the total charge-mode, $(\Phi_{\rho +}, \theta_{\rho +})$, remains gapless.

Analyzing the various correlation functions, one finds, that the superconducting pairing correlation function is the most dominant one [49]; the (singlet) pair-field operator is given by

$$\Delta_j = \Psi_{Rj\uparrow} \Psi_{Lj\downarrow} + \Psi_{Lj\uparrow} \Psi_{Rj\downarrow} = \frac{\eta_j \eta_j}{\pi \alpha} e^{-i2\pi \theta_{\rho j}} \cos(\sqrt{2}\alpha \Phi_{\sigma j}).$$

(3.62)

The dual field $\theta_{\rho j}$ can therefore be interpreted as the phase of the order parameter. Carrying out a similar calculation as for the spinless case, one obtains, that the pairing correlation function decays as

$$\langle \Delta_j^\dagger(x) \Delta_j(0) \rangle \propto \langle e^{i\sqrt{2}\alpha \theta_{\rho +}(x) - \theta_{\rho +}(0)} \rangle \propto x^{-\kappa},$$

(3.63)

where $\kappa \approx 1$. Note that the pairing correlations in the spin-1/2 case decay $\propto x^{-1/2}$, while all the other correlation functions decay approximately $\propto x^{-2}$. This is the reason why (in contrast to the spinless case) the spin-1/2 two-leg ladder is superconducting both for $t_1 < U$ and for $t_1 > U$.

Furthermore, the pair field operator has a different "sign" in band 1 and band 2, i.e., $\langle \Delta_1^\dagger \Delta_2 \rangle < 0$. It is then natural to say that the symmetry is $d$-wave like.

As a result, the hole doped spin-1/2 two-leg Hubbard ladder is a $d$-wave like superconductor. Note that for attractive interactions, the symmetry would be $s$-wave like.

### 3.2.3 Half-filled ladder

We show, that similar to the $t$-$J$ case, the two-leg Hubbard ladder is at half-filling an insulating spin-liquid, i.e., it is Mott insulating and has a spin-gap. Interestingly, the Mott insulator is a "disordered" $d$-wave superconductor;
3.2. SPIN-1/2 TWO-LEG LADDER

Figure 3.3: Strong-coupling value as a function of the initial-value for various interactions ($t = t_\perp$). For a decreasing initial-value $U/t$, the strong-coupling values approach universal ratios, e.g., $c_{12}^\sigma = 4c_{12}^\rho$.

the phase coherence between the two bands is already present, but the total charge-mode is gapped. We first discuss the RG flow in the small $U$ limit; the phase then follows straightforwardly from the bosonized low-energy Hamiltonian.

At half-filling, the Fermi momenta of band 1 and 2 add up to $\pi$, $k_{F1} + k_{F2} = \pi$, such that interband umklapp processes take place and $H_{\text{int}} = H_{I,\alpha} + H_{I,\beta}$. Note that the Fermi velocities are equal, $v_1 = v_2$. Next, we discuss the RG flow (the $N$-leg RGEs are given in Appendix A). Here and in the following chapters, we analyze the RG flow by plotting the strong-coupling value as a function of the initial-value $U/t$, i.e., we integrate the RGEs down to the energy-scale, where the largest coupling has grown up to the bandwidth $t$. As $U/t \to 0$, the couplings then approach universal ratios. Fig. 3.3 shows the strong-coupling value of $c_{12}^\rho/c_{12}^\sigma$, $f_{12}^\sigma/c_{12}^\sigma$, and $u_{1122}^\rho/c_{12}^\sigma$ as a function of the initial-value $U/t$ (we fix the strong-coupling value of $c_{12}^\sigma$ at the bandwidth $t$). We find, that $f_{12}^\sigma$ and $c_{jj}^\sigma$ vanish, while the other couplings grow always up to the bandwidth $t$, approaching fixed ratios, as
$U/t$ decreases,

$$t \sim g = 4c_{12}^0 = 4f_{12}^0 = c_{12}^0 = -c_{11}^0 = -c_{22}^0 = 4u_{1221}^0 = 8u_{1122}^0 = u_{1221}^0 \quad (3.64)$$

and

$$c_{11}^0 = c_{22}^0 \approx f_{12}^0 \approx 0. \quad (3.65)$$

As a result, the effective low-energy Hamiltonian can be rewritten using a single coupling $g$.

The bosonized form of the additional umklapp interactions is

$$\mathcal{H}_{L,u} = -u_{1221}^0 \cos(\beta \Phi_{\rho+})[\cos(\beta \Phi_{\sigma-}) + \cos(\beta \Phi_{\sigma-}) + 2 \cos(\beta \Phi_{\sigma+})]$$

$$-4 \cos(\beta \Phi_{\rho+}) \{ u_{1221}^0 [\cos(\beta \Phi_{\sigma-}) - \cos(\beta \Phi_{\sigma-})] + 4 u_{1122}^0 \cos(\beta \Phi_{\rho-}) \}. \quad (3.66)$$

Using the ratios (3.64) and (3.65), the half-filled low-energy Hamiltonian\(^3\) becomes $H = \hat{H}_0 + \hat{H}_{\text{int}}$, where

$$\hat{H}_0 = \sum_{\nu=\rho,\sigma} \sum_{\mu=\pm} \int dx \frac{\mu \nu}{2} \left[ \frac{1}{K_{\nu\mu}}(\partial_x \Phi_{\nu\mu})^2 + K_{\nu\mu} \Pi_{\nu\mu}^2 \right] \quad (3.67)$$

and

$$\hat{H}_{\text{int}} = \hat{H}_{L,cf} + \hat{H}_{L,u}$$

$$= -2g[\cos(\beta \theta_{\rho-}) + \cos(\beta \Phi_{\rho+})][\cos(\beta \Phi_{\sigma+}) + \cos(\beta \Phi_{\sigma-})]$$

$$-2g \cos(\beta \theta_{\rho-}) \cos(\beta \Phi_{\rho+}) - 2g \cos(\beta \Phi_{\rho+}) \cos(\beta \Phi_{\sigma-}). \quad (3.68)$$

Apart from $\Phi_{\sigma\pm}$ and $\theta_{\rho-}$, at half-filling, also $\Phi_{\rho+}$ appears in a cosine and is pinned. Therefore, all spin and charge modes are gapped. The LLPs are [use Eqs. (3.58) and (3.59)]

$$K_{\rho\pm} = \frac{\pi v_1 - (c_{11}^0 \pm f_{12}^0)}{\pi v_1 + (c_{11}^0 \pm f_{12}^0)} \quad \text{and} \quad K_{\sigma\pm} = \frac{4\pi v_1 + (c_{11}^0 \pm f_{12}^0)}{4\pi v_1 - (c_{11}^0 \pm f_{12}^0)} \quad (3.69)$$

From the ratios (3.64) and (3.65), we find that $K_{\rho+} < 1, K_{\rho-} > 1$, and $K_{\sigma\pm} < 1$, i.e., the values of the LLPs are in agreement with our "quasiclassical" determination of the pinned fields, see the discussion of the quantum sine-Gordon model in chapter 2. A $K_\alpha < 1$ suppresses fluctuations of the field

\(^3\)In fact, the low-energy Hamiltonian is equivalent to the (integrable) SO(8) Gross-Neveu model, see Ref. [80].
3.3. CONCLUSIONS AND DISCUSSION

$\Phi_\alpha$ and enhances fluctuations of the dual field $\theta_\alpha$, while for $K_\alpha > 1$, it is vice versa.

Since all fields are pinned, the half-filled two-leg ladder is a Mott insulator with a spin-gap. Note that the insulating behavior is due to the pinning of $\Phi_{\mu+}$ and $\theta_{\mu-}$, i.e., the phase coherence is already present at half-filling. For this reason, doping the Mott insulating phase, a $d$-wave superconductor appears.

3.3 Conclusions and discussion

In this chapter, we have discussed the weakly interacting two-leg ladder. We have shown that a finite interchain hopping $t_\perp > U$ has the effect of driving the spinless two-leg ladder to a superconducting phase, where left (right) going particles in chain one are paired with right (left) going particles in chain two. The superconducting correlations are largest when the chemical potential is close to the bottom of the antibonding band. Between the superconducting phase at lower and the CDW phase at higher doping, we have found a (new) phase with coexistence of SC and CDW excitations. On the one hand, superconducting correlations are enhanced by the velocity difference $v_1 - v_2 = 2t_\perp / \tan(\pi n)$, but on the other hand, this difference suppresses the coherence between the two bands, when the chemical potential is too close to the bottom of the antibonding band.

Interestingly, the spin-1/2 two-leg ladder has a similar phase diagram as the HTSCs. It is a Mott insulator at half-filling and a $d$-wave superconductor upon hole doping. The key difference is, that the Mott insulator in the two-leg ladder has a spin-gap, while the Mott insulating phase in the HTSCs exhibits two gapless magnon-modes. In the next chapter, we study in detail how the spin-gap vanishes as the number of coupled chains $N$ increases.
Chapter 4

Half-filled $N$-leg ladders

This and the following chapter contain the main part of the thesis. We derive the physical properties of the half-filled and lightly doped, weakly interacting $N$-leg Hubbard ladders, where we use a combination of RG and bosonization techniques. In this chapter, we investigate the half-filled situation. First (Sec. 2), we show that the ladders exhibit in the groundstate an odd-even effect; even-leg ladders are ISLs and odd-leg ladders have one gapless spinon mode — similar to the strongly interacting case, the Heisenberg spin-ladders. We obtain an effective low-energy Hamiltonian which allows us in the next chapter to study the effect of doping away from half-filling, i.e., doping is included as a perturbation (Ledermann, Le Hur, and Rice, Ref. [86]).

In Sec. 3, we treat the crossover from 1D to 2D as $N \to \infty$ (Ledermann, Ref. [87]). From RG studies of the 2D case, the half-filled 2D Hubbard model is expected to be an antiferromagnetic (AFM) Mott insulator with two gapless magnon modes [34, 35, 36], i.e., the spin-gap present in even-leg ladders has to vanish for increasing $N$. We find that the ladders exhibit an odd-even effect (and therefore a spin-gap) only below a crossover energy $E_c \sim t \exp[-a \exp(bN)]$ ($a \ll 1$ and $b \sim 1$); in contrast, above $E_c$, the ladders have the same physical properties as associated with the 2D AFM. In Sec. 4, we investigate in detail the phase present above $E_c$. We obtain an analytical expression for the Hamiltonian where the interacting part is similar to the spin-1/2 Heisenberg case. Since $E_c(N) \to 0$ for $N \to \infty$, the groundstate in the 2D limit is an AFM Mott insulator, in agreement with previous works.
4.1 Hamiltonian

The $N$-leg ($N$-chain) Hubbard model is given by $H = H_0 + H_{\text{int}}$, where the kinetic energy is

$$H_0 = -t \sum_{x,i,s} d_{is}^\dagger(x + 1) d_{is}(x) + \text{H.c.} - t_\perp \sum_{x,i,s} d_{i+1s}^\dagger(x) d_{is}(x) + \text{H.c.}$$

(4.1)

and the interaction term is

$$H_{\text{int}} = U \sum_{x,i} d_{is}^\dagger(x) d_{i\dagger}(x) d_{is}^\dagger(x) d_{i\dagger}(x).$$

(4.2)

The hopping matrix elements along and perpendicular to the chains are denoted by $t$ and $t_\perp$ respectively and $U > 0$ is the on-site repulsion. We proceed in the same way as for the two-leg ladder; we first diagonalize $H_0$ by a canonical transformation. For open boundary conditions$^1$ perpendicular to the legs, the transformation is

$$d_{is} = \sum_m \sqrt{\frac{2}{N+1}} \sin \left( \frac{\pi m i}{N+1} \right) \Psi_{ms},$$

(4.3)

where the $\Psi_{js}$ are the annihilation operators for the band $j$. We then find a $N$-band model (see Fig. 4.1)

$$H_0 = \sum_{j,s} \int dk \epsilon_j(k) \Psi_{js}^\dagger(k) \Psi_{js}(k),$$

(4.4)

where the dispersion relations $\epsilon_j$ are given by

$$\epsilon_j(k) = -2t \cos(k) - 2t_\perp \cos \left[ \pi j/(N+1) \right].$$

(4.5)

The Fermi momenta in each band $k_{Fj}$ are determined by the chemical potential $\mu = \epsilon_j(k_{Fj})$ and the filling $n = 2(\pi N)^{-1} \sum k_{Fj}$. Since we are only interested in the low-energy physics, we linearize the dispersion relation at the FS, resulting in Fermi velocities $v_j = 2t \sin(k_{Fj})$. We introduce operators $\Psi_{R/Ljs}$ for right and left movers at the FS. At half-filling, $\mu = 0$, and

$$v_j = v_g = 2\sqrt{t^2 - t_\perp^2 \cos[\pi j/(N+1)]^2},$$

(4.6)

$^1$Periodic boundary conditions lead to frustration effects and the odd-even effect vanishes.
Figure 4.1: Band structure of the (noninteracting) 4-leg ladder at half-filling. The bands are filled up to the chemical potential $\mu$. The dispersion relation is then linearized at the Fermi momenta $k_{Fj}$. At half-filling, the Fermi momenta of band pairs $(j, \tilde{j})$ add up to $\pi$ allowing interband umklapp processes to take place.

where $\tilde{j} = N + 1 - j$. For $t = t_\perp$, $v_1 = v_N \approx 2\pi t/N$, which leads to a singular behavior for large $N$ (the quasi-1D analog of the van Hove singularities in 2D). For the following, we take $t_\perp \neq t$ in order to avoid singularities. Note that the FS is then not flat [88], but that we still have nesting, i.e., $k_{Fj} + k_{F\tilde{j}} = \pi$. We will call the bands $(j, \tilde{j})$ band pairs. The Fermi velocities fulfill

$$v_1 = v_N < v_2 = v_{N-1} < \ldots$$

These particular values of the velocities will lead to a hierarchy of energy-scales (see below).

The crucial difference between quasi-1D and 2D are the interactions which control the low-energy physics. The system is quasi-1D and only a finite number of different interactions plays a role, provided the energy difference between two neighboring bands is larger than the largest energy-scale of the system, i.e., $te^{-t/U}$. Therefore, we investigate both the groundstate and the crossover between different phases (ISL and AFM) for $U \ll t/\ln N, t_\perp/\ln N$. The 2D limit ($N = \infty$) is then taken within the same phase.
CHAPTER 4. HALF-FILLED N-LEG LADDERS

Since we are interested in the small-$U$ low-energy physics, we have to take into account only the processes, which are present down to arbitrarily small energy-scales. For the half-filled case, this implies that only umklapp processes where the momenta add up exactly to $\pi$, $k_{Fj} + k_{Fm} = \pi$ are relevant. This condition is only fulfilled for the band pairs $(j,j)$, $k_{Fj} + k_{Fj} = \pi$. When doping away from half-filling, the chemical potential introduces a low-energy cutoff for the various umklapp processes. The interactions which have to be taken into account initially are then different.

At half-filling, the interacting part of the Hamiltonian consists of forward ($f$), Cooper ($c$), and umklapp ($u$) processes within a band and between 2 different bands and (non-)umklapp processes between 4 (or 3, for $N$ odd) different bands $H_{\text{int}} = H^{2B} + H^{4B}$. The two-band part, $H^{2B}$, takes the same form as in the half-filled two-leg ladder (for a comparison, see chapter 3)

$$H^{2B} = \sum_{j=1}^{N/2} \int dx \left[ u^p_{j,i,j} \left( I^f_{Rj} I^f_{Lj} + \text{H.c.} \right) - u^p_{j,i,j} \left( I^t_{Rj} \cdot I^t_{Lj} + \text{H.c.} \right) + u^u_{j,i,j} \left( I^t_{Rj} I^u_{Lj} + I^u_{Rj} I^u_{Lj} + \text{H.c.} \right) \right]$$

$$+ \sum_{i \neq j}^{N} \int dx \left( f^p_{ij} J_{Rii} J_{Lij} - f^p_{ij} J_{Rii} \cdot J_{Lij} + c^p_{ij} J_{Rij} J_{Lij} - c^p_{ij} J_{Rij} \cdot J_{Lij} + \text{H.c.} \right)$$

$$+ \sum_{j=1}^{N} \int dx \left( c^p_{ij,j} J_{Rjj} J_{Ljj} - c^p_{ij,j} J_{Rjj} \cdot J_{Ljj} \right). \quad (4.8)$$

For odd $N$, there is 1 umklapp term for the band $r = (N + 1)/2$,

$$u^u_{rr} (I^t_{Rrr} I^t_{Lrr} + \text{H.c.}). \quad (4.9)$$

The 4-band (3-band) interactions combine processes between two different band pairs $(j,j)$ and $(k,k)$,

$$H^{4B} = \sum_{j=1}^{N/2-1} \sum_{k=j+1}^{N-j} \int dx H^4_{jk}, \quad (4.10)$$

where

$$H^4_{jk} = c^p_{j,k,j} \left( J_{Rjk} J_{Lkj} + J_{Rjk} J_{Lkj} + \text{H.c.} \right) - c^p_{j,k,j} \left( J_{Rjk} \cdot J_{Lkj} + J_{Rjk} \cdot J_{Lkj} + \text{H.c.} \right)$$
4.2 Groundstate

Using the RG method (and bosonization), we derive the groundstate properties. We have generalized the RGEs given in Refs. [61, 80] in order to treat ladders with \( N > 2 \) legs at half-filling, see Appendix A.

Integrating the RGEs, we find, that the Fermi velocities \( v_3 \) [see Eqs. (4.6) and (4.7)] lead to a hierarchy of energy-scales by \( E_3 \sim t e^{-\alpha_1 n/U} \), where the band pairs \((j, j)\) become successively frozen out \((\alpha \sim 1)\). As we show below, the low-energy Hamiltonian is then the sum of \( N/2 [\{N - 1\}/2 \) for \( N \) odd\] two-leg ladder Hamiltonians corresponding to the band pairs \((j, j)\) (plus the Hamiltonian of a single chain for \( N \) odd).

