Doctoral Thesis

Embedding problems in symplectic geometry

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Embedding problems in symplectic geometry

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Contents

Abstract i
Zusammenfassung iii
Acknowledgements v
Introduction vii

1 Rigidity 1
1.1 Comparison of the relations $\leq_i$ 1
1.2 Rigidity for ellipsoids 2
1.3 Rigidity for polydiscs? 5

2 Proof of Theorem 2 9
2.1 Reformulation of Theorem 2 9
2.2 The folding construction 18
2.3 End of the proof 26

3 Multiple symplectic folding in four dimensions 31
3.1 Modification of the folding construction 31
3.2 Multiple folding 32
3.3 Embeddings into balls 37
3.4 Embeddings into cubes 54

4 Symplectic folding in higher dimensions 65
4.1 Four types of folding 65
4.2 Embedding polydiscs into cubes 67
4.3 Embedding ellipsoids into balls 74

5 Proof of Theorem 3 95
5.1 Proof of $\lim_{a \to \infty} p_a^p(M, \omega) = 1$ 95
5.2 Proof of $\lim_{a \to \infty} p_a^E(M, \omega) = 1$ 112
5.3 Asymptotic embedding invariants 140

6 Symplectic versus Lagrangian folding 143
6.1 Lagrangian folding 143
6.2 Comparison of symplectic and Lagrangian folding 152
Abstract

Fix a 2n-dimensional symplectic manifold \((M, \omega)\) and an open subset \(U\) of standard symplectic space \((\mathbb{R}^{2n}, \omega_0)\), where

\[
\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.
\]

The symplectic embedding problem associated with \(U\) and \((M, \omega)\) asks for the largest \(\alpha > 0\) for which there exists a symplectic embedding of \(\alpha U = \{\alpha z \mid z \in U\}\) into \((M, \omega)\). We choose \(U\) to be an open symplectic ellipsoid

\[
E(r_1, \ldots, r_n) = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} \frac{|z_i|^2}{r_i^2} < 1\right\}
\]

or a polydisc

\[
P(r_1, \ldots, r_n) = B^2(r_1) \times \cdots \times B^2(r_n).
\]

Here, \(B^{2n}(r)\) denotes the open ball of radius \(r\). We may assume that \(r_1 \leq r_2 \leq \cdots \leq r_n\). We first choose \((M, \omega)\) to be a ball.

**Theorem 1** Assume \(r_2^2 < 2 r_1^2\). Then there does not exist a symplectic embedding of the ellipsoid \(E(r_1, \ldots, r_n)\) into the ball \(B^{2n}(R)\) if \(R < r_n\).

This non-embedding result is sharp and therefore sheds some light on the power of Ekeland-Hofer capacities, which are used in its proof. Indeed, we have the following embedding result.

**Theorem 2** Assume \(r_n^2 > 2 r_1^2\). Then there exists a symplectic embedding of \(E(r_1, \ldots, r_1, r_n)\) into \(B^{2n}(R)\) for all \(\sqrt{r_n^2/2 + r_1^2} < R < r_n\).

The basic ingredient in the proofs of all our embedding results is a refinement of a symplectic folding method first used by F. Lalonde and D. McDuff. In particular, Theorem 2 is proved by folding an ellipsoid once. It can be substantially improved by folding more often. We in particular show that a ball can be asymptotically filled with skinny ellipsoids. More generally, assume that \((M, \omega)\) is a
connected symplectic manifold of finite volume \( \text{Vol} (M^{2n}, \omega) = \frac{1}{n!} \int_M \omega^n \). For \( r > 1 \) set

\[
p_r(M, \omega) = \sup_{a} \frac{\text{Vol}(\alpha E(1, \ldots, 1, r))}{\text{Vol}(M, \omega)}
\]

where the supremum is taken over all \( \alpha \) for which \( \alpha E^{2n}(1, \ldots, 1, r) \) symplectically embeds into \((M, \omega)\).

**Theorem 3** \( \lim_{r \to \infty} p_r(M, \omega) = 1 \).

An analogous result holds for polydiscs.

We finally use the symplectic folding method to answer a question about symplectic embeddings of balls into the standard symplectic cylinder

\[
Z = B^2(1) \times \mathbb{R}^{2n-2} \subset \left( \mathbb{R}^{2n}, \omega_0 \right).
\]

By Gromov’s Nonsqueezing Theorem there is no symplectic embedding of \( B^{2n}(r) \) into \( Z \) if \( r > 1 \). So assume \( r \leq 1 \), and consider the symplectic invariant

\[
c(r) = \inf_{\varphi} \sup_x \text{area} \left( \varphi(B^{2n}(r)) \cap D_x \right)
\]

where \( \varphi \) varies over all symplectic embeddings of \( B^{2n}(r) \) into \( Z \) and where \( D_x \subset Z \) denotes the 2-dimensional open disc \( D_x = B^2(1) \times \{x\}, x \in \mathbb{R}^{2n-2} \). Contrary to expectations we have

**Theorem 4** \( c(r) = 0 \) for all \( r \in ]0, 1[ \).
Zusammenfassung

Sei \((M, \omega)\) eine \(2n\)-dimensionale symplektische Mannigfaltigkeit, und sei \(U\) eine offene Teilmenge des symplektischen Raumes \((\mathbb{R}^{2n}, \omega_0)\), wobei

\[
\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.
\]

Das \(U\) und \((M, \omega)\) zugeordnete symplektische Einbettungsproblem fragt nach der größten Zahl \(\alpha > 0\) für welche die Menge \(\alpha U = \{a \cdot z \mid z \in U\}\) symplektisch in \((M, \omega)\) eingebettet werden kann. Wir wählen \(U\) als ein offenes symplektisches Ellipsoid

\[
E(r_1, \ldots, r_n) = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} \left|z_i\right|^2 / r_i^2 < 1 \right\}
\]

oder als eine Polyscheibe

\[
P(r_1, \ldots, r_n) = B(r_1) \times \ldots \times B(r_n).
\]

Hier bezeichnet \(B^{2n}(r)\) den offenen Ball vom Radius \(r\). Wir können annehmen, dass \(r_1 \leq r_2 \leq \cdots \leq r_n\). Wir betrachten zuerst den Fall, dass \((M, \omega)\) ein Ball ist.

**Satz 1** Sei \(r_n^2 \leq 2r_1^2\). Dann existiert keine symplektische Einbettung des Ellipsoids \(E(r_1, \ldots, r_n)\) in den Ball \(B^{2n}(R)\) falls \(R < r_n\).


**Satz 2** Sei \(r_n^2 > 2r_1^2\). Dann existiert eine symplektische Einbettung des Ellipsoids \(E(r_1, \ldots, r_1, r_n)\) in \(B^{2n}(R)\) für alle \(\sqrt{r_n^2 / 2 + r_1^2} < R < r_n\).

Die grundlegende Technik in den Beweisen aller unserer Einbettungssätze ist eine Verfeinerung einer symplektischen Faltungsmethode, die zuerst von F. Lalonde and D. McDuff verwendet wurde. Wir beweisen Satz 2 denn auch indem wir ein Ellipsoid einmal falten. Satz 2 kann durch mehrfaches Falten wesentlich
verbessert werden. Wir zeigen insbesondere, dass ein Ball asymptotisch mit dünnen Ellipsoïden gefüllt werden kann. Wir betrachten allgemeiner eine zusammenhängende 2n-dimensional symplektische Mannigfaltigkeit $(M, \omega)$ von endlichem Volumen $\text{Vol}(M, \omega) = \frac{1}{n!} \int_M \omega^n$. Für $r > 1$ definieren wir

$$p_r(M, \omega) = \sup_{\alpha} \frac{\text{Vol}(\alpha E(1, \ldots, 1, r))}{\text{Vol}(M, \omega)}$$

wobei das Supremum über all jene $\alpha$ genommen wird für die $\alpha E^{2n}(1, \ldots, 1, r)$ symplektisch in $(M, \omega)$ eingebettet werden kann.

**Satz 3** \( \lim_{r \to \infty} p_r(M, \omega) = 1 \).

Ein ähnlicher Satz gilt für Polyscheiben.

Wir verwenden die symplektische Faltungsmethode schliesslich, um eine Frage über symplektische Einbettungen von Bällen in den symplektischen Zylinder

$$Z = B^2(1) \times \mathbb{R}^{2n-2} \subset (\mathbb{R}^{2n}, \omega_0)$$

zu beantworten. Aufgrund von Gromovs Nichteinbettungssatz existiert keine symplektische Einbettung von $B^{2n}(r)$ in $Z$ falls $r > 1$. Wir nehmen daher $r \leq 1$ an und betrachten die symplektische Invariante

$$c(r) = \inf_{\varphi} \sup_x \text{area} \left( \varphi(B^{2n}(r)) \cap D_x \right)$$

wobei $\varphi$ über alle symplektischen Einbettungen von $B^{2n}(r)$ in $Z$ läuft und $D_x \subset Z$ die 2-dimensionale offene Scheibe $D_x = B^2(1) \times \{x\}, x \in \mathbb{R}^{2n-2}$, bezeichnet. Entgegen den Erwartungen gilt

**Satz 4** \( c(r) = 0 \) für alle $r \in ]0, 1[$. 
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Doing symplectic geometry at ETH has been greatly facilitated through the existence of the symplectic group. When I started this thesis in 1996, this group consisted of Casim Abbas, Michel Andenmatten, Kai Cieliebak, Hansjörg Geiges, Helmut Hofer, Markus Kriener, Torsten Linnemann, Laurent Moatzy, Matthias Schwarz, Karl Friedrich Siburg, Edi Zehnder and myself, and when I finished my thesis, the group consisted of Meike Akveld, Urs Frauenfelder, Ralph Gautschi, Janko Latchev, Thomas Mautsch, Dietmar Salamon, Joa Weber, Katrin Wehrheim, Edi Zehnder and myself.

This thesis is the visible fruit of my studies in mathematics. A more important fruit are the friendships with Rolf Heeb, Laurent Lazzarini, Christian Rüede and Ivo Stalder.

Sana, Selin kedim, sonsuz sabır ve sevgin için teşekkür ederim.
Seite Leer / Blank leaf
**Introduction**

Consider a connected smooth $n$-dimensional manifold $M$. A volume form on $M$ is a smooth nowhere vanishing $n$-form $\Omega$. It follows that $M$ is orientable. We orient $M$ such that $\int_M \Omega$ is positive, and we write $\text{Vol} (M, \Omega) = \int_M \Omega$. We endow each open (not necessarily connected) subset $U$ of $\mathbb{R}^n$ with the Euclidean volume form

$$\Omega_0 = dx_1 \wedge \cdots \wedge dx_n.$$ 

A smooth embedding $\varphi : U \hookrightarrow M$ is called volume preserving if

$$\varphi^* \Omega = \Omega_0.$$

Then $\text{Vol} (U, \Omega_0) \leq \text{Vol} (M, \Omega)$. The following proposition shows that this obvious condition for the existence of a volume preserving embedding is the only one.

**Proposition 1** The set $U$ embeds into $M$ by a smooth volume preserving embedding if and only if $\text{Vol} (U, \Omega_0) \leq \text{Vol} (M, \Omega)$.

A proof of this result can be found in Appendix A.

A symplectic manifold $(M, \omega)$ is a smooth manifold $M$ endowed with a smooth non-degenerate closed 2-form $\omega$. The non-degeneracy of $\omega$ implies that $\frac{1}{n!} \omega^n$ is a volume form, and that $M$ is even dimensional, $\text{dim} M = 2n$. We endow each open subset $U$ of $\mathbb{R}^{2n}$ with the standard symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

A smooth embedding $\varphi : U \hookrightarrow M$ is called symplectic if

$$\varphi^* \omega = \omega_0.$$

In particular, every symplectic embedding preserves the volume forms $\Omega_0 = \frac{1}{n!} \omega_0^n$ and $\frac{1}{n!} \omega^n$ induced by the symplectic forms. Given an open subset $U$ of $\mathbb{R}^{2n}$ and $\lambda > 0$ we set $\lambda U = \{ \lambda z \in \mathbb{R}^{2n} \mid z \in U \}$. In the symplectic world, the question behind Proposition 1 becomes

**Problem 1** What is the largest number $\lambda$ such that $(\lambda U, \omega_0)$ symplectically embeds into $(M, \omega)$?
In dimension 2, an embedding is volume preserving if and only if it is symplectic, and so Problem 1 is completely solved by Proposition 1. In higher dimensions, however, strong symplectic rigidity phenomena appear. Denote the open $2n$-dimensional ball of radius $r$ by $B^{2n}(\pi r^2)$ and the open $2n$-dimensional symplectic cylinder $B^2(a) \times \mathbb{R}^{2n-2}$ by $Z^{2n}(a) = B^2(a) \times \mathbb{R}^{2n-2}$.

**Examples 1.** (Gromov's Nonsqueezing Theorem [12]) For $n \geq 2$, the ball $B^{2n}(a)$ symplectically embeds into the cylinder $Z^{2n}(A)$ only if $A \geq a$.

2. [15] For $n \geq 2$, there exist bounded starshaped domains $U \subset \mathbb{R}^{2n}$ which have arbitrarily small volume but do not symplectically embed into $B^{2n}(\pi)$.

On the other hand, the following two results suggest that the situation in Problem 1 becomes less rigid if $U$ is "thin". We denote by

$$E(a_1, \ldots, a_n) = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^n \frac{\pi |z_i|^2}{a_i} < 1 \right\}$$

the open symplectic ellipsoid in $\mathbb{R}^{2n}$ with radii $\sqrt{a_i/\pi}$.

**Examples 3.** ([13, p. 335] and [10, p. 579]) Consider a $2n$-dimensional symplectic manifold $(M, \omega)$. For any $a > 0$ there exists a (possibly very small!) $\epsilon > 0$ such that the ellipsoid $E^{2n}(\epsilon, \ldots, \epsilon, a)$ symplectically embeds into $M$.

4. (Traynor, [34, Theorem 6.4]) For all $k \geq 1$ and $\epsilon > 0$ there exists a symplectic embedding

$$E\left(\frac{\pi}{k+1}, k\pi\right) \hookrightarrow B^4(\pi + \epsilon).$$

Examples 2 and 4 show that already the following special case of Problem 1 is interesting:

**Problem 2** What is the smallest ball $B^{2n}(A)$ into which $U$ symplectically embeds?

In this work we investigate the zone of transition between rigidity and flexibility in Problems 1 and 2. The main tool of detecting embedding obstructions will be special symplectic invariants, the so called symplectic capacities (see [18] and
Chapter 1). Unfortunately, symplectic capacities can be computed only for very special sets. Therefore, we look at a model situation in which the set $U$ is a symplectic ellipsoid $E(a_1, \ldots, a_n)$. Since a permutation of the symplectic coordinate planes is a (linear) symplectic map, we may assume $a_1 \leq a_2 \leq \cdots \leq a_n$. We first discuss our answers to Problem 2. Our main rigidity result is

**Theorem 1** Assume $a_n \leq 2a_1$. Then there does not exist a smooth symplectic embedding of the ellipsoid $E(a_1, \ldots, a_n)$ into the ball $B^{2n}(A)$ if $A < a_n$.

Our proof uses the $n$'th Ekeland–Hofer capacity. In the special case $n = 2$, Theorem 1 was proved in [10] as an application of symplectic homology. The argument given here is much simpler and works in all dimensions.

Our first embedding result states that Theorem 1 is sharp.

**Theorem 2** Assume $a_n > 2a_1$. Then there exists a smooth symplectic embedding of the ellipsoid $E(a_1, \ldots, a_1, a_n)$ into the ball $B^{2n}(a_n - \delta)$ for every $\delta \in \left]0, \frac{a_n}{2} - a_1\right]$. 

The reader might ask why we assume $a_{n-1} = a_1$ in Theorem 2. This is because a much better result cannot be expected. Indeed, we will show that for $n \geq 3$ the ellipsoid $E^{2n}(a_1, 3a_1, \ldots, 3a_1)$ does not symplectically embed into the ball $B^{2n}(A)$ if $A < 3a_1$.

In the special case $n = 2$, Lalonde and McDuff observed in [21] that Theorem 2 can be proved by their technique of symplectic folding. A refinement of their method will prove Theorem 2 in all dimensions.

Theorem 2 can be substantially improved by multiple symplectic folding. For the sake of clarity we restrict ourselves to dimension 4. We can assume that $a_1 = \pi$. The optimal function for the embedding problem $E(\pi, a) \leftrightarrow B^4(A)$ is the function $f_{EB}$ on $[\pi, \infty]$ defined by

$$f_{EB}(a) = \inf \left\{ A \mid E(\pi, a) \text{ symplectically embeds into } B^4(A) \right\}.$$ 

We illustrate our results with the help of Figure 1. In view of Theorem 1 we have $f_{EB}(a) = a$ for $a \in [\pi, 2\pi]$. For $a > 2\pi$, the second Ekeland–Hofer capacity still implies that $f_{EB}(a) > 2\pi$. This information is vacuous if $a \geq 4\pi$, since the volume condition $\text{Vol} (E(\pi, a)) \leq \text{Vol} (B^4(f_{EB}(a)))$ translates to $f_{EB}(a) \geq \sqrt{\pi a}$. The estimate $f_{EB}(a) \leq a/2 + \pi$ stated in Theorem 2 is obtained by folding.
once. We define the function \( s_{EB} \) on \([\pi, \infty[\) by

\[
s_{EB}(a) = \inf \left\{ A \mid E(\pi, a) \text{ embeds into } B^4(A) \text{ by multiple symplectic folding} \right\}.
\]

It will turn out that \( s_{EB}(a) \) is obtained from folding "infinitely many times". The graph of the function \( s_{EB} \) is computed by a computer program.

\[
A = a \quad A = \frac{a}{2} + \pi
\]

\[
A = \sqrt{\pi a}
\]

\[
f_{EB}(a) \quad s_{EB}(a)
\]

\[
A = 2\pi, \quad 4\pi, \quad 5\pi, \quad 25\pi \quad a
\]

\[
e_{EH}
\]

Figure 1: The result for the embedding problem \( E(\pi, a) \hookrightarrow B^4(A) \).

We are particularly interested in the behaviour of \( f_{EB}(a) \) as \( a \to 2\pi^+ \) and as \( a \to \infty \). We shall prove that

\[
\limsup_{\epsilon \to 0^+} \frac{s_{EB}(2\pi + \epsilon) - 2\pi}{\epsilon} \leq \frac{3}{7},
\]

and so the same estimate holds for \( f_{EB} \).

**Question 1** How does \( f_{EB}(a) \) look like near \( a = 2\pi \)? In particular,

\[
\limsup_{\epsilon \to 0^+} \frac{f_{EB}(2\pi + \epsilon) - 2\pi}{\epsilon} < \frac{3}{7}?
\]

We shall also prove that the difference \( s_{EB}(a) - \sqrt{\pi a} \) is bounded. Therefore, \( f_{EB}(a) - \sqrt{\pi a} \) is bounded. This also follows from Traynor's theorem stated in Example 4, cf. Figure 53 below. We in particular have

\[
\lim_{a \to \infty} \frac{\text{Vol} (E(\pi, a))}{\text{Vol} \left(B^4(f_{EB}(a))\right)} = 1. \quad (\star)
\]
In abuse of notation we denote in the following by $D(a)$ the open disc in $\mathbb{R}^2$ of area $a$ (and not of radius $a$) centered at the origin and by $P(a_1, \ldots, a_n)$ the open symplectic polydisc

$$P(a_1, \ldots, a_n) = D(a_1) \times \cdots \times D(a_n).$$

The "$n$-cube" $P^{2n}(a, \ldots, a)$ will be denoted by $C^{2n}(a)$. Symplectic folding can also be applied to the following variation of Problem 2.

**Problem 3** What is the smallest cube $C^{2n}(A)$ into which $U$ symplectically embeds?

While embedding $U$ into a minimal ball is related to minimizing its diameter (a 1-dimensional, metric quantity), embedding $U$ into a minimal cube amounts to minimizing the areas of its projections to the symplectic coordinate planes (a 2-dimensional, more symplectic quantity); see Appendix B for details. We will therefore also study symplectic embeddings of ellipsoids and polydiscs into cubes. We refer to the body of the thesis for the results.

We now discuss our answer to Problem 1. In view of the identity $(*)$, a four dimensional ball can be asymptotically symplectically filled by skinny ellipsoids. This also follows from Traynor's theorem stated in Example 4, which she obtained from a Lagrangian folding method. Symplectic folding, however, can be used to prove such a result for any connected symplectic manifold $(M^{2n}, \omega)$ of finite volume $\text{Vol}(M, \omega) = \frac{1}{n!} \int_M \omega^n$. For $a \geq \pi$ we define

$$p^E_a(M, \omega) = \sup_\lambda \frac{\text{Vol}(\lambda E(\pi, \ldots, \pi, a))}{\text{Vol}(M, \omega)}$$

where the supremum is taken over all those $\lambda$ for which $\lambda E^{2n}(\pi, \ldots, \pi, a)$ symplectically embeds into $(M, \omega)$. We define $p^P_a(M, \omega)$ in a similar way by replacing the ellipsoid $E^{2n}(\pi, \ldots, \pi, a)$ by the polydisc $P^{2n}(\pi, \ldots, \pi, a)$.

**Theorem 3** Assume that $(M, \omega)$ is a connected symplectic manifold of finite volume. Then

$$\lim_{a \to \infty} p^E_a(M, \omega) = 1 \quad \text{and} \quad \lim_{a \to \infty} p^P_a(M, \omega) = 1.$$
We finally answer a question about symplectic embeddings of balls into the standard symplectic cylinder $Z^{2n}(\pi) = B^2(\pi) \times \mathbb{R}^{2n-2}$. Recall the

**Definition** An *extrinsic symplectic capacity* on $(\mathbb{R}^2, \omega_0)$ is a map $c$ associating with each subset $S$ of $\mathbb{R}^2$ a number $c(S) \in [0, \infty]$ in such a way that the following axioms are satisfied.

A1. **Monotonicity:** $c(S) \leq c(T)$ if there exists a symplectomorphism $\varphi$ of $\mathbb{R}^2$ such that $\varphi(S) \subseteq T$.

A2. **Conformality:** $c(\lambda S) = \lambda^2 c(S)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$.

A3. **Nontriviality:** $0 < c(B^2(\pi)) < \infty$.

After normalizing $c$ we may assume $c(B^2(\pi)) = \pi$. An example of a normalized extrinsic symplectic capacity on $\mathbb{R}^2$ is the outer Lebesgue measure $\mu$ on $\mathbb{R}^2$. There exist normalized extrinsic symplectic capacities on $\mathbb{R}^2$ which are larger than $\mu$, however. E.g., for any subset $S$ of $\mathbb{R}^2$ we define

$$z(S) = \inf \{ a \mid \text{there is a symplectomorphism } \varphi \text{ of } \mathbb{R}^2 \text{ with } \varphi(S) \subseteq B^2(a) \}.$$ 

Then $z$ is a normalized extrinsic symplectic capacity on $\mathbb{R}^2$, and for the circle

$$S^1 = \left\{ (u, v) \in \mathbb{R}^2 \mid u^2 + v^2 = 1 \right\}$$

we find $z(S^1) = \pi > 0 = \mu(S^1)$. Similarly, $c_1(S^1) = \pi$ for the first Ekeland–Hofer capacity $c_1$, see [6], and $e(S^1) = \pi$ for the displacement energy $e$, see [17].

We refer to Appendix B.2 for more information on symplectic capacities on $\mathbb{R}^2$.

By the Nonsqueezing Theorem stated in Example 1, there does not exist a symplectic embedding of the ball $B^{2n}(a)$ into the cylinder $Z^{2n}(\pi)$ if $a > \pi$. So fix $a \in ]0, \pi[$, and fix a normalized extrinsic symplectic capacity $c$ on $\mathbb{R}^2$. As we shall see in Corollary B.10 of the appendix, the Nonsqueezing Theorem is equivalent to the identity

$$\inf_{\varphi} c \left( p \left( \varphi(B^{2n}(a)) \right) \right) = a$$

where $\varphi$ varies over all symplectomorphisms of $\mathbb{R}^{2n}$ which embed $B^{2n}(a)$ into $Z^{2n}(\pi)$, and where $p: Z^{2n}(\pi) \to B^2(\pi)$ is the projection. Following [24] we consider sections of the image $\varphi(B^{2n}(a))$ instead of its projection, and define

$$\zeta_c(a) = \inf_{\varphi} \sup_x c \left( p \left( \varphi(B^{2n}(a)) \cap D_x \right) \right)$$
where $\varphi$ again varies over all symplectomorphisms of $\mathbb{R}^{2n}$ which embed $B^{2n}(a)$ into $Z^{2n}(\pi)$ and where $D_x \subset Z^{2n}(\pi)$ denotes the disc $D_x = B^2(\pi) \times \{x\}$, $x \in \mathbb{R}^{2n-2}$. Clearly $\zeta_c(a) \leq a$. Moreover, Lemma 1.2 in [23] implies that for any normalized extrinsic symplectic capacity $c$ on $\mathbb{R}^2$ for which $c(S^1) = \pi$ the Nonsqueezing Theorem is equivalent to the identity $\zeta_c(\pi) = \pi$. On her search for symplectic rigidity phenomena beyond the Nonsqueezing Theorem, D. McDuff therefore asked for lower bounds of the functions $\zeta_c(\pi)$ and whether $\zeta_c(a) \rightarrow \pi$ as $a \rightarrow \pi$. It was known to L. Polterovich that $\zeta_c(a)/a \rightarrow 0$ as $a \rightarrow 0$, see again [24]. We shall prove

**Theorem 4** $\zeta_c(a) = 0$ for all $a \in ]0, \pi[.$

The thesis is organized as follows: In Chapter 1 we prove Theorem 1 and several other rigidity results for ellipsoids. In Chapter 2 we prove Theorem 2 by symplectic folding. In Chapter 3 we use multiple symplectic folding to obtain rather satisfactory results for embeddings of 4-dimensional ellipsoids and polydiscs into 4-dimensional balls and cubes. In Chapter 4 we look at higher dimensions. We will concentrate on embedding skinny ellipsoids into balls and skinny polydiscs into cubes. The results in this chapter form half of the proof of Theorem 3, which is completed in Chapter 5. In Chapter 5 we shall also notice that for certain symplectic manifolds our embedding methods can be used to improve Theorem 3. In Chapter 6 we recall the Lagrangian folding method invented by Traynor, and compare the results yielded by symplectic and Lagrangian folding. In Chapter 7 we prove Theorem 4 and its generalizations.

Appendix A contains a proof of Proposition 1. In Appendix B we clarify the relations between the invariants defined by Problem 2 and Problem 3 and other symplectic invariants. In Appendix C we review the Extension after Restriction Principle and prove an extension of this principle to unbounded domains. Appendix D provides computer programs necessary to compute the optimal embeddings of 4-dimensional ellipsoids into a 4-ball and a 4-cube obtainable by our methods.

We write $|x|$ for the Euclidean norm of a point $x \in \mathbb{R}^n$ and $|U|$ for the Lebesgue measure of an open set $U \subset \mathbb{R}^n$. We work in the $C^\infty$-category, i.e., all manifolds and diffeomorphisms are assumed to be $C^\infty$-smooth, and so are all symplectic forms and maps.
1 Rigidity

Denote by $\mathcal{O}(n)$ the set of bounded domains in $\mathbb{R}^{2n}$ diffeomorphic to a ball. Endow each $U \in \mathcal{O}(n)$ with the standard symplectic structure $\omega_0 = \sum_{i=1}^{2n} dx_i \wedge dy_i$. Recall that we write $|U|$ for the volume of $U$ with respect to the Euclidean volume form $\Omega_0 = \frac{1}{n!} \omega_0^n$. Let $\mathcal{D}(n)$ be the group of symplectomorphisms of $\mathbb{R}^{2n}$ and $\mathcal{D}_c(n)$ respectively $Sp(n; \mathbb{R})$ the subgroups of compactly supported respectively linear symplectomorphisms of $\mathbb{R}^{2n}$. Define the following relations on $\mathcal{O}(n)$:

\[ U \leq_1 V \iff \text{There exists a } \varphi \in Sp(n; \mathbb{R}) \text{ with } \varphi(U) \subset V. \]
\[ U \leq_2 V \iff \text{There exists a } \varphi \in \mathcal{D}(n) \text{ with } \varphi(U) \subset V. \]
\[ U \leq_3 V \iff \text{There exists a symplectic embedding } \varphi: U \hookrightarrow V. \]

1.1 Comparison of the relations $\leq_i$

Clearly, $\leq_1 \Rightarrow \leq_2 \Rightarrow \leq_3$. It is, however, well known that all the relations $\leq_i$ are different.

**Proposition 1.1** The relations $\leq_i$ are all different.

**Proof.** That the relations $\leq_1$ and $\leq_2$ are different follows from the equivalence (1.3) below and Example 4 of the introduction.

The construction of sets $U$ and $V \in \mathcal{O}(n)$ with $U \leq_3 V$ but $U \not\leq_2 V$ relies on the following simple observation. Suppose that $U$ and $V$ not only fulfill $U \leq_3 V$ but are symplectomorphic, whence, in particular, $|U| = |V|$. Thus, if $U \leq_2 V$ and $\varphi$ is a map realizing $U \leq_2 V$, no point of $\mathbb{C}^n \setminus U$ can be mapped to $V$, and we conclude that $\varphi(\partial U) = \partial V$. In particular, the characteristic foliations on $\partial U$ and $\partial V$ are isomorphic, and if $\partial U$ is of contact type, then so is $\partial V$ (see [18] for basic notions in Hamiltonian dynamics). Let now $U = B^{2n}(\pi)$, let

\[ SD = D(\pi) \setminus \{(x, y) \mid x \geq 0, y = 0\} \]

be the slit disc and set $V = B^{2n}(\pi) \cap (SD \times \cdots \times SD)$. Traynor proved in [34] that for $n \leq 2$, $V$ is symplectomorphic to $B^{2n}(\pi)$. But $\partial U$ and $\partial V$ are not even diffeomorphic. For $n \geq 2$ very different examples were found in [8] and [5]. Theorem 1.1 in [8] and its proof show that there exist $U, V \in \mathcal{O}(n)$ with smooth convex
boundaries such that $U$ and $V$ are symplectomorphic and $C^\infty$-close to $B^{2n}(\pi)$, but the characteristic foliation of $\partial U$ contains an isolated closed orbit while the one of $\partial V$ does not. And Corollary A in [5] and its proof imply that given any $U \in \mathcal{O}(n), n \geq 2,$ with smooth boundary $\partial U$ of contact type, there exists a symplectomorphic and $C^0$-close set $V \in \mathcal{O}(n)$ whose boundary is not of contact type. We in particular see that even for $U$ being a ball, the relation $\leq_3$ does not imply the relation $\leq_2$.

\section*{1.2 Rigidity for ellipsoids}

Proposition 1.1 shows that in order to detect some rigidity via the relations $\leq_i$ we must pass to a small subcategory of sets: Let $\mathcal{E}(n)$ be the collection of symplectic ellipsoids described in the introduction,

$$\mathcal{E}(n) = \{ E(a) = E(a_1, \ldots, a_n), \quad a = (a_1, \ldots, a_n) \},$$

and write $\preceq_i$ for the restrictions of the relations $\leq_i$ to $\mathcal{E}(n)$. Notice again that

$$\preceq_1 \implies \preceq_2 \implies \preceq_3. \quad (1.1)$$

The relations $\preceq_2$ and $\preceq_3$ are actually very similar: Since ellipsoids are starlike, the Extension after Restriction Principle implies

$$E(a) \preceq_3 E(a') \implies E(\delta a) \preceq_2 E(a') \quad \text{for all } \delta \in ]0, 1[ \quad (1.2)$$

(see [6] or Appendix C for details). It is, however, not known whether $\preceq_2$ and $\preceq_3$ are the same: While Theorem 1.4 proves this under an additional condition, the folding construction of Section 2.2 suggests that $\preceq_2$ and $\preceq_3$ are different in general. But let us first prove a general and common rigidity property of these relations:

**Proposition 1.2** The relations $\preceq_i$ are partial orderings on $\mathcal{E}(n)$.

**Proof.** The relations are clearly reflexive and transitive, so we are left with identitivity, i.e.

$$(E(a) \preceq_i E(a') \quad \text{and} \quad E(a') \preceq_i E(a)) \implies E(a) = E(a').$$

Of course, the identitivity of $\preceq_3$ implies the one of $\preceq_2$ which, in turn, implies the one of $\preceq_1$. It might, however, be instructive to give direct proofs.
1.2. Rigidity for ellipsoids

It is well known from linear symplectic algebra [18, p. 40] that

\[ E(a) \preceq_1 E(a') \iff a_i \leq a_i' \text{ for all } i, \]  

(1.3)

in particular \( \preceq_1 \) is identitive.

Given \( U \in \mathcal{O}(n) \) with smooth boundary \( \partial U \), the spectrum \( \sigma(U) \) of \( U \) is defined to be the collection of the actions of closed characteristics on \( \partial U \). It is clearly invariant under \( \mathcal{D}(n) \), and for an ellipsoid it is given by

\[
\sigma(E(a_1, \ldots, a_n)) = \{ d_1(E) \leq d_2(E) \leq \ldots \}
:= \{ ka_i \mid k \in \mathbb{N}, 1 \leq i \leq n \}. \tag{1.4}
\]

Let now \( \varphi \) be a map realizing \( E(a) \preceq_2 E(a') \). The relations \( E(a) \preceq_2 E(a') \preceq_2 E(a) \) imply in particular \( |E(a)| = |E(a')| \), and we conclude as in the proof of Proposition 1.1 that \( \varphi(\partial E(a)) = \partial E(a') \). This implies \( \sigma(E(a)) = \sigma(E(a')) \) and the identitivity of \( \preceq_2 \) follows.

To prove the identitivity of \( \preceq_3 \) we use Ekeland–Hofer capacities [7]. For the readers convenience we recall the

**Definition 1.3** An extrinsic symplectic capacity on \( (\mathbb{R}^{2n}, \omega_0) \) is a map \( c \) associ¬ating with each subset \( S \) of \( \mathbb{R}^{2n} \) a number \( c(S) \in [0, \infty] \) in such a way that the following axioms are satisfied.

A1. **Monotonicity:** \( c(S) \leq c(T) \) if there exists \( \varphi \in \mathcal{D}(n) \) such that \( \varphi(S) \subset T \).

A2. **Conformality:** \( c(\lambda S) = \lambda^2 c(S) \) for all \( \lambda \in \mathbb{R} \setminus \{0\} \).

A3. **Nontriviality:** \( 0 < c(B^{2n}(\pi)) \) and \( c(Z^{2n}(\pi)) < \infty \).

The Ekeland–Hofer capacities form a countable family \( \{c_i\}, i \geq 1 \), of extrinsic symplectic capacities on \( \mathbb{R}^{2n} \). For a symplectic ellipsoid \( E \) these invariants are given by its spectrum (1.4):

\[
\{ c_1(E) \leq c_2(E) \leq \ldots \} = \{ d_1(E) \leq d_2(E) \leq \ldots \}, \tag{1.5}
\]

see [7, Proposition 4]. Observe that for any \( i \in \{1, 2, 3\} \) and \( \lambda > 0 \)

\[
E(a) \preceq_i E(a') \implies E(\lambda a) \preceq_i E(\lambda a'). \tag{1.6}
\]

This is seen by conjugating the given map \( \varphi \) with the dilatation by \( \lambda^{-1} \). Recalling (1.2) we conclude that for any \( \delta_1, \delta_2 \in \{0, 1\} \) the postulated relations

\[
E(a) \preceq_3 E(a') \preceq_3 E(a)
\]
imply
\[ E(\delta_2 \delta_1 a) \preceq_2 E(\delta_1 a') \preceq_2 E(a). \]

Now the monotonicity property (A1) of the capacities and the set of relations in (1.5) immediately imply that \( a = a' \). This completes the proof of Proposition 1.2. \( \square \)

The equivalence (1.3), Example 4 and the Extension after Restriction Principle show that \( \preceq_2 \) does not imply \( \preceq_1 \) in general. However, a suitable pinching condition guarantees that "linear" and "non linear" coincide:

**Theorem 1.4** Let \( \kappa \in ]\frac{b}{2}, b[ \). Then the following statements are equivalent:

(i) \( B^{2n}(\kappa) \preceq_1 E(a) \preceq_1 E(a') \preceq_1 B^{2n}(b) \),

(ii) \( B^{2n}(\kappa) \preceq_2 E(a) \preceq_2 E(a') \preceq_2 B^{2n}(b) \),

(iii) \( B^{2n}(\kappa) \preceq_3 E(a) \preceq_3 E(a') \preceq_3 B^{2n}(b) \).

We should mention that for \( n = 2 \), Theorem 1.4 was proved in [10]. That proof uses a deep result by McDuff, stating that the space of symplectic embeddings of a ball into a larger ball is unknotted, and then uses the isotopy invariance of symplectic homology. However, Ekeland–Hofer capacities provide an easy proof as we shall see. The crucial observation is that capacities have - in contrast to symplectic homology - the monotonicity property.

**Proof of Theorem 1.4:** In view of (1.1) it is enough to show the implication (iii) \( \Rightarrow \) (i). We start with showing the implication (ii) \( \Rightarrow \) (i). By assumption, \( B^{2n}(\kappa) \preceq_2 E(a) \preceq_2 B^{2n}(b) \). Hence, by the monotonicity of the first Ekeland–Hofer capacity \( c_1 \) we obtain
\[ \kappa \leq a_1 \leq b, \]
and by the monotonicity of \( c_n \)
\[ \kappa \leq c_n(E(a)) \leq b. \]

The estimates (1.7) and \( \kappa > b/2 \) imply \( 2a_1 > b \), whence the only elements in \( \sigma(E(a)) \) possibly smaller than \( b \) are \( a_1, \ldots, a_n \). It follows therefore from (1.8) that \( a_n = c_n(E(a)) \), whence \( c_i(E(a)) = a_i \) (\( 1 \leq i \leq n \)). Similarly we find \( c_i(E(a')) = a'_i \) (\( 1 \leq i \leq n \)), and from \( E(a) \preceq_2 E(a') \) we conclude \( a_i \leq a'_i \).
(iii) \(\Rightarrow\) (i) now follows by a similar reasoning as in the proof of the identitivity of \(\preceq_3\). Indeed, starting from

\[
B^{2n}(\kappa) \preceq_3 E(a) \preceq_3 E(a') \preceq_3 B^{2n}(b),
\]

the implication (1.2) shows that for any \(\delta_1, \delta_2, \delta_3 \in ]0, 1[\)

\[
B^{2n}(\delta_3 \delta_2 \delta_1 \kappa) \preceq_2 E(\delta_2 \delta_1 a) \preceq_2 E(\delta_1 a') \preceq_2 B^{2n}(b).
\]

Choosing \(\delta_1, \delta_2, \delta_3\) so large that \(\delta_3 \delta_2 \delta_1 \kappa > b/2\) we can apply the already proved implication to see

\[
B^{2n}(\delta_3 \delta_2 \delta_1 \kappa) \preceq_1 E(\delta_2 \delta_1 a) \preceq_1 E(\delta_1 a) \preceq_1 B^{2n}(b),
\]

and since \(\delta_1, \delta_2, \delta_3\) can be chosen arbitrarily close to 1, the statement (i) follows in view of (1.3). This completes the proof of Theorem 1.4. \(\square\)

We use Theorem 1.4 in order to prove Theorem 1 of the introduction, which in the notation of this section reads

**Theorem 1.5** Assume that \(E(a_1, \ldots, a_n) \preceq_3 B^{2n}(A)\) for some \(A < a_n\). Then \(a_n > 2a_1\).

**Proof.** Arguing by contradiction we assume \(E(a_1, \ldots, a_n) \preceq_3 B^{2n}(A)\) for some \(A < a_n\) and \(a_n \leq 2a_1\). A volume comparison shows \(a_1 < A\). Hence, \(a_1 \in ]\frac{A}{2}, A[\). Therefore, \(B^{2n}(a_1) \preceq_3 E(a_1, \ldots, a_n) \preceq_3 B^{2n}(A)\), Theorem 1.4 and the equivalence (1.3) imply that \(a_n \leq A\). This contradiction shows \(a_n > 2a_1\), as claimed. \(\square\)

We conclude this section by observing that the third Ekeland–Hofer capacity \(c_3\) implies for \(n \geq 3\) that the ellipsoid \(E^{2n}(a_1, 3a_1, \ldots, 3a_1)\) does not symplectically embed into \(B^{2n}(A)\) if \(A < 3a_1\).

### 1.3 Rigidity for polydiscs?

The rigidity results for symplectic embeddings of ellipsoids into ellipsoids found in the previous section were proved with the help of Ekeland–Hofer capacities. Recall that \(P(a_1, \ldots, a_n)\) denotes the open symplectic polydisc. We may again assume \(a_1 \leq a_2 \leq \cdots \leq a_n\). The Ekeland–Hofer capacities of a polydisc are given by

\[
c_j(P(a_1, \ldots, a_n)) = ja_1, \quad j = 1, 2, \ldots,
\]
[7, Proposition 5], and so they only see the smallest area $a_1$. Many of the polydisc analogues of the rigidity results for ellipsoids are therefore either wrong or much harder to prove. It is for instance not true anymore that $P(a_1, \ldots, a_n)$ embeds into $P(A_1, \ldots, A_n)$ by a linear symplectomorphism if and only if $a_i \leq A_i$ for all $i$, as the following example shows.

**Lemma 1.6** Assume $r > 1 + \sqrt{2}$. Then there exists $A < \pi r^2$ such that the polydisc $P^{2n}(\pi, \ldots, \pi, \pi r^2)$ embeds into the cube $C^{2n}(A) = P^{2n}(A, \ldots, A)$ by a linear symplectomorphism.

**Proof.** It is enough to prove the lemma for $n = 2$. Consider the linear symplectomorphism given by

$$
(z_1, z_2) \mapsto (z'_1, z'_2) = \frac{1}{\sqrt{2}}(z_1 + z_2, z_1 - z_2).
$$

For $(z_1, z_2) \in P(\pi, \pi r^2)$ we have for $i = 1, 2$

$$
|z'_i|^2 \leq \frac{1}{2} \left(|z_1|^2 + |z_2|^2 + 2|z_1||z_2|\right) < \frac{1}{2} + \frac{r^2}{2} + r. \tag{1.10}
$$

The right hand side of (1.10) is strictly smaller than $r^2$ provided that $r > 1 + \sqrt{2}$. \qed

Moreover, it is not known whether the full analogue of Proposition 1.2 for polydiscs instead of ellipsoids holds true. Let $\mathcal{P}(n)$ be the collection of polydiscs

$$
\mathcal{P}(n) = \{P(a_1, \ldots, a_n)\}
$$

and write $\preceq_i$ for the restrictions of the relations $\preceq$ to $\mathcal{P}(n)$, $i = 1, 2, 3$. Again $\preceq_2$ and $\preceq_3$ are very similar, again all the relations $\preceq_i$ are clearly reflexive and transitive, and again the identitivity of $\preceq_2$, which again implies the one of $\preceq_1$, follows from the equality of the spectra, which is implied by the equality of the volumes. (Observe that, even though the boundary of a polydisc $P(a_1, \ldots, a_n)$ is not smooth, its spectrum is still well defined and given by $\sigma(P(a_1, \ldots, a_n)) = \{ka_i \mid k \in \mathbb{N}, 1 \leq i \leq n\}$.) For $n = 2$ the identitivity of $\preceq_3$ follows from the monotonicity of any symplectic capacity, which show that the smaller discs are equal, and from the equality of the volumes, which then shows that also the larger discs are equal. For $n \geq 3$, however, we don't know whether the relation $\preceq_3$ is identitive. In particular, we have no answer to the following question.
1.3. Rigidity for polydiscs?

**Question 1.7** Assume that there exist symplectic embeddings

\[ P(a_1, a_2, a_3) \hookrightarrow P(a_1', a_2', a_3') \quad \text{and} \quad P(a_1', a_2', a_3') \hookrightarrow P(a_1, a_2, a_3). \]

Is it then true that \( a_2 = a_2' \) and \( a_3 = a_3' \)?

We also don’t know whether the polydisc-analogue of Theorem 1 or of Theorem 1.4 holds true. The symplectic embedding results proved in the subsequent chapters will suggest, however, that the polydisc-analogue of Theorem 1 holds true, see Conjecture 6.10.
1. Rigidity
2 Proof of Theorem 2

2.1 Reformulation of Theorem 2

Recall from the introduction that the ellipsoid \( E(a_1, \ldots, a_n) \) is defined by

\[
E(a_1, \ldots, a_n) = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \left| \sum_{i=1}^{n} \frac{\pi |z_i|^2}{a_i} < 1 \right. \right\}.
\]  

Theorem 2 of the introduction clearly can be reformulated as follows:

**Theorem 2.1** Assume \( a > 2\pi \). Then \( E^{2n}(\pi, \ldots, \pi, a) \) symplectically embeds into \( B^{2n} \left( \frac{a}{2} + \pi + \epsilon \right) \) for every \( \epsilon > 0 \).

The symplectic folding construction of Lalonde and McDuff considers a 4-ellipsoid as a fibration of discs of varying size over a disc and applies the flexibility of volume preserving maps to both the base and the fibers. It is therefore purely four dimensional in nature. We will refine the method in such a way that it allows to prove Theorem 2.1 for every \( n \geq 2 \).

We shall conclude Theorem 2.1 from the following proposition in dimension 4.

**Proposition 2.2** Assume \( a > 2\pi \). Given \( \epsilon > 0 \) there exists a symplectic embedding

\[
\Phi: E(a, \pi) \hookrightarrow B^4 \left( \frac{a}{2} + \pi + \epsilon \right)
\]

satisfying

\[
\pi |\Phi(z_1, z_2)|^2 < \frac{a}{2} + \epsilon + \frac{\pi^2 |z_1|^2}{a} + \pi |z_2|^2 \quad \text{for all} \quad (z_1, z_2) \in E(a, \pi).
\]

We recall that \( | \cdot | \) denotes the Euclidean norm. Postponing the proof, we first show that Proposition 2.2 implies Theorem 2.1.

**Corollary 2.3** Assume that \( \Phi \) is as in Proposition 2.2. Then the composition of the permutation \( E^{2n}(\pi, \ldots, \pi, a) \rightarrow E^{2n}(a, \pi, \ldots, \pi) \) with the restriction of \( \Phi \times id_{2n-4} \) to \( E^{2n}(a, \pi, \ldots, \pi) \) embeds \( E^{2n}(\pi, \ldots, \pi, a) \) into \( B^{2n} \left( \frac{a}{2} + \pi + \epsilon \right) \).
Proof. Let $z = (z_1, \ldots, z_n) \in E^{2n}(a, \pi, \ldots, \pi)$. By Proposition 2.2 and the definition (2.1) of the ellipsoid,

$$\pi |\Phi \times \text{id}_{2n-4}(z)|^2 = \pi \left( |\Phi(z_1, z_2)|^2 + \sum_{i=3}^{n} |z_i|^2 \right)$$

$$< \frac{a}{2} + \epsilon + \pi^2 \frac{|z_1|^2}{a} + \pi \sum_{i=2}^{n} |z_i|^2$$

$$= \frac{a}{2} + \epsilon + \pi \left( \frac{\pi |z_1|^2}{a} + \sum_{i=2}^{n} \frac{\pi |z_i|^2}{\pi} \right)$$

$$< \frac{a}{2} + \epsilon + \pi,$$

as claimed. \(\square\)

It remains to prove Proposition 2.2. In order to do so, we start with some preparations.

The flexibility of 2-dimensional area preserving maps is crucial for the construction of the map $\Phi$. We now make sure that we may describe such a map by prescribing it on an exhausting and nested family of embedded loops. Recall that $D(a)$ denotes the open disc of area $a$ centered at the origin.

Definition 2.4 A family $\mathcal{L}$ of loops in a simply connected domain $U \subset \mathbb{R}^2$ is called admissible if there is a diffeomorphism $\beta: D(|U|) \setminus \{0\} \to U \setminus \{p\}$ for some point $p \in U$ such that

(i) concentric circles are mapped to elements of $\mathcal{L}$,

(ii) in a neighbourhood of the origin $\beta$ is a translation.

Lemma 2.5 Let $U$ and $V$ be bounded and simply connected domains in $\mathbb{R}^2$ of equal area and let $\mathcal{L}_U$ and $\mathcal{L}_V$ be admissible families of loops in $U$ and $V$, respectively. Then there is a symplectomorphism between $U$ and $V$ mapping loops to loops.

Remark 2.6 The regularity condition (ii) imposed on the families taken into consideration can be weakened. Some condition, however, is necessary. Indeed, if $\mathcal{L}_U$ is a family of concentric circles and $\mathcal{L}_V$ is a family of rectangles with smooth corners and width larger than a positive constant, then no bijection from $U$ to $V$ mapping loops to loops is continuous at the origin. \(\Diamond\)
2.1. Reformulation of Theorem 2

**Proof of Lemma 2.5.** Denote the concentric circle of radius \( r \) by \( C(r) \). We may assume that \( \mathcal{L}_U = \{ C(r) \}, 0 < r < R \). Let \( \beta \) be the diffeomorphism parameterizing \((V \setminus \{ p \}, \mathcal{L}_V)\). After reparametrizing the \( r \)-variable by a diffeomorphism of \( \]0, R[ \) which is the identity near 0 we may assume that \( \beta \) maps the loop \( C(r) \) of radius \( r \) to the loop \( L(r) \) in \( \mathcal{L}_V \) which encloses the domain \( V(r) \) of area \( \pi r^2 \).

We denote the Jacobian of \( \beta \) at \( re^{i\varphi} \) by \( \beta'(re^{i\varphi}) \). Since \( \beta \) is a translation near the origin and \( U \) is connected, \( \det \beta'(re^{i\varphi}) > 0 \). By our choice of \( \beta \),

\[
\pi r^2 = |V(r)| = \int_{D(\pi r^2)} \det \beta' = \int_0^r \rho \; d\rho \int_0^{2\pi} \det \beta'(\rho e^{i\varphi}) \; d\varphi.
\]

Differentiating in \( r \) we obtain

\[
2\pi = \int_0^{2\pi} \det \beta'(re^{i\varphi}) \; d\varphi.
\] (2.2)

Define the smooth function \( h: \]0, R[ \times \mathbb{R} \to \mathbb{R} \) as the unique solution of the initial value problem

\[
\begin{align*}
\frac{\partial}{\partial t} h(r, t) &= \frac{1}{\det \beta'(re^{ih(r,t)})}, \quad t \in \mathbb{R} \\
h(r, t) &= 0, \quad t = 0
\end{align*}
\] (2.3)

depending on the parameter \( r \). We claim that

\[
h(r, t + 2\pi) = h(r, t) + 2\pi.
\] (2.4)

It then follows, since the function \( h \) is strictly increasing in the variable \( t \), that for every \( r \) fixed the map \( h(r, \cdot): \mathbb{R} \to \mathbb{R} \) induces a diffeomorphism of the circle \( \mathbb{R}/2\pi\mathbb{Z} \). In order to prove the claim (2.4) we denote by \( t_0(r) > 0 \) the unique solution of \( h(r, t_0(r)) = 2\pi \). Substituting \( \varphi = h(r, t) \) into formula (2.2) we obtain, using \( \det \beta'(re^{ih(r,t)}) \cdot \frac{\partial}{\partial t} h(r, t) = 1 \), that

\[
2\pi = \int_0^{t_0(r)} dt = t_0(r).
\]

Hence \( h(r, 2\pi) = 2\pi \). Therefore, the two functions in \( t \), \( h(r, t + 2\pi) - 2\pi \) and \( h(r, t) \), solve the same initial value problem (2.3), and so the claim (2.4) follows. The desired diffeomorphism is now defined by

\[
\alpha: U \setminus \{0\} \to V \setminus \{p\}, \quad re^{i\varphi} \mapsto \beta(re^{ih(r,\varphi)}).
\]
2. Proof of Theorem 2

It is area preserving. Indeed, representing \( \alpha \) as the composition

\[
re^{i\varphi} \mapsto (r, \varphi) \mapsto (r, h(r, \varphi)) \mapsto re^{ih(r,\varphi)} \mapsto \beta(re^{ih(r,\varphi)})
\]

we obtain for the determinant of the Jacobian

\[
\frac{1}{r} \cdot \frac{\partial h}{\partial \varphi}(r, \varphi) \cdot r \cdot \det \beta'(re^{ih(r,\varphi)}) = 1,
\]

where we again have used (2.3). Finally, \( \alpha \) is a translation in a punctured neighbourhood of the origin and thus smoothly extends to the origin. This finishes the proof of Lemma 2.5.

Consider a bounded domain \( U \subset \mathbb{C} \) and a continuous function \( f : U \to \mathbb{R}_{>0} \). The set \( \mathcal{F}(U, f) \) in \( \mathbb{C}^2 \) defined by

\[
\mathcal{F}(U, f) = \{(z_1, z_2) \in \mathbb{C}^2 | z_1 \in U, \pi |z_2|^2 < f(z_1)\}
\]

is the trivial fibration over \( U \) having as fiber over \( z_1 \) the disc of capacity \( f(z_1) \). Given two such fibrations \( \mathcal{F}(U, f) \) and \( \mathcal{F}(V, g) \), a symplectic embedding \( \varphi : U \hookrightarrow V \) defines a symplectic embedding \( \varphi \times \text{id} : \mathcal{F}(U, f) \hookrightarrow \mathcal{F}(V, g) \) if and only if

\[
f(z_1) < g(\varphi(z_1)) \quad \text{for all} \quad z_1 \in U.
\]

Examples 2.7

1. The ellipsoid \( E(a, b) \) can be represented as

\[
E(a, b) = \mathcal{F}(D(a), f(z_1) = b\left(1 - \frac{\pi |z_1|^2}{a}\right)).
\]

2. Define the open trapezoid \( T(a, b) \) by \( T(a, b) = \mathcal{F}(R(a), g) \), where

\[
R(a) = \{ z_1 = (u, v) | 0 < u < a, 0 < v < 1 \}
\]

is a rectangle and \( g(z_1) = g(u) = b(1 - u/a) \). We set \( T^4(a) = T(a, a) \). The example is inspired by [22, p. 54]. It will be very useful to think of \( T(a, b) \) as depicted in Figure 2.

In order to reformulate Proposition 2.2 we shall prove the following lemma which later on allows to work with more convenient "shapes".
2.1. Reformulation of Theorem 2

Lemma 2.8 Assume \( \epsilon > 0 \). Then

(i) \( E(a, b) \) symplectically embeds into \( T(a + \epsilon, b + \epsilon) \),

(ii) \( T^4(a) \) symplectically embeds into \( B^4(a + \epsilon) \).

Proof. Set \( \epsilon' = \frac{ae^2}{ab + ae + be} \). We are going to use Lemma 2.5 to construct an area preserving diffeomorphism \( \alpha: D(a) \to R(a) \) such that for the first coordinate in the image \( R(a) \),

\[
\nu(\alpha(z_1)) \leq \pi |z_1|^2 + \epsilon' \quad \text{for all } z_1 \in D(a),
\]

see Figures 3 and 4.

In an “optimal world” we would choose the loops \( \hat{L}_u, 0 < u < a \), in the image \( R(a) \) as the boundaries of the rectangles with corners \((0, 0), (0, 1), (u, 0), (u, 1)\). If the family \( \hat{L} = \{ \hat{L}_u \} \) induced a map \( \hat{\alpha} \), we would then have \( \nu(\hat{\alpha}(z_1)) \leq \pi |z_1|^2 \).
for all \((z_1, z_2) \in R(a)\). The non admissible family \(\mathbf{L}\) can be perturbed to an admissible family \(\mathbf{L}\) in such a way that the induced map \(\alpha\) satisfies the estimate (2.5). Indeed, choose the translation disc appearing in the proof of Lemma 2.5 as the disc of radius \(\varepsilon'/8\) centered at \((u_0, v_0) = \left(\frac{\varepsilon'}{2}, \frac{1}{2}\right)\). For \(r < \varepsilon'/8\) the loops \(L(r)\) are therefore the circles centered at \((u_0, v_0)\). In the following, all rectangles considered have edges parallel to the coordinate axes. We may thus describe a rectangle by specifying its lower left and upper right corner. Let \(L_0\) be the boundary of the rectangle with corners \((\frac{\varepsilon'}{4}, \frac{\varepsilon'}{4a})\) and \((\frac{3\varepsilon'}{4}, 1 - \frac{\varepsilon'}{4a})\), and let \(L_1\) be the boundary of \(R(a)\). We define a family of loops \(L_s\) by linearly interpolating between \(L_0\) and \(L_1\), i.e., \(L_s\) is the boundary of the rectangle with corners

\[
\left(1-s\right)\left(\frac{\varepsilon'}{4}, \frac{\varepsilon'}{4a}\right) \quad \text{and} \quad \left(u_s, 1 - \frac{\varepsilon'}{4a} + \frac{\varepsilon'}{4a}s\right), \quad s \in [0, 1],
\]

where \(u_s = \frac{3\varepsilon'}{4} + s \left(a - \frac{3\varepsilon'}{4}\right)\). Since \(u_s < a\), the area enclosed by \(L_s\) is estimated from below by

\[
\left(u_s - \frac{\varepsilon'}{4}\right) \left(1 - 2\frac{\varepsilon'}{4a}\right) > u_s - \frac{3\varepsilon'}{4}.
\]

(2.6)

Let \(\{L_s\}, s \in [0, 1]\), be the smooth family of smooth loops obtained from \(\{L_s\}\) by smoothing the corners as indicated in Figure 3. By choosing the smooth corners of \(L_s\) more and more rectangular as \(s \to 1\), we can arrange that the set \(\bigsqcup_{0 < s < 1} L_s\) is the domain bounded by \(L_0\) and \(L_1\). Moreover, by choosing all smooth corners rectangular enough, we can arrange that the area enclosed by \(L_s\) and \(L_s\) is less than \(\varepsilon'/4\). In view of (2.6), the area enclosed by \(L_s\) is then at least \(u_s - \varepsilon'\). Complete the families \(\{L(r)\}\) and \(\{L_s\}\) to an admissible family \(\mathbf{L}\) of loops in \(R(a)\) and let \(\alpha: D(a) \to R(a)\) be the map defined by \(\mathbf{L}\). Fix \((z_1, z_2) \in D(a)\). If \(\alpha(z_1)\) lies on a loop in \(\mathbf{L} \setminus \{L_s\}_{0 < s < 1}\), then \(u(\alpha(z_1)) < \frac{3\varepsilon'}{4} \leq \pi |z_1|^2 + \varepsilon'\), and so the required estimate (2.5) is satisfied. If \(\alpha(z_1) \in L_s\) for some \(s \in [0, 1]\), then the area enclosed by \(L_s\) is \(\pi |z_1|^2\), and so \(\pi |z_1|^2 + \varepsilon' > u_s \geq u(\alpha(z_1))\), whence (2.5) is again satisfied. This completes the construction of a symplectomorphism \(\alpha: D(a) \to R(a)\) satisfying (2.5). In the sequel, we will illustrate a map like \(\alpha\) by a picture like in Figure 4.

To continue the proof of (i) we shall show that \((\alpha(z_1), z_2) \in T(a + \varepsilon, b + \varepsilon)\) for every \((z_1, z_2) \in E(a, b)\), so that the symplectic map \(\alpha \times id\) embeds \(E(a, b)\) into \(T(a + \varepsilon, b + \varepsilon)\). Take \((z_1, z_2) \in E(a, b)\). Then, using the definition (2.1) of
2.1. Reformulation of Theorem 2

\( E(a, b) \), the estimate \((2.5)\) and the definition of \(\varepsilon'\) we find

\[
\pi |z_2|^2 < b \left( 1 - \frac{\pi |z_1|^2}{a} \right) \leq b \left( 1 - \frac{u(\alpha(z_1))}{a} + \frac{\varepsilon'}{a} \right)
\]

\[
< b \left( 1 - \frac{u(\alpha(z_1))}{a + \varepsilon} \right) + b \frac{\varepsilon'}{a} = b \left( 1 - \frac{u(\alpha(z_1))}{a + \varepsilon} \right) + \epsilon - \frac{\varepsilon}{a + \varepsilon} (a + \varepsilon') 
\]

\[
\leq b \left( 1 - \frac{u(\alpha(z_1))}{a + \varepsilon} \right) + \epsilon - \frac{\varepsilon}{a + \varepsilon} u(\alpha(z_1)) 
\]

\[
= (b + \varepsilon) \left( 1 - \frac{u(\alpha(z_1))}{a + \varepsilon} \right).
\]

It follows that

\[
(\alpha(z_1), z_2) \in T(a + \varepsilon, b + \varepsilon) = \mathcal{F} \left( R(a + \varepsilon), (b + \varepsilon) \left( 1 - \frac{u}{a + \varepsilon} \right) \right)
\]

as claimed.

In order to prove (ii) we shall construct an area preserving diffeomorphism \(\omega\) from a rectangular neighbourhood of \(R(a)\) having smooth corners and area \(a + \varepsilon\) to \(D(a + \varepsilon)\) such that

\[
\pi |\omega(z_1)|^2 \leq u + \varepsilon \quad \text{for all } z_1 = (u, v) \in R(a).
\]  

(2.7)

Such a map \(\omega\) can again be obtained with the help of Lemma 2.5. In an "optimal world" we would choose the loops \(\tilde{L}_q\) in the domain \(R(a)\) as before. This time, we perturb this non admissible family to an admissible family \(\mathcal{L}\) of loops as illustrated in Figure 4. If the smooth corners of all those loops in \(\mathcal{L}\) which enclose an area greater than \(\varepsilon/2\) lie outside \(R(a)\) and if the upper, left and lower edges of all these loops are close enough, then the induced map \(\omega\) will satisfy (2.7).

Restricting \(\omega\) to \(R(a)\) we obtain a symplectic embedding \(\omega \times id : T^4(a) \hookrightarrow \mathbb{R}^4\). For \((z_1, z_2) \in T^4(a)\) we have \(\pi |z_2|^2 < a \left( 1 - u/a \right)\), where \(z_1 = (u, v) \in R(a)\). In view of (2.7) we conclude that

\[
\pi \left( |\omega(z_1)|^2 + |z_2|^2 \right) \leq u + \varepsilon + a \left( 1 - \frac{u}{a} \right) = u + \varepsilon + a - u = a + \varepsilon,
\]
and so \((\omega \times id)(z_1, z_2) \in B^4(a + \epsilon)\) for all \((z_1, z_2) \in T^4(a)\).

Lemma 2.8 allows to reformulate Proposition 2.2 as follows.

**Proposition 2.9** Assume \(a > 2\pi\). Given \(\epsilon > 0\), there exists a symplectic embedding

\[
\Psi: T(a, \pi) \hookrightarrow \mathcal{B}^4 \bigg( \frac{a}{2} + \pi + \epsilon \bigg), \quad (z_1, z_2) \mapsto (z'_1, z'_2),
\]

\(z_1 = (u, v)\) and \(z'_1 = (u', v')\), satisfying

\[
u' + \pi |z'_2|^2 < \frac{a}{2} + \epsilon + \frac{\pi u}{a} + \pi |z_2|^2 \quad \text{for all } (u, v, z_2) \in T(a, \pi).
\]

Postponing the proof, we first show that Proposition 2.9 implies Proposition 2.2.

**Corollary 2.10** Assume the statement of Proposition 2.9 holds true. Then there exists a symplectic embedding \(\Phi: E(a, \pi) \hookrightarrow \mathcal{B}^4 \big( \frac{a}{2} + \pi + \epsilon \big)\) satisfying

\[
\pi |\Phi(z_1, z_2)|^2 < \frac{a}{2} + \epsilon + \frac{\pi^2 |z_1|^2}{a} + \pi |z_2|^2 \quad \text{for all } (z_1, z_2) \in E(a, \pi).
\]
2.1. Reformulation of Theorem 2

Proof. Let \( \epsilon' > 0 \) be so small that \( ca + \epsilon' > 2\pi \), where \( c = 1 - \epsilon' / \pi \). As in the proof of Lemma 2.8 we can construct a symplectic embedding

\[
\alpha \times \text{id}: E(ca, c\pi) \hookrightarrow T(ca + \epsilon', c\pi + \epsilon') = T(ca + \epsilon', \pi)
\]

satisfying the estimate

\[
u(\alpha(z_1)) \leq \pi |z_1|^2 + \frac{a(\epsilon')^2}{ca\pi + a\epsilon' + \pi\epsilon'} \quad \text{for all } z_1 \in D(ca) \quad (2.10)
\]

and another symplectic embedding

\[
\omega \times \text{id}: T^4(\frac{ca}{2} + \pi + \epsilon') \hookrightarrow B^4(\frac{ca}{2} + \pi + 2\epsilon')
\]

satisfying

\[
\pi |\omega(z_1)|^2 \leq u + \epsilon' \quad \text{for all } z_1 = (u, v) \in R(\frac{ca}{2} + \pi + \epsilon'). \quad (2.11)
\]

Since \( ca + \epsilon' > 2\pi \), Proposition 2.9 applied to \( ca + \epsilon' \) replacing \( a \) and \( \epsilon'/2 \) replacing \( \epsilon \) guarantees a symplectic embedding

\[
\Psi: T(ca + \epsilon', \pi) \hookrightarrow T^4(\frac{ca}{2} + \pi + \epsilon'),
\]

\((z_1, z_2) \mapsto (\Psi_1(z_1, z_2), \Psi_2(z_1, z_2))\), satisfying

\[
u(\Psi_1(\alpha(z_1), z_2)) + \pi |\Psi_2(\alpha(z_1), z_2)|^2 < \frac{ca}{2} + \epsilon' + \pi u(\alpha(z_1)) + \pi |z_2|^2 \quad (2.12)
\]

for all \((u(\alpha(z_1)), v, z_2) \in T(ca + \epsilon', \pi)\). Set \( \hat{\Phi} = (\omega \times \text{id}) \circ \Psi \circ (\alpha \times \text{id}). \) Then \( \hat{\Phi} \) symplectically embeds \( E(ca, c\pi) \) into \( B^4(\frac{ca}{2} + \pi + 2\epsilon') \). Moreover, if \((z_1, z_2) \in E(ca, c\pi)\), then

\[
\pi |\hat{\Phi}(z_1, z_2)|^2 = \pi |\omega(\Psi_1(\alpha(z_1), z_2))|^2 + \pi |\Psi_2(\alpha(z_1), z_2)|^2 \leq \frac{ca}{2} + 2\epsilon' + \frac{\pi u(\alpha(z_1))}{ca + \epsilon'} + \pi |z_2|^2 \leq \frac{ca}{2} + 3\epsilon' + \frac{\pi^2 |z_1|^2}{ca + \epsilon'} + \frac{\pi |z_2|^2}{ca + \epsilon' c\pi + a\epsilon' + \pi\epsilon'} + \pi |z_2|^2
\]
where in the last step we again used \( ca + \epsilon' > 2\pi \). Now choose \( \epsilon' > 0 \) so small that \( \frac{\pi + 3\epsilon'}{c} < \pi + \epsilon \). We denote the dilatation by \( \sqrt{c} \) in \( \mathbb{R}^4 \) also by \( \sqrt{c} \), and define \( \Phi : E(a, \pi) \to \mathbb{R}^4 \) by \( \Phi = (\sqrt{c})^{-1} \circ \hat{\Phi} \circ \sqrt{c} \). Then \( \Phi \) symplectically embeds \( E(a, \pi) \) into \( B^4 \left( \frac{a}{2} + \frac{\pi + 2\epsilon'}{c} \right) \subset B^4 \left( \frac{a}{2} + \pi + \epsilon \right) \), and since \( \pi |z_1|^2 < a \) for all \( (z_1, z_2) \in E(a, \pi) \) and by the choice of \( \epsilon' \),

\[
\pi |\Phi(z_1, z_2)|^2 = \frac{\pi}{c} |\hat{\Phi}(\sqrt{c} z_1, \sqrt{c} z_2)|^2
\]

\[
< \frac{1}{c} \left( \frac{ca}{2} + 3\epsilon' + \frac{\pi^2 |z_1|^2}{a} + \pi c |z_2|^2 \right)
\]

\[
= \frac{a}{2} + \frac{3\epsilon'}{c} + \frac{\pi^2 |z_1|^2}{a} + \pi |z_2|^2
\]

\[
< \frac{a}{2} + \epsilon + \frac{\pi^2 |z_1|^2}{a} + \pi |z_2|^2
\]

for all \( (z_1, z_2) \in E(a, \pi) \). This proves the required estimate (2.9), and so the proof of Corollary 2.10 is complete. \( \square \)

It remains to prove Proposition 2.9. This is done in the following two sections.