4.2.1 Three-leg ladder

It is instructive to start with the RG flow of the 3-leg ladder in the limit \( v_1 = v_3 \ll v_2 \) (i.e., \( t_{11}/t \rightarrow \sqrt{2} \)), where we can neglect in the RGEs all contributions, where contractions over the band 2 take place. We then obtain the following results. First, the renormalization of the two and single-band interactions of the bands 1 and 3 is exactly the same as for a two-leg ladder, where the couplings flow towards universal ratios (for a comparison, see chapter 3)

\[
\begin{align*}
t &\sim g_{13} = 4c_{13}^g = 4f_{13}^g = c_{13}^a = -c_{11}^a = -c_{33}^a \\
&= 4u_{1331}^g = 8u_{1133}^g = u_{1331}^g,
\end{align*}
\]

and

\[
c_{11}^a = c_{33}^a \approx f_{13}^a \approx 0,
\]

and become of the order of the bandwidth \( t \) at the energy-scale \( E_1 \sim t e^{-\alpha_1 n/U} \) \((\alpha_1 \) is a constant of the order of 1). Second, the interactions between the
bands 1 and 2 (and 2 and 3) are either not renormalized or flow to 0. Finally, the scaling of the remaining 3-band interactions can be calculated as follows (note that \( u_{2213} \) stays always small). Defining \( \mathbf{v} = (c_{1223}, c_{2123}, u_{1223}, u_{2123}) \) and

\[
M = \begin{pmatrix}
2 & 0.75 & 3 & -0.75 \\
4 & 0 & 4 & -1 \\
3 & 0.75 & 2 & -0.75 \\
-4 & -1 & -4 & 0
\end{pmatrix},
\]

we rewrite the RGEs as (the scaling variable \( \mathbf{v} \) is related to the energy by \( E \sim t e^{-\pi l} \))

\[
4(v_1 + v_3) \frac{dv}{dl} = g_{13}(l)Mv,
\]

where, using Eq. (4.12), we have expressed the interactions of band 1 and 3 in terms of \( g_{13} \). Diagonalizing \( M \) allows us to solve Eq. (4.15). The eigenvalues of \( M \) are \( (7, -1, -1, -1) \); the 3-band interactions therefore stay in fixed ratios and only combinations of 3-band couplings belonging to the eigenvalue 7 are growing. Next, we integrate the decoupled differential equation (4.15); from the RGE for \( g_{13} \), we find

\[
\frac{1}{4(v_1 + v_3)} \int_0^l g_{13}(l')dl' = \frac{1}{12} \ln \left( \frac{g_{13}(l)}{g_{13}(l_0)} \right),
\]

where \( l_0 < l \) denotes the scale at which the interactions of the bands 1 and 3 are sufficiently close to the two-leg ladder ratios. We then obtain

\[
\frac{c_{1223}(l)}{g_{13}(l)} = \frac{c_{1223}(l_0)}{g_{13}(l_0)} \left[ \frac{g_{13}(l_0)}{g_{13}(l)} \right]^{5/12} \propto \left( \frac{U}{g_{13}(l)} \right)^{5/12}
\]

such that the 3-band interactions are arbitrary small at the energy-scale where \( g_{13} \sim t \).

For other ratios of the velocities \( v_1/v_2 < 1 \), we have performed a numerical integration of the RGEs: Plotting the strong coupling value as a function of the initial-value \( U/t \) (we fix the strong coupling value of \( c_{12}^0 \) at the bandwidth \( t \)), we find that the couplings given in Eq. (4.12) always grow up to the bandwidth \( t \), approaching their universal ratios, while all the other couplings stay an order of magnitude smaller, i.e., the asymptotic behavior of the couplings for \( U/t \to 0 \) is the same as for \( v_1 \ll v_2 \) (see Fig. 4.2). We note that in Ref. [60] a flow to strong coupling of the 3-band interactions has been
4.2. GROUNDSTATE

Figure 4.2: Strong-coupling values as a function of the initial-value $U/t$ (for $t_\perp/t = 0.95$). The 3-band coupling $c_{1223}^\sigma$ decreases to 0 (the flow of the other 3-band couplings is similar), as the initial-value is reduced; the couplings $c_{13}^\sigma$ and $f_{13}^\sigma$ approach the same values as in the half-filled two-leg ladder (for a comparison, see Fig. 3.3).

found for $t_\perp/t < 0.86$; we only obtain that for $v_1 \approx v_2$, i.e., in the strongly anisotropic limit $t_\perp/t < 0.2$.\footnote{The origin of this discrepancy is unclear at present; the author of Ref. [60] has not published his RGEs.}

To conclude our RG analysis of the 3-leg ladder: The couplings of the bands 1 and 3 scale towards the two-leg ladder ratios, become of the order of the bandwidth $t$ at the energy-scale $E_1 \sim t e^{-\alpha_1 v_1/U}$, and at (and below) $E_1$, the bands 1 and 3 are decoupled from the band 2. The Hamiltonian becomes therefore $H = H_{13} + H_s$. The first term, $H_{13}$, is the Hamiltonian of a two-leg ladder (see chapter 3 and Ref. [80]) and the second term, $H_s$, is the Hamiltonian of a single "chain". At the scale $E_1$, the couplings of $H_{13}$ are of the order of $t$ and all the charge and spin-modes of the bands 1 and 3 acquire a gap. The couplings of $H_s$ are still small at $E_1$, i.e., $0 < c_{22}^\sigma, u_{22}^\sigma \ll t$, such that the charge and spin-modes of band 2 remain gapless.

For a (small but) finite $U$, we then have to investigate the system at
energies below $E_1$. The single chain, $H_s$, has at half-filling a charge gap below an energy $E_2 \sim t e^{-\alpha_2 \nu_2 / U}$ (since $\alpha_2 \approx \alpha_1$ we write in the following $\alpha$ for both of them), but the spin-excitations remain gapless [69]. Since the two-leg part has no gapless mode, the final phase is the same as for the strongly interacting limit, i.e., a 3-leg spin-ladder.

### 4.2.2 Four-leg ladder

For the 4-leg ladder, we find a similar behavior. The couplings of the bands 1 and 4 grow, become of the order of the bandwidth $t$ at the energy-scale $E_1 \sim t e^{-\alpha_1 / U}$, and stay in the same ratio as for a two-leg ladder, while all the other couplings remain small, see Fig. 4.3. In the limit $v_1 = v_4 \ll v_2 = v_3$, the RGEs of the 4-band couplings become the same as for the 3-band couplings and consequently, similarly as above, the 4-band interactions stay in fixed ratios and

$$c^\sigma_{1234}/c^\sigma_{14} \propto (U/c^\sigma_{14})^{5/12}.$$  \hspace{1cm} (4.18)
4.2. GROUNDSTATE

At the energy-scale $E_1$, the bands 1 and 4 are therefore decoupled from the bands 2 and 3. Again, for a finite $U$, we then have to study the flow of the couplings of band 2 and 3 at energies below $E_1$. We find that they flow at the scale $E_2 \sim t e^{-\alpha_{2}/U}$ also to a two-leg ladder fixed point (the flow to the two-leg ladder fixed point takes place for a wide range of initial-values, also for non-Hubbard-like). The (bosonized) Hamiltonian is then the sum of two two-leg ladder Hamiltonians, $H = H_{14} + H_{23}$. All charge and spin excitations are therefore gapped (as it is the case in the Heisenberg 4-leg spin ladder).

4.2.3 The case $N > 4$

Analyzing the RGEs for $N > 4$, we conjecture that this successive freezing out of band pairs $(j, j)$ (plus a single band for $N$ odd) at the characteristic energies $E_j \sim t e^{-\alpha_j/U}$ takes place also for $N > 4$. Again, the $E_j$ are a result of the one-loop RGEs: the couplings of the band pair $(j, j)$ scale towards a two-leg ladder fixed point and become of the order of the bandwidth $t$ at $E_j$.

Note that we have a hierarchy of energy-scales

$$E_1 > E_2 > \ldots > E_r,$$  \hspace{1cm} (4.19)

where for $N$ even $r = N/2$ and for $N$ odd $r = (N + 1)/2$.

To conclude this section: In the groundstate, the Hubbard-ladder Hamiltonian becomes for $N$ even the sum of $N/2$ two-leg ladder Hamiltonians and for $N$ odd the sum of $(N - 1)/2$ two-leg ladder Hamiltonians plus the Hamiltonian of a single chain

$$H = \sum_j H_{jj} + \delta_{N,\text{odd}} H_s,$$  \hspace{1cm} (4.20)

where for $N$ odd, the single chain [the band $(N+1)/2$] has the lowest energy-scale and for $N$ even the two-leg ladder Hamiltonian $H_{N/2, N/2+1}$ [corresponding to the band pair $(N/2, N/2+1)$]. The (half-filled) two-leg ladder Hamiltonians have no gapless excitations and the single-chain Hamiltonian present for odd $N$ has only one gapless spinon-mode. The phases at half-filling are thus the same as for the Heisenberg spin-ladders \cite{52}. Since the Hubbard model converges for large $U$ onto the $t$-$J$ model (which is at half-filling equivalent to the Heisenberg model), the phases of the half-filled $N$-leg Hubbard ladders are the same for small and large $U$.

We like to emphasize that we have obtained the final result, Eq. (4.20), describing \textit{decoupled band pairs}, from an explicit analysis of the \textit{coupled} RGEs of the \textit{interacting} $N$-band problem.
4.3 Dimensional crossover

In this section, we investigate how the half-filled, weakly interacting $N$-leg Hubbard ladders evolve towards the 2D case as $N \to \infty$. We do all our calculations for even $N$ and only refer to the odd $N$ case, where it is of interest. Using the RG method, we study the ladders (for $N$ finite) as a function of the energy-scale. We show that there exist two clearly different phases, separated by a crossover energy $E_c = E_c(N) \ll t$. The phase below $E_c$ is an ISL, i.e., 4-band-AFM processes are suppressed and the physics is dominated by 2-band umklapp and Cooper processes. In contrast, in the phase above $E_c$ (but still below the bandwidth $t$) the 4-band-AFM processes are large and dominating.

![Figure 4.4: Strong-coupling value of the ratio of a 4-band-AFM and a 2-band Cooper coupling, $c_{2N-1,N}^c/c_{1,N}^c$, as a function of the initial-value $U/t$. For increasing $N$, the 4-band-AFM couplings are only suppressed for very small initial-values $U/t$ ($t_{1}/t = 0.95$).](image-url)
4.3. DIMENSIONAL Crossover

4.3.1 ISL vs. AFM

As we have seen in the previous section, for a finite and even \( N \), the ground-state at half-filling is an ISL. Integrating the RGEs (see Appendix A), we find, that the 4-band AFM processes are for increasing \( N \) suppressed for subsequently smaller initial-values \( U/t \) (respectively energy-scales \( t e^{-t/U} \)), see Fig. 4.4. In other words, above a certain energy-scale \( E_c = E_c(N) \) (but still below the bandwidth \( t \)), where \( E_c(N) \to 0 \) for \( N \to \infty \), the 4-band AFM processes dominate and determine the physical properties. As we show in the following, the flow of the LLP for spin-triplet pairing, \( K_{\sigma j^+} \), can be used to determine this crossover energy.

For the half-filled \( N \)-leg ladder, the noninteracting Hamiltonian including density-density interactions takes the form

\[
\hat{H}_0 = \frac{N}{2} \sum_{j=1}^{N/2} \sum_{\nu=\rho,\sigma} \sum_{\mu=\pm} \int dx \frac{u_{\nu j\mu}}{2} \left[ \frac{1}{K_{\nu j\mu}} (\partial_x \Phi_{\nu j\mu})^2 + K_{\nu j\mu} \Pi_{\nu j\mu}^2 \right],
\]

where the LLPs are given by Eq. (3.69), generalized to \( N \) bands,

\[
K_{\rho j^\pm} = \sqrt{\frac{\pi v_j - (c_{jj}^\rho \pm f_{jj}^\rho)}{4\pi v_j + (c_{jj}^\rho \pm f_{jj}^\rho)}} \quad \text{and} \quad K_{\sigma j^\pm} = \sqrt{\frac{4\pi v_j + (c_{jj}^\sigma \pm f_{jj}^\sigma)}{4\pi v_j - (c_{jj}^\sigma \pm f_{jj}^\sigma)}}.
\]

The LLPs determine the exponents of correlation functions and — together with cosine-terms — the pinned fields (see chapters 2 and 3). Note that initially (i.e., at high energies) \( K_{\sigma j^+} > 1 \) and \( K_{\sigma j^-} = 1 \) (and \( K_{\rho j^+} < 1, \ K_{\rho j^-} = 1 \)). For the \( N \)-leg ladders, the RG flow of the couplings is such that in the groundstate \( K_{\sigma j^\pm} < 1 \) [use Eq. (4.12)]. Hence, while \( K_{\sigma j^-} \) is always \( \leq 1 \), there is an energy, where \( K_{\sigma j^+} \) becomes smaller than 1, i.e., \( r_j = c_{jj}^\sigma + f_{jj}^\sigma \leq 0 \). The canonical transformation

\[
\Phi_{\sigma jj^+} = \Phi_{\sigma jj^+}/\sqrt{K_{\sigma j^+}}, \quad \bar{\Phi}_{\sigma jj^+} = \bar{\Phi}_{\sigma jj^+}/\sqrt{K_{\sigma j^+}}
\]

eliminates \( K_{\sigma j^+} \) from \( \hat{H}_0 \) (it just renormalizes the velocities) and results (for the interacting part) in

\[
-\cos \left( \sqrt{4\pi K_{\sigma j^+}} \bar{\Phi}_{\sigma jj^+} \right) \times \text{other cosine terms}.
\]

A large \( K_{\sigma j^+} \) therefore tends to depin the field \( \Phi_{\sigma jj^+} \), while a small \( K_{\sigma j^+} \) results in a strong pinning of the field \( \bar{\Phi}_{\sigma jj^+} \) (for the dual field \( \bar{\Phi}_{\sigma jj^+} \), it is
The initial-values which lead to $r_1 < 0$ (i.e., an ISL) become very small as $N$ becomes large ($t_\perp / t = 0.95$); for a comparison, see Fig. 4.4.

The crossover between the two regimes takes place at $K_{\sigma_j^+} = 1$ (since we are at finite energy, it is a crossover and not a transition). The spin-gap in the ISL phase therefore requires $r_j < 0$ ($K_{\sigma_j^+} < 1$) and $r_j$ can be used in the RG flow to determine a crossover energy: For $r_j < 0$ the band pair $(j, j)$ is an ISL and for $r_j > 0$ a 2D-like AFM. We note that for the pure sine-Gordon model (at zero temperature), the value $K_\alpha = 1$ rigorously separates the massive from the massless phase, see chapter 2.

### 4.3.2 Determination of the crossover energy

Performing a numerical integration of the RGEs, we calculate the crossover energy $E_c = E_c(N)$, which separates ladder-like from 2D-AFM-like behavior.

In Fig. 4.5, we have plotted the strong-coupling value of $r_1 = c_{11}^\sigma + f_{1N}^\sigma$ for $N = 4, 8, 12, 16$ (we fix the strong-coupling value of $c_{1N}^\sigma$ at the bandwidth $t$). We take $t_\perp / t = 0.95$, because for $t_\perp / t = 1$, the band pair $(1, N)$ is close to
the van Hove singularities, such that its behavior would no more be typical for the system. For large $N$, the $r_1$ becomes only negative when $U/t$ is very small (for a comparison, see Fig. 4.4). Note that for the other band pairs $(j, j)$, the corresponding energy-scales are lower, since the velocities $v_j$ are larger and the gaps are of the order of $t \exp(-\alpha_j v_j/U)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure4.6}
\caption{The sum of couplings $r_j = c_{jj}^2 + f_{jj}^2$ allows to determine between ISL and AFM: For $r_j < 0$ the band pair $(j, j)$ is an ISL and for $r_j > 0$ an AFM. We have plotted the initial-value $U_c/t$, which leads to $r_1 \approx 0$ at the energy-scale, where the first coupling of the band pair $(1, N)$ has grown up to the bandwidth $t$. The figure shows, that $-\log U_c(N)/t$ is in good approximation a linear function of $N$ (the squares are the calculated initial-values and the lines are a guide to the eye).

Hence, both the 4-band-AFM processes and the $r_j$ have a strong and similar dependence on the energy-scale. For small enough initial-values $U/t$, the differences between the Fermi velocities lead to a decoupling into band pairs. However, since $v_j - v_{j+1} \propto 1/N$, for increasing $N$, this decoupling becomes suppressed and the physics is dominated by 4-band-AFM processes.

We calculate the crossover energy $E_c$ as follows. We determine initial-
values $U_c = U_c(N)$ such that $r_i \approx 0$ at the scale where the first coupling of the band pair $(1, N)$ has grown up to the bandwidth $t$. For initial-values $U < U_c(N)$, a spin-gap then opens, starting in the band pair $(1, A')$, and the 4-band-AFM processes become suppressed. We find that $U_c(N)$ can be fitted by $U_c(0) \exp(-bN)$, where $b \approx 1$ (for $t_\perp/t = 0.8, 0.85, 0.9, 0.95$, we obtain $b = 3.7, 3.2, 2.6, 1.9$) and $U_c(0) \gg t$, see Fig. 4.6. On the other hand, an initial-value $U$ corresponds to an energy-scale (gap) $t \exp(-t/U)$. The crossover energy $E_c$ is therefore given by

$$E_c \sim t \exp[-a \exp(bN)],$$

where $a \approx t/U_c(0) \ll 1$. Note that the energy $E_c$ is also the upper limit for the spin-gap in the ISL phase. For large $U$ (Heisenberg AFM), the corresponding scale (spin-gap) is $J \exp(-0.68N)$, where $J = 4t^2/U$ (see also table 4.1) [68]. While for large $U$ the spin-gap decreases exponentially, in the small-$U$ case the decrease is double-exponentially.

As a result, the “phase” of the 2D system is for finite $N$ present above an energy-scale $E_c = E_c(N)$, where $E_c(N) \to 0$ for $N \to \infty$. Equivalently, below the length-scale $\xi_c \propto 1/E_c$, the dominating correlation functions are the same as in the 2D system.

### 4.4 2D-like AFM phase

First, using the RG method, we calculate the asymptotic ratios of the couplings and the charge-gap of the phase present above $E_c$. Next, we show that the phase above $E_c$ is the quasi-1D counterpart of a 2D insulating AFM. In particular, there is no odd-even effect and the interacting part of the Hamiltonian is similar to the 2D Heisenberg AFM. Using bosonization techniques, we obtain, that the system is Mott insulating and that the insulator is of the same type as in the half-filled two-leg ladder. For the spin-sector, we find the same physical properties as for the Heisenberg AFM, i.e., two gapless magnon modes, spinon confinement, and long-range order in the 2D limit.

#### 4.4.1 Asymptotic ratios of the couplings

For large $N$, the RGEs given in Appendix A reduce to a more simple system and can be solved analytically (see below). We then determine the asymp-


4.4. 2D-LIKE AFM PHASE

Asymptotic ratios of the couplings for Hubbard initial-values and the size of the charge-gap.

Keeping in the RGEs only sums over \( N \) products of couplings and dropping the (2-band) contributions of the order of 1, the RGEs of the 4-band interactions are given by a sum over \( N/2 \) products of 4-band couplings

\[
\frac{dc^\sigma_{j\tilde{k}k\tilde{j}}}{dl} = \sum_{i \neq j,k} \frac{1}{v_i} \left( c^\sigma_{jiy}c^\sigma_{kiy} + u^\rho_{jiy}u^\rho_{kiy} + \frac{3}{16} c^\sigma_{jiy}c^\sigma_{kiy} + \frac{3}{16} u^\sigma_{jiy}u^\sigma_{kiy} \right)
\]

\[
\frac{du^\rho_{j\tilde{k}k\tilde{j}}}{dl} = \sum_{i \neq j,k} \frac{1}{v_i} \left( c^\rho_{jiy}u^\rho_{kiy} - \frac{3}{16} c^\sigma_{jiy}u^\sigma_{kiy} + k \leftrightarrow j \right)
\]

\[
\frac{dc^\sigma_{j\tilde{k}k\tilde{j}}}{dl} = \sum_{i \neq j,k} \frac{1}{v_i} \left( -\frac{1}{2} c^\sigma_{jiy}c^\sigma_{kiy} + c^\sigma_{jiy}c^\sigma_{kiy} + c^\sigma_{kiy}c^\sigma_{jiy} + \frac{1}{2} u^\sigma_{jiy}u^\sigma_{kiy} - u^\sigma_{jiy}u^\sigma_{kiy} - u^\sigma_{kiy}u^\sigma_{jiy} \right)
\]

\[
\frac{du^\sigma_{j\tilde{k}k\tilde{j}}}{dl} = \sum_{i \neq j,k} \frac{1}{v_i} \left( -\frac{1}{2} c^\sigma_{jiy}u^\sigma_{kiy} + u^\sigma_{jiy}c^\sigma_{kiy} - u^\sigma_{jiy}c^\sigma_{kiy} + k \leftrightarrow j \right), \quad (4.26)
\]

where the energy-scale is related to \( l \) by \( E \sim t e^{-\pi l} \). Defining

\[
h^\pm_{jk} = \frac{c^\sigma_{j\tilde{k}k\tilde{j}}}{4} + \frac{u^\rho_{j\tilde{k}k\tilde{j}}}{4} \pm \left( c^\rho_{j\tilde{k}k\tilde{j}} - u^\rho_{j\tilde{k}k\tilde{j}} \right) \quad (4.27)
\]

we find from the above RGEs, that

\[
\frac{d}{dl} h^\pm_{jk} = \sum_{i \neq j,k} \frac{1}{v_i} h^\pm_{ji} h^\pm_{ki} \quad (4.28)
\]

Given the Hubbard initial-values \( 4c^\rho_{j\tilde{k}k\tilde{j}} = c^\sigma_{j\tilde{k}k\tilde{j}} = 2u^\rho_{j\tilde{k}k\tilde{j}} = 2U/(N + 1) \) and \( u^\sigma_{j\tilde{k}k\tilde{j}} = 0 \), we obtain \( h^\pm_{jk}(l) = 0 \) for all \( l \) (this fixed point is stabilized by the 2-band interactions). Since

\[
\frac{d}{dl} h_{jk} = -\sum_{i \neq j,k} \frac{1}{v_i} \left[ h^-_{ji} h^-_{ki} + (h^+_{ji} + h^-_{ji})(h^+_{ki} + h^-_{ki}) \right] \quad (4.29)
\]

and \( h^-_{jk}(0) > 0 \), the \( h^\pm_{jk} \) flow to 0, \( h^-_{jk}(l) \to 0 \) for increasing \( l \). From the flow of the \( h^\pm_{jk} \), we then calculate the flow of the 4-band couplings and obtain the asymptotic ratios

\[
t \sim 3g_{jk} = 4c^\rho_{j\tilde{k}k\tilde{j}} = 3c^\sigma_{j\tilde{k}k\tilde{j}} = 4u^\rho_{j\tilde{k}k\tilde{j}} = -3u^\sigma_{j\tilde{k}k\tilde{j}} \quad (4.30)
\]
Table 4.1: The table shows the correspondence between crossover energy (spin-gap) and energy-scale of the 2D system for small and large $U$. The exponentially small charge-gap for small $U$ leads to a double exponentially suppressed 1D-2D crossover energy (the result for the spin-gap in the large $U$ case is from Ref. [68]).

<table>
<thead>
<tr>
<th>2D energy-scale</th>
<th>crossover energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U \ll t$</td>
<td>$t \exp(-\lambda t/U)$</td>
</tr>
<tr>
<td>$U \gg t$</td>
<td>$J \exp(-0.68N)$</td>
</tr>
</tbody>
</table>

Note that $u^\rho_{jkk}$ remains always small. The 2-band and single-band interactions are for large $N$ dominantly renormalized by the 4-band interactions. The asymptotic ratios of the 2-band and single-band couplings can therefore be calculated by inserting the ratios (4.30) in the RGEs for these couplings. We then find, that the following couplings of the band pairs $(j,j)$ grow and approach fixed ratios:

$$t \sim 3g_{j3} = 4f_{j3} = 3f_{j3} = 3c_{j3} = 4u^\rho_{j3} = 8u^\rho_{j3} = 3u^\sigma_{j3}.$$ (4.31)

The other 2-band and single-band couplings stay small; in particular the single-band SU(2) processes are small, $|c_{j3}| \ll t$, but they have become attractive, $c_{j3} < 0$.

For bands $k$ and $j$, which are close together on the FS, $k \rightarrow j$, the 4-band coupling becomes the same as the corresponding 2-band couplings, $g_{j3} \approx g_{j3} \approx g_{k3}$. In the limit $N \rightarrow \infty$, all the (2 and 4-band) $g$-couplings take the same value, $g_{j3} = g_{k3}$. The gap $\Delta$ is then similarly calculated as the asymptotic ratios; using that

$$\frac{d}{dt}s_{jk} = \sum_{i \neq j,k} \frac{1}{v_i} s_{ji}s_{ki},$$ (4.32)

where

$$s_{jk} = \frac{c_{jkk}^\rho}{4} - \frac{u^\rho_{jkk}}{4} + c_{jkk}^\rho + u^\rho_{jkk},$$ (4.33)

we find for the scale of divergence

$$l_c = \frac{N + 1}{U \sum_j 1/v_j}.$$ (4.34)
4.4. 2D-LIKE AFM PHASE

In particular, \( l_c = 2t/U \) for \( t_\perp/t = 0 \) and

\[
l_c = \frac{\pi t}{U} \frac{1}{\ln(2N/\pi)} \tag{4.35}
\]

for \( t_\perp \to t \), where \( \ln N < t/U \) for the validity of our calculations. The logarithmic corrections come from the fact that \( v_1 = v_N \sim t/N \) (van Hove singularities). The gap becomes

\[
\Delta \sim te^{-l_c} = te^{-\lambda t/U}, \tag{4.36}
\]

where \( \lambda \) is a function of \( t_\perp/t \) and of the order of 1, \( \lambda \sim 1 \) (see also table 4.1).

As a result, the 4-band AFM processes with a weight \( \propto N \) are responsible for a RG instability at the energy-scale \( \Delta \sim te^{-\lambda t/U} \). The van Hove singularities, see Eq. (4.35), lead to a further increase of the gap-size (but are not the reason for the instability).

4.4.2 Effective low-energy Hamiltonian

We show that above \( E_c \), the Hamiltonian of the half-filled \( N \)-leg Hubbard ladders is similar to the 2D Heisenberg AFM.

At low energies (but above \( E_c \)), the ratios of the couplings are given by Eqs. (4.30) and (4.31), such that the \( U(1) \) and \( SU(2) \) 4-band interactions, see Eq. (4.11), simplify to

\[
3J_{Rjk}J_{Lkj} - 4J_{Rjk} \cdot J_{Lkj} = 2\psi_{Rjs}^\dagger \psi_{Ljs}^\dagger \psi_{Lks}^\dagger \psi_{Rks}^\dagger - 2\psi_{Rjs}^\dagger \psi_{Ljs}^\dagger \psi_{Rks} \psi_{Lks}^\dagger - 4\psi_{Rjs}^\dagger \psi_{Ljs}^\dagger \psi_{Rks}^\dagger \psi_{Lks} \tag{4.37}
\]

(for \( s = \uparrow, \bar{s} = \downarrow \) and vice versa) and

\[
3I_{Rjk}I_{Lkj} + 4I_{Rjk} \cdot I_{Lkj} = 2\psi_{Rjs}^\dagger \psi_{Ljs}^\dagger \psi_{Lks}^\dagger \psi_{Rks}^\dagger - 2\psi_{Rjs}^\dagger \psi_{Ljs}^\dagger \psi_{Rks} \psi_{Lks}^\dagger - 4\psi_{Rjs}^\dagger \psi_{Ljs}^\dagger \psi_{Rks}^\dagger \psi_{Lks} \tag{4.38}
\]

Defining

\[
M_j^p = \psi_{Rjs}^\dagger \psi_{Rjs}^\dagger \psi_{Ljs} \psi_{Ljs}^\dagger + \text{H.c.} \tag{4.39}
\]

the Hamiltonian then takes the form

\[
H = H_0 - \frac{1}{2} \sum_{i,j} \int dx \, g_{ij} M_i \cdot M_j. \tag{4.40}
\]
The 2-band couplings $f_{ij}^{p,p}$ and $u_{ij}^{p,p}$ give the contributions $M_j \cdot M_j$ and the 2-band couplings $c_{ij}^{p,p}$ and $u_{ij}^{p,p}$ lead to the products $M_j \cdot M_j$. The 4-band couplings result in the other products $M_i \cdot M_j$. For $N$ large, all couplings take the same value $g_{ij} = g > 0$ (s-wave AFM).

As a result, all band pairs $(j, j)$ are interacting with each other. In particular, for $N$ odd, the band $r = (N + 1)/2$ is interacting with all other band pairs and there is no qualitative difference between odd and even $N$. This contrasts the ladder-case at energies below $E_c$, where only interactions within the band pairs $(j, j)$ are present, respectively for odd $N$, within the band $r$. This then leads to an odd-even effect, i.e., the band $r$ present only for odd $N$ exhibits a gapless spinon-mode.

It is instructive to Fourier transform $M_i \cdot M_j$,

$$\int dx M_i \cdot M_j = \sum_{k,k',q} \left[ \Psi_{R,i}^\dagger(k) \tau_{ss'}^p \Psi_{L,i'}(k + q) + \text{H.c.} \right] \times \left[ \Psi_{R,j}^\dagger(k') \tau_{ss'}^p \Psi_{L,j'}(k' - q) + \text{H.c.} \right].$$

(4.41)

The Fourier transformed 2D AFM Heisenberg Hamiltonian (large-$U$ limit) takes the form

$$H_J = -J \sum_{k,k',q} \left( e^{iql} + e^{iql} \right) \Psi_{s_1}^\dagger(k) \tau_{s_1 s_2}^p \Psi_{s_1}^\dagger(k + (\pi, \pi) + q)$$

$$\times \Psi_{s_2}^\dagger(k') \tau_{s_2 s_1}^p \Psi_{s_2}^\dagger(k' - (\pi, \pi) - q),$$

(4.42)

where we substituted $q$ by $q + (\pi, \pi)$. Therefore, the interacting part of the Hubbard Hamiltonian (4.40) is basically the Heisenberg Hamiltonian with $k$-values restricted to be close to the umklapp surface (the operators $\Psi_{R,j}$ and $\Psi_{L,j}$ have a momentum difference of $\pi$). Since $U/t$ is small (and therefore the energies low), there is no $q$-dependent coupling in Eq. (4.41). In real space, this corresponds to long-range spin-spin interactions. Note that the generalization/crossover of the Hamiltonian (4.41) to 2D is straightforward. Introducing a sum over $q_y$ is sufficient ($q = q_x$).

To conclude this section: The half-filled $N$-leg Hubbard Hamiltonian — with on-site repulsion between electrons — becomes at energies below the gap $\Delta$ a Hamiltonian with purely long-range AFM spin-spin interactions.

### 4.4.3 Physical properties: Bosonization

Using bosonization techniques, we derive the physical properties of the AFM Hamiltonian, Eq. (4.40).
4.4. 2D-LIKE AFM PHASE

Bosonized Hamiltonian

The noninteracting part including density-density interactions is given by Eq. (4.21). At the AFM fixed-point, the interacting part is as follows. The 4-band interactions take the form,

$$\mathcal{H}_{jk}^{4B} = -g_{jk} (2\mathcal{H}_{S1} + \mathcal{H}_{S2}) + (j \leftrightarrow j', k \leftrightarrow k'). \quad (4.43)$$

The $\mathcal{H}_{S1}$ contains the spin-fields $\Phi_{\sigma jj^-}$ and $\theta_{\sigma jj^-}$,

$$\mathcal{H}_{S1} = \cos[\beta(\Phi_{\sigma jj^-} - \Phi_{\sigma kk^-} + \theta_{\sigma kk^-} - \theta_{\sigma jj^-})], \quad (4.44)$$

where $\beta = \sqrt{\pi}$ and $\mathcal{H}_{S2}$ contains the spin-fields $\Phi_{\sigma jj^+}$ and $\theta_{\sigma jj^-}$,

$$\mathcal{H}_{S2} = \cos[\beta(\Phi_{\sigma jj^-} - \Phi_{\sigma kk^-} + \theta_{\sigma kk^-} - \theta_{\sigma jj^-})] + \cos[\beta(\Phi_{\sigma jj^-} + \theta_{\sigma kk^-} - \Phi_{\sigma kk^-} - \Phi_{\sigma jj^-})]. \quad (4.45)$$

The charge-part, $\mathcal{H}_C$, includes the charge-fields $\Phi_{\rho jj^+}$ and $\theta_{\rho jj^-}$,

$$\mathcal{H}_C = \cos[\beta(\Phi_{\rho kk} + \Phi_{\rho jj} + \theta_{\rho jj^-} - \theta_{\rho kk^-})] + \cos[\beta(\Phi_{\rho kk} + \Phi_{\rho jj} - \theta_{\rho jj^-} - \theta_{\rho kk^-})]. \quad (4.46)$$

In $\mathcal{H}_C$, the first cosine comes from the 4-band non-umklapp interactions (4.37), while the second cosine results from the 4-band umklapp interactions (4.38). The bosonized form of the 2-band non-umklapp terms is

$$\mathcal{H}_{j,j}^{2B} = 2g_{jj} \cos(2\beta\theta_{\rho jj^-}) \left[ \cos(2\beta\Phi_{\sigma jj^+}) - \cos(2\beta\Phi_{\sigma jj^-}) - 2 \cos(2\beta\Phi_{\sigma jj^-}) \right] + 2g_{jj} \cos(2\beta\Phi_{\sigma jj^+}) \cos(2\beta\Phi_{\sigma jj^-}) \quad (4.47)$$

and of the 2-band umklapp terms

$$\mathcal{H}_{j,j}^{2B} = 2g_{jj} \cos(2\beta\Phi_{\rho jj^+}) \left[ \cos(2\beta\Phi_{\rho jj^-}) - \cos(2\beta\Phi_{\rho jj^+}) - 2 \cos(2\beta\Phi_{\rho jj^-}) \right] - 3g_{jj} \cos(2\beta\Phi_{\rho jj^+}) \cos(2\beta\Phi_{\rho jj^-}) \quad (4.48)$$

The LLPs at the AFM fixed-point take the form [use Eqs. (4.22) and (4.31)]

$$K_{\rho j \pm} = \sqrt{\frac{\pi v_j \mp 3g_{jj}/4}{\pi v_j \pm 3g_{jj}/4}} \quad (4.49)$$

(implying $K_{\rho j^+} < 1$ and $K_{\rho j^-} > 1$) and

$$K_{\sigma j \pm} = \sqrt{\frac{\pi v_j \pm g_{jj}/4}{\pi v_j \mp g_{jj}/4}}, \quad (4.50)$$
Table 4.2: The phase at a particular energy-scale is related to the value of the LLPs at this scale. The table shows the LLPs in the insulating AFM, ISL, and superconducting (SC) phase. Note that always $K_{pj-} > 1$ and $K_{pj+} < 1$. The differences are, that $K_{pj+} < 1$ for the AFM and ISL and $K_{pj+} < 1$ for the ISL and the SC.