### 2.2 The folding construction

The idea in the construction of an embedding \( \Psi \) as in Proposition 2.9 is to separate the small fibers from the large ones and then to fold the two parts on top of each other. As in the previous section we denote the coordinates in the base and the fiber by \( z_1 = (u, v) \) and \( z_2 = (x, y) \), respectively.

**Step 1.** Following [22, Lemma 2.1] we first separate the "low" regions over \( R(a) \) from the "high" ones. We may do this using Lemma 2.5. We prefer, however, to give an explicit construction.

Let \( \delta > 0 \) be small. Set \( \mathcal{F} = \mathcal{F}(U, f) \), where \( U \) and \( f \) are described in Figure 5, and write

\[
P_1 = U \cap \left\{ u \leq \frac{a}{2} + \delta \right\},
\]

\[
P_2 = U \cap \left\{ u \geq \frac{a + \pi}{2} + 11\delta \right\},
\]

\[
L = U \setminus (P_1 \cup P_2).
\]
2.2. The folding construction

Choose a smooth function $h : [0, a + \delta] \to ]0, 1]$ as in Figure 6, i.e.

(i) $h(w) = 1$ for $w \in [0, \frac{a}{2}]$,
(ii) $h'(w) < 0$ for $w \in ]\frac{a}{2}, \frac{a}{2} + \delta^2[$,
(iii) $h \left( \frac{a}{2} + \delta^2 \right) = \delta$,
(iv) $h(w) = h(a - w)$ for all $w \in [0, a + \delta]$.

By (ii), (iii) and (iv) we have that

$$\int_{\frac{a}{2}}^{\frac{a}{2} + \delta^2} \frac{1}{h(w)} \, dw < \delta \quad \text{and} \quad \int_{\frac{a}{2} + \delta - \delta^2}^{\frac{a}{2} + \delta} \frac{1}{h(w)} \, dw < \delta. \quad (2.13)$$

We may thus further require that

(v) $h(w) < \delta$ for $w \in ]\frac{a}{2} + \delta^2, \frac{a}{2} + \delta - \delta^2[,$
(vi) $\int_{\frac{a}{2}}^{\frac{a}{2} + \delta} \frac{1}{h(w)} \, dw = \frac{a}{2} + 12\delta.$
2. Proof of Theorem 2

Consider the map

$$
\beta : R(a) \to \mathbb{R}^2, \quad (u, v) \mapsto \left(\int_0^u \frac{1}{h(w)} \, dw, h(u)v\right).
$$

Clearly, $\beta$ is a symplectic embedding. We see from (i), (iv) and (vi) that

$$
\beta\{u<\frac{a}{2}\} = \text{id} \quad \text{and} \quad \beta\{u>\frac{a}{2}+\delta\} = \text{id} + \left(\frac{\pi}{2}, 11\delta, 0\right).
$$

These identities and the estimates (2.13) and (v) imply that $\beta$ embeds $R(a)$ into $U$ (cf. Figure 7, where the black region in $R(a)$ is mapped to the black region in $U$, and so on). Finally, by construction, $\beta \times \text{id}$ symplectically embeds $T(a, \pi)$ into $\mathcal{F}$.

**Step 2.** We next map the fibers into a convenient shape. Using Lemma 2.5 in a similar way as it was used in the proof of Lemma 2.8 we find a symplectomorphism $\sigma$ mapping $D(\pi)$ to the rectangle $R_e$ and $D\left(\frac{\pi}{2}\right)$ to the rectangle with
2.2. The folding construction

smooth corners $R_i$ as specified in Figure 8. We require that for $z_2 \in D\left(\frac{\pi}{2}\right)$

\[
\pi|z_2|^2 + 2\delta > y(\sigma(z_2)) - \left(-\frac{\pi}{2} - 2\delta\right),
\]

i.e.

\[
y(\sigma(z_2)) < \pi|z_2|^2 - \frac{\pi}{2} \quad \text{for} \quad z_2 \in D\left(\frac{\pi}{2}\right). \tag{2.15}
\]

![Figure 8: Preparing the fibers.](image)

Write for the resulting bundle $(id \times \sigma)F$ of rectangles with smooth corners

\[
(id \times \sigma)F = \delta = \delta(P_1) \bigsqcup \delta(L) \bigsqcup \delta(P_2).
\]

In order to fold $\delta(P_2)$ over $\delta(P_1)$ we first move $\delta(P_2)$ along the $y$-axis and then turn it in the $z_1$-direction over $\delta(P_1)$.

**Step 3.** In order to move $\delta(P_2)$ along the $y$-axis we follow again [21, p. 355]. Let $c: \mathbb{R} \to [0, 1 - 2\delta]$ be a smooth cut off function as in Figure 9:

\[
c(t) = \begin{cases} 
0, & t \leq \frac{\delta}{2} + 2\delta \quad \text{and} \quad t \geq \frac{\pi}{2} + 10\delta, \\
1 - 2\delta, & \frac{\delta}{2} + 3\delta \leq t \leq \frac{\pi}{2} + 9\delta,
\end{cases}
\]

Set $I(t) = \int_0^t c(s) \, ds$ and define the diffeomorphism $\varphi: \mathbb{R}^4 \to \mathbb{R}^4$ by
2. Proof of Theorem 2

We then find for the derivative

$$d\varphi(u, x, v, y) = \begin{bmatrix} 1 & 0 \\ A & 1 \end{bmatrix}$$

with $A = \begin{bmatrix} * & c(u) \\ c(u) & 0 \end{bmatrix}$.

whence $\varphi$ is a symplectomorphism in view of the criterion in [18, p. 5]. Moreover, defining the number $I_\infty$ by $I_\infty = I \left( \frac{a+\pi}{2} + 10\delta \right)$, we find

$$\varphi|_{\{u \leq \frac{\pi}{2} + 2\delta\}} = id \quad \text{and} \quad \varphi|_{\{u \geq \frac{a+\pi}{2} + 10\delta\}} = id + (0, 0, 0, I_\infty),$$

and assuming that $\delta < \frac{1}{15}$ we compute with the help of Figure 9 that

$$\frac{\pi}{2} + 2\delta < I_\infty < \frac{\pi}{2} + 5\delta.$$

The first inequality in (2.18) implies

$$\varphi(P_2 \times R_{i}) \cap (\mathbb{R}^2 \times R_{e}) = \emptyset.$$  

**Remark 2.11** The map $\varphi$ is the crucial map of the folding construction. Indeed, $\varphi$ is the only map in the construction which does not split as a product of 2-dimensional maps. It is the map which sends the lines $\{v, x, y \text{ constant}\}$ to the characteristics of the hypersurface

$$(u, x, y) \mapsto \left( u, x, c(u) \left( x + \frac{1}{2} \right), y \right)$$

which generates (the cut off of) the obvious flow separating $R_i$ from $R_e$. \qed
2.2. The folding construction

Step 4. From the definition (2.16) of the map \( \varphi \) and Figure 5 and Figure 8 we read off that the projection of \( \varphi(\mathcal{S}) \) onto the \((u, v)\)-plane is contained in the union \( \mathcal{U} \) of \( U \) with the open set bounded by the graph of \( u \mapsto \delta + c(u) \), the \( u \)-axis and the two lines \( \{u = a/2 + \delta\} \) and \( \{u = (a + \pi)/2 + 11\delta\} \), cf. Figure 10. Observe that \( \delta + c(u) \leq 1 - \delta \). Define a local symplectic embedding \( \gamma \) of \( \mathcal{U} \) into \( \{(u, v) \mid 0 < u < (a + \pi)/2 + 11\delta, 0 < v < 1\} \) as follows: On \( P_1 = \mathcal{U} \cap \{u \leq a/2 + \delta\} \) the map \( \gamma \) is the identity, and on \( \mathcal{U} \cap \{u \geq a/2 + 2\delta\} \) it is the orientation preserving isometry which maps the right edge of \( P_2 = \mathcal{U} \cap \{u \geq (a + \pi)/2 + 11\delta\} \) to the left edge of \( P_1 \). In particular, we have for \( z_1 = (u, v) \in P_2 \),

\[
u'(\gamma(z_1)) = a + \frac{\pi}{2} + 12\delta - u.
\]

On the remaining black square \( \mathcal{B} = \mathcal{U} \cap \{a/2 + \delta < u < a/2 + 2\delta\} \) the map \( \gamma \) looks as shown in Figure 10. We then have for \( (u, v) \in \mathcal{U} \setminus (P_1 \cup P_2) \),

\[
u'(\gamma(u, v)) - \left(\frac{a}{2} + \delta\right) < \frac{\pi}{2} + 10\delta - \left(u - \left(\frac{a}{2} + \delta\right)\right) + \delta,
\]
i.e.

\[ u'(\gamma(u, v)) < -u + \frac{\pi}{2} + a + 13\delta. \quad (2.21) \]

By (2.19) the map \( \gamma \times id \) is one-to-one on \( \varphi(\delta) \).

The existence of an area and orientation preserving embedding as proposed in Figure 10 can be proved as follows: Set \( u_0 = a/2 + 2\delta \) and \( u_1 = (a + \pi)/2 + 21\delta/2 \). Moreover, set \( l = \pi/2 + 1 + 39\delta/4 \) and choose \( \lambda_3 > 0 \) so small that \( \lambda_3 l \leq \delta^2/3 \). Similar to Figure 6 we choose a smooth function \( h: \left[ \frac{a}{2} + \delta, \frac{a}{2} + 2\delta \right] \rightarrow [0, 1] \) such that

(i) \( h(u) = 1 \) for \( u \) near \( \frac{a}{2} + \delta \) and \( u \) near \( \frac{a}{2} + 2\delta \),

(ii) \( h(u) = \frac{\lambda_3}{3} \) for \( u \in \left[ \frac{a}{2} + \frac{3\delta}{2}, \frac{a}{2} + \frac{\lambda_3 l}{2} \right] \),

(iii) \( \int_{\frac{a}{2} + \delta}^{\frac{a}{2} + \frac{\lambda_3 l}{2}} \frac{1}{h(w)} dw = \delta \) and \( \int_{\frac{a}{2} + \frac{\lambda_3 l}{2} + \delta}^{\frac{a}{2} + \frac{\lambda_3 l}{2}} \frac{1}{h(w)} dw = \frac{\delta}{2} \).

The embedding \( \gamma_\delta: B \rightarrow \left[ \frac{a}{2} + \delta, u_0 + l + \frac{\delta}{2} \right] \times [0, \delta] \) defined by

\[ (u, v) \mapsto \left( \frac{a}{2} + \delta + \int_{\frac{a}{2} + \delta}^{u} \frac{1}{h(w)} dw, h(u)v \right) \]

and illustrated in Figure 11 is symplectic.

![Figure 11: The map \( \gamma_\delta \).](image)

We now map the image of \( \gamma_\delta \) to a domain \( B' \) in the \((u', v')\)-plane as painted in Figure 10: By the choice of \( l \) we may require that the part of the “outer” boundary of \( B' \) between \((u_0, 0)\) and \((u_1, 1)\), which contains \((u_1, 0)\), is smooth, has length \( l \), and is parametrized by \( \zeta(s) \), where the parameter \( s \in I := [u_0, u_0 + l] \) is arc length and

\[
\begin{align*}
\zeta(s) &= (s, 0) & \text{on} \ [u_0, u_1], \\
\zeta(s) &= (u_1 + u_0 + l - s, 1) & \text{on} \ [u_0 + l - \frac{\delta}{4}, u_0 + l].
\end{align*}
\quad (2.22)
\]
Denote the inward pointing unit normal vector field along $\zeta$ by $\nu$. We choose $\lambda_1 > 0$ so small that

$$\eta: I \times [0, \lambda_1] \to \mathbb{R}^2, \quad (s, t) \mapsto \zeta(s) + t \nu(s)$$

is an embedding. In order to make the map area preserving, we consider the initial value problem

$$\begin{align*}
\frac{df}{dt}(s, t) &= \frac{1}{\det d\eta(s, f(s, t))} \\
\frac{df}{dt}(s, 0) &= 0
\end{align*} \quad (2.23)$$

in which $s \in I$ is a parameter. The existence and uniqueness theorem for ordinary differential equations with parameters yields a smooth solution $f$ on $I \times [0, \lambda_2]$ for some $\lambda_2 > 0$. Then $f(s, t) < \lambda_1$ for all $(s, t) \in I \times [0, \lambda_2]$. This and the second equation in (2.23) imply that the composition

$$\gamma_\zeta: (s, t) \mapsto (s, f(s, t)) \mapsto (u', v')$$

is a diffeomorphism of $I \times [0, \lambda_2]$ onto half of a tubular neighbourhood of $\zeta$. Moreover, by the first equation in (2.23),

$$\det \gamma_\zeta(s, t) = \frac{\partial f}{\partial t}(s, t) \det d\eta(s, f(s, t)) = 1,$$

i.e., $\gamma_\zeta$ is area preserving. In view of the identities (2.22) for $\zeta$, the map $\gamma_\zeta$ is the identity in $\mathbb{R}^2$ for $s$ near $u_0$ and $t \in [0, \lambda_2]$, and $\gamma_\zeta$ is an isometry for $s$ near $u_0 + l$ and $t \in [0, \lambda_2]$.

We now choose the parameter $\lambda_3 > 0$ in the construction of $\gamma_\theta$ smaller than $\lambda_2$. Restrict $\gamma_\zeta$ to the gray region $I \times [0, \lambda_3]$ in the image of $\gamma_\theta$, and let $\overline{\gamma}_\zeta$ be the smooth extension of $\gamma_\zeta$ to the image of $\gamma_\theta$ which is the identity on $\{u \leq u_0\}$ and an isometry on $\{u \geq u_0 + l\}$. By (i), the composition $\overline{\gamma}_\zeta \circ \gamma_\theta$ is the identity near $u = a/2 + \delta$ and an isometry near $u = a/2 + 2\delta$. It thus smoothly fits with the map $\gamma|_{U \setminus B}$ already defined at the beginning of this step.

**Step 5.** We finally adjust the fibers. In view of the constructions in Step 2 and Step 3, the projection of the image $\varphi(\delta)$ onto the $z_2$-plane is contained in a tower shaped domain $T$ (cf. Figure 12), and by the second inequality in (2.18) we have $T \subset \{(x, y) \mid y < \frac{\pi}{2} + 4\delta\}$. Using once more our Lemma 2.5 we construct a symplectomorphism $\tau$ from a neighbourhood of $T$ to a disc such that the preimages of the concentric circles in the image are as in Figure 12. We require that for
2. Proof of Theorem 2

Figure 12: Mapping the tower to a disc.

\[ z_2 = (x, y), \]
\[ \pi |\tau(z_2)|^2 < \frac{\pi}{2} + 3\delta \quad \text{for } y \geq -\frac{\pi}{2} - 2\delta, \]
\[ \pi |\tau(z_2)|^2 < \pi |\sigma^{-1}(z_2)|^2 + \frac{\pi}{2} + 8\delta \quad \text{for } z_2 \in R_e, \]

where \( \sigma : D(\pi) \to R_e \) is the diffeomorphism constructed in Step 2.

Step 1 to Step 5 are the ingredients of our folding construction. The folding map \( \Psi : T(a, \pi) \to \mathbb{R}^4 \) is defined as the composition of maps

\[ \Psi = (id \times \tau) \circ (\gamma \times id) \circ \phi \circ (id \times \sigma) \circ (\beta \times id) = (\gamma \times \tau) \circ \phi \circ (\beta \times \sigma). \]  

2.3 End of the proof

Recall that it remains to prove Proposition 2.9. So let \( \epsilon > 0 \) be as in Proposition 2.9 and set \( \delta = \min\{\frac{1}{15}, \frac{\epsilon}{15}\} \). We define the desired map \( \Psi \) as in (2.26). It remains to verify that \( \Psi \) meets the required estimate (2.8). So let \( z = (z_1, z_2) = (u, v, x, y) \in T(a, \pi) \) and write \( \Psi(z) = (u', v', z'_2) \). By the choice of \( \delta \) it suffices to show that for all \( (u, v, z_2) \in T(a, \pi) \)

\[ u' - \frac{\pi u}{a} + \pi |z'_2|^2 - \pi |z_2|^2 < \frac{a}{2} + 15\delta. \]
2.3. End of the proof

We distinguish three cases according to the locus of the image $\beta(z_1)$ in the set $U = P_1 \cup L \cup P_2$ (see Figure 5 and Figure 7). We denote the $u$-coordinate of $\beta(z_1) = \beta(u, v)$ by $u''(\beta(u, v))$.

**Case 1.** $\beta(z_1) \in P_1$. The first identity in (2.17) implies $\varphi|_{\delta(P_1)} = id$, and Step 4 implies $\gamma|_{\delta(P_1)} = id$. Therefore, $u' = u''(\beta(u, v))$. Moreover, $u''(\beta(u, v)) < u + \delta$. Indeed, the definition of the map $\beta$ illustrated in Figure 7 shows that if $u''(\beta(u, v)) \leq \frac{a}{2}$, then $u''(\beta(u, v)) = u$, and if $u''(\beta(u, v)) \in \left[\frac{a}{2}, \frac{a}{2} + \delta\right]$, then $u > \frac{a}{2}$. Summarizing, we have

$$u' < u + \delta.$$

Using again $\varphi|_{\delta(P_1)} = id$ we find $\sigma(z_2) \in R_e$ and $z'_2 = \tau(\sigma(z_2))$. Hence, the estimate (2.25) for the map $\tau$ yields

$$\pi|z'_2|^2 = \pi|\tau(\sigma(z_2))|^2 < \pi|\sigma^{-1}(\sigma(z_2))|^2 + \frac{\pi}{2} + 8\delta = \pi|z_2|^2 + \frac{\pi}{2} + 8\delta.$$

Finally, we have $u \leq \frac{a}{2} + \delta$. Indeed, if $u > \frac{a}{2} + \delta$, then the second identity in (2.14) implies $\beta(u, v) \in P_2$. Altogether we can estimate

$$u' - \frac{\pi u}{a} + \pi|z'_2|^2 - \pi|z_2|^2 < u \left(1 - \frac{\pi}{a}\right) + \delta + \frac{\pi}{2} + 8\delta$$

$$< \frac{a}{2} \left(1 - \frac{\pi}{a}\right) + \frac{\pi}{2} + 10\delta$$

$$= \frac{a}{2} + 10\delta.$$

**Case 2.** $\beta(z_1) \in P_2$. By the second identity in (2.17) we have $\varphi|_{\delta(P_2)} = id + (0, 0, 0, I_{\infty})$, and so, in view of the identity (2.20), $u' = u''(\gamma(\beta(z_1))) = a + \frac{\pi}{2} + 12\delta - u''(\beta(u, v))$. Moreover, $u''(\beta(u, v)) > u + \frac{\pi}{2} + 10\delta$. Indeed, the definition of $\beta$ shows that if $u''(\beta(u, v)) \geq \frac{a+\pi}{2} + 12\delta$, then $u''(\beta(u, v)) = u + \frac{\pi}{2} + 11\delta$, and if $u''(\beta(u, v)) \in \left[\frac{a+\pi}{2} + 11\delta, \frac{a+\pi}{2} + 12\delta\right]$, then $u < \frac{a}{2} + \delta$. Summarizing, we have

$$u' < a - u + 2\delta.$$

Step 2 shows $\sigma(z_2) \in R_i$, and so $\gamma(\sigma(z_2) + (0, I_{\infty})) \geq -\frac{\pi}{2} - 2\delta$. Hence, the
estimates (2.24), (2.15) and (2.18) imply

\[ \pi |z_2'|^2 = \pi |\tau(\sigma(z_2) + (0, I_\infty))|^2 \]
\[ < \gamma(\sigma(z_2)) + I_\infty + \frac{\pi}{2} + 3\delta \]
\[ < \left( \pi |z_2|^2 - \frac{\pi}{2} \right) + \left( \frac{\pi}{2} + 5\delta \right) + \frac{\pi}{2} + 3\delta \]
\[ = \pi |z_2|^2 + \frac{\pi}{2} + 8\delta. \]

Finally, we have \( u \geq \frac{a}{2} \). Indeed, if \( u < \frac{a}{2} \), then the first identity in (2.14) implies \( \beta(u, v) \in P_1 \). Altogether we can estimate

\[ u' - \frac{\pi u}{a} + \pi |z_2'|^2 - \pi |z_2|^2 < a - u \left( 1 + \frac{\pi}{a} \right) + 2\delta + \frac{\pi}{2} + 8\delta \]
\[ \leq a - \frac{a}{2} \left( 1 + \frac{\pi}{a} \right) + \frac{\pi}{2} + 10\delta \]
\[ = \frac{a}{2} + 10\delta. \]

**Case 3.** \( \beta(z_1) \in L \). Using the definition of \( \varphi \), the estimate (2.21) implies

\[ u' < -u''(\beta(u, v)) + \frac{\pi}{2} + a + 13\delta. \]

Since \( \pi |z_2|^2 < \frac{\pi}{2} \), we have \( \sigma(z_2) \in R_i \), cf. Figure 8. In particular, \( \gamma(\sigma(z_2)) + \left( 0, I(u''(\beta(u, v))) \right) \geq -\frac{\pi}{2} - 2\delta \). Hence, the estimates (2.24) and (2.15) and the estimate \( I(t) < (1 - 2\delta)(t - (\frac{a}{2} + 2\delta)) \) read off from Figure 9 yield

\[ \pi |z_2'|^2 = \pi |\tau(x(\sigma(z_2)), y(\sigma(z_2)) + I(u''(\beta(u, v))))|^2 \]
\[ < \gamma(\sigma(z_2)) + I(u''(\beta(u, v))) + \frac{\pi}{2} + 3\delta \]
\[ < \left( \pi |z_2|^2 - \frac{\pi}{2} \right) + (1 - 2\delta) \left( u''(\beta(u, v)) - \frac{a}{2} - 2\delta \right) + \frac{\pi}{2} + 3\delta \]
\[ = \pi |z_2|^2 + u''(\beta(u, v)) - \frac{a}{2} - 2\delta - 2\delta u''(\beta(u, v)) + \delta a + 4\delta^2 + 3\delta. \]

Finally, we have \( u''(\beta(u, v)) > \frac{a}{2} + \delta \) by the definition of \( L \), and \( u \geq \frac{a}{2} \) by the
first identity in (2.14). Altogether we can estimate

\[
\begin{align*}
    u' - \frac{\pi u}{a} + \pi |z'_2|^2 - \pi |z_2|^2 &< -u''(\beta(u, v)) + \frac{\pi}{2} + a + 13\delta - \frac{\pi a}{2} \\
    &\quad + u''(\beta(u, v)) - \frac{a}{2} - 2\delta - 2\delta \left(\frac{a}{2} + \delta\right) \\
    &= \frac{a}{2} + 14\delta + 2\delta^2 \\
    &< \frac{a}{2} + 15\delta,
\end{align*}
\]

where in the last step we have used that \(2\delta^2 < \delta\) which follows from \(\delta < \frac{1}{15}\).

We have verified that the estimate (2.27) holds for all \((u, v, z_2) \in T(a, \pi)\), and the proof of Proposition 2.9 is complete.

Recall that by Corollary 2.10, Proposition 2.9 implies Proposition 2.2, and so, in view of Corollary 2.3, the proof of Theorem 2.1 is complete.

---

**Remarks 2.12**

1. As the verifications done in this section showed, the specific choice of the maps \(\beta, \sigma, \varphi, \gamma\) and \(\tau\) constructed in the previous section is crucial for obtaining the required estimate (2.8).
2. We recall that the embedding $\Phi: E(a, \pi) \hookrightarrow B^4(\frac{\pi}{2} + \pi + \epsilon)$ in our construction is the composition

$$\Phi = c^{-1} \circ (\omega \times id) \circ \Psi \circ (\alpha \times id) \circ c$$

$$= c^{-1} \circ (\omega \times id) \circ (id \times \tau) \circ (\gamma \times id) \circ \varphi \circ (id \times \sigma) \circ (\beta \times id) \circ (\alpha \times id) \circ c,$$

where $c$ is the dilatation by a number close to 1.

3. The folding map $\Psi: T(a, \pi) \hookrightarrow T^d(A)$ can be visualized as in Figure 13. Essentially, $\Psi$ restricts to the identity on the black rectangle and maps the triangle \{u > \frac{\pi}{2}\} to the light triangle and the triangle \{\pi \mid z_2 \mid^2 > \frac{\pi}{2}\} to the dark triangle.
3 Multiple symplectic folding in four dimensions

In four dimensions we shall exploit the great flexibility of symplectic maps which only depend on the fiber coordinates to provide rather satisfactory embedding results for simple shapes.

We first discuss a modification of the symplectic folding construction described in Section 2.2, then explain multiple folding, and finally calculate the optimal embeddings of ellipsoids and polydiscs into balls and cubes which can be obtained by these methods.

In order not to disturb the exposition unnecessarily with the arbitrarily small δ-terms (arising from “rounding off corners” and so on) we shall skip them in the sequel. Since all the sets under consideration will be bounded and all constructions will involve only finitely many steps, we will not lose control of the δ-terms.

3.1 Modification of the folding construction

The map σ in Step 2 of the folding construction given in Section 2.2 was dictated by the estimate (2.8) used to obtain the 2n-dimensional result, n > 2. As a consequence, the map ϕ of Step 3 had to disjoin the z2-projection of δ(P2) from the one of δ(P1), and we ended up with the unused white sandwiched triangle in Figure 13. In order to use this room for n = 2, in which case the estimate (2.8) is not necessary, we modify the folding construction as follows: Replace the map σ of Step 2 by the map σ given by Figure 14. If we define ϕ as in (2.16), the z2-projection of the image of ϕ will almost coincide with the image of σ. Choose now γ as in Step 4 and define the final map τ on a neighbourhood of the image of σ such that it restricts to σ⁻¹ on the image of σ. If all the δ’s were chosen appropriately, the composite map Ψ defined as in (2.26) will be one-to-one, and the image of Ψ will be contained in T⁴(π/2 + π + ε) for some small ε. The map Ψ can be visualized as in Figure 15. Essentially, Ψ restricts to the identity on the floor F₁ and maps the white triangle with vertices (π/2, 0), (π, 0), (π/2, π/2) to the floor F₂.
3. Multiple symplectic folding in four dimensions

Neither Theorem 2 nor Traynor’s theorem stated in Example 2.2 tells us whether the ellipsoid $E(\pi, 4\pi)$ symplectically embeds into the ball $B^4(A)$ for some $A \leq 3\pi$ (cf. Figure 53). Multiple symplectic folding, which is explained in this section, will provide a symplectic embedding of $E(\pi, 4\pi)$ into $B^4(2.6927\pi)$. To understand the general construction it is enough to look at a 3-fold. Up to the final fiber adjusting map $\tau$, the folding map $\Psi$ is then the composition of maps explained in Figure 16, in which the pictures are to be understood in the same sense as the picture in Figure 2: The horizontal direction refers to the base and the vertical direction indicates the fiber. Here are the details: Fix $a > \pi$ and the ellipsoid $E(a, \pi)$ as the trapezoid $T(a, \pi)$ fibering over the rectangle $R(a) =$

![Figure 14: The modified map $\sigma$.](image)

![Figure 15: Folding in four dimensions.](image)
3.2. Multiple folding

{\( (u, v) \mid 0 < u < a, \ 0 < v < 1 \) }. Pick \( u_1, \ldots, u_4 \in \mathbb{R}_{>0} \) satisfying \( \sum_{i=1}^{4} u_i = a \); the \( u_i \) will be specified in 3.3.1 for embedding \( E(a, \pi) \) into a ball and in 3.4.1 for embedding \( E(a, \pi) \) into a cube. Then define the heights \( l_i, i = 1, 2, 3, \) by

\[
 l_i = \pi - \frac{\pi}{a} \sum_{j=1}^{i} u_j, \quad i = 1, 2, 3.
\]

**Step 1** (Separating smaller fibers from larger ones). Let \( U \) and \( f \) be as in Figure 17. Proceeding as in Step 1 of the folding construction in Section 2.2 we find a symplectic embedding \( \beta : R(a) \hookrightarrow U \) such that \( (\beta \times \text{id})(T(a, \pi)) \subset \mathcal{F}(U, f) \).

**Step 2** (Preparing the fibers). The map \( \sigma \) is explained in Figure 18. More precisely, \( \sigma \) maps the central black disc to the black disc \( D \), and up to some neglected \( \delta \)-terms we have

\[
 y(\sigma(z_2)) = \begin{cases} 
 l_1 - l_2 + l_3 - \pi |z_2|^2 & \text{for most } z_2 \in D(l_3) \setminus D, \\
 l_1 - l_2 + \pi |z_2|^2 & \text{for most } z_2 \in D(l_2) \setminus D(l_3), \\
 l_1 - \pi |z_2|^2 & \text{for most } z_2 \in D(l_1) \setminus D(l_2), \\
 \pi |z_2|^2 & \text{for most } z_2 \in D(\pi) \setminus D(l_1). 
\end{cases}
\]

In general, when folding \( n \) times, \( \sigma \) maps the circles in the \( n + 1 \) annuli around a small central disc alternately to rectangular loops with essentially constant maxi-
3. Multiple symplectic folding in four dimensions

Step 3 (Lifting the fibers). Choose cut off functions $c_i$ over $L_i$, $i = 1, 2, 3$, and abbreviate $c(t) = \sum_{i=1}^3 c_i(t)$ and $I(t) = \int_0^t c(s) \, ds$. The symplectic embedding $\varphi: (\beta \times \sigma)(T(a, \pi)) \hookrightarrow \mathbb{R}^4$ is defined as in (2.16) by

$$\varphi(u, x, v, y) = \left( u, x, v + c(u) \left( x + \frac{1}{2} \right), y + I(u) \right).$$

Step 4 (Folding). Step 4 in Section 2.2 now requires three steps.

1. The folding map $\gamma_1$ is essentially the map $\gamma$ of Section 2.2: On the part of the base denoted by $P_1$ it is the identity, for $u_1 \leq u \leq u_1 + l_1$ it looks like the map in Figure 10, and for $u > u_1 + l_1$ it is an isometry. Observe that by construction, the slope of the stairs $S_2$ is 1, while the slope of the upper edge of the floor $F_1$ is $\pi/a < 1$. The sets $S_2$ and $F_1$ are thus disjoint.
2. The map $\gamma_2 \times id$ is not really a global product map, but restricts to a product on certain pieces of its domain: It is the identity on $F_1 \coprod S_1 \coprod F_2$, and it is the product $\gamma_2 \times id$ on the remaining domain, where $\gamma_2$ is explained in Figure 19: It is the identity on the gray part of its domain, maps the black square to the black part of its image, and is an isometry on $\{u \leq 0\}$. The map $\gamma_2$ is constructed the same way as the map $\gamma$ in Section 2.2.

3. The map $\gamma_3 \times id$, which turns $F_4$ over $F_3$, is analogous to the map $\gamma_1 \times id$: It is an isometry on $F_4$, looks like the map given by Figure 10 on $S_3$, and restricts to the identity everywhere else.

**Step 5** (Adjusting the fibers). The $z_2$-projection of the image of $\varphi$ is a tower shaped domain $\mathcal{T}$. The final map $\tau$ is a symplectomorphism from a small neighbourhood of $\mathcal{T}$ to a disc. It is enough to choose $\tau$ in such a way that up to some neglected $\delta$-term we have for any $z_2 = (x, y)$, $z'_2 = (x', y') \in \mathcal{T}$

$$y \leq y' \implies |\tau(z_2)|^2 < |\tau(z'_2)|^2.$$ 

This finishes the 3-fold folding construction. The $n$-fold folding construction is analogous. Here, we have $u_1, \ldots, u_{n+1}$, heights

$$l_i = \pi - \frac{\pi}{a} \sum_{j=1}^{i} u_j, \quad i = 1, \ldots, n, \quad (3.1)$$
3. Multiple symplectic folding in four dimensions

Floors $F_1, \ldots, F_{n+1}$, and stairs $S_1, \ldots, S_n$. The thin stairs require some locus in space specified in the following lemma, which is proved in Steps 4.1 and 4.2 above.

**Folding Lemma 3.1** Let $S_i$ be the stairs connecting the floors $F_i$ and $F_{i+1}$ of minimal respectively maximal height $l_i$, $i = 1, \ldots, n$.

(i) If $i$ is odd, then the stairs $S_i$ are contained in a trapezoid with horizontal lower edge of length $l_i$, left edge of length $2l_i$, and right edge of length $l_i$, cf. Figure 20 (i); moreover, no smaller trapezoid contains $S_i$.

(ii) If $i$ is even, then the stairs $S_i$ are contained in a trapezoid with horizontal upper edge of length $l_i$, left edge of length $l_i$, and right edge of length $2l_i$, cf. Figure 20 (ii); moreover, no smaller trapezoid contains $S_i$. 

Figure 19: Folding on the left.
3.3 Embeddings into balls

In this section we use multiple symplectic folding to construct good embeddings of four dimensional ellipsoids and polydiscs into four dimensional balls.

3.3.1 Embedding ellipsoids into balls

Recall from Theorem 1 that if \( a < 2\pi \) then the ellipsoid \( E(\pi, a) \), which is symplectomorphic to \( E(a, \pi) \), does not symplectically embed into \( B^4(A) \) if \( A < a \).
We therefore fix $a > 2\pi$. We again think of $E(a, \pi)$ as $T(a, \pi)$ and of $B^4(A)$ as $T^4(A) = T(A, A)$. In order to find the smallest trapezoid $T^4(A)$ into which $T(a, \pi)$ embeds via multiple symplectic folding we have to choose the $u_i$'s, which appeared in Section 3.2, optimally. Our strategy to do so is as follows. We shall describe a procedure which associates with each $u_1 \in ]0, a[$ the number

$$A(a, u_1) = 2\pi + \left(1 - \frac{2\pi}{a}\right) u_1,$$

(3.2)

and either a finite sequence $u_2, u_3, \ldots, u_{N+1}$ and the attribute admissible, or an empty or finite sequence $u_2, u_3, \ldots, u_N$ and the attribute non-admissible. In both cases the number $N = N(u_1)$ will depend only on $u_1$, and the procedure will describe a symplectic embedding of $T(a, \pi)$ into $T^4(A(a, u_1))$ by $(N$-fold) symplectic folding if and only if $u_1$ is admissible. In view of $a > 2\pi$ and equation (3.2) we have to look for the smallest admissible $u_1$. As we shall see, $u_1$ is non-admissible if $u_1 \leq a\pi/(a + \pi)$, and $u_1$ is admissible if $u_1 > a/2$. Moreover, we will show that if $u_1$ is admissible, then $u'_1$ is admissible for any $u'_1 > u_1$. It follows that there is a unique $u_0 = u_0(a)$ such that $u_1$ is admissible if $u_1 > u_0$ and $u_1$ is non-admissible if $u_1 < u_0$. Therefore, our procedure yields a symplectic embedding of $T(a, \pi)$ into $T^4(A(a, u_0) + \epsilon)$ for any $\epsilon > 0$. Finally, we will explain why our procedure is optimal, i.e., multiple symplectic folding does not yield an embedding of $T(a, \pi)$ into $T^4(A)$ if $A < A(a, u_0)$.

We start with describing our procedure. Fix $u_1 \in ]0, a[$ and fold at $u_1$. The minimal height of the first floor $F_1$ is then $l_1 = \pi - (\pi/a)u_1$. As suggested in Figure 21, we define $A(a, u_1)$ by the condition that the second floor $F_2$ touches the “upper right boundary” of $T^4(A(a, u_1))$, i.e.,

$$A(a, u_1) = u_1 + 2l_1 = 2\pi + \left(1 - \frac{2\pi}{a}\right) u_1.$$

We now successively try to choose $u_i$, $i \geq 2$, in such a way that $u_i$ is maximal and the stairs $S_i$ are contained in $T^4(A(a, u_1))$. Define the remaining length $r_1$ by $r_1 = a - u_1$. The image of $T(a, \pi)$ after folding at $u_1$ is contained in $T^4(A(a, u_1))$ if and only if $r_1 < u_1$, i.e., $u_1 > a/2$. If $r_1 < u_1$, we set $u_2 = r_1$, and $u_1$ is admissible. Indeed, the data $u_1, u_2$ then describe a symplectic embedding of $T(a, \pi)$ into $T^4(A(a, u_1))$ obtained by folding once. Since $u_1 > a/2$, these embeddings are not better than the one constructed in Section 2.2. If $r_1 \geq u_1$, we are forced to fold a second time. Assume that we fold at $u_1 - u_2 > 0$, i.e., the second floor $F_2$ has length $u_2$. Then the height of $F_2$ at $u_1 - u_2$ is $l_2 = l_1 - (\pi/a)u_2$. 


If $l_1 \geq u_1$, then $l_2 \geq u_1 - (\pi/a)u_2 > u_1 - u_2$, and so the Folding Lemma 3.1 (ii) shows that the stairs $S_2$ are not contained in $\{u > 0\}$. A necessary condition that $S_2$ is contained in $T^4(A(a, u_1))$ is therefore $l_1 < u_1$. In view of $l_1 = \pi - (\pi/a)u_1$ this condition is equivalent to the condition on $u_1$

$$u_1 > \frac{a\pi}{a + \pi}. \tag{3.3}$$

If this condition is not met, then $u_1$ is non-admissible. If $l_1 < u_1$, then the Folding Lemma 3.1 (ii) shows that for $u_2$ small enough we have $S_2 \subset \{u > 0\}$, and that the maximal such $u_2$ is characterized by the equation $l_2 + u_2 = u_1$, which by (3.1) translates into the formula

$$u_2 = \frac{a + \pi}{a - \pi}u_1 - \frac{a\pi}{a - \pi}. \tag{3.4}$$

We define $u_2$ by (3.4). Then $S_2 \subset \{u > 0\}$, but it is still possible that $S_2$ is not contained in $T^4(A(a, u_1))$, in which case $u_1$ is non-admissible. Assume that $S_2 \subset T^4(A(a, u_1))$. We denote the length $a - u_1 - u_2 = r_1 - u_2$ of the new remainder by $r_2$. If $r_2 < u_2$, we set $u_3 = r_2$, and $u_1$ is admissible. Indeed, the data $u_1, u_2, u_3$ then describe a symplectic embedding of $T(a, \pi)$ into $T^4(A(a, u_1))$ obtained by folding twice. If $r_2 \geq u_2$, we are forced to fold a third time. Assume that we fold at $l_2 + u_3$, i.e., the third floor $F_3$ has length $u_3$, and its height at $l_2 + u_3$ is $l_3 = l_2 - (\pi/a)u_3$. If $u_2 \leq 2l_2$, then $u_2 - u_3 \leq 2l_2 - u_3 < 2l_2 - (2\pi/a)u_3 = 2l_3$, and so the Folding Lemma 3.1 (i) shows that the stairs $S_3$ are not contained in $T^4(A(a, u_1))$. A necessary condition that $S_3$ is contained in $T^4(A(a, u_1))$ is therefore $u_2 > 2l_2$, and if this condition is not met, $u_1$ is non-admissible. If $u_2 > 2l_2$, the Folding Lemma 3.1 (i) shows that $S_3 \subset T^4(A(a, u_1))$ whenever $u_2 - u_3 > 2l_2 - (2\pi/a)u_3 = 2l_3$. Since $a > 2\pi$, the maximal such $u_3$ is characterized by the equation $u_2 - u_3 = 2l_2 - (2\pi/a)u_3$, i.e.,

$$u_3 = \frac{a}{a - 2\pi}(u_2 - 2l_2). \tag{3.5}$$

We define $u_3$ by (3.5).

Assume now that we folded already $i$ times, where $i$ is even, that $u_2, \ldots, u_i$ have been chosen maximal, and that $u_1$ is still possibly admissible. Denote the length $a - \sum_{j=1}^{i} u_j$ of the remainder by $r_i$. If $r_i < u_i$, we set $u_{i+1} = r_i$, and $u_1$ is admissible. Indeed, $u_1, u_2, \ldots, u_{i+1}$ then describe a symplectic embedding of $T(a, \pi)$ into $T^4(A(a, u_1))$ obtained by folding $i$ times. If $r_i \geq u_i$, we are forced to fold again. As in the case $i = 2$, the Folding Lemma 3.1 (i) shows that if
u_i \leq 2 l_i$, then $u_1$ is non-admissible, and that if $u_i > 2 l_i$, then the maximal $u_{i+1}$ for which the stairs $S_{i+1}$ are contained in $T^4(A(a, u_1))$ is

$$u_{i+1} = \frac{a}{a - 2\pi} (u_i - 2 l_i).$$

(3.6)

We define $u_{i+1}$ by (3.6). Denote the length $a - \sum_{j=1}^{i+1} u_j = r_i - u_{i+1}$ of the new remainder by $r_{i+1}$. If $r_{i+1} < u_{i+1} + l_i$, we set $u_{i+2} = r_{i+1}$, and $u_1$ is admissible. Indeed, $u_1, u_2, \ldots, u_{i+2}$ then describe a symplectic embedding of $T(a, \pi)$ into $T^4(A(a, u_1))$ obtained by folding $i + 1$ times. If $r_{i+1} \geq u_{i+1} + l_i$, we are forced to fold again. The Folding Lemma 3.1 (ii) shows that for $u_{i+2}$ small enough we have $S_{i+2} \subset \{ u > 0 \}$, and that the maximal such $u_{i+2}$ is characterized by the equation $u_{i+2} + l_{i+2} = u_{i+1} + l_i$, which by (3.1) translates into the formula

$$u_{i+2} = \frac{a + \pi}{a - \pi} u_{i+1}.$$  

(3.7)

We define $u_{i+2}$ by (3.7). Then $S_{i+2} \subset \{ u > 0 \}$, but possibly $S_{i+2}$ is not contained in $T^4(A(a, u_1))$, in which case $u_1$ in non-admissible. If $S_{i+2} \subset T^4(A(a, u_1))$, we go on as before.

Proceeding this way we try to decide for each $u_1 \in ]0, a[$ whether it is admissible or not. The value $u_1$ is admissible if and only if our procedure leads to an embedding of $T(a, \pi)$ into $T^4(A(a, u_1))$ after a finite number $N(u_1)$ of folds, and $u_1$ is non-admissible if and only if the procedure leads to an embedding obstruction after finitely many folds or if the procedure does not terminate. As we have already seen, $u_1$ is non-admissible if $u_1 \leq a\pi/(a + \pi)$, and $u_1$ is admissible if $u_1 > a/2$. Also recall that we have to look for the smallest admissible $u_1$. The following lemma implies that there exists a unique $u_0 = u_0(a)$ in the interval

$$I(a) := \left[ \frac{a\pi}{a + \pi}, \frac{a}{2} \right]$$

(3.8)

such that $u_1$ is admissible if $u_1 > u_0$ and $u_1$ is non-admissible if $u_1 < u_0$.

**Lemma 3.2** Assume that $u_1, u'_1 \in ]0, a[$ and that $u_1 < u'_1$. If $u_1$ is admissible, then $u'_1$ is admissible, and $N(u'_1) \leq N(u_1)$.

**Proof.** We abbreviate $N = N(u_1)$ and $N' = N(u'_1)$. Assume that the embedding procedure associated with $u_1$ is described by $u_1, \ldots, u_{N+1}$. We shall prove the lemma by going through our procedure and checking that the folding conditions for $u'_1$ are met whenever they are met for $u_1$. 
3.3. Embeddings into balls

If \( u'_1 \) is admissible and \( N' = 1 \), we are done. Otherwise, \( u_1 < u'_1 < a/2 \), and so \( N \geq 2 \). Then \( u'_1 > u_1 > \alpha \pi / (a + \pi) \). Observe that the next condition \( S_2 \subset T^4(A(a, u_1)) \) is equivalent to \( l'_2 < l_2 \). Equation (3.4) shows \( u'_2 > u_2 \), and now equation (3.1) shows \( l'_2 < l_2 \). Therefore, \( l'_2 < l_2 < u_2 < u'_2 \), and so \( S'_2 \subset T^4(A(a, u'_1)) \). If \( u'_1 \) is admissible and \( N' = 2 \), we are done. Otherwise, \( u_2 < u'_2 < r'_2 < r_2 \), and so \( N \geq 3 \). In view of (3.1) we then have \( u'_2 > u_2 > 2l_2 > 2l'_2 \), and equation (3.5) shows \( u'_3 > u_3 \). If \( u'_1 \) is admissible and \( N' = 3 \), we are done. Otherwise, \( r'_3 > u'_3 + l'_2 \), i.e., \( a - \pi > (1 - \pi / a)(u'_1 + u'_2) + 2u'_3 \), whence \( a - \pi > (1 - \pi / a)(u_1 + u_2) + 2u_3 \), i.e., \( r_3 > u_3 + l_2 \), and so \( N > 3 \). We proceed in this way using equations (3.6) and (3.7) and observing that the condition \( S_i \subset T^4(A(a, u_i)) \) for \( i \) even is equivalent to \( l'_i < l_i \). The lemma then follows.

Define the function \( s_{EB}(a) \) on \( ]2\pi, \infty[ \) by

\[
s_{EB}(a) = 2\pi + \left(1 - \frac{2\pi}{a}\right)u_0(a).
\]

Lemma 3.2 shows that the ellipsoid \( E(\pi, a) \) symplectically embeds into the ball \( B^4(s_{EB}(a) + \epsilon) \) for any \( \epsilon > 0 \).

We next explain why our procedure is optimal in the sense that we cannot embed \( E(\pi, a) \) into a ball smaller than \( B^4(s_{EB}(a)) \) by multiple symplectic folding. Indeed, observe that our procedure can equivalently be described as follows: To \( A \in ]2\pi, a[ \) associate the number \( u_1 = u_1(a, A) \) defined by \( A = 2\pi + (1 - 2\pi / a)u_1 \), and then choose \( u_i, i \geq 2 \), maximal. In other words, for each \( A \) we successively choose \( u_1, u_2, \ldots \) maximal with respect to the condition of staying inside \( T^4(A) \). The only possible way of improving our procedure is therefore to choose some of the \( u_i \) smaller. So let \( A < s_{EB}(a) \), let \( u_1, u_2, \ldots, u_N \) be the sequence associated with \( A \) by our procedure, and try to circumvent the embedding obstruction arising after folding \( N \) times by choosing \( u'_i \leq u_i \). Then, however, the proof of Lemma 3.2 shows that \( l'_i \geq l_i \) and \( r'_i \geq r_i \), and so the modified procedure leads to an embedding obstruction after \( N' \leq N \) folds.

**Lemma 3.3** We have \( N(u_1) \to \infty \) as \( u_1 \downarrow u_0 \). The value \( u_0 \) is non-admissible, and the procedure associated with \( u_0 \) does not terminate.

**Proof.** Arguing by contradiction, assume that the first statement of the lemma is wrong. By Lemma 3.2, there exists \( N \) such that \( N(u_1) = N \) for all \( u_1 > u_0 \) with \( u_1 - u_0 \) small enough. In the sequel we assume that \( u_1 > u_0 \) and that \( u_1 - u_0 \) is so small that \( N(u_1) = N \). By the proof of Lemma 3.2, the functions
$u_i(u_1), i = 2, \ldots, N - 1$ are decreasing as $u_1 \searrow u_0$, and the functions $l_i(u_1), i = 1, \ldots, N$, and $u_N(u_1)$ are increasing as $u_1 \searrow u_0$ and bounded. We set

$$u^0_i = \lim_{u_1 \searrow u_0} u_i(u_1), \quad i = 2, \ldots, N, \quad \text{and} \quad l^0_i = \lim_{u_1 \searrow u_0} l_i(u_1), \quad i = 1, \ldots, N.$$ 

Assume first that during the first $N$ folds there is no embedding obstruction at $u_0$.

Case 1. $N = 1$. If $u_0 > u^0_2$, then $T(a, \pi)$ embeds into $T^4(A(a, u_0))$ by folding once, and if $u_0 = u^0_2$, then the Folding Lemma 3.1 (i) shows that $T(a, \pi)$ embeds into $T^4(A(a, u_0))$ by folding twice.

Case 2. $N \geq 3$ and $N$ odd. If $u^0_N + l^0_{N-1} > u^0_{N+1}$, then $T(a, \pi)$ embeds into $T^4(A(a, u_0))$ by folding $N$ times, and if $u^0_N + l^0_{N-1} = u^0_{N+1}$, then the Folding Lemma 3.1 (i) shows that $T(a, \pi)$ embeds into $T^4(A(a, u_0))$ by folding $N + 1$ times.

Case 3. $N$ even. If $u^0_N > u^0_{N+1}$, then $T(a, \pi)$ embeds into $T^4(A(a, u_0))$ by folding $N$ times, and if $u^0_N = u^0_{N+1}$, then the Folding Lemma 3.1 (ii) and $a > 2\pi$ imply that $T(a, \pi)$ embeds into $T^4(A(a, u_0))$ by folding $N + 1$ times.

Therefore, $u_0$ is admissible with $N(u_0) = N$ or $N(u_0) = N + 1$. Since all conditions in our procedure are open conditions, it follows that if $u_1 < u_0$ and $u_0 - u_1$ is small enough, then $u_1$ is admissible with $N(u_1) = N$ or $N(u_1) = N + 1$. This contradicts the definition of $u_0$. So assume that there is an embedding obstruction at $u_0$ appearing at the latest at the $N$th fold. We shall conclude the proof of Lemma 3.3 by showing that an embedding obstruction at $u_0$ appearing at the latest at the $N$th fold implies an embedding obstruction at all $u_i$ near $u_0$.

Case I. $u_0 \leq a\pi/(a + \pi)$. Then $u_0 = a\pi/(a + \pi)$, i.e., $l_1(u_0) = l^0_1 = u_0$. Therefore, $u^0_2 = 0$ and so $N \geq 2$ and $l^0_2 = l^0_1 = u_0$.

If $N = 2$, we find $u_1(u_1) > u_2(u_1)$ for $u_1$ near $u_0$, a contradiction. If $N > 2$, we find $u_2(u_1) < 2l_2(u_1)$ for $u_1$ near $u_0$, i.e., there is an embedding obstruction for $u_1$ near $u_0$; this is another contradiction.

Case II. $S_i(u_0) \not\subseteq T^4(A(a, u_0))$ for some even $i$, i.e., $l^0_i \geq u^0_i$ for some even $i$.

If $i = N$, we find $u_{N+1}(u_1) > u_N(u_1)$ for $u_1$ near $u_0$, a contradiction. If $i < N$, we find $u_i(u_1) < 2l_i(u_1)$ for $u_1$ near $u_0$, i.e., there is an embedding obstruction for $u_1$ near $u_0$; this is another contradiction.

Case III. $u_i(u_0) \leq 2l_i(u_0)$ for some even $i$. Then $u^0_i = 2l^0_i$. If $i = N$, we find in view of $a > 2\pi$ that $u_{N+1}(u_1) > u_N(u_1)$ for $u_1$ near $u_0$, a contradiction.
3.3. Embeddings into balls

If \( i = N - 1 \), then \( u_{N+1}^{0} \leq l_{N-1}^{0} \), and so we find \( u_{N+1}(u_1) < u_{N-1}(u_1) \) for \( u_1 \) near \( u_0 \), contradicting \( N > i \).

If \( i < N - 1 \), we find \( u_{i+1}^{0} = u_{i+2}^{0} = 0 \) and \( l_{i+1}^{0} = l_{i+2}^{0} \), whence \( l_{i+1}^{0} + l_{i+1}^{0} + l_{i+2}^{0} > u_{i}^{0} \). This contradicts \( S_{i+2}(u_1) \subset T^4(A(a, u_1)) \) for \( u_1 \) near \( u_0 \).

We conclude that \( N(u_1) \to \infty \) as \( u_1 \searrow u_0 \). The value \( u_0 \) is therefore non-admissible, and as we have seen in Case I, Case II and Case III above, the procedure associated with \( u_0 \) cannot lead to an embedding obstruction. Lemma 3.3 is thus proved.

\[ \square \]

The computer program in Appendix D.1 computing \( u_0(a) \) is based on the following lemma, which implies that the value \( u_0 \) is the only value for which our procedure does not terminate.

**Lemma 3.4** Assume that \( u_1 < u_0 \). Then the procedure associated with \( u_1 \) leads to an embedding obstruction after finitely many folds.

**Proof.** By Lemma 3.3 our procedure associated with \( u_0 \) generates the infinite sequences \( u_i(u_0) \) and \( l_i(u_0) \), \( i = 1, 2, \ldots \), of lengths and heights of the floors \( F_i(u_0) \). Set \( h(u_0) := \sum_{i=1}^{\infty} l_i(u_0) \). Then \( h(u_0) \leq A(a, u_0) \). We claim that

\[ h(u_0) = A(a, u_0). \] \hspace{1cm} (3.10)

In other words, the image of \( T(a, \pi) \) in \( T^4(A(a, u_0)) \) obtained by folding first at \( u_0 \) and then infinitely many times touches the upper vertex of \( T^4(A(a, u_0)) \), i.e., the sequence \( F_i(u_0) \) of floors creeps into the upper corner of \( T^4(A(a, u_0)) \), cf. Figure 21. In order to prove the identity (3.10), we argue by contradiction and assume \( h(u_0) < A(a, u_0) \). Then \( w := A(a, u_0) - h(u_0) > 0 \). The Folding Lemma 3.1 (ii) shows that \( l_2(u_0) + u_2(u_0) > w \) for all \( i = 1, 2, \ldots \). Since \( h(u_0) < \infty \), there exists \( j \in \mathbb{N} \) such that \( l_2(u_0) < w/2 \) for \( i \geq j \). Then \( u_2(u_0) > w - l_2(u_0) > w/2 \) for \( i \geq j \), and so \( a = \sum_{i=1}^{\infty} u_i(u_0) > \sum_{i=1}^{\infty} u_{2i}(u_0) = \infty \). This contradiction proves the identity (3.10).

Assume now that Lemma 3.4 is wrong for some \( u_1 < u_0 \). Since \( u_1 \) is non-admissible, the procedure associated with \( u_1 \) does not terminate and generates the infinite sequence \( l_i(u_1) \), \( i = 1, 2, \ldots \). The proof of Lemma 3.2 shows that \( l_i(u_1) > l_i(u_0) \), \( i = 1, 2, \ldots \). Therefore,

\[ h(u_1) := \sum_{i=1}^{\infty} l_i(u_1) = h(u_0) = A(a, u_0) > A(a, u_1). \]
3. Multiple symplectic folding in four dimensions

The contradiction \( h(u_1) > A(a, u_1) \) shows that Lemma 3.4 holds true. □

While the computer program in Appendix D.1 computes for each \( a > 2\pi \) and each \( \epsilon > 0 \) the value of \( s_{EB}(a) \) up to accuracy \( \epsilon \), the following lemma gives some qualitative insight into the function \( s_{EB}(a) \).

**Lemma 3.5** The function \( s_{EB} \) on \( ]2\pi, \infty[ \) is strictly increasing and hence almost everywhere differentiable. Moreover, \( s_{EB} \) is Lipschitz continuous with Lipschitz constant at most \( 1 \).

**Proof.** Assume \( 2\pi < a < a' \). In view of the procedures associated with the pairs \((a, u_0(a'))\) and \((a', u_0(a'))\), the inequalities

\[
l_1(a, u_0(a')) = \pi - \frac{\pi}{a} u_0(a') < \pi - \frac{\pi}{a} u_0(a') = l_1(a', u_0(a'))
\]

and \( a - u_0(a') < a' - u_0(a') \) imply that \( u_0(a) \leq u_0(a') \). Therefore,

\[
s_{EB}(a) = 2\pi + \left(1 - \frac{2\pi}{a}\right) u_0(a) < 2\pi + \left(1 - \frac{2\pi}{a'}\right) u_0(a') = s_{EB}(a').
\]

Since \( a < a' \) were arbitrary, we conclude that the function \( s_{EB} \) is strictly increasing, and so, as every increasing real function, almost everywhere differentiable.

Assume again \( 2\pi < a < a' \). We set \( \delta = a' - a \) and \( u_1 = u_0(a) + \delta \). Since \( u_0(a) < a \), we have

\[
\frac{u_0(a)}{a} < \frac{u_1}{a'}.
\]

It follows that

\[
l_1(a', u_1) = \pi - \frac{\pi}{a'} u_1 < \pi - \frac{\pi}{a} u_0(a) = l_1(a, u_0(a)).
\]

In view of the procedures associated with the pairs \((a, u_0(a))\) and \((a', u_1)\), this inequality and the equality \( a' - u_1 = a - u_0(a) \) imply that \( u_1 \geq u_0(a') \). Using definition (3.9) and inequality (3.11), we may therefore estimate

\[
s_{EB}(a') - s_{EB}(a) = \left(1 - \frac{2\pi}{a'}\right) u_0(a') - \left(1 - \frac{2\pi}{a}\right) u_0(a) \\
\leq \left(1 - \frac{2\pi}{a'}\right) u_1 - \left(1 - \frac{2\pi}{a}\right) u_0(a) \\
= u_1 - u_0(a) - 2\pi \left(\frac{u_1}{a'} - \frac{u_0(a)}{a}\right) \\
< u_1 - u_0(a) \\
= a' - a.
\]
Since $a < a'$ were arbitrary, we conclude that the function $s_{EB}$ is Lipschitz continuous with Lipschitz constant at most 1. \hfill \Box

We don’t know whether the function $s_{EB}$ is differentiable on any open interval. We next investigate the behaviour of the function $s_{EB}(a)$ as $a \to 2\pi^+.$

**Proposition 3.6** We have

$$\limsup_{\epsilon \to 0^+} \frac{s_{EB}(2\pi + \epsilon) - 2\pi}{\epsilon} \leq \frac{3}{7}. $$

**Proof.** Fix $a > 2\pi.$ By the proof of Lemma 3.2 and by Lemma 3.3 there exists a unique value $u_{1,2} = u_{1,2}(a)$ such that $N(u_{1,2}) = 2$ and $u_3(u_{1,2}) = u_2(u_{1,2}).$ Then $u_{1,2} + 2u_2(u_{1,2}) = a.$ In view of equation (3.4) we find $u_{1,2} = \frac{a^2 - 3a\pi}{3a + \pi}.$ Therefore,

$$A_2(a) := A(a, u_{1,2}) = 2\pi + (a - 2\pi)\frac{a + \pi}{3a + \pi}, $$

and so $\frac{d}{da}A_2(2\pi) = \frac{3}{7}.$ Since $s_{EB}(a) \leq A_2(a)$ for all $a > 2\pi,$ the proposition follows. \hfill \Box

**Remark 3.7** The function $A_2(a)$ constructed in the above proof describes the optimal embedding of $E(\pi, a)$ into a ball obtainable by folding exactly twice. More generally, we can compute the function $A_N(a)$ describing the optimal embedding obtainable by folding exactly $N$ times as follows. For each $a > 2\pi$ and each $N = 1, 2, \ldots$ there exists a unique value $u_{1,N} = u_{1,N}(a)$ such that $N(u_{1,N}) = N$ and such that $u_{N+1}(u_{1,N}) = u_N(u_{1,N}) + l_{N-1}(u_{1,N})$ if $N$ is odd and $u_{N+1}(u_{1,N}) = u_N(u_{1,N})$ if $N$ is even. If $N$ is odd, we replace $l_{N-1}(u_{1,N})$ by the expression given in formula (3.1). Plugging the expression for $u_{N+1}(u_{1,N})$ into the equation

$$u_{1,N} + u_2(u_{1,N}) + \cdots + u_{N+1}(u_{1,N}) = a$$

and then alternately using equations (3.6) and (3.7), we compute $u_{1,N}$ as a rational function of $a.$ We finally find

$$A_N(a) = A(a, u_{1,N}) = 2\pi + \left(1 - \frac{2\pi}{a}\right)u_{1,N}(a).$$
For instance,
\[ A_1(a) = 2\pi + (a - 2\pi) \frac{1}{2}, \quad A_2(a) = 2\pi + (a - 2\pi) \frac{a + \pi}{3a + \pi} \]
and
\[ A_3(a) = 2\pi + (a - 2\pi) \frac{(a + \pi)(a + 2\pi)}{4(a^2 + a\pi + \pi^2)}. \]

By Lemma 3.2, \( u_{1,N+1}(a) < u_{1,N}(a) \) for every \( N \) and every \( a > 2\pi \), and arguing as in the proof of Lemma 3.5, we see that the function \( u_{1,N}(a) \) is increasing for every \( N \). The family \( \{A_N\}, N = 1, 2, \ldots, \) is therefore a strictly decreasing family of strictly increasing smooth rational functions on \([2\pi, \infty[) converging to \( s_{EB}(a) \).

In view of Dini's Theorem, the convergence is uniform on bounded sets.

One might try to improve the estimate given in Proposition 3.6 by showing \( \frac{d}{da} A_N(2\pi) < \frac{3}{7} \) for some \( N \). However, \( \frac{d}{da} A_N(2\pi) = \frac{3}{7} \) for all \( N \geq 2 \). Indeed, for all \( a > 2\pi \) and \( N > 2 \) we have

\[ u_2(u_{1,2}(a)) > u_2(u_{1,N}(a)) > 2l_2(u_{1,N}(a)) > 2l_2(u_{1,2}(a)) = \frac{2\pi}{a} u_2(u_{1,2}(a)), \]

and so \( \lim_{a \to 2\pi^+} u_2(u_{1,N}(a)) = \lim_{a \to 2\pi^+} u_2(u_{1,2}(a)) \). In view of formula (3.4), we conclude that

\[ \lim_{a \to 2\pi^+} u_{1,N}(a) = \lim_{a \to 2\pi^+} u_{1,2}(a). \]

Next, the identity

\[ A_N(a) = 2\pi + \left(1 - \frac{2\pi}{a}\right) u_{1,N}(a) \quad \text{for all } a > 2\pi \]

implies that

\[ \frac{d}{da} A_N(a) = \frac{2\pi}{a^2} u_{1,N}(a) + \left(1 - \frac{2\pi}{a}\right) \frac{d}{da} u_{1,N}(a) \quad \text{for all } a > 2\pi. \]

The formal expression for \( u_{1,N}(a) \) defines a rational function on \( \mathbb{R} \). Since \( 2\pi \) is not a singularity of \( u_{1,N}(a) \), the rational function \( \frac{d}{da} u_{1,N}(a) \) is bounded near \( a = 2\pi \). Taking the limit \( a \to 2\pi^+ \), we therefore find

\[ \frac{d}{da} A_N(2\pi) = \frac{1}{2\pi} \lim_{a \to 2\pi^+} u_{1,N}(a) = \frac{1}{2\pi} \lim_{a \to 2\pi^+} u_{1,2}(a) = \frac{d}{da} A_2(2\pi) = \frac{3}{7}, \]

as claimed. \( \Box \)
3.3. Embeddings into balls

We do not know how to analyze the asymptotic behaviour of the function $s_{EB}(a)$ as $a \to \infty$ directly. We shall prove the following proposition at the end of the subsequent section by comparing the optimal multiple folding embedding of $E(\pi, a)$ into a ball with the optimal multiple folding embedding of the polydisc $P\left(\pi, \frac{a}{2} + \pi\right)$ into a ball.

**Proposition 3.8** We have

$$s_{EB}(a) - \sqrt{\pi a} \leq 2\pi \quad \text{for all } a > 2\pi.$$  

The function $s_{EB}(a)$ is further discussed and compared with the result yielded by Lagrangian folding in 6.2.1.1.

3.3.2 Embedding polydiscs into balls

As we have seen in the proof of Lemma 2.8, the disc $D(a)$ is symplectomorphic to the rectangle $R(a)$. The polydisc $P(\pi, a) = D(\pi) \times D(a)$ is therefore symplectomorphic to $R(a) \times D(\pi)$. Since the fiber $D(\pi)$ over each point $(u, v) \in R(a)$ is the same, the optimal embedding into a ball obtainable by multiple symplectic folding is easier to compute for a polydisc than for an ellipsoid. In contrast to our optimal embeddings of an ellipsoid into a ball, which were obtained by folding “more and more often”, the optimal embedding of a polydisc into a ball obtainable by multiple folding will turn out to be described by a picture as in Figure 22. Our embedding result stated in Proposition 3.10 below is readily read off from such a picture. We aim, however, to show that this embedding result is the best one obtainable by multiple folding. We therefore proceed in a systematic way.

![Figure 22: The optimal embedding $P(\pi, a) \hookrightarrow B^4(A)$ for $a = 10\pi$.](image)
We again think of the ball $B^4(A)$ as the trapezoid $T^4(A)$. Fix $a \geq \pi$. Folding $R(a) \times D(\pi)$ first at $u_1 \in [0, a]$ determines $T^4(A(a, u_1))$ by the condition that the second floor $F_2$ touches the “upper right boundary” of $T^4(A(a, u_1))$. Then $A(a, u_1) = u_1 + 2\pi$. We then successively choose $u_i$, $i \geq 2$, maximal with respect to the condition of staying inside $T^4(A(a, u_1))$.

The Folding Lemma 3.1 (ii) shows that a condition for folding a second time, if necessary, is $u_1 > \pi$, and that then the stairs $S_2$ are contained in $T^4(A(a, u_1))$ if and only if $u_1 > 2\pi$. The only condition on $u_1$ for folding a second time is therefore $u_1 > 2\pi$. The Folding Lemma 3.1 (i) shows that folding a third time, if necessary, is then possible whenever $u_2 > 2\pi$, i.e., $u_1 > 3\pi$. For $N \geq 2$, the only condition on $u_1$ for folding an $N$'th time, if necessary, is $u_1 > N\pi$.

If our procedure leads to an embedding obstruction after $N$ folds, then choosing $u'_1 \leq u_1$ leads to an embedding obstruction after $N'$ folds. It is therefore enough to compare embeddings obtained from our procedure.

We say that $u_1$ is admissible if the procedure associated with $u_1$ leads to an embedding of $R(a) \times D(\pi)$ into $T^4(A(a, u_1))$. We then write $N(u_1)$ for the number of folds needed. If $u_1 < u'_1$ and $u_1$ is admissible, then $u'_1$ is admissible and $N(u'_1) \leq N(u_1)$.