<table>
<thead>
<tr>
<th></th>
<th>$K_{pj+}$</th>
<th>$K_{pj-}$</th>
<th>$K_{pj+}$</th>
<th>$K_{pj-}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AFM</td>
<td>$&lt;1$</td>
<td>$&gt;1$</td>
<td>$&gt;1$</td>
<td>$&lt;1$</td>
</tr>
<tr>
<td>ISL</td>
<td>$&lt;1$</td>
<td>$&gt;1$</td>
<td>$&lt;1$</td>
<td>$&lt;1$</td>
</tr>
<tr>
<td>SC</td>
<td>$&gt;1$</td>
<td>$&gt;1$</td>
<td>$&lt;1$</td>
<td>$&lt;1$</td>
</tr>
</tbody>
</table>

i.e., $K_{pj+} > 1$ and $K_{pj-} < 1$. Table 4.2 gives an overview about the LLPs in the AFM and ISL phase.

Note that the commutation relation for the field and its dual field hinders the pinning, i.e., the localization of the field $\Phi_\alpha$ and its dual field $\theta_\alpha$ in the minimum of a cosine at the same time.

**Charge-sector**

For the charge-sector, we then find, that the same fields are pinned as in the half-filled two-leg (respectively $N$-leg) Hubbard ladder, i.e., $\Phi_{pj+} \approx 0$ and $\theta_{pj-} \approx 0$, see chapter 3. The type of Mott insulator is thus the same in 1D and 2D. Note that pinning of $\Phi_{pj-}$ instead of $\theta_{pj-}$ leads to another type of insulator. The difference between these two types of insulators becomes physically relevant upon (hole) doping. While the second type most likely becomes a FL, the first type becomes (in case we have a spin-gap) a superconductor, since doping only depins the $\Phi_\rho$-fields but does not destroy the phase coherence, i.e., the $\theta_\rho$-fields remain pinned, see chapter 3.

**Spin-sector**

In contrast, in the spin-sector, both the fields $\Phi_{\sigma+j\pm}$ and the dual fields $\theta_{\sigma+j\pm}$ appear in a cosine, resulting in a competition between different “phases”, i.e., pinning of the field vs. its dual field. However, since $K_{\sigma+j+} > 1$ and $K_{\sigma+j-} < 1$ (for a comparison, see Sec. 4.3), it is more favorable to pin $\theta_{\sigma+j+}$ and $\Phi_{\sigma+j-}$ than the corresponding dual fields.
The physical interpretation is then the following. The pinning of $\Phi_{\sigma jj}$ leads to a spinon confinement, leaving as physical particles spin 1 magnons. Note that only the differences $\theta_{\sigma jj^+} - \theta_{\sigma kk^+}$ appear in $H^B$, such that the total magnon mode(s), given by

$$\theta_T = \sqrt{\frac{2}{N} \sum_{j=1}^{N/2} \theta_{\sigma jj^+}} \quad \text{and} \quad \Phi_T = \sqrt{\frac{2}{N} \sum_{j=1}^{N/2} \Phi_{\sigma jj^+}}, \quad (4.51)$$

remains gapless. The $\pi$ and 0 mode then result from a superposition of left/right going modes along the chains (since we use open boundary conditions, the transverse momentum is always positive).

For the calculation of the spin-spin correlation function, we first rewrite the real-space operators $d_{ij}$ in terms of the band operators $\Psi_{j\sigma}$ and then the band operators in terms of the bose-operators. Using that only products of fields, which contain the pinned charge-fields $\Phi_{\sigma j}^+$ and $\Phi_{\sigma j}^-$ give non-vanishing contributions to the spin-spin correlation function, the real-space spin-operator at the position $(x, i)$ takes the form

$$S_i^p(x) = \frac{1}{2} \sum_j \gamma_{ij} \gamma_{ij} \left( \Psi_{Rj}^\dagger \Psi_{Lj}^\dagger e^{-i(k_F^x + k_F^z)x} + R \leftrightarrow L \right), \quad (4.52)$$

where $\gamma_{jm} = \sqrt{2/(N+1)} \sin[\pi jm/(N+1)]$ [see Eq. (4.3)] and at half-filling, $k_F^x + k_F^z = \pi$. The abelian bosonization scheme used here, breaks non-abelian symmetries, i.e., the SU(2) spin-symmetry is broken down to U(1), see, e.g., Refs. [69, 76]. In our case, only the $x$ and $y$ components of the spin-spin correlation function give then straightforwardly the correct physical result (this “problem” occurs also in a single chain with a spin-gap [69]). The products of fields appearing in Eq. (4.52) are then rewritten in terms of bose-operators according to

$$\Psi_{Rj}^\dagger \Psi_{Lj} \sim e^{i\sqrt{\pi}(-\theta_{\sigma jj^+} - \theta_{\sigma jj^-} + \theta_{\sigma jj^+})}. \quad (4.53)$$

The charge-fields and $\Phi_{\sigma jj^\pm}$ are pinned and can be set to zero. Using that $\theta_{\sigma jj^+} - \theta_{\sigma kk^+} \approx 0$, we express $\theta_{\sigma jj^+}$ in terms of the total spin-mode $\theta_T$, $\theta_{\sigma jj^+} = \sqrt{2/N} \theta_T$. Since $H_0$ is Gaussian in the fields $\Phi_T$ and $\theta_T$, the spin-spin correlation function takes the form (for details about such calculations, see chapter 2)

$$\langle S_i(x) \cdot S_i(0) \rangle \propto (-1)^{i+j} \cos(\pi x)/x^{1/N}, \quad (4.54)$$
where \( x < \xi_c \propto 1/E_c \) (for \( x > \xi_c \) we are in the ISL phase, where the decay is exponentially with coherence length \( \xi_{AF} \sim \xi_c \), respectively, for \( N \) odd, \( \propto 1/x \)).

The energy-scales for charge and spin-excitations are different. The LLPs in the charge-sector, see Eq. (4.49), deviate more from their noninteracting value (= 1) than the LLPs in the spin-sector, Eq. (4.50), i.e.,

\[
|K_{\rho j} - 1| \approx \frac{3g_{ij}}{4\pi v_j}, \quad |K_{\sigma j} - 1| \approx \frac{g_{ij}}{4\pi v_j}.
\]

This implies that the charge-fields are pinned more strongly than the spin-fields and therefore the Mott-gap (we may interpret the RG-scale \( \Delta \) as the charge-gap) is larger than the gap to spin-1/2 (spinon) excitations. The coupling between the different bands makes it difficult to investigate the detailed excitation spectrum, such that we do not go beyond this qualitative result (for the pure sine-Gordon model, there exist rigorous results about the gap dependence on \( K_\alpha \), see Ref. [78]).

### 4.5 Conclusions and discussion

In this chapter, we have investigated the physical properties of half-filled, weakly interacting \( N \)-leg Hubbard ladders. In particular, we have given an analytical derivation of the odd-even effect: In the groundstate, even-leg ladders have a spin-gap and odd-leg ladders one gapless spinon-mode — similar to the strongly interacting limit, the Heisenberg spin-ladders. We have shown, that the spin-gap present in even-leg ladders vanishes double-exponentially as the number of chains \( N \) increases. For large \( N \), we have found an effective Hamiltonian for the half-filled, weakly interacting Hubbard model; generalized to 2D, it reads

\[
H = H_0 - g \sum_{k,k',q} \Psi_{s_1}^\dagger(k) \tau^p_{s_1 s'_1} \Psi_{s'_1}(k + (\pi, \pi) + q) \times \Psi_{s_2}^\dagger(k') \tau^p_{s_2 s'_2} \Psi_{s'_2}(k' - (\pi, \pi) - q),
\]

where \( k \) and \( k' \) are restricted to be close to the 2D FS (i.e., umklapp surface) and \( q \) close to 0. The interacting part is similar to the Hamiltonian of the 2D Heisenberg AFM. Analyzing the 2D-like AFM Hamiltonian by bosonization techniques, we have shown, that the type of Mott insulator is the same as in
the half-filled two-leg ladder, i.e., it is a “disordered” $d$-wave superconductor. It is then not a surprise, that doping this 2D-like AFM phase results in a $d$-wave superconductor, see the next chapter.
Chapter 5

Hole doped $N$-leg ladders

In this chapter, we study the effect of doping away from half-filling. There are three different doping regimes. The lightly doped case can be treated as a perturbation of the half-filled low-energy Hamiltonian (Sec. 1). Similar as at half-filling, we obtain an odd-even effect: even-leg ladders become superconducting (Luther-Emery liquid), while odd-leg ladders are LLs. These were the same phases as have been found in numerical works for the (strongly interacting) $t$-$J$ ladders [55, 56, 57, 58]. Note that in this regime, the pairing is between holes.

For intermediate dopings (Sec. 2), we again investigate the RG flow and show that AFM fluctuations on short length-scales lead to the appearance of superconductivity, where the low-energy Hamiltonian takes the form of a $d$-wave BCS model with pairing between electrons.

When doping increases further, AFM fluctuations vanish and the system becomes a FL; following Ref. [61], this regime is briefly discussed in Sec. 3. Fig. 5.1 gives a schematic overview.

5.1 Lightly doped case: Holes and hole pairs

First, we investigate the doping away from half-filling for the groundstate and show that the hierarchy of energy-scales (4.19) leads to an odd-even effect: even-leg ladders become a Luther-Emery liquid and odd-leg ladders a LL.

We then study the effect of doping on the 2D-like AFM Hamiltonian (4.40), determining the physics above $E_c$. Interestingly, the phase upon doping is the same as below $E_c$; in particular, for $N$ even, a spin-gap opens. We
Figure 5.1: Schematic phase diagram for the (large) N-leg Hubbard ladders (N even). At half-filling, the system is below the energy-scale $E_c = E_c(N)$ an insulating spin-liquid (ISL) and above $E_c$ an insulating, 2D-like, commensurate antiferromagnet (CAFM). Upon doping away from half-filling, the ladders become below the chemical potential $\mu$ (measured from its value at half-filling) a conducting spin-liquid (CSL), i.e., phase coherence is only present within pairs of Fermi momenta $(k_F, \pi-k_F)$ and $(\pi-k_F, k_F)$; above $\mu$, the antiferromagnetic correlations become incommensurate (ICAFM). Upon increasing doping, phase coherence between all Fermi momenta sets in and the system becomes a 2D-like $d$-wave superconductor. For large dopings, the ladders are a FL.

call this phase a conducting spin-liquid (bound hole-pairs). Furthermore, the magnetic correlations become incommensurate.

5.1.1 Hole doping the groundstate

Numerical calculations for the strongly interacting 3 and 4-leg $t$-$J$ ladders have found an odd-even effect upon doping; the lightly doped 3-leg ladder is a LL, while the 4-leg ladder is a 1D superconductor [55, 56, 57, 58]. Previous analytical works for the weakly interacting Hubbard ladders have considered the limit $U \to 0$ first and $\delta > 0$ finite (i.e., neglecting the effect of umklapp interactions). The authors then found rather “exotic” phases upon doping,
5.1. LIGHTLY DOPED CASE: HOLES AND HOLE PAIRS

C2S1 for the 3-leg and C3S2 for the 4-leg ladder (a phase with \( n \) gapless charge and \( m \) gapless spin-modes is denoted as CnSm) [60, 61]. Here, treating the opposite limit \( \delta \to 0 \) and \( U > 0 \) finite, we recover this odd-even effect.

Since the low-energy cutoff (in the RG treatment) is typically of the order of \( te^{-U/2} \), the effect of the umklapp interactions has to be taken into account for low dopings, \( \delta < e^{-U/2} \). It is then advantageous to take the half-filled ladder as a starting point to study the effect of doping. Similarly as done in Refs. [14, 84], we model in the following the doping by adding a chemical potential term \( -\mu Q \) (\( Q \) is the total charge) to the effective (half-filled) Hamiltonian.

In a single chain (LL), a hole is decomposed in two quasiparticles, one with charge but no spin (holon, here the fermionic "kink") and the other with spin but no charge (spinon), see chapter 2. The quasiparticles of a two-leg ladder are bound hole-pairs (singlet of charge 2). At low doping, these hole-pairs behave as hard-core bosons [89], which are equivalent to spinless fermions. First, using bosonization, we show that the lightly doped A-leg ladders can indeed be described by an effective \( \mathcal{N}/2 \)-band [(\( \mathcal{N} + 1 \))/2-band for \( \mathcal{N} \) odd] model of spinless fermions. We then study the case when only one of these bands is doped. Next, we increase the hole doping such that many bands become subsequently doped.

One band (pair) doped

For the following discussion, we use the bosonized form of the groundstate Hamiltonian (4.20). In the bosonization language, the charge-density is expressed as \( \partial_x \Phi_\rho \). This allows us to study doping using an effective Hamiltonian containing only the charge fields \( \Phi_{\rho j} \) and \( \Pi_{\rho j} \). Here, it is more appropriate to use

\[
\Phi_{+j} = \frac{1}{2} (\Phi_{\rho j} + \Phi_{\rho j}) \quad \text{and} \quad \Pi_{+j} = \Pi_{\rho j} + \Pi_{\rho j}
\]

than the symmetric combination. We then introduce doping via a chemical potential term \(-\mu Q\). Dropping bare energy terms (i.e., terms containing fields which remain pinned), the effective Hamiltonian reads

\[
H = \sum_j \int dx \left\{ \frac{u_{\rho j}}{2} \left[ \frac{2}{K_{\rho j}} (\partial_x \Phi_{+j})^2 + \frac{K_{\rho j}}{2} \Pi_{+j}^2 \right] - g_{+j} \cos(\sqrt{\delta_{\rho j} \Phi_{+j}}) - 2\mu \partial_x \Phi_{+j} \right\},
\]

(5.2)
plus for odd $N$ the Hamiltonian of a single chain (dropping the gapless spin-part)

\[
\int dx \left\{ \frac{v_{ps}}{2} \left[ \frac{1}{K_{ps}} (\partial_x \Phi_{ps})^2 + K_{ps} \Pi_{ps}^2 \right] - g_{ps} \cos(\sqrt{8\pi} \Phi_{ps}) - \mu \partial_x \Phi_{ps} \right\}. \tag{5.3}
\]

The coupling $g_{+j}$ is of the order of $g_{jj}$, the $K_\alpha$ are the LLPs, and the $u_{\alpha}$ are the velocities of the charge-modes, see Eq. (4.22). The charge (doping) is given by

\[
Q = \sqrt{\frac{2}{\pi}} \int dx \sum_j 2 \partial_x \Phi_{+j} + \delta_{N,0} \partial_x \Phi_{ps}. \tag{5.4}
\]

Note that the $\Phi_{+j}$-field is multiplied with 2. The fields of the band pair $(j,j)$, $\Phi_{+j}$ (and of the single chain $\Phi_{ps}$), become pinned at the energy-scales $E_j \sim t e^{-\alpha_{jj}/U}$, i.e., the gaps for the band pairs are of the order of $E_j$.

The equations of motion following from the Hamiltonians (5.2) and (5.3) are a set of sine-Gordon equations. Here, the physical solutions of the sine-Gordon equations are the fermionic kink-solutions (see chapter 2). The kinks of the single-chain field $\Phi_{ps}$ have a mass $M_s \sim E_s$ and represent the charge part of single holes, while the kinks of the two-leg ladder fields $\Phi_{+j}$ have a mass $M_j \sim E_j$ and represent paired holes (singlets). The band structure of these kinks reads

\[
\epsilon_j(k) = \sqrt{M_j^2 + (v_j k)^2}. \tag{5.5}
\]

At low doping $\delta < e^{-1/U}$, the half-filled $N$-leg ladders are therefore described by an effective $N/2$-band $[(N+1)/2$-band] model of spinless fermions, where the bottom of the bands are at the energy-scales $E_j$ (for $N = 3$, see Fig. 5.2; the effective bands are labeled with roman letters).

At half-filling, $\mu = 0$ and minimizing the energy gives $\Phi_{+j} \approx \Phi_{ps} \approx 0$. When doping the first hole (pair), the chemical potential $\mu = \mu(\delta)$ jumps from 0 to $E_r$ and a kink-solution with charge $Q > 0$ minimizes the energy, i.e., the lowest-lying effective band of spinless fermions becomes doped. For odd $N$, the lowest-lying band is the single-chain band $(N+1)/2$ and the phase upon doping is a dilute gas of spinons and kinks (LL, C1S1), while for even $N$, the effective band $N/2$, corresponding to the two-leg-ladder band-pair $(N/2, N/2+1)$, is doped first and the phase consists of singlets of paired holes (Luther-Emery liquid, C1S0). Similarly as for the two-leg ladder (see chapter 3), the (singlet) pair-field operator $\Delta_j$, see Eq. (3.62), has a different sign in band $j$ and $\bar{j}$, such that the C1S0 phase has a $d$-wave like symmetry, $\langle \Delta_j^\dagger \Delta_{\bar{j}} \rangle < 0$. 
5.1. LIGHTLY DOPED CASE: HOLES AND HOLE PAIRS

Figure 5.2: The lightly doped 3-leg ladder can be described by a two-band model of spinless fermions, where the lower band II contains the charge part of holes and the upper band I paired holes (singlets). At half-filling the phase is C0S1; upon doping, the charge part of the holes enters the band II and the phase is C1S1. For dopings $\delta > \delta_c$ (here, $\pi \delta_c = |k_c|$), paired holes enter also the band I and the phase becomes C2S1.

It is instructive to rewrite the band operators $\Psi_{hjs}$ respectively the pair-field operators $\Delta_j$ in terms of the chain operators $d_{hjs}$, see Fig. 5.3. For $N = 3$, we have

$$\Psi_{h2s} = \frac{1}{\sqrt{2}} (d_{h1s} - d_{h3s})$$

and we find that the holes are situated on the outer legs. Both the phase and the location of the holes is in agreement with previous numerical treatments [55, 56]. For $N = 4$, using $\Delta_2 \Delta_4' \approx -1$, we obtain

$$\Delta_2 \propto 0.22 (d_{R1\uparrow} d_{L2\downarrow} + d_{R2\uparrow} d_{L1\downarrow} + R \leftrightarrow L)$$
$$+ 0.22 (d_{R3\uparrow} d_{L4\downarrow} + d_{R4\uparrow} d_{L3\downarrow} + R \leftrightarrow L)$$
$$- 0.36 (d_{R1\uparrow} d_{L4\downarrow} + d_{R4\uparrow} d_{L1\downarrow} + R \leftrightarrow L)$$
$$- 0.14 (d_{R2\uparrow} d_{L3\downarrow} + d_{R3\uparrow} d_{L2\downarrow} + R \leftrightarrow L),$$

such that the singlets are on the top two legs, the bottom two legs, on the
Figure 5.3: (a) We find that upon doping away from half-filling for \( N = 3 \), the holes are situated on the outer legs. (b) For \( N = 4 \), we obtain that the singlets are on the top two legs, the bottom two legs, on the legs 1 and 4, and with lowest probability on the legs 2 and 3, similarly as found in numerical works for the \( t\)-\( J \) model.

legs 1 and 4, and with lowest probability on the legs 2 and 3, similarly as found in Ref. [57] for the \( t\)-\( J \) model. Finally, also the phases for \( N = 5, 6 \) are in agreement with numerical works [57]. Since the \( t\)-\( J \) model is the large \( U \) limit of the Hubbard model, we conclude that at and close to half-filling the phases of the \( N\)-leg Hubbard ladders are the same for small and large \( U \). For the two-leg ladder, such a "universal" behavior has already been noted in Ref. [90].

Many bands doped

When increasing doping, the chemical potential \( \mu = \mu(\delta) \) increases too and the hole-pairs enter subsequently also the higher-lying (effective) bands of spinless fermions. We thus obtain a series of (critical) dopings \( \delta_{cj} \), where the first hole-pair enters the band \( j \). For \( N = 3, 4 \), \( \delta_{c1} \) is estimated as

\[
\delta_{c1} \sim \sqrt{e^{-2\alpha v_1/U} - e^{-2\alpha v_2/U}} \approx e^{-\alpha v_1/U}. \tag{5.8}
\]
5.1. LIGHTLY DOPED CASE: HOLES AND HOLE PAIRS

Figure 5.4: The bands of the 3 and 4-leg ladder correspond to 3 respectively 4 different points on a 2D FS (denoted by 1, 2, 3, and 4). The square is the umklapp surface, which is the FS at half-filling (for $t = t_\perp$). The half-circles indicate the opening of the FS, when the first band (pair) is doped. Left (3-leg): Close to the wave vector $(\pi/2, \pi/2)$, the phase is then a LL, while at $(3\pi/4, \pi/4)$ and $(\pi/4, 3\pi/4)$, the phase is an ISL. Right (4-leg): Upon doping, the phase close to the wave vectors $(2\pi/5, 3\pi/5)$ and $(3\pi/5, 2\pi/5)$ is Luther-Emery liquid (LE). At $(4\pi/5, \pi/5)$ and $(\pi/5, 4\pi/5)$, the phase is still an ISL. Similarly as for the two-leg ladder, the order-parameter has a different sign in the Fermi points 2 and 3.

Since for $\delta = \delta_{cij}$, $\partial \mu / \partial \delta = 0$, the compressibility $\kappa \propto \partial \delta / \partial \mu$ diverges. The phase transitions at $\delta_{cij}$ belong to the same universality class as the commensurate-incommensurate transition [91]. We note that our qualitative estimate for $\delta_{cij}$ does not allow for a comparison with numerical works.

The effective band $j$ corresponds to the band pair $(j, \bar{j})$, such that for a given doping, the band pairs $(N/2, N/2 + 1), \ldots, (j, j)$ are conducting, while the band pairs $(j - 1, j + 1), \ldots, (1, N)$ form still an ISL. Next, we interpret this result by mapping each band (pair) on a 2D FS, see Fig. 5.4. Using the dispersion relation (4.5), the longitudinal Fermi momentum of the band $j$ is given by

$$ k_{Fj} = \pi - \arccos \left[ \frac{t_\perp}{t} \cos \left( \frac{\pi j}{N+1} \right) \right] \quad (5.9) $$
and the corresponding transverse Fermi momentum reads

\[ k_{Fy3} = \frac{\pi j}{N + 1} \]  

(5.10)

Upon doping, the holes enter (for \( t_{\perp} \sim t \)) then first near the wave vector \((\pi/2, \pi/2)\) and finally for increasing doping also near \((\pi, 0)\) and \((0, \pi)\). The FS thus becomes truncated.

For increasing doping, a crossover takes place to the situation away from half-filling, where the umklapp processes can be neglected. It is instructive to study the phases for \( N = 3, 4 \), when both effective bands of spinless fermions are doped and to compare with the phase away from half-filling. In the case \( N = 3 \) (see Fig. 5.2), the different charges in the effective bands I and II forbid scattering processes of the form

\[ \Psi_{RI}^{\dagger} \Psi_{RI} \Psi_{LII}^{\dagger} \Psi_{LII} + I \leftrightarrow II, \]  

(5.11)

where \( \Psi_{R/Ib}^{\dagger} \) creates a spinless fermion in band \( b = I/II \), since they break \( U(1) \) invariance and only the interactions

\[ \Psi_{RI}^{\dagger} \Psi_{RI} \Psi_{LII}^{\dagger} \Psi_{LII} + I \leftrightarrow II \]  

(5.12)

are allowed. For weak interactions, we can bosonize Eq. (5.12) resulting in a density term of the form \( \partial_x \Theta_I \partial_x \Theta_{II} \) (plus the same for the dual fields \( \theta_I, \theta_{II} \)). This term does not reduce the number of gapless modes such that the phase for \( \delta > \delta_c \) is C2S1. The same phase has been found by the RG and bosonization methods away from half-filling [60, 61]. This contrasts numerical calculations for the strongly interacting \( t-J \) model [56], where a spin-gapped phase has been found away from half-filling. Such a phase maybe appears also for weak interactions, if the condensation to the low-energy phase takes place for all bands at the same energy-scale (for another idea, see Ref. [92]).

For \( N = 4 \), we have two types of paired holes with the same charge in the ladder, localized along the rungs. In contrast to the three-leg case, scattering processes of the form (5.11) are allowed. We have shown in chapter 3 that in such a two-band model of spinless fermions with repulsive interactions, slightly away from the band edge, binding takes place and the C2S0 phase close to the band edge becomes a 4-hole C1S0 phase. Away from half-filling, by RG and bosonization, the phase C3S2 has been found [61]. Similarly as already noted in Ref. [92], we argue that the C3S2 phase is a result of the
5.1. LIGHTLY DOPED CASE: HOLES AND HOLE PAIRS

$U \to 0$ limit considered in Ref. [61]; and indeed, our phases agree with a numerical treatment of the $t$-$J$ 4-leg ladder [58]. At low doping, the authors obtained a phase of paired holes and at higher doping levels, 4-hole clusters (or two-pair clusters).

We finally note that including next nearest neighbor hopping terms etc., the Fermi momenta of band pairs add up no more exactly to $\pi$, i.e., $v_j \approx v_f$. However, since the phases are controlled by the relative size of gaps, only a sufficiently large perturbation can qualitatively change the corresponding physics.

5.1.2 Hole doping 2D-like AFMs

Doped holes can be mobile (itinerant) or localized (bound to the impurity site). Here, we discuss the effect of mobile holes in 2D-like AFMs, which is relevant for (possible) superconductivity. Similar as done for the ground-state Hamiltonian, we dope the 2D-like AFM Hamiltonian, Eq. (4.40), by a chemical potential term. We find, that a spin-gap opens and that phase coherence between the bands $j$ and $\bar{j}$ is present. Furthermore, the magnetic correlations become incommensurate, i.e., the peak in the magnetic structure factor shifts away from $(\pi, \pi)$.

The ISL phase of the half-filled two-leg Hubbard ladder becomes superconducting when doped, because of the phase coherence and the spin-gap present at half-filling [80]. Here, we show that the doped Hubbard AFM becomes a conducting spin-liquid, formed by bound hole-pairs (this phase may becomes superconducting at very low energies; for the 4-leg ladder, see above).

The 2-band interactions alone, see Eqs. (4.47) and (4.48), would result in an ISL phase, i.e., $\Phi_{\sigma j j^\pm} \approx 0$. It is the presence of interactions between bands $i$ and $j$ which are not "paired" ($j \neq i, \bar{i}$), which renders the system a 2D-like AFM. The effect of a decreasing $N < \infty$ is therefore that interactions between unpaired bands become suppressed and the AFM correlations are reduced. Next, we show that the effect of doping away from half-filling is a similar one, i.e., the doping $\delta$ corresponds to $1/N$.

Similarly as above, we dope the system perturbatively by introducing a chemical potential term $-\mu Q$ in the half-filled Hamiltonian. Doping then introduces kinks in the fields $\Phi_{\sigma j j^+}$, such that the expectation value of cosine terms, which contain this field gradually goes to zero. Here, the kinks are bound hole-pairs of charge 2 and zero spin. All 4-band interactions contain
Figure 5.5: The square is the umklapp surface, which is the FS at half-filling ($t = t_\perp$). Left: There are 2 types of AFM processes; AFM processes which are umklapp processes (dashed arrows), and AFM processes which are not umklapp processes (solid arrows). Right: The (non-umklapp) AFM processes which take place within a band pair $(j, \bar{j})$ are identical to the corresponding $(d$-wave$)$ Cooper processes within this band pair. Upon doping away from half-filling, these processes remain large, open a spin-gap and lead to phase coherence within band pairs $(j, \bar{j})$. Note that we use for the ladders open boundary conditions, whereas the above figures are drawn — for a better comparison with the 2D case — for periodic boundary conditions.

the field $\Phi_{RJ^+}$, see Eq. (4.46), and therefore vanish. The effect of the doping on the 2-band interactions is the same as when doping an ISL, i.e., in the bosonized version, the umklapp term (4.48) vanishes leaving only the term (4.47), which opens a spin-gap, and leads to $d$-wave-like phase coherence between the bands $j$ and $\bar{j}$, corresponding to the Fermi momenta $(k_{Fj}, \pi - k_{Fj})$ and $(\pi - k_{Fj}, k_{Fj})$. The charge-gaps close to $(\pi/2, \pi/2)$ are the smallest ones (note that $g_{N/2,N/2+1}/v_{N/2} < \ldots < g_{1,N}/v_1$), such that the hole-pairs enter first there.

In other words, it is the fact, that the $(s$-wave$)$ AFM processes

$$-M_j \cdot M_j = -\Psi_{RjLj}^\dagger \Psi_{LjRj}^\dagger \Psi_{RjLj} \Psi_{LjRj}^\dagger (\text{+other terms}) \quad (5.13)$$

(partially) coincide with the $(d$-wave$)$ Cooper processes with momentum transfer $\pi$

$$\Delta_j^\dagger \Delta_j = \Psi_{RjLj}^\dagger \Psi_{LjRj}^\dagger \Psi_{LjRj} \Psi_{RjLj}^\dagger (\text{+other terms}), \quad (5.14)$$
which makes a doped AFM a superconductor: These particular AFM/Cooper processes flow at higher energies — driven by AFM processes — to strong coupling and are not suppressed at lower energies by doping (see also Fig. 5.5). We therefore have AFM mediated superconductivity [38]. The full phase coherence around the FS then grows out of the phase coherence between bands $j$ and $\bar{j}$, see below.

Upon doping, the AFM correlations become incommensurate. When doping away from half-filling, $k_{Fj} + k_{F\bar{j}} < \pi$, and the AFM peak in the magnetic structure factor,

$$S(q) \propto \sum_{i,t} \sum_{x,x'} e^{i q(x-x')} \langle S_i(x) \cdot S_i(x') \rangle$$

shifts away from $(\pi, \pi)$ [use Eqs. (4.52) and (4.54)]. In particular, for small $\frac{t_1}{t} \ll 1$, all bands are (almost) equally doped and $k_{Fj} + k_{F\bar{j}} = \pi(1 - \delta)$, i.e., the incommensurability is equal the doping $\delta$. Furthermore, doping leads to a cutoff of the AFM processes at the energy-scale of the chemical potential $\mu$ and therefore to an exponential decay of the spin-spin correlation function, where the coherence-length is given by $\xi_{AF} \propto 1/\mu$.

We note, that for Li doping the 2D AFM La$_2$CuO$_4$, the holes remain localized. Interestingly, both for Sr doping and Li doping, experiments show, that the long-range AFM order is rapidly suppressed [93]. The difference is (apart from the fact that the Li doped compound remains an insulator), that mobile holes lead to incommensurate AFM fluctuations [19], while localized holes give commensurate AFM fluctuations [94], i.e., the peak in the magnetic structure factor remains at $(\pi, \pi)$. The microscopic mechanism which leads to the suppression of AFM order in the Li doped case is unclear at present.

5.2 Intermediate regime: Superconductivity

A series of RG studies for the 2D Hubbard model found dominant $d$-wave superconductivity for sufficient hole doping [34, 35, 36]. Here, investigating the RG flow of the (doped) $N$-leg Hubbard ladders for AFM initial-values, we obtain (for all $N > 2$) a BCS-like low-energy Hamiltonian with $d$-wave pairing between the electrons. This confirms that the superconductivity is due to a Kohn-Luttinger-type attraction mediated by short-range AFM fluctuations.

First, using heuristic arguments, we show that the phase coherence between the band pairs sets in, when the “distance” between neighboring bands
is of the order of the “distance” of the FS to the umklapp surface, i.e., for dopings $\delta > \delta_c(N) \sim (t_\perp/t)/N$.

A finite chemical potential $\mu$ results in a low-energy cutoff for the umklapp and 4-band-AFM interactions. Note that at high energies the RG flow is not affected by doping. However, in the doped case, we have to study the RG flow with all interactions which are large at the cutoff energy-scale. In leading order, this implies that not only the 4-band interactions $c_{jkkj}^{\rho,\sigma}$ and $u_{jkkj}^{\rho,\sigma}$ have to be taken into account, but also “neighbor” processes of the form $c_{jkk\pm\pm1\pm1}^{\rho,\sigma}$ and $u_{jkk\pm\pm1\pm1}^{\rho,\sigma}$. These processes are only present down to an energy-scale $t_\perp/N$. For dopings $0 < \delta < (t_\perp/t)/N$, the “neighbor” processes are cutoff before the main AFM processes, which leads to a decoupling into band pairs $(j, j)$, while for $\delta > (t_\perp/t)/N$ all AFM processes are cutoff by the chemical potential. In this case, not only 2-band Cooper interactions within band pairs $(j, j)$ are large at the cutoff energy-scale, but also the neighboring 2-band Cooper interactions $(j, j \pm 1)$.

Next, we study the RG flow for AFM initial-values. The RGEs without AFM and umklapp processes read (see Ref. [61] and Appendix A; $v_{jk} = v_j + v_k$)

$$\frac{df_{jk}^{\rho}}{dl} = \frac{1}{v_{jk}} \left[ (c_{jk}^{\rho})^2 + \frac{3}{16} (c_{jk}^{\sigma})^2 \right]$$

$$\frac{df_{jk}^{\sigma}}{dl} = \frac{1}{v_{jk}} \left[ 2c_{jk}^{\rho}c_{jk}^{\sigma} - \frac{1}{2} (c_{jk}^{\sigma})^2 - (f_{jk}^{\sigma})^2 \right]$$

$$\frac{dc_{jj}^{\rho}}{dl} = -\sum_{k \neq j} \frac{1}{2v_k} \left[ (c_{jk}^{\rho})^2 + \frac{3}{16} (c_{jk}^{\sigma})^2 \right]$$

$$\frac{dc_{jj}^{\sigma}}{dl} = -\sum_{k \neq j} \frac{1}{2v_k} \left[ \frac{1}{2} (c_{jk}^{\sigma})^2 + 2c_{jk}^{\rho}c_{jk}^{\sigma} \right] - \frac{1}{2v_j} \left( c_{jj}^{\sigma} \right)^2$$

$$\frac{dc_{jk}^{\rho}}{dl} = -\sum_{i=1}^{N} \frac{1}{2v_i} \left( c_{ji}^{\rho}c_{ik}^{\rho} + \frac{3}{16} c_{ji}^{\sigma}c_{ik}^{\sigma} \right) + \frac{2}{v_{jk}} \left( c_{jk}^{\rho}f_{jk}^{\rho} + \frac{3}{16} c_{jk}^{\sigma}f_{jk}^{\sigma} \right)$$

$$\frac{dc_{jk}^{\sigma}}{dl} = -\sum_{i=1}^{N} \frac{1}{2v_i} \left( c_{ji}^{\rho}c_{ik}^{\sigma} + c_{ji}^{\sigma}c_{ik}^{\rho} + \frac{1}{2} c_{ji}^{\sigma}c_{ik}^{\sigma} \right)$$

$$+ \frac{2}{v_{jk}} \left( c_{jk}^{\rho}f_{jk}^{\sigma} + c_{jk}^{\sigma}f_{jk}^{\rho} - \frac{1}{2} c_{jk}^{\sigma}f_{jk}^{\sigma} \right).$$

Taking for the Cooper and forward interactions initial-values as given by the AFM phase, see Eq. (4.31), and similar initial-values for non-leading
5.2. INTERMEDIATE REGIME: SUPERCONDUCTIVITY

Figure 5.6: Matrix element $V_{ij}$ for a $N = 32$-leg ladder. The strongest correlations are between band 1 and $N$, $V_{1N} \approx t$, respectively, within a band, $V_{11} \approx -t$. Apparently, the symmetry of $V_{ij}$ is $d_{x^2-y^2}$.

Cooper processes $c_{j,j+p}^{\rho\sigma}$ ($p = 1, 2, 3, \ldots$), we obtain a $d_{x^2-y^2}$ superconducting instability at lower energies. The reason is, that in the RGEs (5.16), the Cooper processes between bands $j$ and $j$ render the couplings between bands $i$ and $j$ ($i, j \leq N/2$ or $> N/2$) negative, $c_{ij}^{\rho\sigma} < 0$. The crucial point is now, that there is positive feedback from these negative couplings on $c_{j,j+p}^{\rho\sigma}$, e.g.,

$$\frac{dc_{j,j+p}^{\rho\sigma}}{dl} \propto -c_{j,j+p}^{\rho\sigma}c_{j,j+p}^{\rho\sigma} > 0,$$

such that we finally obtain a RG instability.