**Lemma 3.9** Assume that $u_1$ is admissible and that $N(u_1)$ is even or that $N(u_1)$ is odd and the last floor $F_{N(u_1)+1}$ does not touch $\{u = 0\}$. Then there exists an admissible $u'_1$ such that $u'_1 < u_1$ and such that $N(u'_1)$ is odd and $F_{N(u'_1)+1}$ touches $\{u = 0\}$.

**Proof.** Set $N = N(u_1)$. Observe that on those admissible $u_1$'s for which $N(u_1) = N$, the functions $u_2(u_1), \ldots, u_N(u_1)$ are continuous and increasing in $u_1$, and $u_{N+1}(u_1)$ is continuous and decreasing in $u_1$.

Assume first that $N$ is odd and $F_{N+1}$ does not touch $\{u = 0\}$, i.e., $u_2 < u_1$ if $N = 1$ and $u_{N+1} < u_N + \pi$ if $N \geq 3$. If $N = 1$, shrinking $u_1$ leads to an admissible $u'_1$ such that $u'_2 = u'_1$. If $N \geq 3$, shrinking $u_1$ either leads to an admissible $u'_1$ such that $u'_{N+1} = u_N + \pi$, or to $u'_1 = N\pi$. In the second case, however, folding at $u'_1$ would already lead to an embedding after $N - 1$ folds, i.e., $u'_1 < u_1$ would be admissible with $N(u'_1) < N(u_1)$, a contradiction.

Assume next that $N$ is even. After shrinking $u_1$, if necessary, we may assume that $u_{N+1}$ touches the “upper right boundary” of $T^4(A(a, u_1))$, i.e., $u_{N+1} + \pi = u_N$. We have $u_{N+1} > \pi$, since otherwise $N(u_1) \leq N - 1$. Therefore $u_N > 2\pi$, and so we can fold another time at $u = u_N - \pi$ and obtain an embedding of $R(a) \times D(\pi)$ into $T^4(A(a, u_1))$ with $u_{N+2} < u_{N+1} + \pi$. As we have seen above,
shrinking $u_1$ leads to an admissible $u'_1$ such that $u'_{N+2} = u_{N+1} + \pi$. \hfill \Box

Assume that $a \in [\pi, 2\pi]$ and that $u_1$ is admissible. Then $N(u_1) = 1$. By Lemma 3.9, we may assume that $u_1 = a/2$. Since $A(a, a/2) = a/2 + 2\pi \geq a + \pi$, we see that for $a \in [\pi, 2\pi]$ multiple symplectic folding does not provide a better embedding result than the inclusion $P(\pi, a) \hookrightarrow B^4(\pi + a)$.

So assume $a > 2\pi$. By Lemma 3.9 it suffices to analyze those embeddings for which the number of folds is $N = 2k - 1$ and $F_{N+1}$ touches $\{u = 0\}$. The optimal embedding obtainable by folding once is therefore described by $A(a) = a/2 + 2\pi$. If $N \geq 3$, we read off from Figure 22 that
\[
a = u_1 + u_2 + \cdots + u_{N+1} = \pi + 2(u_1 - \pi) + 2(u_1 - 3\pi) + \cdots + 2(u_1 - N\pi) + \pi = 2\pi + 2ku_1 - 2k^2\pi
\]
provided that $u_1 > N\pi$. Solving for $u_1$ and using the formula $A(a, u_1) = u_1 + 2\pi$, we find that the optimal embedding of $R(a) \times D(\pi)$ into a ball obtainable by folding $N = 2k - 1$ times is described by
\[
A_k(a) = \frac{a - 2\pi}{2k} + (k + 2)\pi \quad \text{provided that } a > 2(k^2 - k + 1)\pi.
\]
Observe that this formula also holds true for $N = 1$. Define the function $s_{PB}(a)$ on $]2\pi, \infty[$ by
\[
s_{PB}(a) = \min \left\{ A_k(a) \mid k = 1, 2, \ldots; a > 2(k^2 - k + 1)\pi \right\},
\]
cf. Figure 23. We in particular have proved

**Proposition 3.10** Assume $a > 2\pi$. Then the polydisc $P(\pi, a)$ symplectically embeds into the ball $B^4(s_{PB}(a) + \epsilon)$ for every $\epsilon > 0$, where
\[
s_{PB}(a) = \frac{a - 2\pi}{2k} + (k + 2)\pi
\]
for the unique integer $k$ for which $2(k^2 - k + 1) < a/\pi \leq 2(k^2 + k + 1)$.

**Remark 3.11** Let $d_{PB}(a) = s_{PB}(a) - \sqrt{2\pi a}$ be the difference between $s_{PB}$ and the volume condition. The function $d_{PB}$ attains its local maxima at $a_k = 2(k^2 - k + 1)\pi$, where $d_{PB}(a_k) = (2k + 1)\pi - 2\pi\sqrt{k^2 - k + 1}$. This is an increasing sequence converging to $2\pi$. \hfill \Diamond
Extend the above function $s_{PB}(a)$ to a function on $[\pi, \infty]$ by setting $s_{PB}(a) = a + \pi$ for $a \in [\pi, 2\pi]$. The problem considered in this section was to understand the function $f_{PB}$ on $[\pi, \infty]$ defined by

$$f_{PB}(a) = \inf\left\{ A \mid P(\pi, a) \text{ symplectically embeds into } B^4(A) \right\}.$$ 

The following proposition summarizes what we know about this function.

**Proposition 3.12** The function $f_{PB} : [\pi, \infty] \rightarrow \mathbb{R}$ is bounded from below and above by

$$\max\left(2\pi, \sqrt{2\pi a}\right) \leq f_{PB}(a) \leq sp_{PB}(a),$$

see Figure 23. It is monotone increasing and hence almost everywhere differentiable. Moreover, $f_{PB}$ is Lipschitz continuous with Lipschitz constant at most 2; more precisely,

$$f_{PB}(a') - f_{PB}(a) \leq \frac{sp_{PB}(a)}{a}(a' - a) \quad \text{for all } a' \geq a \geq \pi.$$

**Proof.** In view of the monotonicity axiom for the second Ekeland–Hofer capacity, the identities (1.5) and (1.9) imply $f_{PB}(a) \geq 2\pi$ for all $a \geq \pi$. The volume condition $|P(\pi, a)| \leq |B^4(f_{PB}(a))|$ translates to $f_{PB}(a) \geq \sqrt{2\pi a}$. We conclude that $\max\left(2\pi, \sqrt{2\pi a}\right) \leq f_{PB}(a)$. For $a \leq 2\pi$, the estimate $f_{PB}(a) \leq sp_{PB}(a)$ is provided by the inclusion $P(\pi, a) \hookrightarrow B^4(a + \pi)$, and for $a > 2\pi$ by Proposition 3.10.

Assume $a < a'$. If $\varphi$ symplectically embeds $P(\pi, a')$ into $B^4(A)$, then $\varphi|P(\pi, a)$ symplectically embeds $P(\pi, a)$ into $B^4(A)$. Therefore $f_{PB}$ is increasing, and so, as every increasing real function, almost everywhere differentiable. Denote the dilatation by $a'/a$ by $\lambda$. Assume that $\psi$ symplectically embeds $P(\pi, a)$ into $B^4(A)$. Then the composition

$$P(\pi, a') \xrightarrow{\lambda^{-1}} P\left(\frac{a'}{a'}, \pi, a\right) \xrightarrow{\psi} B^4(A) \xrightarrow{\lambda} B^4\left(\frac{a'}{a}, A\right)$$

is a symplectic embedding. Therefore $f_{PB}(a') \leq \frac{a'}{a} f_{PB}(a)$. We conclude that

$$f_{PB}(a') - f_{PB}(a) \leq f_{PB}(a) \left(\frac{a'}{a} - 1\right) \leq \frac{sp_{PB}(a)}{a}(a' - a) \leq 2(a' - a)$$

as claimed. \qed
3.3. Embeddings into balls

For $a = 2\pi$, multiple symplectic folding does not provide a better upper bound of $f_{PB}(a)$ than the inclusion $P(\pi, a) \hookrightarrow B^4(\pi + a)$, and Ekeland–Hofer capacities do not provide a better lower bound of $f_{PB}(a)$ than the volume condition $|P(\pi, a)| \leq |B^4(f_{PB}(a))|$. We therefore would like to know the answer to

**Question 3.13** $f_{PB}(2\pi) < 3\pi$?

We end this section by deriving Proposition 3.8 from Proposition 3.10.

**Proof of Proposition 3.8:** Computer calculations suggest that $s_{EB}(a) < s_{PB}\left(\frac{a}{2} + \pi\right)$ for all $a > 2\pi$. For our purpose, the following result will be sufficient.

**Lemma 3.14** We have $s_{EB}(a) < s_{PB}\left(\frac{a}{2} + \pi\right)$ for all $a > 2\pi$.

**Proof.** As in 3.3.1 we think of the ellipsoid $E(\pi, a)$ as the trapezoid $T(a, \pi)$ and of the ball $B^4(A)$ as the trapezoid $T^4(A)$. We fix $a > 2\pi$ and let

$$\mathcal{F} = \coprod_{i=1}^{N+1} F_i \cup \coprod_{i=1}^{N} S_i$$

be the image of the “optimal” embedding of $P(\pi, \frac{a}{2} + \pi)$ into $T^4\left(s_{PB}\left(\frac{a}{2} + \pi\right)\right)$. We recall from Lemma 3.9 that the number of folds $N$ is odd and that for $N = 3$ the set $\mathcal{F}$ looks as in Figure 22. We define $A \in [2\pi, \infty[$ as the unique real number for which

$$s_{EB}(A) = s_{PB}\left(\frac{a}{2} + \pi\right)$$

Figure 23: What is known about $f_{PB}(a)$.
Multiple symplectic folding in four dimensions

and we let

$$\mathcal{F}' = \bigsqcup_{i=1}^{\infty} F'_i \cup \bigsqcup_{i=1}^{\infty} S'_i$$

be the image of the “optimal” embedding of $T(A, \pi)$ into $T^4(s_{EB}(A))$, cf. Figure 21.

In the sequel we shall compare the volume of $\mathcal{F}$ with the volume of $\mathcal{F}'$. Since the embeddings of $P(\pi, a/2 + \pi)$ and $T(A, \pi)$ are both “optimal”, the volumes of the stairs $\bigsqcup S_i$ and $\bigsqcup S'_i$ “vanish”. We shall therefore neglect the stairs of both sets.

Recall from 3.3.1 that $l_i$ denotes the minimal height of the floor $F'_i$ and that the width and the height of the $i$’th “triangle” $T'_i$ in $T^4(s_{EB}(A)) \setminus \mathcal{F}'$ is $2l_{2i-1}$.

Also recall that

$$n > h > l_2 > \ldots \quad (3.12)$$

This and the Folding Lemma 3.1 imply that $F'_i \subset \mathcal{F}'$ for each odd $i \geq 3$, and that $\mathcal{F} \setminus \mathcal{F}'$ is the disjoint union of the thin “triangle” $Q_1 = F'_1 \setminus (F'_0 \cup F'_2)$, the “rectangles” $Q_i \subset F'_i$, $i$ even, each contained in a different triangle $T'_{2i(i)}$, and the set $Q_0$ lying in the left end of the floor $F_{N+1}$, see Figure 24.

Using the definition (3.1) of $l_1$ and the estimate (3.3) we find

$$l_2 < l_1 = \pi - \frac{\pi}{A} < \pi - \frac{\pi^2}{A + \pi}$$

and so

$$|Q_0| + |Q_1| \leq l_2\pi + \frac{1}{2}l_2\frac{\pi}{A}l_2 \leq \pi^2. \quad (3.13)$$

We shall prove that

$$|Q_2| + |Q_4| + \cdots + |Q_{N+1}| < |\mathcal{F} \setminus \mathcal{F}'|. \quad (3.14)$$

The estimates (3.13) and (3.14) yield

$$|\mathcal{F}| = |\mathcal{F} \setminus \mathcal{F}'| + |\mathcal{F} \cap \mathcal{F}'|$$

$$< \pi^2 + |\mathcal{F}' \setminus \mathcal{F}| + |\mathcal{F}' \cap \mathcal{F}'|$$

$$= \pi^2 + |\mathcal{F}'|. \quad (7)$$

Therefore,

$$\pi^2 + \frac{\pi a}{2} = \left| P\left(\pi, \frac{a}{2} + \pi\right) \right| = |\mathcal{F}| < \pi^2 + |\mathcal{F}'| = \pi^2 + |T(A, \pi)| = \pi^2 + \frac{\pi A}{2},$$

where $P(\pi, a/2 + \pi)$ is the area under the curve $y = \pi (x - \pi/2)$ from $x = \pi/2$ to $x = \pi$. 


3.3. Embeddings into balls

Figure 24: The sets $Q_i \subset \mathcal{F} \setminus \mathcal{F}'$ and the sets $R_{j(i)} \subset \mathcal{F}' \setminus \mathcal{F}$.

i.e., $a < A$. Since the function $s_{EB}$ is monotone increasing, we conclude that $s_{EB}(a) \leq s_{PB}(\frac{a}{2} + \pi)$ as claimed.

In order to prove the estimate (3.14) we denote the “triangles” in $T^4(s_{EB}(A)) \setminus \mathcal{F}$ of height and width $2\pi$ by $T_i$, $i = 1, 2, \ldots$, and associate with each rectangle $Q_i \subset T'_{j(i)}$, $i = 2, 4, \ldots, N + 1$, the rectangle

$$R_i \subset \mathcal{F}' \cap T_{i/2} \subset \mathcal{F}' \setminus \mathcal{F}$$

whose width $w(R_i)$ is equal to the height $h(Q_i)$ of $Q_i$ and whose height $h(R_i)$ is $2\pi - h(Q_i)$, cf. Figure 24. Since the width $w(Q_i)$ of $Q_i$ is

$$2l_{2j(i)} - w(R_i) = 2l_{2j(i)} - h(Q_i)$$
we find together with the inequalities (3.12) that

\[ |Q_i| = w(Q_i)h(Q_i) = (2l_{2j(i)} - h(Q_i))h(Q_i) < (2\pi - h(Q_i))h(Q_i) = h(R_i)w(R_i) = |R_i|. \]

The estimate (3.14) thus follows, and so the proof of Lemma 3.14 is complete. \(\square\)

Proposition 3.8 follows from Lemma 3.14 and Proposition 3.10. Indeed, in view of Proposition 3.10 the function

\[ d(a) := sp_B \left( \frac{a}{2} + \pi \right) - \sqrt{\pi a} \]

on \([2\pi, \infty[\) has its local maxima at

\[ a_k = 2 \left( 2(k^2 - k + 1) - 1 \right) \pi, \quad k = 1, 2, \ldots, \]

where

\[ d(a_k) = (2k + 1)\pi - \sqrt{\pi a_k}. \]

The sequence \(d(a_k)\) is monotone increasing to \(2\pi\). Together with Lemma 3.14 we conclude that

\[ s_{EB}(a) - \sqrt{\pi a} \leq s_{pB} \left( \frac{a}{2} + \pi \right) - \sqrt{\pi a} \leq 2\pi \]

and so the proof of Proposition 3.8 is complete. \(\square\)

### 3.4 Embeddings into cubes

In this section we use multiple symplectic folding to construct symplectic embeddings of four dimensional ellipsoids and polydiscs into four dimensional cubes. While in Section 1.3 the lack of convenient invariants made it impossible to get good rigidity results for embeddings into cubes, multiple symplectic folding provides us with rather satisfactory flexibility results.
3.4. Embeddings into cubes

3.4.1 Embedding ellipsoids into cubes

Fix \( a > \pi \). We think of the ellipsoid \( E(\pi, a) \) as \( T(a, \pi) \) and of the cube \( C^4(A) \) as \( R(A) \times D(A) \). In order to find the smallest \( A \) for which \( T(a, \pi) \) embeds into \( R(A) \times D(A) \) via multiple symplectic folding, we proceed as follows. We fix \( u_1 \in ]0, a[ \) and fold at \( u_1 \). By the Folding Lemma 3.1 (i), the stairs \( S_1 \) are contained in \( \{ u < w(a, u_1) \} \), where

\[
    w(a, u_1) = u_1 + l_1 = \pi + \left( 1 - \frac{\pi}{a} \right) u_1.
\]

We then choose \( u_i, i \geq 2 \), maximal with respect to the condition of staying inside \( \{ 0 < u < w(a, u_1) \} \). If the remainder \( r_1 = a - u_1 \) is smaller than \( u_1 \), we obtain an embedding of \( T(a, \pi) \) into \( \{ 0 < u < w(a, u_1) \} \) by folding once. If \( r_1 \geq u_1 \), we are forced to fold a second time. The same discussion as in 3.3.1 shows that the only condition for doing so is \( u_1 < m_a \), i.e.,

\[
    u_1 > \frac{a \pi}{a + \pi}.
\]

If this condition is met, we define \( u_2 \) by formula (3.4). The sequence \( l_i, i = 1, 2, \ldots \), of the widths of the stairs \( S_i \) is decreasing. Hence, there are no further conditions at the subsequent folds, and if there is an \( i \)’th fold, then \( u_i > u_{i-1} \) for all \( i \geq 3 \). Under the condition (3.16), our procedure therefore embeds \( T(a, \pi) \) into \( \{ 0 < u < w(a, u_1) \} \) by folding finitely many times. We denote the number of folds needed by \( N(u_1) \). Recall that \( l_i = l_i(a, u_1) \) is the width of the stairs \( S_i \) as well as the minimal height of the floor \( F_i \) as well as the maximal height of \( F_{i+1} \), \( i = 1, \ldots, N(u_1) \). We set \( l_i(a, u_1) = 0 \) if \( i > N(u_1) \). For \( a > \pi \) fixed and
$u_1 \in \left[ \frac{\pi}{a + \pi}, a \right]$, the functions $l_i(a, u_1)$ are decreasing continuous functions of $u_1$. Therefore, the height of the image

$$h(a, u_1) = \sum_{i=1}^{N(u_1)} l_i(a, u_1)$$

is a decreasing continuous function of $u_1$ for $u_1 \in \left[ \frac{\pi}{a + \pi}, a \right]$. For $u_1 \geq a/2$ we have $h(a, u_1) = \pi$, and for $u_1 \not\in a\pi/(a + \pi)$ we have $N(u_1) \to \infty$ and $u_i \to 0$, $l_i(a, u_1) \to l_1(a, u_1)$ for all $i \geq 2$, and so $h(a, u_1) \to \infty$ as $u_1 \not\in a\pi/(a + \pi)$. On the other hand, $w(a, u_1) = \pi + (1 - \pi/a)u_1$ is a strictly increasing continuous function of $u_1$ such that $w(a, 0) = \pi$ and $w(a, a) = a$. It follows that $w(a, u_1) = h(a, u_1)$ for exactly one $u_1 \in \left[ \frac{\pi}{a + \pi}, a \right]$, which we call $u_0 = u_0(a)$. Define the function $s_{EC}(a)$ on $]\pi, \infty[$ by

$$s_{EC}(a) = \pi + \left(1 - \frac{\pi}{a}\right)u_0(a),$$

cf. Figure 26. We have shown that the ellipsoid $E(\pi, a)$ symplectically embeds into the cube $C^4(s_{EC}(a) + \epsilon)$ for any $\epsilon > 0$. A computer program for the function $s_{EC}(a)$ is presented in Appendix D.2.

Our procedure is optimal in the sense that we cannot embed $E(\pi, a)$ into a cube smaller than $C^4(s_{EC}(a))$ by multiple symplectic folding. This follows from an argument similar to the one given in 3.3.1. Indeed, our procedure can equivalently be described as follows: For each $A \in ]\pi, a[$ we successively choose $u_1, u_2, \ldots$ maximal with respect to the condition of staying inside $\{0 < u < A\}$.

Then $s_{EC}(a)$ is the smallest $A$ for which we do not run into an embedding obstruction and for which the height of the image is smaller than $A$. The only way of improving our procedure is therefore to choose some of the $u_i$ smaller. So let $A < s_{EC}(a)$, and choose $u_i' \leq u_i$. We then either run into an embedding obstruction, or the height of the image of the modified embedding is larger than $h(a, u_1)$ and hence larger than $A$.

**Remarks 3.15**

1. We are going to investigate the function $s_{EC}(a)$ in more detail.

We have $N(u_1) = 1$ if and only if $u_1 \in [a/2, a]$. Then $h(a, u_1) = \pi < \pi + (1 - \pi/a)u_1 = w(a, u_1)$, and so $w(a, a/2) = (a + \pi)/2$, and $h(a, u_1) < w(a, u_1)$ for all $u_1 \in [a/2, a]$. It follows that by folding once we can embed $E(\pi, a)$ into $C^4\left(\frac{a + \pi}{2} + \epsilon\right)$ for any $\epsilon > 0$, and that folding only once never yields an optimal embedding, i.e., $s_{EC}(a) < (a + \pi)/2$ for all $a > \pi$. 
3.4. Embeddings into cubes

We have \( N(u_1) = 2 \) if and only if \( u_1 < a/2 \) and \( l_2 + u_3 = l_2 + (a/\pi)l_2 \leq w(a, u_1) \). Using the formulas (3.15), (3.1) and (3.4) for \( w, l_2 \) and \( u_2 \), we find that the second inequality is equivalent to the condition on \( u_1 \)

\[
\frac{a(a^2 + \pi^2)}{3a^2 + \pi^2} \leq u_1. \tag{3.17}
\]

If \( N(u_1) = 2 \), then \( h(a, u_1) = 2l_1 + l_2 \). Plugging the identity \( u_2 = u_1 - l_2 \) into the identity \( a = u_1 + u_2 + u_3 = u_1 + u_2 + (a/\pi)l_2 \) and solving for \( l_2 \) we find

\[
h(a, u_1) = 2\pi - \frac{2\pi}{a}u_1 + \frac{\pi(a - 2u_1)}{a - \pi}.
\]

The equation \( h = w \) thus yields

\[
u_0(a) = \frac{a\pi(2a - \pi)}{a^2 + 2a\pi - \pi^2} \tag{3.18}
\]

provided that \( u_0(a) \) meets condition (3.16), that \( u_0(a) < a/2 \), and that \( u_0(a) \) meets condition (3.17). We compute that \( u_0(a) \) meets condition (3.16) and that \( u_0(a) < a/2 \) whenever \( a > \pi \), and that \( u_0(a) \) meets condition (3.17) if and only if \( \pi \leq a \leq 3\pi \). It follows that (3.18) holds for all \( a \in [\pi, 3\pi] \).

In fact, the identity (3.18) also holds true for all those \( a \) for which the optimal embedding of \( T(a, \pi) \) obtainable by multiple folding is a 3-fold for which the height is still \( h = 2l_1 + l_2 \), i.e., for which \( u_4(u_0(a)) \leq u_3(u_0(a)) \). The largest \( a \) for which (3.18) holds true is characterized by the identity \( u_4(u_0(a)) = u_3(u_0(a)) \). Using the identity \( u_3 = \frac{a + \pi}{a - \pi}u_2 \) we compute that the equation \( a = u_0(a) + u_2(u_0(a)) + 2u_3(u_0(a)) \) translates into

\[
a = \frac{a\pi(5a^2 - 2a\pi + \pi^2)}{(a - \pi)(a^2 + 2a\pi - \pi^2)},
\]

i.e., \( a = \left(2 + \sqrt{5}\right)\pi \). Therefore,

\[
s_{EC}(a) = \frac{a\pi(3a - \pi)}{a^2 + 2a\pi - \pi^2} \quad \text{for} \quad \pi < a \leq (2 + \sqrt{5})\pi.
\]

In general, \( s_{EC}(a) \) is a piecewise rational function. Its singularities are those \( a \) for which \( u_{N(a)}(u_0(a)) = u_{N(a)+1}(u_0(a)) \), \( N(a) = 3, 5, 7, \ldots \), and those \( a \) for which the "endpoint" of \( F_{N(a)+1} \) touches the "axis" \( \{u = 0\} \); here, we set \( N(a) = N(u_0(a)) \). Denoting the two sets of singularities by \( a_3, a_5, a_7, \ldots \) and
3. Symplectic folding in higher dimensions

We have that the singular set of $s_{EC}$ is the strictly increasing, diverging sequence $(a_k), k \geq 3$.

2. Let $d_{EC}(a) = s_{EC}(a) - \sqrt{\pi a/2}$ be the difference between $s_{EC}$ and the volume condition. Set $a_2 = \pi$. Computer calculations suggest that the function $d_{EC}$ attains exactly one local maximum $M_k$ between $a_{2k}$ and $a_{2k+1}$, that $d_{EC}$ attains its local minima $m_k$ at $a_{2k+1}$, and that $d_{EC}$ strictly increases between $a_{2k+1}$ and $a_{2k+2}$, $k \geq 1$. Moreover, they suggest that both $(m_k)$ and $(M_k)$ are strictly increasing and converging to $(2/3)n$. In particular, we seem to have $\lim_{a \to \infty} s_{EC}(a) - \sqrt{\pi a/2} = (2/3)n$. ◦

As in the case of the function $s_{EB}$ studied in 3.3.1 we do not know how to analyze the asymptotic behaviour of the function $s_{EC}(a)$ as $a \to \infty$ directly. We shall prove the following proposition at the end of the subsequent section by comparing the optimal multiple folding embedding of $E(\pi, a)$ into a cube with the optimal multiple folding embedding of the polydisc $P\left(\pi, \frac{\pi}{2} + \pi\right)$ into a cube.

**Proposition 3.16** We have

$$s_{EC}(a) - \sqrt{\frac{\pi a}{2}} \leq \frac{3}{2} \pi \quad \text{for all } a > 2\pi.$$

Figure 26: What is known about $f_{EC}(a)$. 
3.4. Embeddings into cubes

Extend the function $s_{EC}(a)$ to a function on $[\pi, \infty[$ by setting $s_{EC}(\pi) = \pi$. The problem considered in this section was to understand the function $f_{EC}$ on $[\pi, \infty[$ defined by

$$f_{EC}(a) = \inf \left\{ A \mid E(\pi, a) \text{ symplectically embeds into } C^4(A) \right\}.$$

The following proposition summarizes what we know about this function.

**Proposition 3.17** The function $f_{EC} : [\pi, \infty[ \to \mathbb{R}$ is bounded from below and above by

$$\max \left( \pi, \sqrt{\frac{\pi a}{2}} \right) \leq f_{EC}(a) \leq s_{EC}(a),$$

see Figure 26. It is monotone increasing and hence almost everywhere differentiable. Moreover, $f_{EC}$ is Lipschitz continuous with Lipschitz constant at most 1; more precisely,

$$f_{EC}(a') - f_{EC}(a) \leq \frac{s_{EC}(a)}{a} (a' - a) \quad \text{for all } a' \geq a \geq \pi.$$

**Proof.** In view of the monotonicity axiom for the first Ekeland–Hofer capacity, the identities (1.5) and (1.9) imply $f_{EC}(a) \geq \pi$ for all $a \geq \pi$, and the volume condition $|E(\pi, a)| \leq |C^4(f_{EC}(a))|$ translates to $f_{EC}(a) \geq \sqrt{\pi a/2}$. The first claim thus follows. The remaining claims follow as in the proof of Proposition 3.12. \qed 

### 3.4.2 Embedding polydiscs into cubes

Fix $a > \pi$. We think of the polydisc $P(\pi, a)$ as $R(a) \times D(\pi)$ and of the cube $C^4(A)$ as $R(A) \times D(A)$. In order to find the smallest $A$ for which $R(a) \times D(\pi)$ embeds into $R(A) \times D(A)$ via multiple symplectic folding, we proceed as follows. We fix $u_1 \in [0, a[$ and fold at $u_1$. By the Folding Lemma 3.1 (i), the stairs $S_1$ are contained in $\{ u < u_1 + \pi \}$. We then choose $u_i, i \geq 2$, maximal with respect to the condition of staying inside $\{ 0 < u < u_1 + \pi \}$.

The Folding Lemma 3.1 (ii) shows that the only condition for folding a second time, if necessary, is $u_1 > \pi$. For $N \geq 2$, the only condition on $u_1$ for folding an $N$'th time, if necessary, is $u_1 > \pi$.

We say that $u_1$ is admissible if $u_1 \geq a/2$ or $u_1 > \pi$. It follows that if $u_1$ is admissible, then our procedure embeds $R(a) \times D(\pi)$ into $R(A(a, u_1)) \times D(A(a, u_1))$ by a finite number $N(u_1)$ of folds. Here,

$$A(a, u_1) = \max \{ u_1 + \pi, (N(u_1) + 1)\pi \}.$$  (3.19)
Let $u_2, \ldots , u_{N(u_1)+1}$ be the lengths associated with some admissible $u_1$. Choosing some of the $u_i$, $i = 2, \ldots , N(u_1)$, smaller would lead to an embedding by folding at least $N(u_1)$ times. It is therefore enough to compare embeddings obtained from our procedure.

Assume that $a \in ]\pi, 2\pi]$. Then $A(a, u_1) \geq 2\pi$ for every admissible $u_1$. It follows that for $a \in ]\pi, 2\pi]$ multiple symplectic folding does not provide a better embedding result than the inclusion $P(\pi, a) \hookrightarrow C^4(a)$.

So assume $a > 2\pi$. Suppose that $u_1$ is admissible and that $N := N(u_1)$ is even. We claim that if the last floor $F'_{N+1}$ does not touch $\{u = u_1 + \pi\}$, then there exists an admissible $u'_1$ such that $u'_1 < u_1$, $N(u'_1) = N$ and $F'_{N+1}$ touches $\{u = u'_1 + \pi\}$. Indeed, shrinking $u_1$ either leads to a $u'_1$ as claimed or to $u'_1 = \pi$. In the second case, however, we find $a \leq 2\pi$, a contradiction. We may therefore assume that $F'_{N+1}$ touches $\{u = u_1 + \pi\}$. A similar argument shows that we may also assume that $F'_{N(u_1)+1}$ touches $\{u = 0\}$ if $N(u_1)$ is odd.

The optimal embedding obtainable by folding only once is therefore described by $A_1(a) = \max \{\frac{a}{2} + \pi, 2\pi\} = \frac{a}{2} + \pi$, and if the number of folds is $N \geq 2$, we read off from Figure 27 that

$$a = 2\pi + (N + 1)(u_1 - \pi)$$

provided that $u_1 > \pi$. Since $a > 2\pi$, we see that this condition is met. Solving for $u_1$ and using formula (3.19) we then find that the optimal embedding of $R(a) \times D(\pi)$ into a cube obtainable by folding $N$ times is described by

$$A_N(a) = \max \left\{\frac{a + 2N\pi}{N+1}, (N + 1)\pi\right\}.$$
Observe that this formula also holds true for $N = 1$. Define the function $s_{PC}(a)$ on $]2\pi, \infty[$ by

$$s_{PC}(a) = \min \{ A_N(a) \mid N = 1, 2, \ldots \},$$

cf. Figure 54. We in particular have proved

**Proposition 3.18** Assume $a > 2\pi$. Then the polydisc $P(\pi, a)$ symplectically embeds into the cube $C^4(s_{PC}(a) + \epsilon)$ for every $\epsilon > 0$, where

$$s_{PC}(a) = \begin{cases} 
(N + 1)\pi & \text{if } (N - 1)N + 2 < \frac{a}{\pi} \leq N^2 + 1, \\
\frac{a+2N\pi}{N+1} & \text{if } N^2 + 1 < \frac{a}{\pi} \leq N(N + 1) + 2.
\end{cases}$$

The function $s_{PC}(a)$ is compared with the result yielded by Lagrangian folding in 6.2.2.1.

We end this section by deriving Proposition 3.16 from Proposition 3.18.

**Proof of Proposition 3.16:** We proceed as in the proof of Proposition 3.8.

**Lemma 3.19** We have $s_{EC}(a) < s_{PC}\left(\frac{a}{2} + \pi\right)$ for all $a > 2\pi$.

**Proof.** As in 3.4.1 we think of the ellipsoid $E(\pi, a)$ as the trapezoid $T(a, \pi)$. We fix $a > 2\pi$ and let

$$\mathcal{F} = \bigsqcup_{i=1}^{N+1} F_i \cup \bigsqcup_{i=1}^{N} S_i$$

be the image of the "optimal" embedding of $P(\pi, \frac{a}{2} + \pi)$ into $C^4\left(s_{PC}\left(\frac{a}{2} + \pi\right)\right)$. For $N = 3$ the set $\mathcal{F}$ looks as in Figure 27. We define $A \in ]2\pi, \infty[$ as the unique real number for which

$$s_{EC}(A) = s_{PC}\left(\frac{a}{2} + \pi\right)$$

and we let

$$\mathcal{F}' = \bigsqcup_{i=1}^{N'+1} F'_i \cup \bigsqcup_{i=1}^{N'} S'_i$$

be the image of the "optimal" embedding of $T(A, \pi)$ into $C^4(s_{EC}(A))$, cf. Figure 25.

As in the proof of Lemma 3.14 we shall neglect the stairs of $\mathcal{F}$ and $\mathcal{F}'$. Recall from 3.4.1 that $l_i$ denotes the minimal height of the floor $F'_i$ and that

$$\pi > l_1 > l_2 > \ldots.$$
This and the Folding Lemma 3.1 imply that
\[ \mathcal{F} \setminus \mathcal{F}' = F_1 \setminus \mathcal{F}' \cup F_{N+1} \setminus \mathcal{F}'. \]
The set \( Q_1 := F_1 \setminus \mathcal{F}' = F_1 \setminus (F'_1 \cup F'_2) \), which is analogous to the set \( Q_1 \) in Figure 24, has volume
\[ |Q_1| = \frac{1}{2} l_2 \frac{\pi}{A} l_2. \]

We decompose the set \( Q_0 := F_{N+1} \setminus \mathcal{F}' \) into the sets \( Q'_0 \) and \( Q''_0 = Q_0 \setminus Q'_0 \), where
\[ Q'_0 := \begin{cases} 
\{(u, v, x, y) \in Q_0 | u < l_1\} & \text{if } N \text{ is odd}, \\
\{(u, v, x, y) \in Q_0 | u > s_{EC}(A) - l_1\} & \text{if } N \text{ is even}.
\end{cases} \]

Using the definition (3.1) of \( l_1 \) and the estimate (3.16) we find
\[ l_2 < l_1 = \pi - \frac{\pi}{A} u_1 < \pi - \frac{\pi^2}{A + \pi} \]
and so
\[ |Q'_0| + |Q_1| \leq l_1 \pi \frac{1}{2} l_2 \frac{\pi}{A} l_2 \leq \pi^2. \tag{3.20} \]

We shall prove that
\[ |Q''_0| < |\mathcal{F} \setminus \mathcal{F}|. \tag{3.21} \]

The estimates (3.20) and (3.21) and the same argument as in the proof of Lemma 3.14 yield \( a < A \). Since the function \( s_{EC} \) is monotone increasing, we conclude that \( s_{EC}(a) \leq s_{PC} \left( \frac{a}{2} + \pi \right) \) as claimed.

We are left with proving the estimate (3.21). The length of the set \( Q''_0 \) is \( s_{EC}(A) - \pi - l_1 \). We assume first that \( N' \) is even. Recall that \( l_{N'} \) is the height of the floor \( F'_{N'+1} \). Since the length of \( F'_{N'+1} \) is at most \( s_{EC}(A) - l_{N'} \), the height of \( Q''_0 \) is at most \( \frac{3}{A} \left( s_{EC}(A) - l_{N'} \right) \). Therefore,
\[ |Q''_0| \leq (s_{EC}(A) - \pi - l_1) \frac{\pi}{A} (s_{EC}(A) - l_{N'}). \tag{3.22} \]

Let \( R \) be the union of maximal "rectangles" in \( \mathcal{F}' \setminus \mathcal{F} \) based over
\[ \{(u, v) \mid s_{EC}(A) - \pi \leq u < s_{EC}(A) - l_1\}. \]
3.4. Embeddings into cubes

If \( N \) is odd, \( R \) has one component, whose height is at least \( s_{EC}(A) - l_{N'} \). If \( N \) is even, \( R \) has one or two components, whose total height is at least \( s_{EC}(A) - \pi - l_{N'} \). Together with \( \pi - l_1 = \frac{\pi}{A} (s_{EC}(A) - l_1) \) we conclude that

\[
|\mathcal{F}' \setminus \mathcal{F}| > |R| \geq \frac{\pi}{A} (s_{EC}(A) - l_1) (s_{EC}(A) - \pi - l_{N'}).
\]  

(3.23)

Since \( l_1 > l_{N'} \), the right hand side in (3.23) is larger than the one in (3.22), and so the estimate (3.21) follows. Assume now that \( N' \) is odd. Then the above argument with \( l_{N'} \) replaced by \( l_{N'-1} \) goes through. The estimate (3.21) is thus proved, and so the proof of Lemma 3.19 is complete.

Proposition 3.16 follows from Lemma 3.19 and Proposition 3.18. Indeed, in view of Proposition 3.18 the function

\[
d(a) := spC\left(\frac{a}{2} + \pi \right) - \sqrt{\frac{\pi a}{2}}
\]

on \( ]2\pi, \infty[ \) has its local maxima at

\[
a_N = 2 \left( N^2 - N + 1 \right) \pi, \quad N = 1, 2, \ldots,
\]

where

\[
d(a_N) = \left( N + 1 - \sqrt{N^2 - N + 1} \right) \pi.
\]

The sequence \( d(a_N) \) is monotone increasing to \( \frac{3}{2} \pi \). Together with Lemma 3.19 we conclude that

\[
s_{EC}(a) - \sqrt{\frac{\pi a}{2}} < spC\left(\frac{a}{2} + \pi \right) - \sqrt{\frac{\pi a}{2}} \leq \frac{3}{2} \pi
\]

and so the proof of Proposition 3.16 is complete.
3. Symplectic folding in higher dimensions
4 Symplectic folding in higher dimensions

Even though symplectic folding is a four dimensional process, we can use it to prove interesting symplectic embedding results in higher dimensions as well. The reason is that we can fold into \( n-1 \) different symplectic directions of the \((2n-2)\)-dimensional fiber over the 2-dimensional symplectic base. We will concentrate on embedding skinny polydiscs into cubes and skinny ellipsoids into balls. The results of this chapter will be used in Chapter 5 to prove Theorem 3.

4.1 Four types of folding

In Chapter 3 we folded on the right and on the left into the \( y \)-direction. In the multiple folding procedures considered in this chapter we shall also fold into the \((-y)\)-direction. Hence there will be four types of folding. This section reviews these four types. As usual, we shall neglect those terms in the constructions which can be chosen arbitrarily small.

We define \( F \subset \mathbb{R}^4 \) by

\[
F := \{ (u, v, x, y) \in \mathbb{R}^4 \mid u \in \mathbb{R}, 0 < v < 1, 0 < x < 1, 0 < y < \pi \}.
\]

Fix \( u_1 \in \mathbb{R} \) and choose a (right-)cut off function \( c_r : \mathbb{R} \to [0, 1] \) with support \([u_1, u_1 + \pi]\) and a (left-)cut off function \( c_l : \mathbb{R} \to [0, 1] \) with support \([u_1 - \pi, u_1]\).

1. Folding on the right into the \( y \)-direction

We fold \( F \) on the right at \( u_1 \) into the \( y \)-direction by applying the symplectic map

\[
\phi_{r+} := (\gamma_1 \times id) \circ \varphi_1 \circ (\beta_1 \times id).
\]

Here, the maps \( \beta_1 \) and \( \gamma_1 \), which are constructed the same way as the maps \( \beta \) and \( \gamma \) in Step 1 and Step 4 of Section 2.2, are explained in the first column of Figure 28, and the lifting map \( \varphi_1 \) is defined by

\[
\varphi_1(u, v, x, y) = \left( u, x, v + c_r(u)x, y + \int_{u_1}^{u} c_r(s) \, ds \right).
\]
4. Symplectic folding in higher dimensions

2. Folding on the left into the $y$-direction
We fold $F$ on the left at $u = u_1$ into the $y$-direction by applying the map
\[
\phi_{l+} := (\gamma_2 \times \text{id}) \circ \varphi_2 \circ (\beta_2 \times \text{id}).
\]
Here, the maps $\beta_2$ and $\gamma_2$ are explained in the second column of Figure 28, and
\[
\varphi_2(u, v, x, y) = \left( u, x, v - c_l(u)x, y + \int_{u_1}^{u} c_l(s) \, ds \right).
\]

3. Folding on the right into the $(-y)$-direction
We fold $F$ on the right at $u = u_1$ into the $(-y)$-direction by applying the map
\[
\phi_{r-} := (\gamma_3 \times \text{id}) \circ \varphi_3 \circ (\beta_3 \times \text{id}).
\]
Here, the maps $\beta_3$ and $\gamma_3$ are explained in the third column of Figure 28, and
\[
\varphi_3(u, v, x, y) = \left( u, x, v - c_r(u)x, y - \int_{u_1}^{u} c_r(s) \, ds \right).
\]

4. Folding on the left into the $(-y)$-direction
We fold $F$ on the left at $u = u_1$ into the $(-y)$-direction by applying the map
\[
\phi_{l-} := (\gamma_4 \times \text{id}) \circ \varphi_4 \circ (\beta_4 \times \text{id}).
\]
Here, the maps \( \beta_4 \) and \( \gamma_4 \) are explained in the fourth column of Figure 28, and

\[
\varphi_4(u, v, x, y) = \left( u, x, v + c_1(u)x, y - \int_u^{u_1} c_1(s) \, ds \right).
\]

### 4.2 Embedding polydiscs into cubes

In this section we shall study symplectic embeddings of skinny polydiscs

\[
P^{2n}(\pi, \ldots, \pi, a) = D(\pi) \times \cdots \times D(\pi) \times D(a)
\]

into cubes \( C^{2n}(A) \) for \( n \geq 2 \). As before, we shall work with more convenient shapes. Define the rectangle \( R(a, b) \) by

\[
R(a, b) = \{(x, y) \mid 0 < x < a, 0 < y < b\}.
\]

We denote the \( 2n \)-dimensional set \( R(a, 1) \times R(1, b) \times \cdots \times R(1, b) \) by

\[
R^n(a, b) = R(a, 1) \times R(1, b) \times \cdots \times R(1, b).
\]

If \( b = a \), we abbreviate \( R^n(a) = R^n(a, a) \). In view of Lemma 2.5, the disc \( D(a) \) is symplectomorphic to \( R(a, 1) \) and the disc \( D(\pi) \) is symplectomorphic to \( R(1, \pi) \). Therefore, the polydisc \( P^{2n}(\pi, \ldots, \pi, a) \), which is symplectomorphic to \( P^{2n}(a, \pi, \ldots, \pi) \), is symplectomorphic to \( R^n(a, \pi) \). Similarly, the cube \( C^{2n}(A) \) is symplectomorphic to \( R^n(A) \). The symplectic coordinates will be denoted by

\[
(u, v, x_2, y_2, \ldots, x_n, y_n) = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^{2n}
\]

where we set again \((u, v) = (x_1, y_1)\). We abbreviate \( x = (x_2, \ldots, x_n) \) and \( y = (y_2, \ldots, y_n) \) as well as

\[
1_y = (1, \ldots, 1) \in \mathbb{R}^{n-1}(y).
\]

We shall associate with each triple \( a > 2 \pi, N \in \mathbb{N}, n \geq 2 \) a symplectic embedding procedure

\[
\phi_N^n(a) : R^n(a, \pi) \hookrightarrow \mathbb{R}^{2n}.
\]

In the following description we again neglect the arbitrarily small \( \delta \)-terms appearing in the actual construction.
4. Symplectic folding in higher dimensions

If \( n = 2 \), we proceed as in 3.4.2, cf. Figure 27: For \( a > 2\pi \) and \( N \in \mathbb{N} \) we define \( u_1 = u_{1,N}^2(a) \) by

\[
a = 2\pi + (N + 1)(u_1 - \pi). \tag{4.1}
\]

Then \( u_1 > \pi \). We first fold \( R^2(a, \pi) \) on the right into the \( y_2 \)-direction by applying the map

\[
(z_1, z_2) \mapsto \phi_{r+}(z_1, z_2)
\]

at \( u = u_1 \). Here, \( \phi_{r+} \) is the restriction to \( R^2(a, \pi) \) of the map \( \phi_{r+} \) introduced in 4.1.1. If \( N = 1 \), the embedding procedure \( \phi_{N}^2(a) \) terminates at this point. Indeed, in view of definition (4.1), “\( a \) is used up” and the front face of the second floor of the image touches the hyperplane \( \{u = 0\} \). If \( N \geq 2 \), the inequality \( u_1 > \pi \) implies that we can then fold the second floor

\[
\{(u, v, x_2, y_2) \in \phi_{r+}\left(R^2(a, \pi)\right) \mid u < u_1, y_2 > \pi\}
\]

of the image on the left into the \( y_2 \)-direction by applying the map

\[
(z_1, z_2) \mapsto \phi_{l+}(z_1, z_2)
\]

to the second floor at \( u = \pi \). Going on this way, we altogether fold \( N \) times into the \( y_2 \)-direction by alternatingly folding the last floor of the image on the right at \( u = u_1 \) and on the left at \( u = \pi \). At this point the embedding procedure \( \phi_{N}^2(a) \) terminates. Indeed, in view of definition (4.1), “\( a \) is used up” and the front face of the last floor of the image touches the hyperplane \( \{u = u_1 + \pi\} \) if \( N \) is even and the hyperplane \( \{u = 0\} \) if \( N \) is odd.

If \( n = 3 \), the embedding procedure \( \phi_{N}^3(a) \) can be visualized by Figure 29. For \( a > 2\pi \) and \( N \in \mathbb{N} \) we define \( u_1 = u_{1,N}^3(a) \) by

\[
a = 2\pi + (N + 1)^2(u_1 - \pi). \tag{4.2}
\]

Then \( u_1 > \pi \). Set \( a' = 2\pi + (N + 1)(u_1 - \pi) \). The first \( N \) folds of the embedding procedure \( \phi_{N}^3(a) \) yield a symplectic embedding of \( R^3(a, \pi) \) into \( \mathbb{R}^6 \) whose restriction to \( R^3(a', \pi) \) is

\[
\phi_{N}^3(a') \times \text{id} : R^2(a', \pi) \times R(1, \pi) \hookrightarrow \mathbb{R}^4 \times R(1, \pi).
\]

We next fold once into the \( y_3 \)-direction. If \( N \) is even, we do this by applying the map

\[
(z_1, z_2, z_3) \mapsto (z'_1, z'_2, z'_3) \quad \text{where} \quad (z'_1, z'_3) = \phi_{r+}(z_1, z_3) \quad \text{and} \quad z'_2 = z_2
\]
4.2. Embedding polydiscs into cubes

Figure 29: The first 5 folds of an embedding $\phi_N^3(a): R^3(a, \pi) \hookrightarrow \mathbb{R}^{2n}$ with $N = 4$.

to the $N + 1$'st floor of the image at $u = u_1$, and if $N$ is odd, we do this by applying the map

$$(z_1, z_2, z_3) \mapsto (z'_1, z'_2, z'_3) \quad \text{where} \quad (z'_1, z'_3) = \phi_{l+}(z_1, z_3) \quad \text{and} \quad z'_2 = z_2$$

to the $N + 1$'st floor of the image at $u = \pi$, see Figure 29. We then fold the part of the image on which $y_3 > \pi$ exactly $N$ times into the $(-y_2)$-direction by using restrictions of the maps $\phi_{r-}$ and $\phi_{l-}$ and thereby fill a second $z_1$-$z_2$-layer. If $N = 1$, the embedding procedure $\phi_N^3(a)$ terminates at this point. Indeed, in view of definition (4.2), "$a$ is used up" and the front face of the last floor of the image touches the hyperplane $\{u = 0\}$. If $N \geq 2$, we fold a second time into the $y_3$-direction, and fill a third $z_1$-$z_2$-layer. Going on this way, we altogether fold $(N + 1)^2 - 1$ times, in which we fold $N$ times into the $y_3$-direction, and thereby fill $N + 1$ $z_1$-$z_2$-layers. At this point the embedding procedure $\phi_N^3(a)$ terminates. Indeed, in view of definition (4.2), "$a$ is used up" and the front face of the last floor of the image touches the hyperplane $\{u = u_1 + \pi\}$ if $N$ is even and the hyperplane $\{u = 0\}$ if $N$ is odd.

In order to describe the embedding procedure $\phi_N^n(a)$ for $n \geq 4$, we proceed by induction and assume that we have described the symplectic embeddings

$$\phi_N^{n-1}(a') : R^{n-1}(a', \pi) \hookrightarrow \mathbb{R}^{2n-2}, \quad a' > 2\pi.$$
We define $u_1 = u_{1,N}(a)$ by

$$a = 2\pi + (N + 1)^{n-1}(u_1 - \pi).$$

(4.3)

Then $u_1 > \pi$. Set $a' = 2\pi + (N + 1)^{n-2}(u_1 - \pi)$. The first $(N + 1)^{n-2} - 1$ folds of the embedding procedure $\phi_N(a)$ yield a symplectic embedding of $R^n(a, \pi)$ into $\mathbb{R}^{2n}$ whose restriction to $R^n(a', \pi)$ is

$$\phi_N^{-1}(a') \times id : R^{n-1}(a', \pi) \times R(1, \pi) \hookrightarrow \mathbb{R}^{2n-2} \times R(1, \pi).$$

We next fold once into the $y_n$-direction. If $N$ is even, we do this by applying the map

$$(z_1, \ldots, z_n) \mapsto (z_1', \ldots, z_n'), \quad (z_1', z_n') = \phi_{t+}(z_1, z_n), \quad z'_i = z_i, \quad i = 2, \ldots, n-1,$$

to the last floor of the image at $u = u_1$, and if $N$ is odd, we do this by applying the map

$$(z_1, \ldots, z_n) \mapsto (z_1', \ldots, z_n'), \quad (z_1', z_n') = \phi_{t+}(z_1, z_n), \quad z'_i = z_i, \quad i = 2, \ldots, n-1,$$

to the last floor of the image at $u = \pi$. We then fill a second $z_1 \cdots z_{n-1}$-layer by folding the part of the image on which $y_n > \pi$ exactly $(N + 1)^{n-2} - 1$ times. If $N = 1$, the embedding procedure $\phi_N(a)$ terminates at this point in view of definition (4.3). If $N \geq 2$, we fold a second time into the $y_n$-direction, and fill a third $z_1 \cdots z_{n-1}$-layer. Going on this way, we altogether fold $(N + 1)^{n-1} - 1$ times, in which we fold $N$ times into the $y_n$-direction, and thereby fill $N + 1$ $z_1 \cdots z_{n-1}$-layers. At this point the embedding procedure $\phi_N(a)$ terminates in view of definition (4.3).

The following proposition generalizes Proposition 3.18 to arbitrary dimension.

**Proposition 4.1** Assume $a > 2\pi$. Then the polydisc $P^{2n}(\pi, \ldots, \pi, a)$ symplectically embeds into the cube $C^{2n}(s_{PC}^n(a) + \epsilon)$ for every $\epsilon > 0$, where

$$s_{PC}^{2n}(a) = \begin{cases} 
(N + 1)\pi, & (N - 1)N^{n-1} < \frac{a}{\pi} - 2 \leq (N - 1)(N + 1)^{n-1}, \\
\frac{a - 2\pi}{(N + 1)^{n-1}} + 2\pi, & (N - 1)(N + 1)^{n-1} < \frac{a}{\pi} - 2 \leq N(N + 1)^{n-1}.
\end{cases}$$

**Proof.** Fix $a > 2\pi$ and $N \in \mathbb{N}$. We define $u_1$ by equation (4.3). The previously described embedding procedure yields a symplectic embedding

$$\phi_N^n(a) : R^n(a, \pi) \hookrightarrow R^n(A_N(a) + \epsilon).$$
where
\[ A_N(a) := \max \{ u_1 + \pi, (N + 1)\pi \}. \]

Solving equation (4.3) for \( u_1 \) we find that
\[ A_N(a) = \max \left\{ \frac{a - 2\pi}{(N + 1)^n - 1} + 2\pi, (N + 1)\pi \right\}. \]

Optimizing the choice of \( N \in \mathbb{N} \), we conclude that the polydisc \( P^{2n}(\pi, \ldots, \pi, a) \) symplectically embeds into the cube \( C^{2n} \left( s_{PC}^{2n}(a) + \epsilon \right) \) for any \( \epsilon > 0 \), where \( s_{PC}^{2n}(a) \) is defined by
\[ s_{PC}^{2n}(a) = \min \{ A_N(a) \mid N = 1, 2, \ldots \}. \]

This completes the proof of Proposition 4.1. \( \square \)

Remarks 4.2

1. Arguing as in 3.4.2 we see that for \( a \in [\pi, 2\pi] \) multiple symplectic folding does not provide a better embedding result than the inclusion \( P^{2n}(\pi, \ldots, \pi, a) \hookrightarrow C^{2n}(a) \), and that the procedure proving Proposition 4.1 is optimal in the sense that we cannot embed \( P^{2n}(\pi, \ldots, \pi, a) \) into a cube smaller than \( C^{2n} \left( s_{PC}^{2n}(a) \right) \) by multiple symplectic folding.

2. The functions \( s_{PC}^{2n}(a) \), \( n \geq 3 \), are compared with the results yielded by Lagrangian folding in 6.2.2.2. \( \diamond \)

In view of the proof of the second statement in Theorem 3, which will be completed in Section 5.1, we also prove

Proposition 4.3 Fix \( n \geq 2. \) For every \( a > 3\pi \) there exists a natural number \( N(a) \) and a symplectic embedding
\[ \varphi_a: R^n(a, \pi) \hookrightarrow R^n ((N(a) + 1)\pi) \]

such that the following assertions hold.

(i) If \( u < \pi, \)
\[ \varphi_a(u, v, x, y) = (u, v, x, y), \]
and if \( u > a - \pi, \)
\[ \varphi_a(u, v, x, y) = (u - a + (N(a) + 1)\pi, v, x, y + (N(a)\pi) 1_y). \]
4. Symplectic folding in higher dimensions

(ii) \[ \lim_{a \to \infty} \frac{|\varphi_a(R^n(a, \pi))|}{|R^n((N(a) + 1)\pi)|} = 1. \]

**Proof.** Fix \( n \geq 2 \) and \( N \in 2\mathbb{N} \). We set \( u_1 = N\pi \) and

\[ \hat{a}_N = 2\pi + (N + 1)^{n-1}(u_1 - \pi) = 2\pi + (N + 1)^{n-1}(N - 1)\pi. \]

We recall that in the previous description of the symplectic embedding \( \varphi_N^a(\hat{a}_N) \) we have neglected the arbitrarily small \( \delta \)-terms appearing in the actual construction. Define \( \delta_N > 0 \) by

\[ \delta_N = \frac{\pi}{3((N+1)^{n-1} - 1)}. \]

Since \( N \geq 2 \) we find that \( u_1 - 2\delta_N > \pi + 2\delta_N \). In the actual construction associated with \( \hat{a}_N \) and \( N \) we can therefore achieve the \( i \)'th fold as follows. We fold the last floor of the image at \( u = u_1 - 2\delta_N \) if \( i \) is odd and at \( u = \pi + 2\delta_N \) if \( i \) is even in such a way that the \( \delta \)-length of the part of the last floor which is mapped to the \( i \)'th stairs is equal to \( \delta_N \). After folding \( (N + 1)^{n-1} - 1 \) times we thereby obtain a symplectic embedding

\[ \varphi_N^a(\hat{a}_N, \delta_N) : R^n(a_N, \pi) \hookrightarrow R^n((N + 1)\pi)) \]

where

\[
\begin{align*}
a_N &= 2\pi + (N + 1)^{n-1}(u_1 - \pi) - 3((N+1)^{n-1} - 1)\delta_N \\
&= \pi + (N + 1)^{n-1}(N - 1)\pi.
\end{align*}
\]

We abbreviate \( \psi_N = \varphi_N^a(\hat{a}_N, \delta_N) \). In view of the construction of \( \psi_N \) and the inequality \( \pi + 2\delta_N < u_1 - 2\delta_N \) we have

\[
\begin{align*}
\psi_N(u, v, x, y) &= (u, v, x, y) & \text{if } u < \pi + 2\delta_N, \\
\psi_N(u, v, x, y) &= (u - a_N + (N + 1)\pi, v, x, y + N\pi y) & \text{if } u > a_N - \pi - 2\delta_N.
\end{align*}
\]

Notice that

\[ a_N < a_{N+2} \text{ for every } N \in 2\mathbb{N} \quad \text{and} \quad a_N \to \infty \text{ as } N \to \infty. \]

The function \( N : [\pi, \infty] \to \mathbb{N}, \)

\[ N(a) := \min\{ N \in 2\mathbb{N} \mid a_N \geq a \}. \]
4.2. Embedding polydiscs into cubes

is therefore well-defined, and \( N(a_N) = N \). Fix \( a > 3\pi \). Since \( a \leq a_{N(a)} \), we find a symplectic embedding \( \beta_a : \mathbb{R}(a) \hookrightarrow \mathbb{R}(a_{N(a)}) \) which is the identity on \( \{ u < \pi \} \) and the translation \( (u, v) \mapsto (u + a_{N(a)} - a, v) \) on \( \{ u > a - \pi \} \), cf. Figure 7 for the case \( a < a_{N(a)} \). We define the symplectic embedding

\[
\varphi_a : \mathbb{R}^n(a, \pi) \hookrightarrow \mathbb{R}^n((N(a) + 1)\pi)
\]

by

\[
\varphi_a = \psi_{N(a)} \circ (\beta_a \times \text{id}_{2n-2}).
\]

In view of the formulae (4.5) and (4.6) and in view of its definition, \( \varphi_a \) meets assertion (i) in Proposition 4.3.

In order to verify assertion (ii), we first of all observe that

\[
|\varphi_a(\mathbb{R}^n(a, \pi))| = |\mathbb{R}^n(a, \pi)| \quad \text{and} \quad \varphi_a(\mathbb{R}^n(a, \pi)) \subset \mathbb{R}^n((N(a) + 1)\pi)
\]

for all \( a > 3\pi \). Therefore,

\[
1 \geq \frac{|\varphi_a(\mathbb{R}^n(a, \pi))|}{|\mathbb{R}^n((N(a) + 1)\pi)|} = \frac{|\mathbb{R}^n(a, \pi)|}{|\mathbb{R}^n((N(a) + 1)\pi)|} = \frac{a}{(N(a) + 1)^n \pi} \quad (4.9)
\]

for all \( a > 3\pi \). Assume now that \( a > a_2 \). In view of (4.7) and the definition (4.8) of \( N(a) \) we have \( a \in ]a_{N(a)} - 2, a_{N(a)}[ \). Using this and the formula (4.4) for \( a_{N(a)} - 2 \) we can further estimate

\[
\frac{a}{(N(a) + 1)^n \pi} > \frac{a_{N(a)} - 2}{(N(a) + 1)^n \pi} = \frac{(N(a) - 1)^{n-1}(N(a) - 3) + 1}{(N(a) + 1)^n \pi}. \quad (4.10)
\]

The definition (4.8) and (4.7) imply that

\[
N(a) \to \infty \quad \text{as} \quad a \to \infty. \quad (4.11)
\]

Combining the estimates (4.9) and (4.10) we therefore conclude that

\[
\lim_{a \to \infty} \frac{|\varphi_a(\mathbb{R}^n(a, \pi))|}{|\mathbb{R}^n((N(a) + 1)\pi)|} = 1,
\]

and so the proof of Proposition 4.3 is complete. \( \square \)
4. Symplectic folding in higher dimensions

4.3 Embedding ellipsoids into balls

In this section we shall study a problem closely related to symplectically embedding skinny ellipsoids $E^{2n}(\pi, \ldots, \pi, a)$ into balls $B^{2n}(A)$. As in Chapter 2 we start with replacing these sets by symplectomorphic sets which are more convenient to work with. Recall that given a domain $U \subset \mathbb{R}^{2n}$ and $\lambda > 0$ we set

$$\lambda U = \{ \lambda z \in \mathbb{R}^{2n} | z \in U \}.$$ 

Reorganizing the coordinates, we consider the Lagrangian splitting $\mathbb{R}^n(x) \times \mathbb{R}^n(y)$ of $\mathbb{R}^{2n}$. We set

$$\Delta(a_1, \ldots, a_n) = \left\{ 0 < x_1, \ldots, x_n \left| \sum_{i=1}^{n} \frac{x_i}{a_i} < 1 \right. \right\} \subset \mathbb{R}^n(x),$$

$$\Box(b_1, \ldots, b_n) = \{ 0 < y_i < b_i, 1 \leq i \leq n \} \subset \mathbb{R}^n(y),$$

and we abbreviate $\Delta^n(a) = \Delta(a, \ldots, a)$ and $\Box^n(b) = \Box(b, \ldots, b)$.

**Lemma 4.4** Assume $\epsilon > 0$. Then

(i) the ellipsoid $E(a_1, \ldots, a_n)$ symplectically embeds into the Lagrangian product $((1 + \epsilon)\Delta(a_1, \ldots, a_n)) \times \Box^n(1)$,

(ii) the Lagrangian product $\Delta(a_1, \ldots, a_n) \times \Box^n(1)$ symplectically embeds into the ellipsoid $(1 + \epsilon)E(a_1, \ldots, a_n)$.

**Proof.** (i) Define $\epsilon'$ by $\sum_{i=1}^{n} \frac{\epsilon'}{a_i} = \epsilon$. Replacing the parameter $a$ in the proof of Lemma 2.8 (i) by $a_i$, $1 \leq i \leq n$, we obtain area and orientation preserving diffeomorphisms $\alpha_i : D(a_i) \to R(a_i)$ satisfying

$$x_i(\alpha_i(z_i)) \leq \pi |z_i|^2 + \epsilon' \text{ for all } z_i \in D(a_i), \quad 1 \leq i \leq n,$$

cf. Figure 4. For $(z_1, \ldots, z_n) \in E(a_1, \ldots, a_n)$ we then find

$$\sum_{i=1}^{n} \frac{x_i(\alpha_i(z_i))}{a_i} \leq \sum_{i=1}^{n} \frac{\pi |z_i|^2}{a_i} + \frac{\epsilon'}{a_i} < 1 + \epsilon.$$

It follows that the restriction of the symplectic embedding

$$\alpha_1 \times \cdots \times \alpha_n : D(a_1) \times \cdots \times D(a_n) \hookrightarrow \mathbb{R}^{2n}$$
4.3. Embedding ellipsoids into balls

to $E(a_1, \ldots, a_n)$ is as desired.

(ii) Define $\epsilon'$ by $\sum_{i=1}^{n} \frac{\epsilon'}{a_i} = \epsilon^2$. Replacing the parameters $a$ and $\epsilon$ in the proof of Lemma 2.8 (ii) by $a_i$ and $\epsilon'$, $1 \leq i \leq n$, we obtain area and orientation preserving embeddings $\omega_i : R(a_i) \hookrightarrow D(a_i + \epsilon')$ satisfying

$$
\pi |\omega_i(z_i)|^2 \leq x_i + \epsilon' \quad \text{for all } z_i = (x_i, y_i) \in R(a_i), \quad 1 \leq i \leq n,
$$
cf. Figure 4. For $(z_1, \ldots, z_n) \in \Delta(a_1, \ldots, a_n) \times \Box^n(1)$ we then find

$$
\sum_{i=1}^{n} \frac{\pi |\omega_i(z_i)|^2}{a_i} \leq \sum_{i=1}^{n} \frac{x_i}{a_i} + \frac{\epsilon'}{a_i} < 1 + \epsilon^2 < (1 + \epsilon)^2.
$$

It follows that the restriction of the symplectic embedding

$$
\omega_1 \times \cdots \times \omega_n : R(a_1) \times \cdots \times R(a_n) \hookrightarrow \mathbb{R}^{2n}
$$
to $\Delta(a_1, \ldots, a_n) \times \Box^n(1)$ is as desired.

In view of Lemma 4.4 we may think of an ellipsoid as a Lagrangian product of a simplex and a cube. In the setting of symplectic folding, however, we want to work with sets which fiber over a symplectic rectangle. We therefore set again

$$(u, v) = (x_1, y_1)$$

and define the $2n$-dimensional trapezoid $T^n(a, b)$ by

$$
T^n(a, b) = \left\{ (u, v, x, y) \in \mathbb{R}^2 \times \mathbb{R}^{n-1}(x) \times \mathbb{R}^{n-1}(y) \mid (u, v) \in R(a), \ (x, y) \in \left( (1 - \frac{u}{a}) \Delta^{n-1}(b) \right) \times \Box^{n-1}(1) \right\}.
$$

Then

$$
T^n(a, b) = \Delta(a, b, \ldots, b) \times \Box^n(1). \tag{4.12}
$$

If $b = a$, we abbreviate $T^n(a) = T^n(a, a)$.

**Corollary 4.5** Assume $\epsilon > 0$. Then

(i) $E^{2n}(\pi, \ldots, \pi, a)$ symplectically embeds into $T^n(a + \epsilon, \pi + \epsilon)$.

(ii) $T^n(a)$ symplectically embeds into $B^{2n}(a + \epsilon)$.

**Proof.** (i) The ellipsoid $E^{2n}(\pi, \ldots, \pi, a)$ is symplectomorphic to the ellipsoid $E^{2n}(a, \pi, \ldots, \pi)$, and by Lemma 4.4 (i) and the identity (4.12) this ellipsoid symplectically embeds into

$$
((1 + \epsilon')\Delta(a, \pi, \ldots, \pi)) \times \Box^n(1) = T^n(a + \epsilon'a, \pi + \epsilon'\pi)
$$
for every $\epsilon' > 0$. The claim thus follows.

(ii) follows from the identity (4.12) and Lemma 4.4 (ii). \hfill \Box

By Corollary 4.5 the problem of symplectically embedding skinny ellipsoids $E^{2n}(\pi, \ldots, \pi, a)$ into balls $B^{2n}(A)$ is equivalent to the problem of symplectically embedding trapezoids $T^n(\pi, a)$ into trapezoids $T^n(A)$. In view of the proof of the first statement in Theorem 3, which will be completed in Section 5.2, we shall, however, consider a somewhat different embedding problem. Instead of embeddings of $T^n(\pi, a)$ we shall study embeddings of the larger set

$$S_a := R(a) \times \Delta^{n-1}(\pi) \times \Box^{n-1}(1) \subset R^2 \times R^{n-1}(x) \times R^{n-1}(y)$$

into $T^n(A)$.

**Proposition 4.6** Fix $n \geq 2$. For every $a > 3\pi$ there exists a natural number $l(a)$ and a symplectic embedding $\varphi_a : S_a \hookrightarrow T^n(l(a)^2)$ such that the following assertions hold.

(i) If $u < \pi$,

$$\varphi_a(u, v, x, y) = \left( u, v, \frac{(l(a)-1)l(a)}{\pi} x, \frac{\pi}{l(a)-1} y \right),$$

and if $u > a - \pi$,

$$\varphi_a(u, v, x, y) = \left( u - a + (l(a) - 1)l(a), v, \frac{l(a)}{\pi} x, \frac{\pi}{l(a)} y \right).$$

(ii) \( \lim_{a \to \infty} \frac{|\varphi_a(S_a)|}{|T^n(l(a)^2)|} = 1. \)

**Proof.** The proof of Proposition 4.6 is more difficult than the proof of the analogous Proposition 4.3. The reason is that for $n \geq 4$ it is impossible to fill a large $(n-1)$-simplex with small $(n-1)$-simplices. We shall therefore repeatedly rescale the fibers of $S_a$ and fill the cube-factor $\Box^{n-1}(1)$ of the fibers of $T^n(1(a)^2)$ with the small cube-factors of the rescaled fibers.