Integrating the RGEs (5.16), we always find that the scaling of the couplings is exactly such that

$$2V_{ij} = 4c_{ij}^{\rho} = c_{ij}^{\sigma} \text{ and } f_{ij}^{\rho} \approx f_{ij}^{\sigma} \approx 0.$$

Using the (singlet) pair-field operator, $\Delta_j$, see Eq. (3.62), and the relationship

$$2\Delta_{ij}^\dagger \Delta_j = J_{Rij}J_{Lij} - 4J_{Rij} \cdot J_{Lij},$$

(5.19)
the low-energy Hamiltonian takes the form

\[ H = H_0 + \sum_{i,j} \int dx \, V_{ij} \Delta_i^\dagger \Delta_j, \]  

(5.20)

where \( V_{ij} < 0 \) for \( i, j \leq N/2 \) respectively \( i, j > N/2 \) and \( V_{ij} > 0 \) in all other cases, i.e., the symmetry is \( d_{x^2-y^2} \) [the \( k_y \)-values are related to the bands by \( k_{yj} = \pi j/(N + 1) \)]. The size (but not the sign) of \( V_{ij} \) depends on the initial-values (i.e., on the doping). In Fig. 5.6, we show (the strong-coupling value of) \( V_{ij} \) for a \( N = 32 \)-leg ladder, where we have chosen the initial-values as such that the band pairs \((j, j)\) have the largest initial-values \( j + j = N + 1 \).

We find, that the strongest correlations appear between band 1 and \( N \), i.e., between \((0, \pi)\) and \((\pi, 0)\); the correlations vanish when approaching \((\pi/2, \pi/2)\), as required for a \( d_{x^2-y^2} \) symmetry. Similar as at half-filling, the correlation function of the order parameter decays as (use bosonization and the pinning of the phase differences, \( \theta_{pi} - \theta_{pj} \approx 0 \))

\[ \langle \Delta_i^\dagger(x) \Delta_j(0) \rangle \propto x^{-1/N} \]  

(5.22)

Furthermore, the energy-scale (gap) of the system is independent of \( N \). However, the superconducting gap(-function) resulting from \( \Delta_j \) is considerably smaller than the Mott gap.

As a result, for dopings \( \delta > \delta_c(N) \sim (t_\perp/t)/N \), phase coherence between the band pairs sets in and the ladders become a 2D-like \( d \)-wave superconductor. The conducting spin-liquid present at very low dopings, \( \delta < \delta_c(N) \) seems to disappear in the the 2D-limit (see also Fig. 5.1).

### 5.3 Large dopings: Fermi liquid

For large enough dopings, the 4-band interactions are cutoff before they could have flown sufficiently close to the AFM fixed-point, such that the AFM correlations vanish completely. In Ref. [61], Lin et al. discuss the RG flow (for Hubbard initial-values) of the \( N \)-leg ladders including only Cooper and forward scattering. They show, that without umklapp and AFM interactions, the large \( N \) limit is a FL. In particular, the typical gap-size scales exponentially to 0, \( \propto e^{-N} \). The reason is, that the forward scattering (f) processes,
which tend to drive the system towards a RG instability give contributions of the order of 1, while the Cooper processes (c), which drive the system towards a FL, have a weight \( \propto N \).

Here, we briefly revisit the work of Lin et al.; to simplify the analysis, we take equal Fermi velocities in each band, \( v_t = v_j \) (this corresponds to the limit \( t_\perp \ll t \)). Interestingly, integrating (numerically) the RGEs (5.16), we still find a tendency towards d-wave pairing (for a comparison, see the previous section): The flow is such that \( 4c_{ij}^c = c_{ij}^c \sim t \), \( c_{ij}^\sigma \sim -t \), and \( 8f_{ij}^c = -8c_{ij}^c \sim t \). All other couplings are suppressed for small \( U/t \), i.e., there is a spin-gap and phase coherence within band pairs \((j, j)\). However, there is no phase coherence between the band pairs and the gap vanishes exponentially as a function of \( N \); in particular, for \( v_3 = 2t \),

\[
E \sim t \exp(-3.1Nt/U) .
\]

As a result, the AFM processes are required and responsible for phase coherence between the band pairs and for a finite gap (in the 2D case). AFM initial-values lead to an instability in the Cooper channel, while for Hubbard initial-values, the system is only driven by forward scattering — which becomes negligible in the 2D case, see the RGEs (5.16) — such that there is no instability for \( N \) large.

In the (true) 2D limit \((N = \infty)\), the RGEs (5.16) scale on the RGEs of a 2D FL. The reason is [61], that the additional processes which appear in the 2D case have a very small phase space and that forward scattering can be neglected in the RGEs (5.16) when \( N \) becomes large. We then find, that \( 4c_{ij}^c = c_{ij}^\sigma = c_{ij} \) on all energy-scales. Therefore, the RGEs reduce to

\[
\frac{dc_{ij}}{dl} = -\sum_{k=1}^{N} \frac{1}{2v_k} c_{ik}c_{kj} .
\]

This is the same form as in a 2D FL,

\[
\frac{dV(\theta_1, \theta_2)}{dl} = -\frac{1}{v_F} \int d\theta V(\theta_1, \theta)V(\theta, \theta_2) ,
\]

where \( \theta \) is the angle parameterizing the FS. This shows once again, that a RG instability requires at some points in the phase space attractive interactions or highly anisotropic interactions (as caused by AFM fluctuations).
5.4 Conclusions and discussion

We have studied the doping away from half-filling in weakly interacting $N$-leg Hubbard ladders. There are three doping regimes. For the lightly doped case, we have obtained — similar as at half-filling — an odd-even effect: even-leg ladders are a 1D superconductor and odd-leg ladders a LL. The same phases have been found for the strongly interacting $t-J$ ladders. This lightly doped regime is conveniently described as a "liquid" of bound hole-pairs (conducting spin-liquid). It may becomes superconducting at very low energies. For increasing doping, we showed that phase coherence between all bands sets in above a doping $\delta_c \propto 1/N$ and that the system becomes a 2D-like superconductor, where the pairing is between electrons and the low-energy Hamiltonian has the form of a $d$-wave BCS model, generalized to 2D,

$$H = H_0 + \sum_{k,k',q} V(\theta, \theta') \Psi^\dagger_s(k) \Psi^\dagger_s(-k + q) \Psi_s(-k' + q) \Psi_s(k'), \quad (5.26)$$

where $k$ and $k'$ are restricted to be close to the 2D FS, and $q$ close to 0 ($\theta$ and $\theta'$ are the angles parameterizing $k$ and $k'$). The matrix element $V(\theta, \theta')$ corresponds to $V_{ij}$.

The conducting spin-liquid would certainly be a good candidate for the underdoped phase in the HTSCs, but there is strong evidence, that for $N \to \infty$, this phase disappears.

For large dopings the system becomes a FL. Interestingly, for all dopings (from half-filling to the FL), the bands $j$ and $\bar{j}$ are phase coherent. However, phase coherence between all band pairs (and therefore superconductivity) is only present, when there are AFM fluctuations on short length-scales.
Chapter 6

Conclusions and outlook

In this thesis, we have given a careful and systematic investigation of weakly interacting $N$-leg Hubbard ladders and their crossover to 2D as $N \to \infty$. We have disentangled the complicated interplay between antiferromagnetic, umklapp, and Cooper processes, which is present at and close to half-filling. We have used a combination of one-loop RG and bosonization techniques — these two methods are known to give unbiased and (rather) rigorous results for weakly interacting systems.

We have shown, that $N$-leg Hubbard ladders exhibit unusual groundstate properties, i.e., an odd-even effect. At half-filling, the ladders are insulating; there is a spin-gap for $N$ even and one gapless spinon-mode for $N$ odd — similar to the Heisenberg spin-ladders. Upon doping away from half-filling, even-leg ladders become a Luther-Emery liquid and odd-leg ladders a Luttinger liquid. The same phases have been found for the $t$-$J$ ladders. In other words, the phase of the Hubbard ladders at and close to half-filling is the same for small and large $U$.

Next, we have studied what happens, when the system approaches the 2D limit, $N \to \infty$. Since the phases in 1D and 2D are different (the half-filled 2D Hubbard model has two gapless magnon-modes), this is an interesting question for itself. In addition, it allows one to gain some insight in the 2D case. We have found, that the system behaves ladder-like only below a crossover energy $E_c \sim t \exp[-a \exp(bN)]$, i.e., below $E_c$ there is an odd-even effect (in particular, a spin-gap for $N$ even). In contrast, above $E_c$, the odd-even effect vanishes and the physical properties are the same as associated with the 2D Heisenberg AFM. We have derived an analytical expression for the Hamiltonian of this 2D-like AFM phase. Using bosonization, we have
shown, that the charge-sector of the AFM phase is the same as in the half-filled two-leg ladders, i.e., there is $d$-wave like phase coherence between the Fermi momenta $(k_{Fj}, \pi - k_{Fj})$ and $(\pi - k_{Fj}, k_{Fj})$. Doping away from half-filling, the system becomes conducting, a spin-gap opens, and the phase coherence remains. This conducting spin-liquid is conveniently described by bound hole pairs. It is an open question whether it becomes superconducting at very low energies or not.

Finally, we have shown how and why phase coherence between all band pairs sets in above a doping $\delta_c \propto 1/N$ and the system becomes a 2D-like $d$-wave superconductor. Similar as for the 2D case, the superconducting instability is driven by antiferromagnetic fluctuations on short length-scales (Kohn-Luttinger effect). For the low-energy Hamiltonian, we have obtained a BCS-like $d$-wave Hamiltonian with electron pairing. There is no additional $s$-wave etc. component.

We believe, that the 2D-like Hamiltonians for the AFM and the superconducting phase can straightforwardly be generalized to 2D, see Eqs. (4.56) and (5.26). In this sense, we have derived the effective low-energy Hamiltonians of the half-filled and hole-doped 2D Hubbard model in the limit $U \ll t$. The Hamiltonians have the form which one expects; but, we like to emphasize, that we have derived both of them with the same method and without making any assumptions about the low-energy theory.

In the 2D limit, we have not found (as a groundstate) a phase which corresponds to the underdoped phase of the HTSCs. The conducting spin-liquid present for finite $N$ seems to disappear as $N \to \infty$. On the other hand, such a phase may survives at finite energy: Note that the phase depends always on the energy-scale and is related to the value of the LLPs at this scale. Consult table 4.2, AFM and superconductor differ in two LLPs. We therefore conjecture that in the doped case, the flow of the couplings from the AFM (at high energies) to the $d$-wave (in the groundstate) takes place via an intermediate phase. This intermediate (finite energy) phase has then the same LLPs as the ISL — suggesting that the spin-gap opens before phase coherence sets in. It would be interesting to derive an effective Hamiltonian for this phase.
Appendix A

Renormalization group

In this chapter, we derive and give the one-loop RG equations of the half-filled \( N \)-leg Hubbard ladders, where we take \( t_{\perp} \neq t \), such that the FS is not flat. We will use the notation of current algebra which largely simplifies the treatment. The integration for particular cases/limits is done in the main part of this work.

A.1 Current algebra

Current algebra methods allow for a rather simple derivation of one-loop RGES. First, they manifestly take into account the U(1) and SU(2) symmetries and second, once all possible products of two (non)umklapp currents have been calculated, the derivation of the RGES is straightforward.

In the perturbative calculation of the one-loop RGES, only the most singular part of a product of two currents plays a role (see, e.g., Ref. [61]). We can therefore use operator product expansion; a product of normal-ordered currents obeys the following short-distance expansion for \( z \rightarrow 0 \) (Wick-Theorem),

\[
J_R(z)J_R(0) = :\Psi^\dagger_{Rs}(z)\Psi_{Rs}(z) : :\Psi^\dagger_{Rs'}(0)\Psi_{Rs'}(0) :
\]

\[
= :\Psi^\dagger_{Rs}(z)\Psi_{Rs}(z)\Psi^\dagger_{Rs'}(0)\Psi_{Rs'}(0) :
\]

\[
-\frac{i}{2\pi z} :\Psi^\dagger_{Rs}(z)\Psi_{Rs}(0) : + \frac{i}{2\pi z} :\Psi^\dagger_{Rs}(0)\Psi_{Rs}(z) :, \quad (A.1)
\]

where \( z = x + iy \) (for left-moving currents \( z \rightarrow z^* \)). Keeping only the relevant terms and dropping overall prefactors, we then obtain for the non-umklapp
currents
\[ J_{Rij} J_{Rlm} \sim \delta_{ij} J_{Rim} - \delta_{im} J_{Rlj} \]
\[ J_{Rij}^a J_{Rlm}^b \sim \frac{\delta^{ab}}{4} (\delta_{ij} J_{Rim} - \delta_{im} J_{Rlj}) + \frac{i \epsilon^{abc}}{2} (\delta_{ij} J_{Rim}^c + \delta_{im} J_{Rlj}^c) \]
\[ J_{Rij}^a J_{Rlm} \sim \delta_{ij} J_{Rim}^a - \delta_{im} J_{Rlj}^a \] (A.2)

where \( \epsilon^{abc} \) is totally antisymmetric; for the umklapp currents, we find
\[ I_{Rij}^a I_{Rlm} \sim \delta_{il} J_{Rjm} + \delta_{jm} J_{Ril} + \delta_{jl} J_{Rim} + \delta_{im} J_{Rjl} \]
\[ (I_{Rij}^a) I_{Rlm}^b \sim \frac{\delta^{ab}}{4} (\delta_{il} J_{Rjm} + \delta_{jm} J_{Ril} - \delta_{jl} J_{Rim} - \delta_{im} J_{Rjl}) \]
\[ (I_{Rij}^a) I_{Rlm} \sim \delta_{il} J_{Rjm}^a + \delta_{jm} J_{Ril}^a - \delta_{jl} J_{Rim}^a - \delta_{im} J_{Rjl}^a. \] (A.3)

The mixed products become
\[ J_{Rij} I_{Rlm} \sim \delta_{il} I_{Rjm} + \delta_{im} I_{Rlj} \] (A.4)

\[ J_{Rij}^a I_{Rlm}^b \sim \frac{\delta^{ab}}{4} (\delta_{im} I_{Rlj} - \delta_{il} I_{Rjm}) + \frac{i \epsilon^{abc}}{2} (\delta_{im} I_{Rlj}^c - \delta_{il} I_{Rjm}^c) \]
\[ J_{Rij}^a (I_{Rlm}^b) \sim \frac{\delta^{ab}}{4} (\delta_{jm} I_{Ril}^b - \delta_{jl} I_{Rim}^b) - \frac{i \epsilon^{abc}}{2} \left[ \delta_{jm} (I_{Ril}^c) - \delta_{jl} (I_{Rim}^c) \right] \] (A.5)

\[ J_{Rij} I_{Rlm} \sim \delta_{il} I_{Rjm}^a + \delta_{im} I_{Rlj}^a \]
\[ J_{Rij}^a I_{Rlm} \sim \delta_{jl} (I_{Rlm})^a + \delta_{jm} (I_{Rili})^a \]
\[ J_{Rij}^a I_{Rlm} \sim \delta_{il} I_{Rjm}^a - \delta_{jm} I_{Rlj}^a \]
\[ J_{Rij} (I_{Rlm})^a \sim \delta_{jl} (I_{Rlm})^a - \delta_{jm} (I_{Rili})^a. \] (A.6)

### A.2 RG equations

We give the one-loop RGEs of the half-filled \( N \)-leg Hubbard ladders. Using the above operator products, the derivation is straightforward (but still time consuming): In leading (one-loop) order, only products of two couplings
appear — and therefore a product of two left moving currents times a product of two right moving currents. The above rules then tell us which couplings are renormalized (the overall prefactor has to be calculated only once).

There are a series of “rules” which allow to check the correctness of the signs.

- \( u^\sigma \times u^\rho \) or \( \times f^\rho / c^\rho \) results in +
- \( f^\sigma / c^\sigma \times f^\rho / c^\rho \) to \( f^\sigma / c^\sigma \) results in –
- \( f^\sigma / c^\sigma \times f^\rho / c^\rho \) results in + for a particle-hole and in – for a particle-particle diagram.
- \( f^\sigma / c^\sigma \times f^\rho / c^\rho \) to \( f^\rho / c^\rho \) results in + for a particle-hole and in – for a particle-particle diagram.
- \((u^\sigma)^2\) to \( f^\sigma / c^\sigma \) results in – and to \( f^\rho / c^\rho \) results in +
- If \( d_{gb}/dl = s_{gb}g_c \), then permutations of the couplings have the same sign \( s \), i.e., \( d_{gb}/dl = s_{ga}g_c \) and \( d_{gc}/dl = s_{ga}g_b \).

Next, we give the Hubbard initial-values of the various couplings; for their determination, the following relations are useful

\[
J_{Rij}I_{Llm} - 4J_{Rij}J_{Llm} = 2 \left( \Psi^\dagger_{Ris} \Psi^\dagger_{Rjs} \Psi^\dagger_{Lls} \Psi^\dagger_{Lms} - \Psi^\dagger_{Ris} \Psi^\dagger_{Rjs} \Psi^\dagger_{Lls} \Psi^\dagger_{Lms} \right) \quad (A.7)
\]

and

\[
I^\dagger_{Rij}J_{Llm} = \Psi^\dagger_{Ris} \Psi^\dagger_{Lls} \Psi^\dagger_{Rjs} \Psi^\dagger_{Lms} + \Psi^\dagger_{Ris} \Psi^\dagger_{Lms} \Psi^\dagger_{Rjs} \Psi^\dagger_{Lls}. \quad (A.8)
\]

The 2-band U(1) interactions are

\[
2f^\rho_{ij} = 2c^\rho_{ij} = 2c^\rho_{ji} = u^\rho_{ij} = u^\rho_{ji} = \frac{3U}{2(N + 1)}, \quad (A.9)
\]

and for \( i \neq j, j \)

\[
f^\rho_{ij} = c^\rho_{ij} = \frac{U}{2(N + 1)}. \quad (A.10)
\]

For the SU(2) interactions, we obtain \( f^\rho_{ij} = 4f^\rho_{ij}, c^\rho_{ij} = 4c^\rho_{ij}, \) and \( u^\rho_{jij} = 0 \) for all \( i \) and \( j \). Finally, the 4-band interactions are

\[
2c^\rho_{jk\bar{k}j} = u^\rho_{jk\bar{k}j} = u^\rho_{j\bar{k}k} = \frac{U}{N + 1}, \quad c^\rho_{jk\bar{k}j} = 4c^\rho_{jkkj} \quad \text{and} \quad u^\rho_{jkkj} = 0. \quad (A.11)
\]
The RGEs take the following form, where $l$ is related to the energy by $E \sim t e^{-nl}$ and $v_{jk} = v_j + v_k$; note that at half-filling, $v_3 = v$. Forward scattering is renormalized according to

$$ \frac{df_{ij}^{\rho}}{dl} = \frac{1}{v_{ij}} \left[ \left( c_{ij}^{\rho} \right)^2 + \frac{3}{16} \left( c_{ij}^{\sigma} \right)^2 + \left( u_{ijij}^{\rho} \right)^2 + 16 \left( u_{ijij}^{\sigma} \right)^2 + \frac{3}{16} \left( u_{ijij}^{\rho} \right)^2 \right] + \sum_{k \neq ij} \frac{1}{v_{kk}} \left[ \left( c_{jk}^{\rho} \right)^2 + \frac{3}{16} \left( c_{jk}^{\sigma} \right)^2 + \left( u_{jkkj}^{\rho} \right)^2 + \frac{3}{16} \left( u_{jkkj}^{\rho} \right)^2 \right] $$

$$ \frac{df_{jk}^{\rho}}{dl} = \frac{1}{v_{jk}} \left[ \left( c_{jk}^{\rho} \right)^2 + \frac{3}{16} \left( c_{jk}^{\sigma} \right)^2 \right] - \frac{1}{v_{jk}} \left[ \left( c_{jkkj}^{\rho} \right)^2 + \frac{3}{16} \left( c_{jkkj}^{\sigma} \right)^2 \right] + \frac{1}{v_{jk}} \left[ \left( u_{jkkj}^{\rho} \right)^2 + \frac{3}{16} \left( u_{jkkj}^{\rho} \right)^2 \right] $$

$$ \frac{df_{ij}^{\sigma}}{dl} = \frac{1}{v_{ij}} \left[ 2c_{ij}^{\rho} c_{ij}^{\sigma} - \frac{1}{2} \left( c_{ij}^{\sigma} \right)^2 - \left( f_{ij}^{\sigma} \right)^2 + 2u_{ijij}^{\rho} u_{ijij}^{\sigma} - \frac{1}{2} \left( u_{ijij}^{\sigma} \right)^2 \right] + \sum_{k \neq ij} \frac{1}{v_{kk}} \left[ 2c_{jk}^{\rho} c_{jk}^{\sigma} c_{jk}^{\sigma} c_{jkkj} - \frac{1}{2} \left( c_{jkkj}^{\sigma} \right)^2 - 2u_{jkkj}^{\rho} u_{jkkj}^{\sigma} - \frac{1}{2} \left( u_{jkkj}^{\sigma} \right)^2 \right] $$

$$ \frac{df_{jk}^{\sigma}}{dl} = \frac{1}{v_{jk}} \left[ 2c_{jk}^{\rho} c_{jk}^{\sigma} - \frac{1}{2} \left( c_{jk}^{\sigma} \right)^2 - \left( f_{jk}^{\sigma} \right)^2 \right] + \frac{1}{v_{jk}} \left[ -2c_{jkkj}^{\rho} c_{jkkj}^{\sigma} - \frac{1}{2} \left( c_{jkkj}^{\sigma} \right)^2 - 2u_{jkkj}^{\rho} u_{jkkj}^{\sigma} - \frac{1}{2} \left( u_{jkkj}^{\sigma} \right)^2 \right] $$

and the single-band interactions according to

$$ \frac{dc_{ij}^{\rho}}{dl} = -\sum_{k \neq ij} \frac{1}{2v_k} \left[ \left( c_{ij}^{\rho} \right)^2 + \frac{3}{16} \left( c_{ij}^{\sigma} \right)^2 \right] + \frac{1}{2v_j} \left[ \left( u_{ijij}^{\rho} \right)^2 + \frac{3}{16} \left( u_{ijij}^{\rho} \right)^2 \right] $$

$$ \frac{dc_{ij}^{\sigma}}{dl} = -\sum_{k \neq ij} \frac{1}{2v_k} \left[ \frac{1}{2} \left( c_{ij}^{\sigma} \right)^2 + 2c_{jk}^{\rho} c_{jk}^{\sigma} \right] - \frac{1}{2v_j} \left( \left( c_{ij}^{\sigma} \right)^2 + \frac{1}{2v_j} \left[ 2u_{ijij}^{\rho} u_{ijij}^{\sigma} + \frac{1}{2} \left( u_{ijij}^{\sigma} \right)^2 \right] \right). \quad (A.13) $$

For backward scattering, we find

$$ \frac{dc_{ij}^{\rho}}{dl} = -\sum_{k=1}^{N} \frac{1}{2v_k} \left( c_{jk}^{\rho} c_{kij}^{\sigma} + \frac{3}{16} c_{jk}^{\sigma} c_{kij}^{\sigma} \right) + \frac{2}{v_{jj}} \left( c_{jj}^{\rho} f_{jj}^{\sigma} + \frac{3}{16} c_{jj}^{\sigma} f_{jj}^{\sigma} + 4u_{jjjj}^{\rho} u_{jjjj}^{\sigma} \right) $$

The RGEs take the following form, where $l$ is related to the energy by $E \sim t e^{-nl}$ and $v_{jk} = v_j + v_k$; note that at half-filling, $v_3 = v$. Forward scattering is renormalized according to

$$ \frac{df_{ij}^{\rho}}{dl} = \frac{1}{v_{ij}} \left[ \left( c_{ij}^{\rho} \right)^2 + \frac{3}{16} \left( c_{ij}^{\sigma} \right)^2 + \left( u_{ijij}^{\rho} \right)^2 + 16 \left( u_{ijij}^{\sigma} \right)^2 + \frac{3}{16} \left( u_{ijij}^{\rho} \right)^2 \right] + \sum_{k \neq ij} \frac{1}{v_{kk}} \left[ \left( c_{jk}^{\rho} \right)^2 + \frac{3}{16} \left( c_{jk}^{\sigma} \right)^2 + \left( u_{jkkj}^{\rho} \right)^2 + \frac{3}{16} \left( u_{jkkj}^{\rho} \right)^2 \right] $$

$$ \frac{df_{jk}^{\rho}}{dl} = \frac{1}{v_{jk}} \left[ \left( c_{jk}^{\rho} \right)^2 + \frac{3}{16} \left( c_{jk}^{\sigma} \right)^2 \right] - \frac{1}{v_{jk}} \left[ \left( c_{jkkj}^{\rho} \right)^2 + \frac{3}{16} \left( c_{jkkj}^{\sigma} \right)^2 \right] + \frac{1}{v_{jk}} \left[ \left( u_{jkkj}^{\rho} \right)^2 + \frac{3}{16} \left( u_{jkkj}^{\rho} \right)^2 \right] $$

$$ \frac{df_{ij}^{\sigma}}{dl} = \frac{1}{v_{ij}} \left[ 2c_{ij}^{\rho} c_{ij}^{\sigma} - \frac{1}{2} \left( c_{ij}^{\sigma} \right)^2 - \left( f_{ij}^{\sigma} \right)^2 + 2u_{ijij}^{\rho} u_{ijij}^{\sigma} - \frac{1}{2} \left( u_{ijij}^{\sigma} \right)^2 \right] + \sum_{k \neq ij} \frac{1}{v_{kk}} \left[ 2c_{jk}^{\rho} c_{jk}^{\sigma} c_{jk}^{\sigma} c_{jkkj} - \frac{1}{2} \left( c_{jkkj}^{\sigma} \right)^2 - 2u_{jkkj}^{\rho} u_{jkkj}^{\sigma} - \frac{1}{2} \left( u_{jkkj}^{\sigma} \right)^2 \right] $$

$$ \frac{df_{jk}^{\sigma}}{dl} = \frac{1}{v_{jk}} \left[ 2c_{jk}^{\rho} c_{jk}^{\sigma} - \frac{1}{2} \left( c_{jk}^{\sigma} \right)^2 - \left( f_{jk}^{\sigma} \right)^2 \right] + \frac{1}{v_{jk}} \left[ -2c_{jkkj}^{\rho} c_{jkkj}^{\sigma} - \frac{1}{2} \left( c_{jkkj}^{\sigma} \right)^2 - 2u_{jkkj}^{\rho} u_{jkkj}^{\sigma} - \frac{1}{2} \left( u_{jkkj}^{\sigma} \right)^2 \right] $$

and the single-band interactions according to

$$ \frac{dc_{ij}^{\rho}}{dl} = -\sum_{k \neq ij} \frac{1}{2v_k} \left[ \left( c_{ij}^{\rho} \right)^2 + \frac{3}{16} \left( c_{ij}^{\sigma} \right)^2 \right] + \frac{1}{2v_j} \left[ \left( u_{ijij}^{\rho} \right)^2 + \frac{3}{16} \left( u_{ijij}^{\rho} \right)^2 \right] $$

$$ \frac{dc_{ij}^{\sigma}}{dl} = -\sum_{k \neq ij} \frac{1}{2v_k} \left[ \frac{1}{2} \left( c_{ij}^{\sigma} \right)^2 + 2c_{jk}^{\rho} c_{jk}^{\sigma} \right] - \frac{1}{2v_j} \left( \left( c_{ij}^{\sigma} \right)^2 + \frac{1}{2v_j} \left[ 2u_{ijij}^{\rho} u_{ijij}^{\sigma} + \frac{1}{2} \left( u_{ijij}^{\sigma} \right)^2 \right] \right). \quad (A.13) $$

For backward scattering, we find

$$ \frac{dc_{ij}^{\rho}}{dl} = -\sum_{k=1}^{N} \frac{1}{2v_k} \left( c_{jk}^{\rho} c_{kij}^{\sigma} + \frac{3}{16} c_{jk}^{\sigma} c_{kij}^{\sigma} \right) + \frac{2}{v_{jj}} \left( c_{jj}^{\rho} f_{jj}^{\sigma} + \frac{3}{16} c_{jj}^{\sigma} f_{jj}^{\sigma} + 4u_{jjjj}^{\rho} u_{jjjj}^{\sigma} \right) $$
A.2. RG EQUATIONS

\[ d\varepsilon_{jj}^\sigma = \frac{1}{2v_j} \left( c_{jj}^\sigma c_{ik}^\sigma + c_{ij}^\sigma c_{ik}^\sigma + \frac{1}{2} c_{ij}^\sigma c_{ik}^\sigma \right) + \frac{2}{v_j} \left( c_{jj}^\sigma f_{jj}^\sigma + c_{jj}^\sigma f_{jj}^\sigma - \frac{1}{2} c_{jj}^\sigma f_{jj}^\sigma + 4u_{jjj}^\sigma u_{jjj}^\sigma \right) \]

\[ d\varepsilon_{kk}^\sigma = \frac{1}{2v_k} \left( c_{kk}^\sigma c_{ik}^\sigma + c_{ij}^\sigma c_{ik}^\sigma + \frac{1}{2} c_{ij}^\sigma c_{ik}^\sigma \right) + \frac{2}{v_k} \left( c_{kk}^\sigma f_{kk}^\sigma + c_{kk}^\sigma f_{kk}^\sigma - \frac{1}{2} c_{kk}^\sigma f_{kk}^\sigma + 4u_{kkk}^\sigma u_{kkk}^\sigma \right) \]

and for two-band umklapp scattering

\[ du_{jjj}^\sigma = \frac{1}{v_jj} \left( 2f_{jjj}^\sigma + c_{jjj}^\sigma + c_{jjj}^\sigma + \frac{3}{16} u_{jjj}^\sigma \right) + \frac{3}{2} \left( 2f_{jjj}^\sigma + c_{jjj}^\sigma + c_{jjj}^\sigma \right) \]

\[ du_{jjj}^\sigma = \frac{1}{v_j} \left( c_{jjj}^\sigma u_{jjj}^\sigma + \frac{3}{16} c_{jjj}^\sigma u_{jjj}^\sigma + 4f_{jjj}^\sigma u_{jjj}^\sigma \right) \]

\[ du_{jjj}^\sigma = \frac{1}{v_j} \left( 2f_{jjj}^\sigma + c_{jjj}^\sigma + c_{jjj}^\sigma - u_{jjj}^\sigma \right) \]

\[ + u_{jjj}^\sigma \left( \frac{3}{2} f_{jjj}^\sigma + c_{jjj}^\sigma + c_{jjj}^\sigma \right) + \frac{8 c_{jjj}^\sigma u_{jjj}^\sigma}{v_j} \]

\[ + \sum_{k \neq j} \frac{1}{v_{kk}} \left( u_{jkj}^\sigma c_{jkj}^\sigma - \frac{3}{16} u_{jkj}^\sigma c_{jkj}^\sigma \right) \]
The (4-band) AFM processes are renormalized according to

\[
\frac{dc^\rho_{jk\overline{j}}} {dl} = c^\rho_{jk\overline{j}} \left( \frac{c^\sigma_{j\overline{j}}} {v_{j\overline{j}}} + \frac{c^\rho_{kk}} {v_{kk}} \right) + c^\rho_{jk\overline{j}} \left( \frac{f^\sigma_{jj}} {v_{jj}} + \frac{f^\rho_{kk}} {v_{kk}} - \frac{2f^\rho_{jk}} {v_{jk}} \right) + \frac{3c^\sigma_{jk\overline{j}}} {16} \left( \frac{c^\sigma_{j\overline{j}}} {v_{j\overline{j}}} + \frac{c^\sigma_{kk}} {v_{kk}} \right) + \frac{3c^\sigma_{jk\overline{j}}} {16} \left( \frac{f^\sigma_{jj}} {v_{jj}} + \frac{f^\sigma_{kk}} {v_{kk}} - \frac{2f^\sigma_{jk}} {v_{jk}} \right) + \sum_{i\neq j, k} \frac{1} {v_{iit}} \left( u^\sigma_{iit} u^\rho_{jkk} + 4u^\sigma_{iit} u^\rho_{jk\overline{j}} - \frac{3} {16} u^\sigma_{iit} u^\sigma_{jk\overline{j}} \right) + \sum_{i\neq j, k} \frac{2} {v_{iit}} \left( c^\sigma_{j\overline{ij}} c^\sigma_{k\overline{ik}} + \frac{3} {16} c^\sigma_{j\overline{ij}} c^\sigma_{k\overline{ik}} + u^\sigma_{j\overline{ij}} u^\rho_{k\overline{ik}} + \frac{3} {16} u^\sigma_{j\overline{ij}} u^\sigma_{k\overline{ik}} \right) \]

\[
\frac{dc^\sigma_{jk\overline{j}}} {dl} = c^\sigma_{jk\overline{j}} \left( \frac{c^\sigma_{j\overline{j}}} {v_{j\overline{j}}} + \frac{c^\sigma_{kk}} {v_{kk}} \right) + c^\sigma_{jk\overline{j}} \left( \frac{f^\sigma_{jj}} {v_{jj}} + \frac{f^\sigma_{kk}} {v_{kk}} - \frac{2f^\sigma_{jk}} {v_{jk}} \right) - c^\sigma_{jk\overline{j}} \left( \frac{c^\sigma_{j\overline{j}}} {v_{j\overline{j}}} + \frac{c^\sigma_{kk}} {v_{kk}} \right) - c^\sigma_{jk\overline{j}} \left( \frac{f^\sigma_{jj}} {v_{jj}} + \frac{f^\sigma_{kk}} {v_{kk}} - \frac{2f^\sigma_{jk}} {v_{jk}} \right) - \sum_{i\neq j, k} \frac{1} {v_{iit}} \left( u^\sigma_{iit} u^\sigma_{j\overline{jk}} - u^\sigma_{iit} u^\sigma_{jk\overline{j}} + 4u^\sigma_{iit} u^\sigma_{jk\overline{j}} - \frac{1} {2} u^\sigma_{iit} u^\sigma_{jk\overline{j}} \right) + \sum_{i\neq j, k} \frac{2} {v_{iit}} \left( c^\sigma_{j\overline{ij}} c^\sigma_{k\overline{ik}} + c^\sigma_{k\overline{ik}} c^\sigma_{j\overline{ij}} - \frac{1} {2} c^\sigma_{j\overline{ij}} c^\sigma_{k\overline{ik}} \right) - u^\rho_{j\overline{ij}} u^\sigma_{k\overline{ik}} - u^\rho_{k\overline{ik}} u^\sigma_{j\overline{ij}} - \frac{1} {2} u^\sigma_{j\overline{ij}} u^\sigma_{k\overline{ik}} \right) \] (A.16)