Fix $n \geq 2$ and $a > 3\pi$. We prove Proposition 4.6 in six steps. In the first four steps we construct for each odd number $l > 3n\pi$ a symplectic embedding $\psi_l$ of the unbounded set

$$S_{\infty} := \{ 0 < u, 0 < v < 1 \} \times \Delta^{n-1}(\pi) \times \Box^{n-1}(1)$$
4.3. Embedding ellipsoids into balls

into \( \mathbb{R}^{2n} \). The basic idea behind the embeddings \( \psi_l \) is explained in Figure 30. In Step 5 we associate to \( a > 3n \pi \) an odd number \( l(a) \) and use the embeddings \( \psi_l \) to construct symplectic embeddings \( \varphi_a : S_a \hookrightarrow T^n(l(a)^2) \) which meet assertion (i). In Step 6 we verify that these embeddings also meet assertion (ii).

![Figure 30: The embedding \( \psi_l : S_\infty \hookrightarrow \mathbb{R}^{2n} \) for \( l = 7 \).](image)

**Step 1. Preparations**

Fix \( l \in 2\mathbb{N} + 1 \) with \( l > 3n \pi \). We define subsets \( P_i = P_i(l) \) of \( T^n(l^2) \) by

\[
P_i := \left\{ (u, v, x, y) \in T^n(l^2) \mid (i - 1)l < u < il \right\}, \quad 1 \leq i \leq l, \quad (4.13)
\]

cf. Figure 31. Define real numbers \( k_i = k_i(l) \) by

\[
k_i := \frac{1}{\pi} (l - i)l, \quad 1 \leq i \leq l - 1. \quad (4.14)
\]

Since \( l > 3n \pi \), we find that \( k_i - k_i - k_i - k_i - k_i > 3, 1 \leq i \leq l - 2 \), and \( k_{l-1} > 3 \). We may therefore define even numbers \( N_i = N_i(l) \) by

\[
N_i := \max \left\{ N \in 2\mathbb{N} \mid N + 1 < k_i - k_i - k_i - k_i - k_i \right\}, \quad 1 \leq i \leq l - 2, \quad (4.15)
\]

\[
N_{l-1} := \max \left\{ N \in 2\mathbb{N} \mid N + 1 < k_{l-1} \right\}. \quad (4.16)
\]
In view of definition (4.15) we have

$$0 < d_i := 1 - \frac{1}{k_{i+1}} - \frac{N_{i+1}}{k_i} \leq \frac{2}{k_i}, \quad 1 \leq i \leq l - 2,$$

(4.17)

and in view of definition (4.14) we have $k_{i+1} < k_i$, $1 \leq i \leq l - 2$. Then

$$e_i := \frac{2k_{i+1}}{k_i + k_{i+1}} d_i \in [0, d_i[, \quad 1 \leq i \leq l - 2.$$  

(4.18)

The set $S_\infty$ is symplectomorphic to the set

$$F_1 := \{ 0 < u, 0 < v < 1 \} \times \Delta^{n-1}(k_1 \pi) \times \Box^{n-1}(\frac{1}{k_1})$$

via the linear symplectomorphism

$$\sigma : S_\infty \to F_1, \quad (u, v, x, y) \mapsto (u, v, k_1 x, \frac{1}{k_1} y).$$

In view of the definitions of $k_1$ and $P_1$, the fibers $\Delta^{n-1}(k_1 \pi) \times \Box^{n-1}(1/k_1)$ of $F_1$ are contained in the fibers of $P_1$, cf. Figure 31.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure31.png}
\caption{The subset $P_1$ of $T^n(l^2)$.}
\end{figure}

**Step 2. Multiple folding in $P_1$**

We symplectically embed a part of $F_1$ into $P_1$ by the multiple folding procedure described in Section 4.2. In the following description we shall again neglect the arbitrarily small $\delta$-terms appearing in the actual construction. We first fold $F_1$ at $u = l - \pi$ on the right into the $y_2$-direction. The lifting map involved in this folding has the form

$$(u, x_2, v, y_2) \mapsto \left( u, x_2, v + c(u)x_2, y_2 + \int_0^u c(s) \, ds \right)$$
4.3. Embedding ellipsoids into balls

where the cut off function \( c: \mathbb{R} \to [0, 1/(k_1 \pi)] \) has support in \([l - \pi, l]\). We next fold the part of the image on which \( y_2 > 1/k_1 \) at \( u = \pi \) on the left into the \( y_2 \)-direction. This is possible because \( l > 2\pi \). The length of the second floor is \( l - 2\pi \). We fold \( N_1 \) times alternatingly at \( u = l - \pi \) on the right and at \( u = \pi \) on the left into the \( y_2 \)-direction. We then fold once into the \( y_3 \)-direction. Since \( N_1 \) is even, we do this by folding the part of the image on which \( y_2 > N_1/k_1 \) at \( u = l - \pi \) on the right. Going on this way, we altogether fold \((N_1 + 1)^n - 1\) times, in which we fold \( N_1 \) times into the \( y_n \)-direction. Denote the multiple folding embedding \( F_1 \leftrightarrow \mathbb{R}^{2n} \) thus obtained by \( \mu_1 \). The image of the projection of \( \mu_1(F_1) \) onto \( \mathbb{R}^{n-1}(y) \) is contained in the cube \( D_{n-1}(\frac{N_1 + 1}{k_1}) \), cf. Figure 33. Since \( N_1 \) is even, the infinite end of \( \mu_1(F_1) \) points into the \( u \)-direction. More precisely, the last floor of \( \mu_1(F_1) \) is the subset

\[
F'_1 := [\pi, \infty[ \times 0, 1[ \times \Delta^{n-1}(k_1 \pi) \times C_1'
\]

of \( \mathbb{R}^2 \times \mathbb{R}^{n-1}(x) \times \mathbb{R}^{n-1}(y) \) where

\[
C_1' = \{ (y_2, \ldots, y_n) \mid \frac{N_1}{k_1} < y_j < \frac{N_1 + 1}{k_1}, \quad j = 2, \ldots, n \},
\]

cf. Figure 32 and Figure 33.

Define

\[
\delta_1 := d_1 - e_1.
\]

We choose the \( \delta \)-terms in the actual construction of the embedding \( \mu_1 \) in such a way that the \( u \)-length \( u_1 \) of the part of \( F_1 \) mapped to \( \mu_1(F_1) \setminus F'_1 \) is equal to

\[
u_1 = (l - \pi) + ((N_1 + 1)^n - 1)(l - 2\pi) - \delta_1.
\]

By construction, the set \( \mu_1(F_1) \setminus F'_1 \) is contained in \( P_1 \). We next want to pass to \( P_2 \) and fill as much of \( P_2 \) as possible. The fibers \( \Delta^{n-1}(k_1 \pi) \times C_1' \) of \( F'_1 \), however, are not contained in the smallest fiber \( \Delta^{n-1}(k_2 \pi) \times \Box^{n-1}(\pi) \) of \( P_2 \). We therefore need to rescale the fibers of \( F'_1 \).

**Step 3. Rescaling the fibers**

We want to rescale the fibers \( \Delta^{n-1}(k_1 \pi) \times C_1' \) of \( F'_1 \) to fibers \( \Delta^{n-1}(k_2 \pi) \times C_2 \) where

\[
C_2 := \{ (y_2, \ldots, y_n) \mid 1 - \frac{1}{k_2} < y_j < 1, \quad j = 2, \ldots, n \},
\]

cf. Figure 33. In view of the definition (4.14) of \( k_2 \) and the definition (4.13) of \( P_2 \) the fibers \( \Delta^{n-1}(k_2 \pi) \times C_2 \) are contained in the fibers of \( P_2 \).
4. Symplectic folding in higher dimensions

Figure 32: How one can think about the sets $F'_1$, $F''_1$, and $F_2$.

separate the fibers $\Delta^{n-1}(k_1 \pi) \times C'_1$ from themselves by lifting them to the fibers $\Delta^{n-1}(k_1 \pi) \times C''_1$ where

$$C''_1 := \{ (y_2, \ldots, y_n) \mid \frac{N_1+1}{k_1} + e_1 < y_j < \frac{N_1+2}{k_1} + e_1, \quad j = 2, \ldots, n \}$$

and then deform the separated fibers to the fibers $\Delta^{n-1}(k_2 \pi) \times C_2$.

Construction of the lifting $\lambda_1$

We shall separate $F'_1$ from itself by lifting its fibers into each $y_j$-direction, $j = 2, \ldots, n$, by $1/k_1 + e_1$. As in Step 1 of the folding construction described in Section 2.2 we find a symplectic embedding $\beta_1 : [0, \infty[ \times [0, 1[ \rightarrow [0, \infty[ \times [0, 1[$ which is the identity on $\{ u < \pi \}$ and the translation $(u, v) \mapsto (u + l - \pi - \delta_1, v)$ on $\{ u > \pi + \delta_1 \}$, cf. Figure 7. Define $s_1 := \pi + k_1 \pi d_1$. In view of the second inequality in (4.17) and $l > 3n\pi$ we find

$$\pi + (n - 1)s_1 = \pi + (n - 1)\pi(1 + k_1d_1)$$
$$\leq \pi + (n - 1)\pi(1 + 2)$$
$$< 3n\pi$$
$$< l.$$
4.3. Embedding ellipsoids into balls

In view of the definition (4.19) of \( \delta_1 \) we have

\[
\frac{1}{k_1 \pi} \delta_1 = \frac{1}{k_1} + (\frac{1}{k_1} + \epsilon_1).
\]

For \( j = 2, \ldots, n \) we therefore find a cut off function \( c_j : \mathbb{R} \to [0, 1/(k_1 \pi)] \) with support \([\pi + (j - 2)s_1, \pi + (j - 1)s_1]\) and such that \( \int_0^\infty c_j(s) \, ds = 1/k_1 + \epsilon_1 \).

The symplectic embedding

\[
\varphi_1 : \text{Im} \beta_1 \times \Delta^{n-1}(k_1 \pi) \times C'_1 \hookrightarrow \mathbb{R}^{2n}, \quad (u, v, x, y) \mapsto (u', v', x', y')
\]

defined by

\[
u' = u, \quad v' = v + \sum_{j=2}^n c_j(u)x_j, \quad x_j' = x_j, \quad y_j' = y_j + \int_0^u c_j(s) \, ds, \quad j = 2, \ldots, n,
\]

is the identity on \( \{ u < \pi \} \), maps \( \{ u < l \} \) to \( P_1 \) and translates \( \{ u > l \} \) to the set

\[
F''_1 := \{ u > 1, \, 0 < v < 1 \} \times \Delta^{n-1}(k_1 \pi) \times C''_1,
\]

cf. Figure 32. We restrict the symplectic embedding

\[
\varphi_1 \circ (\beta_1 \times id_{n-2}) : \mathbb{R} \times \mathbb{R} \times \Delta^{n-1}(k_1 \pi) \times C'_1 \hookrightarrow \mathbb{R}^{2n}
\]
to the intersection of its domain with $\mu_1(F_1)$ and extend this restriction by the identity to the symplectic embedding $\lambda_1 : \mu_1(F_1) \hookrightarrow \mathbb{R}^{2n}$.

In view of the construction of the "translation" $\beta_1 \times \text{id}_{2n-2}$ the $u$-length of the part of $F_1'$ which $\lambda_1$ embeds into $P_1$ is $\delta_1$. In view of the identity (4.20) we therefore conclude that the $u$-length $u_1'$ of the part of $F_1$ which $\lambda_1 \circ \mu_1$ embeds into $P_1$ is equal to

$$u_1' = (l - \pi) + \left( (N_1 + 1)^{n-1} - 2 \right) (l - 2\pi). \quad (4.21)$$

**Construction of the deformation $\alpha_1$**

The deformation of the fibers $\Delta^{n-1}(k_1\pi) \times C'_1$ of $F''_1$ to fibers $\Delta^{n-1}(k_2\pi) \times C_2$ is based on the following lemma.

**Lemma 4.7** There exists a symplectic embedding

$$\alpha : ]0, k_1\pi[ \times ]0, \frac{N_1+2}{k_1} + e_1[ \hookrightarrow \mathbb{R}^2$$

which restricts to the identity on $\{(x, y) \mid y \leq (N_1 + 1)/k_1\}$, restricts to the affine map

$$(x, y) \mapsto \left( \frac{k_2}{k_1} x, \frac{k_1}{k_2} y + 1 - \frac{1}{k_2}(N_1 + 2 + k_1 e_1) \right) \quad (4.22)$$

on $\{(x, y) \mid y \geq (N_1 + 1)/k_1 + e_1\}$, and is such that

$$x'(\alpha(x, y)) \leq x \quad \text{and} \quad y'(\alpha(x, y)) < 1 \quad (4.23)$$

for all $(x, y) \in ]0, k_1\pi[ \times ]0, (N_1 + 2)/k_1 + e_1[, \text{cf. Figure 34}.$

![Figure 34: The map $\alpha$.](image)

**Proof.** Choose a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that...
4.3. Embedding ellipsoids into balls

(i) $h(w) = 1$ for $w \leq \frac{N_1 + 1}{k_1}$,

(ii) $h'(w) < 0$ for $w \in \left(\frac{N_1 + 1}{k_1}, \frac{N_1 + 1}{k_1} + e_1\right]$,

(iii) $h(w) = \frac{k_2}{k_1}$ for $w \geq \frac{N_1 + 1}{k_1} + e_1$.

In view of the definition (4.18) of $e_1$ and the inequality $k_2 < k_1$ we have

$$e_1 < d_1 < e_1 \frac{k_1}{k_2}.$$ 

We may therefore further require that

(iv) $\int_{\frac{N_1 + 1}{k_1}}^{\frac{N_1 + 1}{k_1} + e_1} \frac{1}{h(w)} \, dw = d_1$.

Then the map

$$\alpha: \left[0, k_1\pi \right] \times 0, \frac{N_1 + 2}{k_1} + e_1 \left[ \rightarrow \mathbb{R}^2, (x, y) \mapsto \left(h(y)x, \int_0^y \frac{1}{h(w)} \, dw\right)\right]$$

is a symplectic embedding which is as required.

Denote by $\alpha_1$ the restriction to $\lambda_1(\mu_1(F_1))$ of the symplectic embedding

$$\lambda_1(\mu_1(F_1)) \cap P_1 \rightarrow P_1$$

where $(x_j, y_j) = \alpha(x_j, y_j)$, $j = 2, \ldots, n$. In view of the inequalities (4.23), $\alpha_1$ maps the set $\lambda_1(\mu_1(F_1)) \cap P_1$ into $P_1$, and in view of (4.22), $\alpha_1$ maps the set $F_1''$ symplectically onto the set

$$F_2 := \{u > l, 0 < v < 1\} \times \Delta^{n-1}(k_2 \pi) \times C_2,$$

cf. Figure 32.

Step 4. Construction of $\psi_t$

The symplectic embedding $\psi_t: S_\infty \rightarrow \mathbb{R}^{2n}$ is the composition of symplectic embeddings

$$\lambda_{l-1} \circ \mu_{l-1} \circ (\alpha_{l-2} \circ \lambda_{l-2} \circ \mu_{l-2}) \circ \cdots \circ (\alpha_2 \circ \lambda_2 \circ \mu_2) \circ (\alpha_1 \circ \lambda_1 \circ \mu_1) \circ \sigma.$$
Here, \( \sigma \) is the map defined in Step 1, \( \mu_1 \) is the map constructed in Step 2, \( \lambda_1 \) and \( \alpha_1 \) are the maps constructed in Step 3, and the maps \( \mu_i, \lambda_i, \alpha_i, i \geq 2 \), are constructed in a similar way as \( \mu_1, \lambda_1, \alpha_1 \). In order to describe the maps \( \mu_i, \lambda_i, \alpha_i, i \geq 2 \), in more detail, we assume by induction that we have already constructed embeddings \( \mu_j, \lambda_j, \alpha_j, j = 1, \ldots, i - 1 \), where \( i \leq l - 2 \), and that the set

\[
\{(u, v, x, y) \in (\alpha_{i-1} \circ \lambda_{i-1} \circ \mu_{i-1} \circ \cdots \circ \alpha_1 \circ \lambda_1 \circ \mu_1)(F_1) | u > (i - 1)l\}
\]

is the set

\[
F_i := \{u > (i - 1)l, 0 < v < 1\} \times \Delta^{n-1}(k_i \pi) \times C_i
\]

where

\[
C_i := \begin{cases} 
\{(y_2, \ldots, y_n) | 0 < y_j < \frac{1}{k_i}, j = 2, \ldots, n\} & \text{if } i \text{ is odd,} \\
\{(y_2, \ldots, y_n) | 1 - \frac{1}{k_i} < y_j < 1, j = 2, \ldots, n\} & \text{if } i \text{ is even.}
\end{cases}
\]

The multiple folding map \( \mu_i \) embeds a part of \( F_i \) into \( P_i \). We fold \( N_i \) times alternatingly at \( u = il - \pi \) on the right and at \( u = (i - 1)l + \pi \) on the left into the \( y_2 \)-direction, and so on. Since \( N_i \) is even, the last floor of the image of \( \mu_i \) is

\[
F'_i := [(i - 1)l + \pi, \infty[ \times ]0, 1[ \times \Delta^{n-1}(k_i \pi) \times C'_i
\]

where

\[
C'_i = \begin{cases} 
\{(y_2, \ldots, y_n) | \frac{N_i}{k_i} < y_j < \frac{N_i + 1}{k_i}, j = 2, \ldots, n\} & \text{if } i \text{ is odd,} \\
\{(y_2, \ldots, y_n) | 1 - \frac{N_i + 1}{k_i} < y_j < 1 - \frac{N_i}{k_i}, j = 2, \ldots, n\} & \text{if } i \text{ is even.}
\end{cases}
\]

We define \( \delta_i := d_i - e_i \) and choose the \( \delta \)-terms in the actual construction of \( \mu_i \) in such a way that the \( u \)-length \( u_i \) of the part of \( F_i \) mapped to \( \mu_i(F_i) \setminus F'_i \) is equal to

\[
u_i = (l - \pi) + ((N_i + 1)^{n-1} - 2) (l - 2\pi) - \delta_i. \tag{4.24}
\]

The maps \( \lambda_i \) and \( \alpha_i \) rescale the fibers \( \Delta^{n-1}(k_i \pi) \times C'_i \) of the floor \( F'_i \) to fibers \( \Delta^{n-1}(k_{i+1} \pi) \times C_{i+1} \)

\[
C_{i+1} := \begin{cases} 
\{(y_2, \ldots, y_n) | 1 - \frac{1}{k_{i+1}} < y_j < 1, j = 2, \ldots, n\} & \text{if } i \text{ is odd,} \\
\{(y_2, \ldots, y_n) | 0 < y_j < \frac{1}{k_{i+1}}, j = 2, \ldots, n\} & \text{if } i \text{ is even.}
\end{cases}
\]
4.3. Embedding ellipsoids into balls

In view of the definition (4.14) of \( k_{i+1} \) and the definition (4.13) of \( P_{i+1} \) the fibers \( \Delta^{n-1}(k_{i+1} \pi) \times C_{i+1} \) are contained in the fibers of \( P_{i+1} \).

The lifting map \( \lambda_i \) separates \( F'_i \) from itself by lifting its fibers into each \( y_j \)-direction, \( j = 2, \ldots, n \), by \((-1)^{i+1}(1/k_i + e_i)\). More precisely, we define \( s_i := \pi + k_i \pi d_i \) and find as in the construction of \( \lambda_1 \) that

\[
(i - 1)l + \pi + (n - 1)s_i < il.
\]

Proceeding as in the construction of \( \lambda_1 \) we can therefore construct a symplectic embedding \( \lambda_i : \text{Im} \mu_i \hookrightarrow \mathbb{R}^{2n} \) which is the identity on \( \text{Im} \mu_i \setminus F_i' \) and translates \( \{(u, v, x, y) \in F_i' \mid u > (i - 1)l + \pi + \delta_i \} \) to the set

\[
F''_i := \{ u > il, 0 < v < 1 \} \times \Delta^{n-1}(k_i \pi) \times C''_i
\]

where

\[
C''_i := \begin{cases} 
\left\{ (y_2, \ldots, y_n) \mid \frac{N_{i+1}}{k_i} + e_i < y_j < \frac{N_{i+2}}{k_i} + e_i, \ j = 2, \ldots, n \right\} & \text{if } i \text{ is odd,} \\
\left\{ (y_2, \ldots, y_n) \mid 1 - \frac{N_{i+2}}{k_i} - e_i < y_j < 1 - \frac{N_{i+1}}{k_i} - e_i, \ j = 2, \ldots, n \right\} & \text{if } i \text{ is even.}
\end{cases}
\]

The \( u \)-length of the part of \( F'_i \) which \( \lambda_i \) embeds into \( P_i \) is \( \delta_i \). In view of the identity (4.24) we therefore conclude that the \( u \)-length \( u'_i \) of the part of \( F_i \) which \( \lambda_i \circ \mu_i \) embeds into \( P_i \) is equal to

\[
u'_i = (l - \pi) + \left( (N_i + 1)^{n-1} - 2 \right) (l - 2\pi).
\]

(4.25)

The symplectic embedding \( \alpha_i : \text{Im} \lambda_i \hookrightarrow \mathbb{R}^{2n} \) maps the set \( \text{Im} \lambda_i \setminus P_i \) into \( P_i \) and maps \( F''_i = \text{Im} \lambda_i \setminus P_i \) onto the set

\[
F_{i+1} := \{ u > il, 0 < v < 1 \} \times \Delta^{n-1}(k_{i+1} \pi) \times C_{i+1}.
\]

The deformation \( \alpha_i \) is constructed the same way as \( \alpha_1 \) if \( i \) is odd, and in a similar way if \( i \) is even.

Next, the multiple folding map \( \mu_{i-1} \) embeds a part of \( F_{i-1} \) into \( P_{i-1} \). We fold \( N_{i-1} \) times alternatingly at \( u = (l - 1)l - \pi \) on the right and at \( u = (l - 2)l + \pi \) on the left into the \( y_2 \)-direction, and so on. Since \( N_{i-1} \) is even and \( l \) is odd, the last floor of the image of \( \mu_{i-1} \) is

\[
F'_{i-1} := \{(l - 2)l + \pi, \infty[ \times 0, 1[ \times \Delta^{n-1}(k_{i-1} \pi) \times C'_{i-1}
\]
where
\[ C'_{l-1} = \{ (y_2, \ldots, y_n) \mid 1 - \frac{N_{l-1}+1}{k_{l-1}} < y_j < 1 - \frac{N_{l-1}}{k_{l-1}}, \ j = 2, \ldots, n \}. \]

We define \( \delta_{l-1} := 1/k_{l-1} \) and choose the \( \delta \)-terms in the actual construction of \( \mu_{l-1} \) in such a way that the \( u \)-length \( u_{l-1} \) of the part of \( F_{l-1} \) mapped to \( \mu_{l-1}(F_{l-1}) \setminus F'_{l-1} \) is equal to
\[ u_{l-1} = (l - \pi) + ((N_{l-1} + 1)^{n-1} - 2)(l - 2 \pi) - \delta_{l-1}. \] (4.26)

Finally, define \( s_{l-1} := k_{l-1} - (N_{l-1} + 1) + \pi \). In view of the definition (4.16) of \( N_{l-1} \) we have
\[ 1 - \frac{N_{l-1}+1}{k_{l-1}} \leq \frac{2}{k_{l-1}}. \]

This and \( l > 3n\pi \) imply that
\[ (l - 2)l + \pi + (n - 1)s_{l-1} < (l - 1)l - \pi. \]

Proceeding as in the construction of \( \lambda_i, \ i \) even, we therefore find a symplectic embedding \( \lambda_{l-1} : \text{Im} \mu_{l-1} \hookrightarrow \mathbb{R}^{2n} \) which is the identity on \( \text{Im} \mu_{l-1} \setminus F'_{l-1} \) and translates \( \{ (u, v, x, y) \in F'_{l-1} \mid u > (l - 2)l + \pi + \delta_{l-1} \} \) to the set
\[ F_l := \{ u > (l - 1)l - \pi, 0 < v < 1 \} \times \Delta^{n-1}(k_{l-1} \pi) \times C_l \]
where
\[ C_l := \{ (y_2, \ldots, y_n) \mid 0 < y_j < \frac{1}{k_{l-1}}, \ j = 2, \ldots, n \}. \]

The \( u \)-length of the part of \( F'_{l-1} \) which \( \lambda_{l-1} \) embeds into \( P_{l-1} \) is \( \delta_{l-1} + \pi \). In view of the identity (4.26) we therefore conclude that the \( u \)-length \( u'_{l-1} \) of the part of \( F_{l-1} \) which \( \lambda_{l-1} \circ \mu_{l-1} \) embeds into \( P_{l-1} \) is equal to
\[ u'_{l-1} = (l - \pi) + ((N_{l-1} + 1)^{n-1} - 2)(l - 2 \pi) + \pi. \] (4.27)

This completes the construction of the symplectic embedding
\[ \psi_l = \lambda_{l-1} \circ \mu_{l-1} \circ \alpha_{l-2} \circ \lambda_{l-2} \circ \mu_{l-2} \circ \cdots \circ \alpha_1 \circ \lambda_1 \circ \mu_1 \circ \sigma : S_{\infty} \hookrightarrow \mathbb{R}^{2n}. \]
4.3. Embedding ellipsoids into balls

Step 5. Construction of $\varphi_a$

In view of the construction of the symplectic embedding $\psi_l : S^\infty \hookrightarrow \mathbb{R}^{2n}$ in the previous four steps and in view of the identities (4.21), (4.25) and (4.27), the $u$-length of the part of $F_l$ embedded into $P_l$ is equal to

$$u'_i = \begin{cases} (l - \pi) + ((N_i + 1)^{n-1} - 2)(l - 2\pi) & \text{if } i \leq l - 2, \\ (l - \pi) + ((N_i + 1)^{n-1} - 2)(l - 2\pi) + \frac{\pi}{l} & \text{if } i = l - 1. \end{cases}$$

Therefore, the $u$-length $a_l := \sum_{i=1}^{l-1} u'_i$ of the part of $F_l$ embedded into $T^n(l^2) \setminus P_l$ is

$$a_l = l\pi + \sum_{i=1}^{l-1} (\frac{(N_i + 1)^{n-1} - 2}{l} - 2\pi).$$  

Moreover, by construction of $\psi_l$,

$$\psi_l(u, v, x, y) = (u, v, k_1 x, \frac{1}{k_1} y) \quad \text{if } u < l - \pi - \delta_1,$$

$$\psi_l(u, v, x, y) = (u - a_l + (l - 1)l, v, k_{l-1} x, \frac{1}{k_{l-1}} y) \quad \text{if } u > a_l - \pi.$$  

Using the definition (4.19) of $\delta_1$ and (4.18), (4.17) and (4.14) we find that

$$\delta_1 = d_1 - e_1 < d_1 \leq \frac{2}{k_l} < 2\pi.$$  

Since $l > 3n\pi$, we therefore find that $\pi < l - \pi - \delta_1$. This and the definition (4.14) of $k_1$ and $k_{l-1}$ imply that

$$\psi_l(u, v, x, y) = (u, v, \frac{u-1}{\pi} x, \frac{\pi}{u-1} y) \quad \text{if } u < \pi,$$

$$\psi_l(u, v, x, y) = (u - a_l + (l - 1)l, v, \frac{1}{\pi} x, \frac{\pi}{l} y) \quad \text{if } u > a_l - \pi.$$  

Before defining the embeddings $\varphi_a$, we further investigate the sequence $(a_l)$.

Lemma 4.8 (i) $a_l < a_{l+2}$ for every $l \in 2N + 1$ with $l > 3n\pi$.

(ii) $a_l \to \infty$ as $l \to \infty$.

Proof. (i) Fix $l \in 2N + 1$ with $l > 3n\pi$. As in Step 1 we abbreviate

$$k_i = k_i(l), \quad N_i = N_i(l), \quad i = 1, \ldots, l - 1.$$

Moreover, we set $l' = l + 2$ and abbreviate

$$k'_i = k_i(l'), \quad N'_i = N_i(l'), \quad i = 1, \ldots, l' - 1.$$
By computation,
\[ \frac{l_i}{l_i - 1} > \frac{l'_i}{l'_i - 1}, \quad i = 1, \ldots, l - 2. \]

Using the definition (4.14) of \( k_i \) and \( k'_i \), we therefore find
\[ k_i - \frac{k_i}{k_{i+1}} = \frac{1}{\pi} (l - i) l - \frac{l_i}{l_i - 1} < \frac{1}{\pi} (l' - i) l' - \frac{l'_i}{l'_i - 1} = k'_i - \frac{k'_i}{k'_{i+1}}, \]
i = 1, \ldots, l - 2. In view of the definition (4.15) of \( N_i \) and \( N'_i \), we conclude that
\[ N_i \leq N'_i, \quad i = 1, \ldots, l - 2. \quad (4.31) \]

Moreover, we read off from definition (4.14) that \( k_{l-1} = \frac{l}{\pi} I < \frac{l'}{\pi} I' = k'_{l-1}, \) and so, in view of definition (4.16),
\[ N_{l-1} \leq N'_{l-1}. \quad (4.32) \]

Using equation (4.28) and the inequalities (4.31) and (4.32), we can now estimate
\[ a_l = l \pi + \sum_{i=1}^{l-2} ((N_i + 1)^{a-1} - 2)(l - 2\pi) + ((N_{l-1} + 1)^{a-1} - 2)(l - 2\pi) \]
\[ < l' \pi + \sum_{i=1}^{l-2} ((N'_i + 1)^{a-1} - 2)(l' - 2\pi) + ((N'_{l-1} + 1)^{a-1} - 2)(l' - 2\pi) \]
\[ < l' \pi + \sum_{i=1}^{l'-1} ((N'_i + 1)^{a-1} - 2)(l' - 2\pi) \]
\[ = a_{l'}. \]

This proves (i).

(ii) follows from equation (4.28).

In view of Lemma 4.8 (ii) the function \( l \colon ]\pi, \infty[ \to \mathbb{N}, \)
\[ l(a) := \min \{ l \in 2\mathbb{N} + 1 \mid l > 3n\pi, \quad a_l \geq a \}, \quad (4.33) \]
is well-defined. Lemma 4.8 (i) shows that \( l(a_l) = l \). Fix \( a > 3\pi \). Since \( a \leq a_{l(a_l)} \), we find a symplectic embedding \( \beta_a \colon R(a) \hookrightarrow R(a_{l(a)}) \) which is the identity on \( \{ u < \pi \} \) and the translation \( (u, v) \mapsto (u + a_{l(a)} - a, v) \) on \( \{ u > a - \pi \} \), cf.
4.3. Embedding ellipsoids into balls

Figure 7 for the case \( a < a_{l(a)} \). We finally define the symplectic embedding \( \varphi_a : S_a \to T^n(l(a)^2) \) by

\[
\varphi_a = \psi_{l(a)} \circ (\beta_a \times i d_{2n-2}).
\]

In view of the formulae (4.29) and (4.30) and in view of its definition, \( \varphi_a \) meets assertion (i) in Proposition 4.6.

**Step 6. Verification of assertion (ii) in Proposition 4.6**

Recall that assertion (ii) in Proposition 4.6 claims that

\[
\frac{|\varphi_a(S_a)|}{|T^n(l(a)^2)|} \to 1 \quad \text{as} \quad a \to \infty. \tag{4.34}
\]

**Lemma 4.9** Assertion (4.34) is a consequence of

\[
\left| \frac{T^n(l^2) \setminus \psi_l(S_{a_l})}{T^n(l^2)} \right| \to 0 \quad \text{as} \quad l \to \infty. \tag{4.35}
\]

**Proof.** Since \( |\psi_l(S_{a_l})| = |S_{a_l}| \) and \( \psi_l(S_{a_l}) \subset T^n(l^2) \), we have

\[
\frac{|S_{a_l}|}{|T^n(l^2)|} = 1 - \frac{|T^n(l^2)| - |\psi_l(S_{a_l})|}{|T^n(l^2)|} = 1 - \frac{|T^n(l^2) \setminus \psi_l(S_{a_l})|}{|T^n(l^2)|}. \tag{4.36}
\]

Fix \( \epsilon \in ]0, 1[ \). The assumption (4.35) and (4.36) imply that there exists \( l_0 \in 2N + 1 \) such that

\[
\frac{|S_{a_l}|}{|T^n(l^2)|} > \sqrt{1 - \epsilon} \quad \text{for all} \quad l \in 2N + 1 \quad \text{with} \quad l \geq l_0. \tag{4.37}
\]

Choosing \( l_0 \) larger if necessary, we may assume that

\[
\frac{l}{l + 2} > \sqrt{1 - \epsilon} \quad \text{for all} \quad l \in 2N + 1 \quad \text{with} \quad l \geq l_0. \tag{4.38}
\]

Assume now that \( a > a_{l_0} \). In view of Lemma 4.8 we have \( a \in ]a_l, a_{l+2}[ \) for some \( l \in 2N + 1 \) with \( l \geq l_0 \). The definition (4.33) of \( l(a) \) implies \( l(a) = l + 2 \). Using
the estimates (4.37) and (4.38), we can therefore estimate
\[
\frac{|S_a|}{|T^n(l(a)^2)|} > \frac{|S_{a_1}|}{|T^n((l + 2)^2)|} = \frac{|T^n(l^2)|}{|T^n((l + 2)^2)|} \frac{|S_{a_1}|}{|T^n(l^2)|} = \frac{l^{2n}}{(l + 2)^{2n}} \frac{|S_{a_1}|}{|T^n(l^2)|} > \sqrt{1 - \epsilon} \sqrt{1 - \epsilon} = 1 - \epsilon.
\]

Since \( \varphi_a(S_a) \subset T^n(l(a)^2) \) and \( |\varphi_a(S_a)| = |S_a| \), we therefore find
\[
1 > \frac{|\varphi_a(S_a)|}{|T^n(l(a)^2)|} = \frac{|S_a|}{|T^n(l(a)^2)|} > 1 - \epsilon.
\]

Since \( \epsilon \in ]0, 1[ \) was arbitrary, Lemma 4.9 follows. \( \square \)

In order to prove assertion (4.35), we fix \( l \in 2\mathbb{N} + 1 \) with \( l > 3n\pi \) and introduce several subsets of \( T^n(l^2) \). We define the subsets \( Q_i(l) \) of \( P_i(l) \) by
\[
Q_i(l) := \{ (u, v, x, y) \in P_i(l) \mid x \in \triangle^{n-1}((l - 1)l) \}, \quad i = 1, \ldots, l,
\]
and we define subsets \( X_i(l) \), \( Y_i(l) \) and \( Z_i(l) \) of \( P_i(l) \) by
\[
X_i(l) := P_i(l) \setminus Q_i(l), \quad i = 1, \ldots, l,
\]
\[
Y_i(l) := \{ (u, v, x, y) \in Q_i(l) \mid u \notin \{(i - 1)l + \pi, il - \pi \} \}, \quad i = 1, \ldots, l - 1,
\]
\[
Z_i(l) := \{ (u, v, x, y) \in Q_i(l) \mid y \notin \triangle^{n-1}(\frac{N_i + 1}{k_i}) \}, \quad i = 1, \ldots, l - 1.
\]

We also set
\[
X(l) = \bigsqcup_{i=1}^l X_i(l), \quad Y(l) = \bigsqcup_{i=1}^{l-1} Y_i(l), \quad Z(l) = \bigsqcup_{i=1}^{l-1} Z_i(l).
\]

The sets \( X(l) \) and \( Y(l) \) are illustrated in Figure 35, and for \( Z_1(l) \) we refer to Figure 33.
4.3. Embedding ellipsoids into balls

We recall from the construction of the embedding \( \psi_l \) that the unfilled space \( X_l(l) \subset P_l(l) \setminus \psi_l(S_{a_l}) \) is caused by the fact that the size of the fibers of \( F_l \) is constant, that the space \( Y(l) \) was needed for folding and contains all stairs, and that the space \( Z_l(l) \) is the union of the space needed to deform the fibers of \( F_l \) and the space caused by the fact that the \( N_l \) have to be even integers. Denote the closure of \( \psi_l(S_{a_l}) \) in \( \mathbb{R}^{2n} \) by \( \overline{\psi_l(S_{a_l})} \). By construction of \( \psi_l \) we have \( |\psi_l(S_{a_l})| = |\overline{\psi_l(S_{a_l})}| \) and

\[
T^n(l^2) \setminus \overline{\psi_l(S_{a_l})} \subset X(l) \cup Y(l) \cup Z(l).
\]

We conclude that

\[
|T^n(l^2) \setminus \psi_l(S_{a_l})| = |T^n(l^2) \setminus \overline{\psi_l(S_{a_l})}| \leq |X(l)| + |Y(l)| + |Z(l)|. \tag{4.39}
\]

**Lemma 4.10**

(i) \(|X(l)| < \frac{\pi}{n} |T^n(l^2)|\).

(ii) \(|Y(l)| < \frac{2\pi}{1} |T^n(l^2)|\).

(iii) \(|Z(l)| < \frac{2\pi n}{1} |T^n(l^2)|\).
Proof. (i) Notice that \( X(l) \subseteq T^n(l^2) \setminus T^n((l-1)^2) \), cf. Figure 35. Since \((l-1)^n > l^n - n \cdot l^{n-1}\), we can therefore estimate

\[
|X(l)| \leq \left| T^n(l^2) \right| - \left| T^n((l-1)^2) \right| \\
= \frac{1}{n^1} \left( l^{2n} - (l-1)^n l^n \right) \\
< \frac{1}{n^1} n^2 l^{2n-1} \\
= \frac{n}{l} \left| T^n(l^2) \right|.
\]

(ii) The definitions of \( Y_i(l) \) and \( Q_i(l) \) yield

\[
\frac{|Y_i(l)|}{|Q_i(l)|} = \frac{2\pi}{l}, \quad i = 1, \ldots, l-1,
\]

cf. Figure 35. Therefore,

\[
|Y(l)| = \sum_{i=1}^{l-1} |Y_i(l)| = \frac{2\pi}{l} \sum_{i=1}^{l-1} |Q_i(l)| < \frac{2\pi}{l} \sum_{i=1}^{l-1} |P_i(l)| < \frac{2\pi}{l} \left| T^n(l^2) \right|.
\]

(iii) Assume first that \( i \leq l - 2 \). In view of definition (4.14) we then find

\[
\frac{k_i}{k_{i+1}} = \frac{l-i}{l-i-1} \leq 2,
\]

and so, in view of definition (4.15), \( N_i + 1 \geq k_i - 4 \), i.e.,

\[
\frac{N_i + 1}{k_i} \geq 1 - \frac{4}{k_i}.
\]

Since \( i \leq l - 2 \) and \( l > 3n\pi \) we have

\[
\frac{4}{k_i} = \frac{4\pi}{(l-i) l} \leq \frac{4\pi}{2l} = \frac{2\pi}{l} < 1.
\]

Applying the formula \((1 - r)^{n-1} > 1 - (n-1)r\) valid for all \( r \in ]0, 1[\) we can therefore estimate

\[
\left( \frac{N_i + 1}{k_i} \right)^{n-1} \geq \left( 1 - \frac{4}{k_i} \right)^{n-1} > 1 - (n-1) \frac{4}{k_i} \geq 1 - (n-1) \frac{2\pi}{l}.
\]

Similarly, the definition (4.16) shows that \( N_{l-1} + 1 \geq k_{l-1} - 2 \), whence

\[
\frac{N_{l-1} + 1}{k_{l-1}} \geq 1 - \frac{2}{k_{l-1}}.
\]
Estimating as before, we find
\[
\left( \frac{N_{i-1} + 1}{k_{i-1}} \right)^{n-1} \geq \left( 1 - \frac{2}{k_{i-1}} \right)^{n-1} > 1 - (n-1) \frac{2\pi}{l} = 1 - (n-1) \frac{2\pi}{l}.
\] (4.41)

The definitions of $Z_i(I)$ and $Q_i(I)$ and the estimates (4.40) and (4.41) now yield
\[
\frac{|Z_i(I)|}{|Q_i(I)|} = \frac{|\Box^{n-1}(1) \setminus \Box^{n-1}(\frac{N_{i+1}}{k_i})|}{|\Box^{n-1}(1)|} = 1 - \left( \frac{N_{i+1}}{k_i} \right)^{n-1} < (n-1) \frac{2\pi}{l} < \frac{2\pi n}{l},
\]
i = 1, \ldots, l - 1, and so
\[
|Z(I)| = \sum_{i=1}^{l-1} |Z_i(I)| < \frac{2\pi n}{l} \sum_{i=1}^{l-1} |Q_i(I)| < \frac{2\pi n}{l} \left| T^n(I^2) \right|.
\]

This completes the proof of Lemma 4.10. \hfill \Box

In view of the estimate (4.39) and Lemma 4.10 we find
\[
\frac{|T^n(I^2) \setminus \psi_1(S_{2l})|}{|T^n(I^2)|} \leq \frac{|X(I)|}{|T^n(I^2)|} + \frac{|Y(I)|}{|T^n(I^2)|} + \frac{|Z(I)|}{|T^n(I^2)|} \leq \frac{n}{l} + \frac{2\pi}{l} + \frac{2\pi n}{l}.
\]

Taking the limit $l \to \infty$, we see that assertion (4.35) holds true, and so the proof of assertion (ii) in Proposition 4.6 is complete.

The proof of Proposition 4.6 is accomplished. \hfill \Box
4. Symplectic folding in higher dimensions
5 Proof of Theorem 3

Throughout this chapter $(M, \omega)$ is a given connected $2n$-dimensional symplectic manifold of finite volume $\text{Vol}(M, \omega) = \frac{1}{n!} \int_M \omega^n$. We denote by $\mu$ the measure on $M$ induced by the volume form $\frac{1}{n!} \omega^n$. As before, $|S|$ denotes the Lebesgue measure of a measurable subset $S$ of $\mathbb{R}^{2n}$. For $w \in \mathbb{R}^{2n}$ we denote the translation $z \mapsto z + w$ of $\mathbb{R}^{2n}$ by $\tau_w$. Of course, $\tau_w$ is a symplectomorphism of $(\mathbb{R}^{2n}, \omega_0)$. In this chapter the symplectic coordinates will again be denoted by

$$(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) = (u, v, x_2, y_2, \ldots, x_n, y_n) \in \mathbb{R}^{2n},$$

and we again abbreviate $x = (x_2, \ldots, x_n)$ and $y = (y_2, \ldots, y_n)$ as well as

$$1_y = (1, \ldots, 1) \in \mathbb{R}^{n-1}(y).$$

5.1 Proof of $\lim_{a \to \infty} p^p_a(M, \omega) = 1$

We recall from the introduction that for every $a > n$ the real number $p^p_a(M, \omega)$ is defined by

$$p^p_a(M, \omega) = \sup_{\lambda} \frac{|\lambda P(\pi, \ldots, \pi, a)|}{\text{Vol}(M, \omega)}$$

where the supremum is taken over all those $\lambda$ for which $\lambda P^{2n}(\pi, \ldots, \pi, a)$ symplectically embeds into $(M, \omega)$. Moreover, we recall from Section 4.2 that the polydisc $P^{2n}(\pi, \ldots, \pi, a)$ is symplectomorphic to the set $\mathbb{R}^n(a, \pi)$. We conclude that the second statement in Theorem 3 of the introduction can be reformulated as

**Theorem 5.1** For every $\epsilon > 0$ there exists a number $a_0 = a_0(\epsilon) > n$ having the following property. For every $a \geq a_0$ there exist a number $\lambda(a) > 0$ and a symplectic embedding $\Phi_a: \lambda(a)\mathbb{R}^n(a, \pi) \hookrightarrow M$ such that

$$\mu(M \setminus \Phi_a(\lambda(a)\mathbb{R}^n(a, \pi))) < \epsilon.$$

**Proof.** We shall proceed along the following lines. We shall first fill almost all of $M$ with finitely many symplectically embedded cubes whose closures are disjoint, and connect these cubes by neighbourhoods of lines. In view of Proposition 4.3
we can then almost fill the cubes with symplectically embedded thin polydiscs, and we shall use the neighbourhoods of the lines to pass from one cube to another, cf. Figure 36.

**Step 1. Filling $M$ by cubes**

We denote by $C(s)$ the $2n$-dimensional open cube

$$C(s) = \{(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n} | 0 < x_i < s, 0 < y_i < s, \ i = 1, \ldots, n\}.$$ 

**Lemma 5.2** For every $\epsilon > 0$ there exist $s \in (0, 1]$, an integer $k$ and a symplectic embedding

$$\gamma: \bigsqcup_{i=1}^{k} C_i(s) \hookrightarrow M$$

of a disjoint union of $k$ translates $C_i(s)$ of $C(s)$ in $\mathbb{R}^{2n}$ such that

$$\mu\left(M \setminus \gamma\left(\bigsqcup_{i=1}^{k} C_i(s)\right)\right) < \epsilon.$$ 

**Proof.** We choose for each point $p \in M$ a Darboux chart $\chi_p: U_p \rightarrow V_p \subset M$. We can assume that the sets $U_p$ are bounded. As every manifold, $M$ satisfies the second axiom of countability, and so $M$ is Lindelöf, i.e., every open covering of $M$ has a countable subcovering. We therefore find a countable subcovering $\{V_{p_i}\}$ of the open covering $\{V_p\}$ of $M$. Since the sets $U_{p_i} \subset \mathbb{R}^{2n}$ are bounded, we find points $w_i \in \mathbb{R}^{2n}$ such that the translates $U_i := \tau_{w_i}(U_{p_i})$, $i \geq 1$, are disjoint. We abbreviate $\chi_i = \chi_{p_i} \circ \tau_{-w_i}$ and $V_i = V_{p_i}$. We have constructed countably many disjoint Darboux charts $\chi_i: U_i \rightarrow V_i$ which cover $M$. 

Figure 36: Filling $M$ with a thin polydisc.
We define subsets $V_i'$ of $M$ by

$$V_1' = V_1 \quad \text{and} \quad V_i' = V_i \setminus \bigcup_{j=1}^{i-1} V_j, \quad i \geq 2.$$ 

Then

$$\bigcup_{i \geq 1} V_i' = \bigcup_{i \geq 1} V_i = M.$$ 

Since the open sets $V_i$ are $\mu$-measurable, the sets $V_i'$ are also $\mu$-measurable. It follows that

$$\sum_{i \geq 1} \mu(V_i') = \mu\left(\bigcup_{i \geq 1} V_i'\right) = \mu(M).$$ 

Since $\mu(M) < \infty$ we therefore find $m \in M$ such that

$$\sum_{i=1}^{m} \mu(V_i') > \mu(M) - \frac{\epsilon}{2}. \quad (5.1)$$

Set $U_i' = \chi_i^{-1}(V_i') \subset U_i$, $i = 1, \ldots, m$. Since $V_i'$ is $\mu$-measurable and $\chi_i^{-1}$ is smooth, $U_i'$ is Lebesgue-measurable, $i = 1, \ldots, m$. We therefore find $s \in ]0, 1[$ and finitely many disjoint translates $C_{i,j}(s) \subset U_i'$, $j = 1, \ldots, j_i$, of the cube $C(s)$ such that

$$\sum_{j=1}^{j_i} |C_{i,j}(s)| > |U_i'| - \frac{\epsilon}{2m}, \quad i = 1, \ldots, m. \quad (5.2)$$

We set $k = \sum_{i=1}^{m} j_i$. Since the sets $U_i'$ are disjoint, the $k$ cubes $C_{i,j}(s)$ are disjoint. Moreover, the embeddings $\chi_i : U_i' \hookrightarrow M$ are symplectic, and so the embedding $\gamma$ defined by

$$\gamma = \bigsqcup_{i,j} \chi_i|_{C_{i,j}(s)} : \bigsqcup_{i,j} C_{i,j}(s) \hookrightarrow M$$

is symplectic, and

$$\mu(\chi_i(C_{i,j}(s))) = |C_{i,j}(s)| \quad \text{and} \quad \mu(V_i') = |U_i'|.$$
In view of the estimates (5.1) and (5.2) we therefore find
\[
\mu \left( M \setminus \gamma \left( \bigsqcup C_{i,j}(s) \right) \right) = \mu(M) - \sum_{i,j} \mu(\chi_i(C_{i,j}(s))) \\
= \mu(M) - \sum_{i,j} \left| C_{i,j}(s) \right| \\
= \mu(M) - \sum_{i=1}^{m} \mu(V_i') + \sum_{i=1}^{m} \left( |U_i'| - \sum_{j=1}^{i} \left| C_{i,j}(s) \right| \right) \\
< \frac{\epsilon}{2} + \sum_{i=1}^{m} \frac{\epsilon}{2m} \\
= \epsilon,
\]
and so the proof of Lemma 5.2 is complete.

Let \( \epsilon > 0 \) be as in Theorem 5.1 and set \( \epsilon' = \epsilon/3 \). In view of Lemma 5.2 we find \( s' \in [0,1] \) and a symplectic embedding
\[
\hat{\gamma} = \bigsqcup_{i=1}^{k} \hat{\gamma}_i : \bigsqcup_{i=1}^{k} C_i(s') \hookrightarrow M
\]
of a disjoint union of \( k \) translates \( \hat{C}_i(s') \) of \( C(s') \) such that
\[
\mu \left( M \setminus \hat{\gamma} \left( \bigsqcup \hat{C}_i(s') \right) \right) < \epsilon'.
\]
(5.3)
We choose \( s \in [0, s'] \) so large that
\[
k \left( (s')^{2n} - s^{2n} \right) < \epsilon'.
\]
(5.4)
We abbreviate \( d := (s' - s)/2 \). For each \( \delta \in [0, d] \) we define
\[
C_i(\delta) = \{ z + ((i-1)s - \delta, -\delta, \ldots, -\delta) \mid z \in C(s + 2\delta) \},
\]
and we abbreviate \( C_i = C_i(0) \) and \( C_i' = C_i(d) \), \( i = 1, \ldots, k \), cf. Figure 37. After choosing \( s \in [0, s'] \) larger if necessary we can assume that \( 2d \leq s \) so that the cubes \( C_i' \) are disjoint. We define \( w_i \in \mathbb{R}^{2n} \) through the identity \( \tau_{w_i}(C_i(s')) = C_i' \), and we define the symplectic embedding \( \gamma_i' : C_i' \hookrightarrow M \) by
\[
\gamma_i' = \hat{\gamma}_i \circ \tau_{-w_i} : C_i' \hookrightarrow M.
\]
We denote the restriction of \( \gamma'_i \) to \( C_i \) by \( \gamma_i \), and we write

\[
\gamma = \bigsqcup_{i=1}^{k} \gamma_i : \bigsqcup_{i=1}^{k} C_i \hookrightarrow M \quad \text{and} \quad \gamma' = \bigsqcup_{i=1}^{k} \gamma'_i : \bigsqcup_{i=1}^{k} C'_i \hookrightarrow M.
\]

Since \( \gamma \) and \( \gamma' \) are symplectic, we have \( \mu \left( \gamma \left( \bigsqcup C_i \right) \right) = \sum |C_i| = ks^{2n} \) and

\[
\mu \left( \gamma' \left( \bigsqcup C'_i \right) \setminus \gamma \left( \bigsqcup C_i \right) \right) = \sum |C'_i \setminus C_i| = k \left( (s')^{2n} - s^{2n} \right).
\]

In view of the inequalities (5.3) and (5.4) we can therefore estimate

\[
\mu(M) = \mu \left( \gamma \left( \bigsqcup C_i \right) \right) + \mu \left( \gamma' \left( \bigsqcup C'_i \right) \setminus \gamma \left( \bigsqcup C_i \right) \right) + \mu \left( M \setminus \gamma' \left( \bigsqcup C'_i(s') \right) \right)
\]

\[
= ks^{2n} + k \left( (s')^{2n} - s^{2n} \right) + \mu \left( M \setminus \gamma' \left( \bigsqcup C_i(s') \right) \right)
\]

\[
< ks^{2n} + \epsilon' + \epsilon'
\]

\[
= ks^{2n} + 2\epsilon'. \quad (5.5)
\]

Figure 37: The cubes \( C_i \) and \( C'_i \), \( i = 1, 2, 3 \), and the lines \( L_i \), \( i = 1, 2 \).

**Step 2. Connecting the cubes**

Our next goal is to extend the embedding \( \gamma : \bigsqcup C_i \hookrightarrow M \) to a symplectic embedding of a connected domain. For \( i = 1, \ldots, k - 1 \) we abbreviate

\[
I_i = [(2i - 1)s, 2is],
\]

and we define straight lines \( L_i(t) : I_i \hookrightarrow \mathbb{R}^2 \times \mathbb{R}^{n-1}(x) \times \mathbb{R}^{n-1}(y) \) by

\[
L_i(t) = \begin{cases} 
(t, 0, 0, 0) & \text{if } i \text{ is odd}, \\
(t, 0, s1_y) & \text{if } i \text{ is even},
\end{cases}
\]
cf. Figure 37. Then
\[ L_i(t) \in C'_i \quad \text{if } t \in \left[ (2i - 1)s, (2i - 1)s + d \right], \]
\[ L_i(t) \in C'_{i+1} \quad \text{if } t \in \left[ 2is - d, 2is \right]. \]
\[ (5.6) \]

For \( \delta > 0 \) we define the "\( \delta \)-neighbourhood" \( N_i(\delta) \) of \( L_i \) in \( \mathbb{R}^2 \) by
\[ N_i(\delta) = \begin{cases} 
I_i \times ] - \delta, \delta [ \times ] - \delta, \delta [^{n-1} & \text{if } i \text{ is odd}, \\
I_i \times ] - \delta, \delta [ \times ] s - \delta, s + \delta [^{n-1} & \text{if } i \text{ is even}.
\end{cases} \]

**Proposition 5.3** There exist \( \delta \in ]0, d/8[ \) and a symplectic embedding
\[ \rho: \bigcup_{i=1}^{k} C_i(d/3) \cup \bigcup_{i=1}^{k-1} N_i(\delta) \hookrightarrow M. \]

**Proof.** By construction of the map \( \gamma' \), the set \( M \setminus \gamma' \left( \bigcup C_i' \right) \) is connected. Using this and the inclusions (5.6) we find a smooth embedding
\[ \bigcup_{i=1}^{k-1} \lambda_i: \bigcup_{i=1}^{k} L_i \hookrightarrow M \setminus \gamma' \left( \bigcup C_i \right) \]
such that
\[ \lambda_i(L_i(t)) = \gamma'_i(L_i(t)) \quad \text{if } t \in \left[ (2i - 1)s, (2i - 1)s + d/2 \right], \]
\[ \lambda_i(L_i(t)) = \gamma'_{i+1}(L_i(t)) \quad \text{if } t \in \left[ 2is - d/2, 2is \right], \]
and such that
\[ \lambda_i(L_i(t)) \in M \setminus \gamma' \left( \bigcup C_i(d/2) \right) \quad \text{if } t \in \left[ (2i - 1)s + d/2, 2is - d/2 \right]. \]
\[ (5.9) \]

We are now going to construct a symplectic extension of \( \lambda_1 \) to a neighbourhood of \( L_1 \). A symplectic extension of \( \lambda_i \) to a neighbourhood of \( L_i \) for \( i \geq 2 \) can be constructed the same way. We denote by
\[ \{ e_1(t), e_2(t), \ldots, e_{2n-1}(t), e_{2n}(t) \} = \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n} \right\} \]
the standard symplectic frame of the tangent space \( T_{L_1(t)} \mathbb{R}^{2n} \).
Lemma 5.4 There exists a smooth 1-parameter family of symplectic frames \( \{ f_j(t) \} \), \( j = 1, \ldots, 2n \), along the curve \( \lambda_1(t) = \lambda_1(L_1(t)) \) such that
\[
f_1(t) = \frac{d}{dt} \lambda_1(t) \quad \text{for all } t \in [s, 2s]
\]
and such that for all \( j = 1, \ldots, 2n \),
\[
\begin{align*}
f_j(t) &= \left( T_{L_1(t)} \gamma_1^j \right)(e_j(t)) \quad \text{if } t \in [s, s + d/2], \quad (5.10) \\
f_j(t) &= \left( T_{L_1(t)} \gamma_2^j \right)(e_j(t)) \quad \text{if } t \in [2s - d/2, 2s]. \quad (5.11)
\end{align*}
\]

Proof. For \( t \in [s, 2s] \) we define \( f_1(t) = \frac{d}{dt} \lambda_1(t) \). In view of the identities (5.7) and (5.8) for \( i = 1 \) the assertions (5.10) and (5.11) for \( j = 1 \) are met. Using (5.10) for \( j = 1 \) and that \( \gamma_1^j \) is symplectic we find a smooth vector field \( f_2(t) \) along \( \lambda_1(t) \) such that
\[
f_2(t) = \left( T_{L_1(t)} \gamma_1^j \right)(e_2(t)) \quad \text{if } t \in [s, s + d/2] \quad (5.12)
\]
and such that \( \omega(f_1(t), f_2(t)) = 1 \) for all \( t \in [s, 2s] \). Choose a smooth function \( c(t) : [s, 2s] \to [0, 1] \) such that
\[
c(t) = \begin{cases} 
0 & \text{if } t \leq 2s - d/2, \\
1 & \text{if } t \geq 2s - d/3,
\end{cases}
\]
and define a second smooth vector field \( f_2''(t) \) along \( \lambda_1(t) \) by
\[
f_2''(t) = \begin{cases} 
0 & \text{if } t \leq 2s - d/2, \\
c(t) \left( T_{L_1(t)} \gamma_2^j \right)(e_2(t)) & \text{if } t > 2s - d/2.
\end{cases}
\]
We define the smooth vector field \( f_2(t) \) along \( \lambda_1(t) \) by
\[
f_2(t) = c(t)f_2''(t) + (1 - c(t))f_1(t).
\]
In view of the formula (5.12) and the definition of \( f_2''(t) \) the assertions (5.10) and (5.11) for \( j = 2 \) are met, and since \( \gamma_2^j \) is symplectic, we find \( \omega(f_1(t), f_2(t)) = 1 \) for all \( t \in [s, 2s] \). The subvector space \( V_t \) of \( T_{\lambda_1(t)}M \) spanned by \( f_1(t) \) and \( f_2(t) \) is therefore a symplectic subspace. Denote the symplectic complement of \( V_t \) in \( T_{\lambda_1(t)}M \) by \( V_t^\perp \). Since the linear symplectic group \( Sp(n - 1; \mathbb{R}) \) is path-connected, we find a smooth 1-parameter family of symplectic frames \( \{ f_3(t), \ldots, f_{2n}(t) \} \) of \( V_t^\perp, t \in [s, 2s] \), such that the assertions (5.10) and (5.11) are met for all \( j = 3, \ldots, 2n \). The family of symplectic frames \( \{ f_1(t), f_2(t), f_3(t), \ldots, f_{2n}(t) \} \) of \( V_t \oplus V_t^\perp = T_{\lambda_1(t)}M \) is as desired. \( \square \)
Lemma 5.5 There exist $\kappa_1 > 0$ and a smooth embedding

$$\sigma_1 : C_1(d/3) \cup N_1(\kappa_1) \cup C_2(d/3) \hookrightarrow M$$

such that

$$\sigma_1|_{C_1(d/3)} = \gamma_1', \quad \sigma_1|_{L_1} = \lambda_1, \quad \sigma_1|_{C_2(d/3)} = \gamma_2'.$$

and such that $(T_{L_1(t)}\sigma_1)(e_j(t)) = f_j(t)$ for all $t \in [s, 2s]$ and $j = 1, \ldots, 2n$.

Proof. Define the smooth embedding $\sigma : C_1(d/2) \cup L_1 \cup C_2(d/2) \hookrightarrow M$ by

$$\sigma|_{C_1(d/2)} = \gamma_1', \quad \sigma|_{L_1} = \lambda_1, \quad \sigma|_{C_2(d/2)} = \gamma_2'.$$

Applying the Whitney extension theorem to the restriction of $\sigma$ to the closed set $C_1(d/3) \cup L_1 \cup C_2(d/3)$ we find $\kappa > 0$ and a smooth map

$$\hat{\sigma} : C_1(d/3) \cup N_1(\kappa) \cup C_2(d/3) \rightarrow M$$

which extends the restriction of $\sigma$ to $C_1(d/3) \cup L_1 \cup C_2(d/3)$. We read off from the identities (5.7) and (5.8) and the inclusions (5.9) for $i = 1$ that the restriction of $\sigma$ to the compact set $C_1(d/3) \cup L_1 \cup C_2(d/3)$ is an embedding. After choosing $\kappa > 0$ smaller if necessary, we can therefore assume that $\hat{\sigma}$ is an embedding.

For each $t \in [s, 2s]$ we define the frame $\{\hat{e}_j(t)\}$ of $T_{L_1(t)}\mathbb{R}^{2n}$ by

$$\hat{e}_j(t) = \hat{\sigma}^* f_j(t), \quad j = 1, \ldots, 2n. \quad (5.13)$$

Since $f_1(t) = \frac{d}{dt} \lambda_1(t)$ and $\hat{\sigma}|_{L_1} = \lambda_1$ we can compute

$$\hat{e}_1(t) = \hat{\sigma}^* f_1(t)$$

$$= \hat{\sigma}^* \left( \frac{d}{dt} \lambda_1(t) \right)$$

$$= \hat{\sigma}^* \left( T_{L_1(t)} \lambda_1 \right)(e_1(t))$$

$$= \hat{\sigma}^* \left( T_{L_1(t)} \hat{\sigma} \right)(e_1(t))$$

$$= e_1(t) \quad (5.14)$$

for all $t \in [s, 2s]$. Similarly, the identities (5.10), (5.11) and $\hat{\sigma}|_{C_1(d/3)} = \gamma_1'$, $\hat{\sigma}|_{C_2(d/3)} = \gamma_2'$ imply that for all $j = 1, \ldots, 2n$,

$$\hat{e}_j(t) = e_j(t) \quad \text{if} \ t \in [s, s + d/3] \cup [2s - d/3, 2s]. \quad (5.15)$$
Define the smooth map \( \alpha : [s, 2s] \times \mathbb{R}^{2n-1} \to \mathbb{R}^{2n} \) by
\[
\alpha \left( L_1(t) + \sum_{j=2}^{2n} b_i e_i(t) \right) = L_1(t) + \sum_{j=2}^{2n} b_i \hat{e}_i(t). \tag{5.16}
\]

In view of the identities (5.14) and (5.15) the map \( \alpha \) restricts to the identity on \( (C_1(d/3) \cap N_1(\hat{k})) \cup L_1 \cup \left( C_2(d/3) \cap N_1(\hat{k}) \right) \). We choose \( \kappa_1 > 0 \) so small that the restriction of \( \alpha \) to \( N_1(\kappa_1) \) is an embedding whose image is contained in \( N_1(\hat{k}) \).

We can now define a smooth embedding
\[
\sigma_1 : C_1(d/3) \cup N_1(\kappa_1) \cup C_2(d/3) \hookrightarrow M
\]
by
\[
\sigma_1 \big|_{C_1(d/3)} = \gamma_1', \quad \sigma_1 \big|_{N_1(\kappa_1)} = \hat{\sigma} \circ \alpha, \quad \sigma_1 \big|_{C_2(d/3)} = \gamma_2'.
\]

For \( z \in L_1 \) we then have \( \sigma_1(z) = \hat{\sigma}(\alpha(z)) = \hat{\sigma}(z) = \lambda_1(z) \).
Moreover, we read off from the identity (5.14) and the definition (5.16) of \( \alpha \) that \( (T_{L_1(t)} \alpha)(e_j(t)) = \hat{e}_j(t) \). Together with the definitions (5.13) we therefore conclude that
\[
(T_{L_1(t)} \sigma_1)(e_j(t)) = \left( T_{L_1(t)} \hat{\sigma} \right) \left( T_{L_1(t)} \alpha \right)(e_j(t)) = \left( T_{L_1(t)} \hat{\sigma} \right)(\hat{e}_j(t)) = f_j(t)
\]
for all \( t \in [s, 2s] \) and \( j = 1, \ldots, 2n \). The proof of Lemma 5.5 is complete. \( \square \)

Since the maps \( \gamma_1' \) and \( \gamma_2' \) are symplectic and since the frames \( \{ f_j(t) \} \) along \( \lambda_1(t) \) are symplectic, the map \( \sigma_1 \) guaranteed by Lemma 5.5 is symplectic on the set \( C_1(d/3) \cup L_1 \cup C_2(d/3) \). Applying the proof of Lemma 3.14 in [26] to \( L_1 \subseteq N_1(\kappa_1) \) and to the symplectic forms \( \omega_0 \) and \( \sigma_1^* \omega \), we therefore find \( \delta_1 > 0 \) and a smooth embedding \( \psi_1 : N_1(\delta_1) \hookrightarrow N_1(\kappa_1) \) such that
\[
\psi_1 \big|_{(N_1(\delta) \cap C_1(d/3)) \cup L_1 \cup (N_1(\delta) \cap C_2(d/3))} = id \quad \text{and} \quad \psi_1^* (\sigma_1^* \omega) = \omega_0. \tag{5.17}
\]

Define the embedding \( \hat{\psi}_1 : N_1(\delta_1) \hookrightarrow M \) by \( \hat{\psi}_1 = \sigma_1 \circ \psi_1 \). In view of the first statement in (5.17) and the identities \( \sigma_1 \big|_{C_1(d/3)} = \gamma_1' \) and \( \sigma_1 \big|_{C_2(d/3)} = \gamma_2' \) we have
\[
\hat{\psi}_1 \big|_{N_1(\delta_1) \cap C_1(d/3)} = \gamma_1' \quad \text{and} \quad \hat{\psi}_1 \big|_{N_1(\delta_1) \cap C_2(d/3)} = \gamma_2',
\]
and in view of the second statement in (5.17) we have
\[
\hat{\psi}_1^* \omega = (\sigma_1 \circ \psi_1)^* \omega = \psi_1^* (\sigma_1^* \omega) = \omega_0.
\]
i.e., $\hat{\rho}_1$ is symplectic. We can therefore define a smooth symplectic embedding

$$\rho_1 : C_1(d/3) \cup N_1(\delta_1) \cup C_2(d/3) \hookrightarrow M$$

by

$$\rho_1|_{C_1(d/3)} = \gamma_1', \quad \rho_1|_{N_1(\delta_1)} = \hat{\rho}_1, \quad \rho_1|_{C_2(d/3)} = \gamma_2'. \quad (5.18)$$

Proceeding as in the construction of $\rho_1$ we find $\delta_i > 0$ and symplectic embeddings

$$\rho_i : C_i(d/3) \cup N_i(\delta_i) \cup C_{i+1}(d/3) \hookrightarrow M$$

such that

$$\rho_i|_{C_i(d/3)} = \gamma_i', \quad \rho_i|_{N_i(\delta_i)} = \hat{\rho}_i, \quad \rho_i|_{C_{i+1}(d/3)} = \gamma_{i+1}' \quad (5.19)$$

where $\hat{\rho}_i : N_i(\delta_i) \hookrightarrow M$ is a symplectic extension of $\lambda_i, i = 1, \ldots, k - 1$. In view of the identities (5.18) and (5.19) the map

$$\rho : \coprod_{i=1}^{k} C_i(d/3) \cup \coprod_{i=1}^{k-1} N_i(\delta_i) \rightarrow M$$

defined by

$$\rho|_{C_i(d/3) \cup N_i(\delta_i)} = \rho_i|_{C_i(d/3) \cup N_i(\delta_i)}, \quad i = 1, \ldots, k - 1, \quad \rho|_{C_k(d/3)} = \rho_{k-1}|_{C_k(d/3)}$$

is smooth and symplectic. In view of the inclusions (5.9) we finally find $\delta \in ]0, \min \{\delta_1, \ldots, \delta_k, d/8\}[$ such that the restriction of $\rho$ to $\coprod C_i(d/3) \cup \coprod N_i(\delta)$
5.1. Proof of \( \lim_{a \to \infty} p_a^P(M, \omega) = 1 \)

is an embedding. The proof of Proposition 5.3 is thus complete. \( \square \)

**Step 3. Replacing \( \bigsqcup C_i \cup \bigsqcup N_i(\delta) \) by a more convenient set**

In view of the previous two steps we are left with filling the set \( \bigsqcup C_i \cup \bigsqcup N_i(\delta) \) with thin polydiscs. If we would embed a part of a polydisc into the cube \( C_1 \) by using the multiple folding technique described in Section 4.2, the \( x \)-width of the last floor of the embedded part of the polydisc would be \( s \), while the \( x \)-width of \( N_1(\delta) \) is \( \delta < s \). In order to pass to \( C_2 \) we would therefore have to deform the fibers of the last floor. This can be done in a similar way as in Step 3 of the proof of Proposition 4.6. For technical reasons we shall take a different route, however, and replace the set \( \bigsqcup C_i \cup \bigsqcup N_i(\delta) \) by a set all of whose fibers have \( x \)-width \( \delta \).