and

\[
\frac{du^\rho_{jk\overline{j}}} {dl} = \frac{2u^\rho_{jk\overline{j}}} {v_{jk}} + u^\rho_{jk\overline{j}} \left( \frac{c^\sigma_{j\overline{j}}} {v_{j\overline{j}}} + \frac{c^\rho_{kk}} {v_{kk}} \right) + u^\rho_{jk\overline{j}} \left( \frac{f^\sigma_{jj}} {v_{jj}} + \frac{f^\rho_{kk}} {v_{kk}} + \frac{2f^\rho_{jk}} {v_{jk}} \right) - \frac{3u^\rho_{jk\overline{j}}} {16} \left( \frac{c^\sigma_{j\overline{j}}} {v_{j\overline{j}}} + \frac{c^\sigma_{kk}} {v_{kk}} \right) + \frac{3u^\rho_{jk\overline{j}}} {16} \left( \frac{f^\sigma_{jj}} {v_{jj}} + \frac{f^\sigma_{kk}} {v_{kk}} + \frac{2f^\sigma_{jk}} {v_{jk}} \right) + \sum_{i\neq j, k} \frac{1} {v_{iit}} \left( u^\sigma_{iit} u^\rho_{jkk} + 4u^\sigma_{iit} u^\rho_{jk\overline{j}} + \frac{3} {16} u^\sigma_{iit} u^\sigma_{jk\overline{j}} \right) + \sum_{i\neq j, k} \frac{2} {v_{iit}} \left( c^\sigma_{j\overline{ij}} u^\rho_{k\overline{ik}} + c^\rho_{k\overline{ik}} u^\sigma_{j\overline{ij}} - \frac{3} {16} c^\sigma_{j\overline{ij}} u^\sigma_{k\overline{ik}} - \frac{3} {16} c^\sigma_{k\overline{ik}} u^\sigma_{j\overline{ij}} \right) \]
A.2. RG EQUATIONS

\[ \frac{du^\rho_{jkk}}{dl} = \frac{4u^\rho_{jkk}f^\rho_{jk}}{v_{jk}} + \frac{4u^\rho_{jkk}f^\rho_{jk}}{v_{jk}} + \frac{3u^\sigma_{jkk}c^\sigma_{jkk}}{4v_{jk}} \]

\[ \frac{du^\sigma_{jkk}}{dl} = \frac{2u^\sigma_{jkk}c^\sigma_{jkk}}{v_{jk}} - u^\sigma_{jkk} \left( \frac{c^\sigma_{jj}}{v_{jj}} + \frac{c^\sigma_{kk}}{v_{kk}} \right) + u^\sigma_{jkk} \left( \frac{f^\sigma_{jj}}{v_{jj}} + \frac{f^\sigma_{kk}}{v_{kk}} + \frac{2f^\sigma_{jk}}{v_{jk}} \right) \]

\[ + u^\sigma_{jkk} \left( \frac{c^\sigma_{jj}}{v_{jj}} + \frac{c^\sigma_{kk}}{v_{kk}} \right) + u^\sigma_{jkk} \left( \frac{f^\sigma_{jj}}{v_{jj}} + \frac{f^\sigma_{kk}}{v_{kk}} + \frac{2f^\sigma_{jk}}{v_{jk}} \right) \]

\[ - \frac{u^\sigma_{jkk}}{2} \left( \frac{c^\sigma_{jj}}{v_{jj}} + \frac{c^\sigma_{kk}}{v_{kk}} \right) - \frac{u^\sigma_{jkk}}{2} \left( \frac{f^\sigma_{jj}}{v_{jj}} + \frac{f^\sigma_{kk}}{v_{kk}} + \frac{2f^\sigma_{jk}}{v_{jk}} \right) \]

\[ - \sum_{i=1}^{N/2} \frac{1}{v_{ik}} \left( u^\rho_{iin}c^\rho_{ijkk} + u^\rho_{iin}c^\rho_{ijkk} + 4u^\rho_{iin}c^\rho_{ijkk} - \frac{1}{2}u^\rho_{iin}c^\rho_{ijkk} \right) \]

\[ + \sum_{i \neq j,k} \frac{2}{v_{ik}} \left( c^\rho_{ijij}u^\rho_{ijik} + c^\rho_{ijij}u^\rho_{ijij} - u^\rho_{ijij}c^\rho_{ijij} \right) \]

\[ - u^\rho_{ijij}c^\rho_{ijij} - \frac{1}{2}u^\rho_{ijij}u^\rho_{ijij} - \frac{1}{2}c^\rho_{ijij}u^\rho_{ijij} \]. \hspace{1cm} (A.17)

For odd \( N \), the renormalization of \( c^\rho_{rr} \) and \( u^\rho_{rr} \) [\( r = (N + 1)/2 \)] is

\[ \frac{dc^\rho_{rr}}{dl} = - \sum_{k \neq r} \frac{1}{2v_{rk}} \left[ \frac{1}{2} \left( c^\rho_{rk} \right)^2 + \frac{3}{16} \left( c^\rho_{kk} \right)^2 \right] + \frac{8}{v_r} \left( u^\rho_{rr} \right)^2 \]

\[ + \sum_{k \neq r} \frac{1}{2v_{rk}} \left[ \frac{1}{2} \left( c^\rho_{rrk} \right)^2 + \frac{3}{16} \left( c^\rho_{kkk} \right)^2 + \left( u^\rho_{rrk} \right)^2 + \frac{3}{16} \left( u^\rho_{rrk} \right)^2 \right] \]

\[ \frac{du^\rho_{rr}}{dl} = - \sum_{k \neq r} \frac{1}{2v_{rk}} \left[ \frac{1}{2} \left( c^\rho_{rrk} \right)^2 + 2c^\rho_{rrk}c^\rho_{rk} \right] - \frac{1}{2v_r} \left( c^\rho_{rr} \right)^2 \]

\[ + \sum_{k \neq r} \frac{1}{2v_{rk}} \left[ 4c^\rho_{rrk}c^\rho_{rrk} - 4u^\rho_{rrk}u^\rho_{rrk} - \left( c^\rho_{rrk} \right)^2 - \left( u^\rho_{rrk} \right)^2 \right] \]

\[ \frac{du^\rho_{rr}}{dl} = \frac{2c^\rho_{rr}u^\rho_{rr}}{v_r} + \sum_{k \neq r} \frac{1}{v_{kk}} \left( c^\rho_{rrk}u^\rho_{rrk} - \frac{3}{16} c^\rho_{rrk}u^\rho_{rrk} \right) \]. \hspace{1cm} (A.18)
Appendix B

Correlation functions

In this chapter, we give an introduction to the calculation of correlation functions. First (Sec. 1), we discuss the massless boson (equivalent to the single chain of spinless fermions) and then (Sec. 2), we derive the pairing correlation function of the spinless two-leg ladder (see chapter 3).

B.1 Massless boson

Correlation functions can be calculated using (imaginary time, $\tau = it$) path integrals. In the bosonized form, the integration is done over the fields $\Phi$ and $\Pi$. For an operator $O$, the correlation function is then given by

$$\langle O^\dagger(x)O(0) \rangle \propto \int \mathcal{D}\Phi\mathcal{D}\Pi O^\dagger(x)O(0)e^{-S},$$  \hspace{1cm} (B.1)

where $S = \int dxd\tau (\Pi \partial_\tau \Phi - \mathcal{H})$ is the action of the system with Hamiltonian-density $\mathcal{H}$.

The action of the massless boson can be expressed as a function of $\Phi$ and reads

$$S = -\frac{1}{2K} \int dxd\tau \Phi \left(\frac{1}{u} \partial_\tau^2 + u \partial_x^2\right) \Phi,$$  \hspace{1cm} (B.2)

where $u$ is the velocity and $K$ the LLP. The Green's function $G$ satisfying

$$-\frac{1}{K} \left(\frac{1}{u} \partial_\tau^2 + u \partial_x^2\right) G(x, \tau) = \delta(x)\delta(\tau)$$ \hspace{1cm} (B.3)

is given by

$$G(x, \tau) = \frac{K}{4\pi} \ln \left(\frac{R^2}{x^2 + u^2\tau^2 + \alpha^2}\right),$$  \hspace{1cm} (B.4)

93
where $\alpha$ is a short-distance cutoff and $R$ the integration boundary in the complex plane. The two-point correlation function takes the form

$$\langle \exp\{i\beta[\Phi(x, \tau) - \Phi(0, 0)]\} \rangle \propto \exp[\beta^2 G(x, \tau)] = (x^2 + u^2)^{-K\beta^2/(4\pi)} \quad (B.5)$$

The one-point correlation function is equal to 0, $\langle e^{i\beta\Phi(x)} \rangle = 0$.

Next, we use these results to calculate the correlation functions of the single chain of spinless fermions. The CDW operator is $O_{\text{CDW}} = \Psi_R^\dagger \Psi_L \propto e^{i\sqrt{4\pi} \Phi}$ and the pairing operator is $O_{\text{SS}} = \Psi_R \Psi_L \propto e^{i\sqrt{4\pi} \theta}$. We then find, that

$$\langle O_{\text{CDW}}^\dagger(x) O_{\text{CDW}}(0) \rangle \propto x^{-2K}. \quad (B.6)$$

For the calculation of the pairing correlation function, we first perform a canonical transformation from $(\Phi, \Pi)$ to $(\tilde{\Phi}, \tilde{\Pi}) = (\theta, \partial_x \Phi)$. In the new fields, the action has the same form as Eq. (B.2) but with $K = 1/K$, i.e.,

$$\langle O_{\text{SS}}^\dagger(x) O_{\text{SS}}(0) \rangle \propto x^{-2/K}. \quad (B.7)$$

For $K > 1$ (attractive interactions), superconducting pairing dominates and for $K < 1$ (repulsive interactions), the chain is a CDW metal. Note that the pairing operator has odd parity, $O_{\text{SS}}(-x) = -O_{\text{SS}}(x)$ (for spin-1/2 fermions, the parity is even).

### B.2 Spinless two-leg ladder

Next, we calculate the correlation functions of the spinless two-leg ladder for the case, when the $\theta_-$ field is pinned, see chapter 3.

The action of the spinless two-leg ladder [resulting from the Hamiltonian (3.25)] is invariant under the shift $\Phi_+(x) \rightarrow \Phi_+(x) + c$, where $c$ is any constant. Similarly as it is the case for the action of the massless boson, we then obtain (for a comparison, see Ref. [78])

$$\langle e^{i\sum_j \beta_j \Phi_+(x_j)} \rangle = 0, \quad (B.8)$$

if $\sum_j \beta_j \neq 0$. In particular, the one-point correlation function of the $\Phi_+$ field is equal to 0, $\langle e^{i\beta \Phi_+(x)} \rangle = 0$, i.e., the $\Phi_+$ field is indeed a free (unpinned) field. The same holds for the dual field $\theta_+$. We conclude that the mixing term (3.28) is for $v_1 \neq v_2$ an analytical perturbation.
B.2. SPINLESS TWO-LEG LADDER

Using the Green's function (B.4), we carry out the integration over the fields $\Pi_+$ and $\Phi_+$ and obtain for the equal time CDW correlation function

$$\left\langle e^{i\theta \Phi_+(x)} e^{-i\theta \Phi_+(0)} \right\rangle \propto x^{-2K_+ \beta^2/4\pi} \left\langle e^{S^n_{\text{mix}}(x)} \right\rangle_{SG}.$$  (B.9)

The expression $S^n_{\text{mix}}$ depends (nonlocally) on the fields $\Phi_-, \Pi_-$ vanishing for $v_1 = v_2$ (for simplicity, we drop in the following the short-distance cutoff $\alpha$),

$$S^n_{\text{mix}} = - \int dz_1 dz_2 \left( v^c \partial^2_{x_1} \Phi_- (1) + \frac{i v^p}{K_+ u_+} \partial_{x_1} \Pi_- (1) \right) \times \frac{K_+}{8\pi} \ln \left( x_{12}^2 + u^2 \tau_{12}^2 \right) (1 \leftrightarrow 2)$$

$$- \frac{\beta i K_+}{4\pi} \int dz_1 \left( v^c \partial^2_{x_1} \Phi_- + \frac{i v^p}{K_+ u_+} \partial_{x_1} \Pi_- \right) \ln \left[ \frac{(x - x_1)^2 + u_+^2 \tau_1^2}{x_1^2 + u_+^2 \tau_1^2} \right].$$  (B.10)

where $(1) = (x_1, \tau_1), \ x_{12} = x_1 - x_2, \ \tau_{12} = \tau_1 - \tau_2$, and $dz_1 = dz_1 d\tau_1$. The average is taken with the sine-Gordon action resulting from Eq. (3.27). Since the $\Phi_-$ field is unpinned, the integration over $\Phi_-$ gives corrections $\propto (v^c)^2$ (and higher order) to the exponent of $x$.

Next, we carry out a canonical transformation from the fields $(\Phi_-, \Pi_-)$ to $(\bar{\Phi}_-, \bar{\Pi}_-) = (\theta_-, \partial_{x_1} \Phi_-)$. Rewriting $S^n_{\text{mix}}$ in terms of the new fields then allows us to carry out the integration over $\bar{\Pi}_-$. The $\bar{\Pi}_-$ part has the form,

$$- \frac{u_-}{2K_-} \int dz_1 \bar{\Pi}_- (1) \left[ \bar{\Pi}_- (1) + \frac{K_+ K_-}{4\pi u_-} (v^c)^2 \right]$$

$$\times \int dz_2 \bar{\Pi}_- (2) \partial_{x_1} \partial_{x_2} \ln \left( x_{12}^2 + u^2 \tau_{12}^2 \right).$$  (B.11)

The inverse of the above operator on $\bar{\Pi}_-$ can be expanded in a power series in $(v^c)^2$. The linear part in $\bar{\Pi}_-$ reads

$$i \int dz_1 \bar{\Pi}_- (1) \left\{ \frac{\beta K_+ v^c}{4\pi} \partial_{x_1} \ln \left[ \frac{(x - x_1)^2 + u_+^2 \tau_1^2}{x_1^2 + u_+^2 \tau_1^2} \right] + \partial_{\tau_1} \Phi_- (1) \right\}$$

$$+ \frac{v^p v^c}{4\pi u_+} \int dz_2 \partial_{x_1} \ln \left( x_{12}^2 + u_+^2 \tau_{12}^2 \right) \partial_{x_2} \partial_{\tau_2} \Phi_- (2).$$  (B.12)

Carrying out the integration over $\bar{\Pi}_-$, we obtain the following contribution to the correlation function in $(v^c)^2$,

$$- \frac{K_-}{2u_-} \left( \frac{\beta K_+ v^c}{4\pi} \right)^2 \int dz_1 \left\{ \partial_{x_1} \ln \left[ \frac{(x - x_1)^2 + u_+^2 \tau_1^2}{x_1^2 + u_+^2 \tau_1^2} \right] \right\}^2.$$  (B.13)
The integral is equal to \( 8\pi (\ln x)/u_+ \). Including all orders in \( v_\tau \), the correlation function finally decays \( \propto x^{-\gamma_c} \), where
\[
\gamma_c = \frac{\beta^2 K_+}{2\pi} \frac{1}{1 - \frac{K_+ K_-}{2u_+ u_-} (v_\tau)^2}.
\] (B.14)

Similarly, the pairing correlation function takes the form
\[
\langle e^{i\beta \theta_+(x)} e^{-i\beta \theta_+(0)} \rangle \propto x^{-2\beta^2/(4\pi K_+)} \langle e^{S_{\text{mix}}^\theta(x)} \rangle_{\text{SG}},
\] (B.15)

where again \( S_{\text{mix}}^\theta \) depends (nonlocally) on the fields \( \Phi_- \), \( \Pi_- \), also vanishing for \( v_1 = v_2 \),
\[
S_{\text{mix}}^\theta = \frac{\beta}{2\pi} \int dz_1 \left( v_\tau \partial^2_{x_1} \Phi_- + \frac{i v_\tau^P}{K_+ u_+} \partial_{\tau_1} \Pi_- \right)
\times \left[ \arctan \left( \frac{x_1 - x}{u_+ \tau_1} \right) - \arctan \left( \frac{x_1}{u_+ \tau_1} \right) \right]
- \int dz_1 dz_2 \left( v_\tau \partial^2_{x_1} \Phi_- (1) + i \frac{v_\tau^P}{K_+ u_+} \partial_{\tau_1} \Pi_- (1) \right)
\times \frac{K_+}{8\pi} \ln \left( x_{12}^2 + u_+^2 \tau_{12}^2 \right) \left( 1 \leftrightarrow 2 \right).
\] (B.16)

The logarithm in (B.10) and the arccosine in (B.16) give after partial integration a similar leading order contribution. The difference comes from the \( iK_+ \) present in (B.10) but not in (B.16). Here, there are only terms \( \propto (v_\tau)^2 \). The combination \( \partial_x \arctan(\cdot) \partial_x \ln(\cdot) \) does not give logarithmic contributions after integration. A similar calculation as above leads to a decay \( \propto x^{-\gamma_p} \), where
\[
\gamma_p = \frac{\beta^2}{2\pi K_+} \left[ 1 - \frac{K_+ K_-}{2u_+ u_-} (v_\tau)^2 \right].
\] (B.17)

In both cases, the remaining part in the \( \hat{\Phi}_- \) fields is either real (and does therefore pin the field) or it is imaginary but multiplied with derivatives of logarithmes being strongly peaked at \( \tau_1 = 0 \) and \( x_1 = x \) or \( x_1 = 0 \), resulting in a effective contribution \( \propto i[\hat{\Phi}_-(x) - \hat{\Phi}_-(0)] \). In both cases, the remaining part then becomes a constant \( (\neq 0) \) for large \( x \).

Comparing (B.14) with (B.17), we see that one can express both correlation functions in terms of a single exponent,
\[
\gamma = \frac{K_+}{1 - \frac{K_+ K_-}{2u_+ u_-} (v_\tau)^2},
\] (B.18)
such that the CDW correlation function decays \( \propto x^{-\frac{\beta^2}{2}} \) and the pairing correlation function \( \propto x^{-\frac{\beta^2}{2\pi}} \). In our case, \( \beta^2 = 2\pi \).
Bibliography


100

BIBLIOGRAPHY


[88] For a related work with flat Fermi surface (but without umklapp interactions), see S. Dusuel, F. Vistulo de Abreu, and B. Douçot, cond-mat/0107548.


Curriculum Vitae

Personen:

Name: Urs Ledermann  
Geburtsdatum: 25.4.1973  
Nationalität: Schweizer  

Ausbildung:

3. Studienjahr in Lyon, Frankreich
Diplomarbeit bei Prof. G. Blatter
11/1998-10/2001 Assistent und Doktorand an der ETH Zürich
Doktorarbeit bei Prof. T.M. Rice