We abbreviate

\[
\nu = \frac{s^2}{\delta}.
\]

We define the subset \( K \) of \( \mathbb{R}^{2n} \) by

\[
K = \{(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n} \mid 0 < x_i < \delta, 0 < y_i < \nu, i = 1, \ldots, n\},
\]

and for \( i = 1, \ldots, k \) we define subsets \( K_i \) of \( \mathbb{R}^{2n} \) by

\[
K_i = \{z + ((i-1)(\nu+s), 0, \ldots, 0) \mid z \in K\}, \quad i = 1, \ldots, k.
\]

Notice that the sets \( K_i \) are symplectomorphic to the cubes \( C_i \). We abbreviate

\[
I_i^s = [(i-1)(\nu+s), 0, \ldots, 0, i(\nu+s)], \quad i = 1, \ldots, k-1,
\]

and define the "\( \delta \)-halfneighbourhood" \( H_i \) in \( \mathbb{R}^{2n} \) by

\[
H_i = \begin{cases} 
I_i^s \times [0, \delta] \times [0, \delta]^{n-1} & \text{if } i \text{ is odd}, \\
I_i^s \times [0, \delta] \times [0, \delta]^{n-1} & \text{if } i \text{ is even},
\end{cases}
\]

cf. Figure 42.

**Proposition 5.6** There exists a symplectic embedding

\[
\xi : \bigsqcup_{i=1}^{k} K_i \cup \bigsqcup_{i=1}^{k-1} H_i \hookrightarrow \bigsqcup_{i=1}^{k} C_i(\delta/3) \cup \bigsqcup_{i=1}^{k} N_i(\delta).
\]

**Proof.** We start with
Lemma 5.7 There exists a symplectic embedding

$$\alpha: ]0, \delta[ \times ]0, v[ \hookrightarrow ]0, s + d/3[ \times ]0, s[$$

which restricts to the identity on \{ (x, y) \mid y < \delta \} and to the translation \((x, y) \mapsto (x, y - v + s)\) on \{ (x, y) \mid y > v - \delta \}, cf. Figure 39.

Proof. Choose a smooth function \(h: ]0, v[ \to [1, (s + d/3)/\delta]\) such that

(i) \(h(w) = 1\) if \(w \in ]0, \delta[ \cup ]v - \delta/3, v[\).

Since \(\delta < d/8\) and \(d < s\) we find by computation that

$$2\delta + \frac{v - 2\delta}{(s + d/3)/\delta} < s < v.$$  

We may therefore further require that

(ii) \(\int_0^v \frac{1}{h(w)} \, dw = s\).

Then the map

$$\alpha: ]0, \delta[ \times ]0, v[ \to \mathbb{R}^2, \quad (x, y) \mapsto (h(y)x, \int_0^y \frac{1}{h(w)} \, dw)$$

is the desired symplectic embedding. \(\Box\)
5.1. Proof of \( \lim_{a \to \infty} p_a^p(M, \omega) = 1 \)

The \((n - 1)\)-fold product \( \alpha \times \cdots \times \alpha \) embeds the fibers \((]0, \delta[ \times ]0, v[)^{n-1}\) of \(K_i\) into the fibers of \(C_i(d/3)\) in such a way that the fibers of \(H_i\) are embedded into the fibers of \(N_i(\delta)\). We next embed the base of \( \bigsqcup K_i \cup \bigsqcup H_i \) into the base of \( \bigsqcup C_i(d/3) \cup \bigsqcup N_i(\delta) \). We denote by \(K\) and \(C\) the projections of the sets \( \bigsqcup K_i \cup \bigsqcup H_i \) and \( \bigsqcup C_i(d/3) \cup \bigsqcup N_i(\delta) \) onto the \((u, v)\)-plane, cf. Figure 40.

**Lemma 5.8** There exists a symplectic embedding \( \overline{\alpha} : K \hookrightarrow C \), cf. Figure 40.

*Proof.* We denote by \( \xi \) the reflection \( (u, v) \mapsto (v, u) \), and we define translations \( \tau_i^+ \) and \( \tau_i^- \) by

\[
\tau_i^+(u, v) = (u + (i - 1)2s, v), \quad \tau_i^-(u, v) = (u - (i - 1)(v + s), v),
\]

\(i = 1, \ldots, k\). Moreover, we abbreviate

\[
I_i^+ = \{(i - 1)(v + s), (i - 1)(v + s) + v \}, \quad I_i^- = \{(i - 1)(v + s) + v, (i - 1)(v + s) + v \},
\]

\(i = 1, \ldots, k - 1\).

In view of the properties of the symplectic embedding \( \alpha \) guaranteed by Lemma 5.7 the map \( \overline{\alpha} : K \to \mathbb{R}^2 \) defined by

\[
\overline{\alpha}(u, v) = \begin{cases} 
(\tau_i^+ \circ \xi \circ \alpha \circ \tau_i^-)(u, v) & \text{if } u \in I_i^+, \\
(\tau_i^+ \circ \tau_i^-)(u, v) & \text{if } u \in I_i^-,
\end{cases}
\]

is a smooth symplectic embedding as desired, cf. Figure 40. \( \square \)

We finally define \( \xi \) to be the restriction to \( \bigsqcup K_i \cup \bigsqcup H_i \) of the symplectic embedding

\( \overline{\alpha} \times \alpha \times \cdots \times \alpha : K \times (]0, \delta[ \times ]0, v[)^{n-1} \hookrightarrow C \times (]0, s + d/3[ \times ]0, \delta[)^{n-1} \).

By construction, \( \xi \left( \bigsqcup K_i \cup \bigsqcup H_i \right) \subset \bigsqcup C_i(d/3) \cup \bigsqcup N_i(\delta) \), and so the proof of Proposition 5.6 is complete. \( \square \)

**Step 4. Filling \( \bigsqcup K_i \cup \bigsqcup H_i \) with thin polydiscs**

Let \(k, s, d\) and \(\delta\) be the numbers found in the previous three steps, and recall that \(\epsilon' = \epsilon/3\) and \(v = s^2/\delta\). For each \(\mathring{a} > 3\pi\) we let \(N(\mathring{a})\) and

\[
\varphi_{\mathring{a}} : R^n(\mathring{a}, \pi) \hookrightarrow R^n((N(\mathring{a}) + 1)\pi)
\]
be the natural number and the symplectic embedding found in Proposition 4.3. By Proposition 4.3 (ii) there exists \( a_0 > 3\pi \) such that for all \( \hat{a} \geq \hat{a}_0 \),

\[
\left| \varphi_{\hat{a}} \left( R^n (\hat{a}, \pi) \right) \right| > 1 - \frac{\epsilon'}{k s^{2n}}.
\]  

(5.20)

In view of the definition (4.8) we have \( N(\hat{a}) \geq N(\hat{a}_0) \) whenever \( \hat{a} \geq \hat{a}_0 \), and in view of (4.11) we have \( N(\hat{a}) \to \infty \) as \( \hat{a} \to \infty \). Choosing \( \hat{a}_0 \) larger if necessary, we can therefore assume that

\[
\frac{\nu}{N(\hat{a}) + 1} \leq \delta \quad \text{for all } \hat{a} \geq \hat{a}_0.
\]  

(5.21)

We set \( a_0 = k \hat{a}_0 \), and we define the function \( \lambda : [a_0, \infty[ \to \mathbb{R} \) by

\[
\lambda(a) = \frac{s}{\sqrt{(N \left( \frac{a}{k} \right) + 1) \pi}}.
\]  

(5.22)

**Proposition 5.9** For each \( a \geq a_0 \) there exists a symplectic embedding

\[
\Psi_{\hat{a}} : \lambda(a) R^n (a, \pi) \hookrightarrow \bigcup_{i=1}^{k} K_i \cup \bigcup_{i=1}^{k-1} H_i.
\]
5.1. Proof of \( \lim_{a \to \infty} p^P_a(M, \omega) = 1 \)

**Proof.** Fix \( a \geq a_0 \). We set \( \hat{a} = a/k \), and we abbreviate \( N = N(\hat{a}) \) and \( \lambda = \lambda(a) \). In abuse of notation we denote the dilatation \( \zeta \mapsto \lambda \zeta, \zeta \in \mathbb{R}^{2n} \), also by \( \lambda \). Moreover, we define the linear symplectomorphism \( \sigma \) of \( \mathbb{R}^{2n} \) by

\[
\sigma(u, v, x, y) = \left( \frac{u}{\lambda}, \frac{v}{\lambda}, x, y \right).
\]

Then \( \sigma(\lambda R^n((N + 1)\pi)) = K \), and so the map \( \psi_{\hat{a}} : \lambda R^n(\hat{a}, \pi) \to \mathbb{R}^{2n} \) defined by

\[
\psi_{\hat{a}} = \sigma \circ \lambda \circ \psi_{\hat{a}} \circ \lambda^{-1}
\]
symplectically embeds \( \lambda R^n(\hat{a}, \pi) \) into \( K \). Using Proposition 4.3 (i), the definitions of \( \psi_{\hat{a}}, \sigma \) and \( \lambda \), and \( v = s^2/\delta \), we find that

\[
\psi_{\hat{a}}(u, v, x, y) = \begin{cases} 
\sigma(u, v, x, y) & \text{if } u < \lambda \pi, \quad (5.23) \\
\sigma(u, v, x, y) + \left( v - \frac{u}{N+1} \hat{a} \right, 0, \left( v - \frac{u}{N+1} \right) \right) & \text{if } u > \lambda(\hat{a} - \pi), \quad (5.24)
\end{cases}
\]
cf. the left picture in Figure 41.

Denote the reflection \( (u, v, x, y) \mapsto (u, v, x, -y) \) by \( \xi_y \). Since \( \psi_{\hat{a}} \) symplectically embeds \( \lambda R^n(\hat{a}, \pi) \) into \( K \), the map \( \overline{\psi}_{\hat{a}} \) defined by

\[
\overline{\psi}_{\hat{a}} = \tau_{v_1 y} \circ \xi_y \circ \psi_{\hat{a}} \circ \tau_{v_1 y} \circ \xi_y
\]
symplectically embeds \( \lambda R^n(\hat{a}, \pi) \) into \( K \) as well, cf. Figure 41. We read off from the identities (5.23) and (5.24) that

\[
\overline{\psi}_{\hat{a}}(u, v, x, y) = \begin{cases} 
\sigma(u, v, x, y) & \text{if } u < \lambda \pi, \quad (5.25) \\
\sigma(u, v, x, y) + \left( v - \frac{u}{N+1} \hat{a} \right, 0, 0 \right) & \text{if } u > \lambda(\hat{a} - \pi), \quad (5.26)
\end{cases}
\]
cf. the right picture in Figure 41.

We are now going to use the embeddings \( \psi_{\hat{a}} \) and \( \overline{\psi}_{\hat{a}} \) of \( \lambda R^n(\hat{a}, \pi) \) into \( K \) to construct a symplectic embedding \( \Psi_{\hat{a}} \) of \( \lambda R^n(a, \pi) \) into \( \bigsqcup_i K_i \cup \bigsqcup_i H_i \), cf. Figure 42. As in Step 1 of Section 2.2 we find a symplectic embedding

\[
\beta : ]0, v[ \times ]0, \delta[ \mapsto ]0, v + s[ \times ]0, \delta[
\]
which restricts to the identity on \( \{ u \leq v - \frac{v}{N+1} \} \) and to the translation \( (u, v) \mapsto (u + s, v) \) on \( \{ u \geq v - \frac{v}{N+1} \} \). For \( i = 1, \ldots, k \) we define

\[
R_i := \{ (u, v, x, y) \in \lambda R^n(a, \pi) \mid (i - 1)\lambda \hat{a} < u \leq i\lambda \hat{a} \}.
\]
5. Proof of Theorem 3

Figure 41: The embeddings $\psi_\tilde{a}$ and $\overline{\psi_\tilde{a}}: \lambda R^n(\tilde{a}, \pi) \hookrightarrow K$.

Then

$$\lambda R^n(a, \pi) = \prod_{i=1}^{k} R_i.$$  

We denote the translation $(u, v, x, y) \mapsto (u + (i - 1)(v + s), v, x, y)$ of $\mathbb{R}^{2n}$ by $\tau_i$. For each odd $i \in \{1, \ldots, k-1\}$ we define the map $\psi_i: R_i \rightarrow \mathbb{R}^{2n}$ by

$$\psi_i(u, v, x, y) = \begin{cases} (\tau_i \circ \overline{\psi_\tilde{a}} \circ \tau_i^{-1})(u, v, x, y) & \text{if } u < \lambda(i\tilde{a} - \pi), \\ (\tau_i \circ (\beta \times i^{2n-2}) \circ \overline{\psi_\tilde{a}} \circ \tau_i^{-1})(u, v, x, y) & \text{if } u \geq \lambda(i\tilde{a} - \pi). \end{cases}$$

The estimate (5.21) and formula (5.26) imply that $\psi_i$ is a smooth symplectic embedding of $R_i$ into $K_i \cup H_i$ for which

$$\psi_i(u, v, x, y) = \sigma(u, v, x, y) + (u_i, 0, 0, 0) \quad \text{if } u \geq \lambda \left(i\tilde{a} - \frac{\pi}{2}\right) \quad (5.27)$$

where $u_i$ is such that the right end of $R_i$ is mapped to the right end of $H_i$, cf. Figure 42. For each even $i \in \{2, \ldots, k-1\}$ we define the map $\psi_i: R_i \rightarrow \mathbb{R}^{2n}$ by

$$\psi_i(u, v, x, y) = \begin{cases} (\tau_i \circ \overline{\psi_\tilde{a}} \circ \tau_i^{-1})(u, v, x, y) & \text{if } u < \lambda(i\tilde{a} - \pi), \\ (\tau_i \circ (\beta \times i^{2n-2}) \circ \overline{\psi_\tilde{a}} \circ \tau_i^{-1})(u, v, x, y) & \text{if } u \geq \lambda(i\tilde{a} - \pi). \end{cases}$$

The estimate (5.21) and formula (5.24) imply that $\psi_i$ is a smooth symplectic embedding of $R_i$ into $K_i \cup H_i$ for which

$$\psi_i(u, v, x, y) = \sigma(u, v, x, y) + (u_i, 0, 0, \left(v - \frac{v}{N+1}\right)_y) \quad \text{if } u \geq \lambda \left(i\tilde{a} - \frac{\pi}{2}\right) \quad (5.28)$$
5.1. Proof of $\lim_{a \to \infty} p_a^P(M, \omega) = 1$

where $u_i$ is such that the right end of $R_i$ is mapped to the right end of $H_i$, cf. Figure 42. We finally define the symplectic embedding $\psi_k : R_k \hookrightarrow K_k$ by

$$
\psi_k(u, v, x, y) = \begin{cases} 
\tau_i \circ \overline{\psi_{\delta}} \circ \tau_i^{-1} (u, v, x, y) & \text{if } k \text{ is odd}, \\
\tau_i \circ \psi_{\delta} \circ \tau_i^{-1} (u, v, x, y) & \text{if } k \text{ is even}.
\end{cases}
$$

![Figure 42: The embedding $\Psi_a : \lambda R^n(a, \pi) \hookrightarrow \bigcup_{i=1}^k K_i \cup \bigcup_{i=1}^{k-1} H_i$.](image)

In view of the identities (5.23), (5.25), (5.27) and (5.28) the embedding

$$
\Psi_a : \lambda R^n(a, \pi) \hookrightarrow \bigcup_{i=1}^k K_i \cup \bigcup_{i=1}^{k-1} H_i
$$

defined by

$$
\Psi_a|_{R_i} = \psi_i, \quad i = 1, \ldots, k,
$$

is a smooth symplectic embedding. The proof of Proposition 5.9 is complete. □

**Step 5. End of the proof of Theorem 5.1**

We let $\epsilon > 0$ be as in Theorem 5.1, set $\epsilon' = \epsilon/3$ and let $k$ and $s$ be as introduced after the proof of Lemma 5.2. We choose $a_0 = a_0(\epsilon)$ as before Proposition 5.9, fix $a \geq a_0$ and define $\lambda(a)$ as in (5.22).

**Lemma 5.10** We have

$$
|\lambda(a) R^n(a, \pi)| > ks^{2n} - \epsilon'.
$$

(5.29)
Proof. We set again \( â = a/k \), \( N = N(â) \) and \( \lambda = \lambda(a) \). Since the embedding \( \varphi_\lambda : R^n(\hat{a}, \pi) \hookrightarrow R^n((N + \pi)n) \) is volume preserving, we have

\[
|\lambda R^n(a, \pi)| = \lambda^{2n} k |R^n(\hat{a}, \pi)| = k\lambda^{2n} |\varphi_\lambda (R^n(\hat{a}, \pi))| \quad (5.30)
\]

and multiplying the inequality (5.20) by

\[
k\lambda^{2n} \left| R^n((N + \pi)n) \right| = k\lambda^{2n} ((N + \pi)n) = ks^{2n}
\]

we find that

\[
k\lambda^{2n} |\varphi_\lambda (R^n(\hat{a}, \pi))| > ks^{2n} - \epsilon'. \quad (5.31)
\]

Lemma 5.10 now follows from combining the identity (5.30) with the estimate (5.31).

Composing the symplectic embeddings \( \Psi_a, \xi \) and \( \rho \) guaranteed by Proposition 5.9, Proposition 5.6 and Proposition 5.3 we obtain the symplectic embedding

\[
\Phi_a := \rho \circ \xi \circ \Psi_a : \lambda(a) R^n(a, \pi) \hookrightarrow M.
\]

Using the estimates (5.5) and (5.29) we find

\[
\mu (M \setminus \Phi_a (\lambda(a) R^n(a, \pi))) = \mu(M) - \mu (\Phi_a (\lambda(a) R^n(a, \pi)))
\]

\[
= \mu(M) - |\lambda(a) R^n(a, \pi)|
\]

\[
< \left( ks^{2n} + 2\epsilon' \right) - \left( ks^{2n} - \epsilon' \right)
\]

\[
= 3\epsilon'
\]

This is the required estimate in Theorem 5.1 and so the proof of Theorem 5.1 is complete.

5.2 Proof of \( \lim_{a \to \infty} \rho^F_a(M, \omega) = 1 \)

We recall from the introduction that for every \( a > \pi \) the real number \( \rho^F_a(M, \omega) \) is defined by

\[
\rho^F_a(M, \omega) = \sup_{\lambda} \frac{|\lambda E(\pi, \ldots, \pi, a)|}{\text{Vol} (M, \omega)}
\]

where the supremum is taken over all those \( \lambda \) for which \( \lambda E^{2n}(\pi, \ldots, \pi, a) \) symplectically embeds into \((M, \omega)\). Corollary 4.5 (i) implies that the first statement in Theorem 3 of the introduction is a consequence of
Theorem 5.11  For every $\epsilon > 0$ there exists a number $a_0 = a_0(\epsilon) > \pi$ having the following property. For every $a \geq a_0$ there exist a number $\lambda(a) > 0$ and a symplectic embedding $\Phi_a : \lambda(a) T^n(a, \pi) \hookrightarrow M$ such that
\[
\mu \left( M \setminus \Phi_a \left( \lambda(a) T^n(a, \pi) \right) \right) < \epsilon.
\]

Proof. We shall proceed along the same lines as in the proof of Theorem 5.1. We shall first fill almost all of $M$ with finitely many symplectically embedded balls whose closures are disjoint, and connect these balls by neighbourhoods of lines. Using Corollary 4.5 (ii) and Proposition 4.6 we can then almost fill the balls with symplectically embedded parts of $\lambda(a) T^n(a, \pi)$, and we shall use the neighbourhoods of the lines to pass from one ball to another. The proof of Theorem 5.11 is substantially more difficult than the proof of Theorem 5.1, however. The first reason is that for $n \geq 3$ there is no elementary method of symplectically filling $M$ with balls. We shall overcome this difficulty by using a result in [25]. The second reason is that symplectically filling a ball with a part of $\lambda(a) T^n(a, \pi)$ is more difficult than symplectically filling a cube with a polydisc. We have overcome this difficulty in Corollary 4.5 (ii) and Proposition 4.6. The third reason is that the fibers of $\lambda(a) T^n(a, \pi)$ are not constant. This will merely cause technical complications.

Step 1. Filling $M$ by balls

We denote by $B(r)$ the $2n$-dimensional open ball
\[
B(r) = \{ z \in \mathbb{R}^{2n} \mid |z| < r \}.
\]

Lemma 5.12  For every $\epsilon > 0$ there exists an integer $k_0$ with the following property. For each integer $k > k_0$ there exist $r = r(k) > 0$ and a symplectic embedding
\[
\eta_k : \bigsqcup_{i=1}^{k} B_i(r) \hookrightarrow M
\]
of a disjoint union of $k$ translates $B_i(r)$ of $B(r)$ in $\mathbb{R}^{2n}$ such that
\[
\mu \left( M \setminus \eta_k \left( \bigsqcup B_i(r) \right) \right) < \epsilon. \tag{5.32}
\]

Proof. We follow [25, Remark 1.5.G]. Fix $\epsilon > 0$. In view of Lemma 5.2 we find $s > 0$ and a symplectic embedding $\gamma : \bigsqcup C_i(s) \hookrightarrow M$ of a disjoint union of $m$ translates of $C(s)$ such that
\[
\mu \left( M \setminus \gamma \left( \bigsqcup C_i(s) \right) \right) < \frac{\epsilon}{3}. \tag{5.33}
\]
Choose \( l_0 \in \mathbb{N} \) so large that for all \( l \geq l_0 \),
\[
\frac{(l - 1)^n}{l^n} > 1 - \frac{\epsilon}{3m^2s^2n}.
\] (5.34)

We define the integer \( k_0 \) by \( k_0 = mn!l_0^n \). We fix \( k > k_0 \) and define the integer \( l > l_0 \) through the inequalities
\[
m n! (l - 1)^n < k \leq m n! l^n.
\] (5.35)

The crucial ingredient of the proof of Lemma 5.12 is the following result which is proved in [25, Corollary 1.5.F]. We abbreviate \( \varsigma = s/\sqrt{n} \).

**Lemma 5.13** (McDuff–Polterovich) There exist \( r_1 > 0 \) and a symplectic embedding
\[
\beta_l : \bigsqcup_{j=1}^{n!l^n} \hat{B}_j(r_1) \hookrightarrow C^{2n}(\pi) = D(\pi) \times \cdots \times D(\pi)
\]
of a disjoint union of \( n! l^n \) translates \( \hat{B}_j(r_1) \) of \( B(r_1) \) such that
\[
\left| C^{2n}(\pi) \setminus \beta_l \left( \bigsqcup_{j=1}^{n!l^n} \hat{B}_j(r_1) \right) \right| < \frac{\epsilon}{3m^2s^2n}.
\]

**Remark 5.14** If \( n = 2 \), Lemma 5.13 follows from Lemma 4.4 (i) and Lemma 2.5, see also [30] and [34]. For general \( n \), however, the only known proof in [25] uses non-elementary, algebro-geometric methods.

Continuing with the proof of Lemma 5.12 we define \( r = \varsigma r_1 \) and \( \hat{B}_j(r) = \varsigma \hat{B}_j(r_1) \). For each \( i = 1, \ldots, m \) we choose \( u_i \in \mathbb{R}^{2n} \) such that the \( m n! l^n \) balls \( B_i(r) = \tau_{-u_i}(\hat{B}_j(r)) \) are disjoint. In view of Lemma 2.5 the disc \( D(\pi) \) is symplectomorphic to the square \([0, \sqrt{n}] \times [0, \sqrt{n}]\). We therefore find a symplectomorphism
\[
\sigma : C^{2n}(\pi) \to C\left(\sqrt{n}\right).
\]

We finally denote the dilatation \( z \mapsto \varsigma z \) of \( \mathbb{R}^{2n} \) also by \( \varsigma \) and the translation \( C(s) \to C_i(s) \) by \( \tau_i \). We can now define a symplectic embedding
\[
\beta : \bigsqcup_{i,j} B_{i,j}(r) \hookrightarrow \bigsqcup_{i} C_i(s)
\]
by
\[
\beta|_{B_{i,j}(r)} = \left( \tau_i \circ \varsigma \circ \sigma \circ \beta_l \circ \varsigma^{-1} \circ \tau_{u_i} \right)|_{B_{i,j}(r)}.
\]
5.2. Proof of \( \lim_{a \to \infty} p^E_a(M, \omega) = 1 \)

In view of the second inequality in (5.35) we can choose \( k \) members \( B_1(r), \ldots, B_k(r) \) of the family \( \{B_i(r)\} \). We define the symplectic embedding

\[
\eta_k: \prod_{i=1}^k B_i(r) \hookrightarrow M
\]

by \( \eta_k = \gamma \circ \beta \). In order to verify the estimate (5.32) we first use Lemma 5.13 to estimate

\[
\left| \prod_i C_i(s) \setminus \beta \left( \prod_{i,j} B_{i,j}(r) \right) \right|
= \sum_{i=1}^m \left| (\zeta \circ \sigma) \left( \mathcal{C}^{2n}(\pi) \setminus \prod_j (\zeta \circ \sigma \circ \beta_i) (\tilde{B}_j(r)) \right) \right|
= \sum_{i=1}^m \xi^{2n} \left| \mathcal{C}^{2n}(\pi) \setminus \prod_j \beta_i(\tilde{B}_j(r)) \right|
< m \xi^{2n} \frac{\epsilon}{3m \xi^{2n}}
= \frac{\epsilon}{3}.
\] (5.36)

Moreover, since \( \beta \) is volume preserving, the inclusion \( \beta \left( \prod B_{i,j}(r) \right) \subset \prod C_i(s) \) implies that

\[
m n! l^n |B(r)| \leq m s^{2n}.
\] (5.37)

The left inequality in (5.35) and the estimates (5.37) and (5.34) now yield

\[
\left| \prod B_{i,j}(r) \setminus \prod B_i(r) \right|
= (m n! l^n - k) |B(r)|
< m n! (l^n - (l - 1)^n) |B(r)|
\leq m s^{2n} \left( 1 - \frac{(l-1)^n}{l^n} \right)
< \frac{\epsilon}{3}.
\] (5.38)

Using the fact that \( \beta \) and \( \gamma \) are both volume preserving and using the estimates (5.33), (5.36) and (5.38) we finally find

\[
\mu \left( M \setminus \eta_k \left( \prod B_i(r) \right) \right)
= \mu \left( M \setminus \gamma \left( \prod C_i(s) \right) \right) + \left| \prod C_i(s) \setminus \beta \left( \prod B_{i,j}(r) \right) \right| + \left| \prod B_{i,j}(r) \setminus \prod B_i(r) \right|
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}
= \epsilon.
\]
The proof of Lemma 5.12 is complete. \(\square\)

Let \(\epsilon > 0\) be as in Theorem 5.11 and set \(\epsilon' = \epsilon/5\). In view of Lemma 5.12 we find an integer \(k_0\) such that for each integer \(k > k_0\) there exists \(r_k > 0\) and a symplectic embedding

\[
\eta_k : \bigcup_{i=1}^{k} B_i(r_k) \hookrightarrow M
\]

of a disjoint union of \(k\) translates \(B_i(r_k)\) of \(B(r_k)\) such that

\[
\mu \left( M \setminus \eta_k \left( \bigcup B_i(r_k) \right) \right) < \epsilon'. \tag{5.39}
\]

Fix \(k > k_0\). In order to bring Proposition 4.6 into play we shall next replace the balls \(B_i(r_k)\) by trapezoids. We choose \(s' \in \left[0, \sqrt{\pi r_k}\right]\) so large that

\[
k \frac{1}{n!} \left( \pi^n r_k^{2n} - (s')^{2n} \right) < \epsilon ', \tag{5.40}
\]

and we choose \(s \in \left[0, s'\right]\) so large that

\[
k \frac{1}{n!} \left( (s')^{2n} - s^{2n} \right) < \epsilon '. \tag{5.41}
\]

We abbreviate \(d := (s' - s)/(2n)\). For each \(\delta \in [0, d]\) we define the trapezoid \(T(\delta)\) by

\[
T(\delta) = \Delta^n(s + 2n\delta) \times \Box^n(s + 2n\delta) \subset \mathbb{R}^n(x) \times \mathbb{R}^n(y).
\]

Moreover, we define the translates \(T_i(\delta)\) by

\[
T_i(\delta) = \{ z + ((i-1)s - \delta, -\delta, \ldots, -\delta) \mid z \in T(\delta) \},
\]

and we abbreviate \(T_i = T_i(0)\) and \(T'_i = T_i(d), i = 1, \ldots, k\), cf. Figure 43 and Figure 44. Notice that \(T_i(\delta) \subset T_i(\delta')\) whenever \(\delta \leq \delta'\). After choosing \(s \in \left[0, s'\right]\) larger if necessary we can assume that \(2nd \leq s\) so that the trapezoids \(T'_i\) are disjoint.

Composing the linear symplectomorphism

\[
T(d) = \Delta^n(s') \times \Box^n(s') \rightarrow \Delta^n(s')^2 \times \Box^n(1) = T^n(s')^2, \quad (x, y) \mapsto (s'x, \frac{1}{s'}y)
\]
with the symplectic embedding \( T^n((s')^2) \hookrightarrow B^{2n}(\pi r_k^n) = B(r_k) \) guaranteed by Corollary 4.5 (ii), we obtain a symplectic embedding

\[
\sigma : T(d) \hookrightarrow B(r_k).
\]

We define the points \( u_i \) and \( w_i \) in \( \mathbb{R}^{2n} \) through the identities \( \tau_{u_i}(T_i') = T(d) \) and \( \tau_{w_i}(B(r_k)) = B_i(r_k) \). We can now define the symplectic embedding \( \vartheta_i' : T_i' \hookrightarrow M \) by

\[
\vartheta_i' = \eta_k \circ \tau_{w_i} \circ \sigma \circ \tau_{u_i} : T_i' \hookrightarrow M.
\]

We denote the restriction of \( \vartheta_i' \) to \( T_i \) by \( \vartheta_i \), and we write

\[
\vartheta = \bigsqcup_{i=1}^k \vartheta_i : \bigsqcup_{i=1}^k T_i \hookrightarrow M \quad \text{and} \quad \vartheta' = \bigsqcup_{i=1}^k \vartheta_i' : \bigsqcup_{i=1}^k T_i' \hookrightarrow M.
\]

Since \( \vartheta \), \( \vartheta' \) and \( \eta_k \) are symplectic, we have

\[
\mu(\vartheta \left( \bigsqcup T_i \right)) = \sum |T_i| = k \frac{1}{n!} s^{2n},
\]

\[
\mu(\vartheta'(\bigsqcup T_i') \setminus \vartheta(\bigsqcup T_i)) = \sum |T_i' \setminus T_i| = k \frac{1}{n!} (s')^{2n} - s^{2n}
\]

and

\[
\mu(\eta_k \left( \bigsqcup B_i(r_k) \right) \setminus \vartheta'(\bigsqcup T_i')) = k (|B(r_k)| - |T(d)|) = k \frac{1}{n!} \left( \pi^n r_k^{2n} - (s')^{2n} \right).
\]
In view of the inequalities (5.39), (5.40) (5.41) we can therefore estimate
\[
\mu(M) = \mu\left(\bigcup T_i\right) + \mu\left(\vartheta'\left(\bigcup T_i'\right) \setminus \vartheta\left(\bigcup T_i\right)\right) + \mu\left(\eta_k\left(\bigcup B_i(r_k)\right)\right) - \mu\left(\bigcup B_i(r_k)\right) + \mu\left(M \setminus \eta_k\left(\bigcup B_i(r_k)\right)\right)
\]
\[
= k \frac{1}{n!} s^{2n} + k \frac{1}{n!} \left(\pi^2 - s^{2n}\right) + k \frac{1}{n!} \left(\pi^2 r_{k,i}^2 - s^{2n}\right) + \mu(M \setminus \eta_k\left(\bigcup B_i(r_k)\right))
\]
\[
< k \frac{1}{n!} s^{2n} + e' + e' + e' + e'.
\]
(5.42)

**Step 2. Connecting the trapezoids**

We next extend the embedding \(\vartheta : \bigcup T_i \hookrightarrow M\) to a symplectic embedding of a connected domain. For \(i = 1, \ldots, k - 1\) we define straight lines
\[
L_i(t) : \{(2i-1)s, 2is\} \hookrightarrow \mathbb{R}^2 \times \mathbb{R}^{n-1}(x) \times \mathbb{R}^{n-1}(y)
\]
by \(L_i(t) = (t, 0, 0, 0)\), cf. Figure 44. Then
\[
L_i(t) \in T_i' \quad \text{if} \quad t \in (2i-1)s, (2i-1)s + (2n-2)d\];
\[
L_i(t) \in T_{i+1}' \quad \text{if} \quad t \in ]2is - d, 2is\].
(5.43)

\[\text{Figure 44: The trapezoids } T_i \text{ and } T_i', \ i = 1, 2, \text{ and the line } L_1.\]

For \(\delta > 0\) we define the "\(\delta\)-neighbourhood" \(N_i(\delta)\) of \(L_i\) in \(\mathbb{R}^{2n}\) by
\[
N_i(\delta) = \{(2i-1)s, 2is\} \times \mathbb{R}^{n-1} - \delta, \delta\}
\]
5.2. Proof of \( \lim_{a \to \infty} p_a^E(M, \omega) = 1 \)

For later use we shall verify that the left end of \( N_i(\delta) \) and the right end of \( T_i(\delta) \) fit together nicely. In the sequel we denote by \( p_{u,v}, p_{u,x} \) and \( p_{v,y} \) the projections of \( \mathbb{R}^{2n} \) onto \( \mathbb{R}^2(u,v), \mathbb{R}^n(u,x) \) and \( \mathbb{R}^n(v,y) \).

**Lemma 5.15** For each \( \delta \in [0, d] \) and each \( i \in \{1, \ldots, k - 1\} \) we have

\[
\{(u, v, x, y) \in N_i(\delta) \mid u = (2i - 1)s + \delta\} \subset T_i(\delta).
\]

**Proof.** We may assume that \( i = 1 \). We compute that

\[
\{ (u, x_2, \ldots, x_n) \mid u = s + \delta, -\delta < x_2, \ldots, x_n < \delta \} 
\subset \{ (u, x_2, \ldots, x_n) \mid u = s + \delta, -\delta < x_2, \ldots, x_n \text{ and } \}
\]

\[
x_2 + \cdots + x_n < (n - 1)\delta \}
\]

\[
\{ (u - \delta, x_2 - \delta, \ldots, x_n - \delta) \mid u - \delta = s + \delta, 0 < x_2, \ldots, x_n \text{ and } \}
\]

\[
u + x_2 + \cdots + x_n < s + 2n\delta \}
\]

\[
\subset \{ (u - \delta, x_2 - \delta, \ldots, x_n - \delta) \mid 0 < u, x_2, \ldots, x_n \text{ and } \}
\]

\[
u + x_2 + \cdots + x_n < s + 2n\delta \},
\]

i.e.,

\[
p_{u,x}(\{(u, v, x, y) \in N_1(\delta) \mid u = s + \delta\}) \subset p_{u,x} (T_1(\delta)).
\]

Moreover, \( ] - \delta, \delta [^n \subset ] - \delta, s + (2n - 1)\delta [^n \), i.e.,

\[
p_{v,y}(\{(u, v, x, y) \in N_1(\delta) \mid u = s + \delta\}) \subset p_{v,y} (T_1(\delta)).
\]

Lemma 5.15 thus follows. \( \square \)

The following proposition parallels Proposition 5.3.

**Proposition 5.16** There exist \( \delta \in [0, d] \) and a symplectic embedding

\[
\rho : \bigcup_{i=1}^k T_i(\delta) \cup \bigcup_{i=1}^{k-1} N_i(\delta) \hookrightarrow M.
\]

**Proof.** By construction of the map \( \vartheta' \) the set \( M \setminus \vartheta'(\bigcup T_i) \) is connected. Using this and the inclusions (5.43) we find a smooth embedding

\[
\bigcup_{i=1}^{k-1} \lambda_i : \bigcup_{i=1}^{k-1} L_i \hookrightarrow M \setminus \vartheta \left( \bigcup T_i \right).
\]
such that
\[
\lambda_i(L_i(t)) = \theta_i'(L_i(t)) \quad \text{if } t \in [(2i - 1)s, (2i - 1)s + (2n - 1)d/2],
\lambda_i(L_i(t)) = \theta_{i+1}'(L_i(t)) \quad \text{if } t \in [2is - d/2, 2is],
\]
and such that
\[
\lambda_i(L_i(t)) \in M \setminus \theta'(\bigsqcup T_i(d/2)) \quad \text{if } t \in [(2i - 1)s + (2n - 1)d/2, 2is - d/2].
\]

For \(i = 1, \ldots, k - 1\) and \(\delta \in [0, d]\) we define the truncated trapezoid
\[
\tilde{T}_i(\delta) = \{ z \in T_i(\delta) \mid u < (2i - 1)s + \delta \},
\]
cf. Figure 45. Replacing the maps \(\gamma'_1, \ldots, \gamma'_k\) and the sets
\[
C_1(d/3), C_1(d/2), \ldots, C_{k-1}(d/3), C_{k-1}(d/2), C_k(d/3), C_k(d/2)
\]
in the proof of Proposition 5.3 by the maps \(\theta_1', \ldots, \theta_k'\) and the sets
\[
\tilde{T}_1(d/3), \tilde{T}_1(d/2), \ldots, \tilde{T}_{k-1}(d/3), \tilde{T}_{k-1}(d/2), T_k(d/3), T_k(d/2)
\]
we find \(\delta > 0\) and a symplectic embedding
\[
\rho: \bigsqcup_{i=1}^{k-1} \left( \tilde{T}_i(d/3) \cup N_i(\delta) \right) \cup T_k(d/3) \hookrightarrow M,
\]
cf. Figure 45. We define \(\delta\) by
\[
\delta = \min \left\{ \hat{\delta}, \frac{d/3}{2n - 1} \right\}.
\]
Then \(\delta < d\) and \(T_i(\delta) \subset \tilde{T}_i(d/3)\) for all \(i = 1, \ldots, k - 1\). The restriction of \(\rho\) to the domain
\[
\bigsqcup_{i=1}^{k} T_i(\delta) \cup \bigsqcup_{i=1}^{k-1} N_i(\delta)
\]
is as desired. \(\square\)
5.2. Proof of \( \lim_{a \to \infty} p_a^E(M, \omega) = 1 \)

![Diagram](image)

**Figure 45:** The set \( \tilde{T}_1(\delta/3) \cup N_1(\delta_1) \).

**Step 3. The choice of \( k \) and of \( a_0 \)**

We recall that \( \epsilon' = \epsilon/5 \), and we choose the integer \( k_0 \) as after the proof of Lemma 5.12. We now choose the integer \( k \) such that \( k \geq 2 \) and \( k > k_0 \) and

\[
\left(1 + \frac{1}{\sqrt{k}} \right)^n - 1 \leq \frac{1}{\mu(M)} \epsilon'. \tag{5.44}
\]

The inequality (5.44) will be crucial for the proof of Lemma 5.19 below. Let \( s \) and \( d \) be the numbers associated with \( k \) after the proof of Lemma 5.12, and let \( \delta \) be as in Proposition 5.16. For each \( \hat{\alpha} > 3\pi \), we let \( l(\hat{\alpha}) \) and

\[
\varphi_{\hat{\alpha}} : S_{\hat{\alpha}} \hookrightarrow T^n \left( l(\hat{\alpha})^2 \right)
\]

be the natural number and the symplectic embedding found in Proposition 4.6. We abbreviate

\[
q = 1 - \frac{\epsilon'}{k} \frac{n!}{s^{2n}}. \tag{5.45}
\]

By Proposition 4.6 (ii) there exists \( \hat{\alpha}_0 > 3\pi \) such that for all \( \hat{\alpha} \geq \hat{\alpha}_0 \),

\[
\frac{|\varphi_{\hat{\alpha}} (S_{\hat{\alpha}})|}{\left| T^n \left( l(\hat{\alpha})^2 \right) \right|} > q. \tag{5.46}
\]

In view of Lemma 4.8 and the definition (4.33) we have that \( l(\hat{\alpha}) \geq l(\hat{\alpha}_0) \) whenever \( \hat{\alpha} \geq \hat{\alpha}_0 \) and that \( l(\hat{\alpha}) \to \infty \) as \( \hat{\alpha} \to \infty \). Choosing \( \hat{\alpha}_0 \) larger if necessary, we
can therefore assume that
\[ l(\hat{a}) \geq \pi \frac{s}{\delta} \quad \text{for all } \hat{a} \geq \hat{a}_0. \] (5.47)

We set \( a_0 = kn\hat{a}_0 \). In the sequel we fix \( a \geq a_0 \). We set \( \hat{a} = \frac{a}{kn} \).

**Step 4. The set \( \bigcup_{i=1}^{k} S_i \) and the choice of \( \lambda(a) \)**

With each \( k \)-tuple \((u_1, \ldots, u_k) \in \mathbb{R}^k \) we associate the \( k \)-tuple \((v_1, \ldots, v_k) \) defined by \( v_i = u_1 + \cdots + u_i \). We say that a \( k \)-tuple \((u_1, \ldots, u_k) \) is admissible if \( u_i > 0 \) for all \( i \) and if \( v_k = a \).

**Lemma 5.17** There exists a unique admissible \( k \)-tuple \((u_1, \ldots, u_k) \) such that
\[ u_i \left(1 - \frac{v_{i-1}}{a}\right)^{n-1} = u_1, \quad i = 2, \ldots, k. \] (5.48)

**Proof.** Fix \( u_1 \in ]0, a[ \). We inductively associate with \( u_1 \) numbers \( u_2, \ldots, u_k \) as follows. Assume that \( i \in \{2, \ldots, k\} \) and that we have already constructed \( u_2, \ldots, u_{i-1} \). We set \( v_j = u_1 + \cdots + u_j, \quad j = 1, \ldots, i-1 \). We define \( u_i \) by
\[ u_i \left(1 - \frac{v_j}{a}\right)^{n-1} = u_1 \]
where \( j_0 = \max\{ j \mid v_j < a \} \). The function \( f : ]0, a[ \to \mathbb{R} \) defined by
\[ f(u_1) = v_k \]
is then continuous and strictly increasing. Since \( f(u_1) \to 0 \) as \( u_1 \to 0 \) and \( f(u_1) \to \infty \) as \( u_1 \to a \) we conclude that there exists a unique \( u_1 \in ]0, a[ \) with \( f(u_1) = a \). By construction, the \( k \)-tuple \((u_1, \ldots, u_k) \) associated with this \( u_1 \) is admissible and meets the identities (5.48). The proof of Lemma 5.17 is complete. \( \square \)

Let \((u_1, \ldots, u_k)\) be the \( k \)-tuple guaranteed by Lemma 5.17. As before we abbreviate \( v_i = u_1 + \cdots + u_i \), and we set \( v_0 = 0 \). We define subsets \( S_i \) of \( \mathbb{R}^2 \times \mathbb{R}^{n-1}(x) \times \mathbb{R}^{n-1}(y) \) by
\[ S_1 = ]0, v_1[ \times ]0, 1[ \times \triangle^{n-1}(\pi) \times \square^{n-1}(1), \]
\[ S_i = [v_{i-1}, v_i[ \times ]0, 1[ \times \triangle^{n-1}(\pi - \frac{\pi}{a} v_{i-1}) \times \square^{n-1}(1), \quad 2 \leq i \leq k. \]
5.2. Proof of \( \lim_{a \to \infty} p_E^a (M, \omega) = 1 \)

Then \( T^n (a, \pi) \subset \bigsqcup S_i \), cf. Figure 46. Notice that
\[
|S_i| = u_i \frac{\pi^{n-1}}{(n-1)!} \left(1 - \frac{v_{i-1}}{a}\right)^{n-1}, \quad 1 \leq i \leq k.
\]
The identities (5.48) therefore imply that
\[
|S_1| = \cdots = |S_k|.
\] (5.49)

Figure 46: The set \( \bigsqcup_{i=1}^k S_i \) for \( k = 4 \).

**Lemma 5.18** We have
\[
\widehat{a}_0 \leq \frac{a}{kn} \leq u_1 < u_2 < \cdots < u_k \leq \frac{a}{\sqrt{k}}.
\] (5.50)

**Proof.** The first inequality in (5.50) is equivalent to our assumption \( a_0 \leq a \). The inclusion \( T^n (a, \pi) \subset \bigsqcup S_i \) and the identities (5.49) yield
\[
\frac{a \pi^{n-1}}{n!} = |T^n (a, \pi)| \leq \bigsqcup S_i = k |S_i| = k \frac{u_1 \pi^{n-1}}{(n-1)!}
\]
and so \( \frac{a}{kn} \leq u_1 \). The inequalities \( u_1 < u_2 < \cdots < u_k \) follow from the identities (5.48). Finally, the identities (5.49) and the inclusion \( \bigsqcup S_i \subset S_a \) yield
\[
k u_k \frac{\pi^{n-1}}{(n-1)!} = k |S_k| = \bigsqcup S_i \leq |S_a| = \frac{a \pi^{n-1}}{(n-1)!}
\]
and so \( u_k \leq \frac{a}{\sqrt{k}} \). \( \Box \)

We now define the real number \( \lambda = \lambda (a) \) by
\[
\frac{|\lambda S_1|}{|T_1|} = q.
\] (5.51)
Lemma 5.19 \[ \lambda \prod_{i=1}^{k} S_i \setminus \lambda T^n(a, \pi) \leq \epsilon'. \]

Proof. In view of Lemma 5.18 we have \( u_1 < u_2 < \cdots < u_k \), and so

\[ \prod_{i=1}^{k} S_i \subset T^n(a + u_k, \pi + \frac{\pi}{a} u_k), \quad (5.52) \]

cf. Figure 47.

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure47.png}
\caption{T^n(a, \pi) \subset \prod_{i=1}^{k} S_i \subset T^n(a + u_k, \pi + \frac{\pi}{a} u_k).}
\end{figure}

Since the embedding \( \vartheta : \bigsqcup T_i \hookrightarrow M \) is symplectic, we have

\[ \mu(M) \geq \mu(\vartheta(\bigsqcup T_i)) = \sum |T_i| = k \frac{1}{n!} s^{2n}. \quad (5.53) \]

In view of the inclusion (5.52), the last inequality in (5.50) and the estimates (5.44) and (5.53) we can now estimate

\[ \prod_{i=1}^{k} S_i \setminus T^n(a, \pi) \leq \left| T^n(a + u_k, \pi + \frac{\pi}{a} u_k) \setminus T^n(a, \pi) \right| \]
\[ = \frac{\pi^{n-1}}{n!} \left( (a + u_k) \left( 1 + \frac{u_k}{a} \right)^{n-1} - a \right) \]
\[ = \frac{\pi^{n-1}}{n!} \left( \left( 1 + \frac{u_k}{a} \right)^{n} - 1 \right) a \]
\[ \leq \frac{\pi^{n-1}}{n!} \left( 1 - \frac{1}{\frac{1}{k} s^{2n}} \right) a \]
\[ \leq \pi^{n-1} \frac{1}{n! \mu(M)} \epsilon' a \]
\[ \leq \pi^{n-1} \frac{1}{k s^{2n}} \epsilon' a. \]
5.2. Proof of $\lim_{a \to \infty} p^E_a(M, \omega) = 1$

Using the definitions (5.51) and (5.45) and the second inequality in (5.50) we find that

$$\lambda^{2n} = q \frac{|T_1|}{|S_1|} < \frac{s^{2n}}{(n-1)!} \leq \frac{k s^{2n}}{a \pi^{n-1}}.$$  

We conclude that

$$\left| \lambda \bigsqcup S_i \setminus \lambda T^n(a, \pi) \right| = \lambda^{2n} \left| \bigsqcup S_i \setminus T^n(a, \pi) \right| < \epsilon'$$

and so the proof of Lemma 5.19 is complete. \hfill \Box

Step 5. Embedding $\lambda \bigsqcup S_i$ into $\bigsqcup T_i(\delta) \cup \bigsqcup N_i(\delta)$

Proposition 5.20 There exists a symplectic embedding

$$\Psi_\lambda: \lambda \bigsqcup S_i \hookrightarrow \bigsqcup T_i(\delta) \cup \bigsqcup N_i(\delta).$$

Proof. We start with introducing several geometric quantities and with replacing the sets $\lambda S_i$ by more convenient sets.

5.A. Preliminaries

For $i = 1, \ldots, k$ we set

$$h_i = 1 - \frac{u_i}{u_i + 1}.$$  

Notice that $h_i \pi$ is the $x$-width of $S_i$ and that

$$1 = h_1 > h_2 > \ldots > h_k.$$  

We define $a_i$ by

$$a_i = \frac{u_i}{h_i}.$$  

Then $a_1 = u_1$, and in view of Lemma 5.18 we have $a_i \geq u_i \geq u_1 > \hat{a}_0$. We denote the natural number $l(a_i)$ associated with $a_i$ in Proposition 4.6 by

$$l_i = l(a_i).$$  

Since $a_i > \hat{a}_0$ the estimate (5.47) shows that

$$l_i \geq \pi \frac{\hat{\xi}}{l_i}.$$  

We finally define $\lambda_i$ by

$$\lambda_i = \frac{\hat{\xi}}{l_i},$$  

(5.55)
Lemma 5.21  \( \lambda_i > \sqrt{h_i} \lambda \) and \( \delta > \sqrt{h_i} \pi \lambda \).

Proof. Since \( |T_i| = \frac{s_2^2}{n!} \) and \( |T^n (i_1^2)| = \frac{t_2^2}{n!} \) we have
\[
|T_i| = \lambda_i^{2n} \left| T^n \left( i_1^2 \right) \right|,
\]
and since \( a_i \geq \hat{a}_0 \), the estimate (5.46) shows that
\[
\frac{|S_{a_i}|}{|T^n (i_1^2)|} > q.
\]
Together with the definition (5.51) of \( \lambda \) and the identities (5.49) we can now estimate
\[
\frac{h_i^{-n} |S_{i}|}{\lambda_i^{2n} |T_{i}|} = \frac{|S_{a_i}|}{|T^n (i_1^2)|} > q = \frac{|\lambda S_{i}|}{|T_{i}|} = \frac{\lambda^{2n} |S_{i}|}{|T_{i}|},
\]
and so the first statement in Lemma 5.21 follows. The second statement follows from the first one, from the definition of \( \lambda_i \) and from the estimate (5.54).

We define the linear symplectomorphism \( \beta \) of \( \mathbb{R}^2 \) by
\[
\beta(u, v) = \left( \frac{2}{\lambda} u, \frac{2}{\lambda} v \right).
\]
Moreover, we set \( \tilde{u}_i = \frac{2}{\lambda} u_i \) and \( \tilde{v}_i = \tilde{u}_1 + \cdots + \tilde{u}_i \), and we abbreviate
\[
\tilde{S}_i := (\beta \times i d_{2n-2}) (\lambda S_i), \quad 1 \leq i \leq k.
\]
Then
\[
\tilde{S}_1 = [0, \tilde{v}_1] \times [0, s] \times \Delta^{n-1} (\pi \lambda) \times \square^{n-1 (\lambda)},
\]
\[
\tilde{S}_i = [\tilde{v}_{i-1}, \tilde{v}_i] \times [0, s] \times \Delta^{n-1} (h_i \pi \lambda) \times \square^{n-1 (\lambda)}, \quad 2 \leq i \leq k.
\]
We shall next embed \( \tilde{S}_1 \) into \( T_1 \) and shall then successively embed \( \tilde{S}_i \) into \( N_{i-1}(\delta) \cup T_i(\delta), \ i = 2, \ldots, k \).

5.B. The embedding \( \psi_1 : \tilde{S}_1 \hookrightarrow T_1 \)

We recall that \( a_1 = u_1 \) and \( s = \lambda_1 l_1 \), and we read off from the first statement in Lemma 5.21 that \( \lambda_1 > \lambda \). Therefore,
\[
t_1 := \frac{h_1}{l_1} a_1 - \tilde{u}_1 > 0.
\]
and so the affine symplectomorphism $\beta_1$ of $\mathbb{R}^2$ defined by

$$\beta_1(u, v) = \left(l_1(u + t_1), \frac{1}{l_1} v\right)$$

embeds $p_{u,v}(\tilde{S}_1)$ into $]0, \lambda_1 a_1[ \times ]0, \lambda_1[$. Using once more that $\lambda_1 > \lambda$ we conclude that

$$\left(\lambda_1^{-1} \circ (\beta_1 \times id_{2n-2})\right)(\tilde{S}_1) \subset S_{a_1}.$$

We define the linear symplectomorphism $\sigma_1$ of $\mathbb{R}^{2n}$ by

$$\sigma_1(u, v, x, y) = \left(\frac{1}{l_1} u, l_1 v, \frac{1}{l_1} x, l_1 y\right).$$

Composing the affine embedding $\lambda_1^{-1} \circ (\beta_1 \times id_{2n-2}) : \tilde{S}_1 \hookrightarrow S_{a_1}$ with the symplectic embedding

$$\varphi_{a_1} : S_{a_1} \hookrightarrow T^n(l_1^2)$$

guaranteed by Proposition 4.6 and with the linear diffeomorphism

$$\sigma_1 \circ \lambda_1 : T^n(l_1^2) \rightarrow T_1$$

we obtain the symplectic embedding

$$\psi_1 := \sigma_1 \circ \lambda_1 \circ \varphi_{a_1} \circ \lambda_1^{-1} \circ (\beta_1 \times id_{2n-2}) : \tilde{S}_1 \hookrightarrow T_1.$$

We abbreviate $s_1 = s - \lambda_1$. Using Proposition 4.6 (i) and the definitions of $\beta_1$, $\sigma_1$ and $\lambda_1$ we find that

$$\psi_1(u, v, x, y) = (u - \tilde{v}_1 + s_1, v, \frac{1}{\pi} x, \pi y) \quad \text{if} \quad u > \tilde{v}_1 - \frac{\lambda_1}{l_1} \pi.$$

**5.C. The embedding**

$$\psi_j : \bigsqcup_{i=1}^j \tilde{S}_i \hookrightarrow \bigsqcup_{i=1}^j T_1(\delta) \cup \bigsqcup_{i=1}^{j-1} N_i(\delta), \quad j = 2, \ldots, k$$

For $i = 1, \ldots, k$ we abbreviate

$$s_i = (2i - 1)s - \lambda_i, \quad d_i = \min\left\{\frac{\tilde{u}_i}{2}, \delta\right\}, \quad \varepsilon_i = \min\{\tilde{u}_i - d_i, \frac{\lambda_i}{l_i} \pi\}. \quad (5.57)$$

Proceeding by induction we fix $j \in \{1, \ldots, k - 1\}$ and assume that we have already constructed a symplectic embedding

$$\psi_j : \bigsqcup_{i=1}^j \tilde{S}_i \hookrightarrow \bigsqcup_{i=1}^j T_1(\delta) \cup \bigsqcup_{i=1}^{j-1} N_i(\delta) \quad (5.58)$$
which is such that \( \text{Im} \psi_j \subset \{(u, v, x, y) \mid u < s_j\} \) and
\[
\psi_j(u, v, x, y) = \left(u - \tilde{v}_j + s_j, v, \frac{1}{\sqrt{h_j} \pi} x, \sqrt{h_j} \pi y\right)
\]
if \( u > \tilde{v}_j - \varepsilon_j \),

cf. Figure 48. We are going to construct a symplectic extension
\[
\psi_{j+1} : \bigcup_{i=1}^{j+1} \mathcal{S}_i \hookrightarrow \bigcup_{i=1}^{j+1} T_i(\delta) \cup \bigcup_{i=1}^{j} N_i(\delta)
\]
of \( \psi_j \) which is such that \( \text{Im} \psi_{j+1} \subset \{(u, v, x, y) \mid u < s_{j+1}\} \) and
\[
\psi_{j+1}(u, v, x, y) = \left(u - \tilde{v}_{j+1} + s_{j+1}, v, \frac{1}{\sqrt{h_{j+1}} \pi} x, \sqrt{h_{j+1}} \pi y\right)
\]
if \( u > \tilde{v}_{j+1} - \varepsilon_{j+1} \).

We extend \( \psi_j \) by formula (5.59) to the smooth symplectic embedding
\[
\tilde{\psi}_j : \bigcup_{i=1}^{j+1} \mathcal{S}_i \hookrightarrow \mathbb{R}^{2n},
\]
and we denote by \( F'_j \) the end
\[
F'_j = \tilde{\psi}_j(\mathcal{S}_{j+1}) = \{(u, v, x, y) \in \text{Im} \tilde{\psi}_j \mid s_j \leq u < s_j + \tilde{u}_{j+1}\},
\]

cf. Figure 48.

5.C.1. The map \( \beta_j \)
We recall that \( u_{j+1} = h_{j+1} a_{j+1} \) and \( s = \lambda_{j+1} l_{j+1} \), and we read off from the first statement in Lemma 5.21 that \( \lambda_{j+1} > \sqrt{h_{j+1}} \lambda \). Therefore,
\[
t_{j+1} := \frac{\lambda_{j+1}}{l_{j+1}} a_{j+1} - \tilde{u}_{j+1} > 0.
\]

As in Step 1 of the folding construction described in Section 2.2 we find a symplectic embedding
\[
\beta_j : [s_j, s_j + \tilde{u}_{j+1} \times 0, s] \hookrightarrow [s_j, 2 s + \frac{\lambda_{j+1}}{l_{j+1}} a_{j+1} \times 0, s]
\]
5.2. Proof of \( \lim_{a \to \infty} p^E_a(M, \omega) = 1 \)

which restricts to the identity on \( \{ (u, v) \mid s_j \leq u \leq s_j + \frac{d_{j+1}}{2} \} \), restricts to the translation

\[
(u, v) \mapsto (u + 2js - s_j + t_{j+1}, v) \text{ on } \{ (u, v) \mid u \geq s_j + d_{j+1} \},
\]

and is such that

\[
\text{Im} \beta_j \cap \{(2j - 1)s + \delta \leq u \leq 2js - \delta\} \subset \{0 < v < \delta\}, \quad (5.64)
\]

\[
\text{Im} \beta_j \cap \{2js - \delta + \frac{\delta}{2n} \leq u \leq 2js - \frac{\delta}{2} \} \subset \{s - \delta < v < s\}, \quad (5.65)
\]

\[
\text{Im} \beta_j \cap \{2js - \frac{\delta}{2} + \frac{\delta}{2n} \leq u \leq 2js\} \subset \{0 < v < \delta\}, \quad (5.66)
\]

cf. Figure 49.

Lemma 5.22 We have

\[
(\beta_j \times id_{2n-2})(F'_j) \cap \{ (u, v, x, y) \mid s_j \leq u \leq 2js - \delta \} \subset T_j(\delta) \cup N_j(\delta).
\]

Proof. In view of the inclusion (5.64) we have

\[
p_{u,v}( (\beta_j \times id_{2n-2})(F'_j) \cap \{ s_j \leq u \leq 2js - \delta \} ) \subset p_{u,v}( T_j(\delta) \cup N_j(\delta)).
\]

Moreover, the formula (5.59) implies that the fibers of \( F'_j \) are equal to

\[
\left( h_{j+1} \frac{1}{\sqrt{h_j}} \pi^{-1} \Delta^{n-1}(\pi) \right) \times \sqrt{h_j} \pi \lambda \Box^{n-1}(1) = \Delta^{n-1}(w_j) \times \Box^{n-1} \left( \sqrt{h_j} \pi \lambda \right)
\]
5. Proof of Theorem 3

\begin{equation}
S_{j} (2j - l) = 8s + 2js - t_{j+1} + t_{j+1}
\end{equation}

Figure 49: The map $\beta_{j}$.

where we abbreviated

\begin{equation}
W_{j} = \frac{1}{\sqrt{h_{j}}}
\end{equation}

In view of the inequality $h_{j+1} < h_{j}$ and the second estimate in Lemma 5.21 these fibers are contained in the fibers of $N_{j}(\delta)$. Lemma 5.22 thus follows in view of Lemma 5.15.

\textbf{5.C.2. Rescaling the fibers}

We define $v_{j}$ by

\begin{equation}
v_{j} = (l_{j+1} - 1)\sqrt{\frac{h_{j}}{h_{j+1}}}
\end{equation}

In view of the inequalities (5.54) we have $l_{j+1} > 3$, and so $v_{j} > 2$. We abbreviate

\begin{equation}
e_{j} = \frac{1}{v_{j}}\sqrt{h_{j} \pi \lambda}
\end{equation}

and we define

\[ C'_j = \{ (y_2, \ldots, y_n) \mid 0 < y_i < \sqrt{h_j \pi}, \quad i = 2, \ldots, n \}, \]
\[ C''_j = \{ (y_2, \ldots, y_n) \mid -\sqrt{h_j \pi} - 2e_j < y_i < -2e_j, \quad i = 2, \ldots, n \}, \]
\[ C'''_j = \{ (y_2, \ldots, y_n) \mid -2e_j < y_i < -e_j, \quad i = 2, \ldots, n \}, \]
\[ C_{j+1} = \{ (y_2, \ldots, y_n) \mid 0 < y_i < e_j, \quad i = 2, \ldots, n \}. \]

cf. Figure 50.

Our next goal is to rescale the fibers \( \Delta^{n-1}(w_j) \times C'_j \) of \((\beta_j \times id_{2n-2})(F'_j)\) over \(\{u > 2js - \delta\}\) to the fibers \(\Delta^{n-1}(v_jw_j) \times C_{j+1}\). We shall use the same method as in Step 3 of the proof of Proposition 4.6 and first lower the fibers \(\Delta^{n-1}(w_j) \times C'_j\) at \(u = 2js - \delta + \frac{\delta}{2n}\) to the fibers \(\Delta^{n-1}(w_j) \times C''_j\), then deform these fibers to the fibers \(\Delta^{n-1}(v_jw_j) \times C'''_j\), and finally lift these fibers at \(u = 2js - \frac{\delta}{2} + \frac{\delta}{2n}\) to the fibers \(\Delta^{n-1}(v_jw_j) \times C_{j+1}\).
The lowering map $\varphi_j$  

Using the definition of $e_j$ and the inequality $v_j > 2$ we estimate

$$2e_j + \sqrt{h_j \pi \lambda} \leq 2e_j v_j,$$  \hspace{1cm} (5.70)

and using the definitions of $e_j$ and $w_j$, the second statement in Lemma 5.21 and $\delta < d \leq \frac{\ell}{2n}$ we estimate

$$2e_j v_j w_j = 2h_j + 1 \pi \lambda^2 < (\sqrt{h_j \pi \lambda})^2 < \delta^2 < \frac{\delta}{2n}s.$$  \hspace{1cm} (5.71)

Combining the estimates (5.70) and (5.71) we find that

$$(2e_j + \sqrt{h_j \pi \lambda}) w_j < \frac{\delta}{2n}s.$$  

For $i = 2, \ldots, n$ we therefore find a cut off function $c_i^- : \mathbb{R} \to \left[0, \frac{s}{w_j}\right]$ with support $\left[2js - \delta + (i - 1)\frac{\delta}{2n}, 2js - \delta + i\frac{\delta}{2n}\right]$ and such that

$$\int_0^\infty c_i^- (t) \, dt = 2e_j + \sqrt{h_j \pi \lambda}.$$  

The symplectic embedding

$$\varphi_j^- : \text{Im} \beta_j \times \Delta^{n-1} (w_j) \times C'_j \hookrightarrow \mathbb{R}^{2n}, \quad (u, v, x, y) \mapsto (u', v', x', y')$$  

defined by

$$u' = u, \quad v' = v - \sum_{i=2}^n c_i^- (u)x_i, \quad x'_i = x_i, \quad y'_i = y_i - \int_0^u c_i^- (t) \, dt, \quad i = 2, \ldots, n,$$

maps the fibers $\Delta^{n-1} (w_j) \times C'_j$ over the base $\{(u, v) \mid u > 2js - \delta\}$ to the fibers $\Delta^{n-1} (w_j) \times C''_j$.

The deformation $\alpha_j$  

The deformation of the fibers $\Delta^{n-1} (w_j) \times C''_j$ to the fibers $\Delta^{n-1} (v_j w_j) \times C''_j$ is based on the following lemma.

**Lemma 5.23** There exists a symplectic embedding

$$\alpha : \left[0, w_j \left[\times \right] - 2e_j - \sqrt{h_j \pi \lambda}, \infty\left[\mapsto \mathbb{R}^2. \right.$$
5.2. Proof of \( \lim_{a \to \infty} p^E_a(M, \omega) = 1 \) which restricts to the identity on \( \{(x, y) \mid y \geq 0\} \), restricts to the affine map
\[
(x, y) \mapsto \left( v_j x, \frac{1}{v_j} y + \frac{1}{v_j} 2e_j - e_j \right)
\]
on \( \{(x, y) \mid y \leq -2e_j\} \), and is such that
\[
x'(\alpha(x, y)) \leq v_j x \quad \text{and} \quad y'(\alpha(x, y)) > -\delta
\]
for all \( (x, y) \in \left[0, w_j \right] \times [-2e_j - \sqrt{h_j} \pi \lambda, \infty\right[ \), cf. Figure 51.

Figure 51: The map \( \alpha \).

Proof. Choose a smooth function \( f : \mathbb{R} \to [1, v_j] \) such that

(i) \( f(w) = v_j \) for \( w \leq -2e_j \),

(ii) \( f(w) = 1 \) for \( w \geq 0 \).

Since \( v_j > 2 \) we have
\[
2e_j \frac{1}{v_j} < e_j < 2e_j.
\]
We may therefore further require that

(iii) \( \int_{-2e_j}^{0} \frac{1}{f(w)} \, dw = e_j \).

Using (i) and (iii), the definition (5.69) of \( e_j \), the inequality \( v_j > 2 \) and the second inequality in Lemma 5.21 we can estimate
\[
\int_{-2e_j - \sqrt{h_j} \pi \lambda}^{0} \frac{1}{f(w)} \, dw = \frac{1}{v_j} \sqrt{h_j} \pi \lambda + e_j = \frac{2}{v_j} \sqrt{h_j} \pi \lambda < \delta.
\]
We conclude that the map 
\[ \alpha: ]0, w_j[ \times ]-2e_j - \sqrt{h_j} \pi \lambda, \infty[ \to \mathbb{R}^2, \quad (x, y) \mapsto \left( f(y)x, \int_0^y \frac{1}{f(w)} \, dw \right) \]
is a symplectic embedding which is as required. \[ \square \]

In view of (5.72) the symplectic embedding
\[ \alpha_j: (\varphi_j^+ \circ (\beta_j \times id_{2n-2})) \left( F_j' \right) \hookrightarrow \mathbb{R}^{2n} \]
defined by
\[ \alpha_j(u, v, x_2, y_2, \ldots, x_n, y_n) = (u, v, \alpha(x_2, y_2), \ldots, \alpha(x_n, y_n)) \]
maps the fibers \( \triangle^{n-1}(w_j) \times C''_j \) over the base \( \{(u, v) | u > 2js - \frac{\delta}{2}\} \) to the fibers \( \triangle^{n-1}(v_j w_j) \times C''_j. \)

The lifting \( \varphi_j^+ \)
In view of the estimate (5.71) we find for \( i = 2, \ldots, n \) a cut off function \( c_i^+: \mathbb{R} \to \left[ 0, \frac{\delta}{v_j w_j} \right] \) with support \( [2js - \frac{\delta}{2} + (i - 1)\frac{\delta}{2n}, 2js - \frac{\delta}{2} + i\frac{\delta}{2n}] \) and such that
\[ \int_0^{\infty} c_i^+(t) \, dt = 2e_j. \]
The symplectic embedding
\[ \varphi_j^+: (\alpha_j \circ \varphi_j^- \circ (\beta_j \times id_{2n-2})) \left( F_j' \right) \hookrightarrow \mathbb{R}^{2n}, \quad (u, v, x, y) \mapsto (u', v', x', y') \]
defined by
\[ u' = u, \quad v' = v + \sum_{i=2}^{n} c_i^+(u)x_i, \quad x'_i = x_i, \quad y'_i = y_i + \int_0^{u} c_i^+(t) \, dt, \quad i = 2, \ldots, n, \]
maps the fibers \( \triangle^{n-1}(v_j w_j) \times C''_j \) over the base \( \{(u, v) | u > 2js\} \) to the fibers \( \triangle^{n-1}(v_j w_j) \times C_{j+1}. \)

We abbreviate the composition
\[ \phi_j = \varphi_j^+ \circ \alpha_j \circ \varphi_j^- \circ (\beta_j \times id_{2n-2}) : F_j' \hookrightarrow \mathbb{R}^{2n}. \]
Lemma 5.24 We have

\[
\phi_j \left( F'_j \right) \cap \{(u, v, x, y) \mid s_j \leq u \leq 2js\} \subset T_j(\delta) \cup N_j(\delta) \cup T_{j+1}(\delta).
\]

Proof. We fix \((u, v, x, y) \in (\beta_j \times id_{2n-2}) \left( F'_j \right) \cap \{s_j \leq u \leq 2js\}\), and we set

\[
(u', v', x', y') = (\varphi_j^+ \circ \alpha_j \circ \varphi_j^-)(u, v, x, y).
\]

Then \(u' = u\).

Assume first that \(m < 2js - \delta\). Then \((u', v', x', y') = (u, v, x, y)\), and so Lemma 5.22 shows that

\[
(u', v', x', y') \in T_j(\delta) \cup N_j(\delta).
\]

Assume now that \(2js - \delta < u \leq 2js\). We read off from the inclusions (5.65) and (5.66) and from the definitions of \(\varphi_j^+\), \(\alpha_j\) and \(\varphi_j^-\) that

\[
v' \in [-\delta, s + \delta[.
\]

cf. Figure 52.

![Figure 52: The set \(\phi_j \left( F'_j \right) \cap \{2js - \delta < u \leq 2js\}\) for \(n = 3\).

Moreover, we have \(x \in \Delta^{n-1}(w_j)\), and so the first inequality in (5.73) implies that \(x' \in \Delta^{n-1}(v_j w_j)\). Using the definitions (5.68) and (5.67) of \(v_j\) and \(w_j\), the first estimate in Lemma 5.21 and the definition (5.55) of \(\lambda_{j+1}\) we compute

\[
v_j w_j = (l_{j+1} - 1)\sqrt{\frac{h_j}{h_{j+1}}} \frac{h_{j+1}}{\sqrt{h_j}} \lambda = (l_{j+1} - 1)\sqrt{h_{j+1}\lambda} < l_{j+1}\lambda_{j+1} = s,
\]
and so
\[ x' \in \Delta^{n-1}(s). \]
Finally, we have \( y \in C'_j = \mathbb{R}^{n-1}(\sqrt{h_j}\pi\lambda) \), and so the definitions of \( \varphi^-_j \) and \( \varphi^+_j \) and the second inequality in (5.73) imply that \( y' \in ]-\delta, \sqrt{h_j}\pi\lambda[^{-1} \). Using the second estimate in Lemma 5.21 we find \( \sqrt{h_j}\pi\lambda < \delta < s \), and so
\[ y' \in ]-\delta, s[^{-1}. \]
We conclude that
\[ (u', v', x', y') \in ]2js - \delta, 2js[ \times ]-\delta, s + \delta[ \times \Delta^{n-1}(s) \times ]-\delta, s[^{-1} \subset T_{j+1}(\delta). \]
The proof of Lemma 5.24 is thus complete. \( \Box \)

Recall that \( \beta_j \) is the identity on \( \{(u, v) \mid s_j \leq u \leq s_j + \frac{d_{j+1}}{2} \} \). We can therefore extend the symplectic embedding \( \phi_j : F'_j = \tilde{\psi}_j(S_{j+1}) \hookrightarrow \mathbb{R}^{2n} \) by the identity to the symplectic embedding
\[
\tilde{\phi}_j : \tilde{\psi}_j \left( \bigcup_{i=1}^{j+1} \tilde{S}_i \right) \hookrightarrow \mathbb{R}^{2n}, \quad \tilde{\phi}_j = \begin{cases} id & \text{on } \tilde{\psi}_j \left( \bigcup_{i=1}^{j} \tilde{S}_i \right), \\ \phi_j & \text{on } \tilde{\psi}_j (S_{j+1}). \end{cases}
\]
In view of the formulae (5.59) and (5.63) and the definition (5.68) of \( v_j \) we find
\[
(\tilde{\phi}_j \circ \tilde{\psi}_j) (u, v, x, y) =
\left( u - \tilde{v}_j + 2js + t_{j+1}, v, (l_{j+1} - 1) \frac{1}{\sqrt{h_{j+1}}\pi}, \frac{1}{\sqrt{h_{j+1}}} \right) \quad \text{if } u \geq \tilde{v}_j + d_{j+1}. \quad (5.74)
\]

5.C.3. End of the construction of \( \psi_{j+1} \)

We denote by \( F_{j+1} \) the set
\[
F_{j+1} = \left[ 2js, 2js + \frac{\lambda_{j+1}}{l_{j+1}} a_{j+1} \right] \times ]0, s[ \times \Delta^{n-1}(v_j w_j) \times C_{j+1}. \]
In view of the construction of \( \phi_j \) we have that
\[
\phi_j \left( F'_j \right) \cap \{(u, v, x, y) \mid u > 2js \} \subset F_{j+1}
\]
and that the right face of \( \phi_j(F_j') \) is equal to the right face of \( F_{j+1} \). We are going to embed \( F_{j+1} \) into \( T_{j+1} \). We denote by \( \tau_j \) the translation \( (u, v, x, y) \mapsto (u + 2js, v, x, y) \). Moreover, we define the linear symplectomorphisms \( \sigma_{j+1} \) and \( \xi_{j+1} \) of \( \mathbb{R}^{2n} \) by
\[
\sigma_{j+1}(u, v, x, y) = \left( \frac{1}{l_{j+1}} u, l_{j+1} v, \frac{1}{l_{j+1}} x, l_{j+1} y \right)
\]
and
\[
\xi_{j+1}(u, v, x, y) = \left( u, v, \frac{\pi}{(l_{j+1}-1)|l_{j+1}|} x, \frac{(l_{j+1}-1)|l_{j+1}|}{|l_{j+1}|} y \right).
\]
Using the definitions (5.68), (5.67), (5.69) and (5.55) of \( v_j, w_j, e_j \) and \( \lambda_{j+1} \) and the first inequality in Lemma 5.21 we find that
\[
\left( \xi_{j+1} \circ \lambda_{j+1}^{-1} \circ \sigma_{j+1}^{-1} \circ \tau_j^{-1} \right) (F_{j+1})
\]
\[
= J_0, a_{j+1}, J_0, 1| \times \Delta^{n-1} \left( \frac{\sqrt{h_{j+1}+\lambda}}{\lambda_{j+1}} \right) \times \Delta_{n-1} \left( \frac{\sqrt{h_{j+1}+\lambda}}{\lambda_{j+1}} \right)
\]
\[
\subset J_0, a_{j+1}, J_0, 1| \times \Delta^{n-1} (\pi) \times \Delta_{n-1} (1)
\]
\[
= S_{a_{j+1}}.
\]
Composing the affine diffeomorphism \( \xi_{j+1} \circ \lambda_{j+1}^{-1} \circ \sigma_{j+1}^{-1} \circ \tau_j^{-1} \) with the symplectic embedding
\[
\varphi_{a_{j+1}} : S_{a_{j+1}} \hookrightarrow T^n (l_{j+1}^2)
\]
guaranteed by Proposition 4.6 and with the affine diffeomorphism
\[
\tau_j \circ \sigma_{j+1} \circ \lambda_{j+1} : T^n (l_{j+1}^2) \rightarrow T_{j+1}
\]
we obtain the symplectic embedding
\[
\rho_j := \tau_j \circ \sigma_{j+1} \circ \lambda_{j+1} \circ \varphi_{a_{j+1}} \circ \xi_{j+1} \circ \lambda_{j+1}^{-1} \circ \sigma_{j+1}^{-1} \circ \tau_j^{-1} : F_{j+1} \hookrightarrow T_{j+1}.
\]
Using Proposition 4.6 (i) we find
\[
\rho_j(u, v, x, y) = (u, v, x, y) \quad \text{if} \quad u < 2js + \frac{\lambda_{j+1}}{l_{j+1}} \pi \quad (5.75)
\]
and
\[
\rho_j(u, v, x, y) = \left( u + s - \lambda_{j+1} - \frac{\lambda_{j+1}}{l_{j+1}} a_{j+1}, v, \frac{1}{l_{j+1}-1} x, (l_{j+1}-1)y \right)
\]
\[
\quad \text{if} \quad u > 2js + \frac{\lambda_{j+1}}{l_{j+1}} (a_{j+1} - \pi). \quad (5.76)
\]
In view of the identity (5.75) we can extend the restriction of $\rho_j$ to the subset $\phi_j(F'_j) \cap \{(u, v, x, y) \mid u > 2js\}$ by the identity to the symplectic embedding

$$\tilde{\rho}_j : (\tilde{\phi}_j \circ \tilde{\psi}_j) \left( \bigcap_{i=1}^{j+1} \tilde{S}_i \right) \hookrightarrow \mathbb{R}^{2n}, \quad \tilde{\rho}_j = \begin{cases} id & \text{on } \tilde{\phi}_j \circ \tilde{\psi}_j \left( \bigcap_{i=1}^{j} \tilde{S}_i \right), \\ \rho_j & \text{on } \tilde{\phi}_j \circ \tilde{\psi}_j (\tilde{S}_{j+1}). \end{cases}$$

We finally define the symplectic embedding $\psi_{j+1}$ by

$$\psi_{j+1} := \tilde{\rho}_j \circ \tilde{\phi}_j \circ \tilde{\psi}_j : \bigcap_{i=1}^{j+1} \tilde{S}_i \hookrightarrow \mathbb{R}^{2n}.$$ 

In view of the induction hypothesis (5.58) and Lemma 5.24 and the inclusion $\rho_j(F_{j+1}) \subset T_{j+1}$ we have

$$\psi_{j+1} : \bigcap_{i=1}^{j+1} \tilde{S}_i \hookrightarrow \bigcap_{i=1}^{j+1} T_i(\delta) \cup \bigcup_{i=1}^{j} N_i(\delta),$$

i.e., (5.60) holds true. Moreover, we deduce from formulae (5.74) and (5.76), from $\tilde{v}_{j+1} = \tilde{v}_j + \tilde{u}_{j+1}$ and from the definitions of $s_{j+1}, t_{j+1}$ and $\varepsilon_{j+1}$ given in (5.57) and (5.62) that

$$\psi_{j+1}(u, v, x, y) = \begin{cases} u - \tilde{v}_{j+1} + s_{j+1}, v, & \frac{1}{\sqrt{h_{j+1}}} x, \sqrt{h_{j+1}} y \\ & \text{if } u > \tilde{v}_{j+1} - \varepsilon_{j+1}, \end{cases}$$

i.e., formula (5.61) holds true. This completes the inductive construction of the symplectic embedding $\psi_{j+1}$.

Composing the linear symplectomorphism

$$\beta \times id_{2n-2} : k \bigcap_{i=1}^{k} S_i \to k \bigcap_{i=1}^{k} \tilde{S}_i$$

given by formula (5.56) with the inductively constructed symplectic embedding

$$\psi_k : k \bigcap_{i=1}^{k} \tilde{S}_i \hookrightarrow k \bigcap_{i=1}^{k} T_i(\delta) \cup k \bigcup_{i=1}^{k-1} N_i(\delta)$$
5.2. Proof of $\lim_{a \to \infty} p_a^E(M, \omega) = 1$

we finally obtain the symplectic embedding

$$\Psi_a := \psi_k \circ (\beta \times id_{2n-2}) : \lambda \prod_{i=1}^{k} S_i \hookrightarrow \bigcup_{i=1}^{k} T_i(\delta) \cup \bigcup_{i=1}^{k-1} N_i(\delta).$$

The proof of Proposition 5.20 is complete. \(\square\)

**Step 6. End of the proof of Theorem 5.11**

We let $\epsilon > 0$ be as in Theorem 5.11, set $\epsilon' = \epsilon/5$, choose $k$ and $a_0 = a_0(\epsilon)$ as in Step 3 and let $s$ be the number associated with $k$ and $\epsilon'$ after the proof of Lemma 5.12. We fix $a \geq a_0$ and define $\lambda = \lambda(a)$ as in (5.51).

Using Lemma 5.19, the identities (5.49) and the definitions (5.51) and (5.45) we can estimate

$$|\lambda T^n(a, \pi)| > |\lambda \prod_{i=1}^{k} S_i| - \epsilon'$$

$$= k |\lambda S_1| - \epsilon'$$

$$= k q |T_1| - \epsilon'$$

$$= k \frac{1}{n} s^{2n} - 2 \epsilon'. \quad (5.77)$$

Composing the inclusion

$$\iota_a : \lambda T^n(a, \pi) \hookrightarrow \lambda \prod_{i=1}^{k} S_i$$

with the symplectic embeddings $\Psi_a$ and $\rho$ guaranteed by Proposition 5.20 and Proposition 5.16 we obtain the symplectic embedding

$$\Phi_a := \rho \circ \Psi_a \circ \iota_a : \lambda T^n(a, \pi) \hookrightarrow M.$$ 

In view of the estimates (5.42) and (5.77) we find

$$\mu \left( M \setminus \Phi_a \left( \lambda T^n(a, \pi) \right) \right) = \mu(M) - \mu \left( \Phi_a \left( \lambda T^n(a, \pi) \right) \right)$$

$$= \mu(M) - |\lambda T^n(a, \pi)|$$

$$< \left( k \frac{1}{n} s^{2n} + 3 \epsilon' \right) - \left( k \frac{1}{n} s^{2n} - 2 \epsilon' \right)$$

$$= 5 \epsilon'$$

$$= \epsilon.$$

This is the required estimate in Theorem 5.11 and so the proof of Theorem 5.11 is complete. \(\square\)
5.3 Asymptotic embedding invariants

Consider again a connected 2n-dimensional symplectic manifold \((M, \omega)\) of finite volume \(\text{Vol}(M, \omega)\). In view of Theorem 3 the asymptotic symplectic invariants

\[
\lim_{a \to \infty} p_a^E(M, \omega) = 1 \quad \text{and} \quad \lim_{a \to \infty} p_a^P(M, \omega) = 1 \quad (5.78)
\]

are uninteresting. In order to recapture some information on the geometry of \((M, \omega)\) one can try to study the convergence speeds in (5.78). We define the symplectic invariants \(\sigma_E(M, \omega)\) and \(\sigma_P(M, \omega)\) in \([0, \infty]\) by

\[
\sigma_E(M, \omega) = \sup \left\{ s \mid \text{there exists a constant } C < \infty \text{ such that } 1 - p_a^E(M, \omega) \leq C a^{-s} \text{ for all } a > \pi \right\},
\]

\[
\sigma_P(M, \omega) = \sup \left\{ s \mid \text{there exists a constant } C < \infty \text{ such that } 1 - p_a^P(M, \omega) \leq C a^{-s} \text{ for all } a > \pi \right\}.
\]

In this section we notice that \(\sigma_E(M, \omega) \geq \frac{1}{n}\) or \(\sigma_P(M, \omega) \geq \frac{1}{n}\) for large classes of 2n-dimensional symplectic manifolds and thereby improve Theorem 3 for many symplectic manifolds.

Consider a domain \(U\) in \(\mathbb{R}^{2n}\). The distance \(d(p, \partial U)\) between a point \(p \in U\) and the boundary \(\partial U\) is

\[
d(p, \partial U) = \inf \{d(p, q) \mid q \in \partial U\}.
\]

We say that \(U\) is very connected if there exists \(\epsilon > 0\) such that the set

\[U \setminus \{p \in U \mid d(p, \partial U) < \epsilon\}\]

is connected.

**Theorem 5.25** Assume that \(U\) is a very connected bounded domain in \(\mathbb{R}^{2n}\) with piecewise smooth boundary and that \((M, \omega)\) is a compact connected 2n-dimensional symplectic manifold.

(i) \(\sigma_E(U) \geq \frac{1}{n}\) if \(n \leq 3\) or if \(U\) is a ball.

(ii) \(\sigma_E(M, \omega) \geq \frac{1}{n}\) if \(n \leq 3\).

(i) \(\sigma_P(U) \geq \frac{1}{n}\).

(ii) \(\sigma_P(M, \omega) \geq \frac{1}{n}\).
Rough outline of the proof of Theorem 5.25: Assertion (i)\(_E\) for a 2n-ball follows from Corollary 6.6 (i) below which is proved by Lagrangian folding. The other assertions can be proved by the symplectic folding methods previously described.

Assume first that \(U\) is a 2n-cube. If \(n = 2\), assertion (i)\(_E\) follows from Proposition 3.16, and if \(n = 3\), assertion (i)\(_E\) can be proved by using that in dimension 2 a cube can be filled with small simplices. Assertion (i)\(_P\) for a cube follows from Proposition 4.1.

Assume next that \(U\) is a very connected bounded domain in \(\mathbb{R}^{2n}\) with piecewise smooth boundary. Then assertions (i)\(_E\) and (i)\(_P\) can be proved by first exhausting \(U\) with an increasing sequence \(U_1 \subset U_2 \subset \ldots\) of connected unions of equal cubes and then extending the embedding techniques used to fill a cube to the sets \(U_i\).

Assume finally that \((M, \omega)\) is a compact connected 2n-dimensional symplectic manifold. Then assertions (ii)\(_E\) and (ii)\(_P\) can be proved by first choosing finitely many Darboux charts \(\varphi_i : U_i \to V_i \subset M\) such that the \(U_i\) are very connected bounded domains with piecewise smooth boundary and such that the \(V_i\) are disjoint and \(\bigcup V_i = M\), then connecting the \(V_i\) by lines, and finally applying the technique used to prove (i)\(_E\) and (i)\(_P\) to the sets \(V_i = \varphi_i(U_i)\) and using thinner and thinner neighbourhoods of the lines to pass from one \(V_i\) to another. \(\square\)
5. Proof of Theorem 3
6 Symplectic versus Lagrangian folding

6.1 Lagrangian folding

There is a Lagrangian version of folding developed by Traynor in [34]. Here, the whole ellipsoid or the whole polydisc is viewed as a Lagrangian product of a cube and a simplex or of a cube and a cuboid, and folding is then simply achieved by wrapping the cube around the base of the cotangent bundle of the torus via a linear map. This version of folding has therefore a more algebraic flavour than symplectic folding.

For the sake of brevity we shall only study Lagrangian folding embeddings of skinny ellipsoids into balls and of skinny polydiscs into cubes.

Theorem 6.1 Assume that $a > 0$ and that $k_1 < \cdots < k_{n-1}$ are relatively prime numbers.

(i) The ellipsoid $E^{2n}(\pi, \ldots, \pi, a)$ symplectically embeds into the ball

$$B^{2n} \left( \max \left\{ (k_{n-1} + 1) \pi, \frac{a}{k_1 \cdots k_{n-1}} \right\} + \epsilon \right)$$

for any $\epsilon > 0$.

(ii) The polydisc $P^{2n}(\pi, \ldots, \pi, a)$ symplectically embeds into the cube

$$C^{2n} \left( \max \left\{ k_{n-1} \pi, (n-1)\pi + \frac{a}{k_1 \cdots k_{n-1}} \right\} \right).$$

Proof. Theorem 6.1 (i) for $n = 2$ is Theorem 6.4 in [34]. We shall closely follow [34]. We consider again the Lagrangian splitting $\mathbb{R}^n(x) \times \mathbb{R}^n(y)$ of $\mathbb{R}^{2n}$, set

$$\Box(a_1, \ldots, a_n) = \{ 0 < x_i < a_i, \ 1 \leq i \leq n \} \subset \mathbb{R}^n(x),$$

$$\Delta(b_1, \ldots, b_n) = \left\{ 0 < y_1, \ldots, y_n \left| \sum_{i=1}^{n} \frac{y_i}{b_i} < 1 \right. \right\} \subset \mathbb{R}^n(y),$$

and abbreviate $\Box^n(a) = \Box(a, \ldots, a)$ and $\Delta^n(b) = \Delta(b, \ldots, b)$. We also set

$$T^n = \mathbb{R}^n(x)/\pi \mathbb{Z}^n.$$
and abbreviate

\[ \kappa = k_1 \cdots k_{n-1} \]

and

\[ A_E = \max \left\{ (k_{n-1} + 1) \pi, \frac{a}{\kappa} \right\}, \]
\[ A_P = \max \left\{ k_{n-1} \pi, (n - 1) \pi + \frac{a}{\kappa} \right\}. \]

Choose \( \epsilon > 0 \). We set \( \epsilon' = \epsilon/A_E \). The embeddings provided by Lagrangian folding are compositions of symplectic embeddings

\[ E(\pi, \ldots, \pi, a) \xrightarrow{\alpha_E} \square^n(1) \times (1 + \epsilon') \Delta(\pi, \ldots, \pi, a) \]
\[ \xrightarrow{\beta} \square \left( \frac{\pi}{k_1}, \ldots, \frac{\pi}{k_{n-1}}, \kappa \pi \right) \times (1 + \epsilon') \Delta \left( k_1, \ldots, k_{n-1}, \frac{a}{\kappa \pi} \right) \]
\[ \xrightarrow{\gamma} T^n \times \triangle^n \left( \frac{A_E + \epsilon}{\pi} \right) \]
\[ \xrightarrow{\delta_E} B^{2n} \left( A_E + \epsilon \right) \]

respectively

\[ P(\pi, \ldots, \pi, a) \xrightarrow{\alpha_P} \square^n(1) \times \square(\pi, \ldots, \pi, a) \]
\[ \xrightarrow{\beta} \square \left( \frac{\pi}{k_1}, \ldots, \frac{\pi}{k_{n-1}}, \kappa \pi \right) \times \square \left( k_1, \ldots, k_{n-1}, \frac{a}{\kappa \pi} \right) \]
\[ \xrightarrow{\gamma} T^n \times \square^n \left( \frac{A_P}{\pi} \right) \]
\[ \xrightarrow{\delta_P} C^{2n} \left( A_P \right). \]

1. The maps \( \alpha_E \) and \( \alpha_P \). Recall from the proof of Lemma 4.4 (i) that there exists a symplectomorphism

\[ \alpha_1 \times \cdots \times \alpha_n : P(\pi, \ldots, \pi, a) \rightarrow \square(\pi, \ldots, \pi, a) \times \square^n(1) \]

which embeds the subset \( E(\pi, \ldots, \pi, a) \) into \( ((1 + \epsilon') \Delta(\pi, \ldots, \pi, a)) \times \square^n(1) \). We denote the reflection \( (x, y) \mapsto (x, -y) \) of \( \mathbb{R}^n(x) \times \mathbb{R}^n(y) \) by \( \rho \) and the permutation \( (x, y) \mapsto (y, x) \) by \( \sigma \). The composition

\[ \alpha_P := \sigma \circ (\alpha_1 \times \cdots \times \alpha_n) \circ \rho \]

symplectomorphically maps \( P(\pi, \ldots, \pi, a) \) to \( \square^n(1) \times \square(\pi, \ldots, \pi, a) \), and its restriction to \( E(\pi, \ldots, \pi, a) \), which we denote by \( \alpha_E \), symplectically embeds \( E(\pi, \ldots, \pi, a) \) into \( \square^n(1) \times (1 + \epsilon') \Delta(\pi, \ldots, \pi, a) \).
6.1. Lagrangian folding

2. The map $\beta$. The map $\beta$ is the linear symplectomorphism of $\mathbb{R}^n(x) \times \mathbb{R}^n(y)$ given by the diagonal matrix

$$
\text{diag} \left[ \frac{\pi}{k_1}, \ldots, \frac{\pi}{k_{n-1}}, \kappa \pi, \frac{k_1}{\pi}, \ldots, \frac{k_{n-1}}{\pi}, \frac{1}{\kappa \pi} \right].
$$

3. The map $\gamma$. In order to describe the wrapping map $\gamma$ we need an elementary lemma.

**Lemma 6.2** Let $M: \mathbb{R}^n(x) \rightarrow \mathbb{R}^n(x)$ be the linear map given by the matrix

$$
\begin{pmatrix}
1 & -\frac{1}{k_1} \\
1 & 0 & -\frac{1}{k_2} \\
& & \ddots & \vdots \\
0 & 1 & & -\frac{1}{k_{n-1}} \\
& & & 1
\end{pmatrix}
$$

and let $p: \mathbb{R}^n(x) \rightarrow \mathbb{R}^n(x)/\pi \mathbb{Z}^n = T^n$ be the projection. The composition $p \circ M: \mathbb{R}^n(x) \rightarrow T^n$ embeds $\square \left( \frac{\pi}{k_1}, \ldots, \frac{\pi}{k_{n-1}}, \kappa \pi \right)$ into $T^n$.

**Proof.** Let $x$ and $x'$ be points in $\square \left( \frac{\pi}{k_1}, \ldots, \frac{\pi}{k_{n-1}}, \kappa \pi \right)$ for which $(p \circ M)(x) = (p \circ M)(x')$. Then

$$|x'_i - x_i| < \frac{\pi}{k_i}, \quad i = 1, \ldots, n - 1, \quad (6.1)$$

and

$$|x'_n - x_n| < \kappa \pi, \quad (6.2)$$

and there exist integers $l_1, \ldots, l_{n-1}$ such that

$$x'_i - \frac{x'_n}{k_i} = x_i - \frac{x_n}{k_i} + l_i \pi \quad (6.3)$$

and an integer $l_n$ such that

$$x'_n = x_n + l_n \pi. \quad (6.4)$$

Inserting the identity (6.4) into the estimate (6.2) we obtain

$$|l_n| < \kappa \quad (6.5)$$
and inserting (6.4) into the identities (6.3) we obtain

\[ x'_i - x_i = \left( \frac{l_k}{k_i} + l_i \right) \pi, \quad i = 1, \ldots, n - 1. \] (6.6)

The identities (6.6) and the estimates (6.1) yield

\[ \left| \frac{l_k}{k_i} + l_i \right| < \frac{1}{k_i}, \quad i = 1, \ldots, n - 1. \] (6.7)

Assume that \( l_n \neq 0 \). Then the estimates (6.7) imply that \( l_n + k_i l_i = 0 \) for \( i = 1, \ldots, n - 1 \). Since the numbers \( k_1, \ldots, k_{n-1} \) are relatively prime, we conclude that \( l_n = m k_1 \cdots k_{n-1} \) for some integer \( m \neq 0 \). In particular,

\[ |l_n| \geq k_1 \cdots k_{n-1} = \kappa \]

in contradiction to (6.5). Therefore \( l_n = 0 \). The estimates (6.7) now imply that also \( l_1 = \cdots = l_{n-1} = 0 \), and so \( x = x' \) in view of the identities (6.4) and (6.3).

The restriction of \( p \circ M \) to \( \left( \frac{\pi}{k_1}, \ldots, \frac{\pi}{k_{n-1}}, \kappa \pi \right) \) is therefore injective, as claimed. \( \Box \)

We denote by \( M^* \) the transpose of the inverse of \( M \). Then the map

\[ M \times M^*: \mathbb{R}^n(x) \times \mathbb{R}^n(y) \to \mathbb{R}^n(x) \times \mathbb{R}^n(y) \]

is a linear symplectomorphism. In view of Lemma 6.2 the map

\[ \gamma: \Box \left( \frac{\pi}{k_1}, \ldots, \frac{\pi}{k_{n-1}}, \kappa \pi \right) \times \mathbb{R}^n(y) \to T^n \times \mathbb{R}^n(y) \]

defined by \( \gamma = (p \circ M) \times M^* \) is a symplectic embedding.

**Lemma 6.3** We have

\[ \gamma \left( \Box \left( \frac{\pi}{k_1}, \ldots, \frac{\pi}{k_{n-1}}, \kappa \pi \right) \times (1 + \varepsilon') \Delta \left( k_1, \ldots, k_{n-1}, \frac{a}{k\pi} \right) \right) \subset T^n \times \Delta^n \left( \frac{A \pi + \epsilon}{\pi} \right) \]

and

\[ \gamma \left( \Box \left( \frac{\pi}{k_1}, \ldots, \frac{\pi}{k_{n-1}}, \kappa \pi \right) \times \Box \left( k_1, \ldots, k_{n-1}, \frac{a}{k\pi} \right) \right) \subset T^n \times \Box^n \left( \frac{A \pi}{\pi} \right). \]
Proof. In view of the definition of $\gamma$ and since $(1 + \epsilon') A_E = A_E + \epsilon$ it is enough to show that

$$M^* \left( \Delta \left( k_1, \ldots, k_{n-1}, \frac{a}{\kappa \pi} \right) \right) \subset \Delta^n \left( \frac{A_E}{\pi} \right)$$

and that

$$M^* \left( \Box \left( k_1, \ldots, k_{n-1}, \frac{a}{\kappa \pi} \right) \right) \subset \Box^n \left( \frac{A_E}{\pi} \right).$$

We compute that $M^*$ is given by the matrix

\[
\begin{pmatrix}
1 & 0 & \\
1 & 0 & \\
0 & 0 & 1 & \\
\frac{1}{k_1} & \frac{1}{k_2} & \cdots & \frac{1}{k_{n-1}} & 1
\end{pmatrix}
\]

We assume first that $\gamma \in \Delta \left( k_1, \ldots, k_{n-1}, \frac{a}{\kappa \pi} \right)$ and set $y' = M^* \gamma$. Using the definitions of $\Delta \left( k_1, \ldots, k_{n-1}, \frac{a}{\kappa \pi} \right)$ and $A_E$ we then find

\[
y_1 + \cdots + y_n = (k_1 + 1) \frac{y_1}{k_1} + \cdots + (k_{n-1} + 1) \frac{y_{n-1}}{k_{n-1}} + \frac{a}{k \pi} \frac{y_n}{k \pi} \leq \max \left\{ k_{n-1} + 1, \frac{a}{k \pi} \right\} = \frac{A_E}{\pi}.
\]

Therefore $y' \in \Delta^n \left( \frac{A_E}{\pi} \right)$ as claimed.

We assume next that $\gamma \in \Box \left( k_1, \ldots, k_{n-1}, \frac{a}{\kappa \pi} \right)$ and set $y' = M^* \gamma$. Using the definitions of $\Box \left( k_1, \ldots, k_{n-1}, \frac{a}{\kappa \pi} \right)$ and $A_P$ we then find

\[
y' \in \Box \left( k_1, \ldots, k_{n-1}, n - 1 + \frac{a}{k \pi} \right) \subset \Box^n \left( \frac{A_E}{\pi} \right)
\]

as claimed. $\square$

4. The maps $\delta_E$ and $\delta_P$. We define symplectic embeddings

\[
\tilde{\delta}_E : \Box^n (\pi) \times \Delta^n \left( \frac{A_E + \epsilon}{\pi} \right) \hookrightarrow B^{2n} (A_E + \epsilon)
\]
and
\[ \delta_P : \mathbb{B}^n(\pi) \times \mathbb{B}^n\left( \frac{A_P}{\pi} \right) \to C^{2n}(A_P) \]
by
\[(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto \left( \sqrt{y_1} \cos 2x_1, \ldots, \sqrt{y_n} \cos 2x_n, -\sqrt{y_1} \sin 2x_1, \ldots, -\sqrt{y_n} \sin 2x_n \right) .\]

The embedding \( \tilde{\delta}_E \) extends to a symplectic embedding
\[ \tilde{\delta}_E : T^n \times \Delta^n \left( \frac{A_E + \epsilon}{\pi} \right) \to B^{2n}(A_E + \epsilon) \]
and the embedding \( \tilde{\delta}_P \) extends to a symplectic embedding
\[ \tilde{\delta}_P : T^n \times \mathbb{B}^n\left( \frac{A_P}{\pi} \right) \to C^{2n}(A_P) .\]

This completes the construction of the symplectic embeddings involved in Lagrangian folding, and Theorem 6.1 is proved. \( \square \)

Assume that \( a > 0 \) and that \( k_1 < \cdots < k_{n-1} \) are relatively prime numbers. As in the proof of Theorem 6.1 we abbreviate
\[ A_E(a; k_1, \ldots, k_{n-1}) = \max \left\{ (k_{n-1} - 1) \pi, \frac{a}{k_1 \cdots k_{n-1}} \right\} , \]
\[ A_P(a; k_1, \ldots, k_{n-1}) = \max \left\{ k_{n-1} \pi, (n - 1)\pi + \frac{a}{k_1 \cdots k_{n-1}} \right\} . \]

In view of Theorem 6.1 we are interested in the functions \( l^{2n}_{EB} : ]2\pi, \infty[ \to \mathbb{R} \) and \( l^{2n}_{PC} : ]2\pi, \infty[ \to \mathbb{R} \) defined by
\[ l^{2n}_{EB}(a) := \inf \{ A_E(a; k_1, \ldots, k_{n-1}) \mid k_1 < \cdots < k_{n-1} \text{ are relatively prime} \} , \]
\[ l^{2n}_{PC}(a) := \inf \{ A_P(a; k_1, \ldots, k_{n-1}) \mid k_1 < \cdots < k_{n-1} \text{ are relatively prime} \} . \]

Notice that the infimum in the definitions of the functions \( l^{2n}_{EB} \) and \( l^{2n}_{PC} \) can be replaced by the minimum. We shall next explicitly compute the functions \( l^{2n}_{EB} \) and \( l^{2n}_{PC} \) for \( n = 2 \) and \( n = 3 \).
Corollary 6.4 Assume that $a > 2\pi$.

(i) The ellipsoid $E(\pi, a)$ symplectically embeds into the ball $B^4 \left( l^4_{EB}(a) + \epsilon \right)$ for any $\epsilon > 0$ where

$$l^4_{EB}(a) = \begin{cases} (k + 1)\pi & \text{if } (k - 1)(k + 1) < \frac{a}{\pi} \leq k(k + 1), \\
\frac{a}{k} & \text{if } k(k + 1) < \frac{a}{\pi} \leq k(k + 2). \end{cases}$$

(ii) The polydisc $P(\pi, a)$ symplectically embeds into the cube $C \left( l^4_{PC}(a) \right)$ where

$$l^4_{PC}(a) = \begin{cases} k\pi & \text{if } (k - 1)^2 < \frac{a}{\pi} \leq (k - 1)k, \\
\frac{a}{k} + \pi & \text{if } (k - 1)k < \frac{a}{\pi} \leq k^2. \end{cases}$$

Proof. (i) follows from Theorem 6.1 (i), from the identity

$$l^4_{EB}(a) = \min_{k \in \mathbb{N}} \max \left\{ (k + 1)\pi, \frac{a}{k} \right\}$$

and from a straightforward computation.

(ii) follows from Theorem 6.1 (ii), from the identity

$$l^4_{PC}(a) = \min_{k \in \mathbb{N}} \max \left\{ k\pi, \frac{a}{k} + \pi \right\}$$

and from a straightforward computation. \qed

Corollary 6.5 Assume that $a > 2\pi$.

(i) The ellipsoid $E(\pi, \pi, a)$ symplectically embeds into the ball $B^6 \left( l^6_{EB}(a) + \epsilon \right)$ for any $\epsilon > 0$ where

$$l^6_{EB}(a) = \begin{cases} (k + 1)\pi & \text{if } (k - 2)(k - 1)(k + 1) < \frac{a}{\pi} \leq (k - 1)k(k + 1), \\
\frac{a}{(k-1)k} & \text{if } (k - 1)k(k + 1) < \frac{a}{\pi} \leq (k - 1)k(k + 2). \end{cases}$$

(ii) The polydisc $P(\pi, \pi, a)$ symplectically embeds into the cube $C^6 \left( l^6_{PC}(a) \right)$ where

$$l^6_{PC}(a) = \begin{cases} (k + 1)\pi & \text{if } (k - 1)^2k < \frac{a}{\pi} \leq (k - 1)k(k + 1), \\
\frac{a}{k(k+1) + 2\pi} & \text{if } (k - 1)k(k + 1) < \frac{a}{\pi} \leq k^2(k + 1). \end{cases}$$
Proof. (i) Fix $a > 2\pi$ and assume that $k_1 < k_2$ are relatively prime numbers. Then $k_2 - 1$ and $k_2$ are also relatively prime, and

$$A_E(a; k_2 - 1, k_2) \leq A_E(a; k_1, k_2).$$

It follows that

$$l_{EB}^6(a) = \min_{k \in \mathbb{N}} \max \left\{ (k + 1)\pi, \frac{a}{(k-1)k} \right\}. \quad (6.8)$$

Assertion (i) follows from Theorem 6.1 (i), from the identity (6.8) and from a straightforward computation.

(ii) The same argument as above implies that for each $a > 2\pi$,

$$l_{PC}^6(a) = \min_{k \in \mathbb{N}} \max \left\{ k\pi, 2\pi + \frac{a}{(k-1)k} \right\} = \min_{k \in \mathbb{N}} \left\{ (k + 1)\pi, \frac{a}{k(k+1)} + 2\pi \right\}. \quad (6.9)$$

Assertion (ii) follows from Theorem 6.1 (ii), from the identity (6.9) and from a straightforward computation.

For $n \geq 4$ the computation of $l_{EB}^{2n}(a)$ and $l_{PC}^{2n}(a)$ is more involved, and we do not know explicit formulae for the functions $l_{EB}^{2n}$ and $l_{PC}^{2n}$. The next corollary describes the asymptotic behaviour of these functions for all $n \geq 2$.

Corollary 6.6

(i) There exists a constant $C_E$ depending only on $n$ such that

$$\frac{|E(\pi, \ldots, \pi, a)|}{|B^{2n}(l_{EB}^{2n}(a))|} \geq 1 - C_E a^{-1/n} \quad \text{for all } a > 2\pi.$$ 

(ii) There exists a constant $C_P$ depending only on $n$ such that

$$\frac{|P(\pi, \ldots, \pi, a)|}{|C^{2n}(l_{PC}^{2n}(a))|} \geq 1 - C_P a^{-1/n} \quad \text{for all } a > 2\pi.$$ 

Proof. (i) We choose $n - 1$ prime numbers $p_1 < p_2 < \cdots < p_{n-1}$ and define $l$ to be the least common multiple of the differences

$$p_j - p_i, \quad 1 \leq i < j \leq n - 1.$$ 

Fix $a > 2\pi$. We set

$$m := \left\lfloor \frac{1}{l} \left( \sqrt[n]{a} \frac{\pi}{k} + p_{n-1} \right) \right\rfloor \quad (6.10)$$

$$m := \left\lfloor \frac{1}{l} \left( \frac{\sqrt[n]{a}}{p_{n-1}} + k \right) \right\rfloor$$

$$m := \left\lfloor \frac{1}{l} \left( \sqrt[n]{a} + p_{n-1} \right) \right\rfloor \quad (6.10)$$
where \([r]\) denotes the minimal integer which is greater or equal to \(r\), and

\[ k_i := m_l - p_{n-i}, \quad i = 1, \ldots, n - 1. \quad (6.11) \]

We claim that the numbers \(k_1 < k_2 < \cdots < k_{n-1}\) are relatively prime. Indeed, assume that

\[ d \mid m_l - p_i \quad \text{and} \quad d \mid m_l - p_j \quad (6.12) \]

for some \(i \neq j\). Then \(d\) divides the difference

\[ (m_l - p_i) - (m_l - p_j) = p_j - p_i \]

and so \(d\) divides the least common multiple \(l\). But then (6.12) implies that \(d\) divides both \(p_i\) and \(p_j\), and so \(d = 1\) as claimed.

Using the definitions (6.11) and (6.10) we find

\[ f_a(\sum_{i=1}^{n} k_i = m_l - p_{n-1} \geq \sqrt[1\pi]{a}. \]

Therefore,

\[ (k_{n-1} + 1)k_{n-1}\cdots k_1 \geq k_1^n \geq \frac{a}{\pi}. \]

We conclude that

\[ (k_{n-1} + 1)\pi \geq \frac{a}{k_1 \cdots k_{n-1}} \]

and so

\[ (k_{n-1} + 1)\pi = A_E(a; k_1, \ldots, k_{n-1}) \geq \ell_{EB}^2(a). \quad (6.13) \]

By definition (6.10) we have

\[ m_l \leq l \left( \frac{1}{l} \left( \sqrt[1\pi]{a} + p_{n-1} \right) + 1 \right) = \sqrt[1\pi]{a} + p_{n-1} + l. \quad (6.14) \]

Using definition (6.11) and the estimate (6.14) we find

\[ k_{n-1} + 1 = m_l - p_1 + 1 \leq \sqrt[1\pi]{a} + p_{n-1} + l. \quad (6.15) \]

The estimates (6.13) and (6.15) now yield

\[ \frac{|B^{2n}(l_{EB}^n(a))|}{|E(\pi, \ldots, \pi, a)|} \leq \frac{(k_{n-1} + 1)^n \pi}{a} \leq \frac{(\sqrt[1\pi]{a} + (p_{n-1} + l) \sqrt[1\pi]{\pi})^n}{a}. \quad (6.16) \]
Recall that the number \((p_{n-1} + l) \sqrt{n}\) only depends on \(n\). In view of the estimate (6.16) we therefore find a constant \(C_E\) depending only on \(n\) such that

\[
\frac{\left| B^{2n} \left( l_{EB}^{2n}(a) \right) \right|}{\left| E(\pi, \ldots, \pi, a) \right|} \leq 1 + C_E a^{-1/n} \quad \text{for all } a > 2n
\]

and so

\[
\frac{\left| E(\pi, \ldots, \pi, a) \right|}{\left| B^{2n} \left( l_{EB}^{2n}(a) \right) \right|} \geq 1 - C_E a^{-1/n} \quad \text{for all } a > 2n.
\]

Assertion (i) is proved.

(ii) can be proved in the same way as (i).

\[
\square
\]

6.2 Comparison of the embedding results obtained by symplectic and Lagrangian folding

Recall from Chapters 3, 4 and 5 that symplectic folding yields good symplectic embedding results of \(2n\)-dimensional ellipsoids and polydiscs into any connected \(2n\)-dimensional symplectic manifold of finite volume. The reason is that symplectic folding is local in nature in the sense that each fold is achieved on a set of small volume. Lagrangian folding described in Section 6.1, however, is global in nature and thus yields good symplectic embedding results of ellipsoids and polydiscs into special symplectic manifolds only. E.g., the best \(2n\)-dimensional Lagrangian folding embedding of an ellipsoid into a cube respectively of a polydisc into a ball fill less than \(\frac{1}{n!}\) respectively \(\frac{n!}{n^n}\) of the volume.

In this section we shall compare the embedding results for embeddings of skinny ellipsoids into balls and skinny polydiscs into cubes yielded by the two methods.

6.2.1 Embeddings \(E^{2n}(\pi, \ldots, \pi, a) \hookrightarrow B^{2n}(A)\).

1. The case \(n = 2\).

Recall from Theorem 1 that for \(a \in [\pi, 2\pi]\) the ellipsoid \(E(\pi, a)\) does not symplectically embed into the ball \(B^4(A)\) if \(A < 2\pi\). Also recall that the functions \(s_{EB} : [2\pi, \infty[ \rightarrow \mathbb{R}\) and \(l_{EB} \equiv l_{EB}^4 : [2\pi, \infty[ \rightarrow \mathbb{R}\) constructed in 3.3.1 and defined in Corollary 6.4 (i) describe our results for the embedding problem \(E(\pi, a) \hookrightarrow B^4(A)\) obtained by multiple symplectic folding and by Lagrangian folding.
According to Proposition 3.8 the difference $s_{EB}(a) - \sqrt{\pi a}$ between $s_{EB}$ and the volume condition is bounded by $2\pi$. Computer calculations suggest that this difference is monotone increasing and converging to $\pi$ as $a \to \infty$. The difference $l_{EB}(a) - \sqrt{\pi a}$ between $l_{EB}$ and the volume condition attains its local minima at $k(k + 1)\pi$, where it is equal to $m_k = (k + 1)\pi - \sqrt{k(k + 1)}\pi$, and it attains its local maxima at $k(k + 2)\pi$, where it is equal to $M_k = (k + 2)\pi - \sqrt{k(k + 2)}\pi$.

The sequence $(m_k)$ strictly decreases to $\frac{\pi}{2}$, and $(M_k)$ strictly decreases to $\pi$. We conclude that the difference $|s_{EB}(a) - l_{EB}(a)|$ is bounded by $2\pi$.

For $a > 2\pi$ small we have $s_{EB}(a) < l_{EB}(a)$. E.g., $s_{EB}(3\pi) = 2.3801 \ldots \pi$ and $s_{EB}(4\pi) = 2.6916 \ldots \pi$, and so

**Fact 6.7** The ellipsoid $E(\pi, 3\pi)$ symplectically embeds into $B^4(2.381\pi)$, and $E(\pi, 4\pi)$ symplectically embeds into $B^4(2.692\pi)$.

The inequality $l_{EB}(a) < s_{EB}(a)$ happens first at $a = 5.1622 \ldots \pi$. In general, the computer calculations for $s_{EB}$ suggest that the functions $l_{EB}$ and $s_{EB}$ yield alternately better estimates: For all $k \in \mathbb{N}$ we seem to have $l_{EB}(a) < s_{EB}(a)$ on an interval around $k(k + 1)\pi$ and $s_{EB}(a) < l_{EB}(a)$ on an interval around $k(k + 2)\pi$; moreover, we seem to have

$$\lim_{k \to \infty} (s_{EB}(k(k + 2)\pi) - l_{EB}(k(k + 2)\pi)) = 0.$$ 

We checked the above statements for $k \leq 5000$.

We extend both functions $s_{EB}(a)$ and $l_{EB}(a)$ to functions on $[\pi, \infty[$ by setting $s_{EB}(a) = a$ and $l_{EB}(a) = a$ for $a \in [\pi, 2\pi[$. The optimal function for the embedding problem $E(\pi, a) \hookrightarrow B^4(A)$ is the function $f_{EB}$ on $[\pi, \infty[$ defined by

$$f_{EB}(a) = \inf \left\{ A \mid E(\pi, a) \text{ symplectically embeds into } B^4(A) \right\}.$$ 

The following proposition summarizes what we know about this function.

**Proposition 6.8** For $a \in [\pi, 2\pi]$ we have $f_{EB}(a) = a$, and on $]2\pi, \infty[$ the function $f_{EB}$ is bounded from below and above by

$$\max \left( 2\pi, \sqrt{\pi a} \right) \leq f_{EB}(a) \leq \min \left( l_{EB}(a), s_{EB}(a) \right),$$

see Figure 53. In particular,

$$\limsup_{\epsilon \to 0^+} \frac{f_{EB}(2\pi + \epsilon) - 2\pi}{\epsilon} \leq \frac{3}{7}, \quad (6.17)$$
The function $f_{EB}$ is monotone increasing and hence almost everywhere differentiable. Moreover, $f_{EB}$ is Lipschitz continuous with Lipschitz constant at most 1; more precisely,

$$f_{EB}(a') - f_{EB}(a) \leq \frac{\min(l_{EB}(a), s_{EB}(a))}{a} (a' - a) \quad \text{for all } a' \geq a \geq \pi.$$

**Proof.** The estimates of $f_{EB}(a)$ from below are provided by the second Ekeland–Hofer capacity, which yields $f_{EB}(a) \geq a$ for $a \in [\pi, 2\pi]$ and $f_{EB}(a) \geq 2\pi$ for $a > 2\pi$, and by the volume condition $|E(n, a)| \leq |B^4(f_{EB}(a))|$, which translates to $f_{EB}(a) \geq \sqrt{\pi a}$. The estimate (6.17) follows from $f_{EB} \leq s_{EB}$ and from Proposition 3.6. The remaining claims follow as in the proof of Proposition 3.12. 

![Figure 53: What is known about $f_{EB}(a)$.](image)

2. **The case $n \geq 3$.**

Recall from Theorem 1 that for $a \in [\pi, 2\pi]$ the ellipsoid $E^{2n}(\pi, \ldots, \pi, a)$ does not symplectically embed into the ball $B^{2n}(A)$ if $A < a$. For $a > 2\pi$, the $n$'th Ekeland–Hofer capacity still implies that $E^{2n}(\pi, \ldots, \pi, a)$ does not symplectically embed into $B^{2n}(A)$ if $A < 2\pi$. This information is vacuous if $a \geq 2^n \pi$ in view of the volume condition $A \geq \sqrt[2^n-1]{\pi a}$. 

If \( a > 2\pi \), Theorem 2.1, which we proved by symplectic folding, shows that 
\( E^{2n}(\pi, \ldots, \pi, a) \) symplectically embeds into \( B^{2n}(\frac{3}{2} + \pi + \epsilon) \) for every \( \epsilon > 0 \).
For \( a \) large, the results obtained by symplectic folding, which are Theorem 5.25 (i) for \( n = 3 \) and the first statement in Theorem 3, are weaker than the results obtained by Lagrangian folding, which are described by the function \( l_{EB}^{2n} : [2\pi, \infty[ \rightarrow \mathbb{R} \) defined before Corollary 6.4. For \( n = 3 \) the difference \( l_{EB}^{2n}(a) - \sqrt{\pi^2a} \) between the function \( l_{EB}^{2n} \) computed in Corollary 6.5 (i) and the volume condition is bounded by \( 2\pi \). For \( n \geq 4 \) it follows from Corollary 6.6 (i) that the difference \( l_{EB}^{2n}(a) - \sqrt{\pi^{n-1}a} \) between \( l_{EB}^{2n} \) and the volume condition is bounded.

### 6.2.2 Embeddings \( P^{2n}(\pi, \ldots, \pi, a) \hookrightarrow C^{2n}(A) \).

#### 1. The case \( n = 2 \).

Recall that the functions \( s_{PC} : [2\pi, \infty[ \rightarrow \mathbb{R} \) and \( l_{PC} \equiv l_{PC}^{2} : [2\pi, \infty[ \rightarrow \mathbb{R} \) defined in Proposition 3.18 and Corollary 6.4 (ii) describe our results for the embedding problem \( P(\pi, a) \hookrightarrow C^{4}(A) \) obtained by multiple symplectic folding and by Lagrangian folding.

Comparing Proposition 3.18 with Corollary 6.4 (ii) we see that \( l_{PC}(a) < s_{PC}(a) \) for all \( a > 2\pi \), cf. Figure 54. Equality only holds for \( a \in [2\pi, 4\pi[ \) and \( a \in [k2\pi, (k^2 + 1)\pi[ \), \( k \geq 2 \). The difference \( l_{PC}(a) - \sqrt{\pi a} \) between \( l_{PC} \) and the volume condition attains its local minima at \( k(k - 1)\pi, k \geq 3 \), where it is equal to \( m_k = k\pi - \sqrt{k(k - 1)} \pi \), and it attains its local maxima at \( k^2\pi \), where it is equal to \( \pi \). The sequence \( (m_k) \) strictly decreases to \( \frac{2\pi}{3} \).

Extend the function \( l_{PC}(a) \) to a function on \( [\pi, \infty[ \) by setting \( l_{PC}(a) = a \) for \( a \in [\pi, 2\pi[ \). The optimal function for the embedding problem \( P(\pi, a) \hookrightarrow C^{4}(A) \) is the function \( f_{PC} \) on \( [\pi, \infty[ \) defined by

\[
 f_{PC}(a) = \inf \left\{ A \mid P(\pi, a) \text{ symplectically embeds into } C^{4}(A) \right\} .
\]

The following proposition summarizes what we know about this function.

**Proposition 6.9** The function \( f_{PC} : [\pi, \infty[ \rightarrow \mathbb{R} \) is bounded from below and above by

\[
 \sqrt{\pi a} \leq f_{PC}(a) \leq l_{PC}(a),
\]

see Figure 54. It is monotone increasing and hence almost everywhere differentiable. Moreover, \( f_{PC} \) is Lipschitz continuous with Lipschitz constant at most 1;
more precisely,
\[ f_{PC}(a') - f_{PC}(a) \leq \frac{l_{PC}(a)}{a} (a' - a) \quad \text{for all } a' \geq a \geq \pi. \]

Proof. The estimate \( \sqrt{\pi a} \leq f_{PC}(a) \) from below is provided by the volume condition \( |P(\pi, a)| \leq |C^4(f_{PC}(a))| \). The remaining claims follow as in the proof of Proposition 3.12.

Figure 54: What is known about \( f_{PC}(a) \).

2. The case \( n \geq 3 \).

Recall that the functions \( s_{PC}^{2n} : ]2\pi, \infty[ \rightarrow \mathbb{R} \) and \( l_{PC}^{2n} : ]2\pi, \infty[ \rightarrow \mathbb{R} \) defined in Proposition 4.1 and before Corollary 6.4 describe our results for the embedding problem \( P^{2n}(\pi, \ldots, \pi, a) \hookrightarrow C^{2n}(A) \) obtained by multiple symplectic folding and by Lagrangian folding.

Comparing the case \( n = 3 \) of Proposition 4.1 with Corollary 6.5 (ii) we see that in contrast to the case \( n = 2 \) we have \( s_{PC}^{6}(a) \leq l_{PC}^{6}(a) \) for all \( a > 2\pi \). Equality only holds for \( a \in \left[\left( (k-1)k^2 + 2 \right) \pi, (k-1)k(k + 1)\pi \right], k \geq 3 \). The difference \( s_{PC}^{6}(a) - \sqrt{\pi^2a} \) between \( s_{PC}^{6} \) and the volume condition is bounded by \( d_3 = \left( 13 - \sqrt[3]{1586} \right) \pi \).

For \( n \geq 4 \) the comparison of the functions \( s_{PC}^{2n} \) and \( l_{PC}^{2n} \) is more involved since we do not know an explicit formula for \( l_{PC}^{2n} \). The difference \( s_{PC}^{2n}(a) - \sqrt{\pi^{n-1}a} \)
6.2. Comparison of symplectic and Lagrangian folding

between \( s_{sc}^{2n} \) and the volume condition is bounded by \( d_n \) where

\[
d_4 = (5 - \sqrt[4]{194}) \pi, \quad d_5 = (4 - \sqrt[5]{164}) \pi, \quad d_n = \left(3 - \sqrt[n]{2^{n-1} + 2}\right) \pi, \quad n \geq 6.
\]

The sequence \( (d_n), \ n \geq 3 \), strictly decreases to \( \pi \). It follows from Corollary 6.6 (ii) that the difference \( l_{sc}^{2n}(a) - \sqrt{n} \pi^{n-1} a \) is also bounded. Therefore, the difference \( |s_{sc}^{2n} - l_{sc}^{2n}| \) is bounded.

We conclude this section by motivating the conjecture alluded to at the end of Section 1.3. We say that a polydisc \( P^{2n}(\pi, \ldots, \pi, a), \ a > \pi \), is reducible if it symplectically embeds into a cube \( C^{2n}(A) \) for some \( A < a \). It follows from Theorem 1.3 (ii) in [23] that a polydisc cannot be reduced by a local squeezing method. A symplectic embedding which reduces \( P^{2n}(\pi, \ldots, \pi, a) \) must therefore be of global nature. In view of Proposition 4.1 and Corollary 6.4 (ii) symplectic and Lagrangian folding both show that \( P^{2n}(\pi, \ldots, \pi, a) \) is reducible if \( a > 2\pi \). However, none of the two folding methods can reduce \( P^{2n}(\pi, \ldots, \pi, a) \) if \( a \leq 2\pi \). Since we believe that only some kind of folding can reduce a polydisc, we conjecture

**Conjecture 6.10** The polydisc-analogue of Theorem 1 holds true. In particular, the polydiscs \( P^{2n}(\pi, \ldots, \pi, a) \) symplectically embeds into the cube \( C^{2n}(A) \) for some \( A < a \) if and only if \( a > 2\pi \).
6. Symplectic versus Lagrangian folding
7 Proof of Theorem 4

Throughout this section, \( c \) will denote a normalized extrinsic symplectic capacity on \( \mathbb{R}^2 \) as defined in the introduction. We refer to Appendix B.2 for results about such capacities and only mention that any normalized intrinsic symplectic capacity on \( \mathbb{R}^2 \) as defined in Definition B.1 is a normalized extrinsic symplectic capacity on \( \mathbb{R}^2 \).

A symplectic embedding of an arbitrary subset \( S \) of \( \mathbb{R}^{2n} \) into another subset \( S' \) of \( \mathbb{R}^{2n} \) is by definition a symplectic embedding of an open neighbourhood of \( S \) into \( \mathbb{R}^{2n} \) which maps \( S \) into \( S' \). As before we denote by \( Z^{2n}(a) \) the open standard symplectic cylinder \( D(a) \times \mathbb{R}^{2n-2} \), and we denote by \( E_z \subset \mathbb{R}^{2n} \) the affine plane

\[
E_z := \mathbb{R}^2 \times \{z\}, \quad z \in \mathbb{R}^{2n-2}.
\]

Given any subset \( T \) of \( Z^{2n}(a) \) we abbreviate

\[
c(T \cap E_z) := c \left( \{(u, v) \in \mathbb{R}^2 \mid (u, v, z) \in T \cap E_z\} \right).
\]

Assume now that the subset \( S \) of \( \mathbb{R}^{2n} \) symplectically embeds into \( Z^{2n}(a) \). We define the symplectic invariant \( \xi_c(S) \in [0, \pi] \) by

\[
\xi_c(S) := \inf_{\varphi} \sup_z c(\varphi(S) \cap E_z)
\]  

(7.1)

where \( \varphi \) varies over all symplectic embeddings of \( S \) into \( Z^{2n}(a) \). The main result of this section is

**Theorem 7.1** Assume that the subset \( S \) of \( \mathbb{R}^{2n} \) symplectically embeds into \( Z^{2n}(a) \) for some \( a < \pi \). Then \( \xi_c(S) = 0 \) for any normalized extrinsic symplectic capacity \( c \) on \( \mathbb{R}^2 \).

Assume now that \( S \) is a subset of \( \mathbb{R}^{2n} \) which embeds into \( Z^{2n}(a) \) by a symplectomorphism of \( \mathbb{R}^{2n} \). We define the ambient symplectic invariant \( \xi_c(S) \in [0, \pi] \) by

\[
\xi_c(S) := \inf_{\varphi} \sup_z c(\varphi(S) \cap E_z)
\]

where now \( \varphi \) varies over all symplectomorphisms of \( \mathbb{R}^{2n} \) which embed \( S \) into \( Z^{2n}(a) \). Then \( \xi_c(S) \leq \xi_c(S) \).
Corollary 7.2  (i) Assume that $S$ is a relatively compact subset of $\mathbb{R}^{2n}$ whose closure embeds into $\mathbb{Z}^{2n}(\pi)$ by a symplectomorphism of $\mathbb{R}^{2n}$. Then $\zeta_c(S) = 0$ for any normalized extrinsic symplectic capacity $c$ on $\mathbb{R}^2$.

(ii) For the unit circle $S^1 \subset \mathbb{R}^2$ we have $\zeta_c(B^{2n}(\pi)) \in [c(S^1), \pi]$ for any normalized extrinsic symplectic capacity $c$ on $\mathbb{R}^2$.

Remarks 7.3

1. Theorem 4 of the introduction is a special case of Corollary 7.2 (i). We shall explain in Remark 7.8.2 below how Corollary 7.2 (i) can be extended to certain unbounded subsets of $\mathbb{R}^{2n}$.

2. According to Corollary B.9 (i) we have $c(S^1) = 0$ for any intrinsic symplectic capacity $c$ on $\mathbb{R}^2$ and so the statement in Corollary 7.2 (ii) is empty for these capacities. On the other hand, Proposition B.11 says that for any $a \in [0, \pi]$ there exists a normalized extrinsic symplectic capacity $c$ on $\mathbb{R}^2$ such that $c(S^1) = a$. Examples of normalized extrinsic symplectic capacities on $\mathbb{R}^2$ with $c(S^1) = \pi$ are the first Ekeland–Hofer capacity [6, Theorem 1], the displacement energy [17, Theorem 1.9] and the outer cylindrical capacity $\tilde{z}$, see Theorem B.7 (iii).

Proof of Corollary 7.2. (i) Fix $\epsilon > 0$. We denote the closure of $S$ by $\overline{S}$. By assumption there exists a symplectomorphism $\varphi$ of $\mathbb{R}^{2n}$ such that $\varphi(\overline{S}) \subset \mathbb{Z}^{2n}(\pi)$. Since $\overline{S}$ is compact, $\varphi(\overline{S})$ is compact in $\mathbb{R}^{2n}$. We therefore find $a \in ]0, \pi[$ and $A > 0$ such that

$$\varphi(\overline{S}) \subset D(a) \times B^{2n-2}(A).$$

Choose $a' \in ]a, \pi[$. Then

$$D(a) \times B^{2n-2}(A) \subset Z^{2n}(a').$$

In view of Theorem 7.1 there exists a symplectic embedding

$$\psi : Z^{2n}(a') \hookrightarrow Z^{2n}(\pi)$$

such that

$$\sup_z c\left(\psi\left(Z^{2n}(a')\right) \cap E_z\right) < \epsilon.$$\hfill (7.5)

Applying Proposition C.1 to the bounded starlike domain $D(a) \times B^{2n-2}(A)$ we find a symplectomorphism $\Psi$ of $\mathbb{R}^{2n}$ such that

$$\Psi|_{D(a) \times B^{2n-2}(A)} = \psi|_{D(a) \times B^{2n-2}(A)}.$$

\hfill (7.6)
The inclusion (7.2), the identity (7.6) and the inclusion (7.3) show that

\[(\Psi \circ \varphi)(\overline{S}) \subset \psi \left( Z^{2n}(a') \right). \tag{7.7} \]

In view of the inclusions (7.7) and (7.4) the symplectomorphism \(\Psi \circ \varphi\) of \(\mathbb{R}^{2n}\) embeds \(\overline{S}\) into \(Z^{2n}(\pi)\). Moreover, the inclusion (7.7), the normalization of the symplectic capacity \(c\) and the estimate (7.5) yield

\[
\sup_{\tilde{z}} c \left( (\Psi \circ \varphi)(\overline{S}) \cap E_{\tilde{z}} \right) \leq \sup_{\tilde{z}} c \left( \psi \left( Z^{2n}(a') \right) \cap E_{\tilde{z}} \right) < \epsilon.
\]

We conclude that \(\zeta_c(S) \leq \zeta_c(\overline{S}) < \epsilon\). Since \(\epsilon > 0\) was arbitrary, assertion (i) follows.

(ii) We abbreviate

\[S^1_z := \left\{ (u, v, z) \in E_z \mid u^2 + v^2 = 1 \right\}, \quad z \in \mathbb{R}^{2n-2}.\]

Let \(\varphi\) be a symplectomorphism of \(\mathbb{R}^{2n}\) which embeds \(B^{2n}(\pi)\) into \(Z^{2n}(\pi)\). According to Lemma 1.2 in [23] there exists \(z_0 \in \mathbb{R}^{2n-2}\) such that

\[S^1_{z_0} \subset \partial \left( \varphi \left( B^{2n}(\pi) \right) \right) \cap E_{z_0}.\]

Since \(\varphi\) is a diffeomorphism of \(\mathbb{R}^{2n}\) and \(\varphi \left( B^{2n}(\pi) \right) \subset Z^{2n}(\pi)\), the boundary \(\partial \left( \varphi \left( B^{2n}(\pi) \right) \right)\) of \(\varphi \left( B^{2n}(\pi) \right)\) is tangent to the boundary \(S^1 \times \mathbb{R}^{2n-2}\) of \(Z^{2n}(\pi)\) at each point of \(S^1_{z_0}\). This, the inclusion \(\varphi \left( B^{2n}(\pi) \right) \subset Z^{2n}(\pi)\) and the compactness of \(S^1_{z_0}\) imply that there exists an \(\epsilon > 0\) such that the annulus

\[A_\epsilon := \left\{ (u, v, z_0) \in E_{z_0} \mid 1 - \epsilon < |(u, v)| < 1 \right\}\]

is contained in \(\varphi \left( B^{2n}(\pi) \right) \cap E_{z_0}\). For each \(r \in ]1 - \epsilon, 1[\) we then have

\[rS^1_{z_0} \subset A_\epsilon \subset \varphi \left( B^{2n}(\pi) \right) \cap E_{\tilde{z}}.\]

In view of the conformality and the monotonicity of the symplectic capacity \(c\) we obtain that

\[r^2 c(S^1) = c(rS^1) = c \left( rS^1_{z_0} \right) \leq c(A_\epsilon) \leq c \left( \varphi \left( B^{2n}(\pi) \right) \cap E_{\tilde{z}} \right)\]
162 7. Proof of Theorem 4

for each $r \in ]1 - \epsilon, 1[$, and so
\[
c\left(\varphi\left(B^{2n}(\pi)\right) \cap E_z\right) \geq c(S^1).
\]

Since this estimate holds for any symplectomorphism $\varphi$ of $\mathbb{R}^{2n}$ which embeds $B^{2n}(\pi)$ into $Z^{2n}(\pi)$ we conclude that
\[
\zeta_c\left(B^{2n}(\pi)\right) \geq c(S^1).
\]

On the other hand, the definition of $\zeta_c$ and the normalization of $c$ yield
\[
\zeta_c\left(B^{2n}(\pi)\right) \leq \sup_{z} c\left(B^{2n}(\pi) \cap E_z\right) = c\left(D(\pi)\right) = \pi.
\]

Assertion (ii) thus follows, and so the proof of Corollary 7.2 is complete. $\square$

Proof of Theorem 7.1. We fix a normalized extrinsic symplectic capacity $c$ on $\mathbb{R}^2$.

Step 1. Reduction to a 4-dimensional problem

Proposition 7.4 Assume that
\[
\zeta_c\left(Z^4(a)\right) = 0 \quad \text{for all } a \in ]0, \pi[.
\]

Then Theorem 7.1 holds true.

Proof. Let $S$ be a subset of $\mathbb{R}^{2n}$ for which there exist $a < \pi$ and a symplectic embedding $\varphi: S \rightarrow Z^{2n}(a)$. We fix $\epsilon > 0$. By assumption we find a symplectic embedding $\psi: Z^4(a) \rightarrow Z^4(\pi)$ such that
\[
\sup_{z \in \mathbb{R}^2} c\left(\psi\left(Z^4(a)\right) \cap E_z\right) < \epsilon. \quad (7.8)
\]

The composition
\[
\rho: S \xrightarrow{\varphi} Z^{2n}(a) = Z^4(a) \times \mathbb{R}^{2n-4} \xrightarrow{\psi \times id_{2n-4}} Z^4(\pi) \times \mathbb{R}^{2n-4} = Z^{2n}(\pi)
\]
symplectically embeds $S$ into $Z^{2n}(\pi)$. Moreover, the monotonicity of the symplectic capacity $c$ and the estimate (7.8) yield
\[
\sup_{z \in \mathbb{R}^{2n-2}} c\left(\rho(S) \cap E_z\right) = \sup_{z \in \mathbb{R}^{2n-2}} c\left((\psi \times id_{2n-4})\left(\varphi(S)\right) \cap E_z\right)
\leq \sup_{z \in \mathbb{R}^{2n-2}} c\left((\psi \times id_{2n-4})\left(Z^{2n}(a)\right) \cap E_z\right)
= \sup_{z \in \mathbb{R}^2} c\left(\psi\left(Z^4(a)\right) \cap E_z\right)
< \epsilon.
\]
Since \( \epsilon > 0 \) was arbitrary, we conclude that \( \zeta_\epsilon(S) = 0 \), as claimed. \( \square \)

**Step 2. Reformulation of the 4-dimensional problem**

We shall use coordinates \( u, v, x, y \) on \((\mathbb{R}^4, du \wedge dv + dx \wedge dy)\). Fix \( a < \pi \) and \( \epsilon > 0 \). We may assume \( a > \pi/2 \) and \( \epsilon < (\pi - a)/9 \). Denote by \( R \subset \{(u, v) \mid 0 < u < \pi, 0 < v < 1\} \) the convex hull of the image of the map \( \gamma \) drawn in Figure 58 below. The domain \( R \) is a rectangle with smooth corners. We set

\[
\mathcal{A} := \{(u, v, x, y) \mid \epsilon < u, 0 < v < 1 - \epsilon, 0 < x < 1, 0 < y < a\},
\]

\[
\mathcal{Z} := R \times \mathbb{R}^2.
\]

**Proposition 7.5** Assume there exists a symplectic embedding \( \Phi : \mathcal{A} \hookrightarrow \mathcal{Z} \) such that

\[
\sup_{z \in \mathbb{R}^2} c \left( \Phi(\mathcal{A}) \cap E_z \right) \leq 2\epsilon. \tag{7.9}
\]

Then there exists a symplectic embedding \( \Psi : Z^4(a) \hookrightarrow Z^4(\pi) \) such that

\[
\sup_{z \in \mathbb{R}^2} c \left( \Psi \left( Z^4(a) \right) \cap E_z \right) \leq 2\epsilon. \tag{7.10}
\]

**Proof.** We start with

**Lemma 7.6**

(i) There exists a symplectomorphism

\[
\alpha : \mathbb{R}^2 \to \{(u, v) \in \mathbb{R}^2 \mid \epsilon < u, 0 < v < 1 - \epsilon\}.
\]

(ii) There exists a symplectomorphism

\[
\sigma : D(a) \to \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, 0 < y < a\}.
\]

(iii) There exists a symplectomorphism \( \omega \) of \( \mathbb{R}^2 \) such that \( \omega(R) \subset D(\pi) \).

**Proof.** (i) follows from the proof of Lemma 2.5 or from Proposition A.6(ii). Here we give an explicit construction. Choose orientation preserving diffeomorphisms
g : \mathbb{R} \to ]0, \infty[ \quad \text{and} \quad h : \mathbb{R} \to ]0, 1 - \varepsilon[,
and denote by \( g' \) and \( h' \) their derivatives.

Then the maps

\[
\alpha_1 : \mathbb{R}^2 \to ]0, \infty[ \times \mathbb{R}, \quad (u, v) \mapsto \left( g(u), \frac{v}{g'(u)} \right),
\]

\[
\alpha_2 : ]0, \infty[ \times \mathbb{R} \to ]e, \infty[ \times ]0, 1 - \varepsilon[,
\]

\[
(u, v) \mapsto \left( \frac{u}{h'(v)} + \varepsilon, h(v) \right)
\]

are symplectomorphisms. The map \( \alpha := \alpha_2 \circ \alpha_1 \) is therefore as desired.

(ii) follows from Lemma 2.5 or from Proposition A.6(i).

(iii) Choose \( r \in \mathbb{R} \) so large that \( R \subseteq D(\pi r^2) \), and choose a diffeomorphism \( \chi \) of \( D(2\pi r^2) \) which is a translation near the origin, maps the boundary of \( D(\text{area}(R)) \) to the boundary of \( R \), and is the identity near the boundary of \( D(2\pi r^2) \). By Lemma 2.5 and its proof, we may assume that \( \chi \) is a symplectomorphism. Extend \( \chi \) to the symplectomorphism of \( \mathbb{R}^2 \) which is the identity outside \( D(2\pi r^2) \), and let \( \omega \) be the inverse of this extension. Then \( \omega(R) = D(\text{area}(R)) \subseteq D(\pi) \), as desired. \[ \square \]

Let \( \Phi : \mathcal{A} \hookrightarrow \mathcal{Z} \) be a symplectic embedding as assumed in the proposition, let \( \alpha, \sigma \) and \( \omega \) be symplectomorphisms as guaranteed by Lemma 7.6, and denote the linear symplectomorphism \( (u, v, x, y) \mapsto (x, y, u, v) \) of \( \mathbb{R}^4 \) by \( \tau \). Then the composition

\[
\Psi : Z^4(\alpha) \xrightarrow{(\alpha \times \sigma) \circ \tau} \mathcal{A} \xrightarrow{\Phi} \mathcal{Z} \xrightarrow{\omega \times id} Z^4(\pi)
\]

is a symplectic embedding. Moreover, the identity \((\alpha \times \sigma)(\tau(Z^4(\alpha))) = \mathcal{A}\), the monotonicity of the symplectic capacity \( c \) on \( \mathbb{R}^2 \) and the assumed estimate (7.9) imply that

\[
\sup_{z \in \mathbb{R}^2} c \left( \Psi \left( Z^4(\alpha) \right) \cap E_z \right) = \sup_{z \in \mathbb{R}^2} c \left( (\omega \times id) \left( \Phi(\mathcal{A}) \right) \cap E_z \right)
\]

\[
= \sup_{z \in \mathbb{R}^2} c \left( \Phi(\mathcal{A}) \cap E_z \right)
\]

\[
\leq 2\varepsilon.
\]

This completes the proof of Proposition 7.5. \[ \square \]

**Step 3. Construction of the embedding \( \Phi \)**

We are going to construct a symplectic embedding \( \Phi : \mathcal{A} \hookrightarrow \mathcal{Z} \) satisfying the estimate (7.9) by a variant of multiple symplectic folding. Instead of folding alternatingly on the right and on the left, we will always fold on the right. If we
neglect all quantities which can be chosen arbitrarily small, the image \( \Phi(A) \subset \mathbb{Z} \) will look as in Figure 55.

![Figure 55: The embedding \( \Phi: \mathcal{A} \hookrightarrow \mathbb{Z} \) for \( a = \frac{3\pi}{3} \).](image)

The symplectic embedding \( \Phi \) will be the composition of the three maps \( \beta \times id, \varphi \) and \( \gamma \times id \). Here, the smooth area preserving embedding

\[
\beta: \{ (u, v) \mid \varepsilon < u, \ 0 < v < 1 - \varepsilon \} \hookrightarrow \mathbb{R}^2
\]

is similar to the map constructed in Step 1 of Section 3.2. Indeed, set \( \delta = \frac{\varepsilon}{2a} \). The map \( \beta \) restricts to the identity on \( \{ \varepsilon < u \leq 2\varepsilon \} \), and it maps \( \{ 2\varepsilon < u \leq 3\varepsilon \} \) to the black region in \( \{ 2\varepsilon < u \leq a + 9\varepsilon \} \) drawn in Figure 56. Moreover,

\[
\beta(u, v) = \beta(u - i2\varepsilon, v) + (i(a + 8\varepsilon), 0)
\]

for \( u \in \lfloor \varepsilon + i2\varepsilon, \varepsilon + (i + 1)2\varepsilon \rfloor \) and \( i = 1, 2, 3, \ldots \), see Figure 56.

The "lifting" map \( \varphi \) is similar to the map constructed in Step 3 of Section 3.2. Indeed, choose a cut off function \( f_0: \mathbb{R} \to [0, 1 - \varepsilon - \delta] \) with support \( [3\varepsilon, a + 8\varepsilon] \) such that

\[
\tilde{a} := \int_{\mathbb{R}} f_0(s) \, ds > a.
\]
Set $f_i(s) = f_0(s - i(a + 8\epsilon))$, $i = 1, 2, 3, \ldots$, and $f(s) = \sum_{i \geq 0} f_i(s)$. The symplectomorphism $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by

$$
\varphi(u, v, x, y) = \left( u, v + f(u)x, x, y + \int_0^u f(s) \, ds \right).
$$

(7.11)

The image $\varphi \left( (\beta \times \text{id})(\mathcal{A}) \right) \subset \mathbb{R}^4$ is illustrated in Figure 57. Denote the projection $(u, v, x, y) \mapsto (u, v)$ by $p$. The left part of the set $p \left( \varphi \left( (\beta \times \text{id})(\mathcal{A}) \right) \right)$ is equal to the upper domain in Figure 58.

We finally wrap the set $\varphi \left( (\beta \times \text{id})(\mathcal{A}) \right)$ into $Z$. The smooth area preserving local embedding $\gamma : p \left( \varphi \left( (\beta \times \text{id})(\mathcal{A}) \right) \right) \to \mathbb{R}$, which is explained by Figure 58, restricts to the identity on $\{ \epsilon < u \leq a + 8\epsilon \}$, and it maps the black region in $\{ a + 8\epsilon < u \leq a + 9\epsilon \}$ to the black region in its image. Moreover,

$$
\gamma(u, v) = \gamma(u - i(a + 8\epsilon), v)
$$

for $u \in [\epsilon + i(a + 8\epsilon), \epsilon + (i + 1)(a + 8\epsilon)]$ and $i = 1, 2, 3, \ldots$, see Figure 58. The map $\gamma$ is constructed in the same way as the map in Step 4 of Section 2.2. Since $\gamma$ is an embedding on each part.
Figure 57: The left part of the image $\varphi (\beta \times \text{id})(A)$.

\[\{(u, v) \in p (\varphi (\beta \times \text{id})(A)) \mid \epsilon + i(a + 8\epsilon) < u \leq \epsilon + (i + 1)(a + 8\epsilon)\}, i = 0, 1, 2, \ldots, \text{of its domain, and since } \hat{a} > a, \text{the map} \]

\[y \times \text{id}: \varphi (\beta \times \text{id})(A) \rightarrow \mathbb{R}^4\]

is a symplectic embedding, cf. Figure 57 and Figure 59. We conclude that the composition

\[\Phi := (y \times \text{id}) \circ \varphi \circ (\beta \times \text{id})\]

is a symplectic embedding of $A$ into $Z$.

**Step 4. Verification of the estimate (7.9) in Proposition 7.5**

We have to show that for any point $(x_0, y_0) \in \mathbb{R}^2$ the estimate

\[c\left(\Phi(A) \cap E(x_0, y_0)\right) \leq 2\epsilon\quad (7.12)\]

holds true. To this end, we may assume that $x_0 \in ]0, 1[$ since otherwise the intersection $\Phi(A) \cap E(x_0, y_0)$ is empty. Moreover, by construction of the embedding $\Phi$ we have

\[\Phi(u + 2\epsilon, v, x, y) = \Phi(u, v, x, y) + (0, 0, 0, \hat{a})\quad (7.13)\]
for all \((u, v, x, y) \in A\). It follows that for each \(y_0 \in \mathbb{R}\) there exists \(i \in \mathbb{Z}\) such that \(y_0 + i\delta \in ]\delta, 2\delta]\) and
\[
p(A) \cap E(x_0, y_0) \subset p(A) \cap E(x_0, y_0 + i\delta),
\]
cf. Figure 59. We may therefore assume that \((x_0, y_0) \in ]0, 1[ \times ]\delta, 2\delta]\). In order to verify the estimate (7.12) we distinguish two cases.

**Case A.** Assume first \(y_0 \in ]\delta, \delta + \check{a}].\) Then \(E(x_0, y_0)\) intersects the “floor”
\[
F_2 := \{ \Phi(u, v, x, y) \mid 3\epsilon < u \leq 4\epsilon \}
\]
and the two “stairs”
\[
S_i := \{ \Phi(u, v, x, y) \mid 2i\epsilon < u \leq (2i + 1)\epsilon \}, \quad i = 1, 2,
\]
only, cf. Figure 59. The set $F := F_2 \cap E(x_0, y_0)$ has area $\epsilon(1 - \epsilon)$. The set $S_1 \cap E(x_0, y_0)$ consists of the black thickened "arc" $A$ drawn in Figure 60 and of the branch $B_1 = B_1(x_0, y_0)$, which for $(x_0, y_0) = \left(\frac{1}{2}, \hat{a} + \frac{\delta}{2}\right)$ looks like in Figure 60. Indeed, since the preimage of $S_1$ under $\Phi$ is contained in $\{0 < y < a\}$, we read off from definition (7.11) that

$$B_1(x_0, y_0) = \{ (u, v + f(u)x_0, x_0, y_0) \mid u_1 < u < a + 8\epsilon, 0 < v < \delta \} \quad (7.14)$$

where $u_1 = u_1(y_0)$ is defined through the equation

$$\int_{3\epsilon}^{u_1} f(s) ds = y_0 - a. \quad (7.15)$$

Similarly, the branch $B_2 = B_2(x_0, y_0) = S_2 \cap E(x_0, y_0)$ looks for $(x_0, y_0) = \left(\frac{1}{2}, \hat{a} + \frac{\delta}{2}\right)$ as in Figure 60. Indeed, since the preimage of $S_2$ under $\Phi$ is contained in $\{\hat{a} < y < a + \delta\}$, we read off from (7.11) that

$$B_2(x_0, y_0) \cap \{u > 3\epsilon\} = \{ (u, v + f(u)x_0, x_0, y_0) \mid 3\epsilon < u < u_2, 0 < v < \delta \} \quad (7.16)$$
where $u_2 = u_2(y_0)$ is defined through the equation

$$
\int_{3\epsilon}^{u_2} f(s) \, ds = y_0 - \hat{a}.
$$

Subtracting (7.17) from (7.15) and using that $f(s) < 1$ for all $s \in \mathbb{R}$ we obtain

$$
0 < \hat{a} - a = \int_{u_2}^{u_1} f(s) \, ds < u_1 - u_2.
$$

Let $\mu$ be the area of the set $\Phi(A) \cap E(x_0, y_0)$. In order to estimate the capacity of $\Phi(A) \cap E(x_0, y_0)$ we next embed $\Phi(A) \cap E(x_0, y_0)$ into a set whose capacity is known.

**Lemma 7.7** There exists a symplectomorphism $\phi$ of $\mathbb{R}^2$ such that

$$
\phi \left( p \left( \Phi(A) \cap E(x_0, y_0) \right) \right) \subset D \left( \mu + \epsilon^2 \right).
$$

**Proof.** The set $p \left( \Phi(A) \cap E(x_0, y_0) \right) \subset \mathbb{R}^2$ is diffeomorphic to an open disc, and in view of the estimate (7.18) its boundary is a piecewise smooth embedded closed curve in $\mathbb{R}^2$, cf. Figure 60. We therefore find a simply connected domain $U \subset \mathbb{R}^2$ such that the boundary of $U$ is a smoothly embedded closed curve in $\mathbb{R}^2$, and such that $p \left( \Phi(A) \cap E(x_0, y_0) \right) \subset U$ and area($U$) = $\mu + \epsilon^2$. Applying the argument given in the proof of Lemma 7.6 (iii) to $U$, we find a symplectomorphism $\phi$
of $\mathbb{R}^2$ such that $\phi(U) = D(\mu + \epsilon^2)$. Then $\phi\left(p \left( \Phi(A) \cap E_{(x_0, y_0)} \right) \right) \subset \phi(U) = \Phi(A) \cap E_{(x_0, y_0)}$)

In view of the monotonicity and the conformality of the normalized symplectic capacity $c$ on $\mathbb{R}^2$, Lemma 7.7 implies

$$c\left( \Phi(A) \cap E_{(x_0, y_0)} \right) = c\left( \Phi\left(p \left( \Phi(A) \cap E_{(x_0, y_0)} \right) \right) \right) \leq c\left( D\left( \mu + \epsilon^2 \right) \right) = \left( \mu + \epsilon^2 \right).$$

(7.19)

It remains to estimate the number $\mu = \text{area}\left( \Phi(A) \cap E_{(x_0, y_0)} \right)$. We abbreviate $Q = \{2 \epsilon < u \leq 3 \epsilon, 0 < v < 1 - \epsilon\}$. Using the definitions of $\beta$ and $\gamma$ and the identities (7.14) and (7.16) we can estimate

$$\text{area}\left( B_2 \cup B_1 \cup A \right) = \text{area}\left( p(B_2) \cup p(B_1) \cup \gamma^{-1}(p(A)) \right) \leq \text{area}\left( \{ (u, v + f(u)x_0) | (u, v) \in \beta(Q) \} \right) = \text{area}\left( \{ (u, v) | (u, v) \in \beta(Q) \} \right) = \text{area}(\beta(Q)) = \text{area}(Q) = \epsilon(1 - \epsilon).

Therefore,

$$\mu = \text{area}\left( \Phi(A) \cap E_{(x_0, y_0)} \right) = \text{area}(F) + \text{area}(B_2 \cup B_1 \cup A) < \epsilon(1 - \epsilon) + \epsilon(1 - \epsilon).$$

In view of (7.19) we conclude that $c\left( \Phi(A) \cap E_{(x_0, y_0)} \right) < 2 \epsilon$.

Case B. Assume now $y_0 \in [\hat{a} + a, 2 \hat{a}]$. Then $E_{(x_0, y_0)}$ intersects the stairs $S_2$ only, and

$$\Phi(A) \cap E_{(x_0, y_0)} = \{ (u, v + f(u)x_0, x_0, y_0) | u_l < u < u_r, 0 < v < \delta \}$$

where $u_l = u_l(y_0)$ and $u_r = u_r(y_0)$ lie in $[3 \epsilon, a + 8 \epsilon]$ and are defined through the equations

$$\int_{3 \epsilon}^{u_l} f(s) \, ds = y_0 - (\hat{a} + a) \quad \text{and} \quad \int_{3 \epsilon}^{u_r} f(s) \, ds = y_0 - \hat{a}.

The set $p\left( \Phi(A) \cap E_{(x_0, y_0)} \right) \subset \mathbb{R}^2$ is diffeomorphic to an open disc, and its boundary is piecewise smooth. Moreover,

$$\text{area}\left( \Phi(A) \cap E_{(x_0, y_0)} \right) = (u_r - u_l) \delta < (a + 5 \epsilon) \frac{\epsilon}{2 \hat{a}} < \epsilon.$$
Arguing as above, we find $c \left( \Phi(\mathcal{A}) \cap E_{(x_0, y_0)} \right) < 2\varepsilon$. This finishes the verification of the estimate (7.12) and hence of the estimate (7.9) in Proposition 7.5.

**Step 5. End of the proof of Theorem 7.1**

We have constructed a symplectic embedding $\Phi : \mathcal{A} \hookrightarrow \mathcal{Z}$ satisfying the estimate (7.9). In view of Proposition 7.5 there exists a symplectic embedding $\Psi : Z^4(a) \hookrightarrow Z^4(\pi)$ satisfying the estimate (7.10). Since $a < \pi$ and $\varepsilon > 0$ were arbitrary, we conclude that $\xi_c(Z^4(a)) = 0$ for all $a \in ]0, \pi[$. Proposition 7.4 now implies that Theorem 7.1 holds true. \[\square\]

**Remarks 7.8**

1. We don’t know $\xi_c(B^{2n}(\pi))$ for any symplectic capacity $c$. Lemma 1.2 in [23] and its proof suggest that $\xi_c(B^{2n}(\pi)) \geq c(S^1)$. It would be interesting to have a proof of this.

2. In view of the equivariance (7.13) of the symplectic embedding $\Phi : \mathcal{A} \hookrightarrow \mathcal{Z}$ there exists a constant $L > 0$ such that

$$|\Phi(z) - \Phi(z')| \geq L |z - z'| \quad \text{for all } z, z' \in \mathcal{A}.$$ 

Combining the methods used in this section with Theorem C.6 in Appendix C one can therefore prove the following generalization of Corollary 7.2 (i).

**Assume that** $S$ is a subset of $\mathbb{R}^{2n}$ such that there exist numbers $a < \pi$ and $A < \infty$ and a symplectomorphism of $\mathbb{R}^{2n}$ which embeds $S$ into the truncated cylinder

$$D(a) \times \mathbb{R}^{2n-3} \times ]-A, A[ \subset Z^{2n}(a).$$

Then $\xi_c(S) = 0$ for any normalized extrinsic symplectic capacity $c$ on $\mathbb{R}^2$.

We don’t know $\xi_c(Z^{2n}(a))$ for any symplectic capacity $c$ and any $a \in ]0, \pi[$. We also don’t know anything about $\xi_c(B^{2n}(\pi))$ or $\xi_c(Z^{2n}(\pi))$ if $c(S^1) = 0$.

3. The invariants $\xi_c$ and $\xi_c$ considered in this section are special constrained symplectic intersection invariants as studied in [31].
Appendix

A Proof of Proposition 1

The purpose of this appendix is to prove Proposition 1 of the introduction which we restate as

**Theorem A.1** Consider an open subset $U$ of $\mathbb{R}^n$ and a smooth connected $n$-dimensional manifold $M$ endowed with a volume form $\Omega$. Then there exists a volume preserving embedding $\varphi: U \hookrightarrow M$ if and only if $\text{Vol}(U, \Omega_0) \leq \text{Vol}(M, \Omega)$.

**Proof.** Assume first that $\varphi: U \hookrightarrow M$ is a smooth embedding such that $\varphi^*\Omega = \Omega_0$. Then

$$\text{Vol}(U, \Omega_0) = \int_U \Omega_0 = \int_U \varphi^*\Omega = \int_{\varphi(U)} \Omega \leq \int_M \Omega = \text{Vol}(M, \Omega).$$

Assume now that $\text{Vol}(U, \Omega_0) \leq \text{Vol}(M, \Omega)$. We are going to construct a smooth embedding $\varphi: U \hookrightarrow M$ such that $\varphi^*\Omega = \Omega_0$.

We orient $\mathbb{R}^n$ in the natural way. The orientations of $\mathbb{R}^n$ and $M$ orient each open subset of $\mathbb{R}^n$ and $M$. We abbreviate the Lebesgue measure $\text{Vol}(V, \Omega_0)$ of a measurable subset $V$ of $\mathbb{R}^n$ by $|V|$, and we write $\overline{V}$ for the closure of $V$ in $\mathbb{R}^n$. Moreover, we denote by $B_r$ the open ball in $\mathbb{R}^n$ of radius $r$ centered at the origin.

**Proposition A.2** Assume that $V$ is a non-empty open subset of $\mathbb{R}^n$. Then there exists a smooth embedding $\sigma: V \hookrightarrow \mathbb{R}^n$ such that $|\mathbb{R}^n \setminus \sigma(V)| = 0$.

**Proof.** We choose an increasing sequence

$$V_1 \subset V_2 \subset \cdots \subset V_k \subset V_{k+1} \subset \cdots$$

of non-empty open subsets of $V$ such that $\overline{V_k} \subset V_{k+1}$ for $k = 1, 2, \ldots$ and $\bigcup_{k=1}^{\infty} V_k = V$. To fix the ideas, we assume that the $V_k$ have smooth boundaries.

Let $\sigma_1: V_2 \hookrightarrow \mathbb{R}^n$ be a smooth embedding such that $\sigma_1(V_1) \subset B_1$ and

$$|B_1 \setminus \sigma_1(V_1)| \leq 2^{-1}.$$
Since \( \overline{V_1} \subset V_2 \) and \( \overline{\sigma_1(V_1)} \subset \overline{B_1} \subset B_2 \), we find a smooth embedding \( \sigma_2 : V_3 \hookrightarrow \mathbb{R}^n \) such that \( \sigma_2|_{V_1} = \sigma_1|_{V_1} \) and \( \sigma_2(V_2) \subset B_2 \) and
\[
|B_2 \setminus \sigma_2(V_2)| \leq 2^{-2}.
\]

Arguing by induction we find smooth embeddings \( \sigma_k : V_{k+1} \hookrightarrow \mathbb{R}^n \) such that \( \sigma_k|_{V_{k-1}} = \sigma_{k-1}|_{V_{k-1}} \) and \( \sigma_k(V_k) \subset B_k \) and
\[
|B_k \setminus \sigma_k(V_k)| \leq 2^{-k}, \tag{A.1}
\]

\( k = 1, 2, \ldots \). The map \( \sigma : V \rightarrow \mathbb{R}^n \) defined by \( \sigma|_{V_k} = \sigma_k|_{V_k} \) is a well defined smooth embedding of \( V \) into \( \mathbb{R}^n \). Moreover, the inclusions \( \sigma_k(V_k) \subset \sigma(V) \) and the estimates (A.1) imply that
\[
|B_k \setminus \sigma(V)| \leq |B_k \setminus \sigma_k(V_k)| \leq 2^{-k},
\]
and so
\[
|\mathbb{R}^n \setminus \sigma(V)| = \lim_{k \to \infty} |B_k \setminus \sigma(V)| = 0.
\]

This completes the proof of Proposition A.2.

Our next goal is to construct a smooth embedding of \( \mathbb{R}^n \) into the connected \( n \)-dimensional manifold \( M \) such that the complement of the image has measure zero. If \( M \) is compact, such an embedding has been obtained by Ozols [29] and Katok [19, Proposition 1.3]. While Ozols combines an engulfing method with tools from Riemannian geometry, Katok successively exhausts a smooth triangulation of \( M \). Both approaches can be generalized to the case of an arbitrary connected manifold \( M \), and we shall follow Ozols.

We abbreviate \( \mathbb{R}_{>0} = \{ r \in \mathbb{R} \mid r > 0 \} \) and \( \mathbb{R}_{>0} = \mathbb{R}_{>0} \cup \{ \infty \} \). We endow \( \mathbb{R}_{>0} \) with the topology whose base of open sets consists of the intervals \( ]a, b[ \subset \mathbb{R}_{>0} \) and the subsets of the form \( ]a, \infty] = ]a, \infty[ \cup \{ \infty \} \). We denote the Euclidean norm on \( \mathbb{R}^n \) by \( \| \cdot \| \) and the unit sphere in \( \mathbb{R}^n \) by \( S_1 \).

**Proposition A.3** Endow \( \mathbb{R}^n \) with its standard smooth structure, let \( \mu : S_1 \rightarrow \mathbb{R}_{>0} \) be a continuous function and let
\[
S = \left\{ x \in \mathbb{R}^n \mid 0 \leq \|x\| < \mu \left( \frac{x}{\|x\|} \right) \right\}
\]
be the starlike domain associated with \( \mu \). Then \( S \) is diffeomorphic to \( \mathbb{R}^n \). 

Remark A.4 The diffeomorphism guaranteed by Proposition A.3 may be chosen such that the rays emanating from the origin are preserved.

Proof of Proposition A.3. If $\mu(S_1) = \{\infty\}$, there is nothing to prove. In the case that $\mu$ is bounded, Proposition A.3 has been proved by Ozols [29]. In the case that neither $\mu(S_1) = \{\infty\}$ nor $\mu$ is bounded, Ozols's proof readily extends to this situation. Using his notation, the only modifications needed are: Require in addition that $r_0 < 1$ and that $\epsilon_1 < 2$, and define continuous functions $\tilde{\mu}_i : S_1 \to \mathbb{R}_{>0}$ by

$$\tilde{\mu}_i = \min \left\{ i, \mu - \epsilon_i + \frac{\delta_i}{2} \right\}.$$ 

With these minor adaptations the proof in [29] applies word by word. □

In the following we shall use some basic Riemannian geometry. We refer to [20] for basic notions and results in Riemannian geometry. Consider an $n$-dimensional complete Riemannian manifold $(N, g)$. We denote the cut locus of a point $p \in N$ by $C(p)$.

Corollary A.5 The maximal normal neighbourhood $N \setminus C(p)$ of any point $p$ in an $n$-dimensional complete Riemannian manifold $(N, g)$ is diffeomorphic to $\mathbb{R}^n$ endowed with its standard smooth structure.

Proof. Fix $p \in N$. We identify the tangent space $(T_p N, g(p))$ with Euclidean space $\mathbb{R}^n$ by a (linear) isometry. Let $\exp_p : \mathbb{R}^n \to N$ be the exponential map at $p$ with respect to $g$, and let $S_1$ be the unit sphere in $\mathbb{R}^n$. We define the function $\mu : S_1 \to \mathbb{R}_{>0}$ by

$$\mu(x) = \inf \{ t > 0 \mid \exp_p(tx) \in C(p) \}. \tag{A.2}$$

Since the Riemannian metric $g$ is complete, the function $\mu$ is continuous [20, VIII, Theorem 7.3]. Let $S \subset \mathbb{R}^n$ be the starlike domain associated with $\mu$. In view of Proposition A.3 the set $S$ is diffeomorphic to $\mathbb{R}^n$, and in view of [20, VIII, Theorem 7.4 (3)] we have $\exp_p(S) = N \setminus C(p)$. Therefore, $N \setminus C(p)$ is diffeomorphic to $\mathbb{R}^n$. □

A main ingredient of our proof of Theorem A.1 are the following two special cases of a theorem of Greene and Shiohama [11].

Proposition A.6 (i) Assume that $\Omega_1$ is a volume form on the connected open subset $U$ of $\mathbb{R}^n$ such that $\text{Vol}(U, \Omega_1) = |U| < \infty$. Then there exists a diffeomorphism $\psi$ of $U$ such that $\psi^*\Omega_1 = \Omega_0$. 
(ii) Assume that $\Omega_1$ is a volume form on $\mathbb{R}^n$ such that $\text{Vol}(\mathbb{R}^n, \Omega_1) = \infty$. Then there exists a diffeomorphism $\psi$ of $\mathbb{R}^n$ such that $\psi^*\Omega_1 = \Omega_0$.

End of the proof of Theorem A.1

Let $U \subset \mathbb{R}^n$ and $(M, \Omega)$ be as in Theorem A.1. After enlarging $U$, if necessary, we can assume that $|U| = \text{Vol}(M, \Omega)$. We set $N = M \setminus \partial M$. Then

$$|U| = \text{Vol}(M, \Omega) = \text{Vol}(N, \Omega).$$

(A.3)

Since $N$ is a connected manifold without boundary, there exists a complete Riemannian metric $g$ on $N$. Indeed, according to a theorem of Whitney [37], $N$ can be embedded as a closed submanifold in some $\mathbb{R}^m$. We can then take the induced Riemannian metric. A direct and elementary proof of the existence of a complete Riemannian metric is given in [28].

Fix a point $p \in N$. As in the proof of Corollary A.5 we identify $(T_p N, g(p))$ with $\mathbb{R}^n$ and define the function $\mu : S_1 \to \mathbb{R}_{>0}$ as in (A.2). Using polar coordinates on $\mathbb{R}^n$ we see from Fubini’s Theorem that the set

$$\tilde{C}(p) = \{\mu(x)x \mid x \in S_1\} \subset \mathbb{R}^n$$

has measure zero, and so $C(p) = \exp_p(\tilde{C}(p))$ also has measure zero (see [3, VI, Corollary 1.14]). It follows that

$$\text{Vol}(N \setminus C(p), \Omega) = \text{Vol}(N, \Omega).$$

(A.4)

According to Corollary A.5 there exists a diffeomorphism

$$\delta : \mathbb{R}^n \to N \setminus C(p).$$

After composing $\delta$ with a reflection of $\mathbb{R}^n$, if necessary, we can assume that $\delta$ is orientation preserving. In view of (A.3) and (A.4) we then have

$$|U| = \text{Vol}(\mathbb{R}^n, \delta^*\Omega).$$

(A.5)

Case 1. $|U| < \infty$.

Let $U_1, U_2, \ldots$ be the countably many components of $U$. Then $0 < |U_i| < \infty$ for each $i$. Given numbers $a$ and $b$ with $-\infty < a < b < \infty$ we abbreviate the “open strip”

$$S_{a,b} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a < x_1 < b\}.$$
In view of the identity (A.5) we have
\[ \sum_{i \geq 1} |U_i| = |U| = \text{Vol}(\mathbb{R}^n, \delta^*\Omega). \]

We can therefore inductively define \( a_0 = -\infty \) and \( a_i \in ]-\infty, \infty[ \) by
\[ \text{Vol}(S_{a_{i-1}, a_i}, \delta^*\Omega) = |U_i|. \]

Abbreviating \( S_i = S_{a_{i-1}, a_i} \) we then have \( \mathbb{R}^n = \bigcup_{i \geq 1} S_i. \)

For each \( i \geq 1 \) we choose an orientation preserving diffeomorphism \( \tau_i : \mathbb{R}^n \to S_i. \) In view of Proposition A.2 we find a smooth embedding \( \sigma_i : U_i \hookrightarrow \mathbb{R}^n \) such that \( \mathbb{R}^n \setminus \sigma_i(U_i) \) has measure zero. After composing \( \sigma_i \) with a reflection of \( \mathbb{R}^n, \) if necessary, we can assume that \( \sigma_i \) is orientation preserving. Using the definition of the volume, we can now conclude that
\[ \text{Vol}(U_i, \sigma_i^* \tau_i^* \delta^*\Omega) = \text{Vol}(\sigma_i(U_i), \tau_i^* \delta^*\Omega) = \text{Vol}(\mathbb{R}^n, \tau_i^* \delta^*\Omega) = \text{Vol}(S_i, \delta^*\Omega) = |U_i|. \]

In view of Proposition A.6 (i) we therefore find a diffeomorphism \( \psi_i \) of \( U_i \) such that
\[ \psi_i^* (\sigma_i^* \tau_i^* \delta^*\Omega) = \Omega_0. \tag{A.6} \]

We define \( \varphi_i : U_i \hookrightarrow M \) to be the composition of diffeomorphisms and smooth embeddings
\[ U_i \xrightarrow{\psi_i} U_i \xrightarrow{\sigma_i} \mathbb{R}^n \xrightarrow{\tau_i} S_i \subset \mathbb{R}^n \delta \rightarrow \mathbb{N} \setminus C(p) \subset M. \]

The identity (A.6) implies that \( \psi_i^* \Omega = \Omega_0. \) The smooth embedding
\[ \varphi = \bigsqcup \varphi_i : U = \bigsqcup U_i \hookrightarrow M \]
therefore satisfies \( \varphi^* \Omega = \Omega_0. \)

**Case 2.** \( |U| = \infty. \)

In view of (A.5) we have \( \text{Vol}(\mathbb{R}^n, \delta^*\Omega) = \infty. \) Proposition A.6 (ii) shows that there exists a diffeomorphism \( \psi \) of \( \mathbb{R}^n \) such that
\[ \psi^* \delta^*\Omega = \Omega_0. \tag{A.7} \]

We define \( \varphi : U \hookrightarrow M \) to be the composition of inclusions and diffeomorphisms
\[ U \subset \mathbb{R}^n \xrightarrow{\psi} \mathbb{R}^n \delta \rightarrow \mathbb{N} \setminus C(p) \subset M. \]

The identity (A.7) implies that \( \varphi^* \Omega = \Omega_0. \) The proof of Theorem A.1 is complete. \( \square \)
B  Symplectic capacities and the invariants $c_B$ and $c_C$

In the first two sections of this appendix we collect and prove those results on symplectic capacities which are relevant for this thesis. We refer to [2, 6, 7, 14, 15, 17, 18, 21, 26, 32, 33, 35] for more information on symplectic capacities. In Section B.3 we compare the invariants $c_B$ and $c_C$ studied in Chapters 1 to 6 with symplectic capacities. In the last two sections we compare $c_B$ with the symplectic diameter and interpret $c_C$ as a symplectic projection invariant.

B.1  Intrinsic and extrinsic symplectic capacities on $\mathbb{R}^{2n}$

We say that a subset $S \subset \mathbb{R}^{2n}$ symplectically embeds into $T \subset \mathbb{R}^{2n}$ if there exists a symplectic embedding $\varphi$ of an open neighbourhood of $S$ into $\mathbb{R}^{2n}$ such that $\varphi(S) \subset T$. We again write $Z^{2n}(a)$ for the symplectic cylinder $D(a) \times \mathbb{R}^{2n-2}$.

**Definition B.1** An intrinsic symplectic capacity on $(\mathbb{R}^{2n}, \omega_0)$ is a map $c$ associating with each subset $S$ of $\mathbb{R}^{2n}$ a number $c(S) \in [0, \infty]$ in such a way that the following axioms are satisfied.

A1. **Monotonicity:** $c(S) \leq c(T)$ if $S$ symplectically embeds into $T$.

A2. **Conformality:** $c(\lambda S) = \lambda^2 c(S)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$.

A3. **Nontriviality:** $0 < c(B^{2n}(\pi))$ and $c(Z^{2n}(\pi)) < \infty$.

An intrinsic symplectic capacity $c$ on $\mathbb{R}^{2n}$ is normalized if

A3’. **Normalization:** $c(B^{2n}(\pi)) = c(Z^{2n}(\pi)) = \pi$.

For any subset $S$ of $\mathbb{R}^{2n}$ we define the Gromov width $w_G(S)$ and the cylindrical capacity $z(S)$ by

$$w_G(S) := \sup \{a \mid B^{2n}(a) \text{ symplectically embeds into } S\},$$

$$z(S) := \inf \{a \mid S \text{ symplectically embeds into } Z^{2n}(a)\}.$$
It follows from Gromov's Nonsqueezing Theorem stated in Example 1 of the introduction that both \( w_G \) and \( z \) are normalized intrinsic symplectic capacities on \( \mathbb{R}^{2n} \). Another example of a normalized intrinsic symplectic capacity on \( \mathbb{R}^{2n} \) is the Hofer–Zehnder capacity [18].

**Proposition B.2** For any subset \( S \) of \( \mathbb{R}^{2n} \) and any normalized intrinsic symplectic capacity \( c \) on \( \mathbb{R}^{2n} \) we have

\[
w_G(S) \leq c(S) \leq z(S).
\]

**Proof.** If \( B^{2n}(a) \hookrightarrow S \) is a symplectic embedding, we conclude from monotonicity, conformality and normalization that \( a = B^{2n}(a) \leq c(S) \). Taking the supremum, we find \( w_G(S) \leq c(S) \). Similarly, if \( S \hookrightarrow Z^{2n}(a) \) is a symplectic embedding, we conclude from the axioms that \( c(S) \leq c(Z^{2n}(a)) = a \). Taking the infimum we find \( c(S) \leq z(S) \). \( \square \)

**Remark B.3** If \( S \) is an open connected subset of \( \mathbb{R}^2 \), the inequalities in Proposition B.2 are equalities, see Corollary B.9(i). If \( n \geq 2 \), however, there exist bounded starshaped domains \( S \subset \mathbb{R}^{2n} \) with arbitrarily small Gromov width and \( z(S) = 1 \) (see [15]). Still, for an ellipsoid and, more generally, a convex Reinhardt domain all normalized symplectic capacities on \( \mathbb{R}^{2n} \) coincide [15]. It is conjectured that this holds for any bounded convex domain [36]. \( \diamond \)

As in Chapter 1 we denote by \( \mathcal{D}(n) \) the group of symplectomorphisms of \( (\mathbb{R}^{2n}, \omega_0) \). We also recall the

**Definition B.4** An extrinsic symplectic capacity on \( (\mathbb{R}^{2n}, \omega_0) \) is a map \( c \) associating with each subset \( S \) of \( \mathbb{R}^{2n} \) a number \( c(S) \in [0, \infty] \) in such a way that the following axioms are satisfied.

A1. **Monotonicity:** \( c(S) \leq c(T) \) if there exists \( \varphi \in \mathcal{D}(n) \) such that \( \varphi(S) \subset T \).

A2. **Conformality:** \( c(\lambda S) = \lambda^2 c(S) \) for all \( \lambda \in \mathbb{R} \setminus \{0\} \).

A3. **Nontriviality:** \( 0 < c(B^{2n}(\pi)) \) and \( c(Z^{2n}(\pi)) < \infty \).

An extrinsic symplectic capacity \( c \) on \( \mathbb{R}^{2n} \) is normalized if

A3'. **Normalization:** \( c(B^{2n}(\pi)) = c(Z^{2n}(\pi)) = \pi \).
Comparing Definitions B.1 and B.4 we see that any intrinsic symplectic capacity on $\mathbb{R}^{2n}$ is an extrinsic symplectic capacity on $\mathbb{R}^{2n}$. The converse is not true as we shall see in Proposition B.6.

For any subset $S$ of $\mathbb{R}^{2n}$ we define

\[ w_G(S) := \sup \{ a \mid \text{there exists } \varphi \in \mathcal{D}(n) \text{ such that } \varphi \left( B^{2n}(a) \right) \subset S \}, \]

\[ z(S) := \inf \{ a \mid \text{there exists } \varphi \in \mathcal{D}(n) \text{ such that } \varphi(S) \subset Z^{2n}(a) \}. \]

It follows again from Gromov's Nonsqueezing Theorem that both $w_G$ and $z$ are normalized extrinsic symplectic capacities on $\mathbb{R}^{2n}$. Other examples of normalized extrinsic symplectic capacities on $\mathbb{R}^{2n}$ are the first Ekeland-Hofer capacity $c_1$, which was introduced in [6] and used in the proof of Theorem 1.4, and the displacement energy $e$ introduced in [17].

**Proposition B.5** For any subset $S$ of $\mathbb{R}^{2n}$ and any normalized extrinsic symplectic capacity $c$ on $\mathbb{R}^{2n}$ we have

\[ w_G(S) \leq c(S) \leq z(S). \]

Moreover, $w_G(S) = w_G(S)$, and $z(S) = z(S)$ where $\delta$ is the union of $\bar{S}$ with the bounded components of $\mathbb{R}^{2n} \setminus \bar{S}$.

**Proof.** The inequalities $w_G(S) \leq c(S) \leq z(S)$ are proved in the same way as Proposition B.2.

The inequality $w_G(S) \leq w_G(S)$ is obvious. In order to prove $w_G(S) \geq w_G(S)$ we fix $\epsilon > 0$ and assume that $\varphi$ symplectically embeds $B^{2n}(a)$ into $S$. According to Proposition C.1 there exists $\Phi \in \mathcal{D}(n)$ such that $\Phi|_{B^{2n}(a-\epsilon)} = \varphi|_{B^{2n}(a-\epsilon)}$. In particular, $\Phi(B^{2n}(a-\epsilon)) \subset S$, and so $w_G(S) \geq a - \epsilon$. Since $\epsilon > 0$ was arbitrary, we conclude that $w_G(S) \geq a$. Taking the supremum over $a$ we find $w_G(S) \geq w_G(S)$.

In view of the monotonicity of $z$ we have $z(S) \leq z(S)$. In order to prove $z(S) \leq z(S)$ we assume that $\varphi \in \mathcal{D}(n)$ embeds $S$ into $Z^{2n}(a)$. Then $\varphi(S) \subset Z^{2n}(a)$. Moreover, if $U$ is a bounded component of $\mathbb{R}^{2n} \setminus S$, then $\varphi(U)$ is a bounded component of $\mathbb{R}^{2n} \setminus \varphi(S)$, and so $\varphi(U) \subset Z^{2n}(a)$. It follows that $\varphi(S) \subset Z^{2n}(a)$, and hence $z(S) \leq a$. Taking the infimum over $a$ we conclude that $z(S) \leq z(S)$. The proof of Proposition B.5 is complete. □

The next proposition shows that neither the first Ekeland–Hofer capacity $c_1$ nor the displacement energy $e$ nor the outer cylindrical capacity $\hat{z}$ are intrinsic symplectic capacities on $\mathbb{R}^{2n}$. Let $S^1 = \{(u, v) \mid u^2 + v^2 = 1\}$ be the unit circle.
Proposition B.6 For the standard Lagrangian torus $T^n = S^1 \times \cdots \times S^1$ in $\mathbb{R}^{2n}$ we have
\[ c_1(T^n) = e(T^n) = \hat{z}(T^n) = \pi \]
while $c(T^n) = 0$ for any intrinsic symplectic capacity $c$ on $\mathbb{R}^{2n}$.

Proof. According to Théorème (b) on page 43 of [33] we have $c_1(T^n) = \pi$, Theorem 1.6 (i) in [17] implies $c_1(T^n) \leq e(T^n)$, and Proposition B.5 shows that $e(T^n) \leq \hat{z}(T^n)$. Since clearly $\hat{z}(T^n) \leq \pi$, we conclude that
\[ \pi = c_1(T^n) \leq e(T^n) \leq \hat{z}(T^n) \leq \pi \]
and so the first assertion follows. In order to prove the second assertion we assume that $c$ is an intrinsic symplectic capacity on $\mathbb{R}^{2n}$. Fix $\varepsilon > 0$. In view of Theorem B.7 (ii) below there exists a symplectic embedding $\varphi: S^1 \hookrightarrow D(\varepsilon)$. The product
\[ \varphi \times \cdots \times \varphi: T^n \hookrightarrow D(\varepsilon) \times \cdots \times D(\varepsilon) \subset Z^{2n}(\varepsilon) \]
symplectically embeds $T^n$ into $Z^{2n}(\varepsilon)$, and so
\[ c(T^n) \leq c\left(Z^{2n}(\varepsilon)\right) = \frac{\varepsilon}{\pi} c\left(Z^{2n}(\pi)\right). \]
Since $\varepsilon > 0$ was arbitrary and since $c\left(Z^{2n}(\pi)\right) < \infty$ it follows that $c(T^n) = 0$ as claimed. \qed

B.2 Symplectic capacities on $\mathbb{R}^2$

If $n \geq 2$ the computation of a capacity of a subset of $\mathbb{R}^{2n}$ is usually a difficult problem. In this section we show that for subsets of $\mathbb{R}^2$ the situation is different.

Since $B^2(\pi) = Z^2(\pi)$ we can assume that any (intrinsic or extrinsic) symplectic capacity $c$ on $\mathbb{R}^2$ is normalized. Given a subset $S$ of $\mathbb{R}^2$ we denote by $\text{Int} \ S$ and $\overline{S}$ its interior and its closure and by $S_i$ the components of $\text{Int} \ S$. As before, $\delta$ is the union of $\overline{S}$ with the bounded components of $\mathbb{R}^2 \setminus \overline{S}$. The outer Lebesgue measure of $S$ is denoted by $\overline{\mu}(S)$, and if $S$ is measurable, $\mu(S)$ denotes its Lebesgue measure.

Theorem B.7 Consider a subset $S$ of $\mathbb{R}^2$.

(i) $w_G(S) = \sup_{i} \mu(S_i)$. 

B.2. Symplectic capacities on \( \mathbb{R}^2 \)

(ii) \( z(S) = \overline{\mu}(S) \).

(iii) If \( S \) is bounded, \( \hat{z}(S) = \mu(\delta) \). If \( S \) is unbounded, \( \hat{z}(S) = \infty \).

Proof. (i) We abbreviate \( s := \sup_i \mu(S_i) \). If \( \varphi \) symplectically embeds \( D(a) \) into \( S_i \), then \( \varphi(D(a)) \subset S_i \) for some \( i \), and so \( w_G(S) \leq s \). In order to prove the reverse inequality we can assume that \( s \in ]0, \infty[ \). If \( s \in ]0, \infty[ \) we fix \( \epsilon \in ]0, s[ \). Then there exists \( S_i \) such that \( \mu(S_i) > s - \epsilon \). According to Proposition 1 of the introduction we find a symplectic embedding of \( D(s - \epsilon) \) into \( S_i \), and so \( w_G(S) \geq s - \epsilon \). Since \( \epsilon \in ]0, s[ \) was arbitrary we conclude that \( w_G(S) \geq s \). A similar argument shows that \( w_G(S) = \infty \) if \( s = \infty \). Assertion (i) thus follows.

(ii) If \( \varphi \) symplectically embeds \( S \) into \( D(a) \), then

\[
\overline{\mu}(S) = \overline{\mu}(\varphi(S)) \leq \overline{\mu}(D(a)) = a.
\]

Taking the infimum over \( a \) we find that \( \overline{\mu}(S) \leq z(S) \). In order to prove the reverse inequality we can assume that \( \overline{\mu}(S) < \infty \). Fix \( \epsilon > 0 \). In view of the definition of \( \overline{\mu}(S) \) there exists an open neighbourhood \( U \) of \( S \) such that \( \mu(U) \leq \overline{\mu}(S) + \epsilon \). By Proposition 1 we find a symplectic embedding \( \varphi \) of \( U \) into \( D(\overline{\mu}(S) + \epsilon) \). Therefore \( z(S) \leq \overline{\mu}(S) + \epsilon \). Since \( \epsilon > 0 \) was arbitrary we conclude that \( z(S) \leq \overline{\mu}(S) \). Assertion (ii) thus follows.

(iii) Using Proposition B.5, the identity in (ii) and the fact that \( \delta \) is measurable we find

\[
\hat{z}(S) = \hat{z}(\delta) \geq z(\delta) = \overline{\mu}(\delta) = \mu(\delta).
\]

In order to prove \( \hat{z}(S) \leq \mu(\delta) \) we therefore need to show \( \hat{z}(\delta) \leq \mu(\delta) \). By construction, \( \delta \) is compact and \( \mathbb{R}^2 \setminus \delta \) is connected. Therefore, each path-component of \( \delta \) is simply connected. Since \( \delta \) is compact, \( \mu(\delta) \) is finite. Fix \( \epsilon > 0 \).

**Lemma B.8** There exists a simply connected open neighbourhood \( W \) of \( \delta \) such that \( \mu(W) \leq \mu(\delta) + 2\epsilon \).

**Proof.** We say that a subset of \( \mathbb{R}^2 \) is **elementary** if it is the union of finitely many open rectangles in \( \mathbb{R}^2 \). By definition of \( \mu \) and since \( \delta \) is compact we find an elementary subset \( R \) of \( \mathbb{R}^2 \) such that \( \delta \subset R \) and \( \mu(R) \leq \mu(\delta) + \epsilon \). Let \( U \) be one of the finitely many components of \( R \). The elementary set \( U \) is diffeomorphic to an open disc from which \( k \) closed discs have been removed. Since \( \delta \cap U \) is a compact subset of \( U \) all of whose path-components are simply connected, the set \( U \setminus \delta \) is open and connected. We therefore find \( k \) curves \( \gamma_i \) in \( \overline{U} \setminus \delta \) and
elementary neighbourhoods $N_i$ of $\gamma_i$ such that the elementary subset $V := U \setminus \bigcup N_i$ is a simply connected neighbourhood of $U \cap \delta$. Applying this construction to each component of $R$ we obtain disjoint simply connected elementary subsets $V_1, \ldots, V_l$ of $R$ whose union contains $\delta$. We finally choose $l-1$ curves $\gamma^i$ in $\mathbb{R}^2 \setminus \bigcup V_i$ and elementary neighbourhoods $N^i$ of $\gamma^i$ such that the elementary set

$$W := \bigcup_{i=1}^l V_i \cup \bigcup_{i=1}^{l-1} N^i$$

is simply connected and such that $\mu\left(\bigcup N^i\right) \leq \varepsilon$. Then $W$ is a simply connected neighbourhood of $\delta$ and

$$\mu(W) \leq \mu\left(\bigcup V_i\right) + \mu\left(\bigcup N^i\right) \leq \mu(R) + \varepsilon \leq \mu(\delta) + 2\varepsilon.$$ 

Lemma B.8 thus follows.

Let $W$ be a simply connected open neighbourhood of $\delta$ as guaranteed by Lemma B.8. Then $W$ is diffeomorphic to $D(\mu(W))$. According to Proposition A.6(i) we therefore find a symplectomorphism $\varphi: W \to D(\mu(W))$. Since $\varphi(\delta)$ is a compact subset of $D(\mu(W))$, we find $a \in [0, \mu(W)]$ such that $\varphi(\delta) \subset D(a)$. According to Proposition C.1 we find a symplectomorphism $\Phi$ of $\mathbb{R}^2$ such that $\Phi|_{D(a)} = \varphi^{-1}|_{D(a)}$. The map $\Phi^{-1}$ is then a symplectomorphism of $\mathbb{R}^2$ which embeds $\delta$ into $D(a)$. It follows that $\tilde{z}(\delta) \leq a \leq \mu(W) \leq \mu(\delta) + 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary we conclude that $\tilde{z}(\delta) \leq \mu(\delta)$. This completes the proof of the first claim in assertion (iii).

If $S$ is unbounded, then $\varphi(S)$ is unbounded for any diffeomorphism $\varphi$ of $\mathbb{R}^2$, and so $\tilde{z}(S) = \infty$. Assertion (iii) thus follows, and so the proof of Theorem B.7 is complete.

**Corollary B.9**

(i) Assume that $S$ is a subset of $\mathbb{R}^2$ such that $\text{Int } S$ is connected and $\mu(\text{Int } S) = \bar{\mu}(S)$. Then $c(S) = \bar{\mu}(S)$ for any normalized intrinsic symplectic capacity $c$ on $\mathbb{R}^2$. If in addition $\mu(\text{Int } S)$ is finite, then $S$ is measurable and so $c(S) = \mu(S)$.

(ii) Assume that $S$ is a bounded subset of $\mathbb{R}^2$ such that $\text{Int } S$ is connected and $\mu(\text{Int } S) = \mu(S)$. Then $S$ is measurable and $c(S) = \mu(S)$ for any normalized extrinsic symplectic capacity $c$ on $\mathbb{R}^2$. 

\[ \square \]
B.2. Symplectic capacities on $\mathbb{R}^2$

**Proof.** Let $c$ be any normalized intrinsic symplectic capacity on $\mathbb{R}^2$. In view of Proposition B.2, (i) and (ii) in Theorem B.7 and the assumptions on $S$ we have

$$w_G(S) \leq c(S) \leq z(S) = \mu(S) = \mu(\text{Int } S) = w_G(S)$$

and so $c(S) = \mu(S)$. If $\mu(\text{Int } S)$ is finite, the identities

$$\mu(\text{Int } S) = \mu(S) = \inf \{ \mu(U) \mid S \subset U, \ U \text{ open} \},$$

the monotonicity of $\mu$ and the additivity of $\mu$ imply that $\mu(S \setminus \text{Int } S) = 0$. Therefore $S = \text{Int } S \cup S \setminus \text{Int } S$ is measurable, and so $c(S) = \mu(S)$.

(ii) Since $S$ is bounded, $\mu(\text{Int } S) = \mu(\delta)$ is finite. This and the inclusion $S \subset \delta$ yield

$$\mu(S \setminus \text{Int } S) \leq \mu(\delta \setminus \text{Int } S) = \mu(\delta) - \mu(\text{Int } S) = 0.$$ 

It follows that $S$ is measurable and that $\mu(S) = \mu(\text{Int } S)$. Let now $c$ be any normalized extrinsic symplectic capacity on $\mathbb{R}^2$. In view of Proposition B.5, (i) and (iii) in Theorem B.7 and the assumptions on $S$ we have

$$w_G(S) = w_G(S) \leq c(S) \leq z(S) = \mu(S) = \mu(\text{Int } S) = w_G(S)$$

and so $c(S) = \mu(\text{Int } S) = \mu(S)$ as claimed. \hfill $\square$

We recall that Gromov's Nonsqueezing Theorem states that a ball $B^{2n}(a)$ symplectically embeds into a cylinder $Z^{2n}(A)$ only if $A \geq a$.

**Corollary B.10** Fix $a \in [0, \pi]$ and a normalized extrinsic symplectic capacity $c$ on $\mathbb{R}^2$. Then the Nonsqueezing Theorem is equivalent to the identity

$$\inf_{\varphi} c\left( p\left( \varphi(B^{2n}(a)) \right) \right) = a \quad \text{(B.1)}$$

where $\varphi$ varies over all symplectomorphisms of $\mathbb{R}^{2n}$ which embed $B^{2n}(a)$ into $Z^{2n}(\pi)$, and where $p: Z^{2n}(\pi) \to B^{2}(\pi)$ is the projection.

**Proof.** Assume first that the Nonsqueezing Theorem does not hold. Then there exist $a' > 0$ and $A < a'$ and a symplectic embedding $\varphi: B^{2n}(a') \to Z^{2n}(A)$. According to Proposition C.1 we find $a'' \in ]A, a'[ and a symplectomorphism $\Phi$ of $\mathbb{R}^{2n}$ such that

$$\Phi\left( B^{2n}(a'') \right) \subset Z^{2n}(A).$$
Notice that \( \frac{a}{a^n} A < a \leq \pi \). We denote the dilatation \( \varrho \mapsto \frac{a^n}{a} \varrho \) of \( \mathbb{R}^{2n} \) by \( \alpha \). The composition \( \alpha^{-1} \circ \Phi \circ \alpha \) is a symplectomorphism of \( \mathbb{R}^{2n} \) which embeds \( B^{2n}(a) \) into the subset \( Z^{2n}(\frac{a^n}{a} A) \) of \( Z^{2n}(\pi) \), i.e.,

\[
p \left( \left( \alpha^{-1} \circ \Phi \circ \alpha \right) \left( B^{2n}(a) \right) \right) \subset D \left( \frac{a^n}{a} A \right).
\]

The axioms for the normalized symplectic capacity \( c \) on \( \mathbb{R}^2 \) imply that

\[
c \left( p \left( \left( \alpha^{-1} \circ \Phi \circ \alpha \right) \left( B^{2n}(a) \right) \right) \right) < \frac{a}{a^n} A.
\]

Since \( \frac{a}{a^n} A < a \) we conclude that (B.1) does not hold.

Assume now that (B.1) does not hold. Since \( c \left( p \left( B^{2n}(a) \right) \right) = c(D(a)) = a \) we then find a symplectomorphism \( \varphi \) of \( \mathbb{R}^{2n} \) and \( \epsilon > 0 \) such that

\[
c \left( p \left( \varphi \left( B^{2n}(a) \right) \right) \right) \leq a - 2\epsilon. \tag{B.2}
\]

We abbreviate \( U := p \left( \varphi \left( B^{2n}(a) \right) \right) \). Since \( U \) is open and connected, Corollary B.9 (i), Proposition B.5 and the estimate (B.2) imply

\[
z(U) = \mu(U) = \omega_G(U) = \hat{\omega}_G(U) \leq c(U) \leq a - 2\epsilon.
\]

We therefore find a symplectic embedding \( \psi : U \hookrightarrow D(a - \epsilon) \). The composition \( (\psi \times i d_{2n-2}) \circ \varphi \) symplectically embeds \( B^{2n}(a) \) into \( Z^{2n}(a - \epsilon) \) and so the Non-squeezing Theorem does not hold. \( \square \)

According to Proposition B.6 we have \( c(S^1) = 0 \) for any intrinsic symplectic capacity \( c \) on \( \mathbb{R}^2 \) and \( c(S^1) = \pi \) for the first Ekeland–Hofer capacity, the displacement energy and the outer cylindrical capacity. The following result, which was pointed out to me by David Hermann, shows that there exist other normalized extrinsic symplectic capacities on \( \mathbb{R}^2 \).

**Proposition B.11** (D. Hermann) *For any \( a \in [0, \pi] \) there exists a normalized extrinsic symplectic capacity \( c \) on \( \mathbb{R}^2 \) such that \( c(S^1) = a \).*

**Proof.** Fix \( a \in [0, \pi] \). For \( r \geq 0 \) we set

\[
c \left( r D(\pi) \right) := r^2 \pi \quad \text{and} \quad c(r S^1) := r^2 a,
\]
and given an arbitrary subset $S \subset \mathbb{R}^2$ we set

$$c(S) := \max \left\{ \hat{w}_G(S), \sigma(S) \right\}$$

where

$$\hat{w}_G(S) := \sup \left\{ r^2 \pi \mid \text{there exists } \varphi \in \mathcal{D}(2) \text{ such that } \varphi(rD(\pi)) \subset S \right\},$$

$$\sigma(S) := \sup \left\{ r^2 a \mid \text{there exists } \varphi \in \mathcal{D}(2) \text{ such that } \varphi(rS^1) \subset S \right\}.$$ 

Then $c$ is a well-defined normalized extrinsic symplectic capacity on $\mathbb{R}^2$, and $c(S^1) = a$. \hfill \Box

### B.3 Comparison of $c_B$ and $c_C$ with symplectic capacities

For any subset $S$ of $\mathbb{R}^{2n}$ we define the symplectic invariants $c_B$ and $c_C$ by

$$c_B(S) := \inf \{ a \mid S \text{ symplectically embeds into } B^{2n}(a) \}, \quad (B.3)$$

$$c_C(S) := \inf \{ a \mid S \text{ symplectically embeds into } C^{2n}(a) \}. \quad (B.4)$$

For $n = 1$, the invariants $c_B$ and $c_C$ coincide with the cylindrical capacity $z$. For $n \geq 2$, the invariants $c_B$ and $c_C$ fulfill all axioms of a normalized intrinsic symplectic capacity on $\mathbb{R}^{2n}$ except that they are infinite for the symplectic cylinder. Recall from Proposition B.2 that the cylindrical capacity $z$ is the largest normalized intrinsic symplectic capacity on $\mathbb{R}^{2n}$.

**Proposition B.12** For any subset $S$ of $\mathbb{R}^{2n}$ we have

$$z(S) \leq c_C(S) \leq c_B(S).$$

**Proof.** The claim follows from the inclusions $B^{2n}(a) \subset C^{2n}(a) \subset Z^{2n}(a)$. \hfill \Box

**Remark B.13** If $n \geq 2$, the inequalities in Proposition B.12 are in general not equalities: For a polydisc $P = P^{2n}(\pi, \ldots, \pi, a)$ with $a > \pi$ we have $z(P) = \pi < c_C(P)$. Moreover, we see from (1.5) and (1.9) that $c_C(C^{2n}(\pi)) = \pi < c_B(C^{2n}(\pi)) = n\pi$. (This example is extremal in the sense that for any $S \subset \mathbb{R}^{2n}$ we have $c_B(S) \leq n c_C(S)$.)
B.4 Comparison of $c_B$ with the symplectic diameter

Recall that for any subset $S$ of $\mathbb{R}^{2n}$ the \textit{diameter} $d(S)$ is defined by

$$d(S) := \sup \{|z - z'| \mid z, z' \in S\}.$$

Symplectifying the Euclidean invariant $d(S)$ we obtain the symplectic invariant $d_s(S)$ defined by

$$d_s(S) := \inf \left\{ d(\varphi(S)) \mid \varphi \text{ symplectically embeds } S \text{ into } \mathbb{R}^{2n} \right\}.$$

For convenience we shall work with the symplectic invariant $\delta(S)$ defined by

$$\delta(S) := \pi \left( \frac{d_s(S)}{2} \right)^2 \quad \text{(B.5)}$$

rather than with $d_s(S)$.

**Theorem B.14** For any subset $S$ of $\mathbb{R}^{2n}$ we have

$$\delta(S) \leq c_B(S) \leq \frac{4n}{2n + 1} \delta(S) \quad \text{(B.6)}$$

and if $n = 1$, then $\delta(S) = c_B(S) = \bar{\mu}(S)$.

**Proof.** Fix $S \subseteq \mathbb{R}^{2n}$. The \textit{circumradius} $R(S)$ of $S$ is the radius of the smallest ball containing $S$, i.e.,

$$R(S) := \inf \left\{ R \mid \text{there exists } w \in \mathbb{R}^{2n} \text{ such that } \tau_w(S) \subseteq B^{2n} \left( \pi R^2 \right) \right\},$$

where $\tau_w$ denotes the translation $z \mapsto z + w$ of $\mathbb{R}^{2n}$. Symplectifying the Euclidean invariant $R(S)$ we obtain the symplectic invariant

$$R_s(S) := \inf \left\{ R(\varphi(S)) \mid \varphi \text{ symplectically embeds } S \text{ into } \mathbb{R}^{2n} \right\}.$$

Notice that each translation $\tau_w$ of $\mathbb{R}^{2n}$ is symplectic. Comparing the definition (B.3) of the invariant $c_B(S)$ with the definitions of $R(S)$ and $R_s(S)$ we therefore find that

$$c_B(S) = \pi \left( R_s(S) \right)^2. \quad \text{(B.7)}$$
The main ingredient of the proof of Theorem B.14 are the inequalities
\[ \frac{1}{2} d(T) \leq R(T) \leq \sqrt{\frac{n}{2n+1}} d(T) \] (B.8)
valid for any subset \( T \) of \( \mathbb{R}^{2n} \). While the left inequality in (B.8) is obvious, the right inequality is the main content of Jung's Theorem [4, Chapter 2, Theorem 11.1.1]. Applying (B.8) to all symplectic images \( \varphi(S) \) of \( S \) in \( \mathbb{R}^{2n} \) and taking the infimum we find that
\[ \frac{1}{2} d_{\varphi}(S) \leq R_{\varphi}(S) \leq \sqrt{\frac{n}{2n+1}} d_{\varphi}(S). \]
These inequalities, the definition (B.5) and the identity (B.7) imply (B.6).

Assume now \( S \subset \mathbb{R}^{2} \). The left inequality in (B.6) and Theorem B.7 (ii) show that
\[ \delta(S) \leq c_B(S) = z(S) = \mu(S). \]
In order to show that \( \delta(S) \geq \mu(S) \) we assume that \( \varphi \) symplectically embeds \( S \) into \( \mathbb{R}^{2} \) and denote the convex hull of \( \varphi(S) \) by \( \text{conv}\varphi(S) \). Applying the Bieberbach inequality [4, Chapter 2, Theorem 11.2.1] to \( \varphi(S) \) we find
\[ \pi \left( \frac{d(\varphi(S))}{2} \right)^{2} \geq \mu(\text{conv}\varphi(S)) \geq \mu(\varphi(S)) = \mu(S). \]
Since \( \varphi \) was arbitrary it follows that \( \delta(S) \geq \mu(S) \). We conclude that \( \delta(S) = c_B(S) = \mu(S) \). The proof of Theorem B.14 is complete. \( \square \)

Remarks B.15

1. Assume that \( n \geq 2 \). It follows from the Bieberbach inequality [4, Chapter 2, Theorem 11.2.1] that \( \delta(B^{2n}(\pi)) = c_B(B^{2n}(\pi)) \) and so the left inequality in (B.6) is sharp. We don’t know, however, whether the right inequality in (B.6) is sharp.

2. The invariants \( \delta = \pi \left( \frac{d}{2} \right)^{2} \) and \( c_B = \pi(R)^{2} \) are examples of “Euclidean invariants for symplectic domains” as studied in [31].

B.5 The invariant \( c_{C} \) as a symplectic projection invariant

As was pointed out in [10, p. 580], symplectic capacities on \( \mathbb{R}^{2n} \) measure to some extent the area of two dimensional symplectic projections of a set. We are now going to make this point precise for the invariants \( z \) and \( c_{C} \).
For $i = 1, \ldots, n$ we set
\[ E_i = \{ z \in \mathbb{C}^n \mid z_j = 0 \text{ for } j \neq i \}, \]
and we denote the orthogonal projection $\mathbb{R}^{2n} \rightarrow E_i$ by $p_i$. The Lebesgue measure $d x_i \wedge d y_i$ on $E_i$ is denoted $\mu_i$, and the outer Lebesgue measure on $E_i$ is denoted $\overline{\mu}_i$. Given any subset $S$ of $\mathbb{R}^{2n}$ we set
\[ s_m(S) = \inf_{\varphi} \min_{1 \leq i \leq n} \overline{\mu}_i (p_i(\varphi(S))), \]
\[ s_M(S) = \inf_{\varphi} \max_{1 \leq i \leq n} \overline{\mu}_i (p_i(\varphi(S))), \]
where $\varphi$ varies over all symplectic embeddings of $S$ into $\mathbb{R}^{2n}$.

**Remarks B.16**

1. If $S \subset \mathbb{R}^{2n}$ is Lebesgue measurable, then so are the sets $p_i(\varphi(S)) \subset E_i$, $i = 1, \ldots, n$, and so
\[ s_m(S) = \inf_{\varphi} \min_{1 \leq i \leq n} \text{area} (p_i(\varphi(S))), \]
\[ s_M(S) = \inf_{\varphi} \max_{1 \leq i \leq n} \text{area} (p_i(\varphi(S))). \]

2. Composing an “infimal” embedding $\varphi : S \hookrightarrow \mathbb{R}^{2n}$ in the definition of $s_m(S)$ with the linear symplectomorphism
\[
(z_1, z_2, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_n) \mapsto (z_i, z_2, \ldots, z_{i-1}, z_1, z_{i+1}, \ldots, z_n)
\]
we find that for all subsets $S$ of $\mathbb{R}^{2n}$,
\[ s_m(S) = \inf_{\varphi} \overline{\mu}_1 (p_1(\varphi(S))). \]

**Theorem B.17** For any subset $S$ of $\mathbb{R}^{2n}$ we have
\[ z(S) = s_m(S) \quad \text{and} \quad c_G(S) = s_M(S). \]

**Proof.** We show $z(S) \leq s_m(S)$. We may assume that $s_m(S)$ is finite. Fix $\epsilon > 0$. By Remark B.16.2 there exists a symplectic embedding $\varphi : S \hookrightarrow \mathbb{R}^{2n}$ such that $\overline{\mu}_1 (p_1(\varphi(S))) \leq s_m(S) + \epsilon$. According to Theorem B.7 (ii) we find a symplectic embedding $\psi$ of $p_1(\varphi(S))$ into $D(s_m(S) + 2\epsilon)$. The composition $(\psi \times id_{2n-2}) \circ \varphi$ symplectically embeds $S$ into $Z^{2n} (s_m(S) + 2\epsilon)$. Since $\epsilon > 0$ was arbitrary, we conclude $z(S) \leq s_m(S)$. The reverse inequality $s_m(S) \leq z(S)$ is obvious.

The equality $c_G(S) = s_M(S)$ is proved in the same way as the equality $z(S) = s_m(S)$. \hfill \Box
**Remark B.18** The invariants $s_m = z$ and $s_M = c_C$ are special symplectic projection invariants as studied in [31].
B. Symplectic capacities and the invariants $c_B$ and $c_C$
C The Extension after Restriction Principle

Recall that a subset $U$ of $\mathbb{R}^n$ is said to be starlike if $U$ contains a point $p$ such that for every point $x \in U$ the straight line between $p$ and $x$ is contained in $U$. The following result is well known [6] and reproved below.

**Proposition C.1** Assume that $\varphi: U \hookrightarrow \mathbb{R}^{2n}$ is a symplectic embedding of a bounded starlike domain $U \subset \mathbb{R}^{2n}$. Then for any subset $A \subset U$ whose closure in $\mathbb{R}^{2n}$ is contained in $U$ there exists a compactly supported symplectomorphism $\Phi_A$ of $\mathbb{R}^{2n}$ such that $\Phi_A|_A = \varphi|_A$.

The purpose of this appendix is to prove a version of Proposition C.1 for unbounded starlike domains.

**Definition C.2** Consider a symplectic embedding $\varphi: U \hookrightarrow \mathbb{R}^{2n}$ of a starlike domain $U \subset \mathbb{R}^{2n}$. We say that the pair $(U, \varphi)$ has the extension property if for each subset $A \subset U$ whose closure in $\mathbb{R}^{2n}$ is contained in $U$ there exists a symplectomorphism $\Phi_A$ of $\mathbb{R}^{2n}$ such that $\Phi_A|_A = \varphi|_A$.

The following example shows that not every pair $(U, \varphi)$ as above has the extension property.

**Example C.3** We let $U \subset \mathbb{R}^2$ be the strip $]1, \infty[ \times ]-1, 1[$. Combining the methods used in Step 1 and Step 4 of Section 2.2 we find a symplectic embedding $\varphi: U \hookrightarrow \mathbb{R}^2$ such that $\varphi(k, 0) = \left(\frac{1}{k}, 0\right)$, $k = 2, 3, \ldots$. Then there does not exist any subset $A$ of $U$ containing the set $\{(k, 0) \mid k = 2, 3, \ldots\}$ for which $\varphi|_A$ extends to a diffeomorphism of $\mathbb{R}^2$. \hfill \diamond

Observe that if $(U, \varphi)$ has the extension property, then $\varphi$ is proper in the sense that each subset $A \subset U$ whose closure in $\mathbb{R}^{2n}$ is contained in $U$ and whose image $\varphi(A)$ is bounded is bounded. The map $\varphi$ in Example C.3 is not proper in this sense. However, the map $\varphi$ in the following example is proper in this sense, and still $(U, \varphi)$ does not have the extension property.
Example C.4 Let $U \subset \mathbb{R}^2$ be the strip $\mathbb{R} \times ]-1, 0[$, and let

$$A = \{(x, y) \in U \mid |y + \frac{1}{2}| \leq f(x)\}$$

where $f: \mathbb{R} \to ]0, \frac{1}{2}[$ is a smooth function such that

$$\int_{\mathbb{R}} \left(\frac{1}{2} - f(x)\right) dx < \infty,$$

Figure 61: A pair $(U, \varphi)$ which does not have the extension property.

cf. Figure 61. Using the method used in Step 4 of Section 2.2 we find a symplectic embedding $\varphi: U \hookrightarrow \mathbb{R}^2$ such that

$$\varphi(x, y) = (x, y) \text{ if } x \geq 1 \quad \text{and} \quad \varphi(x, y) = (-x, -y) \text{ if } x \leq -1,$$

cf. Figure 61. In view of the estimate (C.1) the component $C$ of $\mathbb{R}^2 \setminus \varphi(A)$ which contains the point $(1, 0)$ has finite volume. Any symplectomorphism $\Phi_A$ of $\mathbb{R}^2$ such that $\Phi_A|A = \varphi|A$ would map the "upper" component of $\mathbb{R}^2 \setminus A$, which has infinite volume, to $C$. This is impossible. \[\Diamond\]
Example C.4 shows that the assumption (C.2) on \( \varphi \) in Theorem C.6 below cannot be omitted. For technical reasons in the proof of Theorem C.6 we shall also impose a mild convexity condition on the starlike domain \( U \). The length of a smooth curve \( \gamma : [0, 1] \rightarrow \mathbb{R}^n \) is defined by

\[
\text{length}(\gamma) := \int_0^1 |\gamma'(s)| \, ds.
\]

On any domain \( U \subset \mathbb{R}^n \) we define a distance function \( d_U : U \times U \rightarrow \mathbb{R} \) by

\[
d_U(z, z') := \inf \{ \text{length}(\gamma) \}
\]

where the infimum is taken over all smooth curves \( \gamma : [0, 1] \rightarrow U \) with \( \gamma(0) = z \) and \( \gamma(1) = z' \). Then \( |z - z'| \leq d_U(z, z') \) for all \( z, z' \in U \).

**Definition C.5** We say that a domain \( U \subset \mathbb{R}^n \) is a Lipschitz domain if there exists a constant \( \lambda > 0 \) such that

\[
d_U(z, z') \leq \lambda |z - z'| \quad \text{for all } z, z' \in U.
\]

Each convex domain \( U \subset \mathbb{R}^n \) is a Lipschitz domain with Lipschitz constant \( \lambda = 1 \). It is not hard to see that there do exist starlike domains which are not Lipschitz domains. But we do not know of a starlike domain with smooth boundary which is not a Lipschitz domain.

**Theorem C.6** Assume that \( \varphi : U \hookrightarrow \mathbb{R}^{2n} \) is a symplectic embedding of a starlike Lipschitz domain \( U \subset \mathbb{R}^{2n} \) such that there exists a constant \( L > 0 \) satisfying

\[
|\varphi(z) - \varphi(z')| \geq L |z - z'| \quad \text{for all } z, z' \in U.
\]

Then the pair \( (U, \varphi) \) has the extension property.

**Proof.** We shall first proceed along the lines of [6] and then verify that our assumptions on \( U \) and \( \varphi \) are sufficient to push the arguments through.

**Step 1. Reduction to a simpler case**

We start with observing that we may assume that \( U \) is starlike with respect to the origin and that \( \varphi(0) = 0 \) and \( d\varphi(0) = id \). Indeed, suppose that Theorem C.6 holds in this situation, that \( A \) is a subset of \( U \) whose closure in \( \mathbb{R}^{2n} \) is contained in \( U \), and that \( U \) is starlike with respect to \( p \neq 0 \) or that \( \varphi(p) \neq 0 \) or that
$D := d\varphi(p) \neq id$. For $w \in \mathbb{R}^{2n}$ we denote by $\tau_w$ the translation $z \mapsto z + w$. We define the symplectic embedding $\psi : (D \circ \tau_p)(U) \hookrightarrow \mathbb{R}^{2n}$ by

$$\psi := \tau_{-\varphi(p)} \circ \varphi \circ \tau_p \circ D^{-1}.$$  

Then $\psi(p) = 0$ and $d\psi(p) = id$. Since $U$ is starlike with respect to $p$ and $D$ is linear, the domain $(D \circ \tau_p)(U)$ is starlike with respect to the origin, and $(D \circ \tau_p)(A)$ is a subset of $(D \circ \tau_p)(U)$ whose closure in $\mathbb{R}^{2n}$ is contained in $(D \circ \tau_p)(U)$. Assume next that $U$ is a $\lambda$-Lipschitz domain. We fix $w, w' \in (D \circ \tau_p)(U)$ and set $z = D^{-1}(w) + p, z' = D^{-1}(w') + p$. Given any smooth path $\gamma : [0, 1] \rightarrow U$ with $\gamma(0) = z, \gamma(1) = z'$, the smooth path $D \circ \tau_p \circ \gamma : [0, 1] \rightarrow (D \circ \tau_p)(U)$ runs from $w$ to $w'$, and so

$$d((D \circ \tau_p)(U))(w, w') \leq \int_0^1 |D\gamma'(s)| ds \leq \|D\| \int_0^1 |\gamma'(s)| ds.$$  

It follows that

$$d((D \circ \tau_p)(U))(w, w') \leq \|D\| \|d_U(z, z')\| \leq \|D\| \lambda \|z - z'\| \leq \|D\| \lambda \|D^{-1}\| \|w - w'\|.$$  

Since $w, w' \in (D \circ \tau_p)(U)$ were arbitrary, we conclude that $(D \circ \tau_p)(U)$ is a $\|D\| \|D^{-1}\| \lambda$-Lipschitz domain. Finally, the assumption (C.2) on $\varphi$ yields

$$|\psi(z) - \psi(z')| \geq L \|D^{-1}(z - z')\| \geq \frac{L}{\|D\|} |z - z'|$$  

for all $z, z' \in (D \circ \tau_p)(U)$. By assumption we therefore find a symplectomorphism $\Psi(D \circ \tau_p)(A)$ of $\mathbb{R}^{2n}$ such that $\Psi(D \circ \tau_p)(A) = \psi|(D \circ \tau_p)(A)|$. Define the symplectomorphism $\Phi_A$ of $\mathbb{R}^{2n}$ by

$$\Phi_A := \tau_{\varphi(p)} \circ \Psi(D \circ \tau_p)(A) \circ D \circ \tau_p.$$  

Then $\Phi|_A = \varphi|_A$, as required.
Step 2. The classical approach

So assume that $U$ is starlike with respect to the origin and that $\varphi(0) = 0$ and $d\varphi(0) = id$. We denote the set of symplectic embeddings of $U$ into $\mathbb{R}^{2n}$ by $\text{Symp}(U, \mathbb{R}^{2n})$. Since $U$ is starlike with respect to the origin we can define a continuous path $\varphi_t \subset \text{Symp}(U, \mathbb{R}^{2n})$ by setting

$$
\varphi_t(z) := \begin{cases} 
z & \text{if } t = 0, \\
\frac{1}{t} \varphi(tz) & \text{if } t \in [0, 1].
\end{cases}
$$

(C.3)

The path $\varphi_t$ is smooth except possibly at $t = 0$. In order to smoothen $\varphi_t$, we define the diffeomorphism $\eta$ of $[0, 1]$ by

$$
\eta(t) := \begin{cases} 
0 & \text{if } t = 0, \\
\frac{1}{2} e^{-2/t} & \text{if } t \in [0, 1],
\end{cases}
$$

(C.4)

where $e$ denotes the Euler number, and for $t \in [0, 1]$ and $z \in U$ we set

$$
\phi_t(z) := \varphi_{\eta(t)}(z).
$$

(C.5)

Then $\phi_t$ is a smooth path in $\text{Symp}(U, \mathbb{R}^{2n})$. We have $\phi_0 = id_U$ and $\phi_1 = \varphi$.

Since $U$ is starlike, it is contractible, and so the same holds true for all the open sets $\phi_t(U)$, $t \in [0, 1]$. We therefore find a smooth time-dependent Hamiltonian function

$$
H: \bigcup_{t \in [0, 1]} \{t\} \times \phi_t(U) \rightarrow \mathbb{R}
$$

(C.6)

generating the path $\phi_t$, i.e., $\phi_t$ is the solution of the Hamiltonian system

$$
\begin{align*}
\frac{d}{dt}\phi_t(z) &= J \nabla H_t(\phi_t(z)), & z \in U, & t \in [0, 1], \\
\phi_0(z) &= z, & z \in U.
\end{align*}
$$

(C.7)

Here, $J$ denotes the standard complex structure defined by

$$
\omega_0(z, w) = \langle Jz, w \rangle, \quad z, w \in \mathbb{R}^{2n}.
$$

The function $H(z, t) = H_t(z)$ is determined by the first equation in (C.7) up to a smooth function $h(t): [0, 1] \rightarrow \mathbb{R}$. Notice that $0 \in \phi_t(U)$ for all $t$. We choose $h(t)$ such that

$$
H_t(0) = 0 \quad \text{for all } t \in [0, 1].
$$

(C.8)
Step 3. Intermezzo: End of the proof of Proposition C.1
Before proceeding with the proof of Theorem C.6 we shall prove Proposition C.1. Fix a subset \( A \) of \( U \) whose closure \( \overline{A} \) in \( \mathbb{R}^{2n} \) is contained in \( U \). Since \( U \) is bounded, the set \( \overline{A} \) is compact, and so the set
\[
K = \bigcup_{t \in [0, 1]} \{ t \} \times \text{cpt}(\overline{A}) \subset [0, 1] \times \mathbb{R}^{2n}
\]
is also compact and hence bounded. We therefore find a bounded neighbourhood \( V \) of \( K \) which is open in \([0, 1] \times \mathbb{R}^{2n}\) and is contained in the set \( \bigcup_{t \in [0, 1]} \{ t \} \times \phi_t(U) \). By Whitney’s Theorem, there exists a smooth function \( f \) on \([0, 1] \times \mathbb{R}^{2n}\) which is equal to 1 on \( A \) and vanishes outside \( V \). Since \( V \) is bounded, the function \( fH: [0, 1] \times \mathbb{R}^{2n} \rightarrow \mathbb{R} \) has compact support, and so the Hamiltonian system associated with \( fH \) can be solved for all \( t \in [0, 1] \). We define \( \Phi_A \) to be the resulting time-1-map. Then \( \Phi_A \) is a globally defined symplectomorphism of \( \mathbb{R}^{2n} \) with compact support, and \( \Phi_A|_A = \varphi|_A \). The proof of Proposition C.1 is thus complete.

Step 4. End of the proof of Theorem C.6
If the set \( U \) is not bounded, the subset \( A \subset U \) does not need to be relatively compact, and so there might be no cut off \( fH \) of \( H \) whose Hamiltonian flow exists for all \( t \in [0, 1] \). We therefore need to extend the Hamiltonian \( H \) more carefully. We shall first verify that our assumption (C.2) on \( \varphi \) implies that \( \nabla H \) is linearly bounded. Since we do not know a direct way to extend a linearly bounded gradient field to a linearly bounded gradient field, we shall then pass to the function
\[
G(t, w) = \frac{H(t, w)}{g(|w|)}
\]
where \( g(|w|) = |w| \) for \( |w| \) large. Our assumption that \( U \) is a Lipschitz domain will imply that \( G \) is Lipschitz continuous in \( w \) and can hence be extended to a Lipschitz continuous function \( \widetilde{G} \) on \([0, 1] \times \mathbb{R}^{2n}\) . After smoothing \( \widetilde{G} \) in \( w \) to \( \widetilde{G} \) we shall obtain an extension \( \widetilde{H}(t, w) = g(|w|)\widetilde{G}(t, w) \) whose gradient is linearly bounded.

Lemma C.7 Let \( L > 0 \) be the constant guaranteed by assumption (C.2).

(i) \[ |\phi_t(z) - \phi_t(z')| \geq L|z - z'| \quad \text{for all} \quad t \in [0, 1] \quad \text{and} \quad z, z' \in U. \]

(ii) \[ \|d\phi_t(z)\| \leq \frac{1}{L} \quad \text{for all} \quad t \in [0, 1] \quad \text{and} \quad z \in U. \]
Proof. (i) In view of definitions (C.5) and (C.3) we have

\[ \phi_t(z) = \frac{1}{\eta(t)^r} \varphi(\eta(t)z) \]  

(C.9)

for all \( t \in [0, 1] \) and \( z \in U \). Together with assumption (C.2) we find

\[ \| \phi_t(z) - \phi_t(z') \| = \frac{1}{\eta(t)} \| \varphi(\eta(t)z) - \varphi(\eta(t)z') \| \]
\[ \geq \frac{1}{\eta(t)} L \| \eta(t)z - \eta(t)z' \| \]
\[ = L \| z - z' \|. \]

Assertion (i) thus follows.

(ii) We fix \( t \in [0, 1] \) and \( z \in U \). Following the proof of Proposition 2.20 in [26] we decompose the linear symplectomorphism \( d\phi_t(z) \) as

\[ d\phi_t(z) = PQ \]

where both \( P \) and \( Q \) are symplectic and \( P \) is symmetric and positive definite and \( Q \) is orthogonal. According to [26, Lemma 2.18] the eigenvalues of \( P \) are of the form

\[ 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \lambda_1^{-1} \leq \ldots \leq \lambda_2^{-1} \leq \lambda_1^{-1}. \]

Since \( Q \) is orthogonal, we find

\[ \| d\phi_t(z) \| = \| P \| = \lambda_1^{-1}. \] (C.10)

Let \( v_1 \) be an eigenvector of \( \lambda_1 \). In view of assertion (i) we have

\[ \lambda_1 \| v_1 \| = \| d\phi_t(z)v_1 \| \geq L \| v_1 \| \]

and so \( \lambda_1^{-1} \leq L^{-1} \). This and the identity (C.10) yield \( \| d\phi_t(z) \| \leq L^{-1} \), and so assertion (ii) follows. \( \square \)

For \( r > 0 \) we denote by \( B(r) \) the closed \( r \)-ball around \( 0 \in \mathbb{R}^{2n} \). We choose \( \epsilon > 0 \) so small that \( B(\epsilon) \subset U \). Finally, we abbreviate

\[ U_t = U \cap B \left( \frac{\epsilon}{e} e^{1/t} \right), \quad t \in [0, 1]. \] (C.11)
Lemma C.8  (i) There exists a constant \( C_1 > 0 \) such that
\[
|\nabla H_t(w)| \leq \frac{C_1}{t^2} |w| \quad \text{for all } t \in [0, 1] \text{ and } w \in \phi_t(U).
\]

(ii) There exists a constant \( c_1 > 0 \) such that
\[
|\nabla H_t(w)| \leq \frac{c_1}{t^2} e^{-1/t} |w| \quad \text{for all } t \in [0, 1] \text{ and } w \in \phi_t(U_t).
\]

Proof. (i) Fix \( t \in [0, 1] \) and \( w = \phi_t(z) \). Using the first line in (C.7) and the definitions (C.9) and (C.4) we compute
\[
\begin{align*}
J \nabla H_t(w) &= \frac{d}{dt} \phi_t(z) \\
&= \frac{d}{dt} \left( \frac{1}{\eta(t)} \varphi(\eta(t)z) \right) \\
&= \frac{\eta'(t)}{\eta(t)} \left( -\frac{1}{\eta(t)} \varphi(\eta(t)z) + d\varphi(\eta(t)z)z \right) \\
&= \frac{2}{t^2} (-w + d\varphi(\eta(t)z)z). \quad (C.12)
\end{align*}
\]

Lemma C.7 (ii) with \( t = 1 \) yields
\[
\|d\varphi(z)\| \leq \frac{1}{L} \quad \text{for all } z \in U \quad (C.14)
\]
and the identity \( \phi_t(0) = \frac{1}{\eta(t)} \varphi(0) = 0 \) and Lemma C.7 (i) with \( z' = 0 \) yield
\[
|w| = |\phi_t(z)| \geq L|z|. \quad (C.15)
\]

In view of the identity (C.13) and the estimates (C.14) and (C.15) we conclude
\[
|\nabla H_t(w)| = |J \nabla H_t(w)| \leq \frac{2}{t^2} (|w| + \|d\varphi(\eta(t)z)\||z|) \\
\leq \frac{2}{t^2} \left( |w| + \frac{1}{L^2} |w| \right) \\
= \frac{2}{t^2} \left( 1 + \frac{1}{L^2} \right) |w|.
\]

The constant \( C_1 := 2 \left( 1 + \frac{1}{L^2} \right) \) is as required.
(ii) By the choice of \( \varepsilon \), the smooth map \( \varphi \) is \( C^2 \)-bounded on \( B(\varepsilon) \), and so Taylor's Theorem applied to \( \varphi: B(\varepsilon) \to \mathbb{R}^{2n} \) and \( d\varphi: B(\varepsilon) \to \mathcal{L}(\mathbb{R}^{2n}) \) guarantees constants \( M_1 \) and \( M_2 \) such that for each \( x \in B(\varepsilon) \),

\[
\varphi(x) = \varphi(0) + d\varphi(0)x + r(x) \quad \text{with} \quad |r(x)| \leq M_1 |x|^2, \\
d\varphi(x) = d\varphi(0) + R(x) \quad \text{with} \quad \|R(x)\| \leq M_2 |x|,
\]

where \( \|R(x)\| \) denotes the operator norm of the linear operator \( R(x) \in \mathcal{L}(\mathbb{R}^{2n}) \).

Since \( \varphi(0) = 0 \) and \( d\varphi(0) = id_{\mathbb{R}^{2n}} \) we conclude that

\[
|\varphi(x) - d\varphi(x)x| = |r(x) - R(x)x| \leq (M_1 + M_2) |x|^2 \quad \text{if} \quad |x| \leq \varepsilon
\]

and so, with \( x = \eta(t)z \),

\[
\frac{1}{|\eta(t)|} |\varphi(\eta(t)z) - d\varphi(\eta(t)z)z| \leq (M_1 + M_2)|\eta(t)| |z|^2 \quad \text{if} \quad |\eta(t)| |z| \leq \varepsilon. \quad (C.16)
\]

Assume now \( z \in U_1 \). In view of the definition (C.11) of \( U_t \) we then have

\[
|\eta(t)z| \leq e^2 e^{-2/\varepsilon} e^{1/\varepsilon} = ee^{-1/\varepsilon} \leq \varepsilon.
\]

Inserting the estimate (C.16) into (C.12) and using (C.15) we conclude that

\[
|\nabla H_t(w)| \leq \frac{2}{t^2} (M_1 + M_2)|\eta(t)||z|^2 \\
\leq \frac{2}{t^2} (M_1 + M_2)ee^{-1/\varepsilon} |z| \\
\leq \frac{2}{t^2} ee^{-1/\varepsilon} (M_1 + M_2)ee^{1/L} |w|.
\]

The constant \( c_1 := 2(M_1 + M_2)ee^{1/L} \) is as required. \( \square \)

**Lemma C.9**

(i) There exists a constant \( C_2 > 0 \) such that

\[
|H_t(w)| \leq \frac{C_2}{t^2} |w|^2 \quad \text{for all} \quad t \in ]0, 1[ \quad \text{and} \quad w \in \phi_t(U).
\]

(ii) There exists a constant \( c_2 > 0 \) such that

\[
|H_t(w)| \leq \frac{c_2}{t^2} ee^{-1/\varepsilon} |w|^2 \quad \text{for all} \quad t \in ]0, 1[ \quad \text{and} \quad w \in \phi_t(U_t).
\]
Proof. (i) Fix $t \in [0, 1]$ and $w = \phi_t(z)$. The smooth path
\[ \gamma : [0, 1] \to \phi_t(U), \quad \gamma(s) = \phi_t(sz) \]
joins 0 with $w$. Since $H_t(0) = 0$ we find that
\[
H_t(w) = H_t(0) + \int_0^1 \langle \nabla H_t(\gamma(s)), \gamma'(s) \rangle \, ds \\
= \int_0^1 \langle \nabla H_t(\phi_t(sz)), d\phi_t(sz)z \rangle \, ds. \tag{C.17}
\]
The identity $\phi_t(0) = 0$, the mean value theorem and Lemma C.7 (ii) yield
\[
|\phi_t(sz)| = |\phi_t(sz) - \phi_t(0)| \leq \frac{1}{L} \|z\|. \tag{C.18}
\]
Using the identity (C.17), Lemma C.8 (i), Lemma C.7 (ii) and the estimates (C.18) and (C.15) we can estimate
\[
|H_t(w)| \leq \int_0^1 |\nabla H_t(\phi_t(sz))| |d\phi_t(sz)z| \, ds \\
\leq \frac{C_1}{t^2} \frac{1}{L} \|z\| \int_0^1 |\phi_t(sz)| \, ds \\
\leq \frac{C_1}{t^2} \frac{1}{L^2} \|z\|^2 \frac{1}{2} \\
\leq \frac{1}{2} \frac{C_1}{t^2} \frac{1}{L^2} \|w\|^2.
\]
The constant $C_2 := \frac{1}{2} C_1 \frac{1}{L^2}$ is as required.

(ii) Assume now $z \in U_t$. Using Lemma C.8 (ii) and estimating as above we obtain
\[
|H_t(w)| \leq \frac{1}{2} C_1 \frac{1}{L^2} \frac{1}{t^2} e^{-1/t} \|w\|^2.
\]
The constant $c_2 := \frac{1}{2} C_1 \frac{1}{L^2}$ is as required. \( \square \)

Choose a smooth function $g : [0, \infty[ \to [1, \infty]$ such that
\[
g(r) = \begin{cases} 
1 & \text{if } r \leq \frac{1}{2}, \\
r & \text{if } r \geq 2
\end{cases} \tag{C.19}
\]
and $0 \leq g'(r) \leq 1$ for all $r$. We define the smooth function $G : \bigcup_{t \in [0,1]} [t] \times \phi_t(U) \to \mathbb{R}$ by

$$G(t, w) = G_t(w) := \frac{H_t(w)}{g(|w|)}.$$  \hfill (C.20)

**Lemma C.10** (i) There exists a constant $C_3 > 0$ such that

$$|\nabla G_t(w)| \leq \frac{C_3}{t^2} \quad \text{for all } t \in ]0, 1] \text{ and } w \in \phi_t(U).$$

(ii) There exists a constant $c_3 > 0$ such that

$$|\nabla G_t(w)| \leq \frac{c_3}{t^2} e^{-1/t} \quad \text{for all } t \in ]0, 1] \text{ and } w \in \phi_t(U_t).$$

**Proof.** (i) We have that

$$\nabla \left( \frac{1}{g(|w|)} \right) = -\frac{g'(|w|)}{g(|w|)^2} \frac{w}{|w|}$$

and so

$$\nabla G_t(w) = -\frac{g'(|w|)}{g(|w|)^2} \frac{w}{|w|} H_t(w) + \frac{1}{g(|w|)} \nabla H_t(w).$$
Using \( g'(t) \in [0, 1] \), Lemma C.9(i) and Lemma C.8(i) and \(|w| \leq g(|w|)\) we therefore find that

\[
|\nabla G_t(w)| \leq \frac{1}{g(|w|)^2} |H_t(w)| + \frac{1}{g(|w|)} |\nabla H_t(w)| \\
\leq \frac{1}{g(|w|)^2} \frac{C_2}{t^2} |w|^2 + \frac{1}{g(|w|)} \frac{C_1}{t^2} |w| \\
\leq \frac{(C_1 + C_2)}{t^2}.
\]

The constant \( C_3 := C_1 + C_2 \) is as required.

(ii) Assume now \( z \in U_t \). Using Lemma C.9(ii) and Lemma C.8(ii) and estimating as above we obtain

\[
|\nabla G_t(w)| \leq (c_1 + c_2) \frac{1}{t^2} e^{-1/t}.
\]

The constant \( c_3 := c_1 + c_2 \) is as required.

**Lemma C.11**

(i) There exists a constant \( C_4 > 0 \) such that

\[
|G_t(w) - G_t(w')| \leq \frac{C_4}{t^2} |w - w'| \quad \text{for all } t \in [0, 1] \text{ and } w, w' \in \phi_t(U).
\]

(ii) There exists a constant \( c_4 > 0 \) such that

\[
|G_t(w) - G_t(w')| \leq \frac{c_4}{t^2} e^{-1/t} |w - w'| \quad \text{for all } t \in [0, 1] \text{ and } w, w' \in \phi_t(U_t).
\]

**Proof.** (i) Fix \( t \in [0, 1] \) and \( w = \phi_t(z), w' = \phi_t(z') \), and assume that \( U \) is a Lipschitz domain with Lipschitz constant \( \lambda \). We then find a smooth path \( \gamma: [0, 1] \to U \) such that \( \gamma(0) = z, \gamma(1) = z' \) and such that

\[
\text{length}(\gamma) = \int_0^1 |\gamma'(s)| \, ds \leq 2\lambda |z' - z|.
\]

(C.21)
Using Lemma C.10(i), Lemma C.7(ii), the estimate (C.21) and Lemma C.7(i) we can estimate

\[ |G_t(w') - G_t(w)| = \left| \int_0^1 \langle \nabla G_t(\phi_t(\gamma(s))), d\phi_t(\gamma(s))\gamma'(s) \rangle ds \right| \]

\[ \leq \frac{C_3}{t^2} \frac{1}{L} \int_0^1 |\gamma'(s)| \, ds \]

\[ \leq \frac{C_3}{t^2} \frac{1}{L} 2\lambda |z' - z| \]

\[ \leq \frac{C_3}{t^2} \frac{1}{L^2} 2\lambda |w' - w|. \]

The constant \( C_4 := 2C_3 \frac{1}{L^2} \lambda \) is as required.

(ii) Assume now \( z, z' \in U_t \). Since \( U \) is starshaped, we can assume that the path \( \gamma \) chosen above is contained in \( U_t \). Using Lemma C.10(ii) and estimating as above we obtain

\[ |G_t(w') - G_t(w)| \leq \frac{c_3}{t^2} e^{-1/t} \frac{1}{L^2} 2\lambda |w' - w|. \]

The constant \( c_4 := 2c_3 \frac{1}{L^2} \lambda \) is as required. \( \square \)

Our next goal is to extend the function \( G \) on \( \bigcup_{t \in [0,1]} \phi_t(U) \) to a continuous function \( \widehat{G} \) on \( [0,1] \times \mathbb{R}^{2n} \) having similar properties. We shall need two auxiliary lemmata.

**Lemma C.12** (McShane [27]) Consider a subset \( W \) of the metric space \( (X, d) \) and a function \( f : W \to \mathbb{R} \) which is \( \lambda \)-Lipschitz continuous. Then the function \( \overline{f} : X \to \mathbb{R} \) defined by

\[ \overline{f}(x) := \inf \{ f(w) + \lambda \, d(x, w) \mid w \in W \} \]

is a \( \lambda \)-Lipschitz continuous extension of \( f \).

**Lemma C.13** Assume that \( V \) is a subset of \( \mathbb{R}^{2n} \) which contains the origin and that the function \( h : V \cup B(2r) \to \mathbb{R} \) is \( \lambda_V \)-Lipschitz continuous on \( V \) and \( \lambda_B \)-Lipschitz continuous on \( B(2r) \). Then \( h \) is \( (2\lambda_V + \lambda_B) \)-Lipschitz continuous on \( V \cup B(r) \).

---

*1I'm grateful to Urs Lang for pointing out to me this reference.*
Proof. Fix \( w, w' \in V \cup B(r) \). If \( w, w' \in V \) or \( w, w' \in B(2r) \) then by assumption
\[
|h(w) - h(w')| \leq \max(\lambda_V, \lambda_B) |w - w'|.
\]
So assume that \( w \in V \setminus B(2r) \) and \( w' \in B(r) \). Then \( |w'| \leq r \leq \frac{|w|}{2} \) and so
\[
\left| \frac{|w|}{2} \right| \leq |w| - |w'| \leq |w - w'|.
\]
Since \( 0 \in V \) and \( 0 \in B(2r) \) we can now estimate
\[
|h(w) - h(w')| \leq |h(w) - h(0)| + |h(w') - h(0)| \\
\leq \lambda_V |w| + \lambda_B |w'| \\
\leq \frac{(2\lambda_V + \lambda_B) |w|}{2} \\
\leq (2\lambda_V + \lambda_B) |w - w'|,
\]
and so \( h \) is \((2\lambda_V + \lambda_B)\)-Lipschitz continuous on \( V \cup B(r) \).

Lemma C.14 There exists a continuous function \( \widehat{G} : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R} \) with the following properties.

(i) \( \widehat{G}(t, w) = G(t, w) \) for all \( t \in [0, 1] \) and \( w \in \phi_t(U) \).

(ii) There exists a constant \( C_5 > 0 \) such that
\[
|\widehat{G}_t(w) - \widehat{G}_t(w')| \leq \frac{C_5}{t^2} |w - w'| \quad \text{for all } t \in ]0, 1[ \text{ and } w, w' \in \mathbb{R}^{2n}.
\]

Proof. We shall first construct a function \( \widehat{G} : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R} \) meeting assertions (i) and (ii), and shall then verify that \( \widehat{G} \) is continuous.

We set \( \widehat{G}(0, w) = 0 \) for all \( w \in \mathbb{R}^{2n} \). Since \( H : \bigcup_{t \in [0, 1]} \{t\} \times \phi_t(U) \to \mathbb{R} \) is continuous, Lemma C.9 (ii) and the definition (C.11) of \( U_t \) imply that \( H(0, w) = 0 \) for all \( w \in U \). In view of definition (C.20) we therefore have \( G(0, w) = 0 \) for all \( w \in U \), and so assertion (i) holds for \( t = 0 \).

We now fix \( t \in ]0, 1[ \). We define the number \( R_t \) by
\[
R_t = \frac{L \epsilon}{2 \epsilon} e^{\epsilon t}.
\]
Fix \( w = \phi_t(z) \in B(2R_t) \). In view of the estimate (C.15) and the definition (C.22) we have
\[
|z| \leq \frac{|w|}{L} \leq \frac{2}{L} R_t = e^{1/t},
\]
and so \( z \in U_t \) in view of definition (C.11). Lemma C.11 (ii) therefore implies that the function \( G_t \) is \( \frac{C_4}{t^2} e^{-1/t} \)-Lipschitz continuous on \( \phi_t(U) \cap B(2R_t) \). According to Lemma C.12 the function \( \overline{G}_t : B(2R_t) \to \mathbb{R} \) defined by
\[
\overline{G}_t(x) := \inf \left\{ G_t(w) + \frac{C_4}{t^2} e^{-1/t} \cdot |x - w| \mid w \in \phi_t(U) \cap B(2R_t) \right\}
\]
(C.23)
is a \( \frac{C_4}{t^2} e^{-1/t} \)-Lipschitz extension of \( G_t \) to \( B(2R_t) \). In particular, the function \( \overline{G}_t : \phi_t(U) \cup B(2R_t) \to \mathbb{R} \),
\[
\overline{G}_t(x) := \begin{cases} G_t(x) & \text{if } x \in \phi_t(U), \\ \overline{G}_t(x) & \text{if } x \in B(2R_t), \end{cases}
\]
(C.24)
is well-defined. According to Lemma C.11 (i), \( \overline{G}_t \) is \( \frac{C_4}{t^2} \)-Lipschitz continuous on \( \phi_t(U) \), and according to the above, \( \overline{G}_t \) is \( \frac{C_4}{t^2} \)-Lipschitz continuous on \( B(2R_t) \). According to Lemma C.13, the restriction of \( \overline{G}_t \) to \( \phi_t(U) \cup B(R_t) \) is therefore \( \frac{C_4}{t^2} \)-Lipschitz continuous where we abbreviated
\[
C_5 := 2C_4 + c_4.
\]
Applying Lemma C.12 once more, we find that the function \( \widehat{G}_t : \mathbb{R}^{2n} \to \mathbb{R} \) defined by
\[
\widehat{G}_t(x) := \inf \left\{ \overline{G}_t(w) + \frac{C_5}{t^2} |x - w| \mid w \in \phi_t(U) \cup B(R_t) \right\}
\]
(C.25)
is a \( \frac{C_5}{t^2} \)-Lipschitz extension of the restriction of \( \overline{G}_t \) to \( \phi_t(U) \cup B(R_t) \). In particular,
\[
\widehat{G}(t, w) = G(t, w) \quad \text{for all } w \in \phi_t(U).
\]
The function \( \widehat{G} : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R} \) thus defined therefore meets assertion (i) for \( t \in [0, 1] \) and assertion (ii).

We are left with showing that the function \( \widehat{G} : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R} \) constructed in the previous two steps is continuous. The definitions (C.22), (C.23), (C.24) and (C.25) show that the functions \( \widehat{G}(:, x) : [0, 1] \to \mathbb{R} \) and \( \widehat{G}(t, :) : \mathbb{R}^{2n} \to \mathbb{R} \) are...
Continuous. This and the fact that the functions \( \tilde{G}(t, \cdot) \) are \( \frac{c_4}{t^2} \)-Lipschitz continuous imply that \( \tilde{G} : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R} \) is continuous. In order to show that \( \tilde{G} \) is also continuous at \((0, w)\) for each \( w \in \mathbb{R}^{2n} \) we fix \( w \in \mathbb{R}^{2n} \). We choose an open ball \( B_w \subset \mathbb{R}^{2n} \) centered at \( w \). In view of definition (C.22) we have \( R_t \to \infty \) as \( t \to 0^+ \). We therefore find \( t_0 > 0 \) such that \( B_w \subset B(R_t) \) for all \( t \in [0, t_0] \). We fix \( t \in [0, t_0] \) and \( w' \in B_w \). Recalling the definition of \( \tilde{G}(t, w') = \tilde{G}_t(w') \) we see that

\[
\tilde{G}_t(w') = \overline{G}_t(w') = \overline{G}_t(w') = \inf \left\{ G_t(v) + \frac{c_4}{t^2} e^{-1/t} |w' - v| \mid v \in \phi_t(U) \cap B(2R_t) \right\}.
\]

Since \( 0 = \phi_t(0) \in \phi_t(U) \cap B(2R_t) \) and \( G_t(0) = H_t(0) = 0 \) we conclude that

\[
\tilde{G}_t(w') \leq \frac{c_4}{t^2} e^{-1/t} |w'|.
\]

(C.26)

Moreover, we recall from the beginning of the proof of Lemma C.14 that \( G_t \) is \( \frac{c_4}{t^2} e^{-1/t} \)-Lipschitz continuous on \( \phi_t(U) \cap B(2R_t) \). This and \( G_t(0) = 0 \) yield

\[
|G_t(v)| \leq \frac{c_4}{t^2} e^{-1/t} |v| \quad \text{for all } v \in \phi_t(U) \cap B(2R_t).
\]
Therefore,
\[
G_t(v) + \frac{c_4}{t^2} e^{-1/t} |w' - v| \geq -|G_t(v)| + \frac{c_4}{t^2} e^{-1/t} |w' - v|
\]
\[
\geq \frac{c_4}{t^2} e^{-1/t} (|v| + |w' - v|)
\]
\[
\geq -\frac{c_4}{t^2} e^{-1/t} |w'|
\]
for all \( v \in \phi_t(U) \cap B(2R_t) \). We conclude that
\[
\hat{G}_t(w') \geq -\frac{c_4}{t^2} e^{-1/t} |w'|.
\] (C.27)

The estimates (C.26) and (C.27), which hold for all \( t \in ]0, t_0] \) and \( w' \in B_w \), now imply that
\[
|\hat{G}_t(w')| \leq \frac{c_4}{t^2} e^{-1/t} |w'| \quad \text{for all } t \in ]0, t_0] \text{ and } w' \in B_w
\]
and so \( \hat{G} \) is continuous at \((0, w)\). This completes the proof of Lemma C.14. \( \square \)

Let now \( A \) be a subset of \( U \) whose closure in \( \mathbb{R}^{2n} \) is contained in \( U \). Since also the origin is contained in \( U \), we can assume that \( A \) is closed and \( 0 \in A \). We abbreviate
\[
\mathcal{A} := \bigcup_{t \in [0,1]} \{ t \} \times \phi_t(A).
\]
The next step is to smoothen \( \hat{G} \) in the variable \( w \) in such a way that the smoothened function \( \tilde{G} \) coincides with \( \hat{G} \) on \( \mathcal{A} \). We shall first construct a smooth function \( G^* \) which approximates \( \hat{G} \) very well and shall then obtain \( \tilde{G} \) by interpolating between \( \hat{G} \) and \( G^* \).

Since \( \mathbb{R}^{2n} \) is a normal space, we find an open set \( V \) in \( \mathbb{R}^{2n} \) such that \( A \subset V \subset \overline{V} \subset U \). Then
\[
\phi_t(A) \subset \phi_t(V) \subset \phi_t(\overline{V}) = \overline{\phi_t(V)} \subset \phi_t(U) \quad \text{for all } t \in [0, 1].
\] (C.28)
We abbreviate
\[
\mathcal{V} := \bigcup_{t \in [0,1]} \{ t \} \times \phi_t(V).
\]
Since \( \mathcal{A} \) is closed and \( \mathcal{V} \) is open in \([0,1] \times \mathbb{R}^{2n}\), we find a smooth function \( f : [0, 1] \times \mathbb{R}^{2n} \to [0, 1] \) such that
\[
f|_{\mathcal{A}} = 1 \quad \text{and} \quad f|_{[0,1] \times \mathbb{R}^{2n} \setminus \mathcal{V}} = 0.
\] (C.29)
We say that a continuous function \( F : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R} \) is smooth in the variable \( w \in \mathbb{R}^{2n} \) if all derivatives \( D^k F_t(w) \) of \( F \) with respect to \( w \) exist and are continuous on \([0, 1] \times \mathbb{R}^{2n}\).

**Lemma C.15** There exists a continuous function \( G^* : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R} \) which is smooth in the variable \( w \in \mathbb{R}^{2n} \) and has the following properties.

(i) \( |\nabla f_t(w)| |G^*_t(w) - \hat{G}_t(w)| \leq \frac{C_5}{t^2} \) for all \( t \in [0, 1] \) and \( w \in \mathbb{R}^{2n} \).

(ii) \( |\nabla G^*_t(w)| \leq \frac{2C_5}{t^2} \) for all \( t \in [0, 1] \) and \( w \in \mathbb{R}^{2n} \).

**Proof.** For each \( l \in \mathbb{N} \) we define the open subset \( V_l \) of \([0, 1] \times \mathbb{R}^{2n} \) by

\[
V_l := \left\{(t, w) \in [0, 1] \times \mathbb{R}^{2n} \mid |\nabla f_t(w)| < l \right\}. \tag{C.30}
\]

Then there exists a smooth partition of unity \( \{\theta_i\}_{i \in \mathbb{N}} \) on \([0, 1] \times \mathbb{R}^{2n} \) such that for each \( i \) the support \( \text{supp} \theta_i \) is compact and contained in some \( V_l \). We let \( l_i \) be a number such that \( \text{supp} \theta_i \subset V_{l_i} \). Since \( \{\text{supp} \theta_i\} \) form a locally finite covering of \([0, 1] \times \mathbb{R}^{2n} \), the set

\[
\Theta_i := \{ j \in \mathbb{N} \mid \text{supp} \theta_i \cap \text{supp} \theta_j \neq \emptyset \}
\]

is finite; let its cardinality be \( m_i \). We set

\[
M_i := \max \{ m_j \mid j \in \Theta_i \}. \tag{C.31}
\]

Since the functions \( \theta_i \) have compact support, the numbers

\[
\mu_i := \max \left\{ |\nabla \theta_i^j(w)| \mid (t, w) \in [0, 1] \times \mathbb{R}^{2n} \right\} + 1 \tag{C.32}
\]

are finite. We define positive numbers \( r_i \) by

\[
r_i := \frac{1}{l_i M_i \mu_i}. \tag{C.33}
\]

We next choose a smooth bump function \( K : \mathbb{R}^{2n} \to [0, \infty) \) such that \( \text{supp} K \subset B(1) \) and \( \int_{\mathbb{R}^{2n}} K(v) \, dv = 1 \). We abbreviate

\[
\kappa := \max \left\{ |\nabla K(v)| \mid v \in \mathbb{R}^{2n} \right\}.
\]
For each $i$ we define a smooth function $K_i: \mathbb{R}^{2n} \to [0, \infty]$ by

$$K_i(w) := \frac{1}{r_i^{2n}} K\left(\frac{w}{r_i}\right).$$

Then $\text{supp } K_i \subset B(r_i)$ and $\int_{\mathbb{R}^{2n}} K_i(v) \, dv = 1$, and

$$|\nabla K_i(w)| \leq \frac{1}{r_i^{2n+1}} \kappa \quad \text{for all } w \in \mathbb{R}^{2n}. \quad (C.34)$$

Let $\hat{G}$ be the function guaranteed by Lemma C.14. For each $i$ we define the function $G_i^*: [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ as the convolution

$$G_i^*(t, w) := \left(\hat{G}_t \ast K_i\right)(w) \equiv \int_{\mathbb{R}^{2n}} \hat{G}_t(v) K_i(w - v) \, dv. \quad (C.35)$$

Since for each $t$ the function $\hat{G}_t$ is continuous and since $K_i$ is smooth, the function $w \mapsto G_i^*(t, w)$ is smooth and

$$D^k G_i^*(t, w) = \int_{\mathbb{R}^{2n}} \hat{G}_t(v) D^k K_i(w - v) \, dv, \quad k = 0, 1, 2, \ldots \quad (C.36)$$

(see, e.g., [16, Chapter 2, Theorem 2.3]). The function $\hat{G}$ is continuous, and $D^k K_i$ is continuous and has compact support and is thus uniformly continuous. Formula (C.36) therefore shows that $D^k G_i^*$ is continuous, $k = 0, 1, 2, \ldots$, and so $G_i^*$ is continuous and smooth in $w$. It follows that the function $G^* : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ defined by

$$G^*(t, w) := \sum_i \theta_i(t, w) G_i^*(t, w) \quad (C.37)$$

is continuous and smooth in $w$. In order to prove assertions (i) and (ii) we fix $t \in [0, 1]$ and abbreviate

$$\theta_i(w) = \theta_i(t, w), \quad \hat{G}(w) = \hat{G}(t, w), \quad G_i^*(w) = G_i^*(t, w), \quad G^*(w) = G^*(t, w).$$

**Proof of (i).** Using the definition (C.35) of the function $G_j^*$ and the identity $\int_{\mathbb{R}^{2n}} K_j(v) \, dv = 1$ we find

$$G_j^*(w) - \hat{G}(w) = \int_{\mathbb{R}^{2n}} \left(\hat{G}(v) - \hat{G}(w)\right) K_j(w - v) \, dv$$

$$= \int_{\mathbb{R}^{2n}} \left(\hat{G}(w - v) - \hat{G}(w)\right) K_j(v) \, dv$$
and so, together with Lemma C.14 (ii),
\[
|G^*_j(w) - \hat{G}(w)| \leq \int_{\mathbb{R}^{2n}} |\hat{G}(w - v) - \hat{G}(w)| \, K_j(v) \, dv
\]
\[
\leq \int_{B(r_j)} \frac{C_5}{t^2} |v| \, K_j(v) \, dv
\]
\[
\leq \frac{C_5}{t^2} r_j \int_{B(r_j)} K_j(v) \, dv
\]
\[
= \frac{C_5}{t^2} r_j. \tag{C.38}
\]

If \( \nabla f_t(w) = 0 \), assertion (i) is obvious. So assume \( |\nabla f_t(w)| > 0 \). Recall from the definitions (C.31) and (C.32) that \( M_j \geq 1 \) and \( \mu_j \geq 1 \). This, the definition (C.33) of \( r_j \), the inclusion \( \text{supp} \theta_j \subseteq V_t \) and the definition (C.30) of \( V_t \) yield
\[
r_j = \frac{1}{l_j M_j \mu_j} \leq \frac{1}{l_j} \leq \frac{1}{|\nabla f_t(w)|} \quad \text{for all } w \in \text{supp} \theta_j. \tag{C.39}
\]

The definition (C.37) of \( G^* \) and the estimates (C.38) and (C.39) now yield
\[
|G^*(w) - \hat{G}(w)| = \left| \sum_j \theta_j(w) \left( G^*_j(w) - \hat{G}(w) \right) \right|
\]
\[
\leq \sum_j \theta_j(w) \frac{C_5}{t^2} \frac{1}{|\nabla f_t(w)|}
\]
\[
= \frac{C_5}{t^2} \frac{1}{|\nabla f_t(w)|}
\]

and so assertion (i) follows.

**Proof of (ii).** Using the definition (C.37) of \( G^* \) and the identities \( \sum_j \theta_j(w) = \sum_j \theta_j(w') = 1 \) we compute that for all \( w, w' \in \mathbb{R}^{2n} \),
\[
G^*(w') - G^*(w) = \sum_j \theta_j(w') G^*_j(w') - \sum_j \theta_j(w) G^*_j(w)
\]
\[
= \sum_j \left( \theta_j(w') - \theta_j(w) \right) \left( G^*_j(w') - \hat{G}(w') \right)
\]
\[
+ \sum_j \theta_j(w) \left( G^*_j(w') - G^*_j(w) \right). \tag{C.40}
\]
Fix now \( w \). We choose \( i \) such that \( \theta_i(w) > 0 \), and we choose an open ball \( B_{w_0} \subset \mathbb{R}^{2n} \) centered at \( w \) such that \( B_{w_0} \subset \text{supp} \theta_i \). Fix \( w' \in B_{w_0} \). In view of the mean value theorem and the definition (C.32) of \( \mu_j \) we find that

\[
|\theta_j(w') - \theta_j(w)| \leq \max_{v \in B_{w_0}} |\nabla \theta_j(v)| |w' - w| \leq \mu_j |w' - w| \tag{C.41}
\]

and the estimate (C.38) with \( w \) replaced by \( w' \) yields

\[
\left| G_j^*(w') - \widehat{G}(w') \right| \leq \frac{C_5}{\ell^2} r_j. \tag{C.42}
\]

The definition (C.31) of \( M_j \) implies that \( M_j \geq m_i \) whenever \( j \in \Theta_i \), and so

\[
\sum_{j \in \Theta_i} \frac{1}{M_j} \leq \sum_{j \in \Theta_i} \frac{1}{m_i} = 1 \tag{C.43}
\]

in view of the definition of \( m_i \). The definition (C.33) of \( r_j \) and the inequalities \( l_j \geq 1 \) and (C.43) yield

\[
\sum_{j \in \Theta_i} \mu_j r_j = \sum_{j \in \Theta_i} \mu_j \frac{1}{l_j M_j} \leq \sum_{j \in \Theta_i} \frac{1}{M_j} \leq 1. \tag{C.44}
\]

Since \( w, w' \in B_{w_0} \subset \text{supp} \theta_i \) we have \( \theta_j(w') - \theta_j(w) = 0 \) if \( j \notin \Theta_i \). This and the estimates (C.41), (C.42) and (C.44) now show that

\[
\left| \sum_j (\theta_j(w') - \theta_j(w)) \left( G_j^*(w') - \widehat{G}(w') \right) \right| \leq \sum_{j \in \Theta_i} \mu_j |w' - w| \frac{C_5}{\ell^2} r_j
\]

\[
\leq \frac{C_5}{\ell^2} |w' - w|. \tag{C.45}
\]

Next, the definition (C.35) of \( G_j^* \) and the identity \( \int_{\mathbb{R}^{2n}} K_j(v) \, dv = 1 \) yield

\[
G_j^*(w') - G_j^*(w) = \int_{\mathbb{R}^{2n}} \widehat{G}(v) \left( K_j(w' - v) - K_j(w - v) \right) \, dv
\]

\[
= \int_{\mathbb{R}^{2n}} \left( \widehat{G}(w' - v) - \widehat{G}(w - v) \right) K_j(v) \, dv.
\]

Together with Lemma C.14 (ii) we obtain

\[
\left| G_j^*(w') - G_j^*(w) \right| \leq \frac{C_5}{\ell^2} \int_{\mathbb{R}^{2n}} |w' - w| K_j(v) \, dv = \frac{C_5}{\ell^2} |w' - w|
\]
and so
\[ \left| \sum_j \theta_j(w) \left( G_j^*(w') - G_j^*(w) \right) \right| \leq \frac{C_5}{t^2} \left| w' - w \right|. \] (C.46)

The identity (C.40) and the estimates (C.45) and (C.46) now imply
\[ \left| G^*(w') - G^*(w) \right| \leq \frac{2C_5}{t^2} \left| w' - w \right|. \]

Since \( w' \in B_w \) was arbitrary, we conclude that
\[ \left| \nabla G^*(w) \right| \leq \frac{2C_5}{t^2} \]
and so assertion (ii) follows. The proof of Lemma C.15 is complete. \( \square \)

**Lemma C.16** There exists a continuous function \( \tilde{G} : [0, 1] \times \mathbb{R}^{2n} \rightarrow \mathbb{R} \) which is smooth in the variable \( w \in \mathbb{R}^{2n} \) and has the following properties.

(i) \( \tilde{G}(t, w) = G(t, w) \) for all \( t \in [0, 1] \) and \( w \in \phi_t(A) \).

(ii) There exists a constant \( C_6 > 0 \) such that
\[ \left| \nabla \tilde{G}_t(w) \right| \leq \frac{C_6}{t^2} \quad \text{for all} \quad t \in [0, 1] \quad \text{and} \quad w \in \mathbb{R}^{2n}. \]

**Proof.** Let \( f : [0, 1] \times \mathbb{R}^{2n} \rightarrow [0, 1] \) be the smooth function chosen before Lemma C.15, and let \( G \) and \( G^* \) be the continuous functions on \( [0, 1] \times \mathbb{R}^{2n} \) guaranteed by Lemma C.14 and Lemma C.15. We define a continuous function \( \tilde{G} : [0, 1] \times \mathbb{R}^{2n} \rightarrow \mathbb{R} \) by
\[ \tilde{G}(t, w) := f(t, w)\hat{G}(t, w) + (1 - f(t, w))G^*(t, w). \]

The inclusions (C.28) and the identities (C.29), Lemma C.14 (i) and the fact that \( G^* \) is smooth in \( w \) imply that \( \tilde{G} \) is smooth in \( w \) and that assertion (i) holds true. In order to verify assertion (ii) we fix \( t \in [0, 1] \). We first assume \( w \in \phi_t(U) \). On \( \phi_t(U) \) we have \( \tilde{G}_t = G_t \), and so
\[ \nabla \tilde{G}_t(w) = \nabla f_t(w) \left( G_t(w) - G_t^*(w) \right) + f_t(w)\nabla G_t(w) + (1 - f_t(w)) \nabla G_t^*(w). \]
In view of Lemma C.15 (i), Lemma C.10 (i) and Lemma C.15 (ii) we can therefore estimate
\[
\left| \nabla \tilde{G}_t(w) \right| \leq \left| \nabla f_t(w) \right| \left| G_t^*(w) - \rho_t(w) \right| + \left| \nabla G_t(w) \right| + \left| \nabla G_t^*(w) \right|.
\]
\[
\leq \frac{C_5}{t^2} + \frac{C_3}{t^2} + \frac{2C_5}{t^2}.
\]

We next assume \( w \in \mathbb{R}^{2n} \setminus \overline{\phi_t(V)} \). On \( \mathbb{R}^{2n} \setminus \overline{\phi_t(V)} \) we have \( f_t \equiv 0 \), and so
\[
\left| \nabla \tilde{G}_t(w) \right| = \left| \nabla G_t^*(w) \right| \leq \frac{2C_5}{t^2}.
\]
Setting \( C_6 := C_3 + 3C_5 \), assertion (ii) follows. The proof of Lemma C.16 is complete. \( \square \)

We are now in a position to define the desired extension \( \tilde{H} \) of \( H \). Let \( g \) be the function chosen in (C.19) and let \( \tilde{G} \) be the function guaranteed by Lemma C.16.

**Lemma C.17** The continuous function \( \tilde{H} : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R} \) defined by
\[
\tilde{H}(t, w) = \tilde{H}_t(w) := g(|w|) \tilde{G}_t(w).
\]

is smooth in the variable \( w \in \mathbb{R}^{2n} \) and has the following properties.

(i) \( \tilde{H}(t, w) = H(t, w) \) for all \( t \in [0, 1] \) and \( w \in \phi_t(A) \).

(ii) There exists a constant \( C > 0 \) such that
\[
\nabla \tilde{H}_t(w) \leq \frac{C}{t^2} (|w| + 1) \quad \text{for all } t \in [0, 1] \text{ and } w \in \mathbb{R}^{2n}.
\]

**Proof.** Since \( g : [0, \infty[ \to [1, \infty[ \) is smooth and \( \tilde{G} : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R} \) is continuous, the function \( \tilde{H} \) is indeed continuous, and since \( g(r) = 1 \) if \( r \leq \frac{1}{2} \) and \( \tilde{G} \) is smooth in \( w \), the function \( \tilde{H} \) is smooth in \( w \). Assertion (i) follows from the definition (C.47) of \( \tilde{H} \), from Lemma C.16 (i) and from the definition (C.20) of \( G \).

In order to verify assertion (ii) we fix \( t \in [0, 1] \) and \( w \in \mathbb{R}^{2n} \). Using definition (C.47) we compute
\[
\nabla \tilde{H}_t(w) = g'(|w|) \frac{w}{|w|} \tilde{G}_t(w) + g(|w|) \nabla \tilde{G}_t(w).
\]
Since \(0 \in A\) and \(\phi_t(0) = 0\) we have, together with equation (C.8),
\[
\tilde{G}_t(0) = G_t(0) = H_t(0) = 0.
\]
This, the mean value theorem and Lemma C.16 (ii) yield
\[
|\tilde{G}_t(w)| \leq \frac{C_6}{t^2} |w| \quad \text{and} \quad |\nabla \tilde{G}_t(w)| \leq \frac{C_6}{t^2}. \tag{C.50}
\]
Using the identity (C.49), the estimates (C.50) and the estimates \(|g'(r)| \leq 1\) and \(g(r) \leq r + 2\) holding for all \(r \geq 0\) we can estimate
\[
|\nabla H_t(w)| \leq \frac{C_6}{t^2} |w| + (|w| + 2) \frac{C_6}{t^2} = \frac{2C_6}{t^2} (|w| + 1).
\]
Setting \(C := 2C_6\) assertion (ii) follows. The proof of Lemma C.17 is complete. \(\Box\)

Theorem C.6 is a consequence of Lemma C.17: The time-dependent vector field \(\nabla \tilde{H}_t(w)\) on \([0, 1] \times \mathbb{R}^{2n}\) is continuous, and since it is smooth in \(w\), it is locally Lipschitz continuous in \(w\). This and assertion (ii) of Lemma C.17 imply that the Hamiltonian system associated with \(\tilde{H}\) can be solved for all \(t \in [0, 1]\).
We define \(\Phi_A\) to be the resulting time-1-map. Since \(\nabla \tilde{H}_t(w)\) is continuous and smooth in \(w\), the map \(\Phi_A\) is smooth (see [1, Proposition 9.4]), and so \(\Phi_A\) is a globally defined symplectomorphism of \(\mathbb{R}^{2n}\). Moreover, Lemma C.17 (i) shows that \(\Phi_A|_A = \varphi|_A\). The proof of Theorem C.6 is finally complete. \(\Box\)

Remark C.18 Proceeding as in Step 2 we obtain a smooth Hamiltonian
\[
H_A : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}
\]
which generates the symplectomorphism \(\Phi_A\) and is such that \(H_A|_A = \tilde{H}|_A\). However, \(H_A\) might not be \(C^0\)-close to \(\tilde{H}\), and \(\nabla H_A\) might not be linearly bounded.
D Computer programs

All the Mathematica programs of this appendix can be found under

For convenience, in the programs (but not in the text) both the $u$-axis and the
capacity-axis are rescaled by the factor $1/\pi$.

D.1 The estimate $s_{EB}$

We fix $a$ and $u_1$ and try to embed $E(\pi, a)$ into $B^4(2\pi + (1 - 2\pi/a)u_1)$ by
the multiple folding procedure specified in 3.3.1. If $u_1$ is admissible, we set
$A(a, u_1) = 2\pi + (1 - 2\pi/a)u_1$ and $A(a, u_1) = a$ otherwise.

A[a_, u1_] :=
  Block[{A = 2 + (1 - 2/a)u1},
    i = 2;
    u1 = (a+1)/(a-1)u1 - a/(a-1);
    r1 = a - u1 - u1;
    l1 = r1/a;
    While[True,
      Which[EvenQ[i],
        If[r1 < u1,
          Return[A],
          If[u1 <= 2l1,
            Return[a],
            i++;
            u1 = a/(a-2)(u1 - 2l1);
            r1 = r1 - u1;
            l1 = l1;
            l1 = r1/a
          ],
        ],
        OddQ[i],
        If[r1 < u1 + l1,
      ]
    ]
]
This program just does what we proposed to do in 3.3.1 in order to decide whether $u_i$ is admissible or not. Note, however, that in the OddQ[$i$]-part we do not check whether the stairs $S_{i+1}$ are contained in $T^4(A)$. This negligence does not cause troubles, since if $S_{i+1} \not\subseteq T^4(A)$, then $u_1$ will be recognized to be non-admissible in the subsequent EvenQ[$i+1$]-part. Indeed, recall that $S_{i+1} \subseteq T^4(A)$ is equivalent to $r_{i+1} = (a/\pi)l_{i+1} \geq u_{i+1}$; hence $r_{i+1} = (a/\pi)l_{i+1} > u_{i+1}$ and $u_{i+1} \leq 2l_{i+1}$.

We denote the infimum of the admissible $u_i$'s again by $u_0 = u_0(a)$. Then $A(a, u_1) = a$ for $u_1 < u_0$, and $A(a, u_1) = 2\pi + (1 - 2\pi/a)u_1$ is a linear increasing function for $u_1 > u_0$. Since, by (3.8), we can assume that $u_0 \leq a/2$, we have $A(a, u_0) \leq A(a, a/2) = \pi + a/2 < a$. Therefore, $u_0$ is found up to accuracy $acc/2$ by the following bisectional algorithm.

```mathematica
u0[a_, acc_] :=
Block[{},
b = a/(a+1);
c = a/2;
ul = (b+c)/2;
While[(c-b)/2 > acc/2,
  If[A[a,ul] < a, c=ul, b=ul];
  ul = (b+c)/2]
Return[ul]
]
```

Here the choice $b = a\pi/(a + \pi)$ is also based on (3.8). Up to accuracy $acc$, the resulting estimate $s_{EB}(a)$ is given by

```mathematica
sEB[a_, acc_] := 2 + (1-2/a)u0[a, acc].
```
D.2 The estimate $s_{EC}$

We fix $a > \pi$. As we have seen in Remark 3.15.1, folding only once does not yield an optimal embedding. We therefore fold at least twice and hence choose $u_1 \in \left[\frac{\pi a}{2}, \frac{\pi}{2}\right]$. We first calculate the height of the image of $T(a, \pi)$ determined by $a$ and $u_1$. The following program is best understood by looking at Figure 25.

\begin{verbatim}
  h[a_, ul_] :=
    Block[{ll = 1 - ul/a},
      i = 2;
      ul = (a + 1)/(a - 1) ul - a/(a - 1);
      ri = a - ul - ui;
      li = ri/a;
      hi = 2 ll;
      While[ri > ul + ll - li,
        i++;
        ul = (a + 1)/(a - 1) ul;
        ri = ri - ui;
        lj = li;
        li = ri/a;
        If[EvenQ[i], hi = hi + 2 lj] ];
      Which[EvenQ[i],
        hi = hi + li,
        OddQ[i],
        hi = hi + Max[lj, 2 li] ];
    Return[hi]
    ]

As explained in 3.4.1, the optimal folding point $u_0(a)$ is the $u$-coordinate of the unique intersection point of the graphs of $h(a, u_1)$ and $w(a, u_1)$. It can thus be found by a bisectional algorithm.

\begin{verbatim}
  u0[a_, acc_] :=
    Block[{},
      b = a/(a + 1);
      c = a/2;
      ul = (b + c)/2;
  \end{verbatim}
While[(c-b)/2 > acc/2, 
   If[h[a,u1] > 1+(1-1/a)u1, b=u1, c=u1]; 
   u1 = (b+c)/2 
]; 
Return[u1] 
]

Up to accuracy acc, the resulting estimate $s_{EC}(a)$ is given by

$$s_{EC}[a_, \text{acc}_:] := 1+(1-1/a)u0[a, \text{acc}].$$
References


Lebenslauf